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$E_{7(7)}$ symmetry in perturbatively quantised $\mathcal{N}=8$ supergravity

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Abstract: We study the perturbative quantisation of $\mathcal{N}=8$ supergravity in a formulation where its $E_{7(7)}$ symmetry is realised *off-shell*. Relying on the cancellation of SU(8) current anomalies we show that there are no anomalies for the non-linearly realised $E_{7(7)}$ either; this result extends to all orders in perturbation theory. As a consequence, the $\mathfrak{e}_{7(7)}$ Ward identities can be consistently implemented and imposed at all orders in perturbation theory, and therefore potential divergent counterterms must in particular respect the full non-linear $E_{7(7)}$ symmetry.

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Contents

1 Introduction

Maximally extended $\mathcal{N}=8$ supergravity [1, 2] is the most symmetric field theoretic extension of Einstein's theory in four space-time dimensions. Although long thought to diverge at three loops [3, 4], spectacular computational advances have recently shown that, contrary to many expectations, the theory is finite at least up to and including four loops [5, 6], and thereby fuelled speculations that the theory may actually be finite to *all orders* in perturbation theory. It appears doubtful whether maximal supersymmetry alone could suffice to explain such a far reaching result [7], if true. Rather, it seems plausible that the possible finiteness of $\mathcal{N}=8$ supergravity will hinge on known or unknown 'hidden symmetries' of the theory. Indeed, already the construction of the $\mathcal{N}=8$ Lagrangian itself was only possible thanks to the discovery of the non-linear duality symmetry $E_{7(7)}$ of its equations of motion [1]. This symmetry is expected to be a symmetry of perturbation theory, and to be broken to an arithmetic subgroup of $E_{7(7)}$ by non-perturbative effects when the theory is embedded into string theory (see e.g. [8, 9] for a recent update, and also the comments below). Nevertheless, the status of the non-linear duality symmetry at the level of quantised perturbation theory has remained rather unclear, because $E_{7(7)}$ is not a symmetry of the original $\mathcal{N} = 8$ Lagrangian and the corresponding non-linear functional Ward identities therefore have not been worked out so far.

Inspired by earlier work devoted to the definition of an action for self-dual form fields [10], one of the authors recently was able to set up a formulation of $\mathcal{N} = 8$ supergravity in which the Lagrangian is *manifestly* $E_{7(7)}$-invariant [11].[1] The main peculiarity of the formalism is to replace the 28 vector fields $A_\mu^{\mathpzc{m}}$ of the original formulation by $56 = 28+28$ vector fields $A_{\mathtt{i}}^{m} \equiv (A_{\mathtt{i}}^{\mathpzc{m}}, A_{\mathtt{i}}^{\bar{\mathpzc{m}}})$ *with spatial components only*, whose conjugate momenta are determined by second class constraints in the canonical formulation, in such a way that they represent the same number of physical degrees of freedom as the original 28 vector fields in the conventional formulation of the theory. Although not manifestly diffeomorphism invariant, the theory still admits diffeomorphism and local supersymmetry gauge invariance [11]. By virtue of its manifest off-shell $E_{7(7)}$ invariance, the theory possesses a *bona fide* $E_{7(7)}$ Noether current, unlike the covariant formulation [13], and this is the feature which permits to write down functional Ward identities for the non-linear duality symmetry.

In this paper we will consider the perturbative quantisation of $\mathcal{N} = 8$ supergravity in this duality invariant formulation. As our main result, we will prove that there exists a renormalisation scheme which maintains the full *non-linear* (continuous) $E_{7(7)}$ duality symmetry at all orders in a perturbative expansion of the theory in the gravitational coupling κ. A key element in this proof is the demonstration of the absence of linear SU(8) and non-linear $E_{7(7)}$ anomalies.

As is well known [14], the proper definition of any quantum field theory relies on the *quantum action principle*, according to which the ultra-violet divergences of the 1PI generating functional are always *local* functionals of the fields. Only thanks to this property can one carry out the renormalisation program by consistently modifying the local bare action order by order to eliminate both divergences and trivial anomalies. Because of the non-conventional character of our reformulation of $\mathcal{N} = 8$ supergravity, and its lack of manifest Lorentz invariance in particular, the validity of the quantum action principle is however not automatically guaranteed.

To deal with this problem, we will in a first step prove that the duality invariant path integral of the theory is equivalent to the conventional formulation by means of a

[1] The formalism had been applied earlier to the definition of a manifestly $\mathrm{SL}(2,\mathbb{R})$ bosonic action for $\mathcal{N} = 4$ supergravity [12].

Gaussian integration. In order to ensure the validity of the quantum action principle, we will require the existence of a local regularisation scheme in the two formulations of the theory, which are equivalent modulo a Gaussian integration (but note that the Gaussian integration reduces the manifest $E_{7(7)}$ invariance to an on-shell symmetry). The validity of the quantum action principle in the conventional formulation of the theory then ensures its validity in the duality invariant formulation. We will define a Pauli-Villars regularisation of the theory satisfying these criteria. Although this regularisation would break Lorentz invariance in the covariant formulation as well, it is local and invariant with respect to abelian gauge invariance in the two formulations. We will exhibit the consistency of this regularisation in the explicit computation of the one-loop vector field contribution to the $\mathfrak{su}(8)$-current anomaly.

With a consistent duality invariant formulation at hand, we can address and answer the question of whether the $\mathfrak{e}_{7(7)}$ current Ward identities are anomalous or not in perturbation theory. According to [15], the local $\mathfrak{su}(8)$ gauge invariance in the version of $\mathcal{N} = 8$ supergravity with *linearly realised* $E_{7(7)}$ is anomalous at one-loop. However, as shown in [16] this anomaly can be cancelled by an SU(8) Wess-Zumino term which in turn breaks the manifest $E_{7(7)}$ invariance, whereby the local SU(8) anomaly is converted into an anomaly of the global $E_{7(7)}$ — unless there appear new contributions to the latter, as happens to be the case for $\mathcal{N} = 8$ supergravity. According to [16] one thus has the option of working either with the locally SU(8) invariant version of $\mathcal{N} = 8$ supergravity, or with its gauge-fixed version where $E_{7(7)}$ is realised non-linearly. Here we prefer the second option, that is, we will consider an explicit parametrisation of the scalar manifold $E_{7(7)}/SU_c(8)$[2] in terms of 70 scalar fields $\Phi \in \mathfrak{e}_{7(7)} \ominus \mathfrak{su}(8)$ which coordinatise the coset manifold. A consistent anomaly must then be a non-trivial solution to the Wess-Zumino consistency condition. We will prove that the associated cohomology problem reduces to the cohomology problem associated to the current $\mathfrak{su}(8)$ Ward identities. It follows from this result that, although the non-linear character of the $\mathfrak{e}_{7(7)}$ symmetry is such that the associated anomalies involve infinitely many correlation functions with arbitrarily many scalar field insertions, the Wess-Zumino consistency condition implies that the corresponding coefficients are all determined in function of the linear $\mathfrak{su}(8)$ anomaly coefficient — thereby saving us the labour of having to determine an infinitude of anomalous diagrams! Now, thanks to a crucial insight of [17], it is known that for $\mathcal{N} = 8$ supergravity, the anomalous contributions to the current (rigid) $\mathfrak{su}(8)$ Ward identities from the fermions cancel against the contributions from the vector fields because the latter are also *chiral* under SU(8). Therefore the non-linear $\mathfrak{e}_{7(7)}$ Ward identities are likewise free of anomalies. Moreover, the cohomological arguments of section 3 show that this results extends to all loop orders.

The fact that the consistent $\mathfrak{e}_{7(7)}$ anomalies are in one-to-one correspondence with the

[2]Where throughout the notation $SU_c(8)$ will be used as a shorthand for the quotient of SU(8) by the $\mathbb{Z}_2$ kernel of the representations of even rank.

set of consistent $\mathfrak{su}(8)$ anomalies can also be understood more intuitively, and in a way that makes the result almost look trivial. Namely, in differential geometric terms, this correspondence is based on the homotopy equivalence

$$E_{7(7)} \cong SU_{\rm c}(8) \times \mathbb{R}^{70}\,, \tag{1.1}$$

which implies that the two group manifolds have the same De Rham cohomology. We will show how to extend the algebraic proof of this property by means of equivariant cohomology to the cohomology problem of classifying the $\mathfrak{e}_{7(7)}$ anomalies in $\mathcal{N}=8$ supergravity, and in this way arrive at a very explicit derivation of the *non-linear* $\mathfrak{e}_{7(7)}$ *anomaly* from the corresponding linear $\mathfrak{su}(8)$ anomaly.

$\mathcal{N}=8$ supergravity is a gauge theory, and its first class constraints (associated to diffeomorphisms, local supersymmetry, abelian gauge invariance, and Lorentz invariance) must be taken care of by means of the BRST formalism. This likewise requires the explicit parametrisation of the coset manifold $E_{7(7)}/\mathrm{SU}_{\rm c}(8)$, such that there are no first class constraints associated to SU(8) gauge invariance in the formulation. For the validity of the proof of the $E_{7(7)}$ invariance of the theory, one must therefore establish the compatibility of the latter with the BRST invariance. We will demonstrate in the last section that the theory can be quantised in its duality invariant formulation within the Batalin-Vilkovisky formalism, as it does in the ordinary formulation. It is not difficult to see that one can define a consistent $E_{7(7)}$-invariant fermionic gauge fixing-functional (or 'gauge fermion'). We will explain how the $E_{7(7)}$ Noether current can be coupled consistently to the theory, despite its lack of gauge invariance.

