

Instanton effects in correlation functions on the light-cone

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ABSTRACT: We study instanton corrections to four-point correlation function of half-BPS operators in $\mathcal{N} = 4$ SYM in the light-cone limit when operators become null separated in a sequential manner. We exploit the relation between the correlation function in this limit and light-like rectangular Wilson loop to determine the leading instanton contribution to the former from the semiclassical result for the latter. We verify that the light-like rectangular Wilson loop satisfies anomalous conformal Ward identities nonperturbatively, in the presence of instantons. We then use these identities to compute the leading instanton contribution to the light-like cusp anomalous dimension and to anomalous dimension of twist-two operators with large spin.

KEYWORDS: Extended Supersymmetry, Solitons Monopoles and Instantons, Supersymmetric Gauge Theory, Wilson, 't Hooft and Polyakov loops

ARXIV EPRINT: [1704.00448](https://arxiv.org/abs/1704.00448)

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Contents

1	Introduction	1
2	Correlation functions in the light-cone limit	4
3	Light-like Wilson loop in the instanton background	9
3.1	Conformal Ward identities	9
3.2	Semiclassical approximation	10
3.3	Cusp anomalous dimension	11
4	Instanton contribution to the cusp anomalous dimension	14
4.1	Wilson loop in the fundamental representation	14
4.2	Leading term	15
4.3	Conformal gauge	16
4.4	Results	17
5	Concluding remarks	18
A	Conventions	20
B	Conformal properties	21
C	D-functions in the light-cone limit	23
D	Gauge field in the instanton background	24

1 Introduction

Maximally supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ SYM) possesses a remarkable electric-magnetic duality, also known as S -duality [1–3]. It establishes the equivalence between certain correlation functions computed at weak and at strong coupling. Testing the S -duality proves to be a complicated task as it requires understanding these functions at strong coupling [4, 5].

In the case of $\mathcal{N} = 4$ SYM with the $SU(N)$ gauge group, the S -duality predicts that the correlation functions of half-BPS operators $G_n = \langle O_{20'}(x_1) \dots O_{20'}(x_n) \rangle$ should be invariant under modular $SL(2, \mathbb{Z})$ transformations acting on the complexified coupling constant $\tau = \theta/(2\pi) + 4\pi i/g^2$. Two- and three-point correlation functions of half-BPS operators are protected from quantum corrections and trivially verify the S -duality. For higher number of points, the functions G_n receive quantum corrections and have a nontrivial dependence on the coupling constant. In order to test the S -duality, we have to find

perturbative contribution to G_n for finite N and supplement it with instanton corrections. Although these corrections are exponentially small at large N , they play a crucial role in restoring the S -duality.

In this paper we study instanton effects in four-point correlation function of half-BPS operators. Quantum corrections to G_4 are described by a single function $\mathcal{G}(u, v)$ of two cross ratios $u = x_{12}^2 x_{34}^2 / (x_{13}^2 x_{24}^2)$ and $v = x_{23}^2 x_{14}^2 / (x_{13}^2 x_{24}^2)$ (with $x_{ij}^2 = (x_i - x_j)^2$). It has the following general form at weak coupling

$$\mathcal{G}(u, v) = \Phi_0(u, v; g^2) + \sum_{n \geq 1} \left(e^{2\pi i n \tau} + e^{-2\pi i n \bar{\tau}} \right) \Phi_n(u, v; g^2), \tag{1.1}$$

where the first term $\Phi_0(u, v; g^2)$ is a perturbative correction and the second one is a non-perturbative correction due to n (anti) instantons. The function $\Phi_n(u, v)$ describes the contribution of quantum fluctuations of instantons and runs in powers of g^2 .¹

At present, the instanton corrections to (1.1) are known to the lowest order in g^2 . The corresponding function $\Phi_n^{(0)} = \Phi_n(u, v; 0)$ can be found in the semiclassical approximation following the standard approach (for a review, see [6–8]). In this approximation the quantum fluctuations are frozen and the correlation function is given by a finite-dimensional integral over the collective coordinates of instantons. An explicit expression for the function $\Phi_n^{(0)}(u, v)$ is known in one-instanton sector ($n = 1$) as well as for an arbitrary number of instantons n in the large N limit. To go beyond the semiclassical approximation, we have to include quantum fluctuations of instantons. Their contribution to $\Phi_n(u, v; g^2)$ scales as $O(g^{2\ell})$ where integer positive ℓ counts the number of instanton loops. It is much more difficult to compute such corrections and a little progress has been made over the last decade.

Perturbative corrections to (1.1) are known to have some additional structure [9, 10] which allows us to construct integral representation for the function $\Phi_0(u, v; g^2)$ to any order in g^2 without going through a Feynman diagram calculation. Moreover, $\mathcal{N} = 4$ SYM is believed to be integrable in the planar limit [11]. In application to the correlation function (1.1), this opens up the possibility to determine perturbative contribution $\Phi_0(u, v; g^2)$ for arbitrary 't Hooft coupling at large N [12, 13]. A natural question is whether some of these remarkable properties survive in (1.1) in the instanton sector.

As a first example, we examine behaviour of the correlation function G_4 at short distances $x_{12}^2 \rightarrow 0$, or equivalently for $u \rightarrow 0$ and $v \rightarrow 1$. As follows from the OPE, the leading contribution to (1.1) in this limit, $\mathcal{G}(u, v) \sim C_K^2 u^{\gamma_K/2}$, comes from Konishi operator, un-protected operator with smallest scalling dimension $\Delta = 2 + \gamma_K$. The anomalous dimension of this operator, γ_K , and its structure constant in the OPE of two half-BPS operators, C_K , have expansion at weak coupling similar to (1.1). The leading instanton correction to γ_K and C_K can be found by examining the asymptotic behaviour of the second term on the right-hand side of (1.1). In this way one obtains, using the known results for the function G_4 in the semiclassical approximation, that γ_K and C_K do not receive $O(e^{2\pi i \tau})$ instan-

¹In general, the functions Φ_0 and Φ_n could also receive $O(e^{-16\pi^2/g^2})$ corrections due to instanton-antiinstanton configurations, but their status remains unclear.

ton correction [14–17]. Thus, the leading corrections to γ_K and C_K can only come from quantum instanton corrections to (1.1).

The same quantities can be also extracted from the two- and three-point correlation functions, $\langle K(x)K(0) \rangle \sim 1/(x^2)^{2+\gamma_K}$ and $C_K \sim \langle O_{20'} O_{20'} K \rangle$, respectively. Computation of these correlation functions in the semiclassical approximation yields the following result for the leading instanton corrections $C_K^{(\text{inst})} = O(g^2 e^{2\pi i\tau})$ and $\gamma_K^{(\text{inst})} = O(g^4 e^{2\pi i\tau})$ (explicit expressions can be found in [18]). Notice that both expressions have additional factors of g^2 as compared with the semiclassical $O(e^{2\pi i\tau})$ contribution to (1.1). To get the same expressions for C_K and γ_K from the four-point correlation function G_4 , one would have to take into account one- and two-loop instanton corrections to (1.1), respectively.

This example illustrates a hidden simplicity of instanton effects — finding the leading quantum instanton contribution to the four-point correlation function at short distances, $\mathcal{G}(u, v) \sim C_K^2 u^{\gamma_K/2}$, can be mapped into a semiclassical calculation of two- and three-point correlation functions of the Konishi operator [18].

We show in this paper that analogous phenomenon also happens for G_4 in the light-like limit $x_{i,i+1}^2 \rightarrow 0$ (with $x_{i+4} \equiv x_i$) when four half-BPS operators become light-like separated in a sequential manner. In this limit, the correlation function is expected to have the following form [19]

$$G_4 = \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} \mathcal{G}(u, v), \tag{1.2}$$

where the product of four scalar propagators defines the leading asymptotic behavior and the function $\mathcal{G}(u, v)$ is given by (1.1) for $u, v \rightarrow 0$. At weak coupling, perturbative corrections to (1.1) are enhanced by powers of logarithms of u and v . Such corrections can be summed to all orders in g^2 leading to [19, 20]

$$\mathcal{G}(u, v) = W_4 \times J = \exp \left[-\frac{1}{2} \Gamma_{\text{cusp}}(g^2) \ln u \ln v + \dots \right]. \tag{1.3}$$

Here dots denote subleading corrections and $\Gamma_{\text{cusp}}(g^2)$ is the light-like cusp anomalous dimension in the adjoint representation of the $SU(N)$.

The same anomalous dimension controls divergences of cusped light-like Wilson loops and its appearance in (1.3) is not accidental. As was shown in [19], the leading asymptotics of the function $\mathcal{G}(u, v)$ is described by (an appropriately regularized) rectangular light-like Wilson loop

$$W_4 = \langle 0 | \text{tr}_A P \exp \left(ig \oint_{C_4} dx \cdot A(x) \right) | 0 \rangle, \tag{1.4}$$

evaluated along light-like rectangle C_4 with vertices at points x_i . The subscript A indicates that W_4 is defined in the adjoint representation of the $SU(N)$. The subleading (logarithmically enhanced) corrections to (1.3) come from the so-called jet factor J . Its form is fixed by the crossing symmetry of the four-point correlation function [20].

In this paper, we compute the leading instanton correction to the four-point correlation function (1.1) in the light-cone limit $x_{i,i+1}^2 \rightarrow 0$. Notice that this limit is Minkowskian in

nature whereas instantons are defined in Euclidean signature. To find instanton corrections to (1.1), we shall determine the function $\mathcal{G}(u, v)$ in Euclidean domain of u and v and, then, analytically continue it to $u, v \rightarrow 0$.

As in the previous example, we start with the semiclassical approximation to (1.1). As was shown in [21], the instanton corrections to $\mathcal{G}(u, v)$ scale in this approximation as $O(uv)$ and, therefore, they do not modify the asymptotic behaviour (1.3). To go beyond the semiclassical approximation, we analyze the light-cone asymptotics of G_4 and argue that the relation between $\mathcal{G}(u, v)$ and light-like rectangular Wilson loop mentioned above also holds in the presence of instantons. This relation allows us to establish the correspondence between the leading (quantum) instanton correction to $\mathcal{G}(u, v)$ and the semiclassical result for W_4 .

We show that the resulting expression for $\mathcal{G}(u, v)$ takes the same form as in perturbation theory (1.3) with the important difference that the light-like cusp anomalous dimension in (1.3) is modified by the instanton correction. In the simplest case of the SU(2) gauge group, this correction in one-(anti)instanton sector is given by

$$\Gamma_{\text{cusp}}(g^2) = -\frac{4}{15} \left(\frac{g^2}{4\pi^2} \right)^4 (e^{2\pi i\tau} + e^{-2\pi i\bar{\tau}}). \tag{1.5}$$

Following [7, 22, 23], this result can be generalized to the SU(N) gauge group and to the case of multi-instantons at large N . To obtain the same result (1.5) from the direct calculation of the four-point correlation function, one would have to compute quantum instanton corrections to (1.1) at order $O(g^8)$.

As a byproduct of our analysis, we verify that the light-like rectangular Wilson loop W_4 satisfies the anomalous conformal Ward identities [24] nonperturbatively, in the presence of instantons. We also determine the leading instanton correction to anomalous dimension γ_S of twist-two operators with large spin $S \gg 1$. This anomalous dimension scales logarithmically with the spin (see eq. (2.4) below) and its behaviour is controlled by the cusp anomalous dimension [25]. Making use of (1.5) we find that the leading instanton contribution scales as $\gamma_S \sim g^8 e^{2\pi i\tau} \ln S$. This agrees with the finding of [26] that γ_S does not receive $O(g^4 e^{2\pi i\tau})$ correction for any spin $S > 2$.

The paper is organized as follows. In section 2 we analyze asymptotic behaviour of the four-point correlation function in the light-cone limit $x_{i,i+1}^2 \rightarrow 0$ in the presence of instantons and discuss its relation with the light-like Wilson loop. In section 3 we compute instanton contribution to the light-like rectangular Wilson loop in the semiclassical approximation. We then use it in section 4 to determine the leading instanton correction to the cusp anomalous dimension. Section 5 contains concluding remarks. Details of the calculation are presented in four appendices.

2 Correlation functions in the light-cone limit

In this section we examine instanton corrections to a four-point correlation function of scalar half-BPS operators

$$O_{20'}(x) = Y_{AB} Y_{CD} \text{tr}[\phi^{AB} \phi^{CD}(x)]. \tag{2.1}$$

Here auxiliary SU(4) tensors satisfy $\epsilon^{ABCD}Y_{AB}Y_{CD} = 0$ and serve to project the operator onto representation $\mathbf{20}'$ of the SU(4). In virtue of $\mathcal{N} = 4$ superconformal symmetry, the dependence of the four-point correlation function

$$G_4 = \langle O_{\mathbf{20}'}(x_1)O_{\mathbf{20}'}(x_2)O_{\mathbf{20}'}(x_3)O_{\mathbf{20}'}(x_4) \rangle \tag{2.2}$$

on Y -variables can be factored into a universal kinematical factor independent on the coupling constant [27]. In what follows we do not display this factor and concentrate on the dynamical part $\mathcal{G}(u, v)$ that depends on the cross ratios only.

