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Flavour singlets in gauge theory as permutations

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Abstract: Gauge-invariant operators can be specified by equivalence classes of permutations. We develop this idea concretely for the singlets of the flavour group $\mathrm{SO}(N_f)$ in $\mathrm{U}(N_c)$ gauge theory by using Gelfand pairs and Schur-Weyl duality. The singlet operators, when specialised at $N_f = 6$, belong to the scalar sector of $\mathcal{N} = 4$ SYM. A simple formula is given for the two-point functions in the free field limit of $g_{YM}^2 = 0$. The free two-point functions are shown to be equal to the partition function on a 2-complex with boundaries and a defect, in a topological field theory of permutations. The permutation equivalence classes are Fourier transformed to a representation basis which is orthogonal for the two-point functions at finite N_c, N_f. Counting formulae for the gauge-invariant operators are described. The one-loop mixing matrix is derived as a linear operator on the permutation equivalence classes.

Keywords: 1/N Expansion, AdS-CFT Correspondence, Supersymmetric gauge theory

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1 Introduction

The AdS/CFT [1–3] correspondence has led to detailed studies of local operators in gauge theories. A remarkable success has been the discovery of integrability in planar $\mathcal{N} = 4$ SYM, which allows the computation of conformal dimensions and other quantities at any value of the 't Hooft coupling [4]. In parallel, the study of AdS/CFT at finite N_c, the rank of the gauge group, is making progress. The CFT duals of giant gravitons [5] are local BPS operators, which have been investigated using techniques based on permutation groups and Fourier transformation in representation theory [6–13]. A class of quarter BPS operators, showing finite N_c cutoffs related to Brauer algebras, were constructed [14]. The general quarter BPS for the case where the dimension of the operators is less than N_c were constructed using permutation algebras in [9, 12, 15, 16]. Integrability of excitations around large half-BPS operators has been established [17–19]. A natural direction of investigation is the application of the permutation-based methods to problems in the $1/N_c$ expansion and far from planarity. In this paper we will take a further step in this direction, studying singlet operators in the SO(6) sector made from six scalars. The study of non-planar corrections in this sector was initiated in [20].

We will focus on the sector of hermitian scalar fields in $\mathcal{N} = 4$ SYM with U(N_c) gauge group, and consider "mesonic" gauge-invariant operators. The mesonic SO(6)-invariant operators form a simple closed subsector under the action of one-loop dilatation operator; namely the explicit form of the one-loop dilatation tells that there is no mixing between mesonic operators and other operators like non-scalar singlets with derivatives or "baryonic" operators.[1] It is in fact convenient to replace SO(6) by SO(N_f) and discuss the general N_f theory. Various arguments simplify at large N_f thanks to the absence of finite N_f constraints. We obtain identities relating different ways of counting gauge-invariant operators.

There is an essential distinction between $N_c \geq 2n$ and $N_c < 2n$, where $2n$ is the operator length. For example the planar limit is a limit in the former regime. The latter regime contains interesting limits like $N_c \gg 1$, $n \sim O(N_c)$, which is related to the description of giant gravitons. Conventionally the former is called large N_c and the latter is called finite N_c. Likewise we mean large N_f and finite N_f by $N_f \geq 2n$ and $N_f < 2n$.

In section 2 we show that operators $\mathcal{O}_\alpha$ of length $2n$ can be parametrised by permutations α in S_{2n}, the symmetric group of all permutations of $2n$ distinct objects. Different permutations giving rise to the same mesonic operator are related by conjugation with an element γ in the wreath product subgroup $S_n[S_2] \subset S_{2n}$. This group, of dimension $2^n n!$, contains n copies of S_2 as well as the symmetric group S_n consisting of permutations of n pairs. The two-point function of operators $\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \rangle$ is expressed in terms of a sum of permutations in S_{2n}. The all-orders expansion in N_f and N_c is given in terms of symmetric group data such as cycle types of appropriate permutations. This wreath product group is also used in [21, 22] to organise the colour structure in SO(N_c) and Sp(N_c) gauge theories.

In section 3, we show how to take a linear combination of the operators labelled by permutations to form a new basis of operators labelled by representations of S_{2n}, along with a group-theoretical multiplicity label. This procedure can be thought of as the Fourier

[1] The absence of mixing with baryonic operators is discussed around (7.1) in section 7.

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transform on finite groups, which replaces permutation labels by representation labels. The two-point function for these representation-labelled operators is diagonal, as given in (3.9). The diagonal operators become null when the constraints of finite N_c or N_f are violated. The construction in this section is an explicit realisation for the mesonic sector of a general construction in [12].

In section 4 we take a careful look at the counting of the mesonic operators constructed. We obtain the exact counting formula using representation theory, in particular Schur-Weyl duality. The representation labels are equipped with finite N_c and finite N_f cut-offs, which give the correct counting for general n. We return to the language of permutations, and analyze the large N_c, N_f limits. In this way, we make contact with counting of the graphs via the Burnside Lemma.

In section 5 we consider a topological lattice gauge theory with a discrete gauge group S_{2n} of the type discussed in [23–25]. The counting of mesonic operators in the permutation basis, and their free-field two-point functions can be interpreted as the partition functions of this topological field theory, defined on 2-complexes with boundaries and a defect.

Finally in section 6, we derive formulae for the action of the one-loop dilatation operator acting on the operators constructed above.

In appendices we will explain our notation, collect mathematical statements and details of computation.

Key technical results

Here is a brief summary of the key technical results of this paper.

Mesonic gauge-invariant operators, composite fields made of $2n$ scalars $(\Phi_a)^i_j$, are associated to permutations $\alpha \in S_{2n}$. They are written as $\mathcal{O}_\alpha$. Different permutations α related through conjugation by a permutation $\gamma \in S_n[S_2]$ are identical.

$$\mathcal{O}_\alpha = \mathcal{O}_{\gamma\alpha\gamma^{-1}} \quad \text{for} \quad \gamma \in S_n[S_2] \tag{1.1}$$

We denote the two-point functions of normal-ordered operators $\mathcal{O}_\alpha$ of length $2n$ by

$$\langle \mathcal{O}_{\alpha_1}(x)\mathcal{O}_{\alpha_2}(y)\rangle = \frac{\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle}{|x-y|^{4n}}\,, \tag{1.2}$$

where the bracket $\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle$ contains the sum of all Wick contractions between the two operators. The two-point functions $\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle$ in the free theory will be described by an elegant formula expressed in terms of cycle structures of permutations (2.31). However, $\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle$ viewed as a function of the permutation equivalence classes, is not diagonal.

To find a basis of operators with diagonal two-point functions in the free theory, we use Fourier transformation on finite groups. Ordinary Fourier transformation can be written as

$$f^K = \int d\theta D^K(\theta)\, f(\theta), \qquad D^K(\theta) = e^{iK\theta} \tag{1.3}$$

where $D^K(\theta)$ is the action of the U(1) group element $e^{i\theta}$ in the representation of charge K. Fourier transformation on finite groups is given by

$$f^R_{ij} = \sum_{g\in G} D^R_{ij}(g) f(g), \qquad f(g) \in \mathbb{C} \tag{1.4}$$

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where $D^R_{ij}(g)$ is the matrix element of g in the representation R and i, j run over some basis of the representation, which we will choose to be an orthogonal basis. A similar transformation in the group algebra $\mathbb{C}(G)$, formed by linear combinations of group elements with complex coefficients, is

$$Q^R_{ij} = \sum_{g\in G} D^R_{ij}(g)\ g \tag{1.5}$$

Specializing G to the symmetric group, the irreducible representations R will correspond to Young diagrams.

In this paper, we find a new diagonal basis for the free-field two-point functions, labelled by two irreducible representations R, Λ_1 of the permutation group S_{2n} and a multiplicity label τ,

$$\begin{aligned}\mathcal{O}^{R,\Lambda_1,\tau} &= tr_{2n}(P^{R,\Lambda_1,\tau}\ \Phi_{a_1}\otimes\Phi_{a_1}\otimes\Phi_{a_2}\otimes\Phi_{a_2}\otimes\cdots\otimes\Phi_{a_n}\otimes\Phi_{a_n})\\ &(\tau = 1, 2, \ldots, C(R, R, \Lambda_1))\end{aligned} \tag{1.6}$$

as in (3.1). Here R corresponds to a Young diagrams with $2n$ boxes, Λ_1 to an even Young diagram with $2n$ boxes, i.e. all the row lengths are even numbers, and $C(R, R, \Lambda_1)$ is the number of times Λ_1 appears in the irreducible decomposition of $R\otimes R$. $P^{R,\Lambda_1,\tau}$ is a linear combination of the sums over the equivalence classes of permutations

$$[\alpha] = \frac{1}{|S_n[S_2]|}\sum_{\gamma\in S_n[S_2]}\gamma\alpha\gamma^{-1}, \qquad |S_n[S_2]| = 2^n n! \tag{1.7}$$

which live in the group algebra $\mathbb{C}[S_{2n}]$. The equation (3.1) for $P^{R,\Lambda_1,\tau}$ is a generalisation of (1.5), and the equation (1.6) for $\mathcal{O}^{R,\Lambda_1,\tau}$ is a generalisation of (1.4), where $f(g)$ has been replaced by gauge-invariant polynomials in $(\Phi_a)^i_j$ parametrised by permutations. The coefficients in $P^{R,\Lambda_1,\tau}$ do not depend explicitly on N_c or N_f , though R, Λ_1 are required to have a bound on the number of rows when $N_f, N_c < 2n$.

The number of diagonal operators (1.6) is given by

$$\text{Number of mesonic operators } = \sum_{\substack{R\\ c_1(R)\le N_c}}\ \sum_{\substack{\Lambda_1:even\\ c_1(\Lambda_1)\le N_f}} C(R, R, \Lambda_1). \tag{1.8}$$

The two-point functions of (1.6) are given by (3.9),

$$\left\langle \mathcal{O}^{R,\Lambda_1,\tau}\mathcal{O}^{R',\Lambda_1',\tau'}\right\rangle = \delta^{RR'}\,\delta^{\Lambda_1\Lambda_1'}\,\delta^{\tau\tau'}\left(\frac{(2n)!}{d_R}\right)^2 Dim(R)\,N_f^n\,\omega_{\Lambda_1/2}(\Omega^{(f)}_{2n}) \tag{1.9}$$

and they vanish unless the corresponding representation labels on the two operators are identical. The normalisation factor $\omega_{\Lambda_1/2}(\Omega^{(f)}_{2n})$ is a polynomial in N_f.

The wreath product group $S_n[S_2]$ in (1.7) also appears as a symmetry of the Kronecker delta's used for the contraction of flavour indices. The groups S_{2n} and $S_n[S_2]$ form what is called a Gelfand pair $(S_{2n}, S_n[S_2])$. A notable property of the Gelfand pair is that the reduction of an irreducible representation of S_{2n} into the singlet representation of $S_n[S_2]$ is multiplicity-free. This property plays an essential role in our construction of the diagonal operators (1.6), which is in the trivial representation of $S_n[S_2]$.

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The free-field two-point functions and the number of operators (1.8) are closely related, which become manifest in the large N_c, N_f limit. Indeed, this agreement opens up a novel interpretation as the partition function of a topological field theory with a discrete gauge group.

The representation basis highly constrains the one-loop mixing, shown in (6.11). We then establish that the elements of the mixing matrix are non-zero for operators having a pair of representation labels R, R' related by the move of at most one box. This is a familiar fact from previous studies of one-loop mixing in representation bases.

2 Singlet operators

2.1 Mesonic operators and wreath-product permutation group

We denote a hermitian scalar field of gauge theory by $\Phi = (\Phi_a)^i_j$ with $a = 1, 2, \ldots, N_f$ and $i, j = 1, 2, \ldots N_c$. The $\mathcal{N} = 4$ SYM corresponds to $N_f = 6$. The upper gauge indices are identified with the lower gauge indices, up to an ordering parametrised by a permutation α

$$(\Phi_{a_1})^{i_1}_{i_{\alpha(1)}} (\Phi_{a_2})^{i_2}_{i_{\alpha(2)}} \cdots (\Phi_{a_{2n}})^{i_{2n}}_{i_{\alpha(2n)}} \tag{2.1}$$

Regarding $(\Phi_a)^i_j$ as matrix elements of operators $\Phi_a : V_N \to V_N$ and defining

$$\Phi_{\vec{a}} = \Phi_{a_1} \otimes \Phi_{a_2} \otimes \cdots \otimes \Phi_{a_{2n}} \tag{2.2}$$

which are operators in $V_N^{\otimes 2n}$, we have

$$(\Phi_{a_1})^{i_1}_{i_{\alpha(1)}} (\Phi_{a_2})^{i_2}_{i_{\alpha(2)}} \cdots (\Phi_{a_{2n}})^{i_{2n}}_{i_{\alpha(2n)}} = tr_{V_N^{\otimes 2n}}(\alpha\ \Phi_{\vec{a}}) = tr_{2n}(\alpha\ \Phi_{\vec{a}}) \tag{2.3}$$

with α acting in the standard way on the tensor product $V_N^{\otimes 2n}$. In the second equality, the trace has been abbreviated as tr_{2n}.

Mesonic operators are defined as gauge-invariant operators whose flavour indices are pairwise contracted. The general mesonic operator can be written by a permutation $\alpha \in S_{2n}$,

$$\begin{aligned} \mathcal{O}_\alpha &= \left(\prod_{k=1}^{n} \delta^{a_{2k-1} a_{2k}} \right) tr_{2n}(\alpha\, \Phi_{\vec{a}}) \\ &= tr_{2n}\left(\alpha\ \Phi_{a_1} \otimes \Phi_{a_1} \otimes \Phi_{a_2} \otimes \Phi_{a_2} \otimes \cdots \otimes \Phi_{a_n} \otimes \Phi_{a_n}\right). \end{aligned} \tag{2.4}$$

Generally this is a multi-trace operator, whose trace structure is given by the cycle type of α. If the cycle type of α is p,

$$p = [1^{p_1}, 2^{p_2}, \ldots, (2n)^{p_{2n}}], \qquad \sum_{i=1}^{2n} i\, p_i = 2n\,, \tag{2.5}$$

then the number of traces of $\mathcal{O}_\alpha$ is equal to the number of cycles in α,

$$C(\alpha) = \sum_i p_i\,. \tag{2.6}$$

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The flavour contractions appear in pairs $(2k-1, 2k)$, which is invariant under the permutation $\Sigma_0 = (1,2)(3,4)\cdots(2n-1,2n)$ consisting of pairwise swops. The permutation Σ_0 is invariant when conjugated by another set of permutations γ belonging to a subgroup $S_n[S_2]$ of S_{2n}

$$\gamma\Sigma_0\gamma^{-1} = \Sigma_0 \qquad \text{for} \qquad \gamma \in S_n[S_2]. \tag{2.7}$$

This wreath product group $S_n[S_2]$[2] has order $2^n n!$

$$|S_n[S_2]| = 2^n n! \tag{2.8}$$

It contains each of the n pairwise swops, which form a subgroup $(S_2)^{\times n} \subset S_n[S_2]$, along with $n!$ permutations of the n pairs. The operator $\mathcal{O}_\alpha$ is invariant under conjugation by $S_n[S_2]$,

$$\mathcal{O}_\alpha = \mathcal{O}_{\gamma\alpha\gamma^{-1}} \qquad \text{for} \qquad \gamma \in S_n[S_2] \tag{2.9}$$

because conjugation by a permutation gives re-ordering of the flavour indices, (A.6). Recalling the definition of $[\alpha]$ in (1.7), we observe

$$[\alpha] = [\gamma\alpha\gamma^{-1}] = \gamma[\alpha]\gamma^{-1} \qquad \text{for} \qquad \gamma \in S_n[S_2]. \tag{2.10}$$

We define

$$\mathcal{O}_{[\alpha]} \equiv \frac{1}{|S_n[S_2]|} \sum_{\gamma\in S_n[S_2]} \mathcal{O}_{\gamma\alpha\gamma^{-1}} \tag{2.11}$$

and observe that

$$\mathcal{O}_\alpha = \mathcal{O}_{[\alpha]} \tag{2.12}$$

It follows therefore that gauge-invariant mesonic operators are in 1-1 correspondence with the sums $[\alpha]$ in the group algebra.

For $N_c \geq 2n$ and $N_f \geq 2n$, the mesonic operators of the form (2.12) is uniquely and completely specified by the equivalence classes (2.9). The latter number is given by the Burnside Lemma as

$$\text{Number of mesonic operators } = \frac{1}{|S_n[S_2]|} \sum_{\gamma\in S_n[S_2]} \sum_{\alpha\in S_{2n}} \delta_{2n}(\gamma\alpha\gamma^{-1}\alpha^{-1}), \tag{2.13}$$

where

$$\delta_{2n}(g) = \begin{cases} 1 & (g = \text{identity} \in S_{2n}) \\ 0 & (\text{otherwise}). \end{cases} \tag{2.14}$$

Consider $\mathcal{O}_\alpha$ with fixed trace structure, where the cycle type of α is p in (2.5). The number of such gauge-invariants is

$$\frac{1}{|S_n[S_2]|} \sum_{\gamma\in S_n[S_2]} \sum_{\alpha\in T_p} \delta_{2n}(\gamma\alpha\gamma^{-1}\alpha^{-1}), \tag{2.15}$$

[2] $S_n[S_2]$ is also called the hyperoctahedral group in mathematics.

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where T_p consists of permutations of cycle type p. This is also equal to

$$\frac{1}{|H_p|}\sum_{\sigma\in H_p}\varphi(\sigma), \qquad \varphi(\sigma)=\sum_{\gamma\in[2^n]}\delta_{2n}(\gamma\sigma\gamma^{-1}\sigma^{-1}), \tag{2.16}$$

where the elements of H_p commute with a fixed permutation of cycle type p. A derivation of the equality of these formulae, along with counting at finite N_f is in section 4.3.

When $N_c < 2n$ or $N_f < 2n$, there exist a number of linearly dependent relations among operators. They are called finite N_c constraints or finite N_f constraints. These constraints can be expressed in terms of Young diagrams.[3] In section 3, we will construct a set of operators with representation labels, where the finite N_c and finite N_f constraints are manifest.