In summary, the proof of the duality invariance of the quantised perturbation theory relies on establishing the following results:

1. Existence of a local action Σ depending on the physical fields and sources, well suited for Feynman rules, and satisfying consistent functional identities associated to both $\mathfrak{e}_{7(7)}$ current Ward identities and BRST invariance.

2. Existence of a regularisation prescription consistent with the quantum action principle; as dimensional regularisation appears unsuitable in the present formulation, we will employ a Pauli-Villars regulator.

3. Existence of a unique non-trivial solution to the $E_{7(7)}$ Wess-Zumino consistency condition associated to the one-loop anomaly.

4. Vanishing of the coefficient of the unique anomaly, which implies the absence of any obstruction towards implementing the full nonlinear $E_{7(7)}$ symmetry at each order in perturbation theory via an associated $\mathfrak{e}_{7(7)}$ master equation.

However, our exposition will not follow these steps in this order, i.e. as a successive proof of each of these points. Instead, we chose to postpone the discussion of the first

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point, i.e. the consistency with BRST invariance, to the end and to first discuss other components of the proof that we consider to be more interesting (and perhaps also more easily accessible). As one of our main results we separately derive the master equations (or “Zinn-Justin equations”) for both $\mathcal{N}=8$ supersymmetry and non-linear $E_{7(7)}$. Using standard textbook results (see e.g. [18, 19]) readers may then directly deduce from these any (non-linear) Ward identity of interest if they wish.

Our results confirm the expectation that any divergent counterterm must respect the full non-linear $E_{7(7)}$ symmetry. They may thus be taken as further evidence that divergences of $\mathcal{N}=8$ supergravity, if any, will not make their appearance before seven loops. The strongest evidence so far of the 6-loop finiteness was the absence of logarithm in the string effective action threshold [20]. The chiral invariants associated to potential logarithmic divergences at three, five and six loops are only known in the linear approximation [21], and if they are invariant with respect to the linearised duality transformations, there is no reason to believe that their non-linear completion would be duality invariant. Indeed, it has recently been exhibited through the study of on-shell tree amplitudes in type II string theory that the 1/2 BPS invariant corresponding to the potential 3-loop divergence *is not* $E_{7(7)}$ invariant [22, 23]. The same argument applies to the invariants associated to potential 5 and 6-loop divergences. The manifestly $E_{7(7)}$ invariant 7-loop counterterm is the full superspace integral of the supervielbein determinant. This is known to vanish for lower $\mathcal{N}$ supergravities, suggesting that the first $E_{7(7)}$ invariant counterterm may actually not appear before *eight* loops. As a corollary of our results, we may also point out that $\mathcal{N}\leq 4$ supergravities whose R-symmetry group K possesses a U(1) factor, do exhibit anomalies, and therefore possible divergences need not respect the non-linear duality invariance.

It is important to emphasise that the preservation of the continuous duality symmetry in perturbation theory is not in contradiction with the string theory expectation that only its arithmetic subgroup remains a symmetry at the quantum level. Within supergravity, we expect that only $E_{7(7)}(\mathbb{Z})$ will be preserved by non-perturbative corrections in $\exp(-\kappa^{-2}S^{\text{Instanton}})$. Although the status of instanton corrections in $\mathcal{N}=8$ supergravity is not clear by any means, we will provide some evidence relying on the classical breaking of the $E_{7(7)}$ current conservation in non-trivial gravitational backgrounds, see section 2.4. On the other hand, considering $\mathcal{N}=8$ supergravity as a limit $\ell_s\to 0$ (decoupling the massive string states) of type II string theory compactified on a product of circles of radii r_i (to be taken $\to 0$ to decouple massive Kaluza-Klein states), one cannot avoid non-perturbative string corrections in the four-dimensional effective string coupling constant

$$g_4^{\,2}\equiv\frac{\ell_s^{\,6}g_s^{\,2}}{\prod_{i=1}^{6}r_i}\,, \tag{1.2}$$

while keeping the gravitational coupling constant $\kappa^2=8\pi g_4^{\,2}\ell_s^{\,2}$ fixed, since necessarily $g_4^{\,2}\to\infty$ in this limit. It is therefore clear that the supergravity limit of string theory must involve string theory non-perturbative states [24], and thus defines some non-perturbative completion of the supergravity field theory. If the supergravity limit of the string theory effective action is the effective action in field theory, the latter must necessarily include non-perturbative contributions associated to field theory instantons. The $E_{7(7)}(\mathbb{Z})$ 'Eisenstein series' that multiplies the Bel-Robinson square R^4 term in the string theory effective action is defined in string theory as an expansion in $\exp(-1/g_4^2)$ [8, 9]. This expansion diverges as g_4^6 in the supergravity limit $g_4^2 \to \infty$ [9], see also [20] for an explicit resummation of the eight-dimensional $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(3,\mathbb{Z})$ invariant threshold in the supergravity limit. The result of the present paper suggests that if this limit makes sense in field theory, it should be defined as an expansion in e^{-1/κ^2}, and that the perturbative contribution would vanish.

The paper is organised as follows. We will first recall the duality invariant formulation of the classical theory defined in [11], and exhibit its equivalence with the conventional formulation of the theory [1, 2] by means of a Gaussian integration. Then we will recall the definition of the $E_{7(7)}$ Noether current. In order to deal with the non-linear realisation of the $E_{7(7)}$ symmetry in the symmetric gauge, it will be convenient to define the non-linear transformations in terms of formal power series in Φ in the adjoint representation. We derive such formulas in section 2.5, and we exhibit the commutation relations between local supersymmetry and the $\mathfrak{e}_{7(7)}$ symmetry. More generally, we show that *the BRST operator commutes with the non-linear* $\mathfrak{e}_{7(7)}$ *symmetry*, cf. (2.91), hence is $E_{7(7)}$ invariant.

Section 3 exhibits the well definedness and consistency of the formalism (and particular the validity of the quantum action principle), through the explicit computation of the one-loop vector field contribution to the $\mathfrak{su}(8)$ anomaly. It will therefore provide answers to both 2 and 4. In this section we discuss the Feynman rules for the vector fields in detail, exhibiting the equivalence with the conventional formulation in terms of free photons. It has been shown in [25] that self-dual form fields contribute to (gravitational) anomalies, just like chiral fermion fields, by means of a formal Fujikawa-like path integral derivation. This result can be understood geometrically from the family's index theorem [26], and it has been used in [17] to establish the absence of anomalies for the $\mathfrak{su}(8)$ current Ward identities in $\mathcal{N} = 8$ supergravity. Here we will exploit the duality invariant formulation to provide a full fledged Feynman diagram computation of the vector field contribution which confirms the expected result, and therefore the absence of anomalies in the theory. In this section we also set up the Pauli-Villars regularisation for the vector fields, and exhibit its (non-trivial) compatibility with the quantum action principle.

Section 4 is also very important: it will provide the definition of the non-linear $\mathfrak{e}_{7(7)}$ Slavnov-Taylor identities for the current Ward identities, and define and solve the Wess-Zumino consistency condition, incidentally answering 3.

The last section finally provides an answer to the first point of the above list. We there discuss the solution of the Batalin-Vilkovisky master equation in the duality invariant formulation, including the coupling to the $E_{7(7)}$ Noether current. Using the property that the BRST operator commutes with the $\mathfrak{e}_{7(7)}$ symmetry and considering a duality invariant

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gauge-fixing, we are able to define consistent and mutually compatible master equations for BRST invariance and $\mathfrak{e}_{7(7)}$ symmetry. In this section we also discuss the 'energy Coulomb divergences' in the one-loop insertions of $E_{7(7)}$ currents, which constitute a special subtlety of the formalism. We will exhibit that these divergences can be consistently removed within the Pauli-Villars regularisation.

As this paper is rather heavy on formalism, we here briefly summarise our notational conventions for the reader's convenience. (Curved) space-time indices are μ, ν, ..., (curved) spatial indices are $\mathtt{i}$, $\mathtt{j}$, $\mathtt{k}$, ..., and space-time Lorentz indices are a, b, c, ... Indices in the fundamental representation **56** of $E_{7(7)}$ are $m, n, \ldots = 1, \ldots, 56$; when split into 28+28 they become $\mathpzc{m}$, $\mathpzc{n}$, .. and $\bar{\mathpzc{m}}$, $\bar{\mathpzc{n}}$, ... Rigid SU(8) indices are I, J, K, ... such that the $E_{7(7)}$ adjoint representation **133** decomposes as **63** $\oplus$ **70** with generators $X^{IJ}{}_{KL} \equiv 2\delta^{[I}_{[K} X^{J]}{}_{L]}$, $X^{IJKL} \equiv \frac{1}{2} X^{[IJKL]} + \frac{1}{48} \varepsilon^{IJKLPQMN} X_{PQMN}$, etc. Local SU(8) indices are i, j, $k \ldots = 1, \ldots, 8$, and raising or lowering them corresponds to complex conjugation. Space-time indices are lowered with the metric $g_{\mu\nu}$, and the tensor densities $\varepsilon^{\mathtt{ijk}}$ and $\varepsilon^{\mu\nu\rho\sigma}$ are normalised as $\varepsilon^{123} = \varepsilon^{0123} = 1$. Finally, we will use the letters S for the classical action, Σ for the classical action with sources, ghost and antifield terms included. While both S and Σ are local, the full quantum effective action Γ is not, but obeys $\Gamma = \Sigma + \mathcal{O}(\hbar)$.

