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Form factors and scattering amplitudes in $\mathcal{N}=4$ SYM in dimensional and massive regularizations

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Abstract: The IR-divergent scattering amplitudes of $\mathcal{N}=4$ supersymmetric Yang-Mills theory can be regulated in a variety of ways, including dimensional regularization and massive (or Higgs) regularization. The IR-finite part of an amplitude in different regularizations generally differs by an additive constant at each loop order, due to the ambiguity in separating finite and divergent contributions. We give a prescription for defining an unambiguous, regulator-independent finite part of the amplitude by factoring off a product of IR-divergent "wedge" functions. For the cases of dimensional regularization and the common-mass Higgs regulator, we define the wedge function in terms of a form factor, and demonstrate the regularization independence of the n-point amplitude through two loops. We also deduce the form of the wedge function for the more general differential-mass Higgs regulator, although we lack an explicit operator definition in this case. Finally, using extended dual conformal symmetry, we demonstrate the link between the differential-mass wedge function and the anomalous dual conformal Ward identity for the finite part of the scattering amplitude.

Keywords: Supersymmetric gauge theory, AdS-CFT Correspondence, NLO Computations, 1/N Expansion

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Contents

1 Introduction and overview

Unraveling the pattern of soft and collinear divergences in scattering amplitudes is a critical endeavor to advance our understanding of gauge theories in general and to assist in concrete computations for collider phenomenology, e.g., in massless gauge theories such as Quantum Chromodynamics (QCD). These studies have a long history (see e.g. the early review [1]) and have contributed to our knowledge of the universal infrared (IR) structure of gauge theory amplitudes. Based on the concepts of soft and collinear factorization, non-abelian exponentiation, and the study of collinear limits, significant information about scattering amplitudes is available to all orders in perturbation theory. For precision predictions at modern colliders, especially within QCD and including higher order quantum corrections, these insights are of great practical importance [2, 3].

Quite generally, factorization implies the separation of scales in a given scattering reaction, i.e. the process-dependent hard scale q^2 from those governing the soft and collinear limit, defined for instance by the masses m_i of the scattering particles with $q^2 \gg m_i^2$ or by the regions of soft momenta. Note that the soft and collinear singularities of massless particles (gauge bosons) strictly require the definition of a regulator, which is conveniently performed in $D = 4-2\epsilon$ dimensions. As an immediate consequence of factorization, evolution equations emerge, which depend on the kinematics of the specific process and on the chosen regulator. Their solution leads to non-abelian exponentiation, a result which also arises from an effective field theory formulation based on the ultra-violet (UV) renormalization properties of effective operators and their anomalous dimensions; see e.g. ref. [4]. Moreover, for scattering amplitudes in gauge theories, the underlying factorization imposes strong constraints on the anomalous dimensions and the all-order structure of the IR singularities [5–8].

In the present paper we will specialize our investigations in a number of ways. First, we choose to work in the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, which is the simplest non-abelian gauge theory in four dimensions due to the vanishing of the four-dimensional β function. In our study, we are concerned with form factors and scattering amplitudes in this theory, which allows us to study their IR singularities without interference from UV divergences. Second, we will be working in the so-called planar limit and, for scattering amplitudes A_n of n external particles, we will assume color ordering. Our main focus is on the study of different kinematical regimes, i.e. scattering amplitudes of massless and massive particles and associated form factors, using different regulator schemes.

The general property of factorization prompts us to ask whether one can delineate a well-defined finite part of A_n independent of the chosen IR regularization. While reasoning along these lines has already been employed in the derivation of radiative corrections for heavy-quark hadroproduction at two loops in QCD [9, 10] (see also refs. [11, 12]), this issue is more generally related to the important question whether physical observables in theories with massless particles are independent of the regulator;[1] see e.g. the discussion in ref. [14]. To that end, in this paper we specifically compare dimensional and massive regularizations schemes for n-particle scattering amplitudes A_n in $\mathcal{N} = 4$ SYM theory, an ideal testing ground for these questions due to the simplicity of its loop expansion.[2] In-depth studies of the latter may lead to new insights for gauge theories with massive particles which will eventually also be of interest for applications in collider phenomenology.

Let us start thus, for simplicity, with maximally-helicity-violating (MHV) scattering amplitudes. Factorization implies that the color-ordered amplitude $A_n = A_n^{\rm tree} M_n$ of n particles can be written as (see e.g. ref. [16])

$$M_n = S_n \times J_n \times H_n \,. \tag{1.1}$$

Here S_n and J_n are "soft" and "jet" functions, respectively, and H_n is an IR-finite "hard function". In general, M_n and H_n are vectors in a space of possible color structures, and S_n is a matrix. In the planar limit, S_n is proportional to the identity matrix, and one can combine S_n and J_n into a product of "wedge" functions $W(s_{i-1,i})$ that depend only on two adjacent particles $i-1$ and i of the color-ordered amplitude [17]

$$M_n = H_n \times \prod_{i=1}^{n} W(s_{i-1,i})\,, \qquad s_{i-1,i} = (p_{i-1}+p_i)^2. \tag{1.2}$$

[1]The well-known physical evolution kernels are of course independent of the factorization scale μ^2 by construction; see e.g. ref. [13].

[2]In the context of $\mathcal{N} = 4$ SYM theory, IR-safe inclusive differential cross-sections were studied in ref. [15].

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As will be detailed below, the factorization (1.2) holds not only in dimensional regularization, but also in cases where masses are used to partially or fully regulate the IR divergences. The wedge functions $W(q^2)$ satisfy renormalization group equations which imply that they exponentiate. The factorization (1.2) fixes the hard function H_n only up to finite pieces, but with a suitable definition of the wedge function, we suggest that

$$\log H_n = \log M_n - \sum_{i=1}^{n} \log W(s_{i-1,i}) \tag{1.3}$$

can be used to define a regulator-independent finite part of the amplitude.

For regulators that leave the external particles massless, such as dimensional regularization in $D = 4-2\epsilon$ dimensions or the common-mass Higgs regulator described below, each wedge has half the IR divergences of a Sudakov form factor $\Phi(q^2)$ (see e.g. refs. [18, 19]), so it is natural to define $W(q^2) = \sqrt{\Phi(q^2)}$ [3, 17]. We show in this paper that, with this definition, $\log H_n$ is identical for both of these regulators through two-loop order.

We also analyze a refined version of the Higgs regulator with differential masses, described below. In this case, the external particles have distinct masses, and so the wedge function cannot simply be defined in terms of a Sudakov form factor. Instead, we define the one-loop wedge function in terms of a certain IR-divergent triangle diagram, and then use extended dual conformal invariance to extend this to an all-loop expression for the sum of wedge functions. With this choice for the IR-divergent wedge function, we establish that the IR-finite hard function $\log H_n$ takes precisely the same form for the differential-mass Higgs regulator as for the common-mass regulator. We lack, however, an explicit operator definition for the wedge function in this case.

For $\mathcal{N} = 4$ SYM theory, the regulator-independent hard function $\log H_n$ takes the simple form

$$\log H_n = \frac{\gamma(a)}{4} H_n^{(1)}(x_{ij}^2) + nD(a) + C(a) + R_n(x_{ij}^2, a) \tag{1.4}$$

due to the conjectured duality between the finite part of the MHV scattering amplitudes and the (UV renormalized) expectation value of certain cusped Wilson loops (see refs. [20, 21] for reviews). In eq. (1.4), $\gamma(a)$ is the cusp anomalous dimension [22], for which a prediction to all orders in the coupling constant $a = g^2N/(8\pi^2)$ exists [23], and $D(a)$ and $C(a)$ are kinematic-independent functions. The amplitude is expressed as a function of the dual or region momenta x_i^μ, which are defined by

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu \tag{1.5}$$

with $x_{i+n}^\mu \equiv x_i^\mu$, and $x_{ij}^2 = (x_i - x_j)^2$. The first three terms on the r.h.s. of eq. (1.4), whose kinematical dependence is determined solely by the one-loop contribution $H_n^{(1)}(x_{ij}^2)$, constitute the ABDK/BDS ansatz [17, 24]. The *a priori* undetermined remainder function $R_n(x_{ij}^2, a)$ contains the only non-trivial, i.e. loop-dependent, kinematical dependence. Equation (1.4) follows from a conformal Ward identity for the dual Wilson loop [25, 26]. The first term on the r.h.s. of eq. (1.4) provides a particular solution to this Ward identity. The remainder function R_n is the general homogeneous solution to the Ward identity, and depends only on dual conformal cross-ratios, which take the form $x_{ij}^2 x_{mn}^2/(x_{im}^2 x_{jn}^2)$. Due to the absence of dual conformal cross-ratios for $n = 4$ and $n = 5$, the remainder functions R_4 and R_5 vanish and therefore the corresponding hard functions $\log H_4$ and $\log H_5$ are completely determined by their one-loop value and the kinematic-independent functions $\gamma(a), D(a)$, and $C(a)$. For $n \geq 6$, dual conformal cross-ratios can be built, and the remainder function is known to be non-zero starting from two loops and $n = 6$ external particles [27, 28]. Its higher-loop and higher-point form is under intense investigation; see e.g. refs. [29–33].

The planar MHV n-point amplitude for $\mathcal{N} = 4$ SYM theory has been studied using dimensional regularization and also using an alternative massive IR regulator[3] [38–42]. The latter is motivated by the AdS/CFT correspondence and consists of computing scattering amplitudes on the Coulomb branch of $\mathcal{N} = 4$ SYM theory, i.e. giving a non-trivial vacuum expectation value to some of the scalars. One can achieve a situation where the propagators on the perimeter of any loop diagram are massive, thereby regulating the IR divergences. The simplest case, the "common mass Higgs regulator" in which only one mass m is introduced, corresponds to the breaking of the $\mathrm{U}(N+M)$ gauge group to $\mathrm{U}(N) \times \mathrm{U}(M)$, with fields in the adjoint representation of $\mathrm{U}(M)$ remaining massless.