In the Born approximation, G_4 reduces to the sum of terms each given by the product of free scalar propagators. In the light-like limit, $x_{i,i+1}^2 \rightarrow 0$, the leading contribution to G_4 comes from only one term of the form (1.2) with $\mathcal{G}(u, v) = 1$.² Going beyond this approximation, we apply the OPE to each pair of neighbouring operators in (2.2), e.g.

$$O_{\mathbf{20}'}(x_1)O_{\mathbf{20}'}(x_2) = \sum_S \frac{C_S}{(x_{12}^2)^{2-t_S/2}} x_{12}^{\mu_1} \dots x_{12}^{\mu_S} O_{\mu_1 \dots \mu_S}(x_1), \tag{2.3}$$

where the sum runs over local operators with Lorentz spin S , dimension Δ_S and twist $t_S = \Delta_S - S \geq 2$. For $x_{12}^2 \rightarrow 0$ the dominant contribution to (2.3) comes from twist-two operators with arbitrary spin S and scaling dimension $\Delta_S = 2 + S + \gamma_S$.³ It scales as $O(1/x_{12}^2)$ and yields the expected asymptotic behaviour (1.2). In the similar manner, the remaining $O(1/x_{i,i+1}^2)$ factors in (1.2) come from the twist-two operators propagating in other OPE channels.

A detailed analysis shows [19, 20], that the leading asymptotic behaviour of the function $\mathcal{G}(u, v)$ for $u, v \rightarrow 0$ is governed by twist-two operators with large spin $S = O(1/\sqrt{u})$ or $S = O(1/\sqrt{v})$ depending on the OPE channel. The anomalous dimension of such operators grows logarithmically with the spin,

$$\gamma_S = 2\Gamma_{\text{cusp}} \ln S + O(S^0), \tag{2.4}$$

and generates corrections to $\mathcal{G}(u, v)$ enhanced by powers of $\ln u$ and $\ln v$. As we see in a moment, this observation simplifies the calculation of instanton corrections to (1.2).

Instantons are classical configurations of fields (scalar, gaugino and gauge fields) satisfying equations of motion in Euclidean $\mathcal{N} = 4$ SYM [28]. To compute their contribution to the correlation function (2.2) at weak coupling, we have to go through few steps. First, we decompose all fields (that we denote generically by $\Phi(x)$) into classical, instanton part and quantum fluctuations,

$$\Phi(x) = \frac{1}{g} \Phi_{\text{inst}}(x) + \Phi_q(x). \tag{2.5}$$

Here we introduced the factor of $1/g$ to emphasize that $\Phi_{\text{inst}}(x)$ does not depend on the coupling constant. Then, we substitute (2.5) into (2.2) and integrate over quantum fluctuations Φ_q and over collective coordinates of instantons. Finally, we match the resulting

²In what follows we discard a disconnected part of the correlation function.

³Strictly speaking, this is true only if the anomalous dimension of the operator is small, or equivalently the value of its twist is close to that in a free theory. If the anomalous dimension of twist-two operator is large $\gamma_S \sim 2$, it collides with the twist-four operators.

expression for G_4 into (1.2), identify the function $\mathcal{G}(u, v)$ and analytically continue it to small u and v .

In the semiclassical approximation, we neglect quantum fluctuations in (2.5) and obtain the following expression for G_4

$$G_{4,\text{inst}} = \int d\mu_{\text{phys}} e^{-S_{\text{phys}}} O_{20'}(x_1)O_{20'}(x_2)O_{20'}(x_3)O_{20'}(x_4), \quad (2.6)$$

where the half-BPS operators (2.1) are replaced by their expressions in the instanton background and are integrated over the collective coordinates of the instantons. The relation (2.6) can be represented diagrammatically as shown in figure 1 (a).

For the one-instanton configuration in $\mathcal{N} = 4$ SYM with the $SU(2)$ gauge group the integration measure is given by [29]

$$\int d\mu_{\text{phys}} e^{-S_{\text{phys}}} = \frac{g^8}{2^{34}\pi^{10}} e^{2\pi i\tau} \int d^4x_0 \int \frac{d\rho}{\rho^5} \int d^8\xi \int d^8\bar{\eta}. \quad (2.7)$$

Here bosonic collective coordinates x_0^μ and ρ define the position of the instanton and its size, respectively. Fermionic coordinates ξ_α^A and $\bar{\eta}_{\dot{\alpha}}^A$ (with $A = 1, \dots, 4$ and $\alpha, \dot{\alpha} = 1, 2$) reflect the invariance of $\mathcal{N} = 4$ SYM under superconformal transformations. For the correlation function (2.6) to be different from zero, the product of four half-BPS operators in (2.6) should soak all 16 fermion modes.

For the $SU(N)$ gauge group the instanton $\Phi_{\text{inst}}(x)$ depends on the additional bosonic and fermion modes. In what follows we shall concentrate on the $SU(2)$ case and discuss generalization to the $SU(N)$ later in section 4.4.

In $\mathcal{N} = 4$ SYM with the $SU(2)$ gauge group, the one-instanton solution can be obtained [30] by applying superconformal transformations $\exp(\xi Q + \bar{\eta} \bar{S})$ to the special field configuration consisting of vanishing scalar and gaugino fields and gauge field given by the celebrated BPST instanton.⁴ For gauge field this leads to

$$A_{\text{inst}}(x) = A^{(0)} + A^{(4)} + A^{(8)}, \quad (2.8)$$

where $A^{(0)}$ is the BPST instanton [28] and $A^{(4n)}$ denotes component containing $4n$ fermion modes ξ_α^A and $\bar{\eta}_{\dot{\alpha}}^A$. In virtue of the $SU(4)$ symmetry, each subsequent term of the expansion has four modes more. Expressions for the scalar and gaugino fields have a form similar to (2.8) with the only difference that the lowest term of the expansion has a nonzero number of fermion modes whose value is dictated by the R -charge of the fields. Notice that the expansion (2.8) is shorter than one might expect as the $SU(4)$ symmetry allows for the presence of terms with up to 16 fermion modes. It turns out, however, that all field components with the number of fermion modes exceeding 8 vanish due to $\mathcal{N} = 4$ superconformal symmetry [26]. The explicit expressions for various components in (2.8) can be found in [6, 26].

We recall that relations (2.8) define the classical part of (2.5). Replacing the scalar field in the definition of the half-BPS operator (2.1) with $\phi(x) = g^{-1}\phi_{\text{inst}}(x)$ we find that $O_{20'}(x)$

⁴This field configuration is annihilated by the remaining \bar{Q} and S generators.

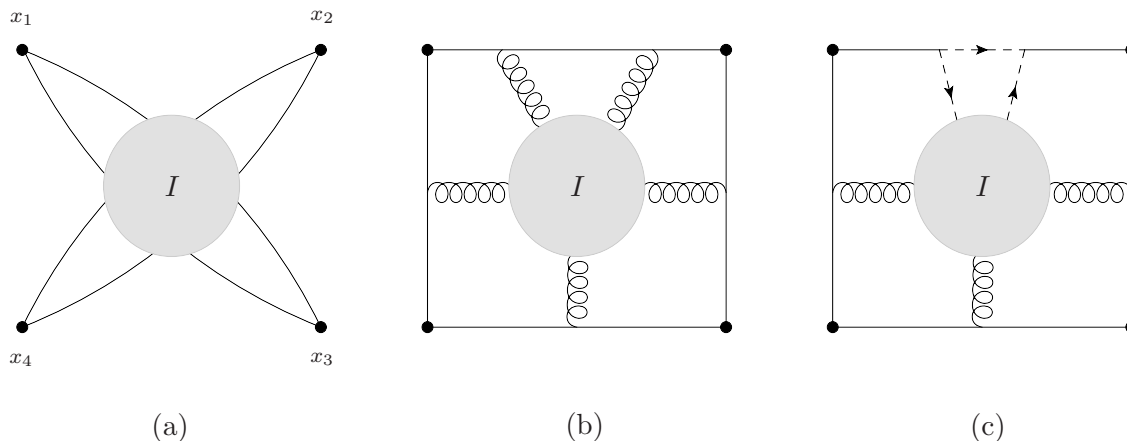


Figure 1. Instanton corrections to the four-point correlation function: (a) contribution in the semiclassical approximation; (b) leading contribution in the light-cone limit; (c) example of subleading contribution. Solid, wavy and dashes lines denote scalars, gauge fields and gauginos, respectively. Lines attached to the central blob represent the instanton background.

scales in the instanton background as $O(1/g^2)$ and contains 4 fermion modes. Then, we can apply (2.6) and (2.7) to arrive at the following result for the four-point correlation function in the semiclassical approximation [29]

$$G_{4,\text{inst}} = \frac{108}{\pi^2} e^{2\pi i\tau} \frac{uv\bar{D}_{44}(u,v)}{x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2}, \tag{2.9}$$

where \bar{D} -function is defined in appendix C. The contribution of anti-instanton is given by a complex conjugated expression. The relation (2.9) holds for arbitrary $x_{i,i+1}^2$. For $x_{i,i+1}^2 \rightarrow 0$ we find from (1.2) that the instanton contribution vanishes as $\mathcal{G}(u,v) = O(uv)$ for $u,v \rightarrow 0$. Thus, the one-instanton correction (2.9) does not modify asymptotic behaviour of G_4 in the light-cone limit [21].

The same result can be obtained using the OPE. In the semiclassical approximation, the product of the operators on the left-hand side of (2.3) reduces to the product of two functions describing the classical profile of half-BPS operators. It is obviously regular for $x_{12}^2 \rightarrow 0$ and, therefore, cannot produce $1/x_{12}^2$ singularity that is needed to get a finite result for $\mathcal{G}(u,v)$ in (1.2). For such singularity to arise, we have to go beyond the semiclassical approximation in (2.5) and exchange quantum fluctuations between the two operators in (2.3). To lowest order in the coupling we have

$$\langle \Phi(x_1)\Phi(x_2) \rangle = \frac{1}{g^2} \Phi_{\text{inst}}(x_1)\Phi_{\text{inst}}(x_2) + \langle \Phi_q(x_1)\Phi_q(x_2) \rangle, \tag{2.10}$$

where $\langle \Phi_q(x_1)\Phi_q(x_2) \rangle \sim 1/x_{12}^2$. Notice that the quantum fluctuation produces $1/x_{12}^2$ singularity but its contribution is suppressed by the factor of g^2 as compared with the semiclassical result.

For the correlation function to have the expected form (1.2) with nonvanishing $\mathcal{G}(u,v)$, at least one quantum fluctuation has to be exchanged between each pair of neighboring

operators in (2.2). As follows from the above analysis, the corresponding contribution to $\mathcal{G}(u, v)$ has the following dependence on the coupling constant

$$\mathcal{G}(u, v) = O(g^8 e^{2\pi i\tau}), \quad (2.11)$$

where each fluctuation brings in the factor of g^2 . Comparing this relation with (1.1) we find that the first three terms in the expansion of the instanton induced function $\Phi_n(u, v; g^2)$ in powers of g^2 should vanish in the light-cone limit $u, v \rightarrow 0$, in the one-instanton sector at least.

To the leading order in g^2 , the dominant contribution to (2.11) comes from Feynman diagrams shown in figure 1 (b). They contain four scalar propagators connecting the points x_i and x_{i+1} . In the first-quantized picture, these diagrams describe a scalar particle propagating between the points x_1, \dots, x_4 in an external instanton gauge field. Notice that the particle can also interact with instanton fields of gaugino and scalar but this leads to a subleading contribution. To show this, consider the diagram shown in figure 1 (c). It contains two Yukawa vertices and its contribution to G_4 has the same dependence on the coupling constant as (2.11). However, in distinction from the diagram shown in figure 1 (b), it does not produce $1/x_{12}^2$ singularity. Indeed, as follows from (2.3), the leading behaviour $G_4 \sim 1/(x_{12}^2)^{2-t/2}$ is controlled by the twist of exchanged operators. For the diagram shown in figure 1 (c) such operators are built from two scalar and two gaugino fields and their twist satisfies $t \geq 4$. For the diagram shown in figure 1 (b), the leading operators have twist two and are of a schematic form $\text{tr}[\Phi D_+^S \Phi]$, where $D_+ = \partial_+ + igA_+$ is a light-cone component of the covariant derivative.