2 $\mathcal{N} = 8$ supergravity with off-shell $E_{7(7)}$ invariance

2.1 Manifestly duality invariant formulation

We start from the usual ADM decomposition of the 4-metric

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{\mathtt{ij}} (dx^{\mathtt{i}} + N^{\mathtt{i}} dt)(dx^{\mathtt{j}} + N^{\mathtt{j}} dt) , \tag{2.1}$$

with the lapse N and the shift $N^{\mathtt{i}}$; $h_{\mathtt{ij}}$ is the metric on the spatial slice. The vector fields $A^m_{\mathtt{i}}$ of the theory appear only with *spatial* indices, and are labeled by internal indices $m, n, \ldots$ which transform in a given representation of the internal symmetry group G with maximal compact subgroup K (for $\mathcal{N} = 8$ supergravity $G \cong E_{7(7)}$ and $K \cong SU_{\mathrm{c}}(8)$, with the vector fields transforming in the **56** of $E_{7(7)}$). In comparison with the usual on-shell formalism this implies a *doubling* of the vector fields, such that the multiplet $A^m_{\mathtt{i}}$ comprises both the (spatial components of the) electric and their dual magnetic vector potentials. To formulate an action we also need the field dependent G-invariant metric G_{mn} on the vector space on which the electromagnetic fields are defined (i.e. the $E_{7(7)}$ invariant metric on $\mathbb{R}^{56}$ for $\mathcal{N} = 8$ supergravity; this metric is explicitly given in (2.30) below). In addition we need the symplectic invariant $\Omega_{mn} = -\Omega_{nm} = \Omega^{mn}$,[3] which is always present, because the generalised duality symmetry is generally a subgroup of a symplectic group acting on the electric and magnetic vector potentials [13] (the group Sp(56, $\mathbb{R}$) $\supset E_{7(7)}$ for $\mathcal{N} = 8$

[3]Hence, with our conventions $\Omega_{mp}\Omega^{pn} = -\delta^n_m$.

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supergravity). Duality invariance implies the following relation for the inverse metric G^{mn}

$$G^{mn} = \Omega^{mp}\Omega^{nq}G_{pq}\,, \qquad (G^{mp}G_{pn} = \delta_n^m)\ . \tag{2.2}$$

For later purposes we also define the 'complex structure' tensor

$$J^m{}_n \equiv G^{mp}\Omega_{pn} \quad \Rightarrow \quad J^m{}_p J^p{}_n = -\delta_n^m\ . \tag{2.3}$$

Note that $J^m{}_n$ depends on the scalar fields via the metric G_{mn}. The maximal compact subgroup K can be characterised as the maximal subgroup in G which commutes with $J^m{}_n(\mathring{\Phi})$ (for some background value $\mathring{\Phi}$ of the scalar fields).

After these preparations we can write down the part of the action containing the vector fields

$$\begin{aligned} S_{\text{vec}} = \frac{1}{2}\int d^4x \Big(&\frac{1}{2}\Omega_{mn}\varepsilon^{\mathtt{ijk}}\left(\partial_0 A^m_{\mathtt{i}} + N^{\mathtt{l}}F^m_{\mathtt{il}}\right)F^n_{\mathtt{jk}} - \frac{1}{2}N\sqrt{h}\,G_{mn}h^{\mathtt{ik}}h^{\mathtt{jl}}F^m_{\mathtt{ij}}F^n_{\mathtt{kl}} \\ &-N\sqrt{h}h^{\mathtt{ik}}h^{\mathtt{jl}}F^m_{\mathtt{ij}}W_{\mathtt{kl}\,m} - \frac{1}{2}N\sqrt{h}\,G^{mn}h^{\mathtt{ik}}h^{\mathtt{jl}}W_{\mathtt{ij}\,m}W_{\mathtt{kl}\,n}\Big)\ . \end{aligned} \tag{2.4}$$

Here $W_{\mathtt{ij}\,m}$ is a bilinear function of the fermion fields, which will be discussed in more detail shortly (see (2.36) below). We also consider the W^2 term which define the non-manifestly diffeomorphism covariant quartic terms in the fermions. For quantisation, the above action must be supplemented by further terms depending on the ghost fields as well as the anti-fields; this will be discussed in more detail below.

As shown in [11], the main advantage of the above reformulation is that it incorporates both the electric and the dual magnetic vector potentials *off-shell*, at the expense of manifest space-time diffeomorphism invariance. In particular, the equation of motion of the 56 vector fields $A^m_{\mathtt{i}}$ can be expressed as a twisted self-duality constraint [1] for the supercovariant field strength $\hat{F}^m_{\mu\nu}$ (see [11] for further details)

$$\hat{F}^m_{\mu\nu} = -\frac{1}{2\sqrt{-g}}\varepsilon_{\mu\nu}{}^{\sigma\rho}J^m{}_n\hat{F}^n_{\sigma\rho}\ , \tag{2.5}$$

where the tensor J takes the place of an imaginary unit. We briefly explain this procedure and why the time-components A^m_0 of the vector fields naturally enter this equation, although they are absent in the original Lagrangian (2.4). The variation of the action functional S_{vec} (2.4) with respect to the 56 vector fields $A^m_{\mathtt{i}}$ leads to the second order equation of motion[4]

$$\varepsilon^{\mathtt{ijk}}\partial_{\mathtt{j}}\mathscr{E}^m_{\mathtt{k}} = 0\,, \tag{2.6}$$

with the abbreviation

$$\mathscr{E}^m_{\mathtt{i}} \equiv \partial_0 A^m_{\mathtt{i}} + N^{\mathtt{j}}F^m_{\mathtt{ij}} - \frac{N}{2\sqrt{h}}h_{\mathtt{ij}}\varepsilon^{\mathtt{jkl}}\left(J^m{}_nF^n_{\mathtt{kl}} + \Omega^{mn}W_{\mathtt{kl}\,n}\right)\ . \tag{2.7}$$

[4]Do not confuse the equation of motion function $\mathscr{E}^m_{\mathtt{i}}$ with the electric potential $\mathcal{E}^m_{\mathtt{i}}$ introduced in [10, 11].

This equation is equivalent to the statement that the one-form $\mathscr{E}^m_k\, dx^k$ is closed. On any contractible open set of the $d = 4$ space-time manifold, every closed form is exact by Poincaré's lemma, which implies the existence of a zero-form A^m_0 satisfying

$$\mathscr{E}^m_{\mathtt{i}} = \partial_{\mathtt{i}} A^m_0 \ . \tag{2.8}$$

It is straightforward to verify that this equation of motion is completely equivalent to the twisted self-duality constraint of equation (2.5). Furthermore, only the identification of the zero-form with the time-component A^m_0 gives rise to an equation of motion that is diffeomorphism covariant in the usual sense.

Before we prove that the action functional (2.4) and the usual second order form of the action are equivalent, and related by functional integration, we briefly explain the realisation of the diffeomorphism algebra on the vector fields. To this aim we recall that the Lie derivative on the vector field in the covariant formulation can be rewritten as

$$\delta A^m_\mu = \partial_\mu \xi^\nu A^m_\nu + \xi^\nu \partial_\nu A^m_\mu = \partial_\mu(\xi^\nu A^m_\nu) + \xi^\nu F^m_{\nu\mu} \ . \tag{2.9}$$

Considering the vector fields A^m as abelian connections, the geometrical action of diffeomorphism is defined via the horizontal lift of the vector ξ^μ to the principle bundle, and is modified by a gauge transformation. We will consider this *covariant (or 'horizontal') diffeomorphism*

$$\delta A^m_\mu = \xi^\nu F^m_{\nu\mu} \ . \tag{2.10}$$

Splitting indices into time and space indices, we get

$$\delta A^m_{\mathtt{i}} = \xi^0 F^m_{0\mathtt{i}} \ + \xi^{\mathtt{j}} F^m_{\mathtt{ji}} \ . \tag{2.11}$$

The recipe for obtaining the correct formula in the present formulation then consists simply in replacing

$$F^m_{0\mathtt{i}} \ \rightarrow \ \partial_0 A^m_{\mathtt{i}} - \mathscr{E}^m_{\mathtt{i}} \tag{2.12}$$

everywhere according to (2.7), such that (2.11) becomes

$$\delta_\xi A^m_{\mathtt{i}} \equiv \xi^\mu \partial_\mu A^m_{\mathtt{i}} - \xi^{\mathtt{j}} \partial_{\mathtt{i}} A^m_{\mathtt{j}} - \xi^0 \mathscr{E}^m_{\mathtt{i}} \ . \tag{2.13}$$

We note that the recipe (2.12) also yields the correct formulas for all other transformations in the manifestly duality invariant formalism, including the modified supersymmetry transformations and the BRST transformations of the ghosts.