In the more general "differential-mass Higgs regulator", one breaks the gauge group further to $\mathrm{U}(N) \times \mathrm{U}(1)^M$, thereby introducing various masses m_i, $i = 1, \ldots, M$. Fields in the adjoint of the broken $\mathrm{U}(M)$, which appear as external states in the scattering amplitudes, now have nonzero masses $|m_i - m_{i+1}| \neq 0$. We use a decomposition of the one-loop MHV n-point amplitude into a sum of IR-divergent triangle diagrams and IR-finite six-dimensional box integrals to define the sum of one-loop wedge functions as[4]

$$\sum_{i=1}^{n} W_{i-1,i,i+1}^{(1)}(x_{i-1,i+1}^2)\Big|_{\text{one-loop}} = \sum_{i=1}^{n} \left[-\frac{1}{4} \log^2\left(\frac{x_{i-1,i+1}^2}{m_i^2}\right) + \frac{1}{8} \log^2\left(\frac{m_{i+1}^2}{m_i^2}\right)\right] \tag{1.6}$$

in the uniform small mass limit (i.e., $m_i = \alpha_i m$, with α_i fixed and $m \to 0$). The one-loop hard function $H_n^{(1)}(x_{ij}^2)$ is then expressed in terms of IR-finite quantities, and thus is manifestly regulator-independent.

A key point is that the massive regulator is closely connected to dual conformal symmetry. The Higgs masses can be interpreted within the AdS/CFT duality as the radial coordinates in a T-dual AdS_5 space. While the isometries of this space yield the usual dual conformal transformations for zero masses, they define a different realization of this symmetry for finite masses, dubbed "extended dual conformal symmetry" [38]. Since no further regulator is needed in the massive setup, the extended dual conformal symmetry is expected to be an exact symmetry of the planar amplitudes. Recently, it was shown that tree-level amplitudes on the Coulomb branch of $\mathcal{N} = 4$ SYM and also all cuts of planar loop amplitudes do indeed have this extended symmetry [43, 44]. Assuming that $\mathcal{N} = 4$ SYM theory remains cut-constructible on the Coulomb branch, this can then be utilized in proving the extended dual conformal symmetry property conjectured in ref. [38].

[3] For earlier applications of a massive IR regulator, see refs. [34–37].

[4] The subscripts on the wedge function refer to its dependence on m_{i-1}, m_i, and m_{i+1}; see eq. (4.9).

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What we wish to emphasize is that, while planar amplitudes have extended dual conformal symmetry, the wedge functions and regulator-independent hard functions separately do not. This is not surprising, since extended dual conformal transformations act on the masses m_i (as well as the dual variables x_i), whereas the hard function is, by definition, independent of the masses in the uniform small mass limit. Nevertheless, extended dual conformal symmetry can be used to determine the all-loop structure of the IR divergences of scattering amplitudes in the case of the differential-mass Higgs regulator. Assuming the exponentiation of Higgs-regulated MHV scattering amplitudes, together with eq. (1.6), we obtain the following expression for the IR-divergent pieces in the differential-mass setup

$$\begin{aligned}
&\sum_{i=1}^{n} \log W_{i-1,i,i+1}(x_{i-1,i+1}^2) \\
&\quad = \sum_{i=1}^{n} \left[-\frac{\gamma(a)}{16} \log^2 \left(\frac{x_{i-1,i+1}^2}{m_i^2} \right) - \frac{\tilde{\mathcal{G}}_0(a)}{2} \log \left(\frac{x_{i-1,i+1}^2}{m_i^2} \right) + \frac{\gamma(a)}{32} \log^2 \left(\frac{m_{i+1}^2}{m_i^2} \right) + \tilde{w}(a) \right]
\end{aligned} \tag{1.7}$$

valid for uniform small masses.

Having deduced the form of the IR divergences of the amplitude for the differential-mass regulator, we turn the argument around and use eq. (1.7) together with extended dual conformal symmetry to deduce the anomalous dual conformal Ward identity, from which the all-loop result (1.4) follows. Hence, a derivation of eq. (1.7) from first principles would constitute a proof of eq. (1.4) without having to rely on the scattering amplitude/Wilson loop duality. It would be very interesting to understand the origin of eq. (1.7) from a renormalization group approach. A first step could be to find a suitable operator definition for the wedge function in the differential-mass regulator case. We leave these questions for future work.

This paper is organized as follows. In section 2, we review the form of color-ordered MHV scattering amplitudes in planar $\mathcal{N}=4$ SYM in dimensional regularization and in the massive regularization of ref. [38]. In section 3 we discuss factorization and exponentiation properties of scattering amplitudes and form factors. We propose a definition, involving Sudakov form factors, for a regulator-independent hard function that can be computed from the IR-divergent scattering amplitudes. We review the result for the form factors up to two loops in dimensional regularization, and compute the analogous quantities to two-loop order in the massive regularization. We then show that the (logarithm of the) hard function defined earlier is the same in both cases. In section 4, we discuss a more general differential-mass Higgs regularization, and compute the IR-divergent terms of the one-loop amplitude in this regularization. We then use extended dual conformal symmetry to derive the all-loop form of the IR-divergent terms, and discuss their relation to the dual conformal Ward identity. Section 5 contains our conclusions, and two appendices contain technical details used in the paper.

2 Review of MHV amplitudes in $\mathcal{N}=4$ SYM

In this section, we briefly review the form of color-ordered MHV amplitudes in planar $\mathcal{N}=4$ SYM theory in different regularization schemes.

The all-loop-order n-point amplitude is given by the tree-level amplitude times a helicity-independent function M_n, which we expand in the 't Hooft parameter

$$M_n = 1 + \sum_{\ell=1}^{\infty} a^{\ell} M_n^{(\ell)}\,, \qquad a = \frac{g^2 N}{8\pi^2}(4\pi e^{-\gamma})^{\epsilon} \tag{2.1}$$

where $\epsilon = (4-D)/2$. Loop-level amplitudes are UV-finite but suffer from IR divergences which can be regulated using either dimensional regularization in D dimensions, or a Higgs regulator in four dimensions. We discuss each of these in turn.

2.1 Dimensional regularization of amplitudes

In dimensional regularization, the n-point amplitude takes the form [17]

$$\begin{aligned}\log M_n = &\sum_{\ell=1}^{\infty} a^{\ell}\left[-\frac{\gamma^{(\ell)}}{8(\ell\epsilon)^2} - \frac{\mathcal{G}_0^{(\ell)}}{4\ell\epsilon}\right]\sum_{i=1}^{n}\left(\frac{\mu^2}{x_{i-1,i+1}^2}\right)^{\ell\epsilon}\\ &+\frac{\gamma(a)}{4}F_n^{(1)}(x_{ij}^2) + nf(a) + C(a) + R_n(x_{ij}^2,a) + \mathcal{O}(\epsilon)\,.\end{aligned} \tag{2.2}$$

The momentum dependence of the amplitude is expressed in terms of dual variables x_i defined via $x_i - x_{i+1} = p_i$, where p_i are the momenta of the external states; we also define $x_{ij}^2 \equiv (x_i - x_j)^2$. The terms on the first line of eq. (2.2) are IR-divergent and are specified in terms of the cusp and collinear anomalous dimensions [22]

$$\gamma(a) = \sum_{\ell=1}^{\infty}\gamma^{(\ell)}a^{\ell} = \sum_{\ell=1}^{\infty}4f_0^{(\ell)}a^{\ell} = 4a - 4\zeta_2 a^2 + 22\zeta_4 a^3 + \mathcal{O}(a^4)\,, \tag{2.3}$$

$$\mathcal{G}_0(a) = \sum_{\ell=1}^{\infty}\mathcal{G}_0^{(\ell)}a^{\ell} = \sum_{\ell=1}^{\infty}\frac{2f_1^{(\ell)}}{\ell}a^{\ell} = -\zeta_3 a^2 + \left(4\zeta_5 + \frac{10}{3}\zeta_2\zeta_3\right)a^3 + \mathcal{O}(a^4)\,. \tag{2.4}$$

The terms on the second line of eq. (2.2) are IR-finite and are determined by the finite part of the one-loop amplitude

$$F_n^{(1)} \equiv M_n^{(1)} + \frac{1}{2\epsilon^2}\sum_{i=1}^{n}\left(\frac{\mu^2}{x_{i-1,i+1}^2}\right)^{\epsilon} \tag{2.5}$$

as well as the constants [17]

$$f(a) = -\sum_{\ell=1}^{\infty}\frac{f_2^{(\ell)}}{2\ell^2}a^{\ell} = \frac{\pi^4}{720}a^2 + \left(-\frac{c_1}{18}\zeta_6 - \frac{c_2}{18}\zeta_3^2\right)a^3 + \mathcal{O}(a^4)\,, \tag{2.6}$$

$$C(a) = -\frac{\pi^4}{72}a^2 + \left[\left(\frac{341}{216} + \frac{2c_1}{9}\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2c_2}{9}\right)\zeta_3^2\right]a^3 + \mathcal{O}(a^4)\,, \tag{2.7}$$

and a remainder function $R_n(x_{ij}^2,a)$ potentially contributing beginning at two loops. The original proposal by Bern, Dixon, and Smirnov [17] conjectured that eq. (2.2) holds with $R_n(x_{ij}^2,a) = 0$. Explicit calculations bore this out for $n=4$ (through four loops) [45] and $n=5$ (through two loops) [46], but the two-loop calculation for $n=6$ [27, 28, 47] revealed the necessity for a non-constant function $R_6(x_{ij}^2,a)$.