2.2 Two-point functions

Consider the free two-point functions of the mesonic operators (2.4). Using the Wick contraction rule (A.1), we obtain

$$\begin{aligned}\langle\mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle &= \delta_{\vec{a}}\,\delta^{\vec{b}}\Big\langle\prod_{k=1}^{2n}(\Phi^{a_k})^{i_k}_{i_{\alpha_1(k)}}\prod_{m=1}^{2n}(\Phi_{b_m})^{j_m}_{j_{\alpha_2(m)}}\Big\rangle = \delta_{\vec{a}}\,\delta^{\vec{b}}\sum_{\sigma\in\mathcal{S}_{2n}}\prod_{k=1}^{2n}\delta^{a_k}_{b_m}\delta^{i_k}_{j_{\alpha_2(m)}}\delta^{j_m}_{i_{\alpha_1(k)}}\Big|_{m=\sigma^{-1}(k)}\\ &= \delta_{\vec{a}}\,\delta^{\vec{b}}\sum_{\sigma\in\mathcal{S}_{2n}}\prod_{k=1}^{2n}\delta^{a_k}_{b_{\sigma^{-1}(k)}}\delta^{i_k}_{i_{\alpha_1\sigma\alpha_2\sigma^{-1}(k)}}\end{aligned} \tag{2.19}$$

where σ represents all possible Wick contractions, and $\delta_{\vec{a}}=\prod_{i=1}^{n}\delta_{a_{2i-1}a_{2i}}$. The results can be expressed by permutations,[4]

$$\langle\mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle = \sum_{\sigma\in S_{2n}}W(\sigma^{-1})\,tr_{2n}(\alpha_1\sigma\alpha_2\sigma^{-1}) = \sum_{\sigma\in S_{2n}}W(\sigma^{-1})\,N_c^{C(\alpha_1\sigma\alpha_2\sigma^{-1})}\,. \tag{2.20}$$

Here $C(\sigma)$ is (2.6), and $W(\sigma)$ is the flavour factor

$$W(\sigma)=\delta_{\vec{a}}\,\delta^{\vec{b}}\,\left(\prod_{k=1}^{2n}\delta^{a_k}_{b_{\sigma(k)}}\right)\equiv\delta_{\vec{a}}\,\delta^{\vec{b}}\,(\sigma)^{a_1,a_2,\cdots,a_{2n}}_{b_1,b_2,\cdots,b_{2n}}\,. \tag{2.21}$$

We have $W(1)=N_f^n$. The σ in (2.21) acts on $V_F^{\otimes 2n}$, with matrix elements equal to Kronecker deltas. We can also write $W(\sigma)$ by introducing contraction operators $C_{12}, C_{34}\cdots$, where C_{12} acts on the first and second tensor factors of $V_F^{\otimes 2n}$ as

$$C_{12}\,e_{a_1}\otimes e_{a_2}=\delta_{a_1a_2}\,e_b\otimes e_b \tag{2.22}$$

[3] Here is a simple example of a finite N_c constraint. For a 2×2 matrix X, we have the identity

$$tr(X^3)=\frac{3}{2}\,trX\,tr(X^2)-\frac{1}{2}(trX)^3. \tag{2.17}$$

We can rewrite this identity in terms of the projection operator associated with the anti-symmetric representation as

$$tr_3(p_{[1^3]}X^{\otimes 3})=0. \tag{2.18}$$

In short, we cannot anti-symmetrise more than $N_c=2$ indices.

[4] $\mathrm{tr}_{2n}(\sigma)=\mathrm{tr}_{2n}(\sigma\,1)$ is a special case of (2.3).

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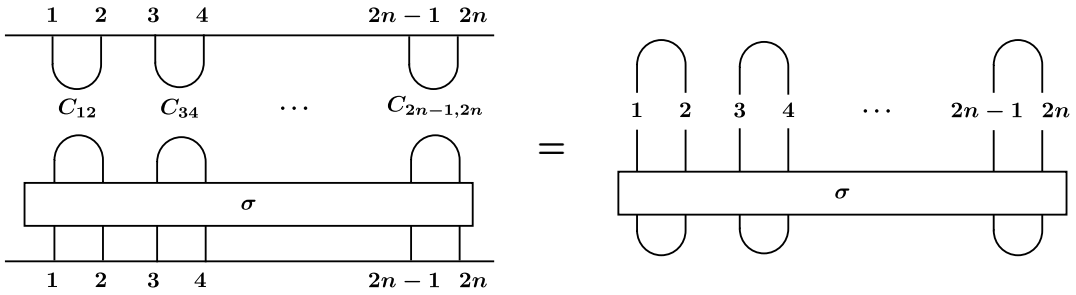

Figure 1. The left figure shows $C_{12}C_{34}\cdots C_{2n-1,2n}\sigma$ acting on $V_F^{\otimes 2n}$. The upper and lower horizontal lines are identified when taking the trace, as in the right figure.

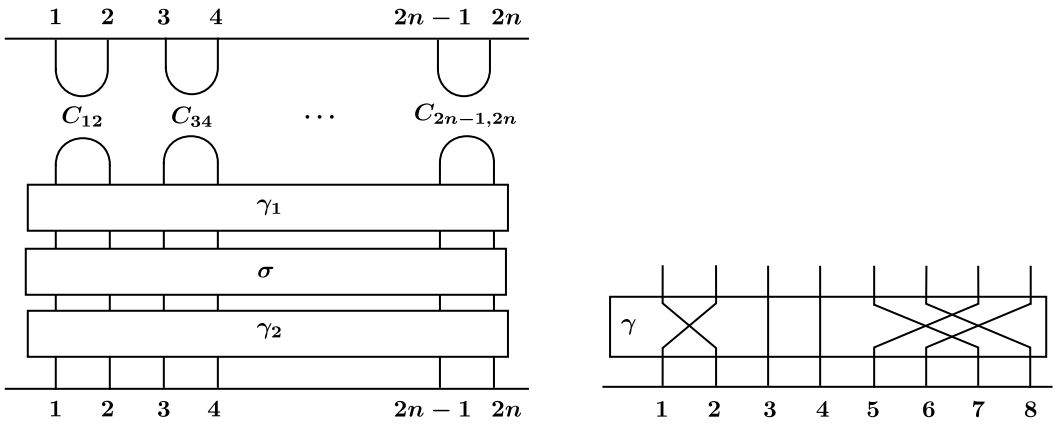

Figure 2. (Left) Diagram for $W(\gamma_1\sigma\gamma_2)$, which counts the number of connected components. (Right) An example of $\gamma \in S_n[S_2]$, showing that $S_n[S_2]$ does not change the number of connected components.

Then

$$W(\sigma) = tr_{V_F^{\otimes 2n}}\left(C_{12}C_{34}C_{56}\cdots C_{2n-1,2n}\sigma\right). \tag{2.23}$$

The contractions form part of the Brauer algebra which is the commutant of $\mathrm{O}(N_f)$ acting on $V_F^{\otimes 2n}$, the $2n$-fold tensor product of the fundamental representations.

The contraction operators obey

$$\gamma_1(C_{12}C_{34}\cdots C_{2n-1,2n})\gamma_2 = (C_{12}C_{34}\cdots C_{2n-1,2n}) \tag{2.24}$$

as can be seen from figure 2. It follows that

$$W(\sigma) = W(\gamma_1\sigma\gamma_2), \qquad \forall\gamma_1,\gamma_2 \in S_n[S_2] \tag{2.25}$$

and from (2.21)

$$W(\sigma) = W(\sigma^{-1}). \tag{2.26}$$

Let us denote by $z(\sigma)$ the number of cycles $W(\sigma)$ represented in figure 1. Then $W(\sigma)$ is given by

$$W(\sigma) = N_f^{z(\sigma)}. \tag{2.27}$$

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Indeed, the function $z(\sigma)$ satisfies

$$z(\sigma) = z(\gamma_1 \sigma \gamma_2), \qquad \forall \gamma_1, \gamma_2 \in S_n[S_2], \tag{2.28}$$

We will show in appendix B that it can be expressed as

$$z(\sigma) = \frac{1}{2} C(\Sigma_0 \sigma \Sigma_0 \sigma^{-1}). \tag{2.29}$$

Since permutations can be multiplied efficiently using group theory software such as GAP or with Mathematica, this is a very useful expression for practical calculations. It does not involve explicitly doing sums over indices ranging from 1 to N_f. The quantity (2.29) also has a nice mathematical meaning. It is equal to the number of cycles in the coset type of σ as explained in appendix A.3.

For later purposes, let us define

$$\Omega_{2n}^{(f)} = \frac{1}{N_f^n} \sum_{\sigma \in S_{2n}} N_f^{z(\sigma)} \sigma^{-1}, \qquad \Omega_{2n} = \frac{1}{N_c^{2n}} \sum_{\sigma \in S_{2n}} N_c^{C(\sigma)} \sigma^{-1}. \tag{2.30}$$

In terms of these quantities the two-point functions (2.20) have the form of

$$\begin{aligned} \langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \rangle &= \sum_{\sigma \in S_{2n}} N_f^{z(\sigma)} N_c^{C(\alpha_1 \sigma \alpha_2 \sigma^{-1})} = \sum_{\sigma \in S_{2n}} N_f^{\frac{1}{2} C(\sigma \Sigma_0 \sigma^{-1} \Sigma_0)} N_c^{C(\alpha_1 \sigma \alpha_2 \sigma^{-1})} \\ &= \sum_{\sigma} N_c^{2n} N_f^n \, \delta_{2n}(\Omega_{2n}^{(f)} \sigma) \, \delta_{2n}(\Omega_{2n} \alpha_1 \sigma \alpha_2 \sigma^{-1}). \end{aligned} \tag{2.31}$$

From (2.28), the $\Omega_{2n}^{(f)}$ satisfies

$$\gamma_1 \Omega_{2n}^{(f)} \gamma_2 = \Omega_{2n}^{(f)}, \tag{2.32}$$

and we recover $W(\sigma)$ by

$$W(\sigma) = N_f^n \, \delta_{2n}(\Omega_{2n}^{(f)} \sigma). \tag{2.33}$$

3 Orthogonal two-point functions

We now propose a set of operators labelled by representations,

$$\begin{aligned} \mathcal{O}^{R,\Lambda_1,\tau} &= tr_{2n}(P^{R,\Lambda_1,\tau} \; \Phi_{a_1} \otimes \Phi_{a_1} \otimes \Phi_{a_2} \otimes \Phi_{a_2} \otimes \cdots \otimes \Phi_{a_n} \otimes \Phi_{a_n}), \\ P^{R,\Lambda_1,\tau} &= \sum_{\alpha \in S_{2n}} B_k^{\Lambda_1} \; S^{\tau,\Lambda_1 R R}_{\quad k \; i \; j} \, D_{ij}^R(\alpha^{-1}) \; [\alpha], \end{aligned} \tag{3.1}$$

where R, Λ_1 are irreducible representations of S_{2n}, τ runs over $1, \ldots, C(R,R,\Lambda_1)$ is the Clebsch-Gordan multiplicity for S_{2n} tensor products (see (A.21)), and the $S^{\tau,\Lambda_1 R R}_{\quad k \; i \; j}$ is the Clebsch-Gordan coefficient (A.20). The use of this group theory data in the construction of covariant bases was developed in [9, 12]. The $B_k^{\Lambda_1}$, called a branching coefficient, is defined in terms of the reduction of the irreducible representation Λ_1 of S_{2n} in terms of a direct sum of irreducible representations of $S_n[S_2] \subset S_{2n}$. A generic state $|\Lambda_1, k\rangle$ can be expanded in terms of irreducible representations of $S_n[S_2]$. The one-dimensional irreducible representation of $S_n[S_2]$ is known to appear in the decomposition of Λ_1 with

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unit multiplicity if Λ_1 is even, i.e. the partition of Λ has the form $(2\lambda_1, 2\lambda_2, \cdots)$ with integers λ_i. Denoting this one-dimensional subspace of Λ_1 as $|\Lambda_1 \to 1_{S_n[S_2]}\rangle$, we have the branching coefficients

$$B_k^{\Lambda_1} = \langle \Lambda_1, k \mid \Lambda_1 \to 1_{S_n[S_2]} \rangle, \tag{3.2}$$

where k runs over $1, \cdots, d_{\Lambda_1}$.[5] The branching coefficient is related to the singlet projector $p_{1_{S_n[S_2]}}$ by

$$D^{\Lambda_1}_{k_1 k_2}(p_{1_{S_n[S_2]}}) = B^{\Lambda_1}_{k_1} B^{\Lambda_1}_{k_2}\,, \qquad p_{1_{S_n[S_2]}} = \frac{1}{|S_n[S_2]|} \sum_{\gamma \in S_n[S_2]} \gamma\,. \tag{3.3}$$

The quantity $P^{R,\Lambda_1,\tau}$ has the symmetry

$$P^{R,\Lambda_1,\tau} = \gamma P^{R,\Lambda_1,\tau} \gamma^{-1} \tag{3.4}$$

for $\gamma \in S_n[S_2]$. From the inverse of (3.1), $[\alpha]$ is expressed by

$$[\alpha] = \frac{1}{(2n)!} \sum_{R,\Lambda_1,\tau} d_R B^{\Lambda_1}_k S^{\tau,\Lambda_1}{}_{k\ i\ j}^{R\ R} D^R_{ji}(\alpha) P^{R,\Lambda_1,\tau}. \tag{3.5}$$

Let us introduce

$$\omega_{\Lambda/2}(\sigma) = \frac{1}{|S_n[S_2]|} \sum_{\gamma \in S_n[S_2]} \chi_\Lambda(\sigma\gamma), \tag{3.6}$$

where Λ is an even Young diagram (A.27). The function $\omega_{\Lambda/2}(\sigma)$ is called the zonal polynomial of the Gelfand pair $(S_{2n}, S_n[S_2])$ [26]. We have other expressions in terms of the projection operator associated with the singlet representation of $S_n[S_2]$,

$$\omega_{\Lambda/2}(\sigma) = \chi_\Lambda(\sigma p_{1_{S_n[S_2]}}) = \langle \Lambda \to 1_{S_n[S_2]} | \sigma | \Lambda \to 1_{S_n[S_2]} \rangle. \tag{3.7}$$

The zonal polynomial has the property

$$\omega_{\Lambda/2}(\sigma) = \omega_{\Lambda/2}(\gamma_1 \sigma \gamma_2), \quad (\gamma_1, \gamma_2 \in S_n[S_2]). \tag{3.8}$$

The prominent property of the operators is that they have diagonal two-point functions

$$\langle \mathcal{O}^{R,\Lambda_1,\tau} O^{S,\Lambda_1',\tau'} \rangle = \delta^{RS}\, \delta^{\Lambda_1\Lambda_1'}\, \delta^{\tau\tau'} \left(\frac{(2n)!}{d_R}\right)^2 Dim(R) N_f^n\, \omega_{\Lambda_1/2}(\Omega^{(f)}_{2n}), \tag{3.9}$$

where $Dim(R)$ is the dimension of $\mathrm{U}(N_c)$ associated with the representation R, and d_R is the dimension of S_{2n} associated with the representation R. $\omega_{\Lambda_1/2}(\Omega^{(f)}_{2n})$ is a specialisation of the zonal spherical function of the Gelfand pair $(\mathrm{GL}(N_f), \mathrm{O}(N_f))$ explained in appendix C.2. It is a polynomial in N_f, which was also encountered in [21, 22]. This equation is derived in appendix C.1. Concrete examples of the operators $\mathcal{O}^{R,\Lambda_1,\tau}$ are given in appendix D.

In section 4.1 we count the number of mesonic operators using Schur-Weyl duality. It will turn out that the representation-labelled operator (3.1) should vanish if

$$c_1(R) < N_c \qquad \text{or} \qquad c_1(\Lambda_1) < N_f\,, \tag{3.10}$$

[5] Due to Frobenius duality between restriction and induction, the state $|\Lambda_1 \to 1_{S_n[S_2]}\rangle$ can also be thought as the Λ_1-component of the induced representation $\mathrm{Ind}_H^G 1_H$ with $G = S_{2n}$ and $H = S_n[S_2]$.

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where $c_1(R)$ denotes the length of the first column of the Young diagram R. This expectation is consistent with the normalisation of our two-point functions (3.9). We can find the vanishing properties from the formulae

$$Dim(R) = \prod_{(i,j)\in R} \frac{(N_c + j - i)}{h(i,j)} \tag{3.11}$$

and

$$\omega_{\Lambda_1/2}(\Omega^{(f)}_{2n}) = \frac{|S_n[S_2]|}{N_f^n} \prod_{(i,j)\in\Lambda_1/2} (N_f + 2j - i - 1), \tag{3.12}$$

where (i,j) labels the row and column of the boxes of the Young diagram, and $h(i,j)$ is the hook-length (A.11). The formula (3.12) was derived in [21, 26]. Both (3.11) and (3.12) depend on N_c and N_f through the product of factors, e.g.

$$Dim(R) \sim \prod_{\text{boxes}} \begin{array}{|c|c|c|c|}\hline N_c & N_c+1 & N_c+2 & N_c+3 \\ \hline N_c-1 & N_c \\ \cline{1-2}\end{array} \tag{3.13}$$

$$\omega_{\Lambda_1/2}(\Omega^{(f)}_{2n}) \sim \prod_{\text{boxes}} \begin{array}{|c|c|c|c|}\hline N_f & N_f+2 & N_f+4 & N_f+6 \\ \hline N_f-1 & N_f+1 \\ \cline{1-2}\end{array} \tag{3.14}$$

In [12] a general formula of diagonal operators is presented, which works for any group G and representation V. However, the formula requires the calculation of the CG coefficients arising from the decomposition of $V_F^{\otimes 2n}$ by $G \times S_{2n}$. It turns out that our operator $\mathcal{O}^{R,\Lambda_1,\tau}$ is a concrete realisation of the general formulae of [12]. The relevant CG coefficients can be obtained explicitly as explained in appendix E. The branching coefficients associated with the Gelfand pair $(S_{2n}, S_n[S_2])$ provide a neat description of these Clebsch-Gordan coefficients, and hence of the space of $\mathrm{O}(N_f)$ singlet operators of length $2n$.

4 Operator counting

In this section we discuss several aspects of the counting formula, taking care of essential distinction between large N_c and finite N_c (i.e. $N_c \geq 2n$ and $N_c < 2n$) as well as large N_f and finite N_f (i.e. $N_f \geq 2n$ and $N_f < 2n$). First we employ Schur-Weyl duality and to obtain group theoretic expressions for the dimension of the space of operators at finite N_c, N_f. Then we rewrite the counting formula in terms of delta functions of permutations in the large N_c and large (N_c, N_f) limits. The latter formulae are recognised as the counting formulae for graphs based on Burnside's Lemma. We will rediscover various ways of associating permutation labels to gauge-invariant operators at large N_c, N_f.

4.1 Schur-Weyl duality and counting for finite N_c, N_f

The construction of our operators can be explained by Schur-Weyl duality, along the lines of [9, 12, 27, 28]. The scalar field $(\Phi^a)^j_i$ has one flavour and two colour indices, and belongs

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to $V_F \otimes V_C \otimes \overline{V}_C$. Thus the tensor product $\Phi^{\otimes 2n}$ belongs to $\left(V_F \otimes V_C \otimes \overline{V}_C\right)^{\otimes 2n}$. There is a natural action of permutations $\sigma \in S_{2n}$ on the tensor product,

$$\sigma \,:\, (\Phi^{a_1})^{j_1}_{i_1}(\Phi^{a_2})^{j_2}_{i_2}\dots(\Phi^{a_{2n}})^{j_{2n}}_{i_{2n}} \;\mapsto\; (\Phi^{a_{\sigma(1)}})^{j_{\sigma(1)}}_{i_{\sigma(1)}}(\Phi^{a_{\sigma(2)}})^{j_{\sigma(2)}}_{i_{\sigma(2)}}\dots(\Phi^{a_{\sigma(2n)}})^{j_{\sigma(2n)}}_{i_{\sigma(2n)}}\,. \tag{4.1}$$

The r.h.s. is a polynomial of bosonic variables $(\Phi^a)^j_i$. Since they commute with each other, this polynomial is invariant under σ.

