Relativistic corrections to light-cone distribution amplitudes of S-wave B_c mesons and heavy quarkonia

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ABSTRACT: In collinear factorization, light-cone distribution amplitudes (LCDAs) are key ingredients to calculate the production rate of a hadron in high energy exclusive processes. For a doubly-heavy meson system (such as B_c , J/ψ , Υ etc), the LCDAs contain perturbative scales that can be integrated out and then are re-factorized into products of perturbatively calculable distribution parts and non-relativistic QCD matrix elements. In this re-factorization scheme, the LCDAs are known at next-to-leading order in the strong coupling constant α_s and at leading order in the velocity expansion. In this work, we calculate the $\mathcal{O}(v^2)$ corrections to twist-2 LCDAs of S-wave B_c mesons. These results are applicable to heavy quarkonia like $\eta_{c,b}$, J/ψ and Υ by setting $m_b = m_c$. We apply these relativistically corrected LCDAs to study their inverse moments and a few Gegenbauer moments which are important for phenomenological study. We point out that the relativistic corrections are sizable, and comparable with the next-to-leading order radiative corrections. These results for LCDAs are useful in future theoretical analyses of the productions of heavy quarkonia and B_c mesons.

KEYWORDS: QCD Phenomenology

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1 Introduction

A typical high energy process with hadrons involves physics from high scales, such as the center-of-mass collision energy, down to very low energy scales such as the mass of a proton. Without disentangling physics associated with these scales, it is nearly hopeless to obtain reliable theoretical predictions for any such process. The disentanglement is often referred to as factorization.

There exist two sorts of factorization schemes to deal with hard exclusive processes involving heavy quarkonia and B_c mesons, if we regard heavy quarkonia and B_c mesons as the non-relativistic bound-states of heavy quark and anti-quark. One is the non-relativistic QCD (NRQCD) factorization [1], in which the amplitude can be factorized into a product of the perturbatively calculable short-distance coefficient and the non-perturbative NRQCD matrix-element. For recent reviews of this approach, one can refer to refs. [2–4]. The other one is called the collinear factorization, which has already been established for a few decades [5, 6]. In this framework, the amplitudes of hard exclusive processes can be expressed as convolutions of the perturbatively calculable hard-kernels and the universal light-cone distribution amplitudes (LCDAs). It is necessary to mention at this moment that the factorization approaches do not always hold, but instead should be proved in a channel-by-channel manner. Once proved, these approaches can both clearly disentangle the short-distance and long-distance contributions.

In literatures, the collinear factorization and its extension k_T factorization have been adopted in refs. [7–17] to study various B_c decays, where the LCDA of B_c has been used to parametrize the non-perturbative physics in the system. Meanwhile the NRQCD factorization has been adopted in refs. [18–26], in which the non-perturbative physics reside in a few NRQCD matrix elements. The LHCb collaboration has started comprehensive experimental studies of the decays [27], and some phenomenological studies can be found in refs. [28–32].

Though the two factorization approaches are different in nature, it has been argued that since the LCDAs of quarkonium and B_c meson contain some perturbatively calculable scales like $\mathcal{O}(m^2)$, it is reasonable to consider the LCDAs can be further factorized into a product of a perturbatively calculable distribution part and a NRQCD matrix-element for the vacuum to quarkonium or B_c state transition [33]. This idea can be called as “re-factorization”. Some other approaches to bridge NRQCD and collinear factorization are proposed in refs. [34, 35].

The re-factorization scheme has been extended to the next-to-leading order $\mathcal{O}(\alpha_s)$ accuracy in refs. [33, 36–38], and the relations between the moments of LCDAs for heavy quarkonia and NRQCD matrix elements are known [39–42]. However, the factorization is only discussed at leading order of the velocity expansion v . In fact, the relativistic corrections are also important for the calculation of LCDAs of quarkonium and B_c meson. Relativistic corrections are often characterized by the relative velocity of heavy quarks within the bound states. It is estimated as $v^2 \sim 0.3$ for $\bar{c}c$ system which is sizable. Thus, in order to test the applications of LCDAs in quarkonium and B_c meson productions precisely, one should consider the relativistic corrections, which is the main focus of this paper. Recently, a new factorization scheme [43] has been proposed to improve the velocity expansion. In this paper, we will calculate the three relativistic corrected LCDAs for the S-wave B_c mesons, and the corresponding results for quarkonia like J/ψ can be easily deduced by setting $m_b = m_c$.

The rest of this paper is organized as follows: in section 2, we give the notations used in this paper, and present the definitions of the leading-twist LCDAs of the S-wave B_c mesons in terms of the matrix-elements of a certain class of non-local QCD operators. In section 3, we present our results of the LCDAs at $\mathcal{O}(\alpha_s^0 v^0)$, $\mathcal{O}(\alpha_s^0 v^2)$ and $\mathcal{O}(\alpha_s^1 v^0)$; in section 4, we will apply the results for the LCDAs and study the inverse moments and Gegenbauer moments of LCDAs. A comparison of relativistic corrections and radiative corrections is made. Finally, we summarize our work in section 5.

2 LCDAs for S-wave B_c mesons

2.1 Light-cone coordinates

As we will see below, LCDAs are defined in the light-cone coordinate frame, in which a four-vector a can be expressed as

$$a^\mu = n_+ \cdot a \frac{n_-^\mu}{2} + n_- \cdot a \frac{n_+^\mu}{2} + a_\perp^\mu. \quad (2.1)$$

Here n_+ and n_- are two light-like vectors, with $n_\pm^2 = 0$, $n_+ \cdot n_- = 2$ and $n_\pm \cdot a_\perp = 0$.

2.2 Definition of LCDAs

To define the LCDAs, one needs to introduce the gauge invariant non-local quark bilinear operator:

$$J[\Gamma](\omega) \equiv (\bar{b}W_c)(\omega n_+) \not{n}_+ \Gamma(W_c^\dagger c)(0), \quad (2.2)$$

where b and c are the quark fields. The Wilson-line

$$W_c(x) = \text{P exp} \left(i g_s \int_{-\infty}^0 ds \, n_+ A(x + s n_+) \right)$$

is a path-ordered exponential with the path along the n_+ direction and g_s is QCD coupling constant and $A_\mu \equiv A_\mu^a(x) T^a$, T^a are the generators of SU(3) group in the fundamental representation.

The leading-twist, i.e. twist-2, LCDAs can be defined from matrix elements of operator $J[\Gamma](\omega)$:

$$\langle B_c(^1S_0, P) | J[\gamma_5](\omega) | 0 \rangle = -i f_P n_+ P \int_0^1 dx \, e^{i\omega n_+ P x} \hat{\phi}_P(x; \mu), \quad (2.3a)$$

$$\langle B_c(^3S_1, P, \epsilon^*) | J[1](\omega) | 0 \rangle = f_V m_V n_+ \epsilon^* \int_0^1 dx \, e^{i\omega n_+ P x} \hat{\phi}_V^\parallel(x; \mu), \quad (2.3b)$$

$$\langle B_c(^3S_1, P, \epsilon^*) | J[\gamma_\perp^\alpha](\omega) | 0 \rangle = f_V^\perp n_+ P \epsilon_\perp^{*\alpha} \int_0^1 dx \, e^{i\omega n_+ P x} \hat{\phi}_V^\perp(x; \mu). \quad (2.3c)$$

Here f_P , f_V and $f_V^\perp$ are decay constants, P is the momentum of the B_c meson, ϵ^* is the polarization vector for the vector B_c meson. $\hat{\phi}_P(x)$, $\hat{\phi}_V(x)$ and $\hat{\phi}_V^\perp(x)$ are LCDAs for pseudo-scalar, longitudinally polarized and transversely polarized vector B_c mesons, respectively. x denotes the light-cone momentum fraction and $\bar{x} \equiv 1 - x$. μ is the renormalization scale. Note that the state $|B_c\rangle$ in eq. (2.3) is relativistically normalized.