Explicit expressions for eq. (2.5) are given in ref. [17]. For $n = 4$ and $n = 5$, they are

$$F_4^{(1)} = \frac{1}{2}\log^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \frac{2\pi^2}{3}\,, \tag{2.8}$$

$$F_5^{(1)} = -\frac{1}{4}\sum_{i=1}^{5}\log\left(\frac{x_{i,i+2}^2}{x_{i+1,i+3}^2}\right)\log\left(\frac{x_{i-1,i+1}^2}{x_{i+2,i+4}^2}\right) + \frac{5\pi^2}{8}\,. \tag{2.9}$$

2.2 Higgs regularization of amplitudes

The four-, five-, and six-point functions have also been computed [38–41] using the common-mass Higgs regulator described in the introduction. These amplitudes exhibit an exponentiation similar to eq. (2.2) which motivated the following analog for Higgs-regulated n-point amplitudes [38, 40]

$$\begin{aligned}\log M_n = \sum_{i=1}^{n}\left[-\frac{\gamma(a)}{16}\log^2\left(\frac{x_{i-1,i+1}^2}{m^2}\right) - \frac{\tilde{\mathcal{G}}_0(a)}{2}\log\left(\frac{x_{i-1,i+1}^2}{m^2}\right)\right] \\ +\frac{\gamma(a)}{4}\tilde{F}_n^{(1)}(x_{ij}^2) + n\tilde{f}(a) + \tilde{C}(a) + \tilde{R}_n(x_{ij}^2, a) + \mathcal{O}(m^2)\,.\end{aligned} \tag{2.10}$$

The terms on the first line of eq. (2.10) are IR-divergent. The cusp anomalous dimension (2.3) is independent of the regularization scheme, but the analog of the collinear anomalous dimension is given by

$$\tilde{\mathcal{G}}_0(a) = -\zeta_3 a^2 + \left(\frac{9}{2}\zeta_5 - \zeta_2\zeta_3\right)a^3 + \mathcal{O}(a^4)\,. \tag{2.11}$$

The terms on the second line of eq. (2.10) are IR-finite and are determined by the finite part of the one-loop amplitude

$$\tilde{F}_n^{(1)} \equiv M_n^{(1)} + \frac{1}{4}\sum_{i=1}^{n}\log^2\left(\frac{x_{i-1,i+1}^2}{m^2}\right) \tag{2.12}$$

as well as the constants [40]

$$\tilde{f}(a) = \frac{\pi^4}{180}a^2 + \mathcal{O}(a^3)\,, \tag{2.13}$$

$$\tilde{C}(a) = -\frac{\pi^4}{72}a^2 + \mathcal{O}(a^3) \tag{2.14}$$

and a remainder function $\tilde{R}_n(x_{ij}^2, a)$. As in the case of dimensional regularization, the remainder function vanishes for four- and five-point amplitudes. For $n = 6$, it was shown [41] that the two-loop remainder function $\tilde{R}_6^{(2)}(x_{ij}^2)$ in the Higgs-regulated amplitude is precisely equal to its value $R_6^{(2)}(x_{ij}^2)$ in dimensional regularization, and this is expected to hold generally.

The one-loop amplitudes may be evaluated to show [40]

$$\tilde{F}_4^{(1)} = \frac{1}{2}\log^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \frac{\pi^2}{2} = F_4^{(1)} - \frac{\pi^2}{6}\,, \tag{2.15}$$

$$\tilde{F}_5^{(1)} = -\frac{1}{4}\sum_{i=1}^{5}\log\left(\frac{x_{i,i+2}^2}{x_{i+1,i+3}^2}\right)\log\left(\frac{x_{i-1,i+1}^2}{x_{i+2,i+4}^2}\right) + \frac{5\pi^2}{12} = F_5^{(1)} - \frac{5\pi^2}{24}\,, \tag{2.16}$$

and more generally [41]

$$\tilde{F}_n^{(1)} = F_n^{(1)} - \frac{n\pi^2}{24}\,. \tag{2.17}$$

3 Defining a regulator-independent IR-finite amplitude

Comparing the known expressions for Higgs-regulated amplitudes (2.10) with those for dimensionally-regulated ones (2.2), one observes that the IR-finite parts of the amplitudes are equal in both regularizations, up to constants. In this section, we make the connection more precise by introducing a regulator-independent expression for the finite part of the amplitude.

In a planar theory, the factorization (see e.g. ref. [16]) of color-ordered amplitudes takes the specific form [17]

$$M_n = \left[\prod_{i=1}^{n} W(x_{i-1,i+1}^2)\right] H_n(x_{ij}^2) \tag{3.1}$$

where $W(x_{i-1,i+1}^2)$ is an IR-divergent "wedge function" depending only on $(p_{i-1}+p_i)^2$ and resulting from the exchange of soft gluons in the wedge between the $(i-1)$th and ith external particles, and $H_n(x_{ij}^2)$ is an IR-finite hard function. With a suitable definition for $W(x_{i-1,i+1}^2)$, we can use

$$\log H_n = \log M_n - \sum_{i=1}^{n} \log W(x_{i-1,i+1}^2) \tag{3.2}$$

to define the IR-finite part of the amplitude. The forms of both M_n and W will depend on the specific regulator, but we will find that $\log H_n$ is regulator-independent.

3.1 Dimensional regularization of the form factor

In dimensional regularization, the wedge function can be defined as the square root of the gluon form factor [3, 17]. Form factors in $\mathcal{N}=4$ SYM have been studied at strong coupling [48, 49], at one loop [50] and at two loops [51, 52], while three-loop results can be inferred from the respective QCD computations [53–58] using the principle of maximal transcendentality; see e.g. ref. [59].

In $\mathcal{N}=4$ SYM we can equivalently use the form factor

$$\Phi(q^2) = \langle J, p_i | \mathcal{O}_{IJ}(q) | I, p_{i-1}\rangle \tag{3.3}$$

for scalars ϕ_I coupling to the operator

$$\mathcal{O}_{IJ} = \mathrm{Tr}\left[\phi_I\phi_J - \frac{1}{6}\delta_{IJ}\sum_{K=1}^{6}\phi_K\phi_K\right] \tag{3.4}$$

with $q^2 = (p_{i-1}+p_i)^2$. The operator $\mathcal{O}_{IJ}$ belongs to the stress-energy multiplet of $\mathcal{N}=4$ SYM and is not UV renormalized. This form factor has been computed to two loops in dimensional regularization [51]

$$\Phi(q^2) = 1 + aq^2[-I_b] + a^2(q^2)^2\left[I_c + \frac{1}{4}I_d\right] + \cdots \tag{3.5}$$

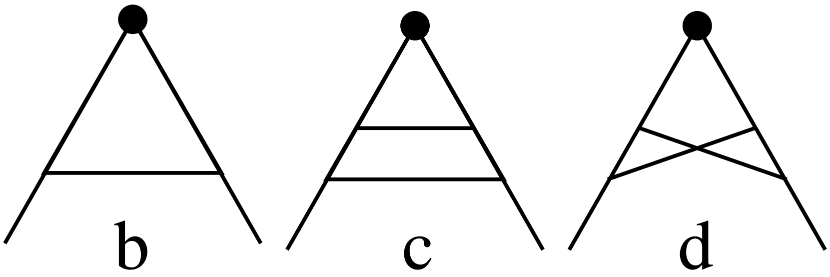

Figure 1. One- and two-loop form factor diagrams in dimensional regularization. All lines represent massless adjoint fields. The dot represents the insertion of $\mathcal{O}_{IJ}$.

where I_i represent the scalar integrals shown in figure 1. (Despite its apparent non-planarity, integral I_d is actually leading order in the $1/N$ expansion as is clear from the corresponding double-line diagram in figure 3.) The explicit expressions for these integrals given in appendix A reveal that the form factor exponentiates to two-loop order

$$\log\Phi(q^2) = a\left(\frac{\mu^2}{q^2}\right)^{\epsilon}\left[-\frac{1}{\epsilon^2}+\frac{\pi^2}{12}+\mathcal{O}(\epsilon)\right]+a^2\left(\frac{\mu^2}{q^2}\right)^{2\epsilon}\left[\frac{\pi^2}{24\epsilon^2}+\frac{\zeta_3}{4\epsilon}+\mathcal{O}(\epsilon)\right]+\mathcal{O}(a^3)\,. \tag{3.6}$$

Equation (3.6) can be promoted to all orders in perturbation theory since the momentum dependence of Φ is governed by an evolution equation (see e.g. refs. [18, 19, 53, 60]). In $D = 4-2\epsilon$ dimensions, the following factorization ansatz holds:

$$q^2\frac{\partial}{\partial q^2}\log\Phi(q^2) = \frac{1}{2}\,K(a,\epsilon)+\frac{1}{2}\,G(q^2/\mu^2,a,\epsilon)\,. \tag{3.7}$$

All dependence of Φ on the hard momentum q^2, which is taken to be Euclidean ($q^2 > 0$ in our mostly-plus metric convention) here and in the sequel, rests inside the function $G(q^2/\mu^2, a, \epsilon)$. The latter is finite in four dimensions and can be considered as a suitable continuation of the collinear anomalous dimension (2.4) to $D = 4-2\epsilon$ dimensions. $K(a, \epsilon)$ on the other hand serves as a pure counterterm. Renormalization group invariance of Φ yields

$$-\,\mu\frac{d}{d\mu}K(a,\epsilon) = \mu\frac{d}{d\mu}G(q^2/\mu^2,a,\epsilon) = \gamma(a) \tag{3.8}$$

i.e., G and K are both governed by the cusp anomalous dimension $\gamma(a)$ of eq. (2.3). The solutions of eqs. (3.7) and (3.8) can be conveniently given with the help of the D-dimensional continuation $\bar{a}$ of the 't Hooft parameter defined in eq. (2.1)

$$\bar{a}(q^2,\epsilon) = a\left(\frac{\mu^2}{q^2}\right)^{\epsilon} \tag{3.9}$$

which exhibits scale dependence on dimensional grounds and vanishes in the IR for $D = 4-2\epsilon$ with $\epsilon < 0$. Using eq. (3.9) and exploiting the fact that K has no explicit scale dependence, which allows one to express it entirely through $\gamma(a)$, one arrives at the following all-order expression for Φ

$$\log \Phi(q^2) = \frac{1}{2}\int_0^{q^2} \frac{d\xi}{\xi}\Big\{K(a,\epsilon) + G\big(1,\bar{a}(\xi,\epsilon),\epsilon\big) - \frac{1}{2}\int_{\mu^2}^{\xi}\frac{d\lambda}{\lambda}\,\gamma\big(\bar{a}(\lambda,\epsilon)\big)\Big\}$$

$$= \frac{1}{2}\int_0^{q^2} \frac{d\xi}{\xi}\Big\{G\big(1,\bar{a}(\xi,\epsilon),\epsilon\big) - \frac{1}{2}\int_0^{\xi}\frac{d\lambda}{\lambda}\,\gamma\big(\bar{a}(\lambda,\epsilon)\big)\Big\} \qquad (3.10)$$

where the explicit solution of eq. (3.8) for the counterterm function K has been used

$$K(a,\epsilon) = -\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda}{\lambda}\,\gamma\big(\bar{a}(\lambda,\epsilon)\big)\,. \qquad (3.11)$$