Let us count the number of gauge-invariant scalar operators. We regard $V_F, V_C, \overline{V}_C$ as $\mathrm{U}(N_f), \mathrm{U}(N_c)$-modules and apply Schur-Weyl (SW) duality. We find

$$\begin{aligned}&\left(V_F \otimes V_C \otimes \overline{V}_C\right)^{\otimes 2n} \\ &= \bigoplus_{\substack{\Lambda\vdash 2n\\ c_1(\Lambda_1)\le N_f}} \bigoplus_{R\vdash 2n} \bigoplus_{\substack{S\vdash 2n\\ c_1(S)\le N_c}} \left(V_\Lambda^{U(N_f)} \otimes V_\Lambda^{S_{2n}}\right) \otimes \left(V_R^{U(N_c)} \otimes V_R^{S_{2n}}\right) \otimes \left(\bar{V}_S^{U(N_c)} \otimes V_S^{S_{2n}}\right).\end{aligned} \tag{4.2}$$

where $c_1(R)$ is defined in (A.8). We impose the condition $R = S$ to select $\mathrm{U}(N_c)$-invariant operators,

$$V_R^{\mathrm{U}(N_c)} \otimes \bar{V}_S^{\mathrm{U}(N_c)}\Big|_{\mathrm{U}(N_c)} = \delta_{R,S}\, V_\emptyset^{\mathrm{U}(N_c)} \tag{4.3}$$

and the operators should also be S_{2n}-invariant

$$V_\Lambda^{S_{2n}} \otimes V_R^{S_{2n}} \otimes V_R^{S_{2n}}\Big|_{S_{2n}} = \bigoplus_{\Lambda'} V_\Lambda^{S_{2n}} \otimes V_{RR}^{\Lambda'} \otimes V_{\Lambda'}^{S_{2n}}\Big|_{S_{2n}} = V_{RR}^{\Lambda} \otimes V_{[2n]}^{S_{2n}}\,. \tag{4.4}$$

where $V_{RR}^{\Lambda'}$ is the multiplicity space of $V_{\Lambda'}^{S_{2n}}$ in the tensor product $V_R^{S_{2n}} \otimes V_R^{S_{2n}}$. Its dimension is the CG multiplicity $C(R, R, \Lambda')$ (A.21). This implies

$$\left(V_F \otimes V_C \otimes \overline{V}_C\right)^{\otimes 2n}\Big|_{\mathrm{U}(N_c)\times S_{2n}} = \bigoplus_{\Lambda} \bigoplus_{R} \left(V_\Lambda^{U(N_f)} \otimes V_{RR}^{\Lambda} \otimes V_{[2n]}^{S_{2n}}\right). \tag{4.5}$$

The number of gauge-invariant scalar operators is given by taking the dimension of both sides.

Now we count the number of mesonic operators (2.4), which contains the Kronecker delta $\prod_{i=1}^n \delta^{a_{2i-1}a_{2i}}$. The mesonic operator is invariant under the following action of permutations γ in the wreath product group $S_n[S_2]$,

$$\begin{aligned}\gamma \,:\, &\left(\prod_{i=1}^n \delta^{a_{2i-1}a_{2i}}\right)(\Phi^{a_1})^{j_1}_{i_1}(\Phi^{a_2})^{j_2}_{i_2}\dots(\Phi^{a_{2n}})^{j_{2n}}_{i_{2n}} \\ &\qquad\mapsto \left(\prod_{i=1}^n \delta^{a_{\gamma(2i-1)}a_{\gamma(2i)}}\right)(\Phi^{a_{\gamma(1)}})^{j_1}_{i_1}(\Phi^{a_{\gamma(2)}})^{j_2}_{i_2}\dots(\Phi^{a_{\gamma(2n)}})^{j_{2n}}_{i_{2n}}\,.\end{aligned} \tag{4.6}$$

This γ is not part of σ in (4.1), because γ does not change the colour indices. Recall that the wreath product subgroup $S_n[S_2]$ is the set of permutations which leaves $\Sigma_0 = (12)(34)\dots(2n-1, 2n) \in S_{2n}$ invariant under action by conjugation

$$\gamma\Sigma_0\gamma^{-1} = \Sigma_0 \quad \text{for} \quad \gamma \in S_n[S_2]. \tag{4.7}$$

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$2n$	2	4	6	8	10	12
I_{2n}	2	8	34	182	1300	12534

Table 1. The number of all mesonic operators of length $2n$ for $N_c \geq 12$ and $N_f \geq 6$.

The contraction of the flavour indices breaks $\mathrm{U}(N_f)$ to $\mathrm{O}(N_f)$ and projects to the invariant representation of $\mathrm{O}(N_f)$. So we can count the number of mesonic operators by restricting (4.2) to the subspace invariant under $\mathrm{O}(N_f)\times S_n[S_2]\times S_{2n}\times \mathrm{U}(N_c)$. We use the fact that

$$\begin{aligned} V_F^{\otimes 2n}\big|_{O(N_f)\times S_n[S_2]} &= \bigoplus_{\Lambda} V_\Lambda^{\mathrm{U}(N_f)}\big|_{\mathrm{O}(N_f)} \otimes V_\Lambda^{S_{2n}}\big|_{S_n[S_2]} \\ &= \bigoplus_{\Lambda_1:\,\text{even}} V_{\Lambda_1}^{U(N_f)}\big|_{\mathrm{O}(N_f)} \otimes V_{\Lambda_1}^{S_{2n}}\big|_{S_n[S_2]}. \end{aligned} \tag{4.8}$$

Both $(\mathrm{GL}(N_f), \mathrm{O}(N_f))$ and $(S_{2n}, S_n[S_2])$ are Gelfand pairs [26], which has the multiplicity-free property. In particular Λ of the parent group contains the trivial of the subgroup with unit multiplicity if Λ_1 is an even partition (A.27). Now we repeat the above argument, taking into account that the two factors in (4.8) are one-dimensional, to obtain

$$\left(V_F \otimes V_C \otimes \overline{V}_C\right)^{\otimes 2n}\Big|_{\mathrm{O}(N_f)\times S_n[S_2]\times \mathrm{U}(N_c)\times S_{2n}} = \bigoplus_{\Lambda_1:\,\text{even}} \bigoplus_{R} V_{RR}^{\Lambda_1} \tag{4.9}$$

The number $I_{2n}(N_c, N_f)$ of mesonic singlet operators of length $2n$ at finite N_c, N_f is thus

$$I_{2n}(N_c, N_f) = \sum_{\substack{\Lambda_1:\,\text{even} \\ c_1(\Lambda_1)\leq N_f}} \sum_{\substack{R \\ c_1(R)\leq N_c}} C(R, R, \Lambda_1). \tag{4.10}$$

This result agrees with the number of diagonal operators (3.1). Table 1 shows some values of $I_{2n}(N_c, N_f)$ at large N_c, N_f.

The formula (4.10) counts only the $\mathrm{O}(N_f)$ singlets. The method to count the $\mathrm{SO}(N_f)$ singlets or other representations are described in [12]. The difference between $\mathrm{O}(N_f)$ and $\mathrm{SO}(N_f)$ lies in the existence of "baryonic" operators. We will explain more about these points in appendix E.2.

4.2 Large N_c

The number of singlet mesonic operators for finite N_c and finite N_f is given by (4.10). We consider simplifications at large N_c; more precisely $N_c \geq 2n$. There we can convert the sum over R into a sum over permutations.

Let us define

$$\varphi_{N_f}(\sigma) := \sum_{\substack{\Lambda_1:even \\ c_1(\Lambda_1)\leq N_f}} \chi_{\Lambda_1}(\sigma), \tag{4.11}$$

and apply the formula (A.12), valid for $N_c \geq 2n$, to the counting formula (4.10). It simplifies as

$$I_{2n}(N_f) \equiv I_{2n}(N_c \geq 2n, N_f) = \frac{1}{(2n)!} \sum_{\sigma\in S_{2n}} \sum_{\gamma\in S_{2n}} \delta_{2n}(\gamma\sigma\gamma^{-1}\sigma^{-1})\varphi_{N_f}(\sigma) \tag{4.12}$$

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p	$[4]$	$[3,1]$	$[2^2]$	$[2,1^2]$	$[1^4]$
H_p	$\mathbb{Z}_4$	$\mathbb{Z}_3 \times \mathbb{Z}_1$	$S_2[\mathbb{Z}_2]$	$\mathbb{Z}_2 \times S_2$	S_4

Table 2. The symmetry group which preserves the cycle type at $2n = 4$.

where $\delta_{2n}(g)$ is defined by (2.14). When the cycle type of σ is $p = [1^{p_1}, 2^{p_2}, \cdots, n^{p_n}]$ ($\sum_{i=1} ip_i = 2n$), define

$$T_p = \{\sigma \in S_{2n} \,|\, \text{cycle type of } \sigma \text{ is } p\} \tag{4.13}$$

$$H_p(\sigma) = \{\gamma \in S_{2n} \,|\, \gamma\sigma = \sigma\gamma,\ \sigma \in T_p\}. \tag{4.14}$$

Note that $H_p(\sigma), H_p(\sigma')$ are conjugate with each other if $\sigma, \sigma' \in T_p$. We define H_p as $H_p(\sigma_*)$ for a fixed $\sigma_* \in T_p$. The order of H_p, namely the number of elements that commute with any permutation of cycle type p, is given by

$$|H_p| = \frac{(2n)!}{|T_p|}. \tag{4.15}$$

See table 2 for examples. In particular, when the cycle type is $[2^n]$, the symmetry group (the stabiliser) is $H_{[2^n]} = S_n[S_2]$. Thus

$$|S_n[S_2]| = \frac{(2n)!}{(2n-1)!!} = 2^n n!. \tag{4.16}$$

Then (4.12) may be written as[6]

$$\begin{aligned} I_{2n}(N_f) &= \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} \sum_{p \vdash 2n} \sum_{\gamma \in T_p} \delta_{2n}(\gamma\sigma\gamma^{-1}\sigma^{-1}) \varphi_{N_f}(\sigma) \\ &= \frac{1}{(2n)!} \sum_p |T_p| \sum_{\sigma \in H_p} \varphi_{N_f}(\sigma) \\ &= \sum_p \frac{1}{|H_p|} \sum_{\sigma \in H_p} \varphi_{N_f}(\sigma) \end{aligned} \tag{4.17}$$

or

$$\begin{aligned} I_{2n}(N_f) &= \frac{1}{(2n)!} \sum_{q \vdash 2n} \sum_{\sigma \in T_q} \sum_{\gamma \in S_{2n}} \delta_{2n}(\gamma\sigma\gamma^{-1}\sigma^{-1}) \varphi_{N_f}(\sigma) \\ &= \frac{1}{(2n)!} \sum_q |H_q| \sum_{\sigma \in T_q} \varphi_{N_f}(\sigma) \\ &= \sum_q \frac{1}{|T_q|} \sum_{\sigma \in T_q} \varphi_{N_f}(\sigma) \end{aligned} \tag{4.18}$$

[6]Here we used the fact that any $\gamma \in T_p$ is written as $\gamma = \nu\gamma_*\nu^{-1}$ with a fixed $\gamma_* \in T_p$ for some $\nu \in S_{2n}$. This ν disappears after the redefinition $\sigma' = \nu^{-1}\sigma\nu$. The sum of $\varphi_{N_f}(\sigma)$ over $H_p(\gamma_*)$ does not depend on the choice of γ_*.

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The two expressions give the large N_c equivalence

$$I_{2n}(N_f) = \sum_p \frac{1}{|H_p|} \sum_{\sigma \in H_p} \varphi_{N_f}(\sigma) = \sum_q \frac{1}{|T_q|} \sum_{\sigma \in T_q} \varphi_{N_f}(\sigma). \tag{4.19}$$

This equivalence can also be derived from the following group theory identity

$$\sum_p \frac{1}{|H_p|} \sum_{\sigma \in H_p} \sum_{\mu \in S_{2n}} \mu\sigma\mu^{-1} = \sum_q \frac{1}{|T_q|} \sum_{\sigma \in T_q} \sum_{\mu \in S_{2n}} \mu\sigma\mu^{-1} \tag{4.20}$$

The expressions (4.17) and (4.18) will be used in the next subsection.

4.3 Large N_c and N_f

We take large N_f and large N_c limits to convert all representations to permutations, and obtain several expressions of the counting.

We define

$$\varphi(\sigma) := \varphi_{N_f \geq 2n}(\sigma) = \sum_{\Lambda_1 : even} \chi_{\Lambda_1}(\sigma) = \sum_{\Lambda_1 \vdash 2n} \chi_{\Lambda_1}(\sigma) M^{\Lambda_1}_{1_{S_n[S_2]}}. \tag{4.21}$$

where we used the formula (A.26) saying that $(S_{2n}, S_n[S_2])$ is a Gelfand pair. We also have

$$\begin{aligned} \varphi(\sigma) &= \frac{1}{|S_n[S_2]|} \sum_{\Lambda_1} \sum_{u \in S_n[S_2]} \chi_{\Lambda_1}(\sigma)\chi_{\Lambda_1}(u) \\ &= \frac{1}{|S_n[S_2]|} \sum_{u \in S_n[S_2]} \sum_{\mu \in S_{2n}} \delta_{2n}(\mu\sigma\mu^{-1}u). \end{aligned} \tag{4.22}$$

Let us define

$$Z_p \equiv \frac{1}{|H_p|} \sum_{\sigma \in H_p} \varphi(\sigma), \tag{4.23}$$

which is also written as

$$Z_p = \sum_{\Lambda_1 \vdash 2n} M^{\Lambda_1}_{1_{H_p}} M^{\Lambda_1}_{1_{S_n[S_2]}} = \sum_{\Lambda_1 \vdash 2n, even} M^{\Lambda_1}_{1_{H_p}} \tag{4.24}$$

From (4.17) the total number of singlets with $2n$ fields is

$$I_{2n} \equiv I_{2n}(N_f \geq 2n) = \sum_{p \vdash 2n} Z_p \tag{4.25}$$

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We will derive three expressions for Z_p below.[7]

The quantity Z_p is the number of equivalence classes in the double coset $S_n[S_2]\backslash S_{2n}/H_p$,

$$Z_p = \frac{1}{|S_n[S_2]|}\frac{1}{|H_p|}\sum_{\sigma\in H_p}\sum_{u\in S_n[S_2]}\sum_{\mu\in S_{2n}}\delta_{2n}(u\mu\sigma\mu^{-1}) \tag{4.29}$$

The double coset space is the set of equivalence classes of permutations in S_{2n}, generated by the left and right multiplications by the subgroups $S_n[S_2]$ and H_p respectively.

$$\mu \sim u\mu\sigma \qquad (u \in S_n[S_2], \quad \sigma \in H_p)\,. \tag{4.30}$$

The above delta-function sum (4.29), counting the number of elements in the double coset space, is the application of the Burnside Lemma, which reduces the counting of orbits under a group action to the counting of fixed points under the group action. This double coset counting is the same as the counting of bi-partite graphs. The bi-partite graphs have vertices in two colours (say black and white), and edges which connect only the vertices of different colours. The black vertices associated with partitions p, have cyclic order and there are p_1 univalent, p_2 bi-valent, p_3 trivalent vertices, etc. These are easy to understand in terms of counting of traces of the scalar fields with global symmetry indices contracted. A cyclic black vertex of valency v corresponds to a trace with v scalar fields. The white vertices correspond to links between pairs of edges emanating from the black vertices, and correspond to flavour indices of the corresponding fields being contracted. The connection between double cosets and graph counting is explained in a physics context in [29]. By going to large N_c, N_f, we see that counting $\mathrm{SO}(N_f)$ invariants is simply counting the graphs.

[7]Each expression of Z_p is related to different ways of writing the mesonic operators. Let us introduce

$$\mathcal{O}_{\alpha,\rho} = \left(\prod_{i=1}^{n}\delta^{a_{2i-1}a_{2i}}\right) tr_{2n}\left(\alpha\ \Phi_{a_{\rho(1)}}\otimes\Phi_{a_{\rho(2)}}\otimes\cdots\otimes\Phi_{a_{\rho(2n-1)}}\otimes\Phi_{a_{\rho(2n)}}\right). \tag{4.26}$$

This description is redundant in the following way,

$$\mathcal{O}_{\alpha,\rho} = O_{\gamma_1\alpha\gamma_1^{-1},\gamma_2\rho\gamma_1^{-1}} \qquad (\gamma_1 \in S_{2n}\,, \quad \gamma_2 \in S_n[S_2])\,. \tag{4.27}$$

The permutation γ_1 comes from the re-ordering (A.6), and γ_2 is the symmetry of the Kronecker delta's. By using this redundancy we can gauge-fix either α or ρ. If we fix ρ, we obtain the operator $\mathcal{O}_\alpha$ in (2.4). The corresponding counting formula is (4.34). If we fix α, then the cycle structure of α, denoted by $p\vdash 2n$, determines the colour (or multi-trace) structure. It corresponds to the counting formula is (4.29). Finally, we use the contraction operators (2.22) and write

$$\mathcal{O}_{\alpha,\tau} = \left(\prod_{i=1}^{n}C_{\rho(2i-1)\rho(2i)}\right) tr_{2n}\left(\alpha\ \Phi_{a_1}\otimes\cdots\otimes\Phi_{a_{2n}}\right) \tag{4.28}$$

We find that the transformation rule of the quantities $\prod_{i=1}^{n}C_{\rho(2i-1)\rho(2i)} = \rho^{-1}\left(\prod_{i=1}^{n}C_{2i-1,2i}\right)\rho$ and $\tau = \rho^{-1}\Sigma_0\rho$ under the map $\rho\to\rho\gamma^{-1}$ are identical. The expression (4.28) is related to the last counting formula (4.33).

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Now observe that the last line in (4.22) can be rewritten as[8]

$$\begin{aligned}\varphi(\sigma) &= \frac{1}{|S_n[S_2]|}\sum_{u\in S_{2n}}\sum_{\mu\in S_{2n}}\delta_{2n}(\mu\sigma\mu^{-1}u)\delta_{2n}(\Sigma_0 u\Sigma_0^{-1}u^{-1}) \\ &= \frac{1}{|S_n[S_2]|}\sum_{\mu\in S_{2n}}\delta_{2n}(\Sigma_0\mu\sigma\mu^{-1}\Sigma_0^{-1}\mu\sigma^{-1}\mu^{-1}) \\ &= \sum_{\tau\in[2^n]}\delta_{2n}(\tau\sigma\tau^{-1}\sigma^{-1}). \end{aligned} \tag{4.32}$$

From (4.12), (4.25) and (4.32),

$$\begin{aligned} Z_p &= \frac{1}{(2n)!}\sum_{\sigma\in T_p}\sum_{\gamma\in S_{2n}}\delta_{2n}(\gamma\sigma\gamma^{-1}\sigma^{-1})\varphi(\gamma) \\ &= \frac{1}{(2n)!}\sum_{\sigma\in T_p}\sum_{\gamma\in S_{2n}}\delta_{2n}(\gamma\sigma\gamma^{-1}\sigma^{-1})\sum_{\tau\in[2^n]}\delta_{2n}(\tau\gamma\tau^{-1}\gamma^{-1}). \end{aligned} \tag{4.33}$$

This can be recognised as the counting of pairs (σ,τ) in conjugacy classes $(T_p,[2^n])$, subject to equivalences $(\sigma,\tau)\sim(\gamma\sigma\gamma^{-1},\gamma\tau\gamma^{-1})$ where $\gamma\in S_{2n}$. Such equivalence classes of pairs form another way of encoding bi-partite graphs. It amounts to choosing a labelling of the edges using integers $\{1,\cdots,2n\}$ and reading off the labels of the edges around the black and white vertices. This is an alternative way to encode graphs, which differs from the encoding by a permutation $\sigma\in S_{2n}$ which links directly with the counting by double cosets. This way of encoding graphs, in the context of Feynman graphs (which have symmetric rather than the cyclic vertices here) is illustrated in figure 7 of [29]). The way that links directly with double cosets is shown in figure 10 there.

