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On the shape dependence of Entanglement Entropy

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Abstract: We study the shape dependence of entanglement entropy (EE) by deforming symmetric entangling surfaces. We show that entangling surfaces with a rotational or translational symmetry extremize (locally) the EE with respect to shape deformations that break some of the symmetry (i.e. the 1st order correction vanishes). This result applies to EE and Renyi entropy for any QFT in any dimension. Using Solodukhin's formula in $4d$ and holography in any d, we calculate the 2nd order correction to the universal EE for CFTs and simple symmetric entangling surfaces. For several entangling surfaces we find that the 2nd order correction is positive for any perturbation, and thus the corresponding symmetric entangling surface is a local minimum. Some of the results are extended to free massive fields and to 4d Renyi entropy.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence

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Contents

1 Introduction

Entanglement entropy is a measure of the quantum correlations of a system. It has a very wide range of applications from condensed matter physics to quantum field theory [1–33] and AdS/CFT [35–62]. The EE in QFT is generally hard to calculate, and most computations have been done in simple setups, such as: free fields, CFTs, and symmetric entangling surfaces (e.g. spheres and planes). Thus there is a need to obtain analytical results for interacting theories, non-CFTs, and less symmetrical entangling surfaces. This work aims to make a step in this direction by studying the shape dependence of EE. Previous works on the shape dependence of EE include [63–81].

The divergent structure of entanglement entropy for a CFT in d-dimensions is:

$$S = c_{d-2}\frac{R^{d-2}}{\delta^{d-2}} + c_{d-4}\frac{R^{d-4}}{\delta^{d-4}} + \ldots + \left\{ \begin{array}{ll} c_1\frac{R}{\delta} + (-1)^{\frac{d-1}{2}} S^{(\mathrm{univ})}\,, & d = \mathrm{odd} \\ c_2\frac{R^2}{\delta^2} + (-1)^{\frac{d-2}{2}} S^{(\mathrm{univ})}\log(\frac{R}{\delta})\,, & d = \mathrm{even} \end{array} \right\} \tag{1.1}$$

where R is the scale of the entangling region, and δ is the UV cutoff. The leading divergence is the area law, and all of the power law divergences are non-universal. We will be interested in the universal term $S^{(\mathrm{univ})}$, which in $d =$ even is the coefficient of the log divergence and in $d =$ odd it is the finite term.

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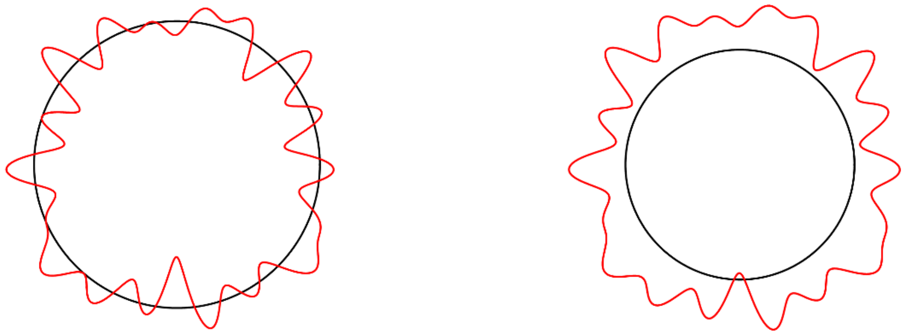

Figure 1. Illustration of a perturbed circle $r(\phi) = 1 + \epsilon \sum_n a_n \cos(n\phi)$. *Left:* a perturbation without a zero mode $a_0 = 0$. *Right:* a perturbation with a zero mode $a_0 = 0.3$.

Consider a QFT parametrized by coordinates (t, y_i, r), where $i = 1 \ldots, d-2$. For instance take r to be the radial coordinate in spherical coordinates, and y_i to be angles parameterizing the entangling surface. We will always work in a constant time slice $t = 0$. Consider a codimension-2 entangling surface defined by:

$$r(y_i) = r_0(y_i) \tag{1.2}$$

where $r_0(y_i)$ is some given function of y_i. Thus we choose r to be the dependent coordinate, and y_i as the independent coordinates. We denote the entanglement entropy corresponding to the entangling surface $r_0(y_i)$ as S_0. Now we slightly perturb the entangling surface:

$$r(y_i) = r_0(y_i) + \epsilon f(y_i) \tag{1.3}$$

where ϵ is a small parameter, and $f(y_i)$ is some arbitrary perturbation function.[1]

The entanglement entropy will change as a result of the perturbation of the entangling surface, and it can generally be written as an expansion in ϵ:

$$S = S_0 + S_1\epsilon + S_2\epsilon^2 + \ldots \tag{1.4}$$

The above procedure was carried out in [63, 64] for the case of a perturbed sphere for a CFT in d dimensions. They start with a sphere entangling surface in a flat space-time background, and perturb the sphere as follows:

$$r(\Omega_{d-2}) = R\left[1 + \epsilon \sum_{l,m_1,\ldots m_{d-3}} a_{l,m_1,\ldots m_{d-3}} Y_{l,m_1,\ldots m_{d-3}}(\Omega_{d-2})\right] \tag{1.5}$$

where R is the sphere radius, the a's are constants, and the Y's are (real) hyper-spherical harmonics. Then they calculate the resulting change in the universal term of the EE. They find that the the change in the universal EE vanishes at 1^{st} order, namely

$$S_1^{(\text{univ})} = 0 \tag{1.6}$$

[1] As an example, a perturbed circle in $d = 3$ is shown in figure 1. In this case, (1.3) is given by: $r(\phi) = R[1 + \epsilon \sum_n a_n \cos(n\phi)]$, where we Fourier expanded the perturbation f. R is the radius of the circle, ϕ is the angle in polar coordinates, and a_n are the Fourier coefficients.

Thus, for a CFT the sphere is a local[2] extremum with respect to perturbations of the entangling surface. Additionally, for holographic CFTs [64] uses the Ryu-Takayanagi formula [35, 36] (more specifically, its generalization to higher derivative gravity [83–86]) to calculate the 2nd order correction $S_2^{(\mathrm{univ})}$:

$$S_2^{(\mathrm{univ})} = C_T \frac{\pi^{\frac{d+2}{2}}(d-1)}{2^{d-2}\Gamma(d+2)\Gamma(\frac{d}{2})} \sum_{l,m_1\ldots m_{d-1}} a^2_{l,m_1\ldots m_{d-1}} \prod_{k=1}^{d}(l+k-2) \times \left\{ \begin{array}{l} \frac{\pi}{2}\,,\ d = \mathrm{odd} \\ 1\,,\ d = \mathrm{even} \end{array} \right\} \tag{1.7}$$

where C_T is the (positive) central charge appearing in the 2-point function: $\langle TT\rangle \sim \frac{C_T}{x^{2d}}$. A priori, $S_2^{(\mathrm{univ})}$ could have depended on the three parameters C_t, t_2, and t_4 of the 3-point function $\langle TTT\rangle$, but it turns out that it depends just on C_T.

$S_2^{(\mathrm{univ})}$ is clearly positive, implying that the sphere is a local minimum for holographic CFTs. (1.7) was compared to $S_2^{(\mathrm{univ})}$ obtained from Solodukhin's formula in 4d CFTs, and precise agreement was found.