The double poles $1/\epsilon^2$ at each loop order are generated by the two λ- and ξ-integrations over γ, while the single poles in ϵ arise from the outer ξ-integration over G. Explicit computation, e.g. along the lines of refs. [17, 53], yields

$$\log \Phi(q^2) = \sum_{\ell=1}^{\infty} a^\ell \left[-\frac{\gamma^{(\ell)}}{4(\ell\epsilon)^2} - \frac{\mathcal{G}_0^{(\ell)}}{2\ell\epsilon} - \frac{\phi^{(\ell)}}{2\ell} + \mathcal{O}(\epsilon)\right]\left(\frac{\mu^2}{q^2}\right)^{\ell\epsilon} \qquad (3.12)$$

where the boundary condition for G has been chosen as

$$G(1,a,\epsilon) = \mathcal{G}_0(a) + \epsilon\phi(a) + \mathcal{O}(\epsilon^2) \qquad (3.13)$$

which is consistent with eq. (3.6). $\mathcal{G}_0(a)$ is given in eq. (2.4) and $\phi(a)$ can be read off from eq. (3.6) as

$$\phi(a) = -\frac{\pi^2}{6}a + \mathcal{O}(a^3)\,. \qquad (3.14)$$

The exponentiation of eq. (3.10) proceeds trivially with the help of the boundary condition for Φ in D dimensions, i.e., $\Phi(q^2=0)=1$, which is implicit also in our choice for $G(1,a,\epsilon)$ in eq. (3.13). Note also that the all-order result for the form factor in eq. (3.10) applies literally to theories with less supersymmetry, e.g., to QCD. There, the coupling constant $\bar{a}$ has to be read as the strong coupling α_s continued to D-dimensions and the respective QCD expressions for the anomalous dimensions $\gamma(a)$ and $\mathcal{G}_0(a)$ are related to eqs. (2.3) and (2.4) by the principle of maximal transcendentality. Moreover, $G(1,a,\epsilon)$ admits a further decomposition [19] into three terms: a universal (spin-independent) eikonal anomalous dimension, (twice) the coefficient of the $\delta(1-x)$-term in the collinear evolution kernel, and a process-dependent term accounting for the running coupling in the coefficient function of the hard scattering. The latter is proportional to the QCD β-function and is, of course, absent in $\mathcal{N}=4$ SYM.

We now introduce the wedge function as announced above. Defining $W(q^2) = \sqrt{\Phi(q^2)}$, we see from eq. (3.12) that

$$\log W(q^2) = \sum_{\ell=1}^{\infty} a^\ell \left[-\frac{\gamma^{(\ell)}}{8(\ell\epsilon)^2} - \frac{\mathcal{G}_0^{(\ell)}}{4\ell\epsilon} + w^{(\ell)} + \mathcal{O}(\epsilon)\right]\left(\frac{\mu^2}{q^2}\right)^{\ell\epsilon}, \qquad (3.15)$$

where

$$w(a) = \frac{\pi^2}{24} a + \mathcal{O}(a^3) \tag{3.16}$$

and $\gamma(a)$ and $\mathcal{G}_0(a)$ are given in eqs. (2.3) and (2.4). With eq. (3.15) at our disposal, and using exponentiation of the n-point amplitude M_n in eq. (2.2), we can now define the finite remainder H_n via eq. (3.2)

$$\log H_n = \frac{\gamma(a)}{4} F_n^{(1)}(x_{ij}^2) + n[f(a) - w(a)] + C(a) + R_n(x_{ij}^2, a) + \mathcal{O}(\epsilon) \tag{3.17}$$

thus the IR-divergent pieces of the wedge function remove all the IR divergences of the n-point amplitude. At one loop, eq. (3.17) gives

$$H_n^{(1)} = F_n^{(1)} - \frac{n\pi^2}{24} \tag{3.18}$$

allowing us to rewrite it in its final form as

$$\log H_n = \frac{\gamma(a)}{4} H_n^{(1)}(x_{ij}^2) + nD(a) + C(a) + R_n(x_{ij}^2, a) + \mathcal{O}(\epsilon) \tag{3.19}$$

where

$$D(a) = f(a) - w(a) + \frac{\pi^2}{96}\gamma(a) = -\frac{\pi^2}{180} a^2 + \mathcal{O}(a^3) \tag{3.20}$$

with $C(a)$ given by eq. (2.7).

Using renormalization group arguments similar to the derivation of eq. (3.10) for the form factor Φ, it is obvious that $\log W$, $\log M_n$, and therefore also $\log H_n$ in eq. (3.19) can be expressed to all orders via (double-)integrals over the respective anomalous dimensions; see e.g. refs. [3, 16].

3.2 Higgs regularization of the form factor

Now we turn to the study of the Higgs-regulated $\mathcal{N} = 4$ SYM form factor $\Phi(q^2)$. We will assume that it is given by the same scalar integrals as in figure 1 except that some of the internal legs are now massive

$$\Phi(q^2) = 1 + aq^2\big[-\tilde{I}_b\big] + a^2(q^2)^2\left[\frac{1}{2}\tilde{I}_{c1} + \frac{1}{2}\tilde{I}_{c2} + \frac{1}{4}\tilde{I}_d\right] + \cdots \tag{3.21}$$

There are several different mass assignments for the two-loop integrals (see figure 2), as can be seen from the double-line representation in figure 3. We have computed these integrals (see appendix A) and they reveal that the Higgs-regulated form factor exponentiates to two-loop order

$$\log\Phi(q^2) = \tag{3.22}$$
$$a\left[-\frac{1}{2}\log^2\left(\frac{q^2}{m^2}\right) + \mathcal{O}(m^2)\right] + a^2\left[\frac{\pi^2}{12}\log^2\left(\frac{q^2}{m^2}\right) + \zeta_3\log\left(\frac{q^2}{m^2}\right) + \frac{\pi^2}{45} + \mathcal{O}(m^2)\right] + \mathcal{O}(a^3)\,.$$

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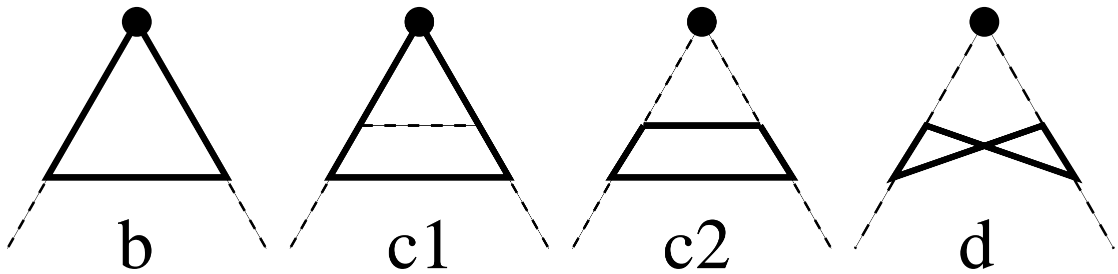

Figure 2. One- and two-loop form factor diagrams for the common-mass Higgs regulator. The solid/dashed lines represent massive/massless adjoint fields.

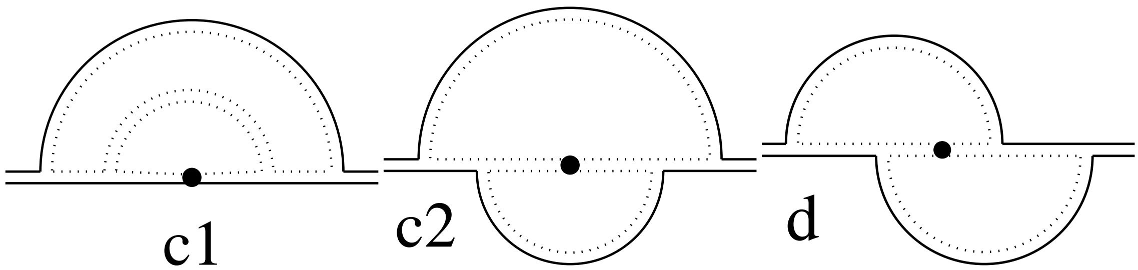

Figure 3. Double-line version of the two-loop diagrams for the common-mass Higgs regulator. The solid/dotted lines represent fundamental fields of $\mathrm{U}(M)/\mathrm{U}(N)$.

The all-loop-order generalization of eq. (3.22) relies on the same factorization ansatz discussed before and the separation of scales, i.e. $q^2 \gg m^2$, so that the momentum dependence of Φ is described by the evolution equation [1, 61]

$$q^2 \frac{\partial}{\partial q^2} \log \Phi(q^2) = \frac{1}{2}\,\tilde{K}(m^2/\mu^2, a) + \frac{1}{2}\,\tilde{G}(q^2/\mu^2, a) \tag{3.23}$$

and, consistent with eq. (3.8), the renormalization group equation for $\tilde{K}$ now reads

$$\lim_{m\to 0} \mu \frac{d}{d\mu} \tilde{K}(m^2/\mu^2, a) = -\gamma(a) \tag{3.24}$$

where the limiting procedure $m \to 0$ indicates that we neglect any power suppressed terms of $\mathcal{O}(m^2)$. It is subject to the following solution

$$\tilde{K}(m^2/\mu^2, a) = \tilde{K}(1, a) - \frac{1}{2} \int\limits_{m^2}^{\mu^2} \frac{d\lambda}{\lambda}\,\gamma(a) \tag{3.25}$$

with a non-vanishing boundary condition $\tilde{K}(1, a)$, since the IR sector has been altered in contrast to eq. (3.11). The solution to the Higgs-regularized form factor Φ then becomes

$$\log \Phi(q^2) = \frac{1}{2} \int\limits_{m^2}^{q^2} \frac{d\xi}{\xi} \left\{ \tilde{K}(1, a) + \tilde{G}(1, a) - \frac{1}{2} \int\limits_{m^2}^{\xi} \frac{d\lambda}{\lambda}\,\gamma(a) \right\} + \tilde{\phi}(a) \tag{3.26}$$

where the integration range is naturally cut off in the IR at m^2, i.e. at the mass scale set by the Higgs regulator, and the function

$$\tilde{\phi}(a) = \frac{\pi^2}{45} a^2 + \mathcal{O}(a^3) \tag{3.27}$$

has been introduced to match the fixed-order computation in eq. (3.22). Note that the evolution equation for the Higgs-regulated form factor in $\mathcal{N} = 4$ SYM depends only on the sum of $\tilde{K}(1, a)$ and $\tilde{G}(1, a)$. To agree with the fixed-order computation in eq. (3.22), we choose (cf. eq. (2.11))

$$\tilde{K}(1, a) + \tilde{G}(1, a) = -2\tilde{\mathcal{G}}_0(a) \tag{3.28}$$

leading to the solution of eq. (3.26)

$$\log \Phi(q^2) = -\frac{\gamma(a)}{8} \log^2\left(\frac{q^2}{m^2}\right) - \tilde{\mathcal{G}}_0(a) \log\left(\frac{q^2}{m^2}\right) + \tilde{\phi}(a) + \mathcal{O}(m^2)\,. \tag{3.29}$$

The exponentiation of eq. (3.26) requires further matching conditions for Φ to be obtained from explicit ℓ-loop computations.