Thus, the leading contribution to G_4 for $x_{i,i+1}^2 \rightarrow 0$ only comes from diagrams shown in figure 1 (b). Denoting the scalar propagator in the instanton background as $D(x_i, x_{i+1})$, we obtain the following result for the correlation function in the light-like limit

$$G_{4,\text{inst}} = \langle \text{tr} [D(x_1, x_2)D(x_2, x_3)D(x_3, x_4)D(x_4, x_1)] \rangle_{\text{inst}}, \quad (2.12)$$

where $\langle \dots \rangle_{\text{inst}}$ denotes integration over the collective coordinates of instantons with the measure (2.7). This result is rather general and it holds for multi-instanton contribution to G_4 in $\mathcal{N} = 4$ SYM with an arbitrary gauge group.

We can argue following [19] that the relation (2.12) leads to the same factorized expression (1.3) for the function $\mathcal{G}(u, v)$ as in perturbation theory. The propagator $D(x_1, x_2)$ depends on two momentum scales, $1/x_{12}^2$ and $1/\rho^2$, which define a proper energy of the scalar particle and its interaction energy with the instanton background, respectively. For $1/x_{12}^2 \gg 1/\rho^2$, or equivalently $x_{12}^2 \ll \rho^2$, the instanton carries small energy and its interaction with the scalar particle can be treated semiclassically. In this limit, $D(x_1, x_2)$ reduces to a free scalar propagator multiplied by the eikonal phase given by the Wilson line evaluated along the light-cone segment $[x_1, x_2]$. Taking the product of four Wilson lines corresponding to four propagators in (2.12), we obtain that the contribution of instanton with $x_{i,i+1}^2 \ll \rho^2$ is described by the rectangular light-like Wilson loop defined in (1.4). The gauge fields in (1.4) are now replaced by the instanton solution (2.8) and integration over its moduli is performed with the measure (2.7). For $x_{i,i+1}^2 \gg \rho^2$ the eikonal approximation

is not applicable. The contribution from this region, denoted by J in (1.3), can be determined from the crossing symmetry of G_4 in the same way as it was done in perturbation theory [20].

3 Light-like Wilson loop in the instanton background

We demonstrated in the previous section that the leading light-cone asymptotics of the correlation function G_4 is described by light-like rectangular Wilson loop W_4 . Due to the presence of cusps on the integration contour, W_4 develops specific ultraviolet divergences. In the expression for the correlation function (1.3), these divergences cancel against those of the J -function in such a way that the UV cut-off of the Wilson loop is effectively replaced with $\mu^2 \sim 1/x_{i,i+1}^2$.

3.1 Conformal Ward identities

Let us start with summarizing the properties of W_4 . As was shown in [24], the conformal symmetry restricts the dependence of W_4 on kinematical invariants

$$W_4 = Z(x_{13}^2\mu^2)Z(x_{24}^2\mu^2)F_4(x_{13}^2/x_{24}^2), \tag{3.1}$$

where Z and F_4 are the divergent and finite parts, respectively. The dependence of W_4 on the UV cut-off μ^2 is described by the evolution equation

$$\left(\mu^2 \frac{\partial}{\partial \mu^2}\right)^2 \ln W_4 = -2\Gamma_{\text{cusp}}(g^2), \tag{3.2}$$

where the light-like cusp anomalous dimension $\Gamma_{\text{cusp}}(g^2)$ depends on the representation of the $SU(N)$ gauge group in which the Wilson loop is defined. The general solution to this equation depends on the so-called collinear anomalous dimension $\Gamma_{\text{col}}(g^2)$. It appears as a coefficient in front of $\ln(x_{i,i+2}^2\mu^2)$ in the expression for $\ln W_4$ and depends on the choice of the regularization. The finite part of (3.1) is uniquely fixed by the conformal symmetry⁵

$$F_4 = \exp\left[\frac{1}{2}\Gamma_{\text{cusp}}(g^2)\ln^2(x_{13}^2/x_{24}^2)\right]. \tag{3.3}$$

Combining together (3.1) and (3.3) we obtain the following relation for W_4

$$\frac{\partial}{\partial \ln x_{13}^2} \frac{\partial}{\partial \ln x_{24}^2} \ln W_4 = -\Gamma_{\text{cusp}}(g^2), \tag{3.4}$$

where the dependence on the UV cut-off disappears since the second derivative annihilates the divergent part of $\ln W_4$. This relation allows us to find $\Gamma_{\text{cusp}}(g^2)$ from W_4 directly, without introducing a regularization, by computing its second derivative (3.4).

We would like to emphasize that relations (3.2)–(3.4) follow from the conformal Ward identities and should hold in the presence of instantons. We shall verify this property below.

⁵Since the Wilson loop (3.1) is defined in the adjoint representation, this expression is the square of the one found in [24].

We recall that the instanton correction to W_4 should match the leading correction to the correlation function $\mathcal{G}(u, v)$ for $u, v \rightarrow 0$. Taking into account (2.11), we expect that the instanton correction to the cusp anomalous dimension should scale as $\Gamma_{\text{cusp}}(g^2) = O(g^8 e^{2\pi i\tau})$.

3.2 Semiclassical approximation

To compute instanton corrections to the light-like Wilson loop, we have to define W_4 in Euclidean signature. This can be achieved by allowing the cusp points x_i^μ to take complex values, such that $x_{i,i+1}^2 = 0$. Having determined W_4 as a function of x_{13}^2 and x_{24}^2 , we shall continue it to Minkowski signature.

To find W_4 in the semiclassical approximation, we have to evaluate the Wilson loop (1.4) in the instanton background and, then, integrate it over the collective coordinates with the measure (2.7)

$$W_{4,\text{inst}} = \int d\mu_{\text{phys}} e^{-S_{\text{phys}}} W_A. \tag{3.5}$$

We recall that W_A is defined in the adjoint representation of the $SU(2)$. It proves convenient to generalize (1.4) and define the Wilson loop in an arbitrary $SU(2)$ representation R

$$W_R = \text{tr}_R [E(x_1, x_2)E(x_2, x_3)E(x_3, x_4)E(x_4, x_1)], \tag{3.6}$$

where $E(x_i, x_{i+1})$ is a light-line Wilson line stretched between the points x_i and x_{i+1}

$$E(x_i, x_{i+1}) = P \exp \left(-ig \int_0^1 dt x_{i,i+1}^\mu A_\mu(x_i - tx_{i,i+1}) \right). \tag{3.7}$$

Here the gauge field $A_\mu(x) = A_\mu^a(x)T^a$ is integrated along the light-cone segment $x(t) = x_i - tx_{i,i+1}$ and T^a are the $SU(2)$ generators in the representation R . Notice that for zero value of the coupling constant, the Wilson loop (3.6) is equal to the dimension of the representation $W_R = d_R$.

In special cases of the fundamental (F) and adjoint (A) representations of the $SU(2)$, the generators are related to Pauli matrices, $T^a = \sigma^a/2$, and completely antisymmetric tensor, $(T^a)^{bc} = i\epsilon^{abc}$, respectively. The corresponding Wilson loops, W_F and W_A , satisfy the fusion relation

$$W_F \overline{W}_F = 1 + W_A, \tag{3.8}$$

where \overline{W}_F is complex conjugated to W_F .

Applying (3.5), we have to replace the gauge field in (3.6) with its expression in the instanton background, $A(x) = g^{-1}A_{\text{inst}}(x)$ (see eqs. (2.5) and (2.8)). As follows from (3.6) and (3.7), the resulting expression for W_R does not depend on the coupling constant. It depends however on 16 fermion modes of the instanton, ξ_α^A and $\overline{\eta}_{\dot{\alpha}}^A$. This dependence has the following general form

$$W_F = W^{(0)} + W^{(4)} + W^{(8)} + W^{(12)} + W^{(16)}, \tag{3.9}$$

where $W^{(4n)}$ denotes a homogenous SU(4) invariant polynomial in ξ_α^A and $\bar{\eta}_\alpha^A$ of degree $4n$. The Wilson loop in the adjoint representation has similar form. The relation (3.8) allows us to express W_A in terms of $W^{(4n)}$.

In order to compute the Wilson loop (3.5) we only need the top component $W_R^{(16)}$ containing 16 fermion modes. The remaining components give vanishing contribution upon integration over fermion modes in (3.5). For the Wilson loop in the fundamental representation, the top component $W_F^{(16)}$ is given by $W^{(16)}$. For the Wilson loop in the adjoint representation we get from (3.8) and (3.9)

$$W_A^{(16)} = W^{(0)}\bar{W}^{(16)} + W^{(4)}\bar{W}^{(12)} + \frac{1}{2}W^{(8)}\bar{W}^{(8)} + \text{c.c.} \quad (3.10)$$

Since $W_R^{(16)}$ contains 16 fermion modes, it has the following form

$$W_R^{(16)} = f_R(x_i; x_0, \rho) \xi^8 \bar{\eta}^8, \quad (3.11)$$

where $\xi^8 = \prod_{\alpha,A} \xi_\alpha^A$ and similar for $\bar{\eta}$. The scalar function $f_R(x_i; x_0, \rho)$ depends on four points x_i (with $x_{i,i+1}^2 = 0$) and on the bosonic collective coordinates x_0 and ρ . It also depends on the representation R of the SU(2) gauge group.

Substituting (3.11) into (3.5) and taking into account (2.7), we obtain the following expression for the instanton correction to W_4 in the semiclassical approximation⁶

$$W_{4,\text{inst}} = \frac{g^8}{2^{34}\pi^{10}} e^{2\pi i\tau} \int d^4x_0 \int \frac{d\rho}{\rho^5} f_R(x_i; x_0, \rho). \quad (3.12)$$

The dependence of this expression on the coupling constant matches (2.11). We expect that the instanton effects should modify the light-like cusp anomalous dimension. For this to happen, the integral in (3.12) has to develop UV divergences. Indeed, as we show below, instantons of small size ($\rho \rightarrow 0$) located in the vicinity of the cusp points ($x_0 \rightarrow x_i$) provide a divergent contribution to (3.12).

3.3 Cusp anomalous dimension

The light-like Wilson loop is invariant under conformal transformations at the classical level. At the quantum level, its conformal symmetry is broken by cusp singularities. In application to (3.12) this implies that if the integral in (3.12) were well-defined, $W_{4,\text{inst}}$ should be conformally invariant. The conformal transformations act nontrivially on the bosonic moduli x_0 and ρ leaving the integration measure in (3.12) invariant. Therefore, invariance of $W_{4,\text{inst}}$ under these transformations translates into conformal invariance of the function $f_R(x_i; x_0, \rho)$. To regularize cusp singularities of (3.12) we can modify the integration measure as

$$\int d^4x_0 \rightarrow \mu^{-2\epsilon} \int d^{4-2\epsilon}x_0, \quad (3.13)$$

leaving the function $f_R(x_i; x_0, \rho)$ intact.⁷

⁶Notice that $W_{4,\text{inst}}$ depends on the choice of the SU(2) representation R .

⁷There are of course different ways to regularize (3.12). The relation (3.4) ensures that the resulting expression for the cusp anomalous dimension is independent on the regularization procedure. This does not apply however to the collinear anomalous dimension.

The conformal symmetry dictates that the function $f_R(x_i; x_0, \rho)$ can depend on the bosonic moduli, ρ and x_0 , and four points x_i through conformal invariants only. The latter have the following form⁸

$$I_{ij} = \frac{x_{ij}^2 \rho^2}{(x_{i0}^2 + \rho^2)(x_{j0}^2 + \rho^2)}. \quad (3.14)$$

We recall that the points x_i define the vertices of light-like rectangle and satisfy $x_{i,i+1}^2 = 0$. As a consequence, $I_{i,i+1} = 0$ and we are left with only two nonvanishing invariants, $f_R = f_R(I_{13}, I_{24})$.

Since f_R is obtained from the Wilson loop (3.11) evaluated in background of instanton field, it is an intrinsically classical quantity. We therefore expect it to be a rational function of I_{13} and I_{24} . In addition, f_R should vanish for $I_{13} = 0$ or $I_{24} = 0$. The reason for this is that for $x_1 \rightarrow x_3$ (or $x_2 \rightarrow x_4$) the rectangular contour in (3.6) collapses into a closed backtracking path. The Wilson lines in (3.6) cancel against each other for such path leading to $W_R = d_R$ or equivalently $f_R = 0$. These properties suggest to look for $f_R(I_{13}, I_{24})$ in the form

$$\begin{aligned} f_R(I_{13}, I_{24}) &= \sum_{\ell_1, \ell_2 \geq 1} f_{\ell_1 \ell_2} (I_{13})^{\ell_1} (I_{24})^{\ell_2} \\ &= \sum_{\ell_1, \ell_2 \geq 1} \frac{f_{\ell_1 \ell_2} \rho^{2(\ell_1 + \ell_2)} (x_{13}^2)^{\ell_1} (x_{24}^2)^{\ell_2}}{[(x_{10}^2 + \rho^2)(x_{30}^2 + \rho^2)]^{\ell_1} [(x_{20}^2 + \rho^2)(x_{40}^2 + \rho^2)]^{\ell_2}}, \end{aligned} \quad (3.15)$$

where expansion coefficients are symmetric $f_{\ell_1 \ell_2} = f_{\ell_2 \ell_1}$ due to the cyclic symmetry of (3.6). Moreover, as we show in appendix D, $f_R(I_{13}, I_{24})$ is actually a polynomial in both variables, so that the sum in (3.15) contains a finite number of terms.