The LCDAs are normalized as

$$\int_0^1 dx \hat{\phi}_P(x) = \int_0^1 dx \hat{\phi}_V^\parallel(x) = \int_0^1 dx \hat{\phi}_V^\perp(x) = 1. \quad (2.4)$$

In the following analysis, it is convenient to employ the Fourier transformed form of eq. (2.3)

$$\langle B_c(^1S_0, P) | Q[\gamma_5](x) | 0 \rangle = -i f_P \hat{\phi}_P(x), \quad (2.5a)$$

$$\langle B_c(^3S_1, P, \epsilon^*) | Q[1](x) | 0 \rangle = f_V \frac{m_V n_+ \epsilon^*}{n_+ P} \hat{\phi}_V^\parallel(x), \quad (2.5b)$$

$$\langle B_c(^3S_1, P, \epsilon^*) | Q[\gamma_\perp^\alpha](x) | 0 \rangle = f_V^\perp \epsilon_\perp^{*\alpha} \hat{\phi}_V^\perp(x), \quad (2.5c)$$

where the Fourier-transformed operator

$$Q[\Gamma](x) \equiv [(\bar{b}W_c)(\omega n_+) \not{n}_+ \Gamma(W_c^\dagger c)(0)]_{\text{F.T.}} = \int \frac{d\omega}{2\pi} e^{-i\omega n_+ P x} (\bar{b}W_c)(\omega n_+) \not{n}_+ \Gamma(W_c^\dagger c)(0). \quad (2.6)$$

It is worth noting that the operator $Q[\Gamma](x)$ is invariant under the longitudinal boost, i.e. $n_+ \rightarrow \alpha n_+$ and $n_- \rightarrow \alpha^{-1} n_-$ where α is an arbitrary positive real number. This leads to the conclusion that both the decay constants and the LCDAs of B_c mesons defined in eq. (2.5) are also boost-invariant. This allows us to calculate these decay constants and LCDAs in any reference frame. In this paper, we will choose the rest frame of B_c mesons, i.e.

$$P^\mu = (m_{B_c}, 0, 0, 0), \quad n_\pm^\mu = (1, 0, 0, \pm 1), \quad (2.7)$$

which is particularly convenient for matching the decay constants and LCDAs to NRQCD matrix-elements.

2.3 Re-factorization of the LCDAs

The re-factorization idea has been proposed in refs. [33, 36–38], where all of the LCDAs of quarkonia or B_c mesons can be factorized into products of perturbatively calculable distribution parts and non-perturbative NRQCD matrix elements. It means that, at the operator level, we have the generic matching equation

$$Q[\Gamma](x, \mu) = \sum_{n=0}^{\infty} \frac{d_n^\Gamma(x, \mu)}{M^{n+1}} O_{\Gamma,n}^{\text{NRQCD}}, \quad (2.8)$$

where n denotes the order of v -expansion, $d_n^\Gamma(x, \mu)$ is the short-distance coefficient as a distribution over the momentum fraction x , $O_{\Gamma,n}^{\text{NRQCD}}$ is the NRQCD operator which scales $\mathcal{O}(v^n)$ in the power-counting, and scale $M \equiv m_b + m_c$ is introduced to balance the mass dimensions of the NRQCD operators so that the short-distance coefficients $d_n^\Gamma(x, \mu)$ are set to be dimensionless. Therefore, the LCDAs of B_c meson can be expressed as

$$\langle B_c | Q[\Gamma](x, \mu) | 0 \rangle \simeq \sum_{n=0}^{\infty} \frac{d_n^\Gamma(x, \mu)}{M^{n+1}} \langle B_c | O_{\Gamma,n}^{\text{NRQCD}} | 0 \rangle. \quad (2.9)$$

In the present work, since we focus on the LCDAs of the S-wave B_c mesons, up to $\mathcal{O}(v^2)$, the NRQCD operators that we will consider are

$$O_0(^1S_0) \equiv \psi_b^\dagger \chi_c, \quad (2.10a)$$

$$O_0(^3S_1) \equiv \psi_b^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \chi_c, \quad (2.10b)$$

$$O_2(^1S_0) \equiv \psi_b^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \chi_c, \quad (2.10c)$$

$$O_2(^3S_1) \equiv \psi_b^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \chi_c. \quad (2.10d)$$

Here ψ_b and χ_c are the two-component effective fields for the b quark and $\bar{c}$ quark in the NRQCD, respectively, and $\psi_b^\dagger \overleftrightarrow{\mathbf{D}} \chi_c \equiv \psi_b^\dagger (\mathbf{D} \chi_c) - (\mathbf{D} \psi_b)^\dagger \chi_c$ with $\mathbf{D} = \boldsymbol{\nabla} - ig_s \mathbf{A}$.

Thus, up to $\mathcal{O}(v^2)$, we have the matching equations

$$-if_P \hat{\phi}_P(x) = \frac{d_0^P(x)}{M} \langle B_c(1S_0, P) | O_0(1S_0) | 0 \rangle + \frac{d_2^P(x)}{M^3} \langle B_c(1S_0, P) | O_2(1S_0) | 0 \rangle, \quad (2.11a)$$

$$f_V \hat{\phi}_V^\parallel(x) = \frac{d_0^V(x)}{M} \langle B_c(3S_1, P, \epsilon^*) | O_0(3S_1) | 0 \rangle + \frac{d_2^V(x)}{M^3} \langle B_c(3S_1, P, \epsilon^*) | O_2(3S_1) | 0 \rangle, \quad (2.11b)$$

$$f_V^\perp \hat{\phi}_V^\perp(x) = \frac{d_0^{V\perp}(x)}{M} \langle B_c(3S_1, P, \epsilon^*) | O_0(3S_1) | 0 \rangle + \frac{d_2^{V\perp}(x)}{M^3} \langle B_c(3S_1, P, \epsilon^*) | O_2(3S_1) | 0 \rangle. \quad (2.11c)$$

Here the ϵ^* is the polarization vector of $3S_1$ state, and $\epsilon \cdot \epsilon^* = 1$. $d_i^{P,V,V\perp}(x)$ ($i = 0, 2$) are perturbatively calculable Wilson coefficients. We should note that the matrix-elements of the NRQCD operators in eq. (2.11) are relativistically normalized.

