

Abelian tensor hierarchy and Chern-Simons actions in 4D $\mathcal{N} = 1$ conformal supergravity

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ABSTRACT: We consider Chern-Simons actions of Abelian tensor hierarchy of p -form gauge fields in four-dimensional $\mathcal{N} = 1$ supergravity. Using conformal superspace formalism, we solve the constraints on the field strengths of the p -form gauge superfields in the presence of the tensor hierarchy. The solutions are expressed by the prepotentials of the p -form gauge superfields. We show the internal and superconformal transformation laws of the prepotentials. The descent formalism for the Chern-Simons actions is exhibited.

KEYWORDS: Supergravity Models, Superspaces

ARXIV EPRINT: [1609.01111](https://arxiv.org/abs/1609.01111)

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1 Introduction

The superstring theory is a candidate for the unified theory of the fundamental interactions including quantum gravity. There are strings and branes in the superstring theory. Our universe can be described by the strings and branes in a unified way at the low energy limit. The stability of the branes is preserved by supersymmetry (SUSY) and conserved Ramond-Ramond charges. We can avoid unstable tachyons by SUSY. Further, the conserved charges guarantee the number of branes. Antisymmetric tensor (p -form) gauge fields are coupled to the conserved charges of the branes.

One of the most important issues in the superstring theory is to construct realistic four-dimensional (4D) effective theories. Since the superstring theory is a ten-dimensional theory, 4D effective theories are obtained by compactifying extra six dimensions. 4D $\mathcal{N} = 1$ supergravity (SUGRA) is a candidate for the effective theories. This theory consists of chiral fermions as well as gravity. Thus, we can embed the standard model particles into the theory. Further, the stability of the theory is ensured by SUSY.

Thus, it is important to consider p -form gauge fields in 4D $\mathcal{N} = 1$ SUGRA [1–11]. In particular, we study the p -form gauge fields which can be regarded as dimensionally reduced ones from higher dimensions. Such p -form gauge fields differ from those of just defined in 4D. Because the original gauge transformation laws of the p -form gauge fields are given in higher dimensions, the gauge transformations of the p -form gauge fields should contain different rank forms in 4D. The structure of the transformations is called a tensor hierarchy [12–15].

In 4D $\mathcal{N} = 1$ global SUSY, Becker et al. constructed such Abelian tensor hierarchy in superspace [16]. They showed Chern-Simons (CS) actions.¹ The CS actions are constructed by integrands which are proportional to p -form gauge fields. Since the different ranked tensors are related each other by the tensor hierarchy, each of the integrands is not independent. The internal gauge invariance requires the relations between the integrands. They showed the relations in a systematic manner, which is called descent formalism. The descent formalism relates the integrands each other by derivatives. The CS actions are important because they are related to the anomaly cancellation in 4D [18–20].

In this paper, we embed the CS actions of Abelian tensor hierarchy obtained in ref. [16] into 4D $\mathcal{N} = 1$ SUGRA. We use 4D $\mathcal{N} = 1$ conformal superspace formalism [21]. This formalism has larger gauge symmetries than superconformal tensor calculus [22–30] and Poincaré superspace formalism [31, 32]. Superconformal tensor calculus and Poincaré superspace formalism are obtained from the conformal superspace formalism by using their correspondences [21, 33, 34]. The CS actions are constructed by the prepotentials of the p -form gauge superfields in the presence of the tensor hierarchy.² We obtain the prepotentials by using so-called covariant approach, which are shown in our previous paper [35]. In the covariant approach, we introduce p -form gauge superfields and their field strength superfields in the superspace. The field strength superfields have some constraints, since they have superfluous degrees of freedom. We obtain the prepotentials as the solutions to the constraints. The CS actions in 4D $\mathcal{N} = 1$ SUGRA would be useful to discuss the roles of the p -form gauge fields, e.g. in cosmology [36, 37].

In the conformal superspace, the derivations of the solutions to the constraints are mostly the same as the case of the global SUSY in ref. [2]. This is because superconformally covariant spinor derivatives satisfy the same anti-commutation relations as those of global SUSY. Moreover, we can naturally extend the descent formalism of the CS actions in global SUSY [16] into the conformal superspace, since the relation between D- and F-term integrations in the conformal superspace are quite similar to the global SUSY case.

This paper is organized as follows. In section 2, we briefly review the covariant approach to Abelian tensor hierarchy in 4D $\mathcal{N} = 1$ conformal superspace. The prepotentials of p -form gauge superfields are obtained in section 3. We show the internal gauge transformation laws of the prepotentials. Section 4 is devoted to constructing the CS actions of the tensor hierarchy. In particular, the descent formalism in the conformal superspace is discussed. Finally, we conclude this paper in section 5. Throughout this paper, we use the terms “gauge superfields”, “field strengths superfields”, and “gauge parameter superfields” are simply written as “gauge fields”, “field strengths”, and “gauge parameters”, respectively.

2 Review of the covariant approach

We briefly review so-called covariant approach to Abelian tensor hierarchy in 4D $\mathcal{N} = 1$ conformal superspace discussed in ref. [35]. Covariant approach is an approach to constructing supersymmetric theories of p -form gauge fields in superspace.

¹They also showed CS actions in the case of non-Abelian tensor hierarchy [17].

²In this paper, we use the term “prepotentials” to refer to superfields which consist of the bosonic gauge fields and field strengths as well as their superpartners.

We use the notations and conventions of ref. [35] except the normalizations of the superfields Y^{I_3} and L^{I_2} , which are the same as G^S and H^M in ref. [16], respectively.

2.1 Conformal superspace

We firstly review conformal superspace formalism to construct SUGRA [21]. Superspace is space which is spanned by the ordinary spacetime coordinates x^m and the Grassmannian coordinates $(\theta^\mu, \bar{\theta}_{\dot{\mu}})$. Here, the indices m, n, \dots are used to refer to curved vector indices. The indices μ, ν, \dots and $\dot{\mu}, \dot{\nu}, \dots$ denote curved undotted and dotted spinor indices, respectively. In the superspace, SUSY transformations are understood as the translations to Grassmannian coordinates. Thus, we simply denote these coordinates at the same time: $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$, where we use Roman capital indices M, N, \dots for both of curved vector and spinor indices.

Conformal superspace is superspace where the superconformal symmetry is introduced as a gauge symmetry. The generators of the superconformal symmetry are spacetime translations P_a , SUSY transformations $(Q_\alpha, \bar{Q}^{\dot{\alpha}})$, Lorentz transformations M_{ab} , dilatation D , chiral rotation A , conformal boosts K_a , and conformal SUSY transformations $(S_\alpha, \bar{S}^{\dot{\alpha}})$. Here, Roman letters a, b, \dots denote flat vector indices. Greek letters α, β, \dots and $\dot{\alpha}, \dot{\beta}, \dots$ denote flat spinor indices. All of the generators of the superconformal symmetry are denoted as $X_{\mathcal{A}}$, where we use calligraphic indices $\mathcal{A}, \mathcal{B}, \dots$ to refer to the generators of the superconformal symmetry. In the conformal superspace, both of P_a and $(Q_\alpha, \bar{Q}^{\dot{\alpha}})$ are understood as the translations. Thus, we simply express P_a and $(Q_\alpha, \bar{Q}^{\dot{\alpha}})$ at the same time: $P_A := (P_a, Q_\alpha, \bar{Q}^{\dot{\alpha}})$. Here, capital Roman letters A, B, \dots are used for both of flat vector and spinor indices. Similarly, we denote both of K_a and $(S_\alpha, \bar{S}^{\dot{\alpha}})$ as $K_A := (K_a, S_\alpha, \bar{S}^{\dot{\alpha}})$. The (anti-)commutation relations of the generators are summarized in ref. [21].