The non-standard representation of the diffeomorphism algebra (2.13) on the vector fields is consistent, because it closes *off-shell* up to a gauge transformation with parameter Λ^m, which cannot be separated from the diffeomorphism action:

$$\begin{aligned} [\delta_{\xi_1}, \delta_{\xi_2}] A^m_{\mathtt{i}} &= \delta_{[\xi_2,\xi_1]} A^m_{\mathtt{i}} + \partial_i \Lambda^m \\ \text{with} \quad \Lambda^m &= \xi_2^{\mathtt{i}} \xi_1^{\mathtt{j}} F^m_{\mathtt{ij}} + (\xi_2^0 \xi_1^{\mathtt{j}} - \xi_1^0 \xi_2^{\mathtt{j}})(\partial_0 A^m_{\mathtt{j}} - \mathscr{E}^m_{\mathtt{j}}) \ . \end{aligned} \tag{2.14}$$

The gauge transformation Λ^m can be obtained from the one that would appear in the covariant formulation by the substitution (2.12).

To sum up: although the equations of motion are covariant under the diffeomorphism action in both formulations of maximal supergravity, the representations of the diffeomorphism algebra on the vector fields do not coincide *off-shell*. Agreement can *a priori* be achieved only *on-shell*, if we impose the equations of motion in their first order form (2.8) with the introduction of the time-component of the 56 vector fields. Nevertheless, the two formulations are also formally equivalent at the quantum level, as we are going to see.

2.2 Equivalence with the covariant formalism

To establish the link with the manifestly diffeomorphism covariant formalism, we must in a first step decompose the electromagnetic fields into Darboux components associated to the symplectic form

$$\Omega_{\boldsymbol{m}\boldsymbol{n}} = \Omega_{\bar{\boldsymbol{m}}\bar{\boldsymbol{n}}} = 0 \qquad \Omega_{\boldsymbol{m}\bar{\boldsymbol{n}}} = -\Omega_{\bar{\boldsymbol{n}}\boldsymbol{m}} = \delta_{\boldsymbol{m}\bar{\boldsymbol{n}}} \tag{2.15}$$

where the indices $m, n, \ldots$ are split into pairs $(\boldsymbol{m}, \bar{\boldsymbol{m}})$ each running over half the range of m, n. For the vector fields this entails the split

$$A_{\mathtt{i}}^{m} \;\rightarrow\; (A_{\mathtt{i}}^{\boldsymbol{m}}, A_{\mathtt{i}}^{\bar{\boldsymbol{m}}}) \tag{2.16}$$

into electric and magnetic vector potentials. With the above split, the manifest off-shell $E_{7(7)}$ symmetry will be lost after the Gaussian integration to be performed below, and is thus reduced to the on-shell symmetry of the standard version. Extending (2.4) by a gauge-fixing term, the action functional becomes

$$\begin{aligned} S_{\text{vec}} = \frac{1}{2}\int d^4x \biggl(& \frac{1}{2}\Omega_{mn}\varepsilon^{\mathtt{ijk}}\left(\partial_0 A_{\mathtt{i}}^{m} + N^{\mathtt{l}} F_{\mathtt{il}}^{m}\right) F_{\mathtt{jk}}^{n} - \frac{1}{2} N\sqrt{h}\, G_{mn} h^{\mathtt{ik}} h^{\mathtt{jl}} F_{\mathtt{ij}}^{m} F_{\mathtt{kl}}^{n} \\ & - N\sqrt{h} h^{\mathtt{ik}} h^{\mathtt{jl}} F_{\mathtt{ij}}^{m} W_{\mathtt{kl}\,m} - \frac{1}{2} N\sqrt{h}\, G^{mn} h^{\mathtt{ik}} h^{\mathtt{jl}} W_{\mathtt{ij}\,m} W_{\mathtt{kl}\,n} + 2 b_m \partial_{\mathtt{i}} A_{\mathtt{i}}^{m} \biggr) . \end{aligned} \tag{2.17}$$

Sums over repeated indices are understood even when they are both down, which only reflects the property that the corresponding terms are not invariant with respect to diffeomorphisms. Performing the split, and up to an irrelevant boundary term, we arrive at the following Lagrange density

$$\begin{aligned} S_{\text{vec}} = \frac{1}{2}\int d^4x \biggl(& \Bigl(\delta_{\boldsymbol{m}\bar{\boldsymbol{n}}}\varepsilon^{\mathtt{ijk}}\left(\partial_0 A_{\mathtt{i}}^{\boldsymbol{m}} + N^{\mathtt{l}} F_{\mathtt{il}}^{\boldsymbol{m}}\right) - N\sqrt{h} G_{\boldsymbol{m}\bar{\boldsymbol{n}}} h^{\mathtt{ik}} h^{\mathtt{jl}} F_{\mathtt{ij}}^{\boldsymbol{m}} - N\sqrt{h} h^{\mathtt{ik}} h^{\mathtt{jl}} W_{\mathtt{ij}\,\bar{\boldsymbol{n}}}\Bigr) F_{\mathtt{kl}}^{\bar{\boldsymbol{n}}} \\ & - \frac{1}{2} N\sqrt{h}\, G_{\bar{\boldsymbol{m}}\bar{\boldsymbol{n}}} h^{\mathtt{ik}} h^{\mathtt{jl}} F_{\mathtt{ij}}^{\bar{\boldsymbol{m}}} F_{\mathtt{kl}}^{\bar{\boldsymbol{n}}} - \frac{1}{2} N\sqrt{h}\, G_{\boldsymbol{m}\boldsymbol{n}} h^{\mathtt{ik}} h^{\mathtt{jl}} F_{\mathtt{ij}}^{\boldsymbol{m}} F_{\mathtt{kl}}^{\boldsymbol{n}} \\ & - N\sqrt{h} h^{\mathtt{ik}} h^{\mathtt{jl}} F_{\mathtt{ij}}^{m} W_{\mathtt{kl}\,m} - \frac{1}{2} N\sqrt{h} G^{mn} h^{\mathtt{ik}} h^{\mathtt{jl}} W_{\mathtt{ij}\,m} W_{\mathtt{kl}\,n} + 2 b_{\boldsymbol{m}} \partial_{\mathtt{i}} A_{\mathtt{i}}^{\boldsymbol{m}} + 2 b_{\bar{\boldsymbol{m}}} \partial_{\mathtt{i}} A_{\mathtt{i}}^{\bar{\boldsymbol{m}}} \biggr) . \end{aligned} \tag{2.18}$$

Integrating out the auxiliary field $b_{\bar{\boldsymbol{m}}}$ enforces the constraint $\partial_{\mathtt{i}} A_{\mathtt{i}}^{\bar{\boldsymbol{m}}} = 0$, and the Lagrangian only depends on $A_{\mathtt{i}}^{\bar{\boldsymbol{m}}}$ through $F_{\mathtt{ij}}^{\bar{\boldsymbol{m}}} = \partial_{\mathtt{i}} A_{\mathtt{j}}^{\bar{\boldsymbol{m}}} - \partial_{\mathtt{j}} A_{\mathtt{i}}^{\bar{\boldsymbol{m}}}$ (note that this is the case even when considering the ghost field terms that we neglect in this discussion). One has then an isomorphism between the square integrable fields $A_{\mathtt{i}}^{\bar{m}}$ satisfying $\partial_{\mathtt{i}} A_{\mathtt{i}}^{\bar{m}} = 0$, and the square integrable fields $\Pi^{\mathtt{i}\,\bar{m}}$ satisfying the same constraint $\partial_{\mathtt{i}}\Pi^{\mathtt{i}\,\bar{m}} = 0$, through

$$\Pi^{\mathtt{i}\,\bar{m}} = \varepsilon^{\mathtt{ijk}}\partial_{\mathtt{j}} A_{\mathtt{k}}^{\bar{m}} \ , \qquad A_{\mathtt{i}}^{\bar{m}} = -[\partial_{\mathtt{l}}\partial_{\mathtt{l}}]^{-1}\varepsilon_{\mathtt{ijk}}\partial_{\mathtt{j}}\Pi^{\mathtt{k}\,\bar{m}} \ , \tag{2.19}$$

where repeated indices are summed (and appropriate boundary conditions assumed). This change of variables leads to a non-trivial functional Jacobian, but the latter does not depend on the fields and can therefore be disregarded.[5] Introducing a Lagrange multiplier $A_{\mathit{0}}^{\bar{m}}$ for the constraint $\partial_{\mathtt{i}}\Pi^{\mathtt{i}\,\bar{m}} = 0$, one has the action