Replacing the function f_R in (3.12) with its general expression (3.15), we find that the integrals over x_0 and ρ can be expressed in terms of D -functions defined in appendix C. These functions are finite for generic $x_{ij}^2 \neq 0$ but develop logarithmic divergences for $x_{i,i+1}^2 = 0$. A close examination shows that divergences arise from integration over $\rho \rightarrow 0$ and $x_0 \rightarrow x_i$ and have a clear UV origin. These are the cusp divergences that were mentioned at the end of the previous subsection. Regularizing divergences according to (3.13), we obtain the following expression for the instanton correction to the light-like Wilson loop (3.12)

$$W_{4,\text{inst}} = \frac{g^8}{2^{34} \pi^{10}} e^{2\pi i \tau} \sum_{\ell_1, \ell_2} f_{\ell_1 \ell_2} \bar{D}_{\ell_1 \ell_2}^{(\epsilon)}. \quad (3.16)$$

Here we introduced notation for the regularized integral

$$\bar{D}_{\ell_1 \ell_2}^{(\epsilon)} = \mu^{-2\epsilon} \int d^{4-2\epsilon} x_0 \int \frac{d\rho}{\rho^5} \frac{\rho^{2(\ell_1 + \ell_2)} (x_{13}^2)^{\ell_1} (x_{24}^2)^{\ell_2}}{[(x_{10}^2 + \rho^2)(x_{30}^2 + \rho^2)]^{\ell_1} [(x_{20}^2 + \rho^2)(x_{40}^2 + \rho^2)]^{\ell_2}}, \quad (3.17)$$

⁸To check conformal properties of I_{ij} we can employ inversions defined in (B.1). It is straightforward to verify that (3.14) is invariant under these transformations.

evaluated for $x_{i,i+1}^2 = 0$. This integral is well-defined for $\epsilon < 0$ and the cusp divergences appear as poles in ϵ . The details of calculation can be found in appendix C.

Substituting the resulting expression for $\bar{D}_{\ell_1 \ell_2}^{(\epsilon)}$ (see (C.6) in appendix C) into (3.16) we find that $W_{4,\text{inst}}$ takes a remarkable simple form

$$\begin{aligned}
 W_{4,\text{inst}}/d_R = & -\Gamma_{\text{cusp}}(g^2) \left[\frac{1}{\epsilon^2} (\mu^2 x_{13}^2)^{-\epsilon} + \frac{1}{\epsilon^2} (\mu^2 x_{24}^2)^{-\epsilon} - \frac{1}{2} \ln^2(x_{13}^2/x_{24}^2) \right] \\
 & - \Gamma_{\text{col}}(g^2) \left[\frac{1}{\epsilon} (\mu^2 x_{13}^2)^{-\epsilon} + \frac{1}{\epsilon} (\mu^2 x_{24}^2)^{-\epsilon} \right] - \Gamma_{\text{h}}(g^2), \tag{3.18}
 \end{aligned}$$

where d_R is the dimension of the SU(2) representation in which the Wilson loop is defined. Here we denoted the residue at the double pole as $\Gamma_{\text{cusp}}(g^2)$ anticipating that the same quantity defines the instanton correction to the cusp anomalous dimension. It is given by the following expression

$$\Gamma_{\text{cusp}}(g^2) = -\frac{g^8}{2^{34} \pi^8 d_R} e^{2\pi i \tau} \sum_{\ell_1, \ell_2 \geq 1} f_{\ell_1 \ell_2} \frac{\Gamma(\ell_1 + \ell_2 - 2)}{\Gamma(\ell_1) \Gamma(\ell_2)}. \tag{3.19}$$

We recall that $f_{\ell_1 \ell_2}$ are coefficients of the expansion of the top component of the Wilson loop (3.11) in powers of the conformal invariants (3.15). Notice that for $\ell_1 = \ell_2 = 1$ the Γ -function in (3.19) develops a pole. For the sum in (3.19) to be finite the corresponding coefficient f_{11} has to vanish.

The residue at the simple pole and the constant term in (3.18), $\Gamma_{\text{col}}(g^2)$ and $\Gamma_{\text{h}}(g^2)$, respectively, are given by expressions similar to (3.19). However, in distinction from $\Gamma_{\text{cusp}}(g^2)$ they depend on the choice of the regularization in (3.17). That is why we do not present their expressions.

Let us now compare (3.18) with the expected properties of light-like Wilson loop. We combine (3.18) with the Born level contribution to the Wilson loop, $W_4 = d_R + W_{4,\text{inst}}$, and require that W_4 has to satisfy (3.2) and (3.4). This leads to the following relations

$$\frac{1}{2} \left(\frac{\partial}{\partial \ln \mu^2} \right)^2 (W_{4,\text{inst}}/d_R) = \frac{\partial}{\partial \ln x_{13}^2} \frac{\partial}{\partial \ln x_{24}^2} (W_{4,\text{inst}}/d_R) = -\Gamma_{\text{cusp}}(g^2). \tag{3.20}$$

It is easy to check that (3.18) verifies these relations. In this way, we find that (3.19) defines indeed the leading instanton correction to the cusp anomalous dimension.

In addition to (3.18), the light-like Wilson loop also receives perturbative corrections that run in powers of g^2 . To lowest order in g^2 , these corrections have exactly the same form (3.18) although expressions for the anomalous dimensions are different [31]. The reason for such universality can be understood as follows. As explained in section 3.1, the conformal symmetry fixes the dependence of the light-like rectangular Wilson loop on kinematical invariants. In particular, it allows us to determine the finite part of the Wilson loop in terms of the cusp anomalous dimension, eq. (3.3). The fact that the instanton corrections (3.18) verify (3.2) and (3.4) implies that the conformal Ward identities found in [24] hold nonperturbatively, in the presence of instanton effects.

In the next section, we apply (3.19) to compute the leading instanton contribution to the cusp anomalous dimension in the fundamental and adjoint representations of the SU(2).

4 Instanton contribution to the cusp anomalous dimension

According to (3.16), the instanton corrections to the light-like Wilson loop are determined by the coefficients $f_{\ell_1 \ell_2}$. To find them, we have to identify the top component of the Wilson loop (3.11) containing 16 fermion modes and, then, expand the corresponding function f_R in powers of the conformal invariants (3.15).

The top component of the Wilson loop $W_R^{(16)}$ depends on the choice of the representation R . Making use of the relation (3.10) (and its generalization for higher spin representations of the SU(2)) we can express $W_R^{(16)}$ in terms of various components $W^{(4n)}$ of the Wilson loop in the fundamental representation (3.9).

4.1 Wilson loop in the fundamental representation

To compute the Wilson loop in the fundamental representation of the SU(2), we have to replace the gauge field in (3.6) and (3.7) with its expression (2.8) in the instanton background, $A^\mu(x) = g^{-1} A_{\text{inst}}^{\mu,a}(x) \sigma^a / 2$.

The resulting expression for the Wilson line (3.7) depends on 16 fermion modes and admits an expansion similar to (3.9)

$$E(x_i, x_{i+1}) = P e^{i \int_{x_i}^{x_{i+1}} dx^\mu A_{\text{inst}}^\mu(x)} = E^{(0)} + E^{(4)} + E^{(8)} + E^{(12)} + E^{(16)}. \quad (4.1)$$

In distinction from (3.9), each term on the right-hand side is gauge dependent. The explicit expressions for the first three terms on the right-hand side of (4.1) are

$$\begin{aligned} E^{(0)}(x_i, x_{i+1}) &= P \exp \left(i \int_0^1 dt \dot{x}^\mu(t) A_\mu^{(0)}(x(t)) \right), \\ E^{(4)}(x_i, x_{i+1}) &= i \int_0^1 dt \dot{x}^\mu(t) E^{(0)}(x_i, x(t)) A_\mu^{(4)}(x(t)) E^{(0)}(x(t), x_{i+1}), \\ E^{(8)}(x_i, x_{i+1}) &= i \int_0^1 dt \dot{x}^\mu(t) E^{(0)}(x_i, x(t)) \\ &\quad \times \left[A_\mu^{(8)}(x(t)) E^{(0)}(x(t), x_{i+1}) + A_\mu^{(4)}(x(t)) E^{(4)}(x(t), x_{i+1}) \right], \end{aligned} \quad (4.2)$$

where $x(t) = (1-t)x_i + tx_{i+1}$ parameterizes the light-like segment $[x_i, x_{i+1}]$ and $\dot{x}(t) = \partial_t x(t) = -x_{i,i+1}$. Expressions for the remaining components of (4.1) are more involved. Going through their calculation we find (see appendix B) that they are proportional to the square of a fermion mode and, therefore, have to vanish

$$E^{(12)}(x_i, x_{i+1}) = E^{(16)}(x_i, x_{i+1}) = 0. \quad (4.3)$$

Thus, the light-like Wilson line (4.1) has at most 8 fermion modes.

Substituting (4.1) into (3.6) we obtain that the Wilson loop (3.9) is given by a linear combination of terms of the form

$$W^{(k_1, k_2, k_3, k_4)} = \text{tr} \left[E^{(k_1)}(x_1, x_2) E^{(k_2)}(x_2, x_3) E^{(k_3)}(x_3, x_4) E^{(k_4)}(x_4, x_1) \right], \quad (4.4)$$

where integers k_1, \dots, k_4 count the number of fermion modes. The explicit expressions for different components of (3.9) are

$$\begin{aligned}
 W^{(0)} &= W^{(0,0,0,0)}, \\
 W^{(4)} &= W^{(4,0,0,0)} + \text{cyclic}, \\
 W^{(8)} &= W^{(8,0,0,0)} + W^{(4,4,0,0)} + \frac{1}{2}W^{(4,0,4,0)} + \text{cyclic}, \\
 W^{(12)} &= W^{(8,4,0,0)} + W^{(8,0,4,0)} + W^{(8,0,0,4)} + W^{(4,4,4,0)} + \text{cyclic}, \\
 W^{(16)} &= W^{(8,4,4,0)} + W^{(8,4,0,4)} + W^{(8,0,4,4)} + W^{(8,8,0,0)} + \frac{1}{2}W^{(8,0,8,0)} + \frac{1}{4}W^{(4,4,4,4)} + \text{cyclic},
 \end{aligned} \tag{4.5}$$

where ‘cyclic’ denotes the additional terms that ensure the invariance of Wilson loop under the cyclic shift of the cusp points (x_1, \dots, x_4) . The additional rational factors are inserted to avoid a double counting.

By definition, $W^{(4n)}$ is a homogenous polynomial in fermion modes ξ_α^A and $\bar{\eta}_{\dot{\alpha}}^A$ of degree $4n$. It depends in addition on four cusp points x_i and bosonic moduli x_0 and ρ . As a function of these variables, it should be invariant under conformal transformations including inversions (see (B.1), (B.2) and (B.7) in appendix B). We shall use this property below to simplify the calculation.

4.2 Leading term

It is convenient to switch from vector to spinor notations and convert $A_\mu(x)$ into 2×2 matrix $A_{\alpha\dot{\alpha}}(x)$ by contracting its Lorentz index with four-dimensional vector of Pauli matrices $\sigma^\mu = (1, i\sigma)$

$$(A_{\alpha\dot{\alpha}})_{i^j} = iA_\mu^a(x)(\sigma^a/2)_{i^j}(\sigma^\mu)_{\alpha\dot{\alpha}}. \tag{4.6}$$

This field carries two spinor indices $(\alpha, \dot{\alpha} = 1, 2)$ and two SU(2) indices $(i, j = 1, 2)$. In addition, we define the gauge field with lower SU(2) indices, $A_{\alpha\dot{\alpha}, ik} = \epsilon_{kj}(A_{\alpha\dot{\alpha}})_{i^j}$, it is symmetric with respect to indices i and k .

The BPST instanton $A^{(0)}$ takes the following form in the spinor notations

$$A_{\alpha\dot{\alpha}, ij}^{(0)}(x) = \frac{\epsilon_{i\alpha}(x-x_0)_{j\dot{\alpha}} + \epsilon_{j\alpha}(x-x_0)_{i\dot{\alpha}}}{(x-x_0)^2 + \rho^2}, \tag{4.7}$$

where $x_{\alpha\dot{\alpha}} = x_\mu(\sigma^\mu)_{\alpha\dot{\alpha}}$ for arbitrary four-dimensional vector x_μ . It intertwines the SU(2) gauge group and the chiral half of the Lorentz group $\text{SO}(4) = \text{SU}(2) \times \text{SU}(2)$.