The gauge fields of the superconformal symmetry are given by

$$h_M{}^A X_{\mathcal{A}} := E_M{}^A P_A + \frac{1}{2} \phi_M{}^{ab} M_{ba} + B_M D + A_M A + f_M{}^A K_A, \quad (2.1)$$

where we assume that the vielbein $E_M{}^A$ is invertible, and the inverse of the vielbein is denoted as $E_A{}^M$: $E_M{}^A E_A{}^N = \delta_M{}^N$ and $E_A{}^M E_M{}^B = \delta_A{}^B$. Note that the gauge fields $h_M{}^A$ are also expressed by differential forms on the conformal superspace as

$$h^A = dz^M h_M{}^A. \quad (2.2)$$

Here, we use the convention of ref. [31] for the differential forms. The differential forms $dz^M = (dx^m, d\theta^\mu, d\bar{\theta}_{\dot{\mu}})$ are bases of the superforms on the conformal superspace. The gauge transformation parameters are denoted as

$$\xi^{\mathcal{A}} X_{\mathcal{A}} = \xi^A P_A + \frac{1}{2} \xi(M)^{ab} M_{ba} + \xi(D) D + \xi(A) A + \xi(K)^A K_A. \quad (2.3)$$

We denote infinitesimal superconformal transformations as $\delta_G(\xi^{\mathcal{A}} X_{\mathcal{A}})$. The transformation laws of the gauge fields $h_M{}^A$ under the superconformal transformations other than P_A are given by

$$\delta_G(\xi^{\mathcal{B}'} X_{\mathcal{B}'}) h_M{}^A = \partial_M \xi^{\mathcal{B}'} \delta_{\mathcal{B}'}{}^A + h_M{}^C \xi^{\mathcal{B}'} f_{\mathcal{B}' C}{}^A. \quad (2.4)$$

Here, primed calligraphic indices $\mathcal{A}', \mathcal{B}', \dots$ are used to refer to the generators of the superconformal symmetry other than P_A : $X_{\mathcal{A}'} = (M_{ab}, D, A, K_A)$. The coefficients $f_{\mathcal{C}\mathcal{B}^A}$ are the structure constants of the superconformal symmetry: $[X_{\mathcal{C}}, X_{\mathcal{B}}] = -f_{\mathcal{C}\mathcal{B}^A} X_{\mathcal{A}}$, where we use the convention of “implicit grading” [21].

We define SUSY transformations and spacetime translations in the conformal superspace. In the conformal superspace, SUSY transformations are regarded as translations to the Grassmannian coordinates. Using field-independent parameters ξ^A , we relate infinitesimal P_A -transformations $\delta_G(\xi^A P_A)$ to the general coordinate transformations $\delta_{GC}(\xi^M)$ as

$$\delta_G(\xi^A P_A) = \delta_{GC}(\xi^M) - \delta_G(\xi^M h_M{}^{B'} X_{B'}). \quad (2.5)$$

Here, the parameters ξ^M are related to ξ^A as $\xi^M = \xi^A E_A{}^M$. The actions of P_A -transformations on a superfield without curved indices Φ define superconformally covariant derivatives ∇_A :

$$\delta_G(\xi^A P_A)\Phi = \xi^A \nabla_A \Phi = \xi^A E_A{}^M (\partial_M - h_M{}^{B'} X_{B'})\Phi. \quad (2.6)$$

2.2 Covariant approach to Abelian tensor hierarchy

Next, we introduce p -form gauge fields in the conformal superspace, where p runs over $p = -1, 0, 1, 2, 3, 4$. We assume that (-1) -forms are zero as in ordinary differential geometry. The p -form gauge fields are denoted as

$$C_{[p]}^{I_p} = \frac{1}{p!} dz^{M_1} \wedge \dots \wedge dz^{M_p} C_{M_p \dots M_1}^{I_p} = \frac{1}{p!} E^{A_1} \wedge \dots \wedge E^{A_p} C_{A_p \dots A_1}^{I_p}. \quad (2.7)$$

Here, I_p are indices of internal degrees of freedom, which run over $I_p = 1, \dots, \dim V_p$. The ranks of the differential forms are represented as $[p]$. The $X_{\mathcal{A}'}$ -transformations of the p -form gauge fields are defined as

$$\delta_G(\xi^{\mathcal{A}'} X_{\mathcal{A}'}) C_{M_p \dots M_1}^{I_p} = 0. \quad (2.8)$$

Thus, the $X_{\mathcal{A}'}$ -transformations of $C_{A_p \dots A_1}^{I_p}$ are given by the $X_{\mathcal{A}'}$ -transformations of vielbein $E_M{}^A$:

$$\begin{aligned} \delta_G(\xi^{\mathcal{A}'} X_{\mathcal{A}'}) C_{A_p \dots A_1}^{I_p} &= -E_{A_p}{}^N (\delta_G(\xi^{\mathcal{A}'} X_{\mathcal{A}'}) E_N{}^B) C_{B A_{p-1} \dots A_1}^{I_p} \\ &\quad - \dots - E_{A_1}{}^N (\delta_G(\xi^{\mathcal{A}'} X_{\mathcal{A}'}) E_N{}^B) C_{A_p \dots A_2 B}^{I_p}. \end{aligned} \quad (2.9)$$

The explicit transformation of the vielbein is summarized in ref. [35]. The infinitesimal internal gauge transformations $\delta_T(\Lambda)$ of the p -form gauge fields are given by

$$\delta_T(\Lambda) C_{[p]}^{I_p} = d\Lambda_{[p-1]}^{I_p} + (q^{(p)} \cdot \Lambda_{[p]})^{I_p}. \quad (2.10)$$

Here, d denotes the exterior derivative in the conformal superspace, and Λ is the set of the real gauge parameter superforms: $\Lambda = (\Lambda_{[0]}^{I_1}, \dots, \Lambda_{[3]}^{I_4})$. We assume that $\Lambda_{M_{p-1} \dots M_1}^{I_p}$ are field independent parameters. Note that $\Lambda_{A_{p-1} \dots A_1}^{I_p} = E_{A_{p-1}}{}^{M_{p-1}} \dots E_{A_1}{}^{M_1} \Lambda_{M_{p-1} \dots M_1}^{I_p}$ are field dependent parameters. Ordinary Abelian gauge transformations are expressed by the first

term in eq. (2.10). Shifts of the gauge fields are represented by the second term due to the tensor hierarchy. $q^{(p)}$ are real linear maps from the vector space V_{p+1} to the vector space V_p . The expressions $(q^{(p)} \cdot \Lambda_{[p]})^{I_p}$ mean $(q^{(p)})_{I_{p+1}}^{I_p} \Lambda_{[p]}^{I_{p+1}}$. Note that $q^{(p)}$ can be understood as the exterior derivative on the extra dimensions [16].