By integrating over x in eq. (2.11), and by imposing the normalization conditions given in eq. (2.4), we can get the matching equations for the decay constants

$$-if_P = \frac{C_0^P}{M} \langle B_c(1S_0, P) | O_0(1S_0) | 0 \rangle + \frac{C_2^P}{M^3} \langle B_c(1S_0, P) | O_2(1S_0) | 0 \rangle, \quad (2.12a)$$

$$f_V = \frac{C_0^V}{M} \langle B_c(3S_1, P, \epsilon^*) | O_0(3S_1) | 0 \rangle + \frac{C_2^V}{M^3} \langle B_c(3S_1, P, \epsilon^*) | O_2(3S_1) | 0 \rangle, \quad (2.12b)$$

$$f_V^\perp = \frac{C_0^{V\perp}}{M} \langle B_c(3S_1, P, \epsilon^*) | O_0(3S_1) | 0 \rangle + \frac{C_2^{V\perp}}{M^3} \langle B_c(3S_1, P, \epsilon^*) | O_2(3S_1) | 0 \rangle, \quad (2.12c)$$

with the short-distance coefficients

$$C_i^\Gamma = \int_0^1 dx d_i^\Gamma(x), \quad i = 0, 2; \quad \Gamma = P, V, V_\perp. \quad (2.13)$$

Then, the LCDAs $\hat{\phi}(x)$ can be derived straight-forwardly. In the following, it is more convenient to express the LCDAs in the following expansions

$$\hat{\phi}_P(x) = \hat{\phi}_P^{(0,0)}(x) + \hat{\phi}_P^{(1,0)}(x) + \hat{\phi}_P^{(0,1)}(x), \quad (2.14a)$$

$$\hat{\phi}_V^\parallel(x) = \hat{\phi}_V^{\parallel(0,0)}(x) + \hat{\phi}_V^{\parallel(1,0)}(x) + \hat{\phi}_V^{\parallel(0,1)}(x), \quad (2.14b)$$

$$\hat{\phi}_V^\perp(x) = \hat{\phi}_V^{\perp(0,0)}(x) + \hat{\phi}_V^{\perp(1,0)}(x) + \hat{\phi}_V^{\perp(0,1)}(x), \quad (2.14c)$$

where the superscript (i, j) denotes the order of α_s and v^2 -expansion. $\hat{\phi}^{(0,0)}(x)$ and $\hat{\phi}^{(1,0)}(x)$ have been given in ref. [38]. The explicit expressions of $\hat{\phi}^{(0,1)}(x)$ which are the main results of this work will be presented in the next section.

3 Calculations of LCDAs for the S-wave B_c mesons

3.1 Perturbative matching

As we describe in the previous section, to get the decay constants and LCDAs of S-wave B_c mesons up to $\mathcal{O}(v^2)$, we need to obtain the short-distance coefficients $d_i^\Gamma(x)$ in eq. (2.11) and C_i^Γ in eq. (2.12) from the perturbative matching.

For the perturbative matching, one is allowed to choose any convenient process. After calculating the matrix elements in both full theory and effective field theory, one can

derive the short-distance coefficients by solving the matching equations. To do so, in the standard NRQCD matching procedure, we usually use the free b -quark and $\bar{c}$ -quark pair to replace the corresponding B_c meson state. Firstly, we calculate the corresponding on-shell amplitude in full QCD, then we expand the amplitude in terms of the relative momentum q^i so that each expanded term has a definite scaling in v -expansion. Finally, we can extract the short-distance coefficients by identifying the corresponding NRQCD matrix-elements in the expanded amplitude.

We set the momenta for on-shell b -quark and $\bar{c}$ -quark as

$$\begin{cases} p_b^\mu = (E_b, \mathbf{q}), & p_c^\mu = (E_c, -\mathbf{q}), & P^\mu = p_b^\mu + p_c^\mu = (E, \mathbf{0}) \\ E_b = m_b + \frac{\mathbf{q}^2}{2m_b} + \mathcal{O}(v^4), & E_c = m_c + \frac{\mathbf{q}^2}{2m_c} + \mathcal{O}(v^4), \\ E = E_b + E_c = M + \frac{\mathbf{q}^2}{2m_b} + \frac{\mathbf{q}^2}{2m_c} + \mathcal{O}(v^4). \end{cases} \quad (3.1)$$

where $M \equiv m_b + m_c$, and we count $|\mathbf{q}| \sim \mathcal{O}(v)$.

At tree level, we have the matrix element

$$\begin{aligned} & \langle b^a(p_b) \bar{c}^b(p_c) | Q[\Gamma](x) | 0 \rangle \\ &= \delta^{ab} \int \frac{d\omega}{2\pi} e^{-i(x - n_+ p_b / n_+ P) \omega n_+ P} \bar{u}_b(p_b) \not{n}_+ \Gamma v_c(p_c) \\ &= \frac{\delta^{ab}}{n_+ P} \delta \left(x - \frac{n_+ p_b}{n_+ P} \right) \bar{u}_b(p_b) \not{n}_+ \Gamma v_c(p_c), \end{aligned} \quad (3.2)$$

where the superscripts a and b are color indices for the b quark and $\bar{c}$ quark, respectively, and $n_\pm^\mu$ are set to have the explicit form in eq. (2.7).

Here it is worth addressing that the quark-states in eq. (3.2) are non-relativistically normalized. Thus, in the Dirac representation for Gamma matrices, the non-relativistically normalized 4-component on-shell spinors for b quark and $\bar{c}$ quark can be written explicitly as

$$u_b(p_b) = \sqrt{\frac{E_b + m_b}{2E_b}} \begin{pmatrix} \xi(\mathbf{q}) \\ \frac{\mathbf{q} \cdot \boldsymbol{\sigma}}{E_b + m_b} \xi(\mathbf{q}) \end{pmatrix}, \quad (3.3a)$$

$$v_c(p_c) = \sqrt{\frac{E_c + m_c}{2E_c}} \begin{pmatrix} -\frac{\mathbf{q} \cdot \boldsymbol{\sigma}}{E_c + m_c} \eta(-\mathbf{q}) \\ \eta(-\mathbf{q}) \end{pmatrix}. \quad (3.3b)$$

Here we suppress the helicity indices for spinors. $\xi(\mathbf{q})$ and $\eta(\mathbf{q})$ are the 2-component Pauli spinors for the b quark and $\bar{c}$ quark, respectively, and they are related to the following NRQCD matrix-elements:

$$\langle 0 | \psi_b^a(0) | b^b(p_b) \rangle = \delta^{ab} \xi(\mathbf{q}), \quad (3.4a)$$

$$\langle \bar{c}^b(p_c) | \chi_c^a(0) | 0 \rangle = \delta^{ab} \eta(-\mathbf{q}). \quad (3.4b)$$

where the superscripts a, b are color indices.

For illustration, we take $\Gamma = \gamma_5$ in eq. (3.2) as an example to show how to match $Q[\gamma_5](x)$ to the NRQCD operators up to $\mathcal{O}(v^2)$. Explicitly, we have

$$\begin{aligned}
 \langle b^a(p_b) \bar{c}^b(p_c) | Q[\gamma_5](x) | 0 \rangle &= \frac{\delta^{ab}}{n_+ P} \delta \left(x - \frac{n_+ p_b}{n_+ P} \right) \bar{u}_b(p_b) \not{n}_+ \gamma_5 v_c(p_c) \\
 &= \frac{\delta^{ab}}{E_b + E_c} \delta \left(x - \frac{E_b - q^3}{E_b + E_c} \right) \sqrt{\frac{(E_b + m_b)(E_c + m_c)}{2E_b 2E_c}} \\
 &\quad \times \left\{ \left[1 - \left(\frac{q^3}{E_b + m_b} - \frac{q^3}{E_c + m_c} \right) - \frac{\mathbf{q}^2}{(E_b + m_b)(E_c + m_c)} \right] \xi^\dagger(\mathbf{q}) \eta(-\mathbf{q}) \right. \\
 &\quad \left. - i \epsilon_{3ij} q^i \left(\frac{1}{E_b + m_b} + \frac{1}{E_c + m_c} \right) \xi^\dagger(\mathbf{q}) \sigma^j \eta(-\mathbf{q}) \right\}. \quad (3.5)
 \end{aligned}$$