To evaluate the leading term $E^{(0)}(x_1, x_2)$ defined in (4.2) we make use of the identity

$$\begin{aligned}
 i \int dt \dot{x}^\mu(t) A_\mu^{(0)}(x(t)) &= -\frac{1}{2} \int dt \dot{x}^{\dot{\alpha}\alpha}(t) A_{\alpha\dot{\alpha}}^{(0)}(x(t)) \\
 &= \frac{1}{2} \int \frac{dt [x_{10}, x_{20}]}{(1-t)x_{10}^2 + tx_{20}^2 + \rho^2},
 \end{aligned} \tag{4.8}$$

where $x(t) = (1-t)x_1 + tx_2$ and the SU(2) indices are suppressed for simplicity. Here in the second relation we substituted (4.7) and contracted Lorentz indices using the conventions described in appendix A. Since the SU(2) matrix part of the integral (4.8) does not

depend on the integration variable, the path-ordered exponential in (4.2) reduces to the conventional exponential. Its calculation yields

$$E^{(0)}(x_1, x_2) = \frac{\rho^2 + x_{10}x_{20}}{[(x_{10}^2 + \rho^2)(x_{20}^2 + \rho^2)]^{1/2}}. \quad (4.9)$$

Here the first term in the numerator is proportional to the SU(2) identity matrix δ_i^j and the second one involves the matrix $(x_{10}x_{20})_i^j = (x_{10})_{i\dot{\alpha}}(x_{20})^{\dot{\alpha}j}$.

We can now use (4.9) to compute the lowest component $W^{(0)}$ of the light-like Wilson loop (4.5). Going through the calculation we find

$$W^{(0)} = 2 - \frac{\rho^4 x_{13}^2 x_{24}^2}{(\rho^2 + x_{01}^2)(\rho^2 + x_{02}^2)(\rho^2 + x_{03}^2)(\rho^2 + x_{04}^2)} = 2 - I_{13}I_{24}, \quad (4.10)$$

where I_{ij} are conformal invariants defined in (3.14). Their appearance is not surprising since $W^{(0)}$ should be invariant under the conformal transformations.

4.3 Conformal gauge

The calculation of the remaining components in (4.2) is more involved since the gauge fields $A^{(4)}$ and $A^{(8)}$ are given by complicated expressions (see appendix D). It can be significantly simplified by making use of the conformal symmetry.

We can exploit this symmetry to choose the cusp points x_i to satisfy the following additional conditions

$$x_{10}^2 = x_{20}^2 = x_{30}^2 = x_{40}^2 = 0. \quad (4.11)$$

The advantage of this gauge is that any point of the light-like rectangle $x(t) = (1-t)x_i + tx_{i+1}$ becomes null separated from the center of instanton x_0

$$(x(t) - x_0)^2 = (1-t)x_{i0}^2 + tx_{i+1,0}^2 = 0, \quad (4.12)$$

where we took into account that $x_{i,i+1}^2 = 0$.

The calculation of the Wilson line (4.2) in the conformal gauge (4.11) is described in details in appendix D. For the component $E^{(4)}(x_1, x_2)$ defined in (4.2) we find

$$E^{(4)}(x_1, x_2) = \frac{2}{3\rho^2} \epsilon_{ABCD} \langle \zeta_1^A | E_{12} | \zeta_2^B \rangle \\ \times \left(|\zeta_1^C \rangle \langle \zeta_2^D | + E_{12} |\zeta_2^C \rangle \langle \zeta_1^D | E_{12} + 2|\zeta_1^C \rangle \langle \zeta_1^D | E_{12} + 2E_{12} |\zeta_2^C \rangle \langle \zeta_2^D | \right), \quad (4.13)$$

where $E_{12} \equiv E^{(0)}(x_1, x_2)$ and ζ_i is a linear combination of fermion modes, $\zeta_i = \xi + x_i \bar{\eta}$. The expression in the second line of (4.13) contains the sum of four SU(2) tensors, each given by the direct product of two pairs of vectors, $|\zeta_1 \rangle, E_{12} |\zeta_2 \rangle$ and $\langle \zeta_2 |, \langle \zeta_1 | E_{12}$.

As we show in appendix B, the general form of (4.13) and of the remaining components of the Wilson line is fixed by the conformal symmetry. In particular, $E^{(8)}(x_1, x_2)$ has the following form in the conformal gauge (4.11)

$$E^{(8)}(x_1, x_2) = \frac{1}{\rho^4} \left(P_{CD} |\zeta_1^C \rangle \langle \zeta_2^D | + Q_{CD} E_{12} |\zeta_2^C \rangle \langle \zeta_1^D | E_{12} \right. \\ \left. + R_{CD} |\zeta_1^C \rangle \langle \zeta_1^D | E_{12} + S_{CD} E_{12} |\zeta_2^C \rangle \langle \zeta_2^D | \right), \quad (4.14)$$

where $P_{CD}, Q_{CD}, R_{CD}, S_{CD}$ are homogenous polynomials in ζ_1 and ζ_2 of degree 6. In virtue of conformal symmetry, the dependence on ζ_1 and ζ_2 can only enter through the following three combinations

$$\langle \zeta_1^A \zeta_1^B \rangle, \quad \langle \zeta_2^A \zeta_2^B \rangle, \quad \langle \zeta_1^A | E_{12} | \zeta_2^B \rangle, \quad (4.15)$$

where we used notations for $\langle \zeta_1^A \zeta_1^B \rangle \equiv \zeta_1^{\alpha A} \zeta_{1,\alpha}^B$ and $\langle \zeta_1^A | E_{12} | \zeta_2^B \rangle \equiv \zeta_1^{iA} (E_{12})_{ij} \zeta_2^B$. The power of ρ in (4.14) is fixed by the condition for $E^{(8)}(x_1, x_2)$ to be dimensionless. The explicit expressions for the polynomials P_{CD}, \dots, S_{CD} are cumbersome, to save space we do not present them here.

4.4 Results

We can use the expressions for the Wilson line obtained in the previous subsection to compute different components of the Wilson loop in the fundamental representation of the SU(2), eqs. (4.4) and (4.5). We recall that in order to find the instanton correction to the Wilson loop we only need the top component $W^{(16)}$. It has the general form (3.11) and (3.15) and is specified by the set of coefficients $f_{\ell_1 \ell_2}$.

The calculation of (4.4) and (4.5) is rather lengthy and can be performed with a help of *Mathematica*. This yields the following result for the coefficients $f_{\ell_1 \ell_2}$ (with $\ell_1, \ell_2 \geq 1$)

$$f_{\ell_1 \ell_2}^{(F)} = 2^{16} \times \begin{pmatrix} 0 & 0 & -288 & 0 & 0 \\ 0 & -144 & 0 & 288 & 0 \\ -288 & 0 & 192 & 0 & -36 \\ 0 & 288 & 0 & -66 & -12 \\ 0 & 0 & -36 & -12 & -1 \end{pmatrix}, \quad (4.16)$$

where we inserted the superscript to indicate that these coefficients define $W_F^{(16)}$ in the fundamental representation.

For the Wilson loop in the adjoint representation of the SU(2) the calculation of (3.10) leads to

$$f_{\ell_1 \ell_2}^{(A)} = 2^{20} \times \begin{pmatrix} 0 & 0 & -72 & 0 & 0 & 0 \\ 0 & 144 & -360 & 312 & 0 & 0 \\ -72 & -360 & 408 & 120 & -144 & 0 \\ 0 & 312 & 120 & -150 & -48 & 16 \\ 0 & 0 & -144 & -48 & 24 & 8 \\ 0 & 0 & 0 & 16 & 8 & 1 \end{pmatrix}. \quad (4.17)$$

We observe that f_{11} vanishes for both matrices. This ensures a finiteness of the sum in (3.19).

Finally, we substitute (4.16) and (4.17) into (3.19), replace $d_F = 2, d_A = 3$ and obtain the one-instanton correction to the cusp anomalous dimension in the fundamental and

adjoint representations of the SU(2)

$$\begin{aligned}\Gamma_{\text{cusp}}^{(F)} &= \frac{387}{8192} \left(\frac{g^2}{4\pi^2} \right)^4 (e^{2\pi i\tau} + e^{-2\pi i\bar{\tau}}), \\ \Gamma_{\text{cusp}}^{(A)} &= -\frac{4}{15} \left(\frac{g^2}{4\pi^2} \right)^4 (e^{2\pi i\tau} + e^{-2\pi i\bar{\tau}}),\end{aligned}\tag{4.18}$$

where we added the contribution of anti-instanton. The following comments are in order.

The relations (4.18) define nonperturbative corrections to the cusp anomalous dimension for two different representations of the SU(2) group. Perturbative corrections to $\Gamma_{\text{cusp}}(g^2)$ are known to verify the so-called Casimir scaling up to order $O(g^6)$. Namely, perturbative contribution to $\Gamma_{\text{cusp}}(g^2)$ depends on the representation R through the quadratic Casimir only, $\Gamma_{\text{cusp}}(g^2) \sim C_R$. This property is violated however at order $O(g^8)$ due to the appearance of higher Casimirs [32]. We can easily check using (4.18) that the instanton corrections do not verify the Casimir scaling. If this property were true, the ratio of two expressions in (4.18) would be equal to the ratio of the quadratic Casimir operators C_F/C_A with $C_F = 3/4$ and $C_A = 2$ in the fundamental and adjoint representations of the SU(2), respectively. Obviously, the expressions (4.18) do not have this property, not to mention that the instanton corrections have an opposite sign for the two representations.

The relations (4.18) describe the leading instanton correction to the cusp anomalous dimension for the SU(2) gauge group. Following [22], we can generalize them to the SU(N) gauge group. In this case, the instantons have the additional $4N - 8$ bosonic modes describing the embedding of the SU(2) instanton into the SU(N) and $8N - 16$ ‘nonexact’ fermionic modes. Their contribution amounts to multiplying (4.18) by the factor of $(2N - 2)!/[2^{2N-3}(N - 1)!(N - 2)!]$. For large N , the relations (4.18) can be also extended to the multi-instanton sector [23]. In this limit, the integral over the moduli space of instantons is dominated by the saddle point in which all instantons are at the same position x_0 , have the same size ρ and lie in commuting SU(2) blocks inside the SU(N). Up to overall $O(g^8)$ factor, the resulting expressions are similar to those given in [23].

5 Concluding remarks

In this paper, we have studied instanton corrections to the four-point correlation function of half-BPS operators in $\mathcal{N} = 4$ SYM in the light-cone limit when operators become null separated in a sequential manner. Perturbative corrections to the correlation function in this limit are enhanced by logarithms of vanishing cross ratios and can be summed to all orders in the coupling. Previous studies revealed that in the semiclassical approximation the instanton corrections are suppressed in the light-cone limit by powers of the cross ratios. It is natural to ask whether this result is an artefact of the approximation or an intrinsic feature of instantons.

To answer this question we exploited the relation between the leading asymptotic behaviour of the correlation function and light-like rectangular Wilson loop. Analysing this relation we found an interesting interplay between semiclassical and quantum instanton

corrections.⁹ Namely, having computed the light-like Wilson loop in the semiclassical approximation, we were able to identify the leading instanton contribution to the correlation function in the light-cone limit. In the conventional approach, the same correction would correspond to taking into account the contribution to the correlation function of quantum fluctuations of instantons to forth order in perturbation theory.

We also demonstrated that the light-like rectangular Wilson loop satisfies conformal Ward identities and identified the leading instanton correction to the cusp anomalous dimension. Making use of this result, we can determine the leading instanton contribution to anomalous dimension of twist-two operators with large spin (2.4) and answer the question of how instantons modify the light-cone asymptotic behaviour of the four-point correlation function (1.3).

At weak coupling, $\mathcal{G}(u, v)$ receives both perturbative and instanton corrections enhanced by powers of $\ln u$ and $\ln v$. They arise due to logarithmic scaling (2.4) of the anomalous dimension of the twist-two operators with large spin, $S \sim u^{-1/2}$ or $S \sim v^{-1/2}$, exchanged in different OPE channels. Following [19, 20], such logarithmically enhanced corrections can be resummed leading to

$$\mathcal{G}(u, v) \sim \exp\left(-\frac{\hat{u} \hat{v}}{2\Gamma_{\text{cusp}}(g^2)}\right) \times e^{-2\Gamma_{\text{cusp}}(g^2)\partial_{\hat{u}}\partial_{\hat{v}}} \left[e^{\gamma_E(\hat{u}+\hat{v})/2} \Gamma\left(1 - \frac{1}{2}\hat{u}\right) \Gamma\left(1 - \frac{1}{2}\hat{v}\right) \right]^2, \tag{5.1}$$

where $\hat{u} = \gamma_{S=1/\sqrt{u}}$ and $\hat{v} = \gamma_{S=1/\sqrt{v}}$ are the anomalous dimensions (2.4) evaluated for the values of spins mentioned above, $\Gamma_{\text{cusp}}(g^2)$ is given by the sum of perturbative and instanton contributions and γ_E is Euler’s constant. The relation (5.1) has the expected factorized form (1.3), the first factor on the right-hand side of (5.1) comes from the rectangular light-like Wilson loop W_4 whereas the second one from the jet function J .