Some further manipulation of (4.33) gives

$$Z_p = \frac{1}{|S_n[S_2]|}\sum_{\sigma\in T_p}\sum_{\gamma\in S_n[S_2]}\delta_{2n}(\gamma\sigma\gamma^{-1}\sigma^{-1}). \tag{4.34}$$

The equation (4.34) establishes the equivalence between (2.15) and (2.16). We can reproduce this result also by applying the Burnside Lemma directly to the equivalence class of (4.27).

Using (4.18) we obtain yet another formula

$$I_{2n} = \sum_q \tilde{Z}_q, \qquad \tilde{Z}_q = \frac{1}{|T_q|}\sum_{\gamma\in T_q}\sum_{\tau\in[2^n]}\delta_{2n}(\tau\gamma\tau^{-1}\gamma^{-1}). \tag{4.35}$$

Note however that $\tilde{Z}_q\neq Z_q$.[9] We now have some physical insight into the two ways of writing I_{2n} as sums over partitions in (4.20). (4.19) In one way we have Z_p. In another, we

[8] At the first line of (4.32) we have used that the elements in $S_n[S_2]$ satisfy

$$\Sigma_0 u\Sigma_0^{-1} = u \tag{4.31}$$

for $\Sigma_0=(12)(34)\cdots(2n-1,2n)$. See also the discussion in section 5.4 of [29].

[9] For example, when $q=\{[4],[3,1],[2^2],[2,1^2],[1^4]\}$, then $\tilde{Z}_q=\{1,0,3,1,3\}$ and $Z_q=\{2,1,2,2,1\}$.

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have $\varphi(\gamma)$ with $\gamma \in T_q$. The partition p is the trace structure. The partition q is the cycle structure of γ which commutes with $\tau \in [2^n]$. Note that we have arrived at the sums over a product of two delta functions in this section by taking large N_c, N_f.

It is instructive to reconsider in reverse what we did in this section. Start from gauge-invariant operators parametrised by permutation equivalence classes. Graph counting associated with gauge-invariant operators can be expressed in terms of permutation sums with a product of delta functions. The Young diagrams R, Λ come from Fourier transforming these two delta functions. In the present context, we have seen that the numbers of rows of R, Λ are cut off by the rank of gauge and global symmetry groups. In a wider context, we may wonder about the physical meaning of introducing extra integers to cut off the numbers of columns.

5 Permutation topological field theory

We explore the connection to a two-dimensional topological field theory (TFT) of permutation groups. This TFT is a topological lattice gauge theory whose gauge group is S_{2n}, defined on a 2-complex (collection of 0- , 1- and 2-cells glued together). The computation of observables of the TFT involves a sum over the group elements of S_{2n} for every edge (1-cell), with a weight equal to a product of delta functions, one for every face (2-cell). The delta function weight ensures that the sum is invariant under refinement of the cell decomposition, so that a continuum limit can be reached. This type of TFT is discussed in the physics literature in e.g. [23–25]. A review of the TFT of permutations and application to a large class of observables in quiver gauge theories is given in [30].

As a first step, we reconsider the number of mesonic states as a partition function in TFT. By summing over $p \vdash 2n$ in (4.33), we find

$$I_{2n} = \frac{1}{|S_n[S_2]|} \sum_{\sigma \in S_{2n}} \sum_{\gamma \in S_{2n}} \delta_{2n}(\gamma\sigma\gamma^{-1}\sigma^{-1}) \delta_{2n}(\Sigma_0 \gamma \Sigma_0 \gamma^{-1}) \tag{5.1}$$

This formula gives the number of mesonic singlets as a partition function for S_{2n} TFT on the 2-complex shown in the upper left of figure 3. The 2-complex consists of two tori (drawn as cylinders with top and bottom boundaries identified) joined along a circle, associated with permutation σ. One of the tori has a cycle with permutation Σ_0, which is fixed rather than being summed. This cycle with constrained permutation is a defect. Note that the delta function $\delta_{2n}(\Sigma_0\gamma\Sigma_0\gamma^{-1})$ can be solved explicitly as in (4.34).

Next we rewrite the free two-point functions (2.31) to make contact with (5.1),

$$\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \rangle = \sum_{\beta_1 \in S_{2n}} \sum_{\beta_2 \in S_{2n}} \sum_{\sigma \in S_{2n}} N_c^{C(\beta_1)} N_f^{\frac{1}{2}C(\beta_2)} \delta_{2n}(\beta_1^{-1}\alpha_1\sigma\alpha_2\sigma^{-1}) \delta_{2n}(\beta_2^{-1}\sigma\Sigma_0\sigma^{-1}\Sigma_0) \tag{5.2}$$

If we take the large N_f, N_c limit, the leading terms only come from $\beta_1 = \beta_2 = 1$. If we set $[\alpha_1] = [\alpha_2^{-1}] = [\alpha]$ and sum over the conjugacy class $[\alpha]$, we reproduce the number of

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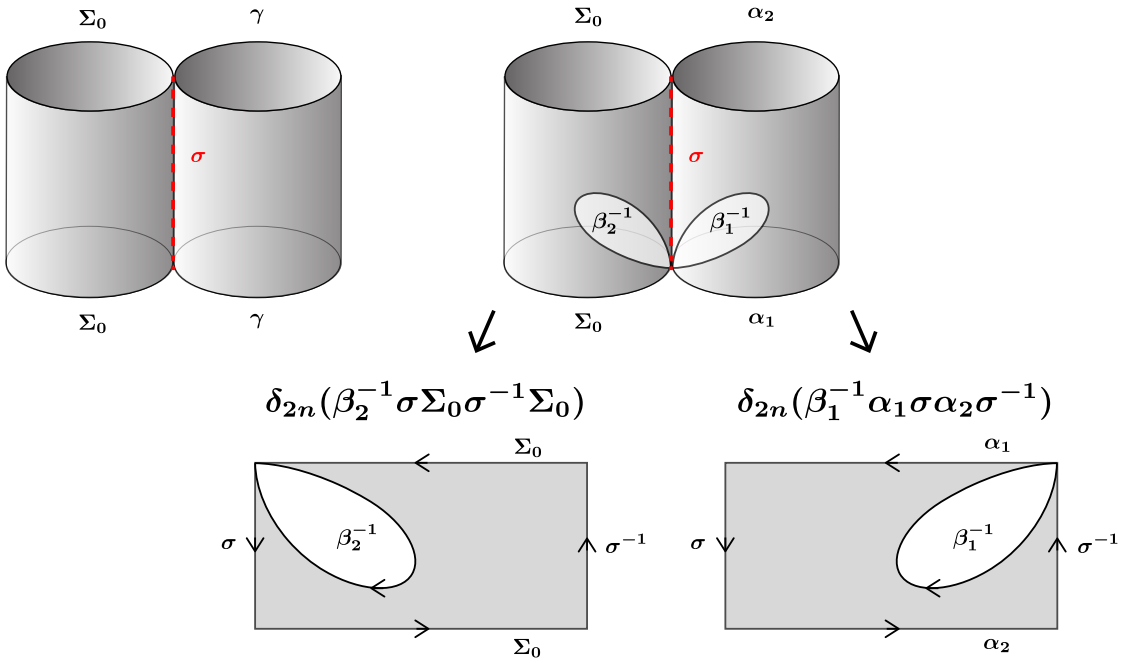

Figure 3. Observables in TFT. The upper left figure is the number of states I_{2n}, where Σ_0, γ and the two ends of σ are identified. The upper right figure is the two-point function $\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \rangle$, where Σ_0 and the two ends of σ are identified. The lower figures represent a pair of 2-cells in the two-point function.

states (5.1) as

$$\begin{aligned} \frac{1}{|S_n[S_2]|} \sum_{\alpha} \langle \mathcal{O}_\alpha \mathcal{O}_\alpha \rangle &= N_c^{2n} N_f^n \sum_{\alpha \in S_{2n}} \sum_{\sigma \in S_{2n}} \delta_{2n}(\alpha^{-1} \sigma \alpha \sigma^{-1}) \delta_{2n}(\sigma \Sigma_0 \sigma^{-1} \Sigma_0) \\ &= N_c^{2n} N_f^n \, I_{2n} \end{aligned} \tag{5.3}$$

Note that the free two-point functions (5.2) become diagonal in the large N_f, N_c limit owing to the symmetry (4.27).

The formula (5.2) can be recognised as the partition function of the S_{2n} TFT on a 2-complex $\mathcal{M}$ with boundaries and a defect, as we now describe. The first delta function is associated to a 2-torus with the disc removed. The boundary of the disc has permutation β_2, summed with $N_f^{\frac{C(\beta_2)}{2}}$ (this forms the Ω_{N_f} factor). One of the cycles of the 2-torus is constrained to be the permutation Σ_0. This constraint can be viewed as a defect. The second delta function is associated to a topological quotient of a cylinder with a disc removed, related to β_1, which is summed with weight $N_c^{C(\beta_2)}$ to give Ω_{N_c}. The permutations α_1, α_2 correspond to the two boundary circles of the cylinder, and a point from each of the two boundary circles is identified by the quotienting. The 2-torus and the quotiented cylinder are glued along a circle, to form the 2-complex which we call $\mathcal{M}$. This is illustrated in figure 3.

The 2-complex $\mathcal{M}$ cannot be the cell decomposition of a 2-manifold (which should locally be $\mathbb{R}^2$), because a 1-cell (the red-dashed circle denoted by σ) is incident on four 2-cells.

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Still, it should be possible to realise it as the 2-skeleton of a higher dimensional manifold. In that case higher dimensional TFT would be the natural setting for the interpretation of the 2-point function. TFT3 has arisen in the context of refined counting formulae for graphs in [31].

6 One-loop operator mixing

In this section we will compute the mixing matrix under the action of the one-loop dilatation operator [32]

$$H = -\frac{1}{2}tr[\Phi_m, \Phi_n][\check{\Phi}^m, \check{\Phi}^n] - \frac{1}{4}tr[\Phi_m, \check{\Phi}^n][\Phi_m, \check{\Phi}^n], \tag{6.1}$$

where $(\check{\Phi}^m)_{ij}(\Phi_n)_{kl} = \delta^m_n \delta_{jk}\delta_{il}$. On the representation basis, the mixing matrix is almost diagonal, where the non-zero components are explained by the repositioning of boxes.

On the permutation basis, the mixing matrix is given by

$$H\mathcal{O}_\sigma = \sum_\rho M_{\sigma,\rho}\mathcal{O}_\rho \tag{6.2}$$

$$\begin{aligned} M_{\sigma,\rho} = &-\sum_{\langle i,j\rangle}\sum_{\beta\in S^{\langle j\rangle}_{2n-1}} \delta_{2n}([\sigma,(ij)]X^{(j)}\beta^{-1})\delta_{2n}([\rho^{-1},(ij)]\beta) \\ &- N_f\sum_{(i,j)}\delta_{2n}(\rho^{-1}(ij)\sigma) - \sum_{\langle i,j\rangle}\delta_{2n}\left(\rho^{-1}(\Sigma_0(i)j)(ij)\sigma(\Sigma_0(i)j)\right) \\ &+ N_f\sum_{(i,j)}\sum_{\beta\in S^{\langle j\rangle}_{2n-1}} \delta_{2n}((ij)\sigma X^{(j)}\beta^{-1})\delta_{2n}(\rho^{-1}(ij)\beta) \\ &+ \sum_{\langle i,j\rangle}\sum_{\beta\in S^{\langle j\rangle}_{2n-1}} \delta_{2n}((ij)\sigma X^{(j)}\beta^{-1})\delta_{2n}(\rho^{-1}(\Sigma_0(i)j)(ij)\beta(\Sigma_0(i)j)) \end{aligned} \tag{6.3}$$

where $\Sigma_0 = (12)(34)\cdots(2n-1,2n)$, and we have defined

$$X^{(j)} = N_c + \sum_{k(\neq j)}(kj). \tag{6.4}$$

The sum $\sum_{(i,j)}$ is over $(1,2)$, $(3,4),\cdots$, while the sum $\sum_{\langle i,j\rangle}$ is over the other pairs. We do not distinguish (i,j) and (j,i). The first line of (6.3) comes from the first term of (6.1), and the remaining lines come from the second term. The derivation of this mixing matrix is presented in appendix F.

Given that we have expressed the mixing matrix purely in terms of permutations, we expect that there will be an interpretation in terms of permutation topological field theory. The construction of 2-complex will be analogous to our interpretation for the free field two-point function given in figure 3, but will involve some new features given the additional complexity apparent here. We will return to this problem in the near future.

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In the latter part of this section we study the mixing matrix on the representation basis. We denote the change of basis by

$$P_{R,\Lambda_1,\tau} = \sum_{\alpha} c_{R,\Lambda_1,\tau}(\alpha)[\alpha],$$
$$[\alpha] = \sum_{R,\Lambda_1,\tau} f_{R,\Lambda_1,\tau}(\alpha) P_{R,\Lambda_1,\tau}. \tag{6.5}$$

See (3.1) and (3.5) for the definition of $c_{R,\Lambda_1,\tau}$ and $f_{R,\Lambda_1,\tau}$. The mixing matrix on the representation basis is related to the mixing matrix on the permutation basis by

$$M_{R,\Lambda_1,\tau}^{R',\Lambda_1',\tau'} = \sum_{\sigma,\rho\in S_{2n}} c_{R,\Lambda_1,\tau}(\sigma) M_{\sigma,\rho} f_{R',\Lambda_1',\tau'}(\rho). \tag{6.6}$$

Let us take one term in the first line of (6.3), and simplify the mixing matrix on the representation basis,

$$\sum_{\sigma,\rho\in S_{2n}} c_{R,\Lambda_1,\tau}(\sigma) \left(\sum_{\beta\in S_{2n-1}} \delta_{2n}(\sigma U\beta)\delta_{2n}(\rho V\beta^{-1}) \right) f_{R',\Lambda_1',\tau'}(\rho)$$
$$U = (ij)X^{(j)}, \quad V = (ij). \tag{6.7}$$

Expand the Kronecker delta's using (A.13). We now use the following formula to remove the sum over β

$$\sum_{\beta\in S_{2n-1}} D_{ij}^{R}(\beta) D_{kl}^{R'}(\beta^{-1}) = (2n-1)! \sum_{r,m,n} \frac{1}{d_r} B_{i,m}^{R,r} B_{j,n}^{R,r} B_{k,n}^{R',r} B_{l,m}^{R',r}$$
$$= (2n-1)! \sum_{r,m,n} \frac{1}{d_r} I_{il}^{RR',r} I_{jk}^{RR',r} \tag{6.8}$$

Here we have introduced the branching coefficient

$$B_{r,m}^{R,i} = \langle R, i | R \to r, m \rangle, \tag{6.9}$$

where the r is a Young diagram with $2n-1$ boxes, and m runs over $1, \cdots, d_r$. The quantity $I_{il}^{RR',r}$ is the intertwiner map

$$I_{il}^{RR',r} = \sum_{m} B_{i,m}^{R,r} B_{l,m}^{R',r} \tag{6.10}$$

of [17, 19], here expressed in terms of branching coefficients. In the formula, the r.h.s. is non-zero when both $g([1], r; R)$ and $g([1], r; R')$ are non-zero.[10] In other words, the r.h.s. of (6.8) is non-zero only when the R is obtained from the S by moving a single box. Performing the sum over σ, ρ in (6.7),

$$\sum_{\sigma,\rho} c_{R,\Lambda_1,\tau}(\sigma) \left(\sum_{\beta\in S_{2n-1}} \frac{1}{((2n)!)^2} d_A d_B D_{ji}^{A}(\sigma U) D_{ij}^{A}(\beta) D_{lk}^{B}(\rho V) D_{kl}^{B}(\beta^{-1}) \right) f_{R',\Lambda_1',\tau'}(\rho)$$
$$= (2n)! B_{k'}^{\Lambda_1} S^{\tau,\Lambda_1}{}_{k'\ s\ j}^{R\ R} D_{si}^{R}(U) \left(\sum_{\beta} D_{ij}^{R}(\beta) D_{kl}^{R'}(\beta^{-1}) \right) D_{qk}^{R'}(V) d_{R'} B_{k''}^{\Lambda_1'} S^{\tau',\Lambda_1'}{}_{k'\ l\ q}^{R'\ R'} \tag{6.11}$$

[10] $g([1], r; R)$ is the Littlewood-Richardson coefficient (A.24).

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This mixing matrix is non-zero if R and R' become identical after the move of a single box. This kind of mixing, re-positioning of boxes, is common for representation bases, which has been studied concretely in [12, 17, 33–36]. The diagonalisation of one-loop mixing is still not trivial, which has been achieved in the SU(2) sector in special cases [18, 19]. This will be an interesting avenue for future investigation.

7 Conclusion and discussion

We summarise the results of this paper. We studied mesonic operators, that is the O(N_f)-singlet scalar operators in U(N_c) gauge theory, and computed free field two-point functions. The two-point functions are expressed them in terms of permutations. We performed a Fourier transform from the permutation basis to the representation basis, which made the two-point functions diagonal. Our mesonic operator provided a concrete realisation of the formula for diagonal operators [12]. We counted the number of operators in both bases by applying inverse Fourier transform. It was important to remove the redundancy of the wreath product group $S_n[S_2]$, noting that $(S_{2n}, S_n[S_2])$ is a Gelfand pair. We computed the one-loop mixing matrix in both bases.

Our expression for the 2-point function on the permutation basis was used to give an interpretation in terms of TFT, based on topological lattice gauge theory of permutations equipped with defects. This extends the connection between TFT and quiver gauge theories [30]. An interesting problem is to connect these results with axiomatic TFT as discussed in [37–40].

An important point is that the R-symmetry group of $\mathcal{N} = 4$ SYM is SO(6) rather than O(6). The SO(N_f)-singlet operators include

$$\epsilon^{a_1 \cdots a_{N_f}} \, tr_{N_f}(\sigma \, \Phi_{\vec{a}}), \quad \delta^{a_1 a_2} \epsilon^{a_3 \cdots a_{N_f+2}} \, tr_{N_f+2}(\sigma \, \Phi_{\vec{a}}), \quad \ldots \tag{7.1}$$

which we call baryonic singlets.[11] The mesonic operators are invariant under O(N_f), while the baryonic operators are invariant under SO(N_f) only. The mesonic and baryonic operators are orthogonal at $g^2_{YM} = 0$. Since the Lagrangian of $\mathcal{N} = 4$ SYM does not contain the ϵ tensor of SO(6), the two-point functions remain orthogonal to all orders of perturbation theory. A systematic generalisation of our study of correlation functions to baryonic operators is left for the future.