In this work we will generalize the above results of [64] to less symmetric entangling surfaces, and (in some cases) to non-CFTs. We will show that entangling surfaces with a rotational or translational symmetry in some direction (see figure 2), extremize the universal EE with respect to shape deformations that break some of the symmetry,[3] i.e.:

$$S_1^{(\mathrm{univ})} = 0 \tag{1.8}$$

The proof of this result will be purely geometrical, and hence will apply to any QFT (and also to Renyi entropy). A simple corollary is that for d = even, (1.8) is true also for multiply connected entangling surfaces.[4]

Additionally, we will calculate the 2nd order correction $S_2^{(\mathrm{univ})}$ (whose sign determines if the local extremum is a minimum or a maximum) for some simple entangling surfaces. We perform this calculation using holography (the Ryu-Takayanagi formula), and also by using Solodukhin's formula (for 4d CFTs). For several examples of symmetric entangling surfaces, we find that the 2nd order correction is positive, and this corresponds to a local minimum. We conjecture this to hold more generally to symmetric entangling surfaces. We also comment on results for free massive fields, and 4d Renyi entropy.

[2]If the topology of the entangling surface is allowed to change, then the EE can become unbounded from below, as shown in 4d in [82].

[3]It should be emphasized that the family of shape perturbations that extremize the EE, are only those which break some of the symmetry of the original entangling surface. For example, for a 3d circle the perturbation in figure 1-left is allowed, whereas the perturbation in figure 1-right is not allowed because it has a component in the radial direction (a zero mode of the Fourier expansion). Such perturbations cannot deform a sphere into a larger/smaller sphere. More generally, such perturbations cannot deform a surface of revolution into a different surface of revolution (containing the same number of symmetries).

[4]For d = even the universal term of the EE is a log divergence which is determined locally by the shape of the entangling surface. Thus for a multiply connected entangling surface the log term is a superposition of the contribution from each separate piece of the entangling surface.

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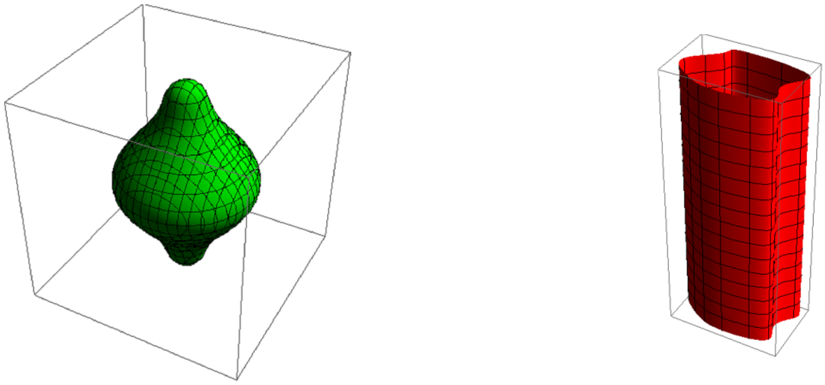

Figure 2. *Left:* an example of a $d = 4$ surface of revolution entangling surface with rotational symmetry. *Right:* a $d = 4$ waveguide entangling surface with translational symmetry. Such symmetric entangling surfaces can obviously be generalized to higher dimensions.

2 The first order correction: stationarity

Consider a QFT parametrized by coordinates $(\phi,\, y_i,\, r)$, where $i = 1\ldots, d-3$. Assume ϕ to be a symmetry direction of the entangling surface. Now perturb the entangling surface (see (1.3)) with a single Fourier mode:[5]

$$r(\phi, y_i) = r_0(y_i) + \epsilon A_n a_n(y_i)\cos(n\phi)\,, \qquad n \neq 0 \tag{2.4}$$

where $r_0(y_i)$ doesn't depend on the symmetry direction ϕ, the $a_n(y_i)$ are functions of y_i, and A_n are constants. The resulting EE can be expanded as:

$$S = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \ldots \tag{2.5}$$

Now lets consider the same perturbation but with negative sign, i.e. $\epsilon \to -\epsilon$:

$$\tilde{r}(\phi, y_i) = r_0(y_i) - \epsilon A_n a_n(y_i)\cos(n\phi)\,, \qquad n \neq 0 \tag{2.6}$$

The resulting EE will be (just flipping the sign of ϵ in (2.5)):

$$\tilde{S} = S_0 - \epsilon S_1 + \epsilon^2 S_2 + \ldots \tag{2.7}$$

[5]A general perturbation (without a zero mode) can be written as:

$$r(\phi, y_i) = r_0(y_i) + \epsilon \sum_{n\neq 0}\Big(A_n a_n(y_i)\cos(n\phi) + B_n b_n(y_i)\sin(n\phi)\Big) \tag{2.1}$$

Then the EE can be expanded:

$$S = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \ldots \tag{2.2}$$

At linear order in ϵ the modes don't mix, and their contributions to S_1 add up linearly:

$$S_1 = \sum_{n\neq 0}\Big(A_n U_1(n) + B_n U_2(n)\Big) \tag{2.3}$$

where $U_1(n)$ and $U_2(n)$ are some functions of n. Because the modes don't mix at linear order, we can compute the contribution of a single mode and then sum over all modes.

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But the two perturbations (2.6) and (2.4) describe precisely the same entangling surface, only rotated. This can be seen by performing $\phi \to \phi + \frac{\pi}{n}$ on (2.6), which gives (2.4). Since the two entangling surfaces are the same, they have same EE and therefore from (2.5) and (2.7) we have:

$$\tilde{S} = S \quad \longrightarrow \quad S_1 = 0 \tag{2.8}$$

and we proved what we wanted.

An important point to make is that the proof above used only the rotation symmetry, and did not use any specific property of entanglement entropy or of the QFT. Therefore stationarity will hold for any quantity which is a function of the surface (e.g. Renyi entropy), and for any QFT.

3 The second order correction in holography

Consider a boundary QFT parametrized by coordinates (t, y_i, r), $i = 1 \ldots, d-2$. z is the holographic coordinate, and we set $t = 0$. The holographic EE according to the Ryu-Takayanagi formula is:

$$S = \int d^{d-2}y_i dz \; \mathcal{L}(z, r, y_i) \tag{3.1}$$

where $\mathcal{L} \equiv \frac{\sqrt{\det g}}{4G_N}$. The corresponding equation of motion for the bulk surface is:

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dz}\frac{\partial \mathcal{L}}{\partial(\partial_z r)} - \frac{d}{dy_i}\frac{\partial \mathcal{L}}{\partial(\partial_{y_i} r)} = 0 \tag{3.2}$$

where there is a summation convention on y_i.

Consider an entangling surface defined by:

$$r(y_i) = r_0(y_i) \tag{3.3}$$

Now perturb the entangling surface:

$$r(y_i) = r_0(y_i) + \epsilon f(y_i) \tag{3.4}$$

where ϵ is a small parameter. Then the bulk surface will also get perturbed:[6]

$$r(z, y_i) = r_0(z, y_i) + \epsilon r_1(z, y_i) + \epsilon^2 r_2(z, y_i) + \ldots \tag{3.5}$$

For the rest of this section we assume $d =$ even dimensions, and we will want to compute the universal log term. We can then write the resulting perturbed $\mathcal{L}$ and S as:

$$S = S_0 + S_1\epsilon + S_2\epsilon^2 + \ldots = \int d^{d-2}y_i dz \mathcal{L} = \int d^{d-2}y_i dz \Big[\mathcal{L}_0 + \mathcal{L}_1\epsilon + \mathcal{L}_2\epsilon^2 + \ldots\Big] \tag{3.6}$$

[6]We use the same letter r for the entangling surface and the bulk surface. They can simply be distinguished by the fact that for the bulk surface there is a z dependence.