The relation (5.1) develops poles at even positive \hat{u} and \hat{v} . These poles have a clear physical meaning and have important consequences for the S –duality properties of the four-point correlation function. We recall that (5.1) takes into account the contribution of twist-two operators only. For $\hat{u} = 2$ (or $\hat{v} = 2$) we encounter a level crossing phenomenon [33] — the twist-two operators acquire anomalous dimension 2 and collide with the twist-four operators. The appearance of spurious poles in (5.1) is a consequence of ignoring the contribution of the latter operators to (5.1). To obtain a reliable prediction for $\mathcal{G}(u, v)$ in the vicinity of the pole we have to resolve the mixing of twist-two and twist-four operators and include the contribution of both to (5.1). To avoid remaining poles of $\mathcal{G}(u, v)$ we have to take into account the mixing with operators of higher twist.

We can arrive at the same conclusion by examining properties of the both sides of (5.1) under the S –duality transformations. The relation between the correlation function and light-like Wilson loop $\mathcal{G}(u, v) \sim W_4$ cannot hold for an arbitrary coupling since the two quantities have different properties. Indeed, the S –duality maps Wilson loop into ’t Hooft loop while the four-point correlation function $\mathcal{G}(u, v)$ remains invariant. To restore the S –duality of $\mathcal{G}(u, v)$, the higher twist contribution has to be added to (5.1). This problem deserves further investigation.

⁹Similar phenomenon has been previously observed for the Konishi operator [18].

The above analysis can be extended to n -point correlation function of half-BPS operators G_n . At weak coupling, the leading asymptotics of G_n in the light-cone limit is described by n -gon light-like Wilson loop W_n . It receives both perturbative and instanton corrections and satisfies the conformal Ward identities [31]. For $n = 5$ the conformal symmetry uniquely fixes the form of W_5 in terms of the cusp anomalous dimension. For $n \geq 6$ it leaves a freedom of adding to W_n a function of cross ratios, the so-called remainder function. This function has been studied in planar $\mathcal{N} = 4$ SYM where it was found to have a number of remarkable properties reflecting integrability of theory [34]. It would be interesting to compute the leading instanton correction to the remainder function and to understand whether some of its symmetries survive in the presence of nonperturbative effects.

The scattering amplitudes are known to be dual to the light-like Wilson loops in planar $\mathcal{N} = 4$ SYM. One may wonder whether the same relation holds for finite N in the presence of instantons. The duality implies that infrared divergences of amplitudes should match ultraviolet (cusp) divergences of Wilson loops. We have shown in this paper that the instanton corrections to light-like Wilson loops have the cusp divergences of the same double logarithmic form as in perturbation theory. This is not the case however for infrared divergences of the scattering amplitudes. These divergences come from integration over instantons with large size ρ and have a power-like dependence on the infrared cutoff. The mismatch in the form of ultraviolet and infrared divergences points towards the breaking of the above mentioned duality between the instanton contribution to scattering amplitudes and light-like Wilson loops.

Acknowledgments

We would like to thank Fernando Alday, Emery Sokatchev and Arkady Tseytlin for useful discussions.

A Conventions

Throughout the paper we use Greek letters, $\alpha, \dot{\alpha}, \dots$ for Lorentz indices and Latin letters i, j, \dots for the $SU(2)$ indices.

We use the Pauli matrices $\sigma_{\alpha\dot{\alpha}}^\mu = (1, i\sigma)$ to convert an arbitrary Euclidean four-vector $x_\mu = (x_1, x_2, x_3, x_4)$ into 2×2 matrix

$$x_{\alpha\dot{\alpha}} = x_\mu \sigma_{\alpha\dot{\alpha}}^\mu = \begin{bmatrix} ix_3 - x_4 & ix_1 - x_2 \\ ix_1 + x_2 & -ix_3 - x_4 \end{bmatrix}. \tag{A.1}$$

Its indices are raised and lowered with a help of an antisymmetric tensor

$$x^\alpha{}_{\dot{\alpha}} = \epsilon^{\alpha\beta} x_{\beta\dot{\alpha}}, \quad x_\alpha{}^{\dot{\alpha}} = x_{\alpha\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}}, \quad x^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} x_{\beta\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}},$$

with $\epsilon_{\alpha\beta} \epsilon^{\gamma\beta} = \delta_\alpha^\gamma$ and $\epsilon_{12} = \epsilon^{12} = 1$. The product of matrices is defined as

$$(x_1 x_2)_{\alpha\beta} = (x_1)_{\alpha\dot{\alpha}} (x_2)_{\dot{\beta}\beta} = (x_1)_{\alpha\dot{\alpha}} (x_2)_{\beta\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}}. \tag{A.2}$$

For the scalar product of Euclidean vectors $(xy) = \sum_i x_i y_i$ we have

$$(xy) = \frac{1}{2} x_{\dot{\alpha}}^{\alpha} y_{\alpha}^{\dot{\alpha}} = \frac{1}{2} x_{\alpha\dot{\alpha}} y_{\beta\dot{\beta}} \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}}. \quad (\text{A.3})$$

Allowing vectors to have complex components, we can define Euclidean analog of light-like vectors n_{μ} satisfying $n^2 = 0$. In spinor notations $n_{\alpha\dot{\alpha}}$ factorizes into the product of commuting spinors $n_{\alpha\dot{\alpha}} = |n\rangle[n|$ leading to

$$(n x n)_{\alpha\dot{\alpha}} = |n\rangle[n|x|n\rangle[n| = -2(xn)n_{\alpha\dot{\alpha}}, \quad (\text{A.4})$$

where $[n|x|n\rangle = n_{\alpha\dot{\alpha}} x^{\alpha\dot{\alpha}} = -2(xn)$. Notice that in Minkowski signature the expression on the right-hand side has an opposite sign. The reason for this is that the definition of the scalar product differs by sign in Minkowski and Euclidean signatures, $(xy)_{\text{M}} = -(xy)_{\text{E}}$ for $x_0 = ix_4$.

For the SU(2) matrices A_i^j we use similar conventions for raising and lowering indices

$$A_i^j = A_{ik} \epsilon^{kj}, \quad A_{ij} = A_i^k \epsilon_{jk}. \quad (\text{A.5})$$

The product of the SU(2) matrices is defined as

$$(AB)_{ik} = A_i^j B_{jk} = A_{ij} \epsilon^{jl} B_{lk}, \quad (ABC)_{in} = A_i^j B_j^k C_{kn}. \quad (\text{A.6})$$

In particular, the rectangular light-like Wilson loop W_4 is given by the product of four SU(2) matrices $E_{i,i+1} \equiv E(x_i, x_{i+1})$ defined as Wilson lines in the fundamental representation of the SU(2) evaluated along the light-like segments $[x_i, x_{i+1}]$

$$\begin{aligned} W_4 &= (E_{12})_{i_1}{}^{i_2} (E_{23})_{i_2}{}^{i_3} (E_{34})_{i_3}{}^{i_4} (E_{41})_{i_4}{}^{i_1} \\ &= (E_{12})_{i_1 k_1} \epsilon^{k_1 i_2} (E_{23})_{i_2 k_2} \epsilon^{k_2 i_3} (E_{34})_{i_3 k_3} \epsilon^{k_3 i_4} (E_{41})_{i_4 k_4} \epsilon^{k_4 i_1}. \end{aligned} \quad (\text{A.7})$$

Computing instanton corrections we encounter matrices in the mixed representation, e.g. $(x_1 x_2)_{ij}$ with matrices x_1 and x_2 given by (A.1). By definition, they are given by expression like (A.2) in which (chiral) Lorentz indices are identified with the SU(2) indices

$$(x_1 x_2)_{ij} \equiv (x_1)_{i\dot{\alpha}} (x_2)_{\dot{\alpha}j}. \quad (\text{A.8})$$

B Conformal properties

Since the equations of motion in $\mathcal{N} = 4$ SYM are invariant under conformal transformations, these transformations should map one classical solution into another one.

To check conformal properties of instantons, we use inversions. They act on both space-time and collective coordinates of instantons

$$I[x_i^{\mu}] = \frac{x_i^{\mu}}{x_i^2}, \quad I[x_0^{\mu}] = \frac{x_0^{\mu}}{x_0^2 + \rho^2}, \quad I[\rho] = \frac{\rho}{x_0^2 + \rho^2}. \quad (\text{B.1})$$

For fermionic modes, the inversion is defined as

$$I[\xi_{\alpha}^A] = \bar{\eta}_{\dot{\alpha}}^A, \quad I[\bar{\eta}_{\dot{\alpha}}^A] = \xi_{\alpha}^A. \quad (\text{B.2})$$

The instanton fields depend on a linear combination of these modes $\zeta = \xi + x\bar{\eta}$. As follows from (B.2), it transforms covariantly under inversions

$$I[\zeta_\alpha^A(x)] = \frac{x_{\dot{\alpha}}^\beta}{x^2} \zeta_\beta^A(x). \quad (\text{B.3})$$

Notice that the inversions change the chirality of Lorentz indices.

We can use explicit expressions for various fields (gauge field, scalar and gaugino) to verify that, up to compensating gauge transformations, they transform under inversions as conformal primary fields. For instance, the gauge field (2.8) transforms as

$$I[A^{\dot{\alpha}\beta}] = x_{\dot{\gamma}}^\alpha x_{\dot{\gamma}}^{\dot{\beta}} (U^\dagger A^{\dot{\gamma}\gamma} U + U^\dagger \partial^{\dot{\gamma}\gamma} U), \quad (\text{B.4})$$

where the SU(2) matrices are multiplied according to (A.6) and the compensating gauge transformation is

$$U_i^{i'} = \delta_i^\alpha \delta_\alpha^{i'} \frac{x_{\dot{\alpha}}^\alpha}{(x^2)^{1/2}}, \quad (U^\dagger)_{j'j} = \delta_\alpha^j \delta_{j'}^{\dot{\alpha}} \frac{x_{\dot{\alpha}}^\alpha}{(x^2)^{1/2}}. \quad (\text{B.5})$$

Here the product of Kronecker delta-functions identifies the SU(2) indices of U with the Lorentz indices of x . Replacing the gauge field in (B.4) with its general expression (2.8), we find that the lowest component $A^{(0)}$ satisfies the same relation (B.4) whereas for higher components we have $I[A^{(n),\dot{\alpha}\beta}] = x_{\dot{\gamma}}^\alpha x_{\dot{\gamma}}^{\dot{\beta}} U^\dagger A^{(n),\dot{\gamma}\gamma} U$. Applying (B.4) we obtain that the light-like Wilson line transforms under the inversions as

$$I[E(x_1, x_2)] = U^\dagger(x_1) E(x_1, x_2) U(x_2). \quad (\text{B.6})$$

An immediate consequence of this relation is that the light-like Wilson loop (A.7) is invariant under the conformal transformations

$$I[W_4] = W_4. \quad (\text{B.7})$$

Substituting (4.1) into (B.6) we find that all components $E^{(4n)}(x_1, x_2)$ of the Wilson line (4.1) have to satisfy (B.6). For the lowest component (4.9), this can be verified by direct calculation. For higher components, we can use (B.6) to argue that they have the general form (4.14).

To show this, we introduce a pair of two-dimensional SU(2) vectors

$$|\tilde{\zeta}_1^A\rangle = \left(\frac{\rho}{x_{10}^2 + \rho^2} \right)^{1/2} \zeta_i^A(x_1), \quad E^{(0)}(x_1, x_2) |\tilde{\zeta}_2^A\rangle, \quad (\text{B.8})$$

where $|\tilde{\zeta}_2^A\rangle$ is obtained from $|\tilde{\zeta}_1^A\rangle$ by replacing x_1 with x_2 . Making use of (B.3) and (B.5) we verify that both vectors transform under the inversions in the same way, e.g. $I[|\tilde{\zeta}_1^A\rangle] = U^\dagger(x_1) |\tilde{\zeta}_1^A\rangle$. In the similar manner, we can show that the vectors

$$\langle \tilde{\zeta}_2^B | = \left(\frac{\rho}{x_{20}^2 + \rho^2} \right)^{1/2} \zeta^{jB}(x_2), \quad \langle \tilde{\zeta}_1^B | E^{(0)}(x_1, x_2) \quad (\text{B.9})$$

transform covariantly under inversions with the same weight, e.g. $I[|\tilde{\zeta}_2^B\rangle] = \langle \tilde{\zeta}_2^B | U(x_2)$. Then, taking the tensor product of vectors (B.8) and (B.9), we can define four 2×2 matrices each satisfying (B.6). These matrices form the basis over which $E^{(4n)}(x_1, x_2)$ can be expanded. The corresponding expansion coefficients depend on the conformal invariants. The latter are given by the scalar product of vectors of the form (4.15). Notice that expressions (B.8) and (B.9) simplify in the conformal gauge (4.11), e.g. $|\tilde{\zeta}_1^A\rangle = \rho^{-1/2}|\zeta_1^A\rangle$ and $|\tilde{\zeta}_2^A\rangle = \rho^{-1/2}|\zeta_2^A\rangle$. For the component $E^{(8)}(x_1, x_2)$, this leads to (4.14).