A few comments are in order here. First, matching to fixed-order computations could, in principle, also impose the condition $\tilde{G} = G$, i.e. demand that the collinear anomalous dimensions coincide. This would proceed at the expense of a non-zero result for $\tilde{K}(1, a)$. Next, the Higgs-regulated form factor is finite, so that eq. (3.26) can be evaluated in four dimensions. In theories with broken supersymmetry, e.g. QCD with massive quarks, collinear singularities are regulated by the heavy quark masses, whereas all soft gluon divergences require dimensional regularization. In such a case, the analogous functions K and G have a clear physical interpretation and are independent (see e.g. ref. [61]). For example, the (electric) form factor of a massive quark-anti-quark pair in QCD is known to two loops [62, 63], and the analogs of collinear anomalous dimensions naturally coincide in this case, i.e. $\tilde{G} = G$.

In the case of the common-mass Higgs regulator (so that the external states are massless), the wedge function can again be defined as the square root of the form factor (3.29) so that

$$\log W(q^2) = -\frac{\gamma(a)}{16} \log^2\left(\frac{q^2}{m^2}\right) - \frac{\tilde{\mathcal{G}}_0(a)}{2} \log\left(\frac{q^2}{m^2}\right) + \tilde{w}(a) + \mathcal{O}(m^2) \tag{3.30}$$

where

$$\tilde{w}(a) = \frac{\pi^2}{90} a^2 + \mathcal{O}(a^3) \tag{3.31}$$

and $\gamma(a)$ and $\tilde{\mathcal{G}}_0(a)$ were given in eqs. (2.3) and (2.11). We now use eqs. (2.10) and (3.30) in eq. (3.2) to define the finite part of the n-point amplitude

$$\begin{aligned} \log \tilde{H}_n &= \log M_n - \sum_{i=1}^{n} \log W(x_{i-1,i+1}^2) \\ &= \frac{\gamma(a)}{4} \tilde{F}_n^{(1)} + n\big[\tilde{f}(a) - \tilde{w}(a)\big] + \tilde{C}(a) + \tilde{R}_n + \mathcal{O}(m^2) \end{aligned} \tag{3.32}$$

again finding that the IR-divergent pieces of the wedge function precisely remove the IR divergences of the n-point amplitude. At one loop, eq. (3.32) gives

$$\tilde{H}_n^{(1)} = \tilde{F}_n^{(1)} \tag{3.33}$$

so that we can rewrite the finite part of the amplitude in its final form

$$\log \tilde{H}_n = \frac{\gamma(a)}{4} \tilde{H}_n^{(1)} + n\tilde{D}(a) + \tilde{C}(a) + \tilde{R}_n(x_{ij}^2) + \mathcal{O}(\epsilon) \tag{3.34}$$

where

$$\tilde{D}(a) = \tilde{f}(a) - \tilde{w}(a) = -\frac{\pi^2}{180} a^2 + \mathcal{O}(a^3) \tag{3.35}$$

with $\tilde{C}(a)$ given by eq. (2.14).

In complete analogy to the previous discussion, it is evidently possible to exploit renormalization group properties to provide expressions for $\log W$, $\log M_n$, and hence $\log \tilde{H}_n$ in eq. (3.34) in terms of (double-)integrals over the anomalous dimensions similar to eq. (3.26).

3.3 Comparison of regulators

By comparing the results of the last two subsections, we can see that $\log H_n(x_{ij}^2)$ as we have defined it in eq. (3.2) is a good candidate for a regularization-independent IR-finite quantity characterizing the planar MHV n-point amplitude. The one-loop hard functions are identical in both dimensional and Higgs regularization (cf. eqs. (2.17), (3.18), and (3.33))

$$H_n^{(1)}(x_{ij}^2) = \tilde{H}_n^{(1)}(x_{ij}^2)\,. \tag{3.36}$$

Moreover, calculations through two loops show the equality of the kinematic-independent functions appearing in the n-point amplitude (3.19) and (3.34)

$$C(a) = \tilde{C}(a)\,, \qquad D(a) = \tilde{D}(a) \tag{3.37}$$

(cf. eqs. (2.7) and (2.14) and eqs. (3.20) and (3.35)). The regulator independence of $C(a)$ was previously observed in ref. [40]. If eq. (3.37) holds to all loops, then the regulator independence of the four- and five-point hard functions necessarily follows. For $n = 6$ at two loops, agreement between the remainder function in dimensional regularization and massive regularization was observed in ref. [41], and this agreement is expected for all n. We thus expect the hard function $\log H_n$ to be regularization-independent for all n-point functions, that is

$$\log \tilde{H}_n(x_{ij}^2) = \log H_n(x_{ij}^2) \tag{3.38}$$

for dimensional and Higgs regularizations.

4 Differential-mass Higgs regulator

In section 3, we defined an IR-finite hard function $\log H_n$ for the n-point amplitude, and showed (through two loops) that it has the same form (including constants) for dimensional regularization and for a common-mass Higgs regulator. In this section, we generalize our

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discussion to a more general class of regulators, viz., the Higgs regulator with arbitrary distinct masses. This is also interesting from the point of view of collider phenomenology. In QCD, amplitudes with different masses have been considered to two loops for the heavy-to-light transitions, i.e., the (axial)-vector form factor with one massive and one massless quark [64–67]. Also, electroweak logarithms in four-fermion processes at high energy arising from loop corrections with massive W- and Z-gauge bosons have been considered to two loops (see e.g. ref. [68]).

Recall that breaking the $\mathrm{U}(N+M)$ symmetry of $\mathcal{N}=4$ SYM theory to $\mathrm{U}(N)\times\mathrm{U}(1)^M$ by assigning distinct vacuum expectation values to one of the scalar fields results in non-zero masses $|m_i-m_j|$ for the off-diagonal adjoint fields and distinct masses m_i for the internal propagators of the scalar integrals that characterize loop amplitudes. (In fact, extended dual conformal invariance requires the freedom to vary the masses.) One can then define a differential-mass Higgs regulator by taking all the masses m_i to zero. More precisely, if $m_i=\alpha_i m$, the "uniform small mass limit" is defined as the limit $m\to 0$ with α_i held fixed. Regulator independence means that the result does not depend on the choice of α_i.

Because the external legs of the n-point amplitude now have distinct masses $|m_i - m_{i+1}|$, it is no longer possible to define the wedge function $W(x^2_{i-1,i+1})$ as the square root of a form factor as we did in section 3. In fact, it is not obvious what the operational definition of $W(x^2_{i-1,i+1})$ for the differential-mass Higgs regulator should be, and we leave this question to the future.

For now, we adopt a different approach by decomposing the one-loop n-point amplitude into an IR-divergent and a manifestly regulator-independent IR-finite piece, and then defining the one-loop wedge function in terms of the former. The extended dual conformal invariance of the BDS ansatz then allows us to generalize this to an all-loop wedge function.

4.1 One-loop amplitude with differential-mass Higgs regulator

As is well known, the one-loop MHV n-point amplitude in dimensional regularization can be written as a sum of two-mass-easy (and one-mass[5]) scalar box integrals [69, 70]

$$M_n^{(1)} = -\frac{1}{8}\sum_{i=1}^{n}\sum_{j=i+2}^{i+n-2} I_{ij}^{\rm 2me}. \tag{4.1}$$

We will assume that the amplitude on the Coulomb branch is given, at least up to $\mathcal{O}(m^2)$, by the same set of integrals, with the mass configuration dictated by dual conformal symmetry. The two-mass-easy diagram in figure 4 corresponds to the integral

$$I_{ij}^{\rm 2me} = \int \frac{d^4x_0}{i\pi^2}\frac{\hat{x}^2_{ij}\hat{x}^2_{i+1,j+1}-\hat{x}^2_{i+1,j}\hat{x}^2_{i,j+1}}{\hat{x}^2_{0i}\hat{x}^2_{0,i+1}\hat{x}^2_{0j}\hat{x}^2_{0,j+1}} \tag{4.2}$$

where $\hat{x}^2_{ij} = x^2_{ij}+(m_i-m_j)^2$ with $m_0=0$. Later, we will take the uniform small mass limit, so henceforth we drop all mass dependence from the numerators, as those terms would only contribute at $\mathcal{O}(m^2)$.

[5]The one-mass integrals are just the special cases $I^{\rm 2me}_{i,i+2}$.

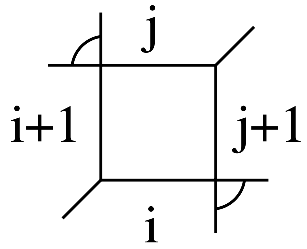

Figure 4. Two-mass-easy diagram corresponding to the integral $I_{ij}^{\rm 2me}$.