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We can derive a formula for the 2nd order correction S_2. Assuming that the entangling surface has a symmetry in all directions (e.g. the cylinder surface in the next section), terms containing r_2 will fall and we get:

$$S_2 = \frac{1}{2}\left\{ \int d^{d-2}y_i dz \left[r_1 \frac{d}{d\epsilon}\frac{d}{dy_i}\frac{\partial \mathcal{L}}{\partial(\partial_{y_i} r)} + \partial_{y_i} r_1 \frac{d}{d\epsilon}\frac{\partial \mathcal{L}}{\partial(\partial_{y_i} r)}\right]_{\epsilon=0} + \int_{\partial \mathcal{M}} d^{d-2}y_i r_1 \frac{d}{d\epsilon}\frac{\partial \mathcal{L}}{\partial(\partial_z r)}\bigg|_{\epsilon=0}\bigg|_{z=\delta}^{z_{\max}} \right\} \tag{3.7}$$

3.1 Cylinder entangling surface

Now let us assume that the entangling surface is a cylinder $S^p \times \mathbb{R}^{d-2-p}$ in flat spacetime. We denote y_j as cartesian directions along $\mathbb{R}^{d-2-p}$ (where $-L \leq y_j \leq L$) and Ω_p as directions along the sphere S^p (of radius R). The metric in cylindrical coordinates is:

$$ds^2 = \frac{\alpha(z)}{z^2}[\beta(z)dz^2 - dt^2 + dy_i^2 + dr^2 + r^2 d\Omega_p^2] \tag{3.8}$$

where $\alpha(z)$ and $\beta(z)$ are some functions of z. The holographic EE written in these cylindrical coordinates is (see also (A.20)):

$$S = \int dz d^{d-2-p}y_j d\Omega_p \; \mathcal{L}(z, r, y_j, \Omega_p) \tag{3.9}$$

where:

$$\mathcal{L}(z, r, y_i, \Omega_p) = \frac{r^p F_1(z)}{z^{d-1}}\sqrt{1 + F_2(z)(\partial_z r)^2 + (\partial_{y_j} r)^2 + \frac{1}{r^2}(\partial_{\Omega_p} r)^2} \tag{3.10}$$

with $F_1(z) \equiv \frac{R_{\text{AdS}}^{d-1}}{4G_N}\alpha^{\frac{d-1}{2}}\beta^{1/2}$ and $F_2(z) \equiv \beta^{-1}$. For an AdS metric we have: $F_2(z) = 1$, and $F_1(z) = \frac{R_{\text{AdS}}^{d-1}}{4G_N}$.

We Fourier expand the perturbation f in (3.4):

$$f(y_j, \Omega_p) = \sum_{\{n_j, l_p\}} \left[a_{\{n_j,l_p\}} Y_{l_p}(\Omega_p) \prod_{i=1}^{d-2-p} \cos(n_i y_i) \right] \tag{3.11}$$

where $Y_{l_p}(\Omega_p)$ are (real) hyperspherical harmonics, the n_j are integers, the $a_{\{n_j,l_p\}}$ are the coefficients of the Fourier expansion, and for conciseness we defined $\{n_j\} \equiv n_1, \ldots, n_{d-2-p}$ and $\{l_p\} \equiv l, m_1 \ldots, m_{p-1}$.

The bulk surface perturbation r_1 of (3.5) can also be Fourier expanded (see (A.25)):

$$r_1(z, y_j, \Omega_p) = \sum_{\{n_j, l_p\}} \left[a_{\{n_j,l_p\}} r^{(1)}_{\{n_j,l_p\}}(z) Y_{l_p}(\Omega_p) \prod_{i=1}^{d-2-p} \cos(n_i y_i) \right] \tag{3.12}$$

where $r^{(1)}_{\{n_j,l_p\}}(z)$ are functions of z, which must obey the boundary condition: $r^{(1)}_{\{n_j,l_p\}}(z = 0) = 1$.[7]

[7]The boundary QFT resides at $z = 0$.

Now we can derive an expression for the boundary term (the last term on the r.h.s. of (3.7)):

$$S_2^{\text{bound.}} = \frac{L^{d-2-p}F_1(z)F_2(z)r_0^{p-1}}{2z^{d-1}[1+F_2(\partial_z r_0)^2]^{1/2}} \sum_{\{n_j,l_p\}} a^2_{\{n_j,l_p\}} \left[\frac{r_0 r^{(1)}_{\{n_j,l_p\}}\partial_z r^{(1)}_{\{n_j,l_p\}}}{1+F_2(\partial_z r_0)^2} + p\partial_z r_0 \left(r^{(1)}_{\{n_j,l_p\}}\right)^2\right]\Bigg|_{z=\delta} \tag{3.13}$$

where L is the length of the $\mathbb{R}^{d-2-p}$ directions, and we take the limit $L\to\infty$. This formula is a boundary term which is evaluated at the boundary $z=\delta$. We will want to extract the universal log divergence from this formula in $d=$ even dimensions.

In principle, $r_0(z)$ and $r^{(1)}_{\{n_j,l_p\}}(z)$ can be obtained by solving the EOM's for the bulk minimal surface (A.23) and (A.26). These solutions will generally have Fefferman-Graham type expansions of the form ([19, 43, 87, 88]):

$$r_0(z) = q_0 + q_2 z^2 + \ldots + q_d z^d + \tilde{q}_d z^d \log(z) + \ldots \tag{3.14}$$

$$r^{(1)}_{\{n_j,l_p\}}(z) = u_0 + u_2 z^2 + \ldots + u_d z^d + \tilde{u}_d z^d \log(z) + \ldots \tag{3.15}$$

The terms in (3.13) can then be written as:

$$\frac{r_0 r^{(1)}_{\{n_j,l_p\}}\partial_z r^{(1)}_{\{n_j,l_p\}}}{[1+F_2(\partial_z r_0)^2]z^{d-1}}\Bigg|_{\log} = dq_0u_0\tilde{u}_d\log z\,, \qquad \frac{p\partial_z r_0 (r^{(1)}_{\{n_j,l_p\}})^2}{z^{d-1}}\Bigg|_{\log} = dpu_0^2\tilde{q}_d\log z \tag{3.16}$$

where we extracted the log terms. Using (3.13), and also the fact that the background is asymptotically AdS: $F_1(0) = \frac{R^{d-1}_{\text{AdS}}}{4G_N} = \frac{2\pi^{\frac{d+2}{2}}(d-1)\Gamma(\frac{d}{2})}{\Gamma(d+2)}C_T$ and $F_2(0)=1$, we get[8]

$$S_2^{\text{bound.}}\big|_{\log} = \frac{(-1)^{\frac{d-2}{2}}\pi^{\frac{d+2}{2}}\Gamma(\frac{d}{2})}{(d+1)\Gamma(d-1)}C_T L^{d-2-p}R^{p-1}\sum_{\{n_j,l_p\}} a^2_{\{n_j,l_p\}}\left(R\tilde{u}_d + p\tilde{q}_d\right)\log\left(\frac{R}{\delta}\right) \tag{3.17}$$

where we used $u_0=1$ and $q_0=R$, and multiplied by $(-1)^{\frac{d-2}{2}}$, see (1.1).