Let us show that the two top components of the Wilson line vanish, $E^{(12)}(x_1, x_2) = E^{(16)}(x_1, x_2) = 0$ (see eq. (4.3)). The top component is proportional to the product of all fermion modes $E^{(16)}(x_1, x_2) \sim \prod_{\alpha,A} \zeta_{1,\alpha}^A \zeta_{2,\alpha}^A$. Since $\langle p_1 \zeta_1 \rangle = \langle p_1 \zeta_2 \rangle$ for $x_{12} = |p_1\rangle\langle p_1|$ (see (D.8)), we have $E^{(16)}(x_1, x_2) \sim \prod_A \langle p_1 \zeta_1^A \rangle^2 = 0$. The component $E^{(12)}(x_1, x_2)$ has the form similar to (4.14) with P_{CD}, \dots, S_{CD} being homogenous polynomials of degree 5 in variables (4.15). As we will see in a moment, these variables are proportional to $\langle p_1 \zeta_1^A \rangle$. Then, P_{CD}, \dots, S_{CD} , being homogenous polynomials in $\langle p_1 \zeta_1^A \rangle$ of degree 5, have to vanish since they are necessarily proportional to the square of a fermion mode leading to $E^{(12)}(x_1, x_2) \sim \langle p_1 \zeta_1^A \rangle^2 = 0$. Indeed, let us choose $|\zeta_1^A\rangle \sim |p_1\rangle$. It follows from $\zeta_1 - \zeta_2 = x_{12}\bar{\eta}$ that $|\zeta_2^A\rangle \sim |p_1\rangle$. The first two expressions in (4.15) obviously vanish in this case whereas the last one reduces to

$$\langle p_1 | E^{(0)}(x_1, x_2) | p_1 \rangle = \frac{1}{\rho^2} \langle p_1 | x_{10} x_{20} | p_1 \rangle = -\frac{1}{\rho^2} \langle p_1 | x_{10} x_{12} | p_1 \rangle = 0. \tag{B.10}$$

Here in the first relation we used (D.9) and in the second one replaced $x_{20} = x_{10} - x_{12}$. Thus, all expressions in (4.15) vanish for $|\zeta_1^A\rangle \sim |p_1\rangle$ and, therefore, they are proportional to $\langle p_1 \zeta_1^A \rangle$.

C D-functions in the light-cone limit

The integral over collective coordinates of instantons can be expressed in terms of D -functions. For $x_{ij}^2 \neq 0$ they are defined as [35]

$$D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} = \int d^4 x_0 \int \frac{d\rho}{\rho^5} \prod_i \left(\frac{\rho}{x_{i0}^2 + \rho^2} \right)^{\Delta_i}. \tag{C.1}$$

For our purposes it is sufficient to consider the special case $\Delta_1 = \Delta_3$ and $\Delta_2 = \Delta_4$. Defining

$$\bar{D}_{\Delta_1 \Delta_2} = (x_{13}^2)^{\Delta_1} (x_{24}^2)^{\Delta_2} D_{\Delta_1 \Delta_2 \Delta_1 \Delta_2}, \tag{C.2}$$

we find that, in virtue of conformal symmetry, $\bar{D}_{\Delta_1 \Delta_2}$ only depends on the cross-ratios u and v . It admits the Mellin integral representation [36]

$$\bar{D}_{\Delta_1 \Delta_2} = K \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} u^{j_1} v^{j_2} \Gamma(j_1 + j_2 + \Delta_1) \Gamma(j_1 + j_2 + \Delta_2) [\Gamma(-j_1) \Gamma(-j_2)]^2, \tag{C.3}$$

where $K = \pi^2 \Gamma(\Delta_1 + \Delta_2 - 2) / [2\Gamma^2(\Delta_1) \Gamma^2(\Delta_2)]$ and integration goes along imaginary axis slightly to the left from the origin, $0 < \delta < 1$.

Let us examine (C.3) in the light-cone limit $x_{i,i+1}^2 \rightarrow 0$, or equivalently $u, v \rightarrow 0$. In this limit, the \bar{D} -function develops logarithmic singularities in u and v . Indeed, closing the integration contour in (C.3) to the right half-plane and picking up the residue at $j_1 = j_2 = 0$ we find

$$\bar{D}_{\Delta_1\Delta_2} = \frac{\pi^2}{2} \frac{\Gamma(\Delta_1 + \Delta_2 - 2)}{\Gamma(\Delta_1)\Gamma(\Delta_2)} \left[\ln u \ln v + (\ln u + \ln v)(C_1 + C_2 - 2\psi(1)) \right] + \dots, \quad (\text{C.4})$$

where $C_i = \psi(\Delta_i)$ is expressed in terms of Euler ψ -function. Here dots denote terms suppressed by powers of u and v .

Computing instanton corrections to light-like Wilson loop we encounter the same integral (C.2) but evaluated for $x_{i,i+1}^2 = 0$. To regularize its divergences we modify the integration measure in (C.1) following (3.13) and arrive at $\bar{D}_{\Delta_1\Delta_2}^{(\epsilon)}$ defined in (3.17). We expect that $\ln u \ln v$ singularity of (C.4) should translate into a double pole, $\bar{D}_{\Delta_1\Delta_2}^{(\epsilon)} \sim 1/\epsilon^2$. Performing integration in (3.17) we obtain

$$\bar{D}_{\Delta_1\Delta_2}^{(\epsilon)} = 2K(x_{13}^2)^{\Delta_1}(x_{24}^2)^{\Delta_2}\mu^{-2\epsilon} \int_0^\infty \prod_{i=1}^4 dt_i t_i^{\Delta_i-1} \left(\sum_i t_i \right)^{2\epsilon} e^{-x_{13}^2 t_1 t_3 - x_{24}^2 t_2 t_4}, \quad (\text{C.5})$$

with $\Delta_3 = \Delta_1$ and $\Delta_4 = \Delta_2$. For small ϵ the calculation of this integral yields

$$\begin{aligned} \bar{D}_{\Delta_1\Delta_2}^{(\epsilon)} = \pi^2 \frac{\Gamma(\Delta_1 + \Delta_2 - 2)}{\Gamma(\Delta_1)\Gamma(\Delta_2)} & \left[\frac{1}{\epsilon^2} (\mu^2 x_{13}^2)^{-\epsilon} + \frac{1}{\epsilon^2} (\mu^2 x_{24}^2)^{-\epsilon} \right. \\ & \left. + \frac{C_2}{\epsilon} (\mu^2 x_{13}^2)^{-\epsilon} + \frac{C_1}{\epsilon} (\mu^2 x_{24}^2)^{-\epsilon} - \frac{1}{2} \ln^2(x_{13}^2/x_{24}^2) + C_1 C_2 - \frac{2}{3} \pi^2 + O(\epsilon) \right], \quad (\text{C.6}) \end{aligned}$$

with C_i the same as in (C.4).

We verify that $\bar{D}_{\Delta_1\Delta_2}^{(\epsilon)}$ satisfies the evolution equation

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} \right)^2 \bar{D}_{\Delta_1\Delta_2}^{(\epsilon)} = 2\pi^2 \frac{\Gamma(\Delta_1 + \Delta_2 - 2)}{\Gamma(\Delta_1)\Gamma(\Delta_2)}, \quad (\text{C.7})$$

which should be compared with (3.2). In distinction from (C.4), the conformal invariance of $\bar{D}_{\Delta_1\Delta_2}^{(\epsilon)}$ is broken by light-cone singularities that appear as poles in ϵ . We notice that the poles depend on one of the kinematical variables, x_{13}^2 or x_{24}^2 , and, therefore, they do not contribute to the mixed derivative of $\bar{D}_{\Delta_1\Delta_2}^{(\epsilon)}$ with respect to both variables. Applying this derivative to (C.4) and (C.6) we arrive at the same result

$$\frac{\partial}{\partial \ln x_{13}^2} \frac{\partial}{\partial \ln x_{24}^2} \bar{D}_{\Delta_1\Delta_2} = \frac{\partial}{\partial \ln x_{13}^2} \frac{\partial}{\partial \ln x_{24}^2} \bar{D}_{\Delta_1\Delta_2}^{(\epsilon)} = \pi^2 \frac{\Gamma(\Delta_1 + \Delta_2 - 2)}{\Gamma(\Delta_1)\Gamma(\Delta_2)}. \quad (\text{C.8})$$

This relation implies that conformal anomaly of $\bar{D}_{\Delta_1\Delta_2}^{(\epsilon)}$ is annihilated by the mixed derivative.

D Gauge field in the instanton background

In this appendix, we present explicit expressions for different components of the gauge field (2.8). The lowest component $A^{(0)}$ is given by the BPST instanton (4.7). The remaining

components $A^{(4)}$ and $A^{(8)}$ were derived in [26]

$$\begin{aligned} A_{\alpha\dot{\alpha}}^{(4)} &= -\frac{1}{12}\epsilon_{ABCD}\zeta_{\alpha}^A\zeta^{\beta B}(\zeta^C D_{\beta\dot{\alpha}}F\zeta^D) - \frac{1}{2}\epsilon_{ABCD}\zeta_{\alpha}^A\bar{\eta}_{\dot{\alpha}}^B(\zeta^C F\zeta^D), \\ A_{\alpha\dot{\alpha}}^{(8)} &= -\frac{3}{2}\zeta^8[D_{\alpha\dot{\alpha}}F^{\beta\gamma}, F_{\beta\gamma}] - \frac{3}{2}(\zeta^7)_{\beta A}\bar{\eta}_{\dot{\alpha}}^A[F^{\beta\gamma}, F_{\gamma\alpha}], \end{aligned} \quad (\text{D.1})$$

where $D_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} + A_{\alpha\dot{\alpha}}^{(0)}$ and $F_{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}}D_{(\alpha\dot{\alpha}}D_{\beta)\dot{\beta}}$ is a self-dual (chiral) part of the gauge strength tensor of the BPST instanton

$$F_{\alpha\beta,ij} = -8\rho^2 \frac{\epsilon_{i\alpha}\epsilon_{j\beta} + \epsilon_{j\alpha}\epsilon_{i\beta}}{[(x-x_0)^2 + \rho^2]^2}. \quad (\text{D.2})$$

The Grassmann variable $\zeta_{\alpha}^A(x)$ is given by a linear combination of fermion modes

$$\zeta_{\alpha}^A(x) = \xi_{\alpha}^A + x_{\alpha\dot{\alpha}}\bar{\eta}^{\dot{\alpha}A}, \quad (\text{D.3})$$

and the following notations are used

$$(\zeta^C F\zeta^D) = \zeta^{\alpha C}F_{\alpha\beta}\zeta^{\beta D}, \quad (\zeta^7)_{\beta A} = \partial\zeta^8/\partial\zeta^{\beta A}, \quad \zeta^8 = \prod_{\alpha A}\zeta^{\alpha A}. \quad (\text{D.4})$$

Notice that the second term on the right-hand side of (D.1) depends on $\bar{\eta}$. As explained in [26], its form is uniquely fixed by the conformal symmetry. Namely, it can be determined from the requirement for (D.1) to satisfy (B.4).