Then we expand the above matrix-element up to the second power of q^i ,

$$\begin{aligned}
 &\langle b^a(p_b) \bar{c}^b(p_c) | Q[\gamma_5](x) | 0 \rangle \\
 &= \frac{\delta^{ab}}{M} \xi^\dagger(\mathbf{q}) \eta(-\mathbf{q}) \left\{ \delta(x - x_0) \left[1 - \frac{q^3}{M} \frac{(1 - 2x_0)}{2x_0 \bar{x}_0} - \frac{\mathbf{q}^2}{M^2} \frac{1 + 4x_0 \bar{x}_0}{8x_0^2 \bar{x}_0^2} \right] \right. \\
 &\quad \left. + \delta'(x - x_0) \frac{q^3}{M} - \delta'(x - x_0) \frac{(q^3)^2 + \mathbf{q}^2}{M^2} \frac{1 - 2x_0}{2x_0 \bar{x}_0} + \delta''(x - x_0) \frac{(q^3)^2}{2M^2} \right\} \\
 &\quad - i \frac{\delta^{ab}}{M} \frac{1}{2x_0 \bar{x}_0} \xi^\dagger(\mathbf{q}) \sigma^j \eta(-\mathbf{q}) \epsilon_{3ij} \frac{q^i}{M} \left\{ \delta(x - x_0) + \frac{q^3}{M} \delta'(x - x_0) \right\} + \mathcal{O}(v^3), \quad (3.6)
 \end{aligned}$$

where $x_0 \equiv m_b/(m_b + m_c)$ and $\bar{x}_0 \equiv 1 - x_0 = m_c/(m_b + m_c)$.

In this work, we are only interested in the S-wave part of the above matrix-element. Due to the standard extraction procedure of the S-wave contribution in the literature, at first we neglect the first order expansion in q^i (which is P-wave part that does not contribute to the vacuum to the S-wave state transition when only leading NRQCD interactions are considered), then replace $q^i q^j$ with $\mathbf{q}^2 \delta^{ij}/3$ in the second order expansion.

Thus, by identifying

$$\langle b^a(p_b) \bar{c}^b(p_c) | O_0(^1S_0) | 0 \rangle = \delta^{ab} \xi^\dagger(\mathbf{q}) \eta(-\mathbf{q}), \quad (3.7a)$$

$$\langle b^a(p_b) \bar{c}^b(p_c) | O_2(^1S_0) | 0 \rangle = \delta^{ab} \mathbf{q}^2 \xi^\dagger(\mathbf{q}) \eta(-\mathbf{q}), \quad (3.7b)$$

we get

$$\begin{aligned}
 &\langle b^a(p_b) \bar{c}^b(p_c) | Q[\gamma_5](x) | 0 \rangle^{\text{S-Wave}} \\
 &= \frac{1}{M} \left\{ \delta(x - x_0) \langle b^a(p_b) \bar{c}^b(p_c) | O_0(^1S_0) | 0 \rangle \right. \\
 &\quad - \left[\frac{1 + 4x_0 \bar{x}_0}{8x_0^2 \bar{x}_0^2} \delta(x - x_0) + \frac{2(1 - 2x_0)}{3x_0 \bar{x}_0} \delta'(x - x_0) - \frac{\delta''(x - x_0)}{6} \right] \frac{\langle b^a(p_b) \bar{c}^b(p_c) | O_2(^1S_0) | 0 \rangle}{M^2} \\
 &\quad \left. + \mathcal{O}(v^3) \right\}. \quad (3.8)
 \end{aligned}$$

Therefore, we can easily extract the short-distance coefficients for S-wave operators defined in eq. (2.11),

$$d_0^P(x) = \delta(x - x_0), \quad (3.9a)$$

$$d_2^P(x) = - \left[\frac{1 + 4x_0\bar{x}_0}{8x_0^2\bar{x}_0^2} \delta(x - x_0) + \frac{2(1 - 2x_0)}{3x_0\bar{x}_0} \delta'(x - x_0) - \frac{1}{6} \delta''(x - x_0) \right], \quad (3.9b)$$

and in turn,

$$C_0^P = 1, \quad C_2^P = - \left[\frac{1 + 4x_0\bar{x}_0}{8x_0^2\bar{x}_0^2} \right]. \quad (3.10)$$

It is worth noting that the above matching procedure by directly decomposing the 4-component Dirac spinors to Pauli spinors is not very convenient and efficient in the cases that the Dirac structures become complicated. In the literatures, the covariant projection approach are commonly used [21, 44–46]. Generally, one applies the replacements

$$\begin{aligned} v(p_c)\bar{u}(p_b) &\rightarrow \Pi_0(p_b, p_c) \\ &= \frac{i}{2\sqrt{2E_bE_c}\sqrt{(E_c + m_c)(E_b + m_b)}} (\not{p}_c - m_c) \gamma_5 \frac{\not{P} + E_b + E_c}{2(E_b + E_c)} (\not{p}_b + m_b), \end{aligned} \quad (3.11a)$$

$$\begin{aligned} v(p_c)\bar{u}(p_b) &\rightarrow \Pi_1(p_b, p_c) \\ &= - \frac{1}{2\sqrt{2E_bE_c}\sqrt{(E_c + m_c)(E_b + m_b)}} (\not{p}_c - m_c) \not{\epsilon}^* \frac{\not{P} + E_b + E_c}{2(E_b + E_c)} (\not{p}_b + m_b), \end{aligned} \quad (3.11b)$$

to project out the spin-singlet and spin-triplet parts, respectively. After expanding the resulting amplitudes in the relative momentum $\mathbf{q}$, one can extract the S-wave contributions by neglecting the first order terms in $\mathbf{q}$ and making the replacement $q^i q^j \rightarrow \mathbf{q}^2 \delta^{ij}/3$. Finally one can obtain the S-wave short-distance coefficients by identifying the NRQCD matrix elements at tree level

$$\langle b\bar{c}(^1S_0) | \psi_b^\dagger \chi_c | 0 \rangle^{(0)} = \sqrt{2N_c}, \quad (3.12a)$$

$$\langle b\bar{c}(^3S_1, \epsilon^*) | \psi_b^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \chi_c | 0 \rangle^{(0)} = \sqrt{2N_c}, \quad (3.12b)$$

$$\langle b\bar{c}(^1S_0) | \psi_b^\dagger (-\frac{i}{2} \overleftrightarrow{\mathbf{D}})^2 \chi_c | 0 \rangle^{(0)} = \sqrt{2N_c} \mathbf{q}^2, \quad (3.12c)$$

$$\langle b\bar{c}(^3S_1, \epsilon^*) | \psi_b^\dagger (-\frac{i}{2} \overleftrightarrow{\mathbf{D}})^2 \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \chi_c | 0 \rangle^{(0)} = \sqrt{2N_c} \mathbf{q}^2, \quad (3.12d)$$

where $|b\bar{c}(^{2S+1}S_J)\rangle$ is formally a non-relativistically normalized color-singlet quark-anti-quark pair state. The factor $\sqrt{2N_c}$ is due to the spin and color factors of the normalized $b\bar{c}(^{2S+1}S_J)$ state.

Since the Dirac structures involved in this work is not very complicated, we use both methods of the direct spinor decomposition and covariant projectors to do the matching for cross-checks. We get the exactly same results as it should be.