The P_A -transformations are redefined with respect to the internal gauge transformations in the presence of the tensor hierarchy. The redefinitions are given by

$$\delta_G(\xi^A P_A) = \delta_{GC}(\xi^M) - \delta_G(\xi^M h_M^{\mathcal{B}'} X_{\mathcal{B}'}) - \delta_T(\Lambda(\xi)). \quad (2.11)$$

Here, $\Lambda(\xi)$ is defined by

$$\Lambda(\xi) = (\iota_\xi C_{[1]}^{I_1}, \dots, \iota_\xi C_{[4]}^{I_4}), \quad (2.12)$$

and ι_ξ is a interior product

$$\iota_\xi C_{[p]}^{I_p} = \frac{1}{(p-1)!} dz^{M_1} \wedge \dots \wedge dz^{M_{p-1}} \xi^{M_p} C_{M_p \dots M_1}^{I_p}. \quad (2.13)$$

In the presence of the tensor hierarchy, the field strengths of the p -form gauge fields are given by using the exterior derivative and q 's. The definitions of the field strengths of the p -form gauge fields are given as follows:

$$F_{[p+1]}^{I_p} = dC_{[p]}^{I_p} - (q^{(p)} \cdot C_{[p+1]})^{I_p}. \quad (2.14)$$

The field strengths are transformed under the internal gauge transformations as

$$\delta_T(\Lambda) F_{[p+1]}^{I_p} = -(q^{(p)} \cdot q^{(p+1)} \cdot \Lambda_{[p+1]})^{I_p}. \quad (2.15)$$

The invariances of the field strengths under the internal transformations require conditions on the q 's as

$$q^{(p)} \cdot q^{(p+1)} = 0. \quad (2.16)$$

The covariant derivatives on the field strengths with Lorentz indices are given by

$$\nabla_B F_{A_{p+1} \dots A_1}^{I_p} = E_B^M (\partial_M - h_M^{\mathcal{A}'} X_{\mathcal{A}'}) F_{A_{p+1} \dots A_1}^{I_p}. \quad (2.17)$$

Note that the covariant derivatives ∇_B on the field strengths $F_{A_{p+1} \dots A_1}^{I_p}$ are superconformally covariant and internally invariant derivatives because $F_{A_{p+1} \dots A_1}^{I_p}$ are invariant under the internal gauge transformations. The Bianchi identities for the field strengths are given by

$$0 = dF_{[p+1]}^{I_p} + (q^{(p)} \cdot F_{[p+2]})^{I_p}. \quad (2.18)$$

We summarize the explicit forms of the gauge fields, field strengths and Bianchi identities in table 1.

We impose some constraints on the field strengths to eliminate degrees of freedoms because there are superfluous degrees of freedoms in the field strengths in the superspace. The constraints are the same ones as the case without the tensor hierarchy [2, 3, 32]. We exhibit the constraints in table 2. In this table, the indices $\underline{\alpha}, \underline{\beta}, \dots$ denote both undotted and dotted spinor indices: $\underline{\alpha} = (\alpha, \dot{\alpha})$. Note that the constraints are covariant under both superconformal and internal gauge transformations.

form	gauge field	field strength	Bianchi identity
4-form	U^{I_4}	$G^{I_4} = dU^{I_4} = 0$	–
3-form	C^{I_3}	$\Sigma^{I_3} = dC^{I_3} - (q^{(3)} \cdot U)^{I_3}$	$d\Sigma^{I_3} = 0$
2-form	B^{I_2}	$H^{I_2} = dB^{I_2} - (q^{(2)} \cdot C)^{I_2}$	$dH = -(q^{(2)} \cdot \Sigma)^{I_2}$
1-form	A^{I_1}	$F^{I_1} = dA^{I_1} - (q^{(1)} \cdot B)^{I_1}$	$dF^{I_1} = -(q^{(1)} \cdot H)^{I_1}$
0-form	f^{I_0}	$g^{I_0} = df^{I_0} - (q^{(0)} \cdot A)^{I_0}$	$dg^{I_0} = -(q^{(0)} \cdot F)^{I_0}$
–1-form	0	$\omega^{I_{-1}} = -(q^{(-1)} \cdot f)^{I_{-1}}$	$d\omega^{I_{-1}} = -(q^{(-1)} \cdot g)^{I_{-1}}$

Table 1. The p -forms, their corresponding field strengths and Bianchi identities. We impose that the field strengths of the 4-form gauge fields are zero as in table 2.

form	constraints
4-form	$G_{EDCBA}^{I_4} = 0$
3-form	$\Sigma_{\underline{\delta}\underline{\gamma}\underline{\beta}A}^{I_3} = \Sigma_{\delta\gamma ba}^{I_3} = 0$
2-form	$H_{\underline{\gamma}\underline{\beta}\underline{\alpha}}^{I_2} = H_{\gamma\beta\alpha}^{I_2} = H_{\dot{\gamma}\dot{\beta}\dot{\alpha}}^{I_2} = 0, \quad H_{\gamma\dot{\beta}\alpha}^{I_2} = i(\sigma_a)_{\gamma\dot{\beta}} L^{I_2}$
1-form	$F_{\underline{\alpha}\underline{\beta}}^{I_1} = 0$
0-form	$g_{\underline{\alpha}}^{I_0} = i\nabla_{\alpha}\Psi^{I_0}, \quad g_{\dot{\beta}}^{I_0} = -i\bar{\nabla}_{\dot{\beta}}\Psi^{I_0}, \quad K_A\Psi^{I_0} = 0$

Table 2. The constraints on the field strengths.

We solve the Bianchi identities under the constraints. As a result, the field strengths are expressed by the irreducible superfields. The irreducible superfields of the 2- and 0-form gauge fields are L^{I_2} and Ψ^{I_0} in table 2. We find the irreducible superfields of 3- and 1-form gauge fields Y^{I_3} and $W_{\underline{\alpha}}^{I_1}$ as follows, respectively:

$$\Sigma^{I_3\delta\dot{\gamma}}{}_{ba} = 4(\bar{\sigma}_{ba}\epsilon)^{\delta\dot{\gamma}} Y^{I_3}, \quad \Sigma_{\delta\gamma ba}^{I_3} = 4(\sigma_{ba}\epsilon)_{\delta\gamma} \bar{Y}^{I_3}. \quad (2.19)$$

$$F_{\dot{\beta},\alpha\dot{\alpha}}^{I_1} = -2\epsilon_{\dot{\beta}\dot{\alpha}} W_{\alpha}^{I_1}, \quad F_{\beta,\alpha\dot{\alpha}}^{I_1} = -2\epsilon_{\beta\alpha} \bar{W}_{\dot{\alpha}}^{I_1}. \quad (2.20)$$

Note that the Weyl weights Δ and chiral weights w of the irreducible superfields are as follows:

$$\begin{aligned} Y^{I_3} &: (\Delta, w) = (3, 2), \\ L^{I_2} &: (\Delta, w) = (2, 0), \\ W_{\alpha}^{I_1} &: (\Delta, w) = (3/2, 1), \\ \Psi^{I_0} &: (\Delta, w) = (0, 0). \end{aligned} \quad (2.21)$$

Here, Weyl and chiral weights of a superfield Φ are given by

$$D\Phi = \Delta\Phi, \quad A\Phi = iw\Phi. \quad (2.22)$$

Hereafter, we use the term ‘‘conformal weights’’ to refer to ‘‘Weyl and chiral weights’’.

The tensor hierarchy deforms the properties of the irreducible superfields such as the linear multiplet conditions for L^{I_2} and reality conditions for $W_{\underline{\alpha}}^{I_1}$:

$$\begin{aligned}
-\frac{1}{4}\bar{\nabla}^2 L^{I_2} &= -(q^{(2)} \cdot Y)^{I_2}, & -\frac{1}{4}\nabla^2 L^{I_2} &= -(q^{(2)} \cdot \bar{Y})^{I_2}, \\
\frac{1}{2i}(\nabla^\alpha W_\alpha^{I_1} - \bar{\nabla}_{\dot{\alpha}} \bar{W}^{I_1 \dot{\alpha}}) &= -(q^{(1)} \cdot L)^{I_1}, & & \\
-\frac{1}{4}\bar{\nabla}^2 \nabla_\alpha \Psi^{I_0} &= -(q^{(0)} \cdot W_\alpha)^{I_0}, & -\frac{1}{4}\nabla^2 \bar{\nabla}_{\dot{\alpha}} \Psi^{I_0} &= -(q^{(0)} \cdot \bar{W}_{\dot{\alpha}})^{I_0}.
\end{aligned} \tag{2.23}$$

Note that the derivatives ∇_A on the superfields Y^{I_3} , L^{I_2} , $W_{\underline{\alpha}}^{I_1}$, and Ψ^{I_0} are superconformally covariant and internally invariant derivatives because of the properties in eq. (2.17).