To compute the Wilson line (4.2), we need the expressions for the projection of the gauge field on the edges of the light-like rectangle. Let us consider the segment $[x_1, x_2]$ and define the SU(2) matrix

$$\mathcal{A}^{(n)}(t) = i\dot{x}^{\mu}(t)A_{\mu}^{(n)}(x(t)) = -\frac{1}{2}x_{12}^{\dot{\alpha}\alpha}A_{\alpha\dot{\alpha}}^{(n)}(x(t)) = -\frac{1}{2}\langle p_1|A^{(n)}(x(t))|p_1\rangle, \quad (\text{D.5})$$

where $x(t) = (1-t)x_1 + tx_2$ and $x_{12}^{\dot{\alpha}\alpha} = |p_1\rangle\langle p_1|$ is a light-like vector. To simplify the calculation we use the gauge (4.11). Replacing the gauge field in (D.5) with (4.7) and (D.1) we find after some algebra

$$\begin{aligned} \mathcal{A}^{(0)} &= \frac{1}{\rho^2}x_{10}x_{20}, \\ \mathcal{A}^{(4)} &= \frac{4}{\rho^2}\epsilon_{ABCD}|\zeta^A(x(t))\rangle\langle\zeta_1^B|E^{(0)}(x_1, x_2)|\zeta_2^C\rangle\langle\zeta^D(x(t))|, \\ \mathcal{A}^{(8)} &= \frac{8}{15\rho^4}\zeta_{AB}^6(x(t))\left[E^{(0)}(x(t), x_1)|\zeta_1^A\rangle\langle\zeta_2^B|E^{(0)}(x_2, x(t))\right. \\ &\quad \left.- E^{(0)}(x(t), x_2)|\zeta_2^A\rangle\langle\zeta_1^B|E^{(0)}(x_1, x(t))\right], \end{aligned} \quad (\text{D.6})$$

where $\zeta_{AB}^6 = \epsilon_{ACDE}\epsilon_{BC'D'E'}(\zeta^2)^{CC'}(\zeta^2)^{DD'}(\zeta^2)^{EE'}$ with $(\zeta^2)^{AB} = \zeta^{\alpha A}\zeta_{\alpha}^B$ and we used notation for the SU(2) matrices $|\zeta_1^A\rangle\langle\zeta_2^B| \equiv \zeta_{1i}^A\zeta_2^{Bj}$.

The dependence on fermion modes ξ_{α}^A and $\bar{\eta}^{\dot{\alpha}A}$ enters into (D.6) through linear combination (D.3)

$$\zeta_{\alpha}^A(x(t)) = (1-t)\zeta_{1\alpha}^A + t\zeta_{2\alpha}^A. \quad (\text{D.7})$$

Its value at the end points is denoted as $\zeta(x_1) = \zeta_1$ and $\zeta(x_2) = \zeta_2$. Since $(\zeta_1 - \zeta_2)_\alpha^A = (x_{12})_{\alpha\dot{\alpha}} \bar{\eta}^{\dot{\alpha}A}$, these variables satisfy the relation $(x_{12})^{\dot{\alpha}\alpha} (\zeta_1 - \zeta_2)_\alpha^A = 0$ or equivalently

$$\langle p_1 \zeta_1^A \rangle - \langle p_1 \zeta_2^A \rangle = 0, \tag{D.8}$$

where spinor $|p_1\rangle$ defines the light-like vector $x_{12}^{\dot{\alpha}\alpha} = |p_1\rangle\langle p_1|$.

We would like to stress that the relations (D.6) hold in the conformal gauge (4.11). The light-like Wilson line (4.9) is given in this gauge by the following expression

$$E^{(0)}(x_1, x_2) = 1 + \frac{x_{10}x_{20}}{\rho^2}. \tag{D.9}$$

The Wilson lines entering the expression for $\mathcal{A}^{(8)}(t)$ in (D.6) are linear functions of t

$$\begin{aligned} E^{(0)}(x_1, x(t)) &= 1 + t \frac{x_{10}x_{20}}{\rho^2}, & E^{(0)}(x(t), x_2) &= 1 + (1-t) \frac{x_{10}x_{20}}{\rho^2}, \\ E^{(0)}(x(t), x_1) &= 1 - t \frac{x_{10}x_{20}}{\rho^2}, & E^{(0)}(x_2, x(t)) &= 1 - (1-t) \frac{x_{10}x_{20}}{\rho^2}. \end{aligned} \tag{D.10}$$

We can use relations (D.6)–(D.10) to compute all components of the light-like Wilson line (4.2) in the conformal gauge (4.11). The lowest component is given by (D.9). For the remaining components we find after some algebra relations (4.13) and (4.14).

Comparing (D.9) with the general covariant expression (4.9) we observe that $E^{(0)}(x_1, x_2)$ is polynomial in x_i in the conformal gauge. Examining the relations (4.13) and (4.14) it is easy to see that the same is true for higher components of the light-like Wilson line. As a consequence, all components of the Wilson loop (4.5) are also polynomial in x_i in the conformal gauge. This leads to important consequences for the function $f_R(I_{13}, I_{24})$ defined in (3.11). It depends on the conformal invariants (3.14) which are given in the gauge (4.11) by

$$I_{13} = \frac{x_{13}^2}{\rho^2}, \quad I_{24} = \frac{x_{24}^2}{\rho^2}. \tag{D.11}$$

The fact that the expression on the right-hand side of (3.15) is polynomial in x_i in the conformal gauge implies that $f_R(I_{13}, I_{24})$ is in fact polynomial in I_{13} and I_{24} . In other words, the sum over ℓ_1 and ℓ_2 in (3.15), (3.16) and (3.19) actually contains a finite number of terms.

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References

- [1] C. Montonen and D.I. Olive, *Magnetic Monopoles as Gauge Particles?*, *Phys. Lett.* **72B** (1977) 117 [[INSPIRE](#)].
- [2] E. Witten and D.I. Olive, *Supersymmetry Algebras That Include Topological Charges*, *Phys. Lett.* **78B** (1978) 97 [[INSPIRE](#)].

- [3] H. Osborn, *Topological Charges for $N = 4$ Supersymmetric Gauge Theories and Monopoles of Spin 1*, *Phys. Lett.* **83B** (1979) 321 [INSPIRE].
- [4] J. Gomis, T. Okuda and D. Trancanelli, *Quantum 't Hooft operators and S-duality in $N = 4$ super Yang-Mills*, *Adv. Theor. Math. Phys.* **13** (2009) 1941 [arXiv:0904.4486] [INSPIRE].
- [5] J. Gomis and T. Okuda, *S-duality, 't Hooft operators and the operator product expansion*, *JHEP* **09** (2009) 072 [arXiv:0906.3011] [INSPIRE].
- [6] A.V. Belitsky, S. Vandoren and P. van Nieuwenhuizen, *Yang-Mills and D instantons*, *Class. Quant. Grav.* **17** (2000) 3521 [hep-th/0004186] [INSPIRE].
- [7] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *The calculus of many instantons*, *Phys. Rept.* **371** (2002) 231 [hep-th/0206063] [INSPIRE].
- [8] M. Bianchi, S. Kovacs and G. Rossi, *Instantons and Supersymmetry*, *Lect. Notes Phys.* **737** (2008) 303 [hep-th/0703142] [INSPIRE].
- [9] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *Hidden symmetry of four-point correlation functions and amplitudes in $N = 4$ SYM*, *Nucl. Phys. B* **862** (2012) 193 [arXiv:1108.3557] [INSPIRE].
- [10] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *Constructing the correlation function of four stress-tensor multiplets and the four-particle amplitude in $N = 4$ SYM*, *Nucl. Phys. B* **862** (2012) 450 [arXiv:1201.5329] [INSPIRE].
- [11] N. Beisert et al., *Review of AdS/CFT Integrability: An Overview*, *Lett. Math. Phys.* **99** (2012) 3 [arXiv:1012.3982] [INSPIRE].
- [12] B. Eden and A. Sfondrini, *Tessellating cushions: four-point functions in $\mathcal{N} = 4$ SYM*, *JHEP* **10** (2017) 098 [arXiv:1611.05436] [INSPIRE].
- [13] T. Fleury and S. Komatsu, *Hexagonalization of Correlation Functions*, *JHEP* **01** (2017) 130 [arXiv:1611.05577] [INSPIRE].
- [14] M. Bianchi, S. Kovacs, G. Rossi and Y.S. Stanev, *On the logarithmic behavior in $N = 4$ SYM theory*, *JHEP* **08** (1999) 020 [hep-th/9906188] [INSPIRE].
- [15] G. Arutyunov, S. Frolov and A. Petkou, *Perturbative and instanton corrections to the OPE of CPOs in $\mathcal{N} = 4$ SYM₄*, *Nucl. Phys. B* **602** (2001) 238 [Erratum *ibid.* **B 609** (2001) 540] [hep-th/0010137] [INSPIRE].
- [16] M. Bianchi, S. Kovacs, G. Rossi and Y.S. Stanev, *Properties of the Konishi multiplet in $N = 4$ SYM theory*, *JHEP* **05** (2001) 042 [hep-th/0104016] [INSPIRE].
- [17] S. Kovacs, *On instanton contributions to anomalous dimensions in $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **684** (2004) 3 [hep-th/0310193] [INSPIRE].
- [18] L.F. Alday and G.P. Korchemsky, *Revisiting instanton corrections to the Konishi multiplet*, *JHEP* **12** (2016) 005 [arXiv:1605.06346] [INSPIRE].
- [19] L.F. Alday, B. Eden, G.P. Korchemsky, J. Maldacena and E. Sokatchev, *From correlation functions to Wilson loops*, *JHEP* **09** (2011) 123 [arXiv:1007.3243] [INSPIRE].
- [20] L.F. Alday and A. Bissi, *Higher-spin correlators*, *JHEP* **10** (2013) 202 [arXiv:1305.4604] [INSPIRE].
- [21] M. Bianchi, A. Brandhuber, G. Travaglini and C. Wen, *Simplifying instanton corrections to $N = 4$ SYM correlators*, *JHEP* **04** (2014) 101 [arXiv:1312.3916] [INSPIRE].

- [22] N. Dorey, V.V. Khoze, M.P. Mattis and S. Vandoren, *Yang-Mills instantons in the large- N limit and the AdS/CFT correspondence*, *Phys. Lett. B* **442** (1998) 145 [[hep-th/9808157](#)] [[INSPIRE](#)].
- [23] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Multi-instanton calculus and the AdS/CFT correspondence in $N = 4$ superconformal field theory*, *Nucl. Phys. B* **552** (1999) 88 [[hep-th/9901128](#)] [[INSPIRE](#)].
- [24] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes*, *Nucl. Phys. B* **826** (2010) 337 [[arXiv:0712.1223](#)] [[INSPIRE](#)].
- [25] G.P. Korchemsky, *Asymptotics of the Altarelli-Parisi-Lipatov Evolution Kernels of Parton Distributions*, *Mod. Phys. Lett. A* **4** (1989) 1257 [[INSPIRE](#)].
- [26] L.F. Alday and G.P. Korchemsky, *Instanton corrections to twist-two operators*, *JHEP* **06** (2017) 008 [[arXiv:1609.08164](#)] [[INSPIRE](#)].
- [27] B. Eden, A.C. Petkou, C. Schubert and E. Sokatchev, *Partial nonrenormalization of the stress tensor four point function in $N = 4$ SYM and AdS/CFT*, *Nucl. Phys. B* **607** (2001) 191 [[hep-th/0009106](#)] [[INSPIRE](#)].
- [28] A.A. Belavin, A.M. Polyakov, A.S. Schwartz and Yu. S. Tyupkin, *Pseudoparticle Solutions of the Yang-Mills Equations*, *Phys. Lett.* **59B** (1975) 85 [[INSPIRE](#)].
- [29] M. Bianchi, M.B. Green, S. Kovacs and G. Rossi, *Instantons in supersymmetric Yang-Mills and D instantons in IIB superstring theory*, *JHEP* **08** (1998) 013 [[hep-th/9807033](#)] [[INSPIRE](#)].
- [30] B. Zumino, *Euclidean Supersymmetry and the Many-Instanton Problem*, *Phys. Lett. B* **69** (1977) 369.
- [31] J.M. Drummond, G.P. Korchemsky and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops*, *Nucl. Phys. B* **795** (2008) 385 [[arXiv:0707.0243](#)] [[INSPIRE](#)].
- [32] J. Frenkel and J.C. Taylor, *Nonabelian Eikonal Exponentiation*, *Nucl. Phys. B* **246** (1984) 231 [[INSPIRE](#)].
- [33] G.P. Korchemsky, *On level crossing in conformal field theories*, *JHEP* **03** (2016) 212 [[arXiv:1512.05362](#)] [[INSPIRE](#)].
- [34] B. Basso, A. Sever and P. Vieira, *Spacetime and Flux Tube S -Matrices at Finite Coupling for $N = 4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **111** (2013) 091602 [[arXiv:1303.1396](#)] [[INSPIRE](#)].
- [35] E. D'Hoker, D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Graviton exchange and complete four point functions in the AdS/CFT correspondence*, *Nucl. Phys. B* **562** (1999) 353 [[hep-th/9903196](#)] [[INSPIRE](#)].
- [36] G. Arutyunov, S. Frolov and A.C. Petkou, *Operator product expansion of the lowest weight CPOs in $\mathcal{N} = 4$ SYM₄ at strong coupling*, *Nucl. Phys. B* **586** (2000) 547 [Erratum *ibid.* **B** **609** (2001) 539] [[hep-th/0005182](#)] [[INSPIRE](#)].