Our explicit construction of operators and free-field 2-point functions has been given for U(N_c) theories. Generalisations to other gauge groups, e.g. SU(N_c) along the lines of [7, 41, 42], and SO(N_c) or Sp(N_c) following [21, 22] will be interesting.

Our results provide a foundation for the systematic studies of planar zero modes (PZM) in the SO(6) singlet sector. Group-theoretical counting methods for the PZM's can be developed, analogously to the construction of general mesonic singlets here. Often the counting formula implies the existence of a basis of PZM's labelled by permutations or representations. An interesting application of such a formalism is to compute the non-planar anomalous dimension of the PZM's [20]. The PZM's are expected to have negative

[11] The term "baryon" refers to the flavour group, and not the colour group.

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anomalous dimensions by $1/N_c$ corrections, according to the results at strong coupling [43–47] and those of conformal bootstrap [48–50]. Following this paper, the general N_f setup can provide a tractable approach to finite $N_f = 6$.

Another application of our results is to determine physical quantities of $\mathcal{N} = 4$ SYM at this level of generality, such as non-planar correlation functions [51, 52], partition functions [53–56], and statistical properties of one-point functions via matrix product states [57].

The sector of SO(6) singlets offers a rich and interesting setting to explore non-planar effects in a non-supersymmetric sector of $\mathcal{N} = 4$ SYM.

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A Notation and formulae

A.1 Notation

We denote by $\Phi = (\Phi_a)^i_j$ a hermitian scalar field, $a = 1, 2, \ldots, N_f$ and $i, j = 1, 2, \ldots N_c$. The flavour group is $\mathrm{SO}(N_f)$ and the colour group is $\mathrm{U}(N_c)$. Φ belongs to the fundamental representation V_F of $\mathrm{SO}(N_f)$, and to the adjoint representation $V_C \otimes \overline{V}_C$ of $\mathrm{U}(N_c)$. The case of $N_f = 6$ describes the six scalars of $\mathcal{N} = 4$ SYM in four dimensions.

The $\mathrm{U}(N_c)$ Wick-contraction rule is

$$\overbrace{(\Phi_a)^j_i\,(\Phi_b)^l_k} = \delta_{ab}\,\delta^l_i\,\delta^j_k\,. \tag{A.1}$$

We introduce a gauge-covariant operator Φ_a, which is related to the component fields by

$$\langle j_1 j_2 \ldots j_{2n} \mid \Phi_{a_1} \otimes \Phi_{a_2} \otimes \cdots \otimes \Phi_{a_{2n}} \mid i_1 i_2 \ldots i_{2n} \rangle = (\Phi_{a_1})^{j_1}_{i_1} (\Phi_{a_2})^{j_2}_{i_2} \cdots (\Phi_{a_{2n}})^{j_{2n}}_{i_{2n}} \tag{A.2}$$

Permutations act on the bases as

$$\langle \vec{j}\,|\alpha \equiv \langle j_1 j_2 \ldots j_{2n}|\alpha = \langle j_{\alpha^{-1}(1)} j_{\alpha^{-1}(2)} \cdots j_{\alpha^{-1}(2n)}| \tag{A.3}$$

$$\alpha|\vec{i}\,\rangle \equiv \alpha|i_1 i_2 \ldots i_{2n}\rangle = |i_{\alpha(1)} i_{\alpha(2)} \ldots i_{\alpha(2n)}\rangle \tag{A.4}$$

Thus, permutations act on SYM fields by

$$\langle \vec{j} \mid \alpha\,(\Phi_{a_1} \otimes \Phi_{a_2} \otimes \cdots \otimes \Phi_{a_{2n}})\,\beta \mid \vec{i}\,\rangle = (\Phi_{a_1})^{j_{\alpha^{-1}(1)}}_{i_{\beta(1)}} (\Phi_{a_2})^{j_{\alpha^{-1}(2)}}_{i_{\beta(2)}} \cdots (\Phi_{a_{2n}})^{j_{\alpha^{-1}(2n)}}_{i_{\beta(2n)}}\,, \tag{A.5}$$

and thus

$$\rho(\Phi_{a_1} \otimes \Phi_{a_2} \otimes \cdots \otimes \Phi_{a_{2n}})\rho^{-1} = \Phi_{a_{\rho(1)}} \otimes \Phi_{a_{\rho(2)}} \otimes \cdots \otimes \Phi_{a_{\rho(2n)}}\,. \tag{A.6}$$

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The order of a finite group G is denoted by $|G|$. The words "representations of S_L", "Young diagrams", and "partitions of L" are used interchangeably. A partition of L is expressed in two ways. The first expression is

$$Q = [\mathtt{q}_1, \mathtt{q}_2, \ldots, \mathtt{q}_\ell] \vdash L, \qquad \mathtt{q}_1 \geq \mathtt{q}_2 \geq \cdots \geq \mathtt{q}_\ell\,, \qquad \sum_{i=1}^{\ell} \mathtt{q}_i = L. \tag{A.7}$$

If we collect the same $\mathtt{q}$'s together, we obtain the second expression (2.5). The symbol $c_1(Y)$ is the length of the first column of the Young diagram Y. In (A.7),

$$c_1(Q) = \ell. \tag{A.8}$$

Clearly $c_1(Q) \geq c_2(Q) \geq \cdots \geq c_{\mathtt{q}_1}(Q)$.

A.2 Formulae

In this subsection the definition of group theory quantities, and group theory formulas are collected.

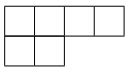
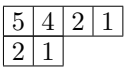
The matrix elements (i, j) of the group element $\sigma \in S_L$ in the representation R are denoted by $D^R_{ij}(\sigma)$. We assume all representations of S_L are real and unitary, and thus $D^R_{ij}(\sigma) = D^R_{ji}(\sigma^{-1})$. They satisfy the so-called grand orthogonality relation

$$\sum_{\sigma \in S_L} D^R_{ij}(\sigma) D^S_{kl}(\sigma^{-1}) = \frac{L!}{d_R} \delta_{il}\delta_{jk}\delta^{RS}, \tag{A.9}$$

where d_R is the dimension of R of symmetric group S_L,

$$d_R = \frac{L!}{\prod_{i,j} h(i,j)} \tag{A.10}$$

The product is over the boxes of the Young diagram R with i, j labelling the rows and columns. The quantity $h(i, j)$ is the hook-length associated with the box at (i, j), namely the number of boxes intersecting the hook which extends from (i, j) toward the right and bottom. For example,


$\Rightarrow$

$$, \quad h(1,1) = 5,\ h(1,2) = 4,\ h(1,3) = 2,\ h(1,4) = 1,\ \ldots \tag{A.11}$$

The character of σ in the representation R is denoted by $\chi^R(\sigma)$. The characters satisfy the orthogonality

$$\sum_{R \vdash L} \chi^R(\sigma)\chi^R(\rho) = \sum_{\gamma \in S_L} \delta_L(\gamma\sigma\gamma^{-1}\rho) \tag{A.12}$$

where the delta function is defined by

$$\delta_L(\sigma) = \frac{1}{L!} \sum_{R \vdash L} d_R\, \chi^R(\sigma) = \begin{cases} 1 \ (\sigma = 1) \\ 0 \ (\sigma \neq 1) \end{cases} \tag{A.13}$$

Consider the tensor space $V^{\otimes L}$, where V is the fundamental representation of the unitary group $\mathrm{U}(N)$. The symmetric group acts on the tensor space by permuting the L

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factors. From the fact that these two actions commute each other, the tensor space can be decomposed as

$$V^{\otimes L} = \bigoplus_R (V_R^{\mathrm{U}(N)} \otimes V_R^{S_L}) \tag{A.14}$$

where the sum is over the Young diagrams with at most N rows, which is expressed in terms of the length of the first column $c_1(R)$ by $c_1(R) \leq N$. This equation is the Schur-Weyl duality between $\mathrm{U}(N)$ and S_L.

A trace over this tensor space is denoted by tr_L,

$$tr_L(\sigma) = N^{C(\sigma)} = N^L \delta_L(\Omega_L \sigma) \tag{A.15}$$

where we have defined the quantity

$$\Omega_L = \sum_{\sigma \in S_L} \sigma N^{C(\sigma)-L}. \tag{A.16}$$

According to the Schur-Weyl duality, a trace of an element $\sigma \in S_L$ can be written as

$$tr_L(\sigma) = \sum_{R \vdash L, c_1(R) \leq N} Dim(R)\, \chi^R(\sigma) \tag{A.17}$$

where $Dim(R)$ is the dimension of R of Lie group $\mathrm{U}(N)$

$$Dim(R) = \prod_{i,j} \frac{N-j+i}{h(i,j)} \tag{A.18}$$

Setting $\sigma = 1$ in (A.17) gives the identity

$$N^L = \sum_{R \vdash L, c_1(R) \leq N} Dim(R) d_R \tag{A.19}$$

Let $R_1\,, R_2$ be the irreducible representations of S_L . $S^{\tau, \Lambda_1 R_1 R_2}_{\ k\ i_1\ i_2}$ is the Clebsch-Gordan (CG) coefficients, defined by the irreducible decomposition of the tensor product $R_1 \otimes R_2 = \oplus S$, with the multiplicity label τ,

$$|\Lambda, \tau, k\rangle = \sum_{i_1, i_2 = 1}^{d_R} S^{\tau, \Lambda_1 R_1 R_2}_{\ k\ i_1\ i_2}\ |R_1, i_1\rangle \otimes |R_2, i_2\rangle\,. \tag{A.20}$$

The indices (k, i_1, i_2) specifies the elements of (R, R_1, R_2). The multiplicity label τ runs over $1, \cdots, C(R, R, \Lambda)$, where $C(R, S, T)$ is called the CG number (also known as CG multiplicity or Kronecker coefficient)

$$C(R, S, T) = \frac{1}{|S_L|} \sum_{\sigma \in S_L} \chi^R(\sigma) \chi^S(\sigma) \chi^T(\sigma). \tag{A.21}$$

The CG coefficients satisfy the following properties

$$\sum_{\sigma \in S_L} D^{\Lambda_1}_{ba}(\sigma) D^R_{jk}(\sigma) D^R_{li}(\sigma^{-1}) = \frac{L!}{d_{\Lambda_1}} \sum_\tau S^{\tau, \Lambda_1 R R}_{\ b\ i\ j}\, S^{\tau, \Lambda_1 R R}_{\ a\ l\ k} \tag{A.22}$$

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and

$$\sum_{ij} S^{\tau,\Lambda_1}{}_{a\ i}^{\ R\ R'}{}_{\ j}\, S^{\tau',\Lambda_1'}{}_{b\ i}^{\ R\ R'}{}_{\ j} = \delta^{\Lambda_1\Lambda_1'}\,\delta^{\tau\tau'}\delta_{ab}\,, \qquad \sum_{\tau,\Lambda_1,a} S^{\tau,\Lambda_1}{}_{a\ i}^{\ R\ R'}{}_{\ j}\, S^{\tau,\Lambda_1}{}_{a\ k}^{\ R\ R'}{}_{\ l} = \delta_{ik}\,\delta_{jl} \tag{A.23}$$

Let R_1, R_2, R be the irreducible representations of S_m, S_n, S_{m+n}, respectively. The Littlewood-Richardson coefficient $g(R_1, R_2; R)$ counts the number of $R_1 \otimes R_2$ appearing in the decomposition of R under $S_m \times S_n$,

$$g(R_1, R_2; R) = \frac{1}{m!\, n!} \sum_{\sigma_1 \in S_m, \sigma_2 \in S_n} \chi_{R_1}(\sigma_1^{-1})\chi_{R_2}(\sigma_2^{-1})\chi_R(\sigma_1 \circ \sigma_2). \tag{A.24}$$

A.3 Gelfand pair and coset type

A pair of finite groups (G, H) with $G \supset H$ is called Gelfand pair if they satisfy either of the following conditions:

(i) Any irreducible representation of G contains at most one singlet representation of H.

(ii) Consider a set of functions on the double coset $H\backslash G/H$,

$$\mathcal{C}(G,H) := \left\{ w : G \to G \,\middle|\, w(g) = w(\gamma_1 g \gamma_2), \quad (g \in G,\ \gamma_1, \gamma_2 \in H) \right\} \tag{A.25}$$

where the multiplication is defined by convolution. The algebra $\mathcal{C}(G, H)$ is commutative.

The two conditions are equivalent [26]. The Gelfand pair for Lie groups is defined similarly.

An important example of the Gelfand pair is $(S_{2n}, S_n[S_2])$. From (i), an irreducible representation R of S_{2n} satisfies

$$M^R_{1_{S_n[S_2]}} := \frac{1}{|S_n[S_2]|} \sum_{\gamma \in S_n[S_2]} \chi^R(\gamma) = \begin{cases} 1 & (R \text{ is even}) \\ 0 & (R \text{ is odd}) \end{cases}. \tag{A.26}$$

An even representation R corresponds to even Young diagram

$$R = [2\mathtt{r}_1, 2\mathtt{r}_2, \cdots] \vdash 2n, \qquad \mathtt{r_i} \in \mathbb{Z}_{>0} \tag{A.27}$$

in (A.7).

The double coset function in the condition (ii) corresponds to $W(\sigma)$ defined in (2.21), which satisfies the equivalence relation,

$$W(\sigma) = W(\gamma_1 \sigma \gamma_2), \quad (\sigma \in S_{2n}, \quad \gamma_1, \gamma_2 \in S_n[S_2]). \tag{A.28}$$

The function $W(\sigma)$ defines the unique *coset type* for each permutation σ,

$$\tilde{p} = [2^{\tilde{p}_1}, 4^{\tilde{p}_2}, \ldots, (2n)^{\tilde{p}_n}], \qquad \sum_{i=1}^{n} i\,\tilde{p}_i = n\,. \tag{A.29}$$

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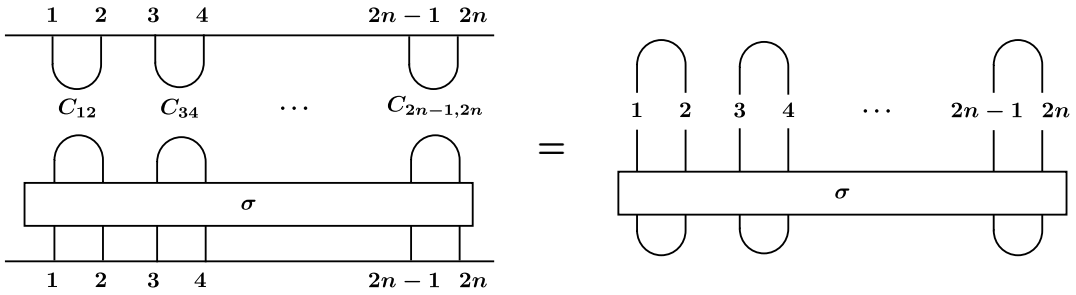

Figure 4. This is identical to figure 1. The left figure shows $C_{12}C_{34}\cdots C_{2n-1,2n}\sigma$ acting on $V_F^{\otimes 2n}$. The upper and lower horizontal lines are identified when taking the trace, as in the right figure.

where $\tilde{p}_l$ in (A.29) is the number of length-$2l$ loops in $W(\sigma)$ shown in figure 4.[12] The number of all loops is equal to the power of N_f in (2.27),

$$z(\sigma) = \sum_l \tilde{p}_l. \tag{A.30}$$

Two permutations $\sigma_1, \sigma_2 \in S_{2n}$ have the same coset type if and only if they are related by $\sigma_1 = \gamma_1\sigma_2\gamma_2$ for $\gamma_1, \gamma_2 \in S_n[S_2]$.

B Powers of N_f

We denote the number of loops in the graph of $W(\sigma)$ by $z(\sigma)$. We will show the identity

$$z(\sigma) = \frac{1}{2}C(\Sigma_0\sigma^{-1}\Sigma_0\sigma), \tag{B.1}$$

where $\Sigma_0 = (12)(34)\ldots(2n-1, 2n)$ and $C(\sigma)$ counts the number of cycles in $\sigma \in S_{2n}$.

The structure of $W(\sigma)$ is depicted in figure 4. Consider the labelled points $\{1, 2, \cdots, 2n\}$ in that diagram. If these points move along the lines toward the upper arcs, they will return to the labelled points $\{1, 2, \cdots, 2n\}$ after a permutation

$$\Sigma_0 = (12)(34)\cdots(2n-1\ 2n) \tag{B.2}$$

If the points went down through σ, the lower arcs, and back up σ again, they will undergo the permutation

$$\Sigma_1 \equiv \sigma^{-1}\Sigma_0\sigma = (\sigma(1)\sigma(2))(\sigma(3)\sigma(4))\cdots(\sigma(2n-1)\sigma(2n)). \tag{B.3}$$

The graph of $W(\sigma)$ contains flavour loops. A flavour loop is a sequence of transitions of the form

$$\Sigma_0\Sigma_1\,, \quad \Sigma_0\Sigma_1\Sigma_0\Sigma_1\,, \quad \Sigma_0\Sigma_1\Sigma_0\Sigma_1\Sigma_0\Sigma_1\,, \quad \cdots \tag{B.4}$$

or their inverses. Note that $\Sigma_0^2 = \Sigma_1^2 = 1$. We cannot return to the original point by after an odd number of Σ's, because both Σ_0 and Σ_1 have cycle type $[2^n]$. In other words, Σ_0

[12] A loop $W(1)$ at $n = 2$ is defined to have length 2.

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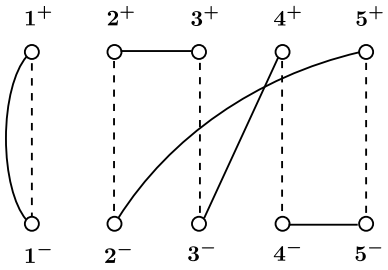

Figure 5. Another graph of $W(\sigma)$. The dashed edges represent the elements of Σ_0 , and the sold ones those of Σ_1 . This graph contains two loops, $(N\Sigma_0)$ and $(H\Sigma_0 N\Sigma_0 H\Sigma_0 N\Sigma_0)$.

and Σ_1 acting on i behaves as a permutation of odd signature for any i. The number of flavour loops, i.e. the power of N_f denoted by $z(\sigma)$, is same as the number of orbits in the subgroup of S_{2n} generated by $\langle\Sigma_0, \Sigma_1\rangle$. This is also the number of connected components in the ribbon graph determined by the permutations Σ_0, Σ_1.

Let us introduce the notation

$$2i - 1 = i^-, \qquad 2i = i^+, \qquad (i = 1, 2, \ldots, n). \tag{B.5}$$

Recall that $S_n[S_2]$ is the stabiliser of Σ_0 ,

$$\xi^{-1}\Sigma_0\,\xi = \Sigma_0\,, \qquad \forall\xi \in S_n[S_2]. \tag{B.6}$$

By using this "gauge degree of freedom" of $S_n[S_2]$, we may transform Σ_1 to $\Sigma_1' = \xi^{-1}\Sigma_1\,\xi$ without changing $C(\Sigma_0\Sigma_1)$. There exists a useful gauge:

Lemma 1. *By a gauge transformation in $S_n[S_2]$, we can transform Σ_1 to the form*

$$\Sigma_1' = (1^-\,\tau(1)^+)\ldots(n^-\,\tau(n)^+), \qquad \tau \in S_n\,. \tag{B.7}$$

Proof of 1. We draw another graph of $W(\sigma)$ emphasising the structure of loops, with $1^+, 2^+, \ldots$ along the upper line and $1^-, 2^-, \ldots$ along the lower line. We connect the points $i^\pm$ and $j^\pm$ when $(i^\pm, j^\pm)$ belong to Σ_0 or Σ_1 as shown in figure 5. The horizontal edges of Σ_1 , namely those connecting $(+,+)$ or $(-,-)$, will be called H-edges. The other edges of Σ_1 , $(+,-)$ or $(-,+)$, will be called N-edges.