It can be shown that for the two special cases $p=0,1$, the boundary term above is the only contribution to $S_2^{(\text{univ})}$. Thus the $p=1$ case gives:

$$S_2^{(\text{univ})} = \frac{(-1)^{\frac{d-2}{2}}\pi^{\frac{d+2}{2}}\Gamma(\frac{d}{2})}{(d+1)\Gamma(d-1)}C_T L^{d-3}\sum_{\{n_j,l\}} a^2_{\{n_j,l\}}\left(R\tilde{u}_d + \tilde{q}_d\right) \tag{3.18}$$

We see from (3.18) that the sign of $S_2^{(\text{univ})}$ depends solely on the coefficients $\tilde{u}_d$ and $\tilde{q}_d$ of the log term in the FG expansions. These coefficients can be obtained (in principle) by solving the bulk EOMs (A.23) and (A.26). It might be interesting to understand if generally these coefficients are constrained to have a definite sign. The $p=0$ case (the plane) is examined in the following subsection.

[8] Note that for a CFT, dimensional analysis (and (A.23), (A.26)) dictate the functional dependence of $\tilde{q}_d$ and $\tilde{u}_d$ such that: $\tilde{q}_d = f_1(d)R^{-d+1}$ and $\tilde{u}_d = f_2(l,\tilde{n},R,d)$, where f_1, f_2 are some functions and $\tilde{n}^2 \equiv \sum_i n_i^2$. Note that there is no dependence on $m_1,\ldots,m_{p-1}$, since the bulk EOM's (A.23), (A.26) do not depend on them. Likewise it can be seen that the sphere result (1.7) doesn't depend on the m's.

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3.2 Plane entangling surface

The plane entangling surface $\mathbb{R}^{d-2}$ is a special case of the cylinder with $p = 0$. Therefore (3.17) becomes:[9]

$$S_2^{(\text{univ})} = \frac{(-1)^{\frac{d-2}{2}}\pi^{\frac{d+2}{2}}\Gamma(\frac{d}{2})}{(d+1)\Gamma(d-1)} C_T L^{d-2} \sum_{\{n_j\}} a^2_{\{n_j\}} \tilde{u}_d \tag{3.19}$$

So the sign of $S_2^{(\text{univ})}$ depends solely on the coefficient $\tilde{u}_d$, which we shall now determine. For a plane entangling surface and Einstein gravity in the bulk, we will find an explicit solution[10] (A.19):

$$x^{(1)}_{\{n_i\}}(z) = \frac{1}{\mathcal{N}} z^{\frac{d}{2}} K_{\frac{d}{2}}(\tilde{n}z) \tag{3.20}$$

where $\mathcal{N}$ is a normalization constant, and $\tilde{n}^2 \equiv \sum_i n_i^2$. The small z expansion of this function gives (see (A.16)) $\tilde{u}_d = \frac{(-1)^{\frac{d-2}{2}}\tilde{n}^d}{2^{d-2}d(\frac{d}{2}-1)!^2}$. Therefore we have:

$$S_2^{(\text{univ})} = \frac{\pi^{\frac{d-2}{2}}(d-1)}{2^{d-2}\Gamma(d+2)(\frac{d}{2}-1)!} C_T L^{d-2} \sum_{\{n_j\}}^{\infty} \tilde{n}^d a^2_{\{n_j\}} \tag{3.21}$$

Since the above expression is positive, we have proved that a plane is a local minimum for Einstein gravity in the bulk. It is now natural to conjecture that (3.21) holds for any CFT. (3.21) was derived for $d =$ even, but it would be simple task to obtain the analogous $d =$ odd result. The $d =$ odd result will differ from (3.21) only by its d dependence, and for $d = 3$ it was obtained in [77].

For the case $d = 4$, (3.21) gives:

$$S_2^{(\text{univ})} = C_T \frac{\pi^3 L^2}{160} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} (n_2^2 + n_3^2)^2 a^2_{n_2,n_3} \tag{3.22}$$

This precisely matches (4.13) which we will obtain in the next section for 4d CFTs via Solodukhin's formula.

4 The second order correction in field theory

4.1 4d CFT: Solodukhin's formula

For a $d =$ even CFT there is a universal log term (1.1):

$$S\big|_{\log} = (-1)^{\frac{d-2}{2}} S^{(\text{univ})} \log\left(\frac{R}{\delta}\right) \tag{4.1}$$

and as in (1.4), we can expand in small ϵ:

$$S^{(\text{univ})} = \sum_{k=0}^{\infty} S_k^{(\text{univ})} \epsilon^k \tag{4.2}$$

[9]Note that (3.17), (3.18), and (3.19) apply to any asymptotically AdS background.
[10]This result was also derived in [52] in the context of "entanglement density".

Solodukhin's formula [65] (which applies to a CFT in 4d) is:[11]

$$S^{(\mathrm{univ})} = \frac{a_4}{180}\int_\Sigma d^2\sigma\sqrt{\gamma}E_2 + \frac{c_4}{240\pi}\int_\Sigma d^2\sigma\sqrt{\gamma}I_2 \tag{4.3}$$

where E_2 is the Euler density, and the integrals are over the entangling surface Σ, and

$$I_2 \equiv Tr(k^2) - \frac{1}{2}k^a k^a = \left(k^a_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}k^a\right)^2 \geq 0 \tag{4.4}$$

The second fundamental form and extrinsic curvature are defined as:

$$k^a_{\mu\nu} = \gamma^\alpha_\mu\gamma^\beta_\nu\nabla_\alpha\hat{n}^a_\beta\,, \quad k^a = Tr(k^a_{\mu\nu}) = \gamma^{\mu\nu}k^a_{\mu\nu}\,, \qquad Tr(k^2) = \gamma^{\mu\nu}\gamma^{\rho\sigma}k^a_{\nu\rho}k^a_{\sigma\mu} \tag{4.5}$$

where $\gamma_{\mu\nu}$ is the induced metric on the entangling surface.

We consider shape perturbations that leave the topology of the entangling surface fixed, therefore the change in the Euler term above is zero, and $S_2^{(\mathrm{univ})}$ of (4.2) will be given by the integral of I_2:

$$S_2^{(\mathrm{univ})} = \frac{c_4}{240\pi}\int_\Sigma d^2\sigma\sqrt{\gamma}I_2\Big|_{\epsilon^2} \tag{4.6}$$

In (4.4) I_2 is always positive [78, 82] and is zero for the sphere. Therefore the sphere locally minimizes I_2 and $S^{(\mathrm{univ})}$. This argument also works for flat entangling surfaces (plane, strips) since these have $I_2 = 0$.

In the following, we calculate $S_2^{(\mathrm{univ})}$ for several examples. See also [63, 64].