3.2 Results of the S-wave LCDAs up to $\mathcal{O}(\alpha_s^0 v^2)$

Following the matching procedure described in the previous subsection, we get the rest of the necessary short-distance coefficients up to $\mathcal{O}(v^2)$ at tree level,

$$d_0^V(x) = d_0^{V\perp}(x) = \delta(x - x_0), \quad (3.13a)$$

$$d_2^V(x) = - \left[\frac{3 + 4x_0\bar{x}_0}{24x_0^2\bar{x}_0^2} \delta(x - x_0) + \frac{2(1 - 2x_0)}{3x_0\bar{x}_0} \delta'(x - x_0) - \frac{1}{6} \delta''(x - x_0) \right], \quad (3.13b)$$

$$d_2^{V\perp}(x) = - \left[\frac{3 + 8x_0\bar{x}_0}{24x_0^2\bar{x}_0^2} \delta(x - x_0) + \frac{2(1 - 2x_0)}{3x_0\bar{x}_0} \delta'(x - x_0) - \frac{1}{6} \delta''(x - x_0) \right], \quad (3.13c)$$

and consequently,

$$C_0^V = C_0^{V\perp} = 1, \quad (3.14a)$$

$$C_2^V = - \frac{3 + 4x_0\bar{x}_0}{24x_0^2\bar{x}_0^2}, \quad (3.14b)$$

$$C_2^{V\perp} = - \frac{3 + 8x_0\bar{x}_0}{24x_0^2\bar{x}_0^2}. \quad (3.14c)$$

In turn, with eqs. (2.11) and (2.12), at tree-level and up to $\mathcal{O}(v^2)$, we have the decay constants in terms of NRQCD matrix elements as

$$-if_P = \frac{\langle B_c(1S_0, P) | O_0(1S_0) | 0 \rangle}{M} \left\{ 1 - \left[\frac{1 + 4x_0\bar{x}_0}{8x_0^2\bar{x}_0^2} \right] \frac{\langle \mathbf{q}^2 \rangle_P}{M^2} \right\}, \quad (3.15a)$$

$$f_V = \frac{\langle B_c(3S_1, P, \epsilon^*) | O_0(3S_1) | 0 \rangle}{M} \left\{ 1 - \left[\frac{3 + 4x_0\bar{x}_0}{24x_0^2\bar{x}_0^2} \right] \frac{\langle \mathbf{q}^2 \rangle_V}{M^2} \right\}, \quad (3.15b)$$

$$f_V^\perp = \frac{\langle B_c(3S_1, P, \epsilon^*) | O_0(3S_1) | 0 \rangle}{M} \left\{ 1 - \left[\frac{3 + 8x_0\bar{x}_0}{24x_0^2\bar{x}_0^2} \right] \frac{\langle \mathbf{q}^2 \rangle_V}{M^2} \right\}, \quad (3.15c)$$

and the LCDAs as

$$\hat{\phi}_P^{(0,0)}(x) = \hat{\phi}_V^{\parallel(0,0)}(x) = \hat{\phi}_V^{\perp(0,0)}(x) = \delta(x - x_0), \quad (3.16a)$$

$$\hat{\phi}_P^{(0,1)}(x) = - \frac{\langle \mathbf{q}^2 \rangle_P}{M^2} \left[\frac{2(1 - 2x_0)}{3x_0\bar{x}_0} \delta'(x - x_0) - \frac{1}{6} \delta''(x - x_0) \right], \quad (3.16b)$$

$$\hat{\phi}_V^{\parallel(0,1)}(x) = \hat{\phi}_V^{\perp(0,1)}(x) = - \frac{\langle \mathbf{q}^2 \rangle_V}{M^2} \left[\frac{2(1 - 2x_0)}{3x_0\bar{x}_0} \delta'(x - x_0) - \frac{1}{6} \delta''(x - x_0) \right], \quad (3.16c)$$

where $\langle \mathbf{q}^2 \rangle_{P,V}$ are the mean values of $\mathbf{q}^2$ in pseudo-scalar and vector B_c mesons respectively:

$$\langle \mathbf{q}^2 \rangle_P \equiv \frac{\langle B_c(1S_0, P) | \psi_b^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \chi_c | 0 \rangle}{\langle B_c(1S_0, P) | \psi_b^\dagger \chi_c | 0 \rangle}, \quad (3.17a)$$

$$\langle \mathbf{q}^2 \rangle_V \equiv \frac{\langle B_c(3S_1, P, \epsilon^*) | \psi_b^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \chi_c | 0 \rangle}{\langle B_c(3S_1, P, \epsilon^*) | \psi_b^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \chi_c | 0 \rangle}, \quad (3.17b)$$

which characterize $\mathcal{O}(v^2)$ relativistic corrections to the LCDAs.

By setting $m_b = m_c = m$, i.e. $x_0 = 1/2$, we recover the f , $\hat{\phi}(x)$ for heavy quarkonia at tree-level as given in ref. [37]; and also eq. (3.15c) returns to the result for $f_{J/\psi}^T$ in ref. [47]. Note that the matrix elements in (3.15) are relativistically normalized which are different from the widely-used non-relativistic normalization. To get the results in the latter form, one can multiply the factor $\sqrt{2m_{B_c}}$ in the right hand side of eq. (3.15) since it is assumed that the states are non-relativistically normalized, as shown by, e.g., eq. (3) in ref. [48].

3.3 Results of the S-wave LCDAs at $\mathcal{O}(\alpha_s^1 v^0)$

$\mathcal{O}(\alpha_s)$ radiative corrections are usually counted as important as $\mathcal{O}(v^2)$ relativistic corrections. For completeness, we quote the results of $\mathcal{O}(\alpha_s)$ radiative corrections from ref. [38].

We have

$$C_0^P = 1 + \frac{\alpha_s}{4\pi} C_F \left[3(x_0 - \bar{x}_0) \ln \frac{x_0}{\bar{x}_0} - 6 + 4\Delta \right], \quad (3.18a)$$

$$C_0^V = 1 + \frac{\alpha_s}{4\pi} C_F \left[3(x_0 - \bar{x}_0) \ln \frac{x_0}{\bar{x}_0} - 8 \right], \quad (3.18b)$$

$$C_0^{V\perp} = 1 + \frac{\alpha_s}{4\pi} C_F \left[-\ln \frac{\mu^2}{M^2} - (3 - 8x_0) \ln x_0 - (3 - 8\bar{x}_0) \ln \bar{x}_0 - 8 \right], \quad (3.18c)$$

for the decay constants, and

$$\hat{\phi}_P^{(1,0)}(x; \mu) = \frac{\alpha_s}{4\pi} C_F \left\{ \Phi_1(x, x_0) + 8\Delta \left[\frac{x}{x_0} \theta(x_0 - x) + (x \leftrightarrow \bar{x}, x_0 \leftrightarrow \bar{x}_0) \right]_+ \right\}, \quad (3.19a)$$

$$\hat{\phi}_V^{\parallel(1,0)}(x; \mu) = \frac{\alpha_s}{4\pi} C_F \left\{ \Phi_1(x, x_0) - 4 \left[\frac{x}{x_0} \theta(x_0 - x) + (x \leftrightarrow \bar{x}, x_0 \leftrightarrow \bar{x}_0) \right]_+ \right\}, \quad (3.19b)$$

$$\begin{aligned} \hat{\phi}_V^{\perp(1,0)}(x; \mu) = & \frac{\alpha_s}{4\pi} C_F \left\{ \Phi_1(x, x_0) \right. \\ & \left. - 2 \left[\left(\ln \frac{\mu^2}{M^2(x_0 - x)^2} - 1 \right) \left(\frac{x}{x_0} \theta(x_0 - x) + (x \leftrightarrow \bar{x}, x_0 \leftrightarrow \bar{x}_0) \right) \right]_+ \right\}, \end{aligned} \quad (3.19c)$$

with

$$\begin{aligned} \Phi_1(x, x_0) = & 2 \left[\left(\ln \frac{\mu^2}{M^2(x_0 - x)^2} - 1 \right) \left(\frac{x_0 + \bar{x}}{x_0 - x} \frac{x}{x_0} \theta(x_0 - x) + (x \leftrightarrow \bar{x}, x_0 \leftrightarrow \bar{x}_0) \right) \right]_+ \\ & + \left[\frac{4x\bar{x}}{(x_0 - x)^2} \right]_{++} + \left[4x_0\bar{x}_0 \ln \frac{x_0}{\bar{x}_0} + 2(2x_0 - 1) \right] \delta'(x - x_0), \end{aligned} \quad (3.20)$$

for the LCDAs. Here $\Delta = 0$ is for NDR scheme [49], and $\Delta = 1$ for the HV scheme [50]. The $++$ - and $+$ -distributions are defined as

$$\int_0^1 dx \left[f(x) \right]_{++} g(x) = \int_0^1 dx f(x) (g(x) - g(x_0) - g'(x_0)(x - x_0)), \quad (3.21a)$$

$$\int_0^1 dx \left[f(x) \right]_+ g(x) = \int_0^1 dx f(x) (g(x) - g(x_0)), \quad (3.21b)$$

where $g(x)$ is a smooth test function.