3 Prepotentials

In this section, we construct the prepotentials of the p -form gauge fields in the presence of the tensor hierarchy. The prepotentials and their gauge transformation laws are needed to construct CS actions. The prepotentials are obtained by solving the constraints on the field strengths in certain gauge-fixing conditions. The relations between the prepotentials and the irreducible superfields are also obtained by the relations of the gauge fields and field strengths in eq. (2.14). The gauge transformations of the prepotentials are determined by the gauge transformations which leave the gauge-fixing conditions invariant.

3.1 Gauge-fixing conditions for the p -form gauge fields

We solve the constraints on the field strengths. Since the constraints in table 2 are gauge covariant, we solve the constraints under the gauge-fixing conditions where some components of the gauge fields are gauged away by using the definitions of the field strengths

$$\begin{aligned}
F_{[p+1]}^{I_p} &= \frac{1}{p!} E^{A_1} \wedge \dots \wedge E^{A_p} \wedge E^B \nabla_B C_{A_p \dots A_1}^{I_p} \\
&+ \frac{1}{p!2!} E^{A_1} \wedge \dots \wedge E^{A_{p-2}} \wedge E^B \wedge E^C T_{CB}{}^{A_p} C_{A_p \dots A_1}^{I_p} \\
&+ \frac{1}{(p+1)!} E^{A_1} \wedge \dots \wedge E^{A_{p+1}} (q^{(p)} \cdot C_{A_{p+1} \dots A_1})^{I_p},
\end{aligned} \tag{3.1}$$

and the internal gauge transformation laws of the gauge fields

$$\begin{aligned}
\delta_T(\Lambda) C_{[p]}^{I_p} &= \frac{1}{(p-1)!} E^{A_1} \wedge \dots \wedge E^{A_{p-1}} \wedge E^B \nabla_B \Lambda_{A_{p-1} \dots A_1}^{I_p} \\
&+ \frac{1}{(p-1)!2!} E^{A_1} \wedge \dots \wedge E^{A_{p-2}} \wedge E^B \wedge E^C T_{CB}{}^{A_{p-1}} \Lambda_{A_{p-1} \dots A_1}^{I_p} \\
&+ \frac{1}{p!} E^{A_1} \wedge \dots \wedge E^{A_p} (q^{(p)} \cdot \Lambda_{A_p \dots A_1})^{I_p}.
\end{aligned} \tag{3.2}$$

Here, ∇_A are covariant with respect to only the superconformal symmetry, and $T_{CB}{}^A$ are the coefficients of torsion 2-form defined by

$$T^A = \frac{1}{2} E^B \wedge E^C T_{CB}{}^A = dE^A - E^C \wedge h^{B'} f_{B'C}{}^A. \tag{3.3}$$

form	conditions on the gauge fields
4-form	$U_{\underline{\delta}\underline{\gamma}\underline{\beta}A}^{I_4} = U_{\delta\dot{\gamma}ba}^{I_4} = 0$
3-form	$C_{\underline{\gamma}\underline{\beta}\underline{\alpha}}^{I_3} = C_{\gamma\beta a}^{I_3} = C_{\dot{\gamma}\dot{\beta}a}^{I_3} = 0, \quad C_{\gamma\beta a}^{I_3} = i(\sigma_a)_{\gamma\dot{\beta}} X^{I_3}$
2-form	$B_{\underline{\beta}\underline{\alpha}}^{I_2} = 0$
1-form	$A_{\underline{\alpha}}^{I_1} = i\nabla_{\alpha} V^{I_1}, \quad A_{\dot{\alpha}}^{I_1} = -i\bar{\nabla}_{\dot{\alpha}} V^{I_1}$

Table 3. The gauge-fixing conditions on the gauge fields. The gauge-fixing conditions are imposed in order of 4-, 3-, 2- and 1-form gauge fields.

The gauge-fixing conditions take the same form as the case of global SUSY without the tensor hierarchy [2] because of the following three reasons. First, the constraints on the following components of the torsion are the same as those of global SUSY (see ref. [21]):

$$T_{\gamma\beta}^A = 0, \quad T_{\dot{\gamma}\dot{\beta}}^A = 0, \quad T_{\gamma\dot{\beta}}^a = 2i(\sigma^a)_{\gamma\dot{\beta}}, \quad T_{\gamma\dot{\beta}}^{\alpha} = 0, \quad T_{\underline{\gamma}b}^A = 0, \quad T_{cb}^a = 0. \quad (3.4)$$

Second, as announced in section 1, the superconformally covariant spinor derivatives obey the same anti-commutation relations as those of global SUSY:

$$\{\nabla_{\alpha}, \nabla_{\beta}\} = 0, \quad \{\bar{\nabla}_{\dot{\alpha}}, \bar{\nabla}_{\dot{\beta}}\} = 0, \quad \{\nabla_{\alpha}, \bar{\nabla}_{\dot{\beta}}\} = -2i\nabla_{\alpha\dot{\beta}}. \quad (3.5)$$

Third, if we impose the gauge-fixing conditions and solve the constraints in order of 4-, 3-, 2-, and 1-form, the gauge-fixing conditions are not deformed from the case of the absence of the tensor hierarchy in ref. [2]. For example, we discuss the gauge-fixing conditions for $C_{\gamma\beta\alpha}^{I_3}$. Since the field strengths of the 4-form gauge fields are the same as the case of the absence of the tensor hierarchy, we fix some of the 4-form gauge fields, e.g., $U_{\delta\gamma\beta\alpha}^{I_4} = 0$. Under the gauge-fixing conditions $U_{\delta\gamma\beta\alpha}^{I_4} = 0$, the field strengths of the 3-form gauge fields $\Sigma_{\delta\gamma\beta\alpha}^{I_3}$ are written as $\Sigma_{\delta\gamma\beta\alpha}^{I_3} = \nabla_{\delta} C_{\gamma\beta\alpha}^{I_3} + \nabla_{\gamma} C_{\delta\beta\alpha}^{I_3} + \nabla_{\beta} C_{\gamma\delta\alpha}^{I_3} + \nabla_{\alpha} C_{\delta\gamma\beta}^{I_3}$. We find that the terms $(q^{(3)} \cdot U_{\delta\gamma\beta\alpha})^{I_3}$ do not appear in the field strengths $\Sigma_{\delta\gamma\beta\alpha}^{I_3}$ in this gauge. Thus, we impose the same gauge-fixing conditions as the case of global SUSY without the tensor hierarchy: $C_{\gamma\beta\alpha}^{I_3} = 0$, which are derived from the constraints $\Sigma_{\delta\gamma\beta\alpha}^{I_3} = 0$.

Therefore, the gauge-fixing conditions take the same ones as the case in which the tensor hierarchy does not exist in global SUSY. The explicit forms are summarized in table 3. In this table, X^{I_3} and V^{I_1} are real superfields, which are the prepotentials of the 3- and 1-form gauge fields, respectively.

3.2 Prepotentials: the solutions to the constraints

In this subsection, we show the prepotentials for the p -form gauge fields. Under the gauge-fixing conditions and the constraints on the field strengths, the gauge fields are expressed in terms of the prepotentials. We remark that the gauge-fixing conditions of p -form gauge fields in table 3 have the same spinor structure as the constraints on the field strengths of $(p-1)$ -form gauge fields in table 2. Thus, we solve the constraints by the same procedure as the Bianchi identities for the field strengths [35]. The conformal weights of the prepotentials

are also determined by using eq. (2.9). We exhibit the expressions of the gauge fields in terms of the prepotentials as follows.