Example 1: plane entangling surface. The metric in cartesian coordinates is:

$$ds^2 = dx^2 + dy_2^2 + dy_3^2 \tag{4.7}$$

Consider a plane at $x = 0$, where y_2, y_3 are coordinates along the surface. Now perturb it as follows:

$$x = 0 + \epsilon f(y_2, y_3) \tag{4.8}$$

The vector normal to the surface is:

$$\hat{n}_\mu = \frac{1}{\sqrt{1+\epsilon^2(f_{y_2}^2 + f_{y_3}^2)}}(1\,, -\epsilon f_{y_2}, -\epsilon f_{y_3})\,, \qquad \text{where} \qquad f_{y_i} \equiv \partial_{y_i} f \tag{4.9}$$

The second fundamental form at order $O(\epsilon^2)$ is:

$$k^a_{\mu\nu} = \begin{bmatrix} 0 & -\epsilon^2(f_{y_3}f_{y_2y_3}+f_{y_2}f_{y_2y_2}) & -\epsilon^2(f_{y_3}f_{y_3y_3}+f_{y_2}f_{y_2y_3}) \\ -\epsilon^2(f_{y_3}f_{y_2y_3}+f_{y_2}f_{y_2y_2}) & -\epsilon f_{y_2y_2} & -\epsilon f_{y_2y_3} \\ -\epsilon^2(f_{y_3}f_{y_3y_3}+f_{y_2}f_{y_2y_3}) & -\epsilon f_{y_2y_3} & -\epsilon f_{y_3y_3} \end{bmatrix} \tag{4.10}$$

$k^a_{\mu\nu}$ starts at order ϵ since the unperturbed plane is flat. Plugging this in (4.4) gives:

$$\sqrt{\gamma}I_2 = I_2 + O(\epsilon^3) = \frac{\epsilon^2}{2}\Big[(f_{y_2y_2} - f_{y_3y_3})^2 + 4f_{y_2y_3}^2\Big](f_{y_2}^2 + f_{y_3}^3) + O(\epsilon^3) \tag{4.11}$$

[11] a_4 and c_4 are the a and c anomalies in 4d. Both are normalized such that for a real scalar field their value is 1. We will sometimes use $C_T = \frac{c_4}{3\pi^4}$ instead of c_4.

Note that I_2 starts at order ϵ^2, therefore the order ϵ correction vanishes even before integration over the surface. Now Fourier expand the perturbation:

$$f(y_2, y_3) = \sum_{n_2,n_3=1}^{\infty} a_{n_2 n_3} \cos(n_2 y_2) \cos(n_3 y_3) \tag{4.12}$$

and plug (4.11), (4.12) in (4.6):

$$S_2^{(\mathrm{univ})} = C_T \frac{\pi^3 L^2}{160} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} (n_2^2 + n_3^2)^2 a_{n_2 n_3}^2 \tag{4.13}$$

where we used $c_4 = 3\pi^4 C_T$, and the integral $\int_{-L}^{L} dy \cos^2(ny) = L$, where the width of the plane is very large $L \to \infty$. So we got a positive result, and therefore the universal term of a plane in 4d is a local minimum. We see that (4.13) precisely matches the result (3.22) obtained in holography.

We can compute higher orders of ϵ in (4.2), and we note that for the plane all odd terms vanish: $S_{2k+1}^{(\mathrm{univ})} = 0$. At 4^{th} order we get:

$$S_4^{(\mathrm{univ})} = -C_T \frac{\pi^3 L^2}{11520} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} a_{n_2 n_3}^4 \left(15 n_2^6 + 15 n_3^6 + 13 n_2^4 n_3^2 + 13 n_3^4 n_2^2\right) \tag{4.14}$$

Example 2: sphere entangling surface. This case was calculated in [63, 64], and we write their result.

The perturbed sphere entangling surface is:

$$r(\theta,\phi) = R[1 + \epsilon f(\theta,\phi)] = R\left[1 + \epsilon \sum_{l,m} a_{lm} Y_{lm}(\theta,\phi)\right] \tag{4.15}$$

where R is the radius of the sphere. Plugging in (4.6) gives:

$$S_2^{(\mathrm{univ})} = C_T \frac{\pi^3}{160} \sum_{l,m} a_{lm}^2 (l-1) l (l+1)(l+2) \tag{4.16}$$

This matches the holographic result (1.7) [64].

Example 3: cylinder entangling surface. The metric in cylindrical coordinates is:

$$ds^2 = dr^2 + dy^2 + r^2 d\phi^2 \tag{4.17}$$

Consider a perturbed cylinder entangling surface with radius R:

$$r(y,\phi) = R + \epsilon f(y,\phi) \tag{4.18}$$

We get:

$$\sqrt{\gamma} I_2\big|_{\epsilon^2} = \frac{1}{4R^3}[2f^2 + 3f_\phi^2 + 2f_{\phi\phi}^2 + 8f f_{\phi\phi} - R^2 f_y^2 + 8R^2 f_{y\phi}^2 - 4R^2 f_{yy} f_{\phi\phi} + 2R^4 f_{yy}^2]\epsilon^2 \tag{4.19}$$

Plugging $f(y,\phi) = \sum_{n,m} a_{n,m} \cos(m\phi)\cos(ny)$,

$$S_2^{(\mathrm{univ})} = \frac{c_4}{240\pi}\int_\Sigma dyd\phi\sqrt{\gamma}I_2\Big|_{\epsilon^2} =$$
$$C_T\frac{\pi^4 L}{320R^3}\sum_{n,m}a_{nm}^2\Big[2-5m^2+2m^4+(nR)^2(4m^2-1)+2(nR)^4\Big] \tag{4.20}$$

Thus $S_2^{(\mathrm{univ})} \geq 0$ for $m \geq 2$ and for all n, and the cylinder is a local minimum.

Example 4: 4d surface of revolution. Consider a surface of revolution entangling surface (e.g. figure 2-left) given by:

$$r(\theta,\phi) = r_0(\theta) \tag{4.21}$$

where we use spherical coordinates (r,θ,ϕ). This surface has rotational symmetry in the ϕ direction. Now perturb the surface as follows:

$$r(\theta,\phi) = r_0(\theta)[1+\epsilon f_2(\theta)\cdot f_3(\phi)] \tag{4.22}$$

with $f_3(\phi) = \sum_{m\neq 0} a_m \cos(m\phi)$. (4.3) and (4.4) give at most 4 derivatives of ϕ, thus the result can be written as a polynomial in m (after integrating over ϕ):

$$S_2^{(\mathrm{univ})} = \frac{c_4}{240\pi}\int_\Sigma d\theta d\phi\sqrt{\gamma}I_2\Big|_{\epsilon^2} =$$
$$\frac{\pi^3 C_T}{80}\int d\theta\sum_{m\neq 0}a_m^2\Big[G_1(r_0,f_2,\theta)m^4+G_2(r_0,f_2,\theta)m^2+G_3(r_0,f_2,\theta)\Big] \tag{4.23}$$

where the $G_i(r_0,f_2,\theta)$ are some functions of r_0, f_2 and their derivatives. Explicit calculation gives the m^4 coefficient:

$$G_1(r_0,f_2,\theta) = \frac{\pi f_2^2\Big(4+4\csc^2(\theta)r_0^2r_0'^8+16\csc^2(\theta)r_0^4r_0'^6+24\csc^2(\theta)r_0^6r_0'^4+16r_0^8r_0'^2\Big)}{8r_0^3\sin\theta(r_0^2+r_0'^2)^{\frac{9}{2}}} \tag{4.24}$$

Since $r_0 \geq 0$, we see that $G_1(r_0,f_2,\theta) \geq 0$. Therefore for perturbations with large enough m (short wave-length perturbations) (4.23) is positive, and thus all 4d surfaces of revolution are local minima. We are not able to show that the functions $G_2(r_0,f_2,\theta)$, $G_3(r_0,f_2,\theta)$ are positive (though the final integrated result $S_2^{(\mathrm{univ})}$ may still turn out to be positive).