Let us prove that every loop has to have an even number of H-edges. As discussed in (B.4), every loop consists of even number of edges. Within the loop, only Σ_0 and N-edges change the parity $\pm$. If we circle around the loop, then there should be no change in parity. Thus, every loop $(\Sigma_1\Sigma_0\ldots\Sigma_1\Sigma_0)$ satisfies

$$\text{Parity}\,(\Sigma_1\Sigma_0\ldots\Sigma_1\Sigma_0) = (-1)^{\#(N)}(-1)^{\#(\Sigma_0)} = +1. \tag{B.8}$$

Since $\#(H) + \#(N) + \#(\Sigma_0)$ is even for each loop, the number of H-edges is also even.

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The statement (B.7) is equivalent to saying that we can remove all H-edges by gauge transformations $S_n[S_2]$. Consider how the flip $(i^- i^+) \in S_n[S_2]$ acts on the Σ_1 edges connected to the points $i^\pm$. Inspecting figure 5, we find

$$(i^- i^+) : \begin{cases} H(\Sigma_0)_{i^\pm} H & \to \; N(\Sigma_0)_{i^\pm} N \\ H(\Sigma_0)_{i^\pm} N & \to \; N(\Sigma_0)_{i^\pm} H \\ N(\Sigma_0)_{i^\pm} H & \to \; H(\Sigma_0)_{i^\pm} N \end{cases} . \tag{B.9}$$

The flip cannot change $H\Sigma_0 N$ into $N\Sigma_0 N$ or $H\Sigma_0 H$, because it violates the parity rule (B.8). The same is true for $N\Sigma_0 H$. Now, if we have a flavour loop $(H\Sigma_0 N \cdots \Sigma_0 H \dots)$, flipping all the edges in Σ_0 between two H-edges will convert the loop to $(N\Sigma_0 N \cdots \Sigma_0 N \dots)$. Since the number of H-edges is even, applying this process repeatedly will remove all H-edges. This means that there is a gauge transformation which converts Σ_1 to Σ_1' of the form (B.7). □

The permutation τ is itself defined up to conjugation in S_n. This is in fact a way to understand the correspondence between partitions of n and the double coset space $S_n[S_2] \setminus S_{2n}/S_n[S_2]$, as we will explain subsequently.

Let us denote by $(\ell_1, \ell_2, \dots)$ the number of edges in the loops of the graph $W(\sigma)$. These $\{\ell_i\}$ are all even, and satisfy $\sum_i \ell_i = 2n$. Thus, $\lambda_i \equiv \ell_i/2$ defines a partition of n. This partition λ is same as the cycle decomposition of τ. We are going to relate the number of loops with the number of cycles in $\Sigma_{\rm tot} \equiv \Sigma_0 \Sigma_1'$.

Corollary 2. $\Sigma_{\rm tot}$ *maps minus variables to minus variables, plus to plus.*

Proof of 2. From (B.7) we find $\Sigma_{\rm tot}(i^-) = \Sigma_0(\tau(i)^+) = \tau(i)^-$ and $\Sigma_{\rm tot}(i^+) = \Sigma_0(\tau^{-1}(i)^-) = \tau^{-1}(i)^+$. □

Thus, $\Sigma_{\rm tot}$ splits into two disjoint actions $\Sigma_{\rm tot}^- \times \Sigma_{\rm tot}^+$, where $\Sigma_{\rm tot}^\pm$ acts on the set $V_n^\pm = (1^\pm, \dots, n^\pm)$. Then, the number of cycles is equal to

$$C(\Sigma_0 \Sigma_1) = C(\Sigma_{\rm tot}) = C(\tau) + C(\tau^{-1}) = 2C(\tau). \tag{B.10}$$

Lemma 3. *The number of loops in* $\Sigma_{\rm tot}^-$ *acting on* V_n^- *is equal to* $C(\tau)$*, and similarly for* $\Sigma_{\rm tot}^+$ *acting on* V_n^+ *.*

Proof of 3. Let us define

$$\Sigma_{\rm tot} = \Sigma_{\rm tot}^- \, \Sigma_{\rm tot}^+, \qquad \Sigma_{\rm tot}^- \equiv \prod_{i=1}^n (i^- \tau(i)^-), \qquad \Sigma_{\rm tot}^+ \equiv \prod_{i=1}^n (i^+ \tau^{-1}(i)^+). \tag{B.11}$$

We can express the number of loops in $\Sigma_{\rm tot}^\pm$ as

$$N_f^{\#(\text{loops})} = \prod_{h=1}^n \delta^{c_{h^-}}_{c_{\tau(h)^-}} = \prod_{h=1}^n \delta^{c_{h^+}}_{c_{\tau(h)^+}} = N_f^{C(\tau)}, \tag{B.12}$$

showing that $\#(\text{loops}) = C(\tau)$. It can also be derived graphically as in figure 6. □

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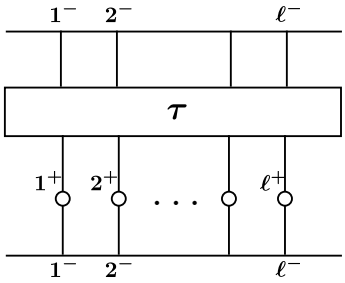

Figure 6. $\Sigma^-_{\rm tot}$ acting on $(1^-\dots n^-)$. Σ_0 interchanges (i^-i^+) and Σ'_1 permutes by τ. We can draw a similar graph for $(1^+\dots n^+)$.

The identity (B.1) follows from (B.12) and (B.10),

$$W(\sigma) = N_f^{z(\sigma)} = N_f^{\#(\text{loops})} = N_f^{C(\tau)}, \qquad z(\sigma) = \frac{1}{2}C(\Sigma_0\sigma^{-1}\Sigma_0\sigma). \tag{B.13}$$

The above discussion gives a concrete insight into the coset type (A.29); a coset type is a partition of n which parametrises the elements of double coset $S_n[S_2]\backslash S_{2n}/S_n[S_2]$. Lemma 1 says that $\Sigma_1 = \sigma^{-1}\Sigma_0\sigma$ can be gauge transformed by $\xi \in S_n[S_2]$ to the form (B.7). We take $\tilde\tau$ to be a permutation in S_{2n} which leaves i^- fixed and acts nontrivially on i^+ $(i = 1, 2, \dots n)$. The gauge transformation of ξ can be written as

$$\begin{aligned}\xi^{-1}\sigma^{-1}\Sigma_0\sigma\xi &= \tilde\tau^{-1}\Sigma_0\tilde\tau \\ &= (\tilde\tau(1^-)\tilde\tau(1^+))(\tilde\tau(2^-)\tilde\tau(2^+))\cdots(\tilde\tau(n^-)\tilde\tau(n^+)) \\ &\equiv (1^-\tau(1)^+)(2^-\tau(2)^+)\cdots(n^-\tau(n)^+).\end{aligned} \tag{B.14}$$

So $\tilde\tau\xi^{-1}\sigma^{-1}$ is in the stabiliser of Σ_0, and $\tilde\tau\xi^{-1}\sigma^{-1} = \eta \in S_n[S_2]$ for some η. Hence any σ can be written as

$$\tilde\tau = \eta\sigma\xi, \qquad \eta, \xi \in S_n[S_2] \tag{B.15}$$

Therefore, the elements of the double coset $S_n[S_2]\backslash S_{2n}/S_n[S_2]$ correspond to the permutations $\tilde\tau \in S_{2n}$, or the permutations $\tau \in S_n$.

The condition (B.14) does not completely fix the gauge. The residual gauge freedom is conjugation of τ by $\xi \in S_n \subset S_n[S_2]$, which should not change the double coset element. As a result, the double coset elements are in 1-1 correspondence with the conjugacy classes in S_n, i.e. partitions of n called coset types.

C Diagonal two-point functions

In this appendix, we will derive (3.9) and compute the normalisation factor.

C.1 Proof of diagonality

Let us first rewrite the two-point functions of the permutation basis

$$\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle = \sum_{\sigma\in S_{2n}} W(\sigma)\, tr_{2n}(\alpha_1\sigma\alpha_2\sigma^{-1}). \tag{C.1}$$

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The colour factor can be expanded using (A.17) as

$$tr_{2n}(\alpha_1\sigma\alpha_2\sigma^{-1}) = \sum_R Dim(R)\,\chi^R(\alpha_1\sigma\alpha_2\sigma^{-1}) \\ = \sum_R Dim(R) D^R_{ij}(\alpha_1) D^R_{jk}(\sigma) D^R_{kl}(\alpha_2) D^R_{li}(\sigma^{-1}), \tag{C.2}$$

where $Dim(R)$ is the dimension of R associated with $\mathrm{U}(N_c)$. The flavour factor can be written from (2.33) as

$$W(\sigma) = \frac{N_f^n}{(2n)!} \sum_{\Lambda_1} d_{\Lambda_1} D^{\Lambda_1}_{ab}(\Omega^{(f)}_{2n}) D^{\Lambda_1}_{ba}(\sigma). \tag{C.3}$$

The sum over σ in (C.1) can be removed using (A.22) to obtain

$$\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle = \sum_{R,\Lambda_1} \sum_{i,j,k,l,a,b} Dim(R) D^R_{ij}(\alpha_1) D^R_{kl}(\alpha_2) N_f^n D^{\Lambda_1}_{ab}(\Omega^{(f)}_{2n}) \sum_\tau S^{\tau,\Lambda_1 RR}_{\ b\ i\ j} S^{\tau,\Lambda_1 RR}_{\ a\ l\ k} \tag{C.4}$$

We now compute the two-point functions of the representation basis

$$\langle \mathcal{O}^{R,\Lambda_1,\tau} O^{S,\Lambda_1',\tau'}\rangle = B^{\Lambda_1}_k B^{\Lambda_1'}_{k'} S^{\tau,\Lambda_1 RR}_{\ k\ i\ j} S^{\tau',\Lambda_1' R'R'}_{\ k'\ i'\ j'} \sum_{\alpha_1,\alpha_2\in S_{2n}} D^R_{ij}(\alpha_1^{-1}) D^{R'}_{i'j'}(\alpha_2^{-1}) \langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle. \tag{C.5}$$

Substituting (C.4) into this and using the relation (A.23),

$$\langle \mathcal{O}^{R,\Lambda_1,\tau} O^{S,\Lambda_1',\tau'}\rangle = \delta_{RS}\delta_{\tau\tau'}\delta_{\Lambda_1\Lambda_1'} \left(\frac{(2n)!}{d_R}\right)^2 Dim(R) N_f^n \langle \Lambda_1 \to 1_{S_n[S_2]} \mid \Omega^{(f)}_{2n} \mid \Lambda_1 \to 1_{S_n[S_2]}\rangle. \tag{C.6}$$

The last factor has several expressions

$$\langle \Lambda_1 \to 1_{S_n[S_2]} \mid \Omega^{(f)}_{2n} \mid \Lambda_1 \to 1_{S_n[S_2]}\rangle = B^{\Lambda_1}_{k_1} B^{\Lambda_1}_{k_2} D^{\Lambda_1}_{k_1k_2}(\Omega^{(f)}_{2n}) \\ = \chi_{\Lambda_1}(\Omega^{(f)}_{2n}\, p_{1_{S_n[S_2]}}) \\ = \omega_{\Lambda_1/2}(\Omega^{(f)}_{2n}). \tag{C.7}$$

The second equality comes from $\gamma_1\Omega^{(f)}_{2n}\gamma_2 = \Omega^{(f)}_{2n}$ for $\gamma_1,\gamma_2 \in S_n[S_2]$. It is non-zero only for the case Λ_1, Λ_1' are even Young diagrams.

C.2 Twisting Wick-contraction rules

We derive the formula (3.12) by developing the connection with [26]. The function $\omega_{\Lambda_1/2}(\Omega^{(f)}_{2n})$ which appears in the normalisation of the diagonal two-point functions (C.7) is equal to the zonal spherical function of the Gelfand pair $(\mathrm{GL}(N_f), \mathrm{O}(N_f))$, introduced in [26].[13] In developing this connection, it is instructive to introduce a twist of the two-point functions parametrised by matrices $T_{ab} \in \mathrm{GL}(N_f)$ and $Y^i_j \in \mathrm{GL}(N_c)$.

Since $P^{R,\Lambda_1,\tau}$ does not depend explicitly on N_c or N_f, our construction of diagonal operators can be readily generalised to the case where the Wick-contraction rules are twisted,

$$\overbrace{(\Phi_a)^j_i\,(\Phi_b)^l_k} = T_{ab}\, Y^l_i\, Y^j_k \tag{C.8}$$

[13] The colour factor $Dim(R)$ in (A.17) is replaced by the Schur polynomial of the eigenvalues of Y^2 [26].

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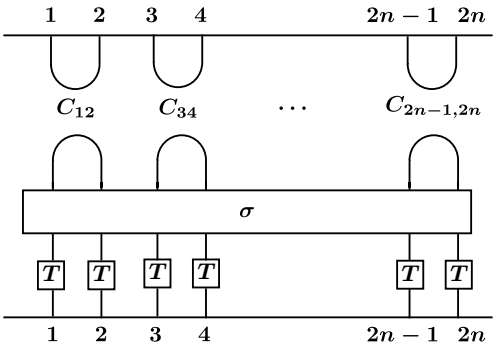

Figure 7. Generalisation of $W(\sigma)$ in figure 1 including the twist T.

The two-point functions (2.19) become

$$\begin{aligned}\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle &= \delta_{\vec{a}}\,\delta^{\vec{b}}\sum_{\sigma\in\mathcal{S}_{2n}}\prod_{k=1}^{2n} T^{a_k}_{b_{\sigma(k)}}\,Y^{i_{\sigma(k)}}_{j_{\alpha_2(k)}}\,Y^{j_k}_{i_{\alpha_1\sigma(k)}} = \sum_{\sigma\in\mathcal{S}_{2n}}\left(\delta_{\vec{a}}\,\delta^{\vec{b}}\prod_{k=1}^{2n} T^{a_k}_{b_{\sigma(k)}}\right)\left(\prod_{k=1}^{2n}(Y^2)^{i_k}_{i_{\alpha_1\sigma\alpha_2\sigma^{-1}(k)}}\right)\\ &\equiv \sum_{\sigma\in\mathcal{S}_{2n}} W_T(\sigma)P_Y(\alpha_1\sigma\alpha_2\sigma^{-1}),\end{aligned} \tag{C.9}$$

where W_T collects the flavour factors and P_Y the colour factor. They can be written as the power sum,

$$P_Y(\rho)=\prod_{\ell=1}^{2n} tr(Y^{2\ell})^{p_\ell},\qquad W_T(\sigma)=\prod_{k=1}^{n}\pi_k(T^TT)^{\tilde{p}_k},\qquad \pi_k(X)\equiv tr(X^k) \tag{C.10}$$

where $p\vdash 2n$ is the cycle type of $\rho=\alpha_1\sigma\alpha_2\sigma^{-1}$, and $\tilde{p}\vdash n$ is the coset type of σ. When we untwist as $T=Y=1$, both factors reduce to

$$P_Y(\alpha_1\sigma\alpha_2\sigma^{-1})\big|_{Y=1}=\prod_{i=1}^{2n}N_c^{p_i}=N_c^{C(\alpha_1\sigma\alpha_2\sigma^{-1})} \tag{C.11}$$

$$W_T(\sigma)\big|_{T=1}=\prod_{k=1}^{n}N_f^{\tilde{p}_k}=N_f^{z(\sigma)}\,. \tag{C.12}$$

A graphical representation of $W_T(\sigma)$ is shown in figure 7.

Owing to (C.10), the operator $\Omega^{(f)}_{2n}$ in (2.30) is twisted as

$$\Omega^{(f_T)}_{2n}=\frac{1}{N_f^n}\sum_{\sigma\in S_{2n}}\left(\prod_{k=1}^{n}\pi_k(T^TT)^{\tilde{p}_k}\right)\sigma^{-1}. \tag{C.13}$$

The flavour factor in the diagonal two-point functions in (3.9) becomes

$$\begin{aligned}\omega_{\Lambda_1/2}(\Omega^{(f_T)}_{2n}) &= \chi_{\Lambda_1}\left(\Omega^{(f_T)}_{2n}p_{1_{S_n[S_2]}}\right)=\frac{1}{N_f^n}\sum_{\sigma\in S_{2n}}\left(\prod_{k=1}^{n}\pi_k(T^TT)^{\tilde{p}_k}\right)\chi_{\Lambda_1}\left(\sigma p_{1_{S_n[S_2]}}\right),\\ &= |S_n[S_2]|\,Z_\Lambda(T^TT)\end{aligned} \tag{C.14}$$

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where we used $\chi_{\Lambda_1}\left(\sigma^{-1}\right) = \chi_{\Lambda_1}(\sigma)$. It turns out that the function $\omega_{\Lambda_1/2}(\Omega_{2n}^{(f_T)})$ is identical, up to normalisation, to the zonal spherical function $Z_\Lambda(T^T T)$ of the Gelfand pair $(\mathrm{GL}(N_f), \mathrm{O}(N_f))$ introduced in [26]. In [26] it is shown that

$$Z_\Lambda(1) = \prod_{(i,j)\in\Lambda} (N_f + 2j - i - 1), \tag{C.15}$$

where (i, j) specifies the position of the Young tableau Λ. From (C.15), we reproduce the formula (3.12).

The overall normalisation of $\omega_{\Lambda_1/2}(\Omega_{2n}^{(f_T)})$ is determined as follows. If we take $T = 1$ and $N_f \gg 1$ in (C.14), the leading terms come from $\sigma \in S_n[S_2]$, whose coset type is $[2^n]$. Since $(S_{2n}, S_n[S_2])$ is a Gelfand pair, the restriction to $S_n[S_2]$-invariant subspace is multiplicity-free; $\chi_{\Lambda_1}(p_{1_{S_n[S_2]}}) = 1$.

D Examples of diagonal operators

We will explain how to construct the diagonal operators (3.1) in $\mathrm{U}(N_c)$ theories

$$\mathcal{O}^{R,\Lambda_1,\tau} = \sum_{k=1}^{d_{\Lambda_1}} B_k^{\Lambda_1} \sum_{\alpha\in S_{2n}} \sum_{i,j=1}^{d_R} S^{\tau,\Lambda_1 R R}_{\ \ k\ i\ j} D^R_{ij}(\alpha^{-1})\, \mathcal{O}_\alpha\,, \tag{D.1}$$

and give explicit examples at $2n = 2, 4$.

D.1 Generality

Let us explain our strategy. First, we classify all irreducible representations R such that $R \otimes R$ contains an even partition Λ_1. The irreducible decomposition of $R \otimes R$ can be computed from the character table by using (A.21).