It is known that one can decompose eq. (4.2) into a sum of IR-divergent triangle integrals and an IR-finite six-dimensional integral (see e.g. ref. [71])

$$I_{ij}^{\rm 2me} = I_{ij}^{\rm 2me}\big|_{\rm tri} + I_{ij}^{\rm 2me,6D} + \mathcal{O}(m^2) \tag{4.3}$$

where

$$I_{ij}^{\rm 2me}\big|_{\rm tri} = \int \frac{d^4x_0}{i\pi^2}\left[\frac{x_{i+1,j+1}^2 - x_{i+1,j}^2}{\hat{x}_{0,i+1}^2\hat{x}_{0,j}^2\hat{x}_{0,j+1}^2} + \frac{x_{i,j}^2 - x_{i,j+1}^2}{\hat{x}_{0,i}^2\hat{x}_{0,j}^2\hat{x}_{0,j+1}^2} + \frac{x_{i+1,j+1}^2 - x_{i,j+1}^2}{\hat{x}_{0,i}^2\hat{x}_{0,i+1}^2\hat{x}_{0,j+1}^2} + \frac{x_{i,j}^2 - x_{i+1,j}^2}{\hat{x}_{0,i}^2\hat{x}_{0,i+1}^2\hat{x}_{0,j}^2}\right] \tag{4.4}$$

and

$$I_{ij}^{\rm 2me,6D} = \frac{2}{x_{ab}^2}\int \frac{d^6x_0}{i\pi^3}\frac{x_{ij}^2 x_{i+1,j+1}^2 - x_{i+1,j}^2 x_{i,j+1}^2}{x_{0i}^2 x_{0,i+1}^2 x_{0j}^2 x_{0,j+1}^2} \tag{4.5}$$

with x_a and x_b the two solutions of the equations $x_{0i}^2 = x_{0,i+1}^2 = x_{0j}^2 = x_{0,j+1}^2 = 0$. In appendix B of this paper, we review the derivation of this decomposition using twistor methods. Since the six-dimensional integral (4.5) is IR-finite, it is independent of which IR-regulator we employ to regulate the n-point amplitude. Therefore in the decomposition of the differential-mass Higgs-regulated amplitude

$$M_n^{(1)} = \sum_{i=1}^{n} W_{i-1,i,i+1}^{(1)}(x_{i-1,i+1}^2) + H_n^{(1)}(x_{ij}^2) + \mathcal{O}(m^2) \tag{4.6}$$

a natural candidate for the regulator-independent hard function is

$$H_n^{(1)} = -\frac{1}{8}\sum_{i=1}^{n}\sum_{j=i+2}^{i+n-2} I_{ij}^{\rm 2me,6D}. \tag{4.7}$$

Moreover, the sum of the one-loop wedge functions in eq. (4.6) will then be given by the sum of triangle diagrams (4.4) that contribute to the n-point amplitude. Most of the triangle diagrams in this sum cancel, leaving

$$\sum_{i=1}^{n} W_{i-1,i,i+1}^{(1)}(x_{i-1,i+1}^2) = -\frac{1}{2}\sum_{i=1}^{n}\int \frac{d^4x_0}{i\pi^2}\frac{x_{i-1,i+1}^2}{\hat{x}_{0,i-1}^2\hat{x}_{0i}^2\hat{x}_{0,i+1}^2}\,. \tag{4.8}$$

This suggests the following one-loop expression for the wedge function[6]

$$W_{i-1,i,i+1}^{(1)}(x_{i-1,i+1}^2) = -\frac{1}{2}\int \frac{d^4x_0}{i\pi^2}\frac{x_{i-1,i+1}^2}{\hat{x}_{0,i-1}^2\hat{x}_{0i}^2\hat{x}_{0,i+1}^2} \tag{4.9}$$

[6]The subscripts on the wedge function refer to its dependence on m_{i-1}, m_i, and m_{i+1}.

although we could in principle add a contribution that vanishes upon summing over i. Equation (4.9) reduces to our previous definition $W(q^2) = \sqrt{\Phi(q^2)}$ when $m_{i-1} = m_i = m_{i+1}$. Evaluating eq. (4.9) in the uniform small mass limit, we obtain

$$W^{(1)}_{i-1,i,i+1}(x^2_{i-1,i+1}) = -\frac{1}{4}\log^2\left(\frac{x^2_{i-1,i+1}}{m_i^2}\right) - \mathrm{Li}_2\left(1-\frac{m_{i-1}}{m_i}\right) - \mathrm{Li}_2\left(1-\frac{m_{i+1}}{m_i}\right) + \mathcal{O}(m^2)\,. \tag{4.10}$$

Substituting eq. (4.10) into eq. (4.6) and using the identity $\mathrm{Li}_2(1-z) + \mathrm{Li}_2(1-z^{-1}) + \frac{1}{2}\log^2 z = 0$, we finally obtain the differential-mass Higgs-regulated n-point amplitude

$$M_n^{(1)} = -\frac{1}{4}\sum_{i=1}^{n}\log^2\left(\frac{x^2_{i-1,i+1}}{m_i^2}\right) + \frac{1}{8}\sum_{i=1}^{n}\log^2\left(\frac{m^2_{i+1}}{m_i^2}\right) + H_n^{(1)}(x^2_{ij}) + \mathcal{O}(m^2) \tag{4.11}$$

where, as discussed above, $H_n^{(1)}(x^2_{ij})$ is IR-finite and regulator-independent; in particular, it does not depend on α_i in the uniform small mass limit $m \to 0$, where $m_i = \alpha_i m$ with α_i fixed.

Note that although $M_n^{(1)}$ has extended dual conformal invariance, the decomposition into $H_n^{(1)}$ and the specific IR-divergent pieces in eq. (4.11) breaks this symmetry. This is not surprising, as triangle integrals manifestly violate dual conformal symmetry.

4.2 Higher loops

In the previous subsection, we derived an expression (4.10) for the one-loop wedge function valid for the differential-mass Higgs regulator. We now use the extended dual conformal invariance of the amplitude and the BDS ansatz to derive the explicit form of the wedge function at higher loops.

Recall that extended dual conformal invariance implies [38] that the amplitude can only be a function of

$$u_{ij} = \frac{m_i m_j}{x^2_{ij} + (m_i - m_j)^2}\,. \tag{4.12}$$

For the common-mass Higgs regulator, this reduces to $u_{ij} = m^2/x^2_{ij}$. Hence, assuming the validity of the all-loop expression (2.10) for the common-mass Higgs-regulated amplitude, its unique generalization is obtained by replacing x^2_{ij} with m^2/u_{ij} everywhere to obtain

$$\begin{aligned}\log M_n = &\sum_{i=1}^{n}\left[-\frac{\gamma(a)}{16}\log^2\left(\frac{1}{u_{i-1,i+1}}\right) - \frac{\tilde{\mathcal{G}}_0(a)}{2}\log\left(\frac{1}{u_{i-1,i+1}}\right)\right] \\ &+\frac{\gamma(a)}{4}H_n^{(1)}(m^2/u_{ij}) + n\tilde{f}(a) + \tilde{C}(a) + \tilde{R}_n(m^2/u_{ij}, a) + \mathcal{O}(m^2)\end{aligned} \tag{4.13}$$

where we have also used eqs. (3.33) and (3.36). In the uniform small mass limit, we can neglect $(m_i - m_j)^2$ relative to x^2_{ij} in eq. (4.12) so that u_{ij} becomes $m_i m_j / x^2_{ij}$ yielding

$$\begin{aligned}\log M_n = &\sum_{i=1}^{n}\left[-\frac{\gamma(a)}{16}\log^2\left(\frac{x^2_{i-1,i+1}}{m_{i-1}m_{i+1}}\right) - \frac{\tilde{\mathcal{G}}_0(a)}{2}\log\left(\frac{x^2_{i-1,i+1}}{m_{i-1}m_{i+1}}\right)\right] \\ &+\frac{\gamma(a)}{4}H_n^{(1)}\left(\frac{m^2 x^2_{ij}}{m_i m_j}\right) + n\tilde{f}(a) + \tilde{C}(a) + \tilde{R}_n\left(\frac{m^2 x^2_{ij}}{m_i m_j}, a\right) + \mathcal{O}(m^2)\,.\end{aligned} \tag{4.14}$$

The apparent dependence of $\tilde{R}_n$ on m_i is illusory since the mass dependence cancels out in the dual conformal cross ratios on which $\tilde{R}_n$ only depends, so that $\tilde{R}_n(m^2 x_{ij}^2/m_i m_j, a) = \tilde{R}_n(x_{ij}^2, a)$. There does, however, remain some dependence on m_i in $H_n^{(1)}$.

Applying the same reasoning as above to the one-loop amplitude, we obtain

$$M_n^{(1)} = -\frac{1}{4}\sum_{i=1}^{n} \log^2\left(\frac{x_{i-1,i+1}^2}{m_{i-1}m_{i+1}}\right) + H_n^{(1)}\left(\frac{m^2 x_{ij}^2}{m_i m_j}\right) + \mathcal{O}(m^2)\,. \tag{4.15}$$

Now since eqs. (4.11) and (4.15) are both valid expressions for the differentially-regulated one-loop amplitude, we deduce that

$$\begin{aligned} H_n^{(1)}\left(\frac{m^2 x_{ij}^2}{m_i m_j}\right) &= H_n^{(1)}(x_{ij}^2) + \frac{1}{4}\sum_{i=1}^{n}\left[\log^2\left(\frac{x_{i-1,i+1}^2}{m_{i-1}m_{i+1}}\right) - \log^2\left(\frac{x_{i-1,i+1}^2}{m_i^2}\right)\right] \\ &\quad + \frac{1}{8}\sum_{i=1}^{n}\log^2\left(\frac{m_{i+1}^2}{m_i^2}\right) + \mathcal{O}(m^2)\,. \end{aligned} \tag{4.16}$$

Substituting eq. (4.16) into eq. (4.14) and using eq. (3.35), we obtain[7]

$$\begin{aligned} \log M_n &= \sum_{i=1}^{n}\left[-\frac{\gamma(a)}{16}\log^2\left(\frac{x_{i-1,i+1}^2}{m_i^2}\right) - \frac{\tilde{\mathcal{G}}_0(a)}{2}\log\left(\frac{x_{i-1,i+1}^2}{m_i^2}\right) + \frac{\gamma(a)}{32}\log^2\left(\frac{m_{i+1}^2}{m_i^2}\right) + \tilde{w}(a)\right] \\ &\quad + \frac{\gamma(a)}{4} H_n^{(1)}(x_{ij}^2) + n\tilde{D}(a) + \tilde{C}(a) + \tilde{R}_n(x_{ij}^2, a) + \mathcal{O}(m^2) \end{aligned} \tag{4.17}$$

where now only the terms in the sum on the first line depend on the regulator, while the pieces on the second line are all regulator-independent. Recalling that

$$\log M_n = \sum_{i=1}^{n}\log W_{i-1,i,i+1}(x_{i-1,i+1}^2) + \log H_n \tag{4.18}$$

we deduce from eq. (4.17) that the all-order expression for the sum of wedge functions in differential-mass Higgs regularization is[8]

$$\begin{aligned} &\sum_{i=1}^{n}\log W_{i-1,i,i+1}(x_{i-1,i+1}^2) \\ &\quad = \sum_{i=1}^{n}\left[-\frac{\gamma(a)}{16}\log^2\left(\frac{x_{i-1,i+1}^2}{m_i^2}\right) - \frac{\tilde{\mathcal{G}}_0(a)}{2}\log\left(\frac{x_{i-1,i+1}^2}{m_i^2}\right) + \frac{\gamma(a)}{32}\log^2\left(\frac{m_{i+1}^2}{m_i^2}\right) + \tilde{w}(a)\right] \end{aligned} \tag{4.19}$$

and the regulator-independent IR-finite piece is, as before

$$\log H_n = \frac{\gamma(a)}{4} H_n^{(1)}(x_{ij}^2) + nD(a) + C(a) + R_n(x_{ij}^2, a) \tag{4.20}$$

where we have dropped all the tildes.