Example 5: 4d waveguide surface. Consider a general waveguide entangling surface (e.g. figure 2-right) given by:

$$r(y,\phi) = r_0(\phi) \tag{4.25}$$

where we use cylindrical coordinates (r,y,ϕ). This surface has translational symmetry in the y direction. Now we perturb the surface as follows:

$$r = r_0(\phi)[1+\epsilon f_2(\phi)f_3(y)] \tag{4.26}$$

with $f_3(y) = \sum_{m\neq 0} a_m \cos(my)$. The result can be written as a polynomial in m (after integrating over y):

$$
\begin{aligned}
S_2^{(\mathrm{univ})} &= \frac{c_4}{240\pi}\int_\Sigma d\phi dy \sqrt{\gamma} I_2\Big|_{\epsilon^2} = \\
&\frac{\pi^3 C_T}{80}\int d\phi \sum_{m\neq 0} a_m^2 \Big[G_1(r_0,f_2)m^4 + G_2(r_0,f_2)m^2 + G_3(r_0,f_2)\Big]
\end{aligned}
\tag{4.27}
$$

where the $G_i(r_0, f_2)$ are some functions of r_0, f_2 and their derivatives. Explicit calculation gives the m^4 coefficient:

$$
G_1(r_0,f_2) = \frac{\pi f_2^2\left(2r_0^{10} + 6r_0^8 r_0'^2 + 6r_0^6 r_0'^4 + 2r_0^4 r_0'^6\right)}{4(r_0^2 + r_0'^2)^{\frac{7}{2}}} \geq 0 \tag{4.28}
$$

Therefore for perturbations with large enough m (short wave-length perturbations) (4.27) is positive, and thus all 4d waveguide surfaces are local minima. We are not able to show that the functions $G_2(r_0, f_2)$, $G_3(r_0, f_2)$ are positive (though the final integrated result $S_2^{(\mathrm{univ})}$ may still turn out to be positive).

4.2 Renyi entropy

There is a generalization of (4.3) to Renyi entropy in a 4d a CFT [70, 79]:

$$
\mathcal{S}_q^{(\mathrm{univ})} = Q_1(q)\frac{a_4}{180}\int_\Sigma d^2\sigma\sqrt{\gamma}E_2 + Q_2(q)\frac{c_4}{240\pi}\int_\Sigma d^2\sigma\sqrt{\gamma}I_2 \tag{4.29}
$$

where $\mathcal{S}_q^{(\mathrm{univ})}$ is the universal log term for the q-th Renyi entropy. $Q_{1,2}(q)$ are functions of q such that $Q_1(1) = Q_2(1) = 1$, so it matches (4.3) in the EE limit $q \to 1$. Up to these functions, (4.29) and (4.3) are the same, hence the results in the previous sections can be used. In particular, if $Q_2(q)$ is positive then for any entangling surface the Renyi entropy will have the same sign as the EE. Note that $Q_2(q)$ is positive for free fields, and there is strong evidence that it is positive for holographic CFTs [70, 82, 89]. If this turns out to be correct, then the Renyi entropy will be a local minimum whenever the EE is.

4.3 Free massive fields

Certain universal EE terms for free massive field theories have been found [16–19]. The so called "universal area law" for free scalars or fermions with mass m has the following form:

$$
S = \left\{\begin{matrix} (-1)^{\#}\gamma_d A_\Sigma m^{d-2}\,, & d = \mathrm{odd} \\ (-1)^{\#}\tilde{\gamma}_d A_\Sigma m^{d-2}\log(m\delta)\,, & d = \mathrm{even}\end{matrix}\right\} \tag{4.30}
$$

where γ_d is a positive constant that depends only on the dimension d. All of the shape dependence is contained in A_Σ, the area of the entangling surface. Table 1 lists the sign factors $(-1)^{\#}$ in (4.30) for a Dirac fermion, a conformally coupled scalar, and a minimally coupled scalar.

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	Dirac Fermion	Conformal scalar	Minimal scalar
$d = \text{odd}$	$(-1)^{\frac{d-1}{2}}$	$(-1)^{\frac{d+1}{2}}$	$(-1)^{\frac{d-1}{2}}$
$d = \text{even}$	$(-1)^{\frac{d}{2}}$	$(-1)^{\frac{d-2}{2}}$	$(-1)^{\frac{d}{2}}$

Table 1. The sign factor in the "universal area law" (4.30).

As an example, let us now compute the "universal area law" term for a deformed sphere entangling surface:

$$r(\Omega_{d-2}) = R\left[1 + \epsilon \sum_{\{lm\}} a_{\{lm\}} Y_{\{lm\}}(\Omega_{d-2})\right] \tag{4.31}$$

The area of the deformed sphere is:

$$A_\Sigma = \int d\Omega_{d-2} r^{d-2} = \frac{(d-1)\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})} R^{d-2} + \epsilon^2 \frac{(d-2)(d-3)}{2} R^{d-2} \sum_{\{lm\}} a^2_{\{lm\}} + O(\epsilon^3) \tag{4.32}$$

where we plugged (4.31) and performed the integrals.

The first term on the r.h.s. is the area of the undeformed sphere, the $O(\epsilon)$ correction vanishes as expected, and the $O(\epsilon^2)$ correction is positive. Therefore the $O(\epsilon^2)$ correction in S has the same sign as the zeroth order, which can be read from table 1. Generalizing to non-spheres, it is easy to see that the area of a perturbed surface of revolution is larger than that of the unperturbed surface of revolution. It would be interesting to perform a similar analysis to higher curvature terms for free massive fields (i.e to curvature corrections to the "universal area law").

A similar analysis applies to "universal area law" terms in interacting theories [19, 21, 69], with a shape dependence that comes only from the area A_Σ. The EE for a CFT perturbed by a relevent operator of dimension $\Delta = \frac{d+2}{2}$, contains the following term:

$$S = N\lambda^2 \frac{d-2}{4(d-1)} \frac{\pi^{\frac{d+2}{2}}}{\Gamma(\frac{d+2}{2})} A_\Sigma \log\left(\frac{R}{\delta}\right) \tag{4.33}$$

where λ is the coupling constant. Such log terms occur both in odd and even dimensions.

5 Discussion

In this work we studied the shape dependence of entanglement entropy by deforming symmetric entangling surfaces. We showed that entangling surfaces with a rotational or translational symmetry locally extremize the EE with respect to shape deformations that break some of the symmetry. This result applies to EE and Renyi entropy for any QFT in any dimension. Using Solodukhin's formula and holography, we calculated the 2nd order correction to the EE for CFTs and simple symmetric entangling surfaces. In several cases we found that the 2nd order correction is positive, and thus the corresponding symmetric entangling surface is a local minimum. Perhaps this result can be shown to hold more generally for other symmetric entangling surfaces.

Let us mention some possible future directions.

- The calculation in section 3 considered only Einstein gravity in the bulk, and it would be interesting to consider also higher derivative gravity. For spheres, [64] found that $S_2^{(\text{univ})}$ depends just on C_T and not on t_2 or t_4 (these are the three parameters in the 3-point function of stress tensors). It would be nice to check if this continues to hold for other entangling surfaces.

- Computing the FG coefficients $\tilde{u}_d$ and $\tilde{q}_d$ in (3.17) by solving the bulk EOMs for the cylinder (A.23) and (A.26). Maybe it is also possible to show, using a more general principle, that $\tilde{q}_d$ and $\tilde{u}_d$ in (3.18) must have a definite sign. It would also be interesting to consider (in holography) more general entangling surfaces with a symmetry.

- The work of [69] attempted to compute S_2^{univ} for a plane entangling surface in a $d = 4$ CFT using the perturbative formalism of EE [66]. They were not able to obtain the I_2 term of Solodukhin's formula (4.3), and the current situation is somewhat puzzling. Our result (3.21) as well as (1.7) (obtained in [64]) might help in resolving this puzzle.

- It would be interesting to repeat the analysis of section 4.1 for 6d CFTs using the results of [80, 81].