$$\begin{aligned} S_{\text{vec}} =& \frac{1}{2}\int d^4x \Big(\Big(2\delta_{m\bar{n}}\left(\partial_{\mathit{0}} A_{\mathtt{i}}^{m} - \partial_{\mathtt{i}} A_{\mathit{0}}^{m} + N^{\mathtt{l}} F_{\mathtt{il}}^{m}\right) - N\sqrt{h}\varepsilon_{\mathtt{ilh}} h^{\mathtt{lj}} h^{\mathtt{hk}}\left(G_{m\bar{n}} F_{\mathtt{jk}}^{m} + W_{\mathtt{jk}\,\bar{n}}\right)\Big)\Pi^{\mathtt{i}\bar{n}} \\ & - \frac{N}{\sqrt{h}} G_{\bar{m}\bar{n}} h_{\mathtt{ij}}\Pi^{\mathtt{i}\,\bar{m}}\Pi^{\mathtt{j}\,\bar{n}} - \frac{1}{2}N\sqrt{h}\, G_{mn} h^{\mathtt{ik}} h^{\mathtt{jl}} F_{\mathtt{ij}}^{m} F_{\mathtt{kl}}^{n} \\ & - N\sqrt{h} h^{\mathtt{ik}} h^{\mathtt{jl}} F_{\mathtt{ij}}^{m} W_{\mathtt{kl}\,m} - \frac{1}{2}N\sqrt{h} G^{mn} h^{\mathtt{ik}} h^{\mathtt{jl}} W_{\mathtt{ij}\,m} W_{\mathtt{kl}\,n} + 2b_m\partial_{\mathtt{i}} A_{\mathtt{i}}^{m}\Big), \end{aligned} \tag{2.20}$$

where we normalised A_{0}^{m} such that it can be identified as the time component of the vector field, and

$$F_{\mathit{0}\mathtt{i}}^{m} = \partial_{\mathit{0}} A_{\mathtt{i}}^{m} - \partial_{\mathtt{i}} A_{\mathit{0}}^{m} \ . \tag{2.21}$$

Note that this is the form of the action that one would obtain by deriving the path integral formulation from the Hamiltonian quantisation in the Coulomb gauge, such that $\Pi^{\bar{m}\,\mathtt{i}}$ define the momentum conjugate to the vector fields $A_{\mathtt{i}}^{m}$. One then sees that (2.19) actually corresponds to a *second class constraint*, as one would expect in a first order formalism. We also emphasise that, when the equations of motion are satisfied, the Lagrange multiplier field A_{0}^{m} in the path integral can be identified with the corresponding component of A_{0}^{m} appearing in (2.8), which is the classical field resulting from rewriting a given expression $\mathscr{E}_{\mathtt{i}}^{m}$ as a curl.

One can now integrate the momentum variables $\Pi^{\mathtt{i}\bar{m}}$ through formal Gaussian inte-

[5]Note that this is only true in the specific metric independent Coulomb gauge we used, in which the ghosts decouple. For a metric dependent gauge, the functional Jacobian would depend non-trivially on the metric, but this field dependence would be exactly compensated by the functional determinant generated by the Gaussian integration over the ghosts $\bar{c}_{\bar{m}}$ and $c^{\bar{m}}$, as is ensured by BRST invariance.

gration, the remaining action is

$$
\begin{aligned}
S_{\text{vec}} =& \frac{1}{2}\int d^4x \Biggl(\frac{\sqrt{h}}{N}\delta_{m\bar{m}}\delta_{n\bar{n}}H^{\bar{m}\bar{n}}h^{\mathtt{ij}}\left(F^m_{0\mathtt{i}}+N^{\mathtt{k}}F^m_{\mathtt{ik}}\right)\left(F^n_{0\mathtt{j}}+N^{\mathtt{l}}F^n_{\mathtt{jl}}\right) \\
&-\frac{1}{2}N\sqrt{h}\Bigl(G_{mn}-G_{m\bar{m}}H^{\bar{m}\bar{n}}G_{\bar{n}n}\Bigr)h^{\mathtt{ik}}h^{\mathtt{jl}}F^m_{\mathtt{ij}}F^n_{\mathtt{kl}}-\varepsilon^{\mathtt{ijk}}\delta_{m\bar{m}}H^{\bar{m}\bar{n}}G_{\bar{n}n}\left(F^m_{0\mathtt{i}}+N^{\mathtt{l}}F^m_{\mathtt{il}}\right)F^n_{\mathtt{jk}} \\
&-\varepsilon^{\mathtt{ijk}}\delta_{m\bar{m}}H^{\bar{m}\bar{n}}\left(F^m_{0\mathtt{i}}+N^{\mathtt{l}}F^m_{\mathtt{il}}\right)W_{\mathtt{kl}\bar{n}}+H^{\bar{n}\bar{m}}G_{\bar{m}m}N\sqrt{h}\,h^{\mathtt{ik}}h^{\mathtt{jl}}F^m_{\mathtt{ij}}W_{\mathtt{kl}\bar{n}} \\
&-N\sqrt{h}\,h^{\mathtt{ik}}h^{\mathtt{jl}}F^m_{\mathtt{ij}}W_{\mathtt{kl}m}+\frac{1}{2}N\sqrt{h}\,H^{\bar{m}\bar{n}}h^{\mathtt{ik}}h^{\mathtt{jl}}W_{\mathtt{ij}\,\bar{m}}W_{\mathtt{kl}\,\bar{n}} \\
&-\frac{1}{2}N\sqrt{h}\,G^{mn}h^{\mathtt{ik}}h^{\mathtt{jl}}W_{\mathtt{ij}\,m}W_{\mathtt{kl}\,n}+2b_m\partial_{\mathtt{i}}A^m_{\mathtt{i}}\Biggr) \quad ,
\end{aligned}
\tag{2.22}
$$

where $H^{\bar{m}\bar{n}}$ is the inverse of $G_{\bar{m}\bar{n}}$ (not to be confused with the component $G^{\bar{m}\bar{n}}$ of the inverse metric G^{mn}). We will discuss the functional determinant afterward. First note that, by (2.2), duality invariance implies

$$
G_{mn}-G_{m\bar{m}}H^{\bar{m}\bar{n}}G_{\bar{n}n}=\delta_{m\bar{m}}\delta_{n\bar{n}}H^{\bar{m}\bar{n}}\ , \qquad \delta_{m\bar{m}}H^{\bar{m}\bar{n}}G_{\bar{n}n}=\delta_{n\bar{m}}H^{\bar{m}\bar{n}}G_{\bar{n}m}\ , \tag{2.23}
$$

and therefore the bosonic component is manifestly diffeomorphism invariant

$$
S_{\text{vec}}=\frac{1}{4}\int d^4x\Bigl(-\sqrt{-g}\delta_{m\bar{m}}\delta_{n\bar{n}}H^{\bar{m}\bar{n}}g^{\mu\sigma}g^{\nu\rho}F^m_{\mu\nu}F^n_{\sigma\rho}-\frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\delta_{m\bar{m}}H^{\bar{m}\bar{n}}G_{\bar{n}n}F^m_{\mu\nu}F^n_{\sigma\rho}+\mathcal{O}(W)\Bigr). \tag{2.24}
$$

The formal Gaussian integration over the momentum variables $\Pi^{i\,\bar{m}}$ also produces a functional determinant

$$
\mathrm{Det}^{-\frac{1}{2}}\left[\frac{N}{\sqrt{h}}G_{\bar{m}\bar{n}}h_{\mathtt{ij}}\delta^4(x-y)\right]=\prod_x\left(\det{}^{-\frac{3}{2}}[G_{\bar{m}\bar{n}}]N^{-42}h^7\right)\ , \tag{2.25}
$$

which defines a one-loop local divergence quartic in the cutoff $\sim\Lambda^4$. This determinant defines in particular the modification of the diffeomorphism invariant measure of the metric field from the duality invariant formulation to the conventional one [27], and respectively for the $E_{7(7)}$ invariant scalar field measure. This kind of volume divergence is in fact a general property of (super)gravity theories [28].

2.3 $\mathcal{N}=8$ supergravity

The discussion was rather general so far, and we now turn to the specific case of maximal $\mathcal{N}=8$ supergravity, where the formalism developed in the foregoing section leads to a formulation of the theory with *manifest and off-shell* $E_{7(7)}$ *invariance.* Here we show that the formalism reproduces the vector Lagrangian as well as the couplings of the vector fields to the fermions and the scalar field dependent quartic fermionic terms in the form given in [2] (the remaining quartic terms in the Lagrangian are manifestly $E_{7(7)}$ invariant). In this case the choice of Darboux coordinates amounts to decomposing the 28 complex vector fields $A_{\mathtt{i}}^{IJ}$ into imaginary and real (or 'electric' and 'magnetic') components[6]

$$A_{\mathtt{i}}^{m} \mathrel{\hat{=}} \mathrm{Im}[A_{\mathtt{i}\,IJ}]\ , \qquad A_{\mathtt{i}}^{\bar{m}} \mathrel{\hat{=}} \mathrm{Re}[A_{\mathtt{i}\,IJ}]\ \ . \tag{2.26}$$