We observe again that, although the amplitude $\log M_n$ has extended dual conformal invariance, the separate terms in the decomposition (4.18) do not.

[7] The apparent difference between the $\tilde{\mathcal{G}}_0(a)$ terms disappears on performing the sum over i.

[8] Equation (4.19) does not allow us, however, to identify the individual terms of the sum over i.

4.3 Relation to anomalous dual conformal Ward identity

In the previous section, we obtained the all-loop expression (4.19) for the sum of wedge functions by assuming eq. (2.10). In this section, we show inversely that eq. (2.10) follows from eq. (4.19). Therefore, it would be interesting to have a first-principles derivation of eq. (4.19).

The n-point amplitude has exact extended dual conformal symmetry, and so is annihilated by the generator of dual special conformal transformations

$$\hat{K}^\mu \log M_n \equiv \sum_{i=1}^{n} \left[2x_i{}^\mu \left(x_i^\nu \frac{\partial}{\partial x_i^\nu} + m_i \frac{\partial}{\partial m_i} \right) - (x_i^2 + m_i^2) \frac{\partial}{\partial x_{i\mu}} \right] \log M_n = 0\,. \tag{4.21}$$

In ref. [38], it was suggested that the IR-divergent properties of the Higgs-regulated amplitude provides a relation between this exact Ward identity and the anomalous dual conformal Ward identity for the IR-finite part of the n-point amplitude that was originally derived in a Wilson loop context [25, 26]. We will see that this is indeed the case.

As we have seen, the n-point amplitude can be written as

$$\log M_n = \sum_{i=1}^{n} \log W_{i-1,i,i+1}(x_{i-1,i+1}^2) + \log H_n\,. \tag{4.22}$$

Using the expression (4.19) for the sum of the wedge functions, one can easily show that

$$\hat{K}^\mu \sum_{i=1}^{n} \log W_{i-1,i,i+1}(x_{i-1,i+1}^2) = -\frac{\gamma(a)}{4} \sum_{i=1}^{n} \left[x_{i-1}^\mu - 2x_i^\mu + x_{i+1}^\mu \right] \log(x_{i-1,i+1}^2) \tag{4.23}$$

which by virtue of eq. (4.21) implies that

$$\hat{K}^\mu \log H_n = \frac{\gamma(a)}{4} \sum_{i=1}^{n} \left[x_{i-1}^\mu - 2x_i^\mu + x_{i+1}^\mu \right] \log(x_{i-1,i+1}^2)\,. \tag{4.24}$$

But $\log H_n$ is regulator-independent, i.e., has no dependence on m_i in the uniform small mass limit, so the m-dependent pieces in $\hat{K}^\mu$ drop out when acting on $\log H_n$ and we have

$$K^\mu \log H_n \equiv \sum_{i=1}^{n} \left[2x_i{}^\mu x_i^\nu \frac{\partial}{\partial x_i^\nu} - x_i^2 \frac{\partial}{\partial x_{i\mu}} \right] \log H_n = \frac{\gamma(a)}{4} \sum_{i=1}^{n} \left[x_{i,i+1}^\mu \log \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} \right] \tag{4.25}$$

which is precisely the anomalous dual conformal Ward identity [25, 26]. This in turn implies eq. (2.10).

We thus see that the decomposition of the amplitude into contributions which separately do not possess extended dual conformal invariance was necessary to obtain the anomalous dual conformal Ward identity for the finite (regulator-independent) part of the amplitude.

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5 Discussion

In this paper, we have given a prescription for defining an unambiguous, regulator-independent IR-finite part of the MHV n-point scattering amplitude in planar $\mathcal{N}=4$ SYM theory. This prescription involves the definition of an IR-divergent wedge function associated with a pair of adjacent external legs of the amplitude. The IR-finite part of the amplitude is then defined as the quotient of the $n-$point amplitude by the product of wedge functions, cf. eq. (1.3).

For regulators that leave the external legs massless (e.g., dimensional regularization or the common-mass Higgs regulator), the wedge function can be naturally defined in terms of a form factor Φ which has the same IR-divergences. Computation of this form factor in dimensional regularization and in the common-mass Higgs regularization through two loops shows that the IR-finite part of the amplitude is identical for these two regularizations. For the more general differential-mass Higgs regulator, which gives (small) masses to the external legs, a wedge function that results in a regulator-independent hard function can still be calculated, but an operator definition in this case is still lacking.

We remark that the idea of defining a regulator-independent finite hard function can also be applied to other objects, e.g. Wilson loops. This is particularly interesting in the context of the Wilson loop/scattering amplitudes duality, since the two objects have different types of divergences, viz., UV and IR divergences respectively. Although these divergences are related, defining hard functions for both objects could be useful for stating the duality in a regulator-independent way.

There exist in the literature other procedures for removing the IR divergences of scattering amplitudes. For example, for non-MHV amplitudes, one can define an IR-finite ratio function [72] by factoring out the entire MHV amplitude, using the universality of IR divergences, i.e. that they do not depend on the helicity configuration. Another example involves MHV amplitudes with $n \geq 6$ external legs. Since the four- and five-point amplitudes (or, equivalently, Wilson loops) are known up to kinematic-independent functions, they can be used to remove the divergences of higher-point amplitudes by defining suitable ratios [73]. This latter procedure preserves dual conformal symmetry.

The hard functions defined in this paper are not dual conformal invariant; they have the advantage, however, of allowing us to study the $n=4$ and $n=5$ cases as well. In particular, it would be interesting to understand better the systematics of how the BES equation [23] for $\gamma(a)$ arises from the loop expansion of the four-point amplitude.

The breaking of dual conformal invariance by the hard function also implies an intimate connection between the anomalous dual conformal Ward identity it satisfies and the IR divergences (wedge functions) of differential-mass regulated amplitudes. A first-principles derivation of the latter would therefore be most interesting.

Finally, we believe that, although our investigation has been specialized to $\mathcal{N}=4$ SYM theory, the insight into the interplay between regulator, kinematics, and soft and collinear momentum configurations applies to many gauge field theories, including those with broken supersymmetry, such as QCD, and also to electroweak radiative corrections in the Standard Model.

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A Results for one- and two-loop integrals

In this appendix, we list the results for various massless and massive three-point integrals that contribute to the form factors computed in this paper. We use the mostly-plus metric, the propagators are of the form k^2+m^2, and the measure of each internal loop momentum is multiplied by a factor of $(\mu^2\,\mathrm{e}^{-\gamma})^\epsilon/(i\pi^{d/2})$. The massless integrals shown in figure 1 are dimensionally-regulated, giving rise to the following Laurent expansions [74–76]

$$I_b = (\mu^2\,\mathrm{e}^{-\gamma})^\epsilon \int \frac{d^d x_0}{i\pi^{d/2}} \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2} = \frac{1}{q^2}\left(\frac{\mu^2}{q^2}\right)^\epsilon \left[\frac{1}{\epsilon^2} - \frac{\pi^2}{12} - \frac{7\zeta_3}{3}\epsilon - \frac{47\pi^4}{1440}\epsilon^2 + \mathcal{O}(\epsilon^3)\right], \quad \text{(A.1)}$$

$$I_c = \frac{1}{(q^2)^2}\left(\frac{\mu^2}{q^2}\right)^{2\epsilon}\left[\frac{1}{4\epsilon^4} + \frac{5\pi^2}{24\epsilon^2} + \frac{29\zeta_3}{6\epsilon} + \frac{3\pi^4}{32} + \mathcal{O}(\epsilon)\right], \quad \text{(A.2)}$$

$$I_d = \frac{1}{(q^2)^2}\left(\frac{\mu^2}{q^2}\right)^{2\epsilon}\left[\frac{1}{\epsilon^4} - \frac{\pi^2}{\epsilon^2} - \frac{83\zeta_3}{3\epsilon} - \frac{59\pi^4}{120} + \mathcal{O}(\epsilon)\right], \quad \text{(A.3)}$$

where $q^2 = x_{13}^2$. The integrals shown in figure 2 use a common-mass Higgs regulator, and can be evaluated to give

$$\tilde{I}_b = \int \frac{d^4 x_0}{i\pi^2} \frac{1}{(x_{01}^2+m^2)(x_{02}^2+m^2)(x_{03}^2+m^2)} = \frac{1}{q^2}\left[\frac{1}{2}\log^2\left(\frac{q^2}{m^2}\right) + \mathcal{O}(m^2)\right], \quad \text{(A.4)}$$

$$\tilde{I}_{c1} = \frac{1}{(q^2)^2}\left[\frac{1}{24}\log^4\left(\frac{q^2}{m^2}\right) + \frac{\pi^2}{3}\log^2\left(\frac{q^2}{m^2}\right) - 8\zeta_3\log\left(\frac{q^2}{m^2}\right) + \frac{\pi^4}{10} + \mathcal{O}(m^2)\right], \quad \text{(A.5)}$$

$$\tilde{I}_{c2} = \frac{1}{(q^2)^2}\left[\frac{1}{24}\log^4\left(\frac{q^2}{m^2}\right) + \frac{\pi^2}{3}\log^2\left(\frac{q^2}{m^2}\right) - 10\zeta_3\log\left(\frac{q^2}{m^2}\right) + \frac{13\pi^4}{60} + \mathcal{O}(m^2)\right], \quad \text{(A.6)}$$

$$\tilde{I}_d = \frac{1}{(q^2)^2}\left[\frac{1}{3}\log^4\left(\frac{q^2}{m^2}\right) - \pi^2\log^2\left(\frac{q^2}{m^2}\right) + 40\zeta_3\log\left(\frac{q^2}{m^2}\right) - \frac{49\pi^4}{90} + \mathcal{O}(m^2)\right]. \quad \text{(A.7)}$$

B Decomposition of the 2me box integral

In this appendix, we derive the decomposition of the 2me box integral into triangle integrals and an IR-finite six-dimensional integral that was used in section 4.1.