- Other possible extensions are to compute higher orders $S_j^{(\text{univ})}$ for $j > 2$, and also to perform computations in a curved space-time background.

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A Equations of Motion for the minimal bulk surface

In this section, we obtain EOMs for the bulk minimal surfaces of cylinder, strip, and plane. We solve the 1st EOM for the plane, and obtain $\tilde{u}_d$ in (3.19).

A.1 Strip entangling surface

Consider a strip entangling surface and a bulk metric in Poincare coordinates:

$$ds^2 = \frac{R_{\text{AdS}}^2}{z^2}\left[\frac{1}{\tilde{\beta}(z)}dz^2 - dt^2 + dx^2 + dy_i^2\right] \tag{A.1}$$

Where $\tilde{\beta}(z)$ is some function of the holographic coordinate z, which for AdS: $\tilde{\beta}(z) = 1$. x is the direction perpendicular to the strip entangling surface, y_i are directions along the strip entangling surface, and $i = 1, \ldots, d-2$. The holographic EE is (see (3.1), (3.2)):

$$S = \frac{R_{\text{AdS}}^{d-1}}{4G_N} \int d^{d-2}y_i \int_{\delta}^{z_{\max}} dz \frac{1}{z^{d-1}} \sqrt{\frac{1}{\tilde{\beta}(z)} \left[1 + (\partial_{y_i} x)^2\right] + (\partial_z x)^2} \tag{A.2}$$

In the ansatz of (3.10) this corresponds to $p = 0$, $F_1 = \frac{R_{\text{AdS}}^{d-1}}{4G_N \sqrt{\tilde{\beta}(z)}}$ and $F_2 = \tilde{\beta}(z)$. The corresponding EOM is:

$$\frac{d}{dz}\left(\frac{1}{z^{d-1}} \frac{\partial_z x}{\sqrt{\frac{1}{\tilde{\beta}(z)}\left[1+(\partial_{y_i}x)^2\right]+(\partial_z x)^2}}\right) + \frac{1}{z^{d-1}} \frac{d}{dy_i}\left(\frac{\partial_{y_i} x}{\sqrt{\frac{1}{\tilde{\beta}(z)}\left[1+(\partial_{y_i}x)^2\right]+(\partial_z x)^2}}\right) = 0 \tag{A.3}$$

The bulk surface can be expanded in ϵ:

$$x(z, y_i) = x_0(z) + \epsilon x_1(z, y_i) + \epsilon^2 x_2(z, y_i) + \ldots \tag{A.4}$$

where because of the translational symmetry of the strip, $x_0(z)$ does not depend on y_i. The EOM (A.3) at 0^{th} order in ϵ is:

$$\frac{d}{dz}\left(\frac{\partial_z x_0}{z^{d-1}\sqrt{\frac{1}{\tilde{\beta}(z)} + (\partial_z x_0)^2}}\right) = 0 \tag{A.5}$$

The solution to this is:

$$(\partial_z x_0)^2 = \frac{z^{2d-2}}{\tilde{\beta}(z)(z_{\max}^{2d-2} - z^{2d-2})} \tag{A.6}$$

where $z_{\max}$ is the turning point of the bulk surface. The EOM at 1^{st} order in ϵ is:

$$\frac{d}{dz}\left(\frac{1}{z^{d-1}} \frac{\frac{1}{\tilde{\beta}(z)}\partial_z x_1}{\left(\frac{1}{\tilde{\beta}(z)} + (\partial_z x_0)^2\right)^{3/2}}\right) + \frac{1}{z^{d-1}} \frac{\partial_{y_i}^2 x_1}{\sqrt{\frac{1}{\tilde{\beta}(z)} + (\partial_z x_0)^2}} = 0 \tag{A.7}$$

Plugging (A.6) in (A.7):

$$\frac{d}{dz}\left(\frac{\tilde{\beta}^{\frac{1}{2}}}{z^{d-1}} \frac{(z_{\max}^{2d-2} - z^{2d-2})^{3/2}\partial_z x_1}{z_{\max}^{2d-2}}\right) + \frac{\tilde{\beta}^{\frac{1}{2}}}{z^{d-1}}(z_{\max}^{2d-2} - z^{2d-2})^{1/2}\partial_{y_i}^2 x_1 = 0 \tag{A.8}$$

Simplifying, we get:

$$\partial_z^2 x_1 + \left[\frac{\tilde{\beta}'(z)}{2\tilde{\beta}(z)} - \frac{(d-1)}{z} \cdot \frac{1 + 2\left(\frac{z}{z_{\max}}\right)^{2d-2}}{1 - \left(\frac{z}{z_{\max}}\right)^{2d-2}}\right]\partial_z x_1 + \frac{\partial_{y_i}^2 x_1}{1 - \left(\frac{z}{z_{\max}}\right)^{2d-2}} = 0 \tag{A.9}$$

This equation seems hard to solve analytically. In the next section we consider the simpler case of a plane entangling surface.

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A.2 Plane entangling surface

For a plane entangling surface situated at $x = 0$, the bulk minimal surface goes straight down in the bulk: $x_0(z) = 0$. We have (see (A.4)):

$$x(z, y_i) = \epsilon x_1(z, y_i) + \epsilon^2 x_2(z, y_i) + \dots \tag{A.10}$$

The turning point of the bulk surface is at $z_{\max} \to \infty$, therefore (A.9) becomes:

$$\partial_z^2 x_1 + \left[\frac{\tilde{\beta}'(z)}{2\tilde{\beta}(z)} - \frac{(d-1)}{z}\right]\partial_z x_1 + \partial_{y_i}^2 x_1 = 0 \tag{A.11}$$

We Fourier expand x_1:

$$x_1(z, y) = \sum_{\{n_i\}} a_{\{n_i\}} x^{(1)}_{\{n_i\}}(z) \prod_{i=1}^{d-2} \cos(n_i y_i) \tag{A.12}$$

where the n_i are integers, and we defined the shorthand notation: $\{n_i\} \equiv n_1, \dots, n_{d-2}$. Plugging this in (A.11) gives:

$$\partial_z^2 x^{(1)}_{\{n_i\}} + \left[\frac{\tilde{\beta}'(z)}{2\tilde{\beta}(z)} - \frac{(d-1)}{z}\right]\partial_z x^{(1)}_{\{n_i\}} - \tilde{n}^2 x^{(1)}_{\{n_i\}} = 0 \tag{A.13}$$

where we defined $\tilde{n}^2 = \sum_i n_i^2$. Now we consider the ansatz $\tilde{\beta}(z) = 1 + \tilde{\alpha} z^k$ for the bulk metric, then:

$$\partial_z^2 x^{(1)}_{\{n_i\}} + \left[\frac{\tilde{\alpha} k z^{k-1}}{2(1+\tilde{\alpha} z^k)} - \frac{d-1}{z}\right]\partial_z x^{(1)}_{\{n_i\}} - \tilde{n}^2 x^{(1)}_{\{n_i\}} = 0 \tag{A.14}$$

The CFT case $\tilde{\beta}(z) = 1$ can be recovered by plugging $\tilde{\alpha} = 0$:

$$\partial_z^2 x^{(1)}_{\{n_i\}} - \frac{d-1}{z}\partial_z x^{(1)}_{\{n_i\}} - \tilde{n}^2 x^{(1)}_{\{n_i\}} = 0 \tag{A.15}$$

This equation is solved by modified Bessel functions:[12]

$$x^{(1)}_{\{n_i\}}(z) = C_1 z^{\frac{d}{2}} K_{\frac{d}{2}}(\tilde{n} z) + C_2 z^{\frac{d}{2}} I_{\frac{d}{2}}(\tilde{n} z) \tag{A.18}$$