For the coset representative $E_{7(7)}/SU_c(8)$, this corresponds to the passage from the SU(8) basis in which

$$\mathcal{V} \mathrel{\hat{=}} \begin{pmatrix} u_{ij}{}^{IJ} & v_{ijKL} \\ v^{klIJ} & u^{kl}{}_{KL} \end{pmatrix} \tag{2.27}$$

to an SL$(8,\mathbb{R})$ basis in which[7]

$$\widetilde{\mathcal{V}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \mathcal{V} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}\ \ , \tag{2.28}$$

or, written out in components,

$$\widetilde{\mathcal{V}} \mathrel{\hat{=}} \begin{pmatrix} \mathrm{Re}\left(u_{ij}{}^{IJ} + v_{ijIJ}\right) & \mathrm{Im}\left(-u_{ij}{}^{KL} + v_{ijKL}\right) \\ \mathrm{Im}\left(u_{kl}{}^{IJ} + v_{klIJ}\right) & \mathrm{Re}\left(u_{kl}{}^{KL} - v_{klKL}\right) \end{pmatrix}\ \ . \tag{2.29}$$

Then one computes that

$$\begin{aligned} G = \widetilde{\mathcal{V}}^T\widetilde{\mathcal{V}} &\mathrel{\hat{=}} \begin{pmatrix} \left(u^{ij}{}_{IJ} + v^{ijIJ}\right)\left(u_{ij}{}^{KL} + v_{ijKL}\right) & 2\mathrm{Im}\big[\left(u_{ij}{}^{IJ} + v_{ijIJ}\right)u^{ij}{}_{KL}\big] \\ 2\mathrm{Im}\big[u^{ij}{}_{IJ}\left(u_{ij}{}^{KL} + v_{ijKL}\right)\big] & \left(u^{ij}{}_{IJ} - v^{ijIJ}\right)\left(u_{ij}{}^{KL} - v_{ijKL}\right) \end{pmatrix} \\ &= \begin{pmatrix} \left(\mathrm{Re}\big[2S - \mathbb{1}\big]\right)^{-1} & \left(\mathrm{Re}\big[2S - \mathbb{1}\big]\right)^{-1}\mathrm{Im}\big[2S\big] \\ \mathrm{Im}\big[2S\big]\left(\mathrm{Re}\big[2S - \mathbb{1}\big]\right)^{-1} & \left(\mathrm{Re}\big[2S - \mathbb{1}\big]\right)^{-1} + \mathrm{Im}\big[2S\big]\left(\mathrm{Re}\big[2S - \mathbb{1}\big]\right)^{-1}\mathrm{Im}\big[2S\big] \end{pmatrix} \end{aligned} \tag{2.30}$$

where we used

$$\mathrm{Im}\big[\left(u^{ij}{}_{IJ} + v^{ijIJ}\right)\left(u_{ij}{}^{KL} + v_{ijKL}\right)\big] = 0\ \ , \tag{2.31}$$

to compute the first matrix, and where the symmetric matrix S is defined such that

$$\left(u^{ij}{}_{IJ} + v^{ijIJ}\right)S^{IJ,KL} = u^{ij}{}_{KL}\ \ . \tag{2.32}$$

To prove the equality of the two matrices in (2.30), one uses again (2.31) to show that

$$2\,\mathrm{Im}\big[u^{ij}{}_{IJ}\left(u_{ij}{}^{KL} + v_{ijKL}\right)\big] = \mathrm{Im}\left[2S\right]^{IJ,PQ}\left(u^{ij}{}_{PQ} + v^{ijPQ}\right)\left(u_{ij}{}^{KL} + v_{ijKL}\right)\ \ , \tag{2.33}$$

[6]With our usual convention $A_{\mathtt{i}}^{IJ} = (A_{\mathtt{i}IJ})^*$. Recall that the standard formulation of $\mathcal{N} = 8$ supergravity has 28 *real* vectors, for which there is no need to distinguish between upper and lower indices.

[7]This transformation is analogous to the Möbius transformation mapping the unit (Poincaré) disk to the upper half plane, and relating SU(1, 1) to SL$(2,\mathbb{R})$.

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and

$$
\begin{aligned}
\mathrm{Re}\big[2S-\mathbb{1}\big]^{IJ,PQ}\,\big(u^{ij}{}_{PQ}+v^{ijPQ}\big)\,\big(u_{ij}{}^{KL}+v_{ijKL}\big) &= \\
=\mathrm{Re}\big[\big(u^{ij}{}_{IJ}-v^{ijIJ}\big)\,\big(u_{ij}{}^{KL}+v_{ijKL}\big)\big] &= \delta^{KL}_{IJ}\ , \qquad (2.34)
\end{aligned}
$$

which establishes the equality for the first column in (2.30). The equality in the second column then follows by using the property that the matrix is symmetric and symplectic. Identifying

$$
H^{\bar{m}\bar{n}} \mathrel{\hat=} \mathrm{Re}\big[2S-\mathbb{1}\big]^{IJ,KL}\ , \qquad H^{\bar{m}\bar{n}}G_{\bar{n}m} \mathrel{\hat=} \mathrm{Im}\big[2S\big]^{IJ,KL}\ , \qquad (2.35)
$$

one recovers the conventional form of the action (2.24) as given in [2].

To investigate the couplings of the vectors to the fermions, we recall from [11] that the fermionic bilinears $W_{\mathtt{ij}m}$ in (2.18) are determined by[8]

$$
W^{IJ}_{\mathtt{ij}} = e^a_{\mathtt{i}} e^b_{\mathtt{j}} \left(u_{ij}{}^{IJ} O^{+\,ij}_{ab} + v^{ijIJ} O^{-}_{ab\,ij} \right)\ , \qquad (2.36)
$$

via the identification (analogous to (2.26))

$$
W_{\mathtt{ij}\,m} \mathrel{\hat=} \mathrm{Im}[W^{IJ}_{\mathtt{ij}}]\ , \qquad W_{\mathtt{ij}\,\bar{m}} \mathrel{\hat=} \mathrm{Re}[W^{IJ}_{\mathtt{ij}}]\ . \qquad (2.37)
$$

Here, $O^{+\,ij}_{ab}$ and its complex conjugate $O^{-}_{ab\,ij}$ are the fermionic bilinears defined in [2]

$$
O^{+\,ij}_{ab} = \bar{\psi}^i_c \gamma^{[c}\gamma_{ab}\gamma^{d]}\psi^j_d - \frac{1}{4}\bar{\psi}_{kc}\gamma_{ab}\gamma^c\chi^{ijk} - \frac{1}{(4!)^2}\varepsilon^{ijklmnpq}\bar{\chi}_{klm}\gamma_{ab}\chi_{npq}\ . \qquad (2.38)
$$

modulo normalisations (our coefficients here are chosen to agree with [11]). By complex self-duality they satisfy

$$
O^{+\,ij}_{ab} = \frac{i}{2}\varepsilon_{ab}{}^{cd}O^{+\,ij}_{cd}\ , \qquad O^{-}_{ab\,ij} = -\frac{i}{2}\varepsilon_{ab}{}^{cd}O^{-}_{cd\,ij}\ . \qquad (2.39)
$$

These relations allow us to express the 'timelike' components $W^{IJ}_{\mathtt{0i}}$ in terms of the purely spatial components $W^{IJ}_{\mathtt{ij}}$, and thereby to recover the full fermionic Lagrangian of the covariant formulation in terms of just the purely spatial components $W^{IJ}_{\mathtt{ij}}$.

After these preparations we return to the Lagrangian (2.22), from which we read off the couplings of the vector fields to the fermions

$$
\begin{aligned}
&\varepsilon^{\mathtt{ijk}}\mathrm{Im}\big[F^{IJ}_{\mathtt{0i}} + N^{\mathtt{l}}F^{IJ}_{\mathtt{il}}\big]\,\mathrm{Re}\big[2S-\mathbb{1}\big]^{IJ,KL}\,\mathrm{Re}\big[W^{KL}_{\mathtt{kl}}\big] \\
&\qquad + N\sqrt{h}\,h^{\mathtt{ik}}h^{\mathtt{jl}}\,\mathrm{Im}\,[F^{IJ}_{\mathtt{ij}}]\,\Big(\mathrm{Im}\big[W^{IJ}_{\mathtt{kl}}\big] - \mathrm{Im}\big[2S\big]^{IJ,KL}\mathrm{Re}\big[W^{KL}_{\mathtt{kl}}\big]\Big)\ . \qquad (2.40)
\end{aligned}
$$

Using the properties of $S^{IJ,KL}$ one computes that

$$
\begin{aligned}
&\mathrm{Re}\big[2S-\mathbb{1}\big]^{IJ,KL}\,\mathrm{Re}\big[u_{ij}{}^{KL}O^{+\,ij}_{ab} + v^{ijKL}O^{-}_{ab\,ij}\big] \\
&\quad = \mathrm{Re}\big[(2S-\mathbb{1})^{IJ,KL}\,\big(u^{ij}{}_{KL}+v^{ijKL}\big)\,O^{-}_{ab\,ij}\big] + \mathrm{Im}[2S]^{IJ,KL}\,\mathrm{Im}\big[\big(u_{ij}{}^{KL}+v_{ijKL}\big)\,O^{+\,ij}_{ab}\big] \\
&\quad = \mathrm{Re}\big[\big(u_{ij}{}^{IJ}-v_{ijIJ}\big)\,O^{+\,ij}_{ab}\big] + \mathrm{Im}[2S]^{IJ,KL}\,\mathrm{Im}\big[\big(u_{ij}{}^{KL}+v_{ijKL}\big)\,O^{+\,ij}_{ab}\big]\ . \qquad (2.41)
\end{aligned}
$$

[8] Readers should keep in mind the different meanings of the letters $\mathtt{i},\mathtt{j},\ldots$ and $i,j,\ldots$ in this and other equations of this section (with apologies from the authors for the proliferation of different fonts!).