We begin by rewriting the dual conformal invariant integral of equation (4.2) in terms of momentum twistors [77]; see ref. [78] for a pedagogical introduction to this topic. A point

x_i in dual space corresponds to a (projective) line $Z_{i-1}^A Z_i^B$ in momentum twistor space. The invariant x_{ij}^2 can be expressed as

$$x_{ij}^2 = \frac{\langle i-1, i, j-1, j\rangle}{\langle i-1, i\rangle\langle j-1, j\rangle} \tag{B.1}$$

where the twistor four-bracket is

$$\langle \alpha, \beta, \gamma, \delta\rangle = \epsilon_{ABCD} Z_\alpha^A Z_\beta^B Z_\gamma^C Z_\delta^D \,. \tag{B.2}$$

We introduce the infinity bitwistor I^{AB}, which when contracted with $Z_\alpha^C Z_\beta^D$ gives the two-bracket

$$\langle \alpha, \beta\rangle = \epsilon_{ABCD} I^{AB} Z_\alpha^C Z_\beta^D \,. \tag{B.3}$$

Finally, we introduce a modified mass-regulated four-bracket

$$\langle \alpha, \beta, \gamma, \delta\rangle_i \equiv \langle \alpha, \beta, \gamma, \delta\rangle + m_i^2 \langle \alpha, \beta\rangle\langle \gamma, \delta\rangle \,. \tag{B.4}$$

Rewriting eq. (4.2) using eqs. (B.1) and (B.4), we obtain

$$I_{ij}^{\text{2me}} = \int \frac{d^4 Z_{\alpha\beta}}{i\pi^2} \frac{N_{ij}}{\langle \alpha, \beta, i-1, i\rangle_i \langle \alpha, \beta, i, i-1\rangle_{i+1} \langle \alpha, \beta, j-1, j\rangle_j \langle \alpha, \beta, j, j-1\rangle_{j+1}} + \mathcal{O}(m^2) \tag{B.5}$$

where

$$N_{ij} \equiv \langle i-1, i, j-1, j\rangle\langle i, i+1, j, j+1\rangle - \langle i, i+1, j-1, j\rangle\langle i-1, i, j, j+1\rangle \tag{B.6}$$

and where $d^4 x_0 \to d^4 Z_{\alpha\beta}/\langle \alpha, \beta\rangle^4$.

We now decompose the two-mass-easy integral into a sum of IR-divergent and IR-finite contributions using a twistor identity. To derive this identity, consider the infinity bitwistor I^{AB} and expand it in the basis spanned by the six simple bitwistors

$$I^{AB} = c_{i-1,i} Z_{i-1}^A Z_i^B + c_{i,i+1} Z_i^A Z_{i+1}^B + c_{j-1,j} Z_{j-1}^A Z_j^B + c_{j,j+1} Z_j^A Z_{j+1}^B + c_{i,j} Z_i^A Z_j^B + c_{\bar{\imath},\bar{\jmath}} Z_{\bar{\imath}}^A Z_{\bar{\jmath}}^B \tag{B.7}$$

where $Z_{\bar{\imath}}^A Z_{\bar{\jmath}}^B$ denotes the line in momentum twistor space formed by the intersection of $(i-1, i, i+1)$ and $(j-1, j, j+1)$. Contracting eq. (B.7) with $Z_\alpha^C Z_\beta^D$, we obtain the identity

$$\begin{aligned} \langle \alpha\beta\rangle = {} & c_{i-1,i}\langle \alpha, \beta, i-1, i\rangle + c_{i,i+1}\langle \alpha, \beta, i, i+1\rangle + c_{j-1,j}\langle \alpha, \beta, j-1, j\rangle \\ & + c_{j,j+1}\langle \alpha, \beta, j, j+1\rangle + c_{i,j}\langle \alpha, \beta, ij\rangle + c_{\bar{\imath},\bar{\jmath}}\langle \alpha, \beta, \bar{\imath}\bar{\jmath}\rangle \,. \end{aligned} \tag{B.8}$$

We multiply and divide the integrand of eq. (B.5) by $\langle \alpha, \beta\rangle$, using eq. (B.8) to rewrite the numerator. The first four terms will each cancel one of the propagators[9] resulting in four triangle integrals, whereas the last two terms remain box integrals

$$I_{ij}^{\text{2me}} = I_{ij}^{\text{2me}}\big|_{\text{tri}} + I_{ij}^{\text{2me}}\big|_{\text{box}} + \mathcal{O}(m^2) \,. \tag{B.9}$$

[9]We rewrite the numerator term $\langle \alpha, \beta, i-1, i\rangle$ as $\langle \alpha, \beta, i-1, i\rangle_i - m_i^2\langle \alpha\beta\rangle\langle i-1, i\rangle$ and then drop the $\mathcal{O}(m^2)$ piece.

The coefficients $c_{i-1,i}$, etc. may be determined by contracting eq. (B.7) with each of the basis elements in turn, e.g.,

$$\langle i-1,i\rangle = c_{j-1,j}\langle i-1,i,j-1,j\rangle + c_{j,j+1}\langle i-1,i,j,j+1\rangle \tag{B.10}$$

and five other equations. Thus we find

$$\begin{aligned} I_{ij}^{\rm 2me}\big|_{\rm tri} &= \int \frac{d^4 Z_{\alpha\beta}}{i\pi^2}\frac{1}{\langle\alpha,\beta\rangle}\bigg[\frac{\langle j-1,j\rangle\langle i,i+1,j,j+1\rangle - \langle j,j+1\rangle\langle i,i+1,j-1,j\rangle}{\langle\alpha,\beta,i,i+1\rangle_{i+1}\langle\alpha,\beta,j-1,j\rangle_j\langle\alpha,\beta,j,j+1\rangle_{j+1}} \\ &\qquad + \frac{\langle j,j+1\rangle\langle i-1,i,j-1,j\rangle - \langle j-1,j\rangle\langle i-1,i,j,j+1\rangle}{\langle\alpha,\beta,i-1,i\rangle_i\langle\alpha,\beta,j-1,j\rangle_j\langle\alpha,\beta,j,j+1\rangle_{j+1}} \\ &\qquad + \frac{\langle i-1,i\rangle\langle i,i+1,j,j+1\rangle - \langle i,i+1\rangle\langle i-1,i,j,j+1\rangle}{\langle\alpha,\beta,i-1,i\rangle_i\langle\alpha,\beta,i,i+1\rangle_{i+1}\langle\alpha,\beta,j,j+1\rangle_{j+1}} \\ &\qquad + \frac{\langle i,i+1\rangle\langle i-1,i,j-1,j\rangle - \langle i-1,i\rangle\langle i,i+1,j-1,j\rangle}{\langle\alpha,\beta,i-1,i\rangle_i\langle\alpha,\beta,i,i+1\rangle_{i+1}\langle\alpha,\beta,j-1,j\rangle_j}\bigg] \\ &= \int\frac{d^4x_0}{i\pi^2}\left[\frac{x^2_{i+1,j+1}-x^2_{i+1,j}}{\hat{x}^2_{0,i+1}\hat{x}^2_{0,j}\hat{x}^2_{0,j+1}} + \frac{x^2_{i,j}-x^2_{i,j+1}}{\hat{x}^2_{0,i}\hat{x}^2_{0,j}\hat{x}^2_{0,j+1}} + \frac{x^2_{i+1,j+1}-x^2_{i,j+1}}{\hat{x}^2_{0,i}\hat{x}^2_{0,i+1}\hat{x}^2_{0,j+1}} + \frac{x^2_{i,j}-x^2_{i+1,j}}{\hat{x}^2_{0,i}\hat{x}^2_{0,i+1}\hat{x}^2_{0,j}}\right] \end{aligned} \tag{B.11}$$

and

$$\begin{aligned} I_{ij}^{\rm 2me}\big|_{\rm box} &= \int\frac{d^4 Z_{\alpha\beta}}{i\pi^2}\frac{\langle i,j\rangle\langle\alpha,\beta,\bar{\imath},\bar{\jmath}\rangle + \langle\bar{\imath},\bar{\jmath}\rangle\langle\alpha,\beta,i,j\rangle}{\langle\alpha,\beta\rangle\langle\alpha,\beta,i-1,i\rangle\langle\alpha,\beta,i,i+1\rangle\langle\alpha,\beta,j-1,j\rangle\langle\alpha,\beta,j,j+1\rangle} \\ &= \frac{\langle i,j\rangle\langle\bar{\imath},\bar{\jmath}\rangle}{\langle i-1,i\rangle\langle i,i+1\rangle\langle j-1,j\rangle\langle j,j+1\rangle}\int\frac{d^4x_0}{i\pi^2}\frac{x^2_{0a}+x^2_{0b}}{x^2_{0i}x^2_{0,i+1}x^2_{0j}x^2_{0,j+1}} \end{aligned} \tag{B.12}$$

where x_a and x_b are the two solutions of the equations $x^2_{0i} = x^2_{0,i+1} = x^2_{0j} = x^2_{0,j+1} = 0$, and where in eq. (B.12) we have used the identities

$$N_{ij} = \langle i-1,i,i+1,j\rangle\langle i,j-1,j,j+1\rangle = \langle i,j,\bar{\imath},\bar{\jmath}\rangle\,. \tag{B.13}$$

The presence of two-brackets in $I_{ij}^{\rm 2me}\big|_{\rm tri}$ and $I_{ij}^{\rm 2me}\big|_{\rm box}$ indicate that these expressions are not individually dual conformal invariant. This is not surprising, as scalar triangle diagrams violate dual conformal invariance.

We observe that the integrands in eq. (B.12) contain "magic" numerators, which render the resulting integrals IR-finite. Hence we have dropped the mass dependence in the denominator. One can show that this integral is in fact equivalent to the scalar two-mass easy integral in six dimensions [71]

$$I_{ij}^{\rm 2me}\big|_{\rm box} = I_{ij}^{\rm 2me,6D} \equiv \frac{2}{x^2_{ab}}\int\frac{d^6x_0}{i\pi^3}\frac{x^2_{ij}x^2_{i+1,j+1} - x^2_{i+1,j}x^2_{i,j+1}}{x^2_{0i}x^2_{0,i+1}x^2_{0j}x^2_{0,j+1}}\,. \tag{B.14}$$

Since this integral is IR-finite, it is manifestly independent of which IR regulator is used to regulate the amplitude. In particular, it has no dependence on α_i in the uniform small mass limit of the differential-mass Higgs regulator introduced in section 4.

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