The 4-form gauge fields. The solutions to the gauge-fixing conditions and constraints for the field strengths are the same as the case of the absence of the tensor hierarchy. The prepotentials of the 4-form gauge fields are given as the 2-spinor/2-vector components:

$$U^{I_4 \dot{\delta}\dot{\gamma}}{}_{ba} = 4(\bar{\sigma}_{ba}\epsilon)^{\dot{\delta}\dot{\gamma}}\Gamma^{I_4}, \quad U_{\dot{\delta}\dot{\gamma}ba}^{I_4} = 4(\sigma_{ba}\epsilon)_{\dot{\delta}\dot{\gamma}}\bar{\Gamma}^{I_4}. \quad (3.6)$$

The prepotentials Γ^{I_4} are primary superfields with conformal weights $(\Delta, w) = (3, 2)$, which are derived from the superconformal transformation laws of $U_{\dot{\delta}\dot{\gamma}ba}^{I_4}$ in eq. (2.9). The prepotential Γ^{I_4} and $\bar{\Gamma}^{I_4}$ are chiral and anti-chiral superfields, respectively:

$$\bar{\nabla}_{\dot{\alpha}}\Gamma^{I_4} = 0, \quad \nabla_{\alpha}\bar{\Gamma}^{I_4} = 0. \quad (3.7)$$

The other components the 4-form gauge fields are expressed in terms of the prepotentials

$$U^{I_4 \dot{\delta}}{}_{cba} = +\frac{1}{2}(\bar{\sigma}^d)^{\dot{\delta}\delta}\epsilon_{dcba}\nabla_{\delta}\Gamma^{I_4}, \quad U_{\dot{\delta}cba}^{I_4} = -\frac{1}{2}(\sigma^d)_{\delta\dot{\delta}}\epsilon_{dcba}\bar{\nabla}^{\dot{\delta}}\bar{\Gamma}^{I_4}, \quad (3.8)$$

$$U_{dcb\dot{a}}^{I_4} = \frac{i}{8}\epsilon_{dcb\dot{a}}(\nabla^2\Gamma^{I_4} - \bar{\nabla}^2\bar{\Gamma}^{I_4}). \quad (3.9)$$

The 3-form gauge fields. We find the prepotentials of the 3-form gauge fields X^{I_3} in the 2-spinor/1-vector component, where X^{I_3} are real primary superfields with conformal weights $(\Delta, w) = (2, 0)$. The derivatives of the prepotentials give the other components of the gauge fields as

$$C_{\gamma ba}^{I_3} = (\sigma_{ba})_{\gamma}{}^{\delta}\nabla_{\delta}X^{I_3}, \quad C^{I_3 \dot{\gamma}}{}_{ba} = (\bar{\sigma}_{ba})^{\dot{\gamma}}{}_{\dot{\delta}}\bar{\nabla}^{\dot{\delta}}X^{I_3}, \quad (3.10)$$

$$C_{cba}^{I_3} = \frac{1}{8}\epsilon_{cbad}(\bar{\sigma}^d)^{\dot{\delta}\delta}[\nabla_{\delta}, \bar{\nabla}_{\dot{\delta}}]X^{I_3}. \quad (3.11)$$

The 2-form gauge fields. The prepotentials of the 2-form gauge fields are primary superfields $\Sigma_{\alpha}^{I_2}$ and their conjugates $\bar{\Sigma}_{\dot{\alpha}}^{I_2}$. The prepotentials are found in the spinor/vector components:

$$B_{\beta, \alpha\dot{\alpha}}^{I_2} = -2\epsilon_{\beta\alpha}\bar{\Sigma}_{\dot{\alpha}}^{I_2}, \quad B_{\dot{\beta}, \alpha\dot{\alpha}}^{I_2} = -2\epsilon_{\dot{\beta}\dot{\alpha}}\Sigma_{\alpha}^{I_2}. \quad (3.12)$$

Here, $\Sigma_{\alpha}^{I_2}$ are primary superfields with conformal weights $(\Delta, w) = (3/2, 1)$. The prepotential $\Sigma_{\alpha}^{I_2}$ and $\bar{\Sigma}_{\dot{\alpha}}^{I_2}$ are chiral and anti-chiral superfields, respectively:

$$\bar{\nabla}_{\dot{\beta}}\Sigma_{\alpha}^{I_2} = 0, \quad \nabla_{\beta}\bar{\Sigma}_{\dot{\alpha}}^{I_2} = 0. \quad (3.13)$$

The 2-vector components are as follows:

$$B_{ba}^{I_2} = \frac{1}{2i} \left((\sigma_{ba})_{\beta}{}^{\alpha}\nabla^{\beta}\Sigma_{\alpha}^{I_2} - (\bar{\sigma}_{ba})^{\dot{\beta}}{}_{\dot{\alpha}}\bar{\nabla}_{\dot{\beta}}\bar{\Sigma}^{I_2\dot{\alpha}} \right). \quad (3.14)$$

The 1-form gauge fields. As in ordinary super QED case, the spinor components of 1-form gauge fields are given by real primary superfields V^{I_1} in table 3. The conformal weights of V^{I_1} are $(\Delta, w) = (0, 0)$. The vector components are expressed by

$$A_{\alpha\dot{\alpha}}^{I_1} = \frac{1}{2}[\nabla_{\alpha}, \bar{\nabla}_{\dot{\alpha}}]V^{I_1}. \quad (3.15)$$

We assume that V^{I_1} are primary superfields: $K_A V^{I_1} = 0$. This assumption and conformal weights of V^{I_1} are consistent with the K_A -invariances of $A_{\alpha}^{I_1}$ [29].

form	prepotentials and irreducible superfields
3-form	$Y^{I_3} = -\frac{1}{4}\bar{\nabla}^2 X^{I_3} - (q^{(3)} \cdot \Gamma)^{I_3}, \quad \bar{Y}^{I_3} = -\frac{1}{4}\nabla^2 X^{I_3} - (q^{(3)} \cdot \bar{\Gamma})^{I_3}$
2-form	$L^{I_2} = \frac{1}{2i}(\nabla^\alpha \Sigma_\alpha^{I_2} - \bar{\nabla}_{\dot{\alpha}} \bar{\Sigma}^{I_2 \dot{\alpha}}) - (q^{(2)} \cdot X)^{I_2}$
1-form	$W_\alpha^{I_1} = -\frac{1}{4}\bar{\nabla}^2 \nabla_\alpha V^{I_1} - (q^{(1)} \cdot \Sigma_\alpha)^{I_1}, \quad \bar{W}_{\dot{\alpha}}^{I_1} = -\frac{1}{4}\nabla^2 \bar{\nabla}_{\dot{\alpha}} V^{I_1} - (q^{(1)} \cdot \bar{\Sigma}_{\dot{\alpha}})^{I_1}$
0-form	$\Psi^{I_0} = \frac{1}{2i}(\Phi^{I_0} - \bar{\Phi}^{I_0}) - (q^{(0)} \cdot V)^{I_0}$
(-1)-form	$J^{I_{-1}} = -(q^{(-1)} \cdot \Phi)^{I_{-1}}$

Table 4. The relations between the prepotentials and the irreducible superfields.

The 0-form gauge fields. The constraints on the field strengths of the 0-form are satisfied if the gauge fields are real parts of chiral superfields Φ^{I_0} , which are the prepotentials of 0-form gauge fields:

$$f^{I_0} = \frac{1}{2}(\Phi^{I_0} + \bar{\Phi}^{I_0}). \tag{3.16}$$

Here, the conformal weights of Φ^{I_0} are $(\Delta, w) = (0, 0)$, and Φ^{I_0} are assumed to be primary superfields.