Second, we specify an orthonormal basis of the irreducible representations of S_{2n} explicitly. We use the Young-Yamanouchi orthonormal form for this purpose [58, 59]. The Young-Yamanouchi basis is labelled by the standard Young tableaux λ of shape R, and the transposition $(j, j+1)$ acts on them as

$$(j, j+1)\ |R, \lambda\rangle = \frac{1}{\rho_\lambda(j+1, j)}\ |R, \lambda\rangle + \sqrt{1 - \frac{1}{\rho_\lambda(j+1, j)^2}}\ |R, (j, j+1)\lambda\rangle\ , \tag{D.2}$$

where $\rho_\lambda(j+1, j)$ is the axial distance from $j+1$ to j in the standard Young tableau λ.[14] The matrix representation of all other elements follows from (D.2). It is straightforward to compute the branching coefficient from (3.3).

The final ingredient to obtain the diagonal operators (D.1) is the CG coefficient defined by (A.20). To extract $S^{\tau,\Lambda RR'}_{kij}$, we apply $\sigma \in S_{2n}$ to (A.20) and compare both sides.

[14] The axial distance between a and b is defined by counting the number of boxes we need to pass through from a to b in the Young tableau. We add $+1$ by going left or down, and -1 by going right or up. For example, $\rho_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}}(3, 4) = -\rho_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}}(4, 3) = +3$.

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Let us explain the computation of the CG coefficients in detail. We rewrite the matrix representation of the product $D^R \otimes D^R(\sigma)$ into a block diagonal form by a similarity transformation,

$$(U_R^T)^{-1} \left(D^R \otimes D^R(\sigma) \right) U_R = \begin{pmatrix} D^{r_1}(\sigma) & & \\ & D^{r_2}(\sigma) & \\ & & \ddots \end{pmatrix}. \tag{D.3}$$

The CG coefficients are equal to the elements of the rotation matrix U_R. We compute each column of U_R by using the Young symmetriser,

$$P_\lambda = \sum_{\sigma \in S_{2n}} p_\lambda(\sigma)\, \sigma \equiv N_\lambda \prod_{k \in \mathrm{Column}(\lambda)} A_k \prod_{\ell \in \mathrm{Row}(\lambda)} S_\ell \,, \tag{D.4}$$

where $p_\lambda(\sigma)$ is a coefficient, N_λ is a normalisation constant, and A_k (or S_ℓ) is anti-symmetric (or symmetric) combination of the entries in the k-th column (or ℓ-th row) of the standard Young tableau λ, respectively. For example,

$$P_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} = N_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} [\mathrm{id} - (13)]\, [\mathrm{id} + (12)]\,, \qquad P_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} = N_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} [\mathrm{id} - (12)]\, [\mathrm{id} + (13)]\,. \tag{D.5}$$

The Young symmetriser projects $D^{R\otimes R}$ onto the state corresponding to $\{\tilde{e}_\lambda\}$.[15] Thus, the combination $\sum_\sigma p_\lambda(\sigma)\, D^{R\otimes R}(\sigma)$ becomes a rank-one matrix corresponding to a single eigenvector $\tilde{e}_\lambda$. By collecting all eigenvectors (and orthogonalizing them appropriately), we obtain the rotation matrix U_R.

D.2 Explicit operators

Length two. The group S_2 has two irreducible representations, symmetric and anti-symmetric. Their tensor products decompose as

$$[1,1] \otimes [1,1] = [2], \qquad [2] \otimes [2] = [2], \tag{D.6}$$

The symmetric representation $[2]$ is an even partition. The representation matrices are

$$D^{[2]}(\alpha) = 1, \qquad D^{[1,1]}(\alpha) = \mathrm{sign}\,(\alpha). \tag{D.7}$$

The CG coefficients and the branching coefficients are trivial. The operators (D.1) are given by

$$\begin{aligned} O^{[2],[2]} &= O_{\mathrm{id}} + O_{(12)} = \mathrm{tr}(\Phi^a)\,\mathrm{tr}(\Phi^a) + \mathrm{tr}(\Phi^a\Phi^a), \\ O^{[2],[1,1]} &= O_{\mathrm{id}} - O_{(12)} = \mathrm{tr}(\Phi^a)\,\mathrm{tr}(\Phi^a) - \mathrm{tr}(\Phi^a\Phi^a). \end{aligned} \tag{D.8}$$

[15] The new basis $\{\tilde{e}_\lambda\}$ is not orthogonal. The new basis is related to the Young-Yamanouchi basis in a trivial way after orthogonalization, in the simple cases studied here.

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Length four. We need the following irreducible representations of S_4

$$\begin{aligned} R &= \{[1^4],[2,1^2],[2^2],[3,1],[4]\}, & d_R &= \{1,3,2,3,1\}, \\ \Lambda_1 &= \{[2^2],[4]\}, & d_{\Lambda_1} &= \{2,1\}. \end{aligned} \tag{D.9}$$

The tensor products decompose as

$$\begin{gathered} [1^4]\otimes[1^4]=[4], \qquad [2^2]\otimes[2^2]=[1^4]\oplus[2^2]\oplus[4], \qquad [4]\otimes[4]=[4], \\ [2,1^2]\otimes[2,1^2]=[3,1]\otimes[3,1]=[2,1^2]\oplus[2^2]\oplus[3,1]\oplus[4]. \end{gathered} \tag{D.10}$$

This decomposition is multiplicity-free, so we can drop the index τ in $\mathcal{O}^{R,\Lambda_1,\tau}$.

After the procedures of appendix D.1, we obtain

$$\begin{aligned} O^{[4],[4]} &= \operatorname{tr}(a_1)^2\operatorname{tr}(a_2)^2+2\operatorname{tr}(a_1)^2\operatorname{tr}(a_2a_2)+4\operatorname{tr}(a_1)\operatorname{tr}(a_2)\operatorname{tr}(a_1a_2)+8\operatorname{tr}(a_1)\operatorname{tr}(a_1a_2a_2) \\ &\quad +2\operatorname{tr}(a_1a_2)^2+\operatorname{tr}(a_1a_1)\operatorname{tr}(a_2a_2)+4\operatorname{tr}(a_1a_1a_2a_2)+2\operatorname{tr}(a_1a_2a_1a_2), \\ O^{[1^4],[4]} &= \operatorname{tr}(a_1)^2\operatorname{tr}(a_2)^2-2\operatorname{tr}(a_1)^2\operatorname{tr}(a_2a_2)-4\operatorname{tr}(a_1)\operatorname{tr}(a_2)\operatorname{tr}(a_1a_2)+8\operatorname{tr}(a_1)\operatorname{tr}(a_1a_2a_2) \\ &\quad +2\operatorname{tr}(a_1a_2)^2+\operatorname{tr}(a_1a_1)\operatorname{tr}(a_2a_2)-4\operatorname{tr}(a_1a_1a_2a_2)-2\operatorname{tr}(a_1a_2a_1a_2), \\ O^{[2^2],[4]} &= -\frac{2}{\sqrt{2}}\Big\{\operatorname{tr}(a_1)^2\operatorname{tr}(a_2)^2-4\operatorname{tr}(a_1)\operatorname{tr}(a_1a_2a_2)+\operatorname{tr}(a_1a_1)\operatorname{tr}(a_2a_2)+2\operatorname{tr}(a_1a_2)\operatorname{tr}(a_1a_2)\Big\}, \\ O^{[2^2],[2^2]} &= -\frac{4}{\sqrt{2}}\Big\{\operatorname{tr}(a_1)^2\operatorname{tr}(a_2a_2)-\operatorname{tr}(a_1)\operatorname{tr}(a_2)\operatorname{tr}(a_1a_2)+\operatorname{tr}(a_1a_2a_1a_2)-\operatorname{tr}(a_1a_1a_2a_2)\Big\}, \\ O^{[3,1],[4]} &= \frac{1}{\sqrt{3}}\Big\{3\operatorname{tr}(a_1)^2\operatorname{tr}(a_2)^2+2\operatorname{tr}(a_1)^2\operatorname{tr}(a_2a_2)+4\operatorname{tr}(a_1)\operatorname{tr}(a_2)\operatorname{tr}(a_1a_2) \\ &\quad -2\operatorname{tr}(a_1a_2)^2-\operatorname{tr}(a_1a_1)\operatorname{tr}(a_2a_2)-4\operatorname{tr}(a_1a_1a_2a_2)-2\operatorname{tr}(a_1a_2a_1a_2)\Big\}, \\ O^{[2,1^2],[4]} &= \frac{1}{\sqrt{3}}\Big\{3\operatorname{tr}(a_1)^2\operatorname{tr}(a_2)^2-2\operatorname{tr}(a_1)^2\operatorname{tr}(a_2a_2)-4\operatorname{tr}(a_1)\operatorname{tr}(a_2)\operatorname{tr}(a_1a_2) \\ &\quad -2\operatorname{tr}(a_1a_2)^2-\operatorname{tr}(a_1a_1)\operatorname{tr}(a_2a_2)+4\operatorname{tr}(a_1a_1a_2a_2)+2\operatorname{tr}(a_1a_2a_1a_2)\Big\}, \\ O^{[3,1],[2^2]} &= \frac{4}{\sqrt{6}}\Big\{\operatorname{tr}(a_1)^2\operatorname{tr}(a_2a_2)-\operatorname{tr}(a_1)\operatorname{tr}(a_2)\operatorname{tr}(a_1a_2) \\ &\quad -\operatorname{tr}(a_1a_2)^2+\operatorname{tr}(a_1a_1)\operatorname{tr}(a_2a_2)+\operatorname{tr}(a_1a_1a_2a_2)-\operatorname{tr}(a_1a_2a_1a_2)\Big\}, \\ O^{[2,1^2],[2^2]} &= -\frac{4}{\sqrt{6}}\Big\{\operatorname{tr}(a_1)^2\operatorname{tr}(a_2a_2)-\operatorname{tr}(a_1)\operatorname{tr}(a_2)\operatorname{tr}(a_1a_2) \\ &\quad +\operatorname{tr}(a_1a_2)^2-\operatorname{tr}(a_1a_1)\operatorname{tr}(a_2a_2)+\operatorname{tr}(a_1a_1a_2a_2)-\operatorname{tr}(a_1a_2a_1a_2)\Big\}, \end{aligned} \tag{D.11}$$

where we used the notation $\operatorname{tr}(a_1a_2a_3a_4)=\operatorname{tr}(\Phi^{a_1}\Phi^{a_2}\Phi^{a_3}\Phi^{a_4})$. Their free two-point functions are given by (3.9).

E Relation to covariant approach

E.1 $\mathbf{O}(N_f)\times S_{2n}$ CG coefficients

In [12] a general construction of free-field diagonal operators was given, where the operators are built out of fields transforming in a general representation V of a general global

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symmetry group G. This general construction requires the explicit computation of the Clebsch-Gordan coefficients, which decompose the tensor products $V^{\otimes m}$ in terms of irreducible representations of $G \times S_m$. Our approach to the free-field diagonal operators in the current paper is similar to [12] since both diagonal operators carry the same representation labels. However, the two operators look slightly different, since the mesonic operators discussed here do not involve the CG coefficients.

We will show that the two operators are identical, by giving an explicit formula for the relevant CG coefficients. Recall that in [12] the colour and flavour indices are treated separately. A gauge-covariant operators turn into gauge-invariant operators by combining indices appropriately. Instead, we may fix a gauge in the gauge-covariant form, and then combine them into gauge-invariant operators. In this way we can reproduce the mesonic operators. The CG coefficients are related to the product of branching coefficients and Kronecker delta's.

Let us just focus on the CG problem for the $2n$-fold tensor product of V_F with $G = \mathrm{O}(N_f)$. In particular, we are interested in the one-dimensional representation of both S_{2n} and $\mathrm{SO}(N_f)$. Let v_a be basis vectors in V_F. Consider

$$v_{a_1} \otimes v_{a_2} \otimes \cdots \otimes v_{a_{2n}} \tag{E.1}$$

A vector v_ρ, parametrised by a permutation $\rho \in S_{2n}$ which controls the pairwise contractions, is invariant under $\mathrm{O}(N_f)$:

$$v_\rho = \delta^{a_{\rho(1)}a_{\rho(2)}} \cdots \delta^{a_{\rho(2n-1)}a_{\rho(2n)}} \quad v_{a_1} \otimes \cdots \otimes v_{a_{2n}} \tag{E.2}$$

Following a theme we have seen repeatedly, whenever we have some invariants parametrised by permutations, in the present case tensor products of vectors invariant under $\mathrm{O}(N_f)$, we must ask about the redundancy in the description. Here the redundancy is

$$v_\rho = v_{\gamma\rho} \tag{E.3}$$

for $\gamma \in S_n[S_2]$. So we can also write

$$v_\rho = \frac{1}{2^n n!} \sum_{\gamma \in S_n[S_2]} v_{\gamma\rho} \tag{E.4}$$

Again, following a familiar theme, disentangle these equivalence classes by using representation theory. The first step is to define

$$v_{IJ}^{\Lambda_1} = \frac{1}{(2n)!} \sum_{\rho \in S_{2n}} D_{IJ}^{\Lambda_1}(\rho)\ v_\rho \tag{E.5}$$

Exploiting the invariance in the Fourier transformed basis,

$$\begin{aligned} v_{IJ}^{\Lambda_1} &= \frac{1}{(2n)!} \frac{1}{|S_n[S_2]|} \sum_{\rho \in S_{2n}} D_{IJ}^{\Lambda_1}(\rho) \sum_{\gamma \in S_n[S_2]} v_{\gamma\rho} \\ &= \frac{1}{(2n)!} \frac{1}{|S_n[S_2]|} \sum_{\gamma \in S_n[S_2]} \sum_{\rho \in S_{2n}} D_{IJ}^{\Lambda_1}(\rho\gamma^{-1})\ v_\rho \end{aligned}$$

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$$= \frac{1}{(2n)!} \sum_{\rho \in S_{2n}} D^{\Lambda_1}_{IJ}(\rho\, p_{1_{S_n[S_2]}})\ v_\rho$$
$$= \frac{1}{(2n)!} \sum_{\rho \in S_{2n}} D^{\Lambda_1}_{IK}(\rho) B^{\Lambda_1 \to 1}_K B^{\Lambda_1 \to 1}_J v_\rho \tag{E.6}$$

The branching coefficient is zero if Λ_1 is not even. The invariant vectors are therefore[16]

$$v^{\Lambda_1}_I \equiv \frac{1}{(2n)!} \sum_{\rho \in S_{2n}} D^{\Lambda_1}_{IK}(\rho) B^{\Lambda_1 \to 1}_K\ v_\rho \tag{E.7}$$

Recalling the definition of v_ρ

$$v^{\Lambda_1}_I = \frac{1}{(2n)!} \sum_{\rho \in S_{2n}} D^{\Lambda_1}_{IK}(\rho) B^{\Lambda_1 \to 1}_K\ \delta^{a_{\rho(1)} a_{\rho(2)}} \cdots \delta^{a_{\rho(2n-1)} a_{\rho(2n)}}\ \ v_{a_1} \otimes \cdots \otimes v_{a_{2n}} \tag{E.8}$$

This can be written in terms of a CG coefficient coupling $V_F^{\otimes 2n}$ to the state $(\Lambda_1, I) \times \emptyset$ of $S_{2n} \times \mathrm{O}(N_f)$, where Λ_1 refers to the representation of S_{2n}, I is the state label of Λ_1, and $\emptyset$ the one-dimensional representation of $\mathrm{O}(N_f)$:

$$v^{\Lambda_1}_I = C^{\vec{a}}_{\Lambda_1, I} v_{a_1} \otimes \cdots \otimes v_{a_{2n}} \tag{E.9}$$

Multiplicity labels are not needed.

It is tempting to identify the CG coefficients as

$$C^{\vec{a}}_{\Lambda_1, I} = \frac{1}{(2n)!} \sum_{\rho \in S_{2n}} D^{\Lambda_1}_{IK}(\rho) B^{\Lambda_1 \to 1}_K\ \delta^{a_{\rho(1)} a_{\rho(2)}} \cdots \delta^{a_{\rho(2n-1)} a_{\rho(2n)}} \tag{E.10}$$

There is an important subtlety of normalisation which has to be considered when comparing to [12]. The key point is that the normalisation of the above-defined $C^{\vec{a}}_{\Lambda_1, I}$ is

$$\sum_{\vec{a}} C^{\vec{a}}_{\Lambda_1, I, \tau_\Lambda} C^{\vec{a}}_{\Lambda_1', I', \tau_\Lambda'} = N_{CG}\, \delta_{\Lambda_1 \Lambda_1'} \delta_{II'} \delta_{\tau_\Lambda \tau_\Lambda'} \tag{E.11}$$

where N_{CG} is determined below. Let us compute $v^\Lambda_I v^\Lambda_J$ (no sum over Λ_1)

$$\langle v^{\Lambda_1}_I, v^{\Lambda_1}_J \rangle = \frac{1}{(2n)!^2} \sum_{\vec{a}, \vec{b}} C^{\vec{a}}_{\Lambda_1, I} C^{\vec{b}}_{\Lambda_1, J} \langle v_{\vec{a}}, v_{\vec{b}} \rangle \tag{E.12}$$
$$= \frac{1}{(2n)!^2} \sum_{\rho_1} \sum_{\rho_2} D^{\Lambda_1}_{IK}(\rho_1) D^{\Lambda_1}_{JL}(\rho_2) B^{\Lambda_1 \to 1}_K B^{\Lambda_1 \to 1}_L \langle v_{\rho_1}, v_{\rho_2} \rangle. \tag{E.13}$$

Since $C^{\vec{a}}_{\Lambda_1, I} = C^{\sigma(\vec{a})}_{\Lambda_1, I}$ for any $\sigma \in S_{2n}$, the first line is also written as

$$\langle v^{\Lambda_1}_I, v^{\Lambda_1}_J \rangle = \frac{N_{CG}}{(2n)!^2}\, \delta_{IJ}\,. \tag{E.14}$$

[16] See e.g. appendix B of [9] of the relevant fact from linear algebra.

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Note we assumed, as in [12], that the vectors in V_F are unit normalised

$$\langle v_a, v_b\rangle = \delta_{ab}\,, \qquad \langle v_{\vec{a}}, v_{\vec{b}}\rangle = \prod_{k=1}^{n} \delta_{a_k b_k}\,. \tag{E.15}$$

The norm of the permutation-parametrised vectors is

$$\begin{aligned}\langle v_{\rho_1}, v_{\rho_2}\rangle &= \delta^{a_{\rho_1(1)} a_{\rho_1(2)}} \cdots \delta^{a_{\rho_1(2n-1)} a_{\rho_1(2n)}} \delta^{b_{\rho_2(1)} b_{\rho_2(2)}} \cdots \delta^{b_{\rho_2(2n-1)} b_{\rho_2(2n)}} \langle v_{a_1}\cdots v_{a_{2n}}, v_{b_1}\cdots v_{b_{2n}}\rangle \\ &= W(\rho_2^{-1}\rho_1)\end{aligned} \tag{E.16}$$

This determines N_{CG} via

$$\langle v_I^{\Lambda_1}, v_J^{\Lambda_1}\rangle = \frac{\delta_{IJ}}{(2n)!\, d_{\Lambda_1}} \sum_{\rho} W(\rho)\chi_{\Lambda_1}(\rho^{-1}\, p_{1_{S_n[S_2]}}) = \delta_{IJ}\, \frac{N_f^n \omega_{\Lambda_1/2}(\Omega_{2n}^{(f)})}{(2n)!\, d_{\Lambda_1}}\,. \tag{E.17}$$

If we substitute the CG coefficients (E.10) to the general diagonal operator of [12], we obtain gauge-invariant operators involving a sum over permutations ρ, α. After doing the sum over ρ, our diagonal operators in section 3 can be recovered.