4 Phenomenological results

4.1 Inverse moment

In the production or decay processes of mesons where collinear factorization can be applied, the inverse moment of the LCDA is crucial because the hard kernels are often in form of $1/x$ and $1/\bar{x}$. Thus it is of phenomenological importance to apply the results in last section to study the inverse moment. The inverse moment of the LCDAs is defined by

$$\left\langle \frac{1}{x} \right\rangle_{\Gamma} \equiv \int_0^1 dx \frac{\hat{\phi}_{\Gamma}(x; \mu)}{x}. \quad (4.1)$$

The inverse moment can be calculated from the LCDA as given in the above section. The results are listed below,

$$\begin{aligned} \left\langle \frac{1}{x} \right\rangle_P = \frac{1}{x_0} \left\{ 1 + \frac{\alpha_s}{4\pi} C_F \left[(3 + 2\ln x_0) \ln \frac{\mu^2}{M^2} - 6\bar{x}_0 \ln \bar{x}_0 - 6x_0 \ln x_0 - 2\ln x_0 - 8\Delta \frac{x_0}{\bar{x}_0} \ln x_0 \right. \right. \\ \left. \left. + 4\text{Li}_2(x_0) - 2\ln^2 x_0 - \frac{2\pi^2}{3} + 6 - 4\Delta \right] - \frac{(1 - 3x_0) \langle \mathbf{q}^2 \rangle_P}{3x_0^2 \bar{x}_0} \frac{1}{M^2} \right\}, \end{aligned} \quad (4.2a)$$

$$\begin{aligned} \left\langle \frac{1}{x} \right\rangle_V = \frac{1}{x_0} \left\{ 1 + \frac{\alpha_s}{4\pi} C_F \left[(3 + 2\ln x_0) \ln \frac{\mu^2}{M^2} - 6\bar{x}_0 \ln \bar{x}_0 - 6x_0 \ln x_0 - 2\ln x_0 + 4\frac{x_0}{\bar{x}_0} \ln x_0 \right. \right. \\ \left. \left. + 4\text{Li}_2(x_0) - 2\ln^2 x_0 - \frac{2\pi^2}{3} + 8 \right] - \frac{(1 - 3x_0) \langle \mathbf{q}^2 \rangle_V}{3x_0^2 \bar{x}_0} \frac{1}{M^2} \right\}, \end{aligned} \quad (4.2b)$$

$$\begin{aligned} \left\langle \frac{1}{x} \right\rangle_{V_{\perp}} = \frac{1}{x_0} \left\{ 1 + \frac{\alpha_s}{4\pi} C_F \left[\left(4 + \frac{2}{\bar{x}_0} \ln x_0 \right) \ln \frac{\mu^2}{M^2} + 4(x_0 - 2) \ln \bar{x}_0 - 2(1 + 2x_0) \ln x_0 \right. \right. \\ \left. \left. - 2\frac{x_0}{\bar{x}_0} \ln x_0 + \frac{1}{\bar{x}_0} \left(4\text{Li}_2(x_0) - 2\ln^2 x_0 - \frac{2\pi^2}{3} \right) + 8 \right] - \frac{(1 - 3x_0) \langle \mathbf{q}^2 \rangle_V}{3x_0^2 \bar{x}_0} \frac{1}{M^2} \right\}. \end{aligned} \quad (4.2c)$$

Here $\Delta = 0$ is for the NDR scheme, and $\Delta = 1$ is for the HV scheme.

4.2 Gegenbauer moments

The Gegenbauer moments are also commonly used, which are defined by

$$(a_n)_{\Gamma} \equiv \frac{2(2n+3)}{3(2+n)(1+n)} \int_0^1 dx \hat{\phi}_{\Gamma}(x) C_n^{(3/2)}(2x-1). \quad (4.3)$$

With the results of LCDA derived above, one can work out the first two Gegenbauer moments, and the results are given below. For the a_1 , we have

$$(a_1)_P = \frac{5}{3}(2x_0 - 1) \left\{ 1 + \frac{\alpha_s}{4\pi} C_F \frac{4}{9(2x_0 - 1)} \left[6(1 - 2x_0) \ln \frac{\mu^2}{M^2} + 3\bar{x}_0(3 - 7\bar{x}_0) \ln \bar{x}_0 \right. \right. \\ \left. \left. - 3x_0(3 - 7x_0) \ln x_0 + (1 - 2x_0)(19 + 6\Delta) \right] - \frac{4}{3x_0\bar{x}_0} \frac{\langle \mathbf{q}^2 \rangle_P}{M^2} \right\}, \quad (4.4a)$$

$$(a_1)_V = \frac{5}{3}(2x_0 - 1) \left\{ 1 + \frac{\alpha_s}{4\pi} C_F \frac{4}{9(2x_0 - 1)} \left[6(1 - 2x_0) \ln \frac{\mu^2}{M^2} + 3\bar{x}_0(3 - 7\bar{x}_0) \ln \bar{x}_0 \right. \right. \\ \left. \left. - 3x_0(3 - 7x_0) \ln x_0 + 16(1 - 2x_0) \right] - \frac{4}{3x_0\bar{x}_0} \frac{\langle \mathbf{q}^2 \rangle_V}{M^2} \right\}, \quad (4.4b)$$

$$(a_1)_{V_\perp} = \frac{5}{3}(2x_0 - 1) \left\{ 1 + \frac{\alpha_s}{4\pi} C_F \frac{1}{2x_0 - 1} \left[-2(1 - 2x_0)^2 \ln \frac{\mu^2}{M^2} + 4\bar{x}_0 \ln \bar{x}_0 \right. \right. \\ \left. \left. + 4x_0(4x_0 - 3) \ln x_0 - 8(1 - 2x_0)^2 \right] - \frac{4}{3x_0\bar{x}_0} \frac{\langle \mathbf{q}^2 \rangle_{V_\perp}}{M^2} \right\}. \quad (4.4c)$$