The relations between the prepotentials and the irreducible superfields. We then find the relations between the prepotentials and the irreducible superfields. The relations are found as follows. On the one hand, the irreducible superfields are given by the components of the field strengths $\Sigma_{\delta\gamma ba}^{I_3}, \Sigma_{\delta\dot{\gamma} ba}^{I_3}, H_{\gamma\beta a}^{I_2}, F_{\beta a}^{I_1}$, and $g_a^{I_0}$. On the other hand, the field strengths are expressed by the derivatives of the gauge fields in eq. (2.14), which are now written in terms of the prepotentials. In addition, the field strengths of (-1)-form gauge fields $\omega^{I_{-1}}$ are given by the 0-form gauge fields f^{I_0} as in table 1: $\omega^{I_{-1}} = -(q^{(-1)} \cdot f)^{I_{-1}}$. Since the 0-form gauge fields are expressed by the prepotential Φ^{I_0} , the field strengths $\omega^{I_{-1}}$ are now given by the real parts of chiral superfields $J^{I_{-1}} = -(q^{(-1)} \cdot \Phi)^{I_{-1}}$:

$$\omega^{I_{-1}} = \frac{1}{2}(J^{I_{-1}} + \bar{J}^{I_{-1}}). \tag{3.17}$$

Thus, we find the relations by using the definitions of the field strengths in terms of gauge fields (2.14), the definitions of the superfields in eqs. (2.19), (2.20) and table 2.

The results are summarized in table 4. Note that the irreducible superfields for p -form gauge fields are expressed by the prepotentials of p - and $(p + 1)$ -form gauge fields due to the tensor hierarchy.

3.3 The gauge transformation laws of the prepotentials

In this subsection, we show internal transformation laws of the prepotentials. The transformation laws are important when we construct CS actions. We have solved the gauge fields in terms of the prepotentials under the set of the gauge-fixing conditions. Although

it seems that the gauge parameters are exhausted to fix the gauge fields, there are remaining gauge parameters which preserve the gauge-fixing conditions in table 3 invariant. The remaining gauge transformation laws are determined by the conditions for the gauge fields which are gauged away in table 3:

$$0 = \delta_T(\Lambda)C_{[p]}^{I_p} = d\Lambda_{[p-1]}^{I_p} + (q^{(p)} \cdot \Lambda_{[p]})^{I_p}. \quad (3.18)$$

We denote the remaining parameters as $\Theta = (\Theta^{I_1}, \Theta^{I_2}, \Theta_{\underline{\alpha}}^{I_3}, \Theta^{I_4})$. We determine the properties of Θ 's and the gauge transformation laws of the prepotentials as follows.

The 4-form gauge fields. The gauge parameters are determined by the conditions so that the gauge-fixing conditions in table 3 are invariant:

$$\delta_T(\Lambda)U_{\underline{\delta}\underline{\gamma}\underline{\beta}A}^{I_4} = 0, \quad \delta_T(\Lambda)U_{\delta\dot{\gamma}ba}^{I_4} = 0. \quad (3.19)$$

The gauge transformations which preserve the gauge-fixing conditions are given by

$$\Lambda_{\underline{\gamma}\underline{\beta}\underline{\alpha}}^{I_4} = 0, \quad \Lambda_{\gamma\beta a}^{I_4} = 0, \quad \Lambda_{\dot{\gamma}\dot{\beta}a}^{I_4} = 0, \quad \Lambda_{\gamma\dot{\beta}a}^{I_4} = i(\sigma_a)_{\gamma\dot{\beta}}\Theta^{I_4}. \quad (3.20)$$

Here, Θ^{I_4} are real superfields. The prepotentials Γ^{I_4} and $\bar{\Gamma}^{I_4}$ are transformed by Θ^{I_4} as

$$\delta_T(\Lambda^{I_1}, \Lambda^{I_2}, \Lambda^{I_3}, \Theta^{I_4})\Gamma^{I_4} = -\frac{1}{4}\bar{\nabla}^2\Theta^{I_4}, \quad \delta_T(\Lambda^{I_1}, \Lambda^{I_2}, \Lambda^{I_3}, \Theta^{I_4})\bar{\Gamma}^{I_4} = -\frac{1}{4}\nabla^2\Theta^{I_4}, \quad (3.21)$$

which are determined by the gauge transformation laws of $U_{\delta\dot{\gamma}ba}^{I_4}$ and $U_{\underline{\delta}\underline{\gamma}\underline{\beta}A}^{I_4}$, respectively.

We can impose Wess-Zumino (WZ) gauge for the prepotentials Γ^{I_4} by using Θ^{I_4} as follows:

$$\Gamma^{I_4}| = 0, \quad \nabla_{\alpha}\Gamma^{I_4}| = 0, \quad \bar{\nabla}_{\dot{\alpha}}\bar{\Gamma}^{I_4}| = 0, \quad (\nabla^2\Gamma^{I_4} + \bar{\nabla}^2\bar{\Gamma}^{I_4})| = 0. \quad (3.22)$$

Here, the symbol of “|” means $\theta = \bar{\theta} = 0$ projection.

The 3-form gauge fields. We determine the remaining gauge parameters so that the gauge-fixing conditions in table 3 are invariant as follows:

$$\begin{aligned} \delta_T(\Lambda^{I_1}, \Lambda^{I_2}, \Lambda^{I_3}, \Theta^{I_4})C_{\underline{\gamma}\underline{\beta}\underline{\alpha}}^{I_3} &= 0, \\ \delta_T(\Lambda^{I_1}, \Lambda^{I_2}, \Lambda^{I_3}, \Theta^{I_4})C_{\gamma\beta a}^{I_3} &= 0, \quad \delta_T(\Lambda^{I_1}, \Lambda^{I_2}, \Lambda^{I_3}, \Theta^{I_4})C_{\dot{\gamma}\dot{\beta}a}^{I_3} = 0. \end{aligned} \quad (3.23)$$

The invariances are preserved by the following conditions for the gauge parameters:

$$\Lambda_{\underline{\beta}\underline{\alpha}}^{I_3} = 0. \quad (3.24)$$

Note that the gauge parameters Θ^{I_4} do not change the gauge-fixing conditions in eq. (3.23) under the conditions for the gauge parameters in eq. (3.20) even if the tensor hierarchy exists. Solving the constraints on the parameters, we obtain that the remaining gauge parameters are

$$\Lambda_{\dot{\beta},\alpha\dot{\alpha}}^{I_3} = -2\epsilon_{\dot{\beta}\dot{\alpha}}\Theta_{\alpha}^{I_3}, \quad \Lambda_{\beta,\alpha\dot{\alpha}}^{I_3} = -2\epsilon_{\beta\alpha}\bar{\Theta}_{\dot{\alpha}}^{I_3}. \quad (3.25)$$

Here, $\Theta_\alpha^{I_3}$ and $\bar{\Theta}_{\dot{\alpha}}^{I_3}$ are chiral and anti-chiral superfields, respectively:

$$\bar{\nabla}_{\dot{\beta}}\Theta_\alpha^{I_3} = 0, \quad \nabla_\beta\bar{\Theta}_{\dot{\alpha}}^{I_3} = 0. \quad (3.26)$$

The gauge transformation laws of the prepotential X^{I_3} are determined by those of $C_{\gamma\beta a}^{I_3}$:

$$\delta_T(\Lambda^{I_1}, \Lambda^{I_2}, \Theta_{\underline{\beta}}^{I_3}, \Theta^{I_4})X^{I_3} = \frac{1}{2i}(\nabla^\alpha\Theta_\alpha^{I_3} - \bar{\nabla}_{\dot{\alpha}}\bar{\Theta}^{I_3\dot{\alpha}}) + (q^{(3)} \cdot \Theta)^{I_3}. \quad (3.27)$$

We find that X^{I_3} are also transformed by the remaining gauge parameters Θ^{I_4} due to the tensor hierarchy.

The WZ gauge conditions for the prepotentials X^{I_3} can be imposed by the parameters $\Theta_\alpha^{I_3}$ as follows:

$$X^{I_3}| = 0, \quad \nabla_\alpha X^{I_3}| = 0, \quad \bar{\nabla}_{\dot{\alpha}} X^{I_3}| = 0. \quad (3.28)$$

Note that the WZ conditions in eq. (3.28) are imposed under the WZ gauge conditions for the prepotentials of 4-form gauge fields in eq. (3.22).