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Invoking the complex self-duality of $O_{ab}^{+\,ij}$ one recovers the manifest diffeomorphism invariant coupling

$$\begin{aligned}
&e\,e^{a\,\mu}e^{b\,\nu}\mathrm{Im}[F_{\mu\nu}^{IJ}]\,\Big(\mathrm{Im}\big[\big(u_{ij}{}^{IJ}-v_{ijIJ}\big)\,O_{ab}^{+\,ij}\big]-\mathrm{Im}\big[2S\big]^{IJ,KL}\mathrm{Re}\big[\big(u_{ij}{}^{KL}+v_{ijKL}\big)\,O_{ab}^{+\,ij}\big]\Big)\\
&\quad=e\,e^{a\,\mu}e^{b\,\nu}\mathrm{Im}[F_{\mu\nu}^{IJ}]\,\mathrm{Re}\big[2S-\mathbb{1}\big]^{IJ,KL}\mathrm{Im}\big[\big(u_{ij}{}^{KL}+v_{ijKL}\big)\,O_{ab}^{+\,ij}\big]\\
&\quad=e\,e^{a\,\mu}e^{b\,\nu}\,\mathrm{Im}\,[F_{\mu\nu}^{IJ}]\,\mathrm{Im}\big[S^{IJ,KL}(u^{-1})^{KL}{}_{ij}O_{ab}^{+\,ij}\big]\ \ .
\end{aligned}\tag{2.42}$$

Next we consider the quartic terms in the fermions. They read

$$\begin{aligned}
&\frac{1}{2}N\sqrt{h}\,H^{\bar{m}\bar{n}}h^{\mathtt{ik}}h^{\mathtt{jl}}W_{\mathtt{ij}\,\bar{m}}W_{\mathtt{kl}\,\bar{n}}\\
&\quad=\frac{1}{2}eh^{\mathtt{ik}}h^{\mathtt{jl}}e_{\mathtt{i}}^{a}e_{\mathtt{j}}^{b}e_{\mathtt{k}}^{c}e_{\mathtt{l}}^{d}\mathrm{Re}\big[\big(u_{ij}{}^{IJ}+v_{ijIJ}\big)\,O_{ab}^{+\,ij}\big]\mathrm{Re}\big[2S-\mathbb{1}\big]^{IJ,KL}\mathrm{Re}\big[\big(u_{kl}{}^{KL}+v_{klKL}\big)\,O_{cd}^{+\,kl}\big]
\end{aligned}\tag{2.43}$$

and

$$-\frac{1}{2}N\sqrt{h}\,G^{mn}h^{\mathtt{ik}}h^{\mathtt{jl}}W_{\mathtt{ij}\,m}W_{\mathtt{kl}\,n}\;=\;-\frac{1}{4}e\,h^{\mathtt{ik}}h^{\mathtt{jl}}e_{\mathtt{i}}^{a}e_{\mathtt{j}}^{b}e_{\mathtt{k}}^{c}e_{\mathtt{l}}^{d}\,O^{-}_{ab\,ij}O_{cd}^{+\,ij}\ \ ,\tag{2.44}$$

where in the last equation the dependence of $W_{\mathtt{ij}m}$ on scalar fields in (2.36) is eliminated through the contraction with G^{mn}. Using (2.34) and

$$\begin{aligned}
&\mathrm{Re}\big[2S-\mathbb{1}\big]^{IJ,KL}\ \big(u_{ij}{}^{IJ}+v_{ijIJ}\big)\,\big(u_{kl}{}^{KL}+v_{klKL}\big)\\
&\qquad\qquad=\ (u^{-1})^{IJ}{}_{ij}\,\big(S^{IJ,KL}+u^{pq}{}_{IJ}v_{pqIJ}\big)\,(u^{-1})^{KL}{}_{kl}\ \ ,
\end{aligned}\tag{2.45}$$

we obtain

$$\begin{aligned}
\frac{1}{2}N\sqrt{h}\,H^{\bar{m}\bar{n}}h^{\mathtt{ik}}h^{\mathtt{jl}}W_{\mathtt{ij}\,\bar{m}}W_{\mathtt{kl}\,\bar{n}}\ =\ &\frac{1}{4}e\,h^{\mathtt{ik}}h^{\mathtt{jl}}e_{\mathtt{i}}^{a}e_{\mathtt{j}}^{b}e_{\mathtt{k}}^{c}e_{\mathtt{l}}^{d}\,\Big(O^{-}_{ab\,ij}O_{cd}^{+\,ij}\\
&+\frac{1}{2}\Big[O_{ab}^{+\,ij}(u^{-1})^{IJ}{}_{ij}\Big(S^{IJ,KL}+u^{pq}{}_{IJ}v_{pqKL}\Big)(u^{-1})^{KL}{}_{kl}O_{cd}^{+\,kl}\ +\mathrm{c.c.}\Big]\Big)\ \ .
\end{aligned}\tag{2.46}$$

The first term in parentheses cancels the (manifestly $E_{7(7)}$ invariant) expression (2.44) — as must be the case because any Lorentz invariant extension of type $O^{+ij}O^{-}_{ij}$ must necessarily vanish because of the opposite duality phases. Altogether we have shown that the relevant part of the Lagrangian agrees with the corresponding one from [2] which reads, in the present notations and conventions[9]

$$\begin{aligned}
\mathcal{L}_{\mathrm{VF}}=\frac{e}{4}\Big(&-\big[2S-\mathbb{1}\big]^{IJ,KL}\mathrm{Im}[F_{\mu\nu}^{IJ}]^{-}\mathrm{Im}[F^{\mu\nu\,KL}]^{-}-ie^{a\mu}e^{b\nu}O_{ab}^{+\,ij}(u^{-1})^{IJ}{}_{ij}S^{IJ,KL}\mathrm{Im}[F_{\mu\nu}^{KL}]\\
&+\frac{1}{8}O_{ab}^{+\,ij}(u^{-1})^{IJ}{}_{ij}\big(S^{IJ,KL}+u^{pq}{}_{IJ}v_{pqKL}\big)(u^{-1})^{KL}{}_{kl}O^{+\,abkl}+\mathrm{c.c.}\Big)\ \ .
\end{aligned}\tag{2.47}$$

[9]The conventions of [2] are recovered with the identifications ${}^{[2]}A_{\mu}^{IJ}\ \equiv\ \sqrt{2}\,\mathrm{Im}[A_{\mu}^{IJ}]$, ${}^{[2]}\psi_{\mu}^{i}\ \equiv\ \frac{1}{\sqrt{2}}\psi_{\mu}^{i}$, ${}^{[2]}\chi^{ijk}\ \equiv\ \frac{1}{4}\chi^{ijk}$. The charge conjugation matrix of [2] is related to ours by, ${}^{[2]}\mathcal{C}\ \equiv\ i\,\mathcal{C}$, such that, for instance $O_{ab}^{+\,ij}=-i2\sqrt{2}\,{}^{[2]}O_{ab}^{+\,ij}$ and the complex self-duality convention is reversed.

Because the vector fields only appear through the field strength $F^{IJ}_{\mathtt{ij}}$ in the BRST transformations of the fields, the Gaussian integration can be carried out for the complete Batalin-Vilkovisky action which will be discussed in the last section. The validity of the BRST master equation all along the process of carrying out the Gaussian path integrals to pass from one formalism to the other ensures the validity of the above formal argument, by fixing all possible ambiguities associated to the regularisation scheme.

2.4 The classical $E_{7(7)}$ current

A main advantage of the present formulation is that the $E_{7(7)}$ current can be derived as a *bona fide* Noether current [11]. It consists of two pieces

$$J^\mu = J^{(1)\mu} + J^{(2)\mu} \quad . \tag{2.48}$$

Here the first piece $J^{(1)}$ does not depend on the vector fields and has the standard form as in any σ-model with fermions (see also [29]). The more important piece for our discussion here is the second term $J^{(2)}$, which depends on the 56 electric and magnetic vector fields and is of Chern-Simons type; this part of the current does not exist off-shell in the usual formulation [13], where it would be given by a non-local expression on-shell. The current J^μ is an axial vector which defines the current three-form

$$J = J^{(1)} + J^{(2)} \equiv \frac{1}{3!}\varepsilon_{\mu\nu\rho\sigma} J^\mu \, dx^\nu{}_\wedge dx^\rho{}_\wedge dx^\sigma \quad , \tag{2.49}$$

in terms of which the classical current conservation simply reads $dJ = 0$.