Notice that for heavy quarkonia, the first Gegenbauer moment a_1 vanishes, since the a_1 reflects the asymmetry in the momentum distribution. For the a_2 , we have

$$(a_2)_P = \frac{7}{18} C_2^{(3/2)} (2x_0 - 1) \left\{ 1 + \frac{\alpha_s}{4\pi} C_F \frac{5}{36(1 - 5x_0\bar{x}_0)} \left[-30(1 - 5x_0\bar{x}_0) \ln \frac{\mu^2}{M^2} \right. \right. \\ + 12\bar{x}_0(5 + x_0(-26 + 27x_0)) \ln \bar{x}_0 + 12x_0(6 + x_0(-28 + 27x_0)) \ln x_0 \\ \left. \left. + x_0\bar{x}_0(589 + 120\Delta) - 119 - 24\Delta \right] - \frac{5(2 - 9x_0\bar{x}_0)}{3x_0\bar{x}_0(1 - 5x_0\bar{x}_0)} \frac{\langle \mathbf{q}^2 \rangle_P}{M^2} \right\}, \quad (4.5a)$$

$$(a_2)_V = \frac{7}{18} C_2^{(3/2)} (2x_0 - 1) \left\{ 1 + \frac{\alpha_s}{4\pi} C_F \frac{5}{36(1 - 5x_0\bar{x}_0)} \left[-30(1 - 5x_0\bar{x}_0) \ln \frac{\mu^2}{M^2} \right. \right. \\ + 12\bar{x}_0(5 + x_0(-26 + 27x_0)) \ln \bar{x}_0 + 12x_0(6 + x_0(-28 + 27x_0)) \ln x_0 \\ \left. \left. - 107 + 529x_0\bar{x}_0 \right] - \frac{5(2 - 9x_0\bar{x}_0)}{3x_0\bar{x}_0(1 - 5x_0\bar{x}_0)} \frac{\langle \mathbf{q}^2 \rangle_V}{M^2} \right\}, \quad (4.5b)$$

$$(a_2)_{V_\perp} = \frac{7}{18} C_2^{(3/2)} (2x_0 - 1) \left\{ 1 + \frac{\alpha_s}{4\pi} C_F \frac{10}{9(1 - 5x_0\bar{x}_0)} \left[-3(1 - 5x_0\bar{x}_0) \ln \frac{\mu^2}{M^2} \right. \right. \\ + 3\bar{x}_0(3x_0 - 2)(4x_0 - 1) \ln \bar{x}_0 + 3x_0(3x_0 - 1)(4x_0 - 3) \ln x_0 - 14(1 - 5x_0\bar{x}_0) \\ \left. \left. - \frac{5(2 - 9x_0\bar{x}_0)}{3x_0\bar{x}_0(1 - 5x_0\bar{x}_0)} \frac{\langle \mathbf{q}^2 \rangle_{V_\perp}}{M^2} \right] \right\}. \quad (4.5c)$$

4.3 Numerical results

In this subsection, we will show the numerical results of the inverse moments and the first two Gegenbauer moments.

In order to get the numerical results, we will first estimate $\langle \mathbf{q}^2 \rangle_{P,V}$ using the Gremm-Kapuskun (G-K) relation [51]

$$m_{B_c} = m_b + m_c + \frac{\langle \mathbf{q}^2 \rangle}{2m_b} + \frac{\langle \mathbf{q}^2 \rangle}{2m_c}, \quad (4.6)$$

which is deduced by implementing the equations of motion from the leading order NRQCD Lagrangian. It is necessary to point out that as one can see that from eq. (3.17), $\langle \mathbf{q}^2 \rangle$ in the above equation is the ratio of two matrix elements, which is not identically the same as the relative momentum of quarks.

It is necessary to stress that the masses in the potential model can only be viewed as an effective mass parameters. Different choices of quark mass parameters can result in different values of $\langle \mathbf{q}^2 \rangle_{P,V}$. Two widely used masses in the literature are the running masses in the $\overline{\text{MS}}$ scheme and the pole masses with the relation:

$$m_Q = \overline{m}_Q(\overline{m}_Q) \left\{ 1 + \frac{4\overline{\alpha}_s(\overline{m}_Q)}{3\pi} + \left[-1.0414 \sum_k \left(1 - \frac{4}{3} \frac{\overline{m}_{Q_k}}{\overline{m}_Q} \right) + 13.4434 \right] \left[\frac{\overline{\alpha}_s(\overline{m}_Q)}{\pi} \right]^2 \right\}, \quad (4.7)$$

where $\overline{\alpha}_s(\mu)$ is the strong interaction coupling constant in the $\overline{\text{MS}}$ scheme, and the sum over k extends over the N_L flavors Q_k lighter than Q . The running masses

$$m_c(\overline{\text{MS}}) = 1.28 \text{ GeV}, \quad m_b(\overline{\text{MS}}) = 4.18 \text{ GeV},$$

correspond to the pole masses:

$$m_c(\text{pole mass}) = 1.67 \text{ GeV}, \quad m_b(\text{pole mass}) = 4.78 \text{ GeV}.$$

In NRQCD, the quark mass parameter is most naturally identified with the quark pole mass. But as pointed out by ref. [52], due to the intrinsic ambiguity of the charm quark pole mass, the G-K relation can not be utilized to nail down the precise value of $\langle \mathbf{v}^2 \rangle_{J/\psi}$. In table 1, we give the corresponding values of $\langle \mathbf{q}^2 \rangle$ and $\langle \mathbf{v}^2 \rangle$ in the two mass schemes. It is expected through the phenomenological studies that $\langle \mathbf{v}^2 \rangle \sim 0.3$ for $\bar{c}c$ system and $\langle \mathbf{v}^2 \rangle \sim 0.1$ for $\bar{b}b$ system. This is consistent with the results with $\overline{\text{MS}}$ masses. So despite the fact that the masses appearing in the NRQCD expansion are the pole masses, we use the masses in $\overline{\text{MS}}$ scheme in the determination of $\langle \mathbf{q}^2 \rangle$ and $\langle \mathbf{v}^2 \rangle$. We have also checked that the effects caused by the pole masses and $\overline{\text{MS}}$ masses in the expansions for the inverse moments in eq. (4.2) and Gegenbauer moments in eqs. (4.4) and (4.5) are moderate.

The inverse moments and first two Gegenbauer moments of $B_c, J/\psi, \Upsilon$ are considered, the parameters are collected in table 2. The values of $\langle \mathbf{q}^2 \rangle_P$ for B_c , $\langle \mathbf{q}^2 \rangle_V$ for J/ψ and Υ deduced from G-K relation are also listed. Numerical results for the inverse moments

	$\langle \mathbf{q}^2 \rangle$		$\langle \mathbf{v}^2 \rangle$	
	$\overline{\text{MS}}$	Pole mass	$\overline{\text{MS}}$	Pole mass
J/ψ	0.69 GeV ²	-0.40 GeV ²	0.42	-0.14
B_c	1.59 GeV ²	-0.45 GeV ²	0.41	-0.07
Υ	4.6 GeV ²	-0.48 GeV ²	0.26	-0.02

Table 1. Values of $\langle \mathbf{q}^2 \rangle$ and $\langle \mathbf{v}^2 \rangle$ using the running quark masses in the $\overline{\text{MS}}$ scheme and pole masses.

	Meson mass	Quark mass	α_s	μ	$\langle \mathbf{q}^2 \rangle$
J/ψ	3.10 GeV	$m_c = 1.28$ GeV	$\alpha_s(2m_c) = 0.26$	2.56 GeV	$\langle \mathbf{q}^2 \rangle_V = 0.69$ GeV ²
B_c	6.27 GeV	$m_c = 1.28$ GeV; $m_b = 4.18$ GeV	$\alpha_s(m_b + m_c) = 0.21$	5.46 GeV	$\langle \mathbf{q}^2 \rangle_P = 1.59$ GeV ²
Υ	9.46 GeV	$m_b = 4.18$ GeV	$\alpha_s(2m_b) = 0.19$	8.36 GeV	$\langle \mathbf{q}^2 \rangle_V = 4.6$ GeV ²

Table 2. Input parameters.