[12] The modified Bessel function:

$$\begin{aligned} K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^{-\nu} \sum_{j=0}^{\nu-1} (-1)^j \frac{(\nu-j-1)!}{4^j j!} z^{2j} \;+\; (-1)^{\nu+1}\log(z/2) I_\nu(z) \\ + (-1)^\nu \frac{1}{2}\left(\frac{z}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{\psi(j+1) + \psi(n+j+1)}{4^j j!(\nu+j)!} z^{2j} \end{aligned} \tag{A.16}$$

where ψ is the digamma function, and

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{1}{4^j j! \Gamma(\nu+j+1)} z^{2j} \tag{A.17}$$

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Choosing a boundary condition such that $x^{(1)}_{\{n_i\}}(z)$ does not explode at $z \to \infty$, leaves only $K_{\frac{d}{2}}$. Therefore the final solution is:

$$x^{(1)}_{\{n_i\}}(z) = \frac{1}{\mathcal{N}} z^{\frac{d}{2}} K_{\frac{d}{2}}(\tilde{n} z) \tag{A.19}$$

where the normalization constant is $\mathcal{N} = 2^{\frac{d}{2}-1}(\frac{d}{2}-1)!\tilde{n}^{-\frac{d}{2}}$. We will use the above solution in (3.20). This result was also derived in [52] in the context of "entanglement density".

A.3 Cylinder entangling surface

Consider a cylinder entangling surface $S^p \times \mathbb{R}^{d-2-p}$ in flat space-time $\mathbb{R}^d$. The holographic EE in cylindrical coordinates is (see (3.9), (3.10)):

$$S = \frac{R^{d-1}_{\text{AdS}}}{4G_N} \int dz d^{d-2-p} y d\Omega_p \frac{r^p}{z^{d-1}} \sqrt{1 + (\partial_z r)^2 + (\partial_{y_j} r)^2 + \frac{1}{r^2}(\partial_{\Omega_p} r)^2} \tag{A.20}$$

where we assumed the AdS_{d+1} metric. The corresponding EOM is:

$$\begin{aligned} &\frac{d}{dz}\left(\frac{r^p}{z^{d-1}} \frac{\partial_z r}{\sqrt{1+(\partial_z r)^2+(\partial_{y_j} r)^2+\frac{1}{r^2}(\partial_{\Omega_p} r)^2}}\right) + \frac{1}{z^{d-1}} \frac{d}{dy_i}\left(\frac{r^p \partial_{y_i} r}{\sqrt{1+(\partial_z r)^2+(\partial_{y_j} r)^2+\frac{1}{r^2}(\partial_{\Omega_p} r)^2}}\right) \\ &\frac{1}{z^{d-1}} \frac{d}{d\Omega_p}\left(\frac{r^{p-2}\partial_{\Omega_p} r}{\sqrt{1+(\partial_z r)^2+(\partial_{y_j} r)^2+\frac{1}{r^2}(\partial_{\Omega_p} r)^2}}\right) - \frac{1}{z^{d-1}} \frac{d}{dr}\left(r^p \sqrt{1+(\partial_z r)^2+(\partial_{y_j} r)^2+\frac{1}{r^2}(\partial_{\Omega_p} r)^2}\right) = 0 \end{aligned} \tag{A.21}$$

The bulk surface expanded around the 0^{th} order cylinder is:

$$r = r_0(z) + \epsilon r_1(z, y_j, \Omega_p) + \epsilon^2 r_2(z, y_j, \Omega_p) + \ldots \tag{A.22}$$

The 0^{th} order EOM is:

$$\frac{d}{dz}\left(\frac{r_0^p}{z^{d-1}} \frac{\partial_z r_0}{\sqrt{1+(\partial_z r_0)^2}}\right) - \frac{1}{z^{d-1}} p r_0^{p-1} \sqrt{1+(\partial_z r_0)^2} = 0 \tag{A.23}$$

The 1^{st} order EOM is:

$$\partial_z^2 r_1 + \left[\frac{\frac{d}{dz}\left(\frac{r_0^p}{z^{d-1}} \frac{1}{(1+(\partial_z r_0)^2)^{3/2}}\right)}{\left(\frac{r_0^p}{z^{d-1}} \frac{1}{(1+(\partial_z r_0)^2)^{3/2}}\right)}\right] \partial_z r_1 + \left[1+(\partial_z r_0)^2\right]\left(\frac{p}{r_0^2} r_1 + \frac{1}{r_0^2} \partial^2_{\Omega_p} r_1 + \partial^2_{y_j} r_1\right) = 0 \tag{A.24}$$

We Fourier expand r_1 (see (3.12)):

$$r_1(z, y_j, \Omega_p) = \sum_{\{n_j, l_p\}} \left[a_{\{n_j, l_p\}} r^{(1)}_{\{n_j, l_p\}}(z) Y_{l_p}(\Omega_p) \prod_{i=1}^{d-2-p} \cos(n_i y_i)\right] \tag{A.25}$$

where the n_j are integers, and we defined the shorthand notation: $\{n_j\} \equiv n_1, \ldots, n_{d-2-p}$ and $\{l_p\} \equiv l, m_1 \ldots, m_{p-1}$. We plug (A.25) in (A.24):

$$\partial_z^2 r^{(1)}_{\{n_j,l_p\}} + \left[\frac{\frac{d}{dz}\left(\frac{r_0^p}{z^{d-1}}\frac{1}{(1+(\partial_z r_0)^2)^{3/2}}\right)}{\left(\frac{r_0^p}{z^{d-1}}\frac{1}{(1+(\partial_z r_0)^2)^{3/2}}\right)}\right]\partial_z r^{(1)}_{\{n_j,l_p\}} \\ + \left[1+(\partial_z r_0)^2\right]\left(\frac{p-l(l+p-1)}{r_0^2} - \tilde{n}^2\right) r^{(1)}_{\{n_j,l_p\}} = 0 \tag{A.26}$$

where we defined $\tilde{n}^2 \equiv \sum_j n_j^2$, and used $\partial^2_{\Omega_p} Y_{l_p} = -l(l+p-1)Y_{l_p}$ [64]. Since the parameters $m_1, \ldots, m_{p-1}$ do not appear in (A.26), the solution $r^{(1)}_{\{n_j,l_p\}}$ will not depend on them (and neither will $S_2^{\text{bound.}}\big|_{\log}$ in (3.17)).

For a plane entangling surface we have $p = 0$ and $r \equiv x$, and $\partial_z x_0 = 0$, and in this case (A.26) reduces to (A.15). On the other hand, for a sphere entangling surface we have $p = d-2$ and $r_0^2 = R^2 - z^2$. Thus for a sphere the EOM is:

$$\partial_z^2 r_l^{(1)} - \frac{1}{z}\frac{(d-1)R^2+2z^2}{R^2-z^2}\partial_z r_l^{(1)} + \frac{R^2[d-2-l(l+d-3)]}{(R^2-z^2)^2} r_l^{(1)} = 0 \tag{A.27}$$

We find a solution to this equation in terms of hypergeometric functions. For $d = 3$ the solution is:

$$r_l^{(1)}(z) = \frac{1}{\mathcal{N}}\left(\frac{z-R}{z+R}\right)^{\frac{l}{2}}\left(\frac{R+lz}{\sqrt{z^2-R^2}}\right) \tag{A.28}$$

which agrees with eq. 43 of [63].

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