E.2 Baryonic operators

We explain how to count $\mathrm{SO}(N_f)$ singlets following [12]. Let us take the flavour part of the Schur-Weyl duality

$$V_F^{\otimes 2n} = \bigoplus_{\substack{\Lambda_1 \\ c_1(\Lambda_1)\le N_f}} \left(V_{\Lambda_1}^{\mathrm{GL}(N_f)} \otimes V_{\Lambda_1}^{S_{2n}}\right). \tag{E.18}$$

We restrict $\mathrm{GL}(N_f)$ to $\mathrm{O}(N_f)$, and further to $\mathrm{SO}(N_f)$ by the projection π [60, 61],[17]

$$V_{\Lambda_1}^{\mathrm{GL}(N_f)} = \bigoplus_{\Lambda_2} V_{\Lambda_1,\Lambda_2} \otimes V_{\Lambda_2}^{\mathrm{O}(N_f)} = \bigoplus_{\Lambda_2} \otimes V_{\pi(\Lambda_2)}^{\mathrm{SO}(N_f)}\,, \tag{E.19}$$

$$\dim V_{\Lambda_1,\Lambda_2} = \sum_{\beta:\mathrm{even}} g(\Lambda_2,\beta;\Lambda_1), \tag{E.20}$$

where $g(A,B;C)$ is the LR coefficient (A.24). The singlet representations of $\mathrm{SO}(N_f)$ have two origins. The first origin is an $\mathrm{O}(N_f)$ singlet. The other is a non-singlet of $\mathrm{O}(N_f)$ projected by π. An example is $\pi([1^{N_f}]) = \emptyset$, corresponding to

$$\Phi_{[a_1}\cdots\Phi_{a_{N_f}]} = \frac{1}{N_f!}\epsilon_{a_1\cdots a_{N_f}}\epsilon^{b_1\cdots b_{N_f}}\Phi_{[b_1}\cdots\Phi_{b_{N_f}]}. \tag{E.21}$$

Following the arguments in section 4.1, the number of $\mathrm{SO}(N_f)$ singlet operators is counted by

$$\sum_{\substack{R \\ c_1(R)\le N_c}} \sum_{\substack{\Lambda_1 \\ c_1(\Lambda_1)\le N_f}} \sum_{p=0}^{n} \sum_{\beta\vdash 2p} \sum_{\Lambda_2\vdash(2n-2p)} C(R,R,\Lambda_1) g(\Lambda_2,\beta;\Lambda_1)\, \delta_{\pi(\Lambda_2),\emptyset}\,. \tag{E.22}$$

The mesonic operators are counted by setting $\Lambda_2 = \emptyset$ in the above formula, yielding (4.10). The baryonic operators correspond to $\Lambda_2 \neq \emptyset$.

[17] The unitary irreducible representations of $\mathrm{O}(N_f)$ should satisfy $c_1(\Lambda_2) + c_2(\Lambda_2) \le N_f$.

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F Mixing matrix in detail

In this appendix we derive the mixing matrix on the permutation basis (6.3).

For our convenience we call each term of the following dilatation operator H_i

$$H = H_1 + H_2 = -\frac{1}{2}tr[\Phi_m, \Phi_n][\check{\Phi}^m, \check{\Phi}^n] - \frac{1}{4}tr[\Phi_m, \check{\Phi}^n][\Phi_m, \check{\Phi}^n] \tag{F.1}$$

where[18]

$$\begin{aligned} H_2 = -\frac{1}{4}tr[\Phi_m, \check{\Phi}^n][\Phi_m, \check{\Phi}^n] &= -\frac{1}{2}tr(\Phi_m\check{\Phi}^n\Phi_m\check{\Phi}^n) + \frac{1}{2}tr(\Phi_m\Phi_m\check{\Phi}^n\check{\Phi}^n) \\ &= H_{21} + H_{22}. \end{aligned} \tag{F.2}$$

The following formulae are useful,

$$tr([\Phi_m, \Phi_n][\check{\Phi}^m, \check{\Phi}^n])(\Phi_a)_{ij}(\Phi_b)_{kl} = 2([\Phi_a, \Phi_b]_{kj}\delta_{il} - [\Phi_a, \Phi_b]_{il}\delta_{kj}) \tag{F.3}$$

$$tr(\Phi_m\check{\Phi}^n\Phi_m\check{\Phi}^n)(\Phi_a)_{ij}(\Phi_b)_{kl} = 2\delta_{ab}(\Phi_m)_{il}(\Phi_m)_{kj} \tag{F.4}$$

$$tr(\Phi_m\Phi_m\check{\Phi}^n\check{\Phi}^n)(\Phi_a)_{ij}(\Phi_b)_{kl} = \delta_{ab}(\Phi_m\Phi_m)_{il}\delta_{jk} + \delta_{ab}(\Phi_m\Phi_m)_{kj}\delta_{il}. \tag{F.5}$$

It is convenient to consider the dilatation operator acting on general operators built from $\mathrm{SO}(N_f)$ scalars in (2.3). The action of H_1 is given by

$$\begin{aligned} &tr([\Phi_m, \Phi_n][\check{\Phi}^m, \check{\Phi}^n])\, tr_{2n}(\sigma\Phi_{\vec{a}}) \\ &= 2\sum_{i\neq j} tr_{2n}([\sigma, (ij)]\Phi_{a_1} \otimes \cdots \otimes [\Phi_{a_i}, \Phi_{a_j}] \otimes \cdots \otimes 1 \otimes \cdots \otimes \Phi_{a_{2n}}) \\ &= 2\sum_{i\neq j}\sum_{\alpha\in S_{2n}} \delta_{2n}([\sigma, (ij)]\alpha^{-1})tr_{2n}(\alpha\Phi_{a_1} \otimes \cdots \otimes [\Phi_{a_i}, \Phi_{a_j}] \otimes \cdots \otimes 1 \otimes \cdots \otimes \Phi_{a_{2n}}) \end{aligned} \tag{F.6}$$

where $[\Phi_{a_i}, \Phi_{a_j}]$ is in the i-th slot and 1 is in the j-th slot.[19] Here the sum $\sum_{i\neq j}$ is over different pairs (i, j), i.e. we do not distinguish $(i, j) = (1, 2)$ and $(2, 1)$.

In order to express the above operator in terms of (2.3), we consider the decomposition $S_{2n} \to S_{2n-1} \times S_1$ [17, 62]. Elements in S_{2n} can be expressed in terms of elements in S_{2n-1} as

$$\{\alpha \mid \alpha \in S_{2n}\} = \{\beta \,|\, \beta \in S_{2n-1}^{\langle j\rangle}\} \cup \{\beta(jk) \mid k = 1, 2, j-1, j+1, \cdots, 2n;\ \beta \in S_{2n-1}^{\langle j\rangle}\}, \tag{F.7}$$

where $S_{2n-1}^{\langle j\rangle}$ is the subgroup obtained by removing the j-th slot from S_{2n}. We illustrate how it works for the case $2n = 3$, $j = 3$. Take $(i, k) = (2, 1)$. When $\alpha = \beta$,

$$tr_3(\beta\Phi_{a_1} \otimes [\Phi_{a_2}, \Phi_{a_3}] \otimes 1) = N_c\, tr_2(\beta\Phi_{a_1} \otimes [\Phi_{a_2}, \Phi_{a_3}]) = N_c\, tr_3([(23), \beta]\Phi_{a_1} \otimes \Phi_{a_2} \otimes \Phi_{a_3}) \tag{F.8}$$

and when $\alpha = \beta(jk)$,

$$tr_3(\beta(31)\Phi_{a_1} \otimes [\Phi_{a_2}, \Phi_{a_3}] \otimes 1) = tr_2(\beta\Phi_{a_1} \otimes [\Phi_{a_2}, \Phi_{a_3}]) = tr_3([(23), \beta]\Phi_{a_1} \otimes \Phi_{a_2} \otimes \Phi_{a_3}) \tag{F.9}$$

[18] The Hamiltonian of integrable $\mathrm{SO}(N_f)$ spin chain is obtained by changing the coefficient of H_2 to $-1/(N_f - 2)$ and taking the planar limit [32].

[19] In the planar limit, only the terms $j = \sigma(i)$ and $j = \sigma^{-1}(i)$ survive.

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The case $(i,k)=(2,2)$ is same as above.

We then find that (F.6) can be written as

$$\begin{aligned}
&2N_c\sum_{i\neq j}\sum_{\beta\in S_{2n-1}^{\langle j\rangle}}\delta_{2n}([\sigma,(ij)]\beta^{-1})tr_{2n}([(ij),\beta]\Phi_{\vec{a}})\\
&+2\sum_{i\neq j}\sum_{k(k\neq j)}\sum_{\beta\in S_{2n-1}^{\langle j\rangle}}\delta_{2n}([\sigma,(ij)](jk)\beta^{-1})tr_{2n}([(ij),\beta]\Phi_{\vec{a}})\\
&\qquad=2\sum_{i\neq j}\sum_{\beta\in S_{2n-1}^{\langle j\rangle}}\delta_{2n}([\sigma,(ij)]X^{(j)}\beta^{-1})tr_{2n}([(ij),\beta]\Phi_{\vec{a}}),
\end{aligned}\tag{F.10}$$

where we have introduced

$$X^{(j)}=N_c+\sum_{k(\neq j)}(kj).\tag{F.11}$$

Next study the action of H_{21},

$$tr(\Phi_m\check{\Phi}^n\Phi_m\check{\Phi}^n)\,tr_{2n}(\sigma\Phi_{\vec{a}})=2\sum_{i\neq j}\delta_{a_i,a_j}tr_{2n}((ij)\sigma\Phi_{a_1}\otimes\cdots\otimes\Phi_m\otimes\cdots\otimes\Phi_m\otimes\cdots\otimes\Phi_{a_{2n}})\tag{F.12}$$

where two Φ_m's are in the i-th position and the j-th position. Introducing the flavour contraction operator acting on two Φ's at (i,j),

$$C^f_{(ij)}\Phi_a\otimes\Phi_b=\delta_{ab}\Phi_m\otimes\Phi_m,\tag{F.13}$$

we have

$$tr(\Phi_m\check{\Phi}^n\Phi_m\check{\Phi}^n)\,tr_{2n}(\sigma\Phi_{\vec{a}})=2\sum_{i\neq j}C^f_{(ij)}tr_{2n}((ij)\sigma\Phi_{\vec{a}}).\tag{F.14}$$

The action of H_{22} is computed using (F.5) as

$$\begin{aligned}
&tr(\Phi_m\Phi_m\check{\Phi}^n\check{\Phi}^n)\,tr_{2n}(\sigma\Phi_{\vec{a}})\\
&=2\sum_{i\neq j}\delta_{a_ia_j}tr_{2n}((ij)\sigma\Phi_{a_1}\otimes\cdots\otimes\Phi_m\Phi_m\otimes\cdots\cdots\otimes 1\otimes\cdots\Phi_{a_{2n}})\\
&=2\sum_{i\neq j}\delta_{a_ia_j}\sum_{\alpha\in S_{2n}}\delta_{2n}((ij)\sigma\alpha^{-1})S^{(ij)}(\alpha),
\end{aligned}\tag{F.15}$$

$$S^{(ij)}(\alpha):=tr_{2n}(\alpha\Phi_{a_1}\otimes\cdots\otimes\underbrace{\Phi_m\Phi_m}_{i}\otimes\cdots\cdots\otimes\underbrace{1}_{j}\otimes\cdots\Phi_{a_{2n}})\tag{F.16}$$

where i,j represent the site of $\Phi_m\Phi_m$ and that of 1. We now apply the reduction $S_{2n}\to S_{2n-1}^{\langle j\rangle}\times S_1$ to $S^{(ij)}(\alpha)$. For example for $i=2,j=1$, we have

$$\begin{aligned}
S^{(21)}(\beta)&=N_c\,tr_{2n-1}(\beta\Phi_m\Phi_m\otimes\Phi_{a_3}\otimes\cdots\otimes\Phi_{a_{2n}})\\
S^{(21)}(\beta(1k))&=tr_{2n-1}(\beta\Phi_m\Phi_m\otimes\Phi_{a_3}\otimes\cdots\otimes\Phi_{a_{2n}})
\end{aligned}\tag{F.17}$$

where $\beta\in S_{2n-1}^{\langle 1\rangle}$, and we can use the following formula,

$$tr_{2n-1}(\beta\Phi_m\Phi_m\otimes\Phi_{a_3}\otimes\cdots\otimes\Phi_{a_{2n}})=tr_{2n}((12)\beta\Phi_m\otimes\Phi_m\otimes\Phi_{a_3}\otimes\cdots\otimes\Phi_{a_{2n}}).\tag{F.18}$$

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We then find that (F.15) can be expressed by

$$tr(\Phi_m\Phi_m\check{\Phi}^n\check{\Phi}^n)\,tr_{2n}(\sigma\Phi_{\vec{a}}) = 2\sum_{i\neq j}C_{(ij)}\sum_{\beta\in S^{\langle j\rangle}_{2n-1}}\delta_{2n}((ij)\sigma X^{(j)}\beta^{-1})tr_{2n}((ij)\beta\Phi_{\vec{a}}). \quad \text{(F.19)}$$

Collecting these results,

$$\begin{aligned}
H_1 tr_{2n}(\sigma\Phi_{\vec{a}}) &= -\sum_{i\neq j}\sum_{\beta\in S^{\langle j\rangle}_{2n-1}}\delta_{2n}([\sigma,(ij)]X^{(j)}\beta^{-1})tr_{2n}([(ij),\beta]\Phi_{\vec{a}}),\\
H_{21} tr_{2n}(\sigma\Phi_{\vec{a}}) &= -\sum_{i\neq j}C_{(ij)}tr_{2n}((ij)\sigma\Phi_{\vec{a}}),\\
H_{22} tr_{2n}(\sigma\Phi_{\vec{a}}) &= \sum_{i\neq j}C_{(ij)}\sum_{\beta\in S^{\langle j\rangle}_{2n-1}}\delta_{2n}((ij)\sigma X^{(j)}\beta^{-1})tr_{2n}((ij)\beta\Phi_{\vec{a}}).
\end{aligned} \quad \text{(F.20)}$$

Note that the N_c-dependence appears only in $X^{(j)}$.

Let us next focus on the mesonic singlet operators

$$\mathcal{O}(\sigma)\equiv\mathcal{O}_\sigma = tr_{2n}(\sigma\Phi_{a_1}\otimes\Phi_{a_1}\otimes\Phi_{a_2}\otimes\Phi_{a_2}\cdots\otimes\Phi_{a_n}\otimes\Phi_{a_n}). \quad \text{(F.21)}$$

We decompose the sum over i,j into

$$\sum_{i\neq j} = \sum_{(i,j)} + \sum_{\langle i,j\rangle} \quad \text{(F.22)}$$

where (i,j) run over $(1,2),(3,4),\cdots$, and $\langle i,j\rangle$ over the other pairs.[20] We have

$$\begin{aligned}
H_1\mathcal{O}(\sigma) &= -\sum_{\langle i,j\rangle}\sum_{\beta\in S^{\langle j\rangle}_{2n-1}}\delta_{2n}([\sigma,(ij)]X^{(j)}\beta^{-1})\mathcal{O}([(ij),\beta]),\\
H_{21}\mathcal{O}(\sigma) &= -N_f\sum_{(i,j)}\mathcal{O}((ij)\sigma) - \sum_{\langle i,j\rangle}C_{(ij)}\mathcal{O}((ij)\sigma),\\
H_{22}\mathcal{O}(\sigma) &= N_f\sum_{(i,j)}\sum_{\beta\in S^{\langle j\rangle}_{2n-1}}\delta_{2n}((ij)\sigma X^{(j)}\beta^{-1})\mathcal{O}((ij)\beta)\\
&\quad + \sum_{\langle i,j\rangle}C_{(ij)}\sum_{\beta\in S^{\langle j\rangle}_{2n-1}}\delta_{n}((ij)\sigma X^{(j)}\beta^{-1})\mathcal{O}((ij)\beta).
\end{aligned} \quad \text{(F.23)}$$

The mixing matrix

$$H_i\mathcal{O}(\sigma) = \sum_{\tau\in S_{2n}}M^{(i)}_{\sigma,\tau}O(\tau) \quad \text{(F.24)}$$

[20] $i=\Sigma_0(j)$ in the sum over (i,j).

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is given by

$$M^{(1)}_{\sigma,\tau} = -\sum_{\langle i,j\rangle}\sum_{\beta\in S^{\langle j\rangle}_{2n-1}} \delta_{2n}([[\sigma],(ij)]X^{(j)}\beta^{-1})\delta_{2n}([[\tau^{-1}],(ij)]\beta),$$
$$= -2n\,(n-1)\sum_{\beta\in S^{\langle 2n\rangle}_{2n-1}} \delta_{n}([[\sigma],(1,2n)]X^{(2n)}\beta^{-1})\delta_{n}([[\tau^{-1}],(1,2n)]\beta), \tag{F.25}$$

$$M^{(21)}_{\sigma,\tau} = -N_f\sum_{(i,j)}\delta_{2n}([\tau^{-1}](ij)[\sigma]) - \sum_{\langle i,j\rangle}\delta_{2n}\left([\tau^{-1}](\Sigma_0(i)j)(ij)[\sigma](\Sigma_0(i)j)\right),$$
$$= -N_f n\delta_{2n}([\tau^{-1}](12)[\sigma]) - 2n\,(n-1)\,\delta_{2n}\left([\tau^{-1}](2,2n)(1,2n)[\sigma](2,2n)\right), \tag{F.26}$$

where $\Sigma_0 = (12)(34)\cdots(2n-1,2n)$,

$$M^{(22)}_{\sigma,\tau} = N_f\sum_{(i,j)}\sum_{\beta\in S^{\langle j\rangle}_{2n-1}} \delta_{2n}((ij)[\sigma]X^{(j)}\beta^{-1})\delta_{2n}([\tau^{-1}](ij)\beta) \tag{F.27}$$
$$+\sum_{\langle i,j\rangle}\sum_{\beta\in S^{\langle j\rangle}_{2n-1}} \delta_{2n}((ij)[\sigma]X^{(j)}\beta^{-1})\delta_{2n}([\tau^{-1}](\Sigma_0(i)j)(ij)\beta(\Sigma_0(i)j))$$
$$= N_f n\sum_{\beta\in S^{\langle 2n\rangle}_{2n-1}} \delta_{2n}((2n-1,2n)[\sigma]X^{(n)}\beta^{-1})\delta_{2n}([\tau^{-1}](n-1,n)\beta)$$
$$+\,2n\,(n-1)\sum_{\beta\in S^{\langle n\rangle}_{2n-1}} \delta_{2n}((1,2n)[\sigma]X^{(2n)}\beta^{-1})\delta_{2n}([\tau^{-1}](2,2n)(1,2n)\beta(2,2n)).$$

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