$\langle x^{-1} \rangle$	LO	LO + α_s	LO + v^2	LO + $\alpha_s + v^2$
B_c	1.31	1.51	1.53	1.73
J/ψ	2.00	2.31	2.28	2.59
Υ	2.00	2.22	2.18	2.40
a_1	LO	LO + α_s	LO + v^2	LO + $\alpha_s + v^2$
B_c	0.89	0.67	0.53	0.32
a_2	LO	LO + α_s	LO + v^2	LO + $\alpha_s + v^2$
B_c	0.24	0.11	-0.21	-0.34
J/ψ	-0.58	-0.26	-0.17	0.14
Υ	-0.58	-0.35	-0.33	-0.09

Table 3. Inverse moments and Gegenbauer moments of LCDAs for the B_c , J/ψ and Υ states, the $\langle \mathbf{q}^2 \rangle$ used here are deduced by G-K relation. Leading-order results, one-loop QCD radiative corrections, relativistic corrections and the total results are shown, respectively.

and Gegenbauer moments can be calculated with these input parameters, we have used the NDR scheme with $\Delta = 0$. They are shown in table 3.

Table 3 shows that the relativistic corrections to the inverse moments and first two Gegenbauer moments are comparable in magnitude with the radiative corrections, if not more important. In figure 1, the LCDA $\hat{\phi}_P$ for the B_c meson is plotted. In this figure, the dashed line denotes the asymptotic form $\phi(x) = 6x(1-x)$, while the dot-dashed line denotes the position $x_0 = m_b/(m_b + m_c) = 0.77$. The solid line is obtained by including two Gegenbauer moments derived in eqs. (4.4a) and (4.5a).

At last, we should note that the analyses of $\langle \mathbf{q}^2 \rangle_V$ based on Cornell potential model in refs. [47, 53] have suggested $\langle \mathbf{q}^2 \rangle_V = 0.44$ GeV² for J/ψ and $\langle \mathbf{q}^2 \rangle_V = -0.19$ GeV² for Υ . The Cornell potential parametrizes the $\bar{Q}Q$ potential as a linear combination of the

Figure 1. LCDA $\hat{\phi}_P(x)$ for the B_c meson, the $\langle \mathbf{q}^2 \rangle_P$ used here is deduced by G-K relation. The dashed line denotes the asymptotic form $\phi(x) = 6x(1-x)$, while the dot-dashed line denotes the position $x_0 = m_b/(m_b + m_c) = 0.77$. The solid line is obtained by including two Gegenbauer moments derived in eqs. (4.4a) and (4.5a).

Coulomb and linear potentials:

$$V(r) = -\frac{\kappa}{r} + \sigma r, \quad (4.8)$$

where κ is a dimensionless model parameter for the Coulomb strength and σ is the string tension. The Schrödinger equation for the radial wave function $R_{nS}(r)$ with the radial and orbital angular-momentum quantum numbers n and S is

$$\left[-\frac{1}{mr^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + V(r) \right] R_{nS}(r) = \epsilon_{nS} R_{nS}(r), \quad (4.9)$$

where ϵ_{nS} is the binding energy of the nS state. Introducing the scaled radius ρ and scaled coupling λ ,

$$\rho = (\sigma m)^{1/3} r, \quad \lambda = \frac{\kappa}{(\sigma/m^2)^{1/3}}, \quad (4.10)$$

one can rewrite the Schrödinger equation as

$$\left[\frac{d^2}{d\rho^2} + \frac{\lambda}{\rho} - \rho + \zeta_{nS} \right] u_{nS}(\rho) = 0, \quad (4.11)$$

where $u_{nS}(\rho)$ and ζ_{nS} are the dimensionless radial wave function and the dimensionless energy eigenvalue of the nl state. Then, the $\langle \mathbf{q}^2 \rangle_H$ can be expressed as [47, 53]

$$\langle \mathbf{q}^2 \rangle_H = m(\lambda) \epsilon_{1S}(\lambda) = [\sigma m(\lambda)]^{2/3} \zeta_{1S}(\lambda). \quad (4.12)$$

Solving the Schrödinger equation with the parameters $m_c = 1.84 \text{ GeV}$, $m_b = 5.18 \text{ GeV}$, $\kappa = 0.52$, and $\sigma = 1/(2.34)^2 \text{ GeV}^2$ as ref. [54] did, we obtain $\zeta_{1S}(\lambda) = 0.425$, which leads to $\langle \mathbf{q}^2 \rangle_{B_c} = 0.25 \text{ GeV}^2$.

$\langle x^{-1} \rangle$	LO	LO + α_s	LO + v^2	LO + $\alpha_s + v^2$
B_c	1.31	1.51	1.34	1.54
J/ψ	2.00	2.31	2.18	2.49
Υ	2.00	2.22	1.99	2.21
a_1	LO	LO + α_s	LO + v^2	LO + $\alpha_s + v^2$
B_c	0.89	0.67	0.83	0.61
a_2	LO	LO + α_s	LO + v^2	LO + $\alpha_s + v^2$
B_c	0.24	0.11	0.17	0.04
J/ψ	-0.58	-0.26	-0.32	0.00
Υ	-0.58	-0.35	-0.59	-0.36

Table 4. Inverse moments and Gegenbauer moments of LCDAs for the B_c , J/ψ and Υ states, the $\langle \mathbf{q}^2 \rangle$ used here are deduced by potential model. Leading-order results, one-loop QCD radiative corrections, relativistic corrections and the total results are shown, respectively.

Figure 2. LCDA $\hat{\phi}_P(x)$ for the B_c meson, where the $\langle \mathbf{q}^2 \rangle_P$ is deduced by potential model as given in eq. (4.13). The dashed line denotes the asymptotic form $\phi(x) = 6x(1-x)$, while the dot-dashed line denotes the position $x_0 = m_b/(m_b + m_c) = 0.77$. The solid line is obtained by including two Gegenbauer moments derived in eqs. (4.4a) and (4.5a).

With the results for $\langle \mathbf{q}^2 \rangle$

$$\langle \mathbf{q}^2 \rangle_{J/\psi} = 0.44 \text{ GeV}^2, \quad \langle \mathbf{q}^2 \rangle_{\Upsilon} = -0.19 \text{ GeV}^2, \quad \langle \mathbf{q}^2 \rangle_{B_c} = 0.25 \text{ GeV}^2, \quad (4.13)$$

we give our numerical results for the LCDAs in table 4 and figure 2. In this case, since the $\langle \mathbf{q}^2 \rangle$ are smaller compared to the results derived from the G-K relation, the relativistic corrections would be less significant, and the total results would be affected sizably. For instance, the first Gegenbauer moment of the B_c LCDA is enhanced from 0.32 to 0.61. On the one hand, these differences will introduce uncertainties to the LCDAs and the corresponding production rates, while on the other hand, this sensitivity can be used to pin down the uncertainties in $\langle \mathbf{q}^2 \rangle$ once the relevant data can be obtained in the future.

5 Conclusion

A high energy process may involve several perturbative scales. It is very important to handle these different scales, since QCD radiative corrections will induce large logarithms $\alpha_s^n \ln^n m_Q^2/s$. In collinear factorization, one can relegate the nonperturbative degrees of freedom into LCDAs, while the logarithms $\alpha_s^n \ln^n m_Q^2/s$ can be handled using the renormalization group equation, running from the scale $\sqrt{s}$ down to m_Q . For the heavy quarkonium and B_c system, the refactorization scheme allows one to further reduce the nonperturbative inputs into only a few NRQCD matrix elements. LCDAs of heavy quarkonia and B_c mesons are known at next-to-leading order (NLO) in the strong coupling constant α_s and at leading order in the velocity expansion.

In this paper, we have calculated the relativistic corrections of three twist-2 LCDAs for the S-wave B_c mesons. The corresponding results for heavy quarkonia such as J/ψ and Υ can be easily deduced by setting $m_b = m_c$. With the results for these relativistically corrected LCDAs, we have studied a few inverse/Gegenbauer moments. We find that the relativistic corrections are comparable with the next-to-leading order radiative corrections.

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