The 2-form gauge fields. The remaining parameters are found by the conditions so that the gauge-fixing conditions in table 3 are invariant:

$$\delta_T(\Lambda^{I_1}, \Lambda^{I_2}, \Theta_{\underline{\gamma}}^{I_3}, \Theta^{I_4})B_{\underline{\beta}\alpha}^{I_2} = 0. \quad (3.29)$$

By using the gauge invariances, we determine the remaining parameters Θ^{I_2} as

$$\Lambda_\alpha^{I_2} = i\nabla_\alpha\Theta^{I_2}, \quad \Lambda_{\dot{\alpha}}^{I_2} = -i\bar{\nabla}_{\dot{\alpha}}\Theta^{I_2}, \quad \Lambda_{\alpha\dot{\alpha}}^{I_2} = \frac{1}{2}[\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}]\Theta^{I_2}, \quad (3.30)$$

where Θ^{I_2} are real superfields. Again, $\Theta_{\underline{\alpha}}^{I_3}$ do not affect the gauge-fixing conditions in eq. (3.29) in the presence of the tensor hierarchy. The gauge transformation laws of the prepotential $\Sigma_{\underline{\alpha}}^{I_2}$ are given by

$$\begin{aligned} \delta_T(\Lambda^{I_1}, \Theta^{I_2}, \Theta_{\underline{\beta}}^{I_3}, \Theta^{I_4})\Sigma_\alpha^{I_2} &= -\frac{1}{4}\bar{\nabla}^2\nabla_\alpha\Theta^{I_2} + (q^{(2)} \cdot \Theta_\alpha)^{I_2}, \\ \delta_T(\Lambda^{I_1}, \Theta^{I_2}, \Theta_{\underline{\beta}}^{I_3}, \Theta^{I_4})\bar{\Sigma}_{\dot{\alpha}}^{I_2} &= -\frac{1}{4}\nabla^2\bar{\nabla}_{\dot{\alpha}}\Theta^{I_2} + (q^{(2)} \cdot \bar{\Theta}_{\dot{\alpha}})^{I_2}. \end{aligned} \quad (3.31)$$

Under the conditions in eqs. (3.22) and (3.28), we can go to the WZ gauge conditions for $\Sigma_\alpha^{I_2}$:

$$\Sigma_\alpha^{I_2}| = 0, \quad \bar{\Sigma}_{\dot{\alpha}}^{I_2}| = 0, \quad (\nabla^\alpha\Sigma_\alpha^{I_2} + \bar{\nabla}_{\dot{\alpha}}\bar{\Sigma}^{I_2\dot{\alpha}})| = 0. \quad (3.32)$$

The 1-form gauge fields. The gauge transformations for the 1-form gauge fields are the same as in ordinary super QED case except the shifts due to the tensor hierarchy. We find that the gauge transformations which leave the gauge-fixing conditions in table 3 invariant are given by

$$\Lambda^{I_1} = \frac{1}{2}(\Theta^{I_1} + \bar{\Theta}^{I_1}), \quad (3.33)$$

Here, Θ^{I_1} and $\bar{\Theta}^{I_1}$ are chiral and anti-chiral superfields, respectively:

$$\bar{\nabla}_{\dot{\alpha}}\Theta^{I_1} = 0, \quad \nabla_\alpha\bar{\Theta}^{I_1} = 0. \quad (3.34)$$

The gauge transformations of the 1-form prepotentials are given by the imaginary parts of Θ^{I_1} and the shifts by the gauge parameters of 2-form gauge fields Θ^{I_2} :

$$\delta_T(\Theta^{I_1}, \Theta^{I_2}, \Theta_{\alpha}^{I_3}, \Theta^{I_4})V^{I_1} = \frac{1}{2i}(\Theta^{I_1} - \bar{\Theta}^{I_1}) + (q^{(1)} \cdot \Theta)^{I_1}. \quad (3.35)$$

We can impose the WZ gauge conditions for the prepotentials V^{I_1} under the conditions in eqs. (3.22), (3.28) and (3.32):

$$V^{I_1}| = 0, \quad \nabla_{\alpha}V^{I_1}| = 0, \quad \bar{\nabla}_{\dot{\alpha}}V^{I_1}| = 0, \quad \nabla^2V^{I_1}| = 0, \quad \bar{\nabla}^2V^{I_1}| = 0. \quad (3.36)$$

The 0-form gauge fields. The gauge transformation laws of the prepotentials of 0-form are given by the chiral shifts by the gauge parameters Θ^{I_1} :

$$\delta_T(\Theta^{I_1}, \Theta^{I_2}, \Theta_{\alpha}^{I_3}, \Theta^{I_4})\Phi^{I_0} = (q^{(0)} \cdot \Theta)^{I_0}. \quad (3.37)$$

Again, the shifts come from the tensor hierarchy.

4 Chern-Simons actions

In this section, we construct CS actions in the conformal superspace. The CS actions of the tensor hierarchy is related to anomaly cancellations in low energy effective theories. The construction of the CS actions in the conformal superspace is quite similar to the global SUSY case [16]. CS actions are constructed by the combinations of the prepotentials and irreducible superfields $Y^{I_3}, L^{I_2}, W_{\alpha}^{I_1}, \Psi^{I_0}, J^{I_{-1}}$ and their conjugates.

To construct the CS actions, we use the descent formalism. This formalism systematically gives the CS actions from the internal transformation laws of the prepotentials. We show that the descent formalism that was given in ref. [16] is straightforwardly extended in the case of the conformal superspace.

Descent formalism in global SUSY. We briefly review the descent formalism in global SUSY in ref. [16]. The descent formalism in global SUSY is given by the combinations of the prepotentials and irreducible field strengths as

$$S_{\text{CS}} = \int d^4x d^4\theta (V^{I_1}c_{I_1} - X^{I_3}c_{I_3}) + \text{Re} \left(i \int d^4x d^2\theta (\Phi^{I_0}c_{I_0} + \Sigma^{I_2\alpha}c_{I_2\alpha} + \Gamma^{I_4}c_{I_4}) \right). \quad (4.1)$$

Here, c 's are polynomials of the irreducible superfields $Y^{I_3}, L^{I_2}, W_{\alpha}^{I_1}, \Psi^{I_0}, J^{I_{-1}}$ and their conjugates. The superfields c_{I_1} and c_{I_3} are real superfields, and $c_{I_0}, c_{I_2\alpha}$, and c_{I_4} are chiral superfields. The internal gauge invariance requires that c 's are related each other as

$$\begin{aligned} -\frac{1}{4}\bar{D}^2c_{I_1} &= (q^{(0)})_{I_1}^{I_0}c_{I_0}, \\ \frac{1}{2i}(D^{\alpha}c_{I_2\alpha} - \bar{D}_{\dot{\alpha}}\bar{c}_{I_2}^{\dot{\alpha}}) &= -(q^{(1)})_{I_2}^{I_1}c_{I_1}, \\ -\frac{1}{4}\bar{D}^2D_{\alpha}c_{I_3} &= (q^{(2)})_{I_3}^{I_2}c_{I_2\alpha}, \\ \frac{1}{2i}(c_{I_4} - \bar{c}_{I_4}) &= -(q^{(3)})_{I_4}^{I_3}c_{I_3}. \end{aligned} \quad (4.2)$$

Here, the derivatives D_α and $\bar{D}_{\dot{\alpha}}$ are the covariant spinor derivatives in global SUSY: $D_\alpha = \partial_\alpha + i(\sigma^a)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_a$ and $\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\partial_a$. The internal gauge invariances are obtained by the relation between the superspace integrations:

$$\int d^4x d^4\theta V = -\frac{1}{4} \int d^4x d^2\theta \bar{D}^2 V = -\frac{1}{4} \int d^4x d^2\bar{\theta} D^2 V, \quad (4.3)$$

where V is a real superfield.

Descent formalism in the conformal superspace. We now discuss the descent formalism in the conformal superspace. The descent formalism in the conformal superspace is given by a natural extension of global SUSY case as

$$S_{\text{CS}} = \int d^4x d^4\theta E (V^{I_1} c_{I_1} - X^{I_3} c_{I_3}) + \text{Re} \left(i \int d^4x d^2\theta \mathcal{E} (\Phi^{I_0} c_{I_0} + \Sigma^{I_2\alpha} c_{I_2\alpha} + \Gamma^{I_4} c_{I_4}) \right), \quad (4.4)$$

where E and \mathcal{E} are the density of the whole superspace and chiral subspace, respectively. The integrations $\int d^4x d^4\theta E$ and $\int d^4x d^2\theta \mathcal{E}$ are called D- and F-term integration, respectively [21]. The superfields c 's are polynomials of the irreducible superfields Y^{I_3} , L^{I_2} , $W_{\dot{\alpha}}^{I_1}$, Ψ^{I_0} , J^{I-1} and their conjugates. Again, c_{I_1} and c_{I_3} are real superfields, and c_{I_0} , $c_{I_2\alpha}$, and c_{I_4} are chiral superfields. The c 's have two type of conditions. One is the condition that is required by the superconformal invariance. The conditions are that all the c 's are primary superfields, and the conformal weights of them are as follows:

$$\begin{aligned} c_{I_0} &: (\Delta, w) = (3, 2), \\ c_{I_1} &: (\Delta, w) = (2, 0), \\ c_{I_2\alpha} &: (\Delta, w) = (3/2, 1), \\ c_{I_3} &: (\Delta, w) = (0, 0), \\ c_{I_4} &: (\Delta, w) = (0, 0). \end{aligned} \quad (4.5)$$

The other is the condition that is required by the internal gauge invariance of the tensor hierarchy as in the global SUSY case. The internal gauge invariance requires the same conditions as those of ref. [16]:

$$\begin{aligned} -\frac{1}{4} \bar{\nabla}^2 c_{I_1} &= (q^{(0)})_{I_1}^{I_0} c_{I_0}, \\ \frac{1}{2i} (\nabla^\alpha c_{I_2\alpha} - \bar{\nabla}_{\dot{\alpha}} \bar{c}_{I_2}^{\dot{\alpha}}) &= -(q^{(1)})_{I_2}^{I_1} c_{I_1}, \\ -\frac{1}{4} \bar{\nabla}^2 \nabla_\alpha c_{I_3} &= (q^{(2)})_{I_3}^{I_2} c_{I_2\alpha}, \\ \frac{1}{2i} (c_{I_4} - \bar{c}_{I_4}) &= -(q^{(3)})_{I_4}^{I_3} c_{I_3}. \end{aligned} \quad (4.6)$$

The internal gauge invariances are obtained by superspace partial integrations of the integrands. In the conformal superspace, the relation between F-term and D-term actions is

$$\int d^4x d^4\theta EV = -\frac{1}{4} \int d^4x d^2\theta \mathcal{E} \bar{\nabla}^2 V = -\frac{1}{4} \int d^4x d^2\bar{\theta} \bar{\mathcal{E}} \nabla^2 V. \quad (4.7)$$

Here, V is a primary scalar superfield with the conformal weight $(\Delta, w) = (2, 0)$ [21]. Although the derivation of the relation between D- and F-term integrations is a bit nontrivial (see ref. [21]), the relation is obtained by replacing $d^4x d^4\theta$, $d^4x d^2\theta$, D_α and $\bar{D}_{\dot{\alpha}}$ in eq. (4.3) with $d^4x d^4\theta E$, $d^4x d^2\theta \mathcal{E}$, ∇_α and $\bar{\nabla}_{\dot{\alpha}}$, respectively. This is a strong point of the conformal superspace approach: the relations of the integrals are quite similar to the case of the global SUSY.

We can go to Poincaré SUGRA by imposing the superconformal gauge-fixing [21, 29]. Because the CS actions are superconformally invariant without a compensator, the CS actions are not changed by the superconformal gauge-fixing conditions.

We finally show an example of the CS actions. We consider an action which is a natural extension of the action proposed in ref. [16]:

$$\begin{aligned}
 S_{\text{CS}} := & \int d^4x d^4\theta E (\alpha_{I_1 I_2} V^{I_1} L^{I_2} - \alpha_{I_3 I_0} X^{I_3} \Psi^{I_0}) \\
 & + \text{Re} \left(i \int d^4x d^2\theta \mathcal{E} (\alpha_{I_0 I_3} \Phi^{I_0} Y^{I_3} + \alpha_{I_2 I_1} \Sigma^{I_2 \alpha} W_\alpha^{I_1} + \alpha_{I_4 I_{-1}} \Gamma^{I_4} J^{I_{-1}}) \right).
 \end{aligned} \tag{4.8}$$

Here, α 's are constant parameters. This action is obtained by choosing c 's as follows:

$$c_{I_0} = \alpha_{I_0 I_3} Y^{I_3}, \quad c_{I_1} = \alpha_{I_1 I_2} L^{I_2}, \quad c_{I_2 \alpha} = \alpha_{I_2 I_1} W_\alpha^{I_1}, \quad c_{I_3} = \alpha_{I_3 I_0} \Psi^{I_0}, \quad c_{I_4} = \alpha_{I_4 I_{-1}} J^{I_{-1}}. \tag{4.9}$$

This action satisfies the conformal weight conditions in eq. (4.5) by using the conformal weights of the irreducible superfields in eq. (2.21) and those of Φ^{I_0} (for $J^{I_{-1}}$). The internal invariance in eq. (4.6) requires the same conditions as the case of global SUSY [16]:

$$\begin{aligned}
 \alpha_{I_1 I_2} (q^{(2)})_{I_3}^{I_2} &= -\alpha_{I_0 I_3} (q^{(0)})_{I_1}^{I_0}, \\
 \alpha_{I_2 I_1} (q^{(1)})_{J_2}^{I_1} &= \alpha_{I_1 J_2} (q^{(1)})_{I_2}^{I_1}, \\
 \alpha_{I_3 I_0} (q^{(0)})_{I_1}^{I_0} &= -\alpha_{I_2 I_1} (q^{(2)})_{I_3}^{I_2}, \\
 \alpha_{I_4 I_{-1}} (q^{(-1)})_{I_0}^{I_{-1}} &= \alpha_{I_3 I_0} (q^{(3)})_{I_4}^{I_3}.
 \end{aligned} \tag{4.10}$$

5 Conclusion

In this paper, we have constructed the CS actions of Abelian tensor hierarchy in 4D $\mathcal{N} = 1$ conformal superspace. In section 3, the constraints on the field strengths have been solved in terms of the prepotentials with the gauge-fixing conditions. The explicit forms are given in eqs. (3.6), (3.12), (3.16) and table 3. The conformal weights have also been determined by the conformal weights of the vielbein. We have obtained the relations between the prepotentials and irreducible superfields in table 4. We have also obtained the gauge transformation laws of the prepotentials in eqs. (3.21), (3.27), (3.31), (3.35) and (3.37). The CS actions have been constructed in the conformal superspace by using prepotentials in section 4. The conformal weights of the c 's are determined in eq. (4.5). We have shown that the descent formalism is mostly the same as the case of global SUSY as in eq. (4.6). Finally, the examples of CS couplings are exhibited in eq. (4.8). These examples are natural extensions of global SUSY case.

The CS actions in 4D $\mathcal{N} = 1$ SUGRA, in particular the action in eq. (4.8), would be useful to discuss phenomenology such as inflation of the early universe [36, 37]. It would be interesting to embed the approach which was proposed in ref. [38] into the conformal superspace.

Acknowledgments

The author thanks Shuntaro Aoki, Tetsutaro Higaki and Yusuke Yamada for useful discussions. This work is supported by Research Fellowships of Japan Society for the Promotion of Science for Young Scientists Grant Number 16J03226.

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