Following the standard Noether procedure, the $E_{7(7)}$-current J^μ was computed in [11] by an infinitesimal displacement along $\Lambda \in \mathfrak{e}_{7(7)}$. Under the $\mathrm{SU_c}(8)$ subgroup of $E_{7(7)}$, J^μ decomposes into 63 components $(J^\mu)^I{}_K$ and 70 components $(J^\mu)^{IJKL}$:

$$J^\mu(\Lambda) = (J^\mu)^I{}_K \Lambda_I{}^K + (J^\mu)^{IJKL}\, \Lambda_{IJKL} \quad . \tag{2.50}$$

The easiest way to write the first piece $J^{(1)}$ is in terms of matrices:

$$J^{(1)\mu}(\Lambda) = -\frac{1}{24}\mathrm{tr}\left(\mathcal{V}^{-1} R^\mu \mathcal{V}\Lambda\right) \quad . \tag{2.51}$$

Here, we are using the matrix form of the scalar coset $\mathcal{V}$ (2.27) and the matrices Λ and R^μ in $\mathfrak{e}_{7(7)}$ that are defined as usual

$$\Lambda \triangleq \begin{pmatrix} 2\delta^{[M}_{[I}\Lambda_{J]}{}^{N]} & \Lambda_{IJOP} \\ \Lambda^{KLMN} & -2\delta^{[K}_{[O}\Lambda_{P]}{}^{L]} \end{pmatrix} \, , \qquad R^\mu \triangleq \begin{pmatrix} -2\delta^{[m}_{[i} R^{\mu\, n]}{}_{j]} & R^\mu{}_{ijop} \\ R^{\mu\, klmn} & 2\delta^{[k}_{[o} R^{\mu\, l]}{}_{p]} \end{pmatrix} \, . \tag{2.52}$$

The components $R^{\mu\, i}{}_j$ and $R^\mu{}_{ijkl} = \frac{1}{4!}\varepsilon_{ijklmnop} R^{\mu\, mnop}$ have the form

$$\begin{aligned} R^{\mu\, i}{}_j &\equiv 2i\varepsilon^{\mu\nu\sigma\rho}\left(\bar\psi^i_\nu \gamma_\sigma \psi_{\rho j} - \frac{1}{8}\delta^i_j\, \bar\psi^k_\nu \gamma_\sigma \psi_{\rho k}\right) + \frac{\sqrt{-g}}{8}\left(\bar\chi^{ikl}\gamma^\mu \chi_{jkl} - \frac{1}{8}\delta^i_j\, \bar\chi^{klm}\gamma^\mu \chi_{klm}\right) \\ R^\mu{}_{ijkl} &\equiv \sqrt{-g}\hat{\mathcal{A}}^\mu_{ijkl} - \frac{i}{2}\varepsilon^{\mu\nu\sigma\rho}\left(\bar\chi_{[ijk}\gamma_{\sigma\rho}\psi_{\nu l]} - \frac{1}{4!}\varepsilon_{ijklmnop}\bar\chi^{mno}\gamma_{\sigma\rho}\psi_\nu{}^p\right) \quad , \end{aligned} \tag{2.53}$$

where $\hat{\mathcal{A}}_\mu^{ijkl}$ is the supercovariant derivative of the scalar coset

$$\hat{\mathcal{A}}_\mu^{ijkl} \equiv u^{ij}{}_{IJ}\partial_\mu v^{klIJ} - v^{ijIJ}\partial_\mu u^{kl}{}_{IJ} - \bar{\psi}_\mu^{[i}\chi^{jkl]} - \frac{1}{4!}\varepsilon^{ijklmnop}\bar{\psi}_{\mu\, m}\chi_{nop}\,. \tag{2.54}$$

Since the second part $J^{(2)\mu}$ of the current contains the 56 vector fields $A_{\mathtt{i}}^m$, it necessarily lacks manifest covariance. With spatial indices $\mathtt{i},\mathtt{j} = 1,2,3$, it has the form:[10]

$$\begin{aligned} J^{(2)\mathtt{k}} &= -\frac{1}{2}\varepsilon^{\mathtt{ijk}}A_{\mathtt{i}}^m\left(\partial_0 A_{\mathtt{j}}^n - 2N^{\mathtt{l}}F_{\mathtt{lj}}^n\right)\Omega_{pm}\Lambda_n{}^p + \sqrt{-g}h^{\mathtt{jk}}h^{\mathtt{il}}A_{\mathtt{i}}^n\Big(G_{pm}F_{\mathtt{jl}}^m + W_{\mathtt{jl}\,p}\Big)\Lambda_n{}^p \\ J^{(2)0} &= \frac{1}{4}\varepsilon^{\mathtt{ijk}}A_{\mathtt{i}}^m F_{\mathtt{jk}}^n\Omega_{pm}\Lambda_n{}^p\,. \end{aligned} \tag{2.55}$$

Like for $J^{(1)}$ in eq. (2.51), the independent components of $J^{(2)}$ are provided by the 133 independent components of Λ within the 56×56-matrix $\Lambda_n{}^p$. For instance, the time-like components in the $\mathfrak{su}(8)$ basis are given by

$$\begin{aligned} J^{(2)0\,I}{}_J &= \frac{i}{4}\varepsilon^{\mathtt{ijk}}\left(A_{\mathtt{i}}^{IK}F_{\mathtt{jk}\,JK} + A_{\mathtt{i}\,JK}F_{\mathtt{jk}}^{JK} - \frac{1}{8}\delta_J^I\left(A_{\mathtt{i}}^{KL}F_{\mathtt{jk}\,KL} + A_{\mathtt{i}\,KL}F_{\mathtt{jk}}^{KL}\right)\right) \\ J^{(2)0\,IJKL} &= -\frac{i}{2}\varepsilon^{\mathtt{ijk}}\left(A_{\mathtt{i}}^{[IJ}F_{\mathtt{jk}}^{KL]} - \frac{1}{4!}\varepsilon^{IJKLMNOP}A_{\mathtt{i}\,MN}F_{\mathtt{jk}\,OP}\right) \end{aligned} \tag{2.56}$$

The space-like components admit a similar form that can straightforwardly be obtained from (2.55). However, the explicit expressions are rather complicated, and would not provide any further insight in this discussion.

As a next step, we want to rewrite the vector field part (2.55) in a way that allows a direct comparison with the current constructed in [13]. A simple computation reveals the identity [11]

$$J^{(2)\mu} = \left(\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\nu\rho}^m A_\sigma^n + \frac{1}{2}\delta_{\mathtt{k}}^\mu\varepsilon^{\mathtt{ijk}}\partial_{\mathtt{i}}\left(A_{\mathtt{j}}^m A_0{}^n\right) - \delta_{\mathtt{k}}^\mu\varepsilon^{\mathtt{ijk}}A_{\mathtt{i}}^n\left(\mathscr{E}_j^m - \partial_{\mathtt{j}}A_0^m\right)\right)\Lambda_n{}^p\Omega_{pm}. \tag{2.57}$$

where a spurious dependence in the component A_0^m has been introduced, such that all the A_0^m dependent terms add up to zero. This form of the current decomposes into three terms:

1. The first term in $J^{(2)\mu}$ is a Chern-Simons three-form. It is manifestly diffeomorphism covariant in the usual sense.

2. The second term is a 'curl', and thus does not affect current conservation.[11]

3. The third term is proportional to the integrated equation of motion of the vector field (2.8) with $\mathscr{E}_{\mathtt{k}}^m$ defined in (2.7).

[10] Note that the normalisation of the vector fields here differs from the one in [11] by a factor 2.

[11] Although the Noether procedure only determines the current J up to a 'curl', this term cannot be avoided in (2.57), because A_0^m is not a fundamental field in the duality invariant formulation.

Let us now recall the procedure of [13] for obtaining the conserved current associated to the duality invariance of the equations of motion. The idea is to supplement the manifestly covariant part of the current $J^{(1)\mu}(\Lambda)$ by a further term $J^{(2)\mu}_{\rm GZ}$ in such a way that the complete current (which we will henceforth refer to as the *Gaillard-Zumino current*) is conserved

$$\partial_\mu \Big(J^{(1)\mu}(\Lambda) + J^{(2)\mu}_{\rm GZ}(\Lambda) \Big) = 0 \ , \tag{2.58}$$

if the equations of motion are enforced. Therefore, it is clear that $J^{(2)\mu}_{\rm GZ}(\Lambda)$ is only defined up to a curl, and modulo terms proportional to the equations of motion. From the complete Noether current (2.57), we thus deduce

$$J^{(2)\mu}_{\rm GZ}(\Lambda) = \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} A^m_\nu \, F^n_{\rho\sigma} \, \Lambda_m{}^p \Omega_{pn} \ . \tag{2.59}$$

This Chern-Simons three-form exhibits manifest diffeomorphism covariance and it depends *only* on the 56 vector fields, unlike the current (2.55). The explicit form of $J^{(2)\mu}_{\rm GZ}(\Lambda)$ as given in [29] is indeed equivalent to the decomposition of (2.59) in a Darboux basis for the 56 electromagnetic fields A^m_μ into $A^{\mathfrak{m}}_\mu$ and $A^{\bar{\mathfrak{m}}}_\mu$ (2.26). The usual covariant formulation of [1] contains only the 28 vector fields $A^{\mathfrak{m}}_\mu$ off-shell, whereas the dual fields $A^{\bar{\mathfrak{m}}}_\mu$ are non-local functionals of all the other fields satisfying the equations of motion.

In a non-trivial background, the Chern-Simons like component (2.59) is not globally defined in general. For a non-trivial connection, one must introduce a reference connection $\mathring{A}^m$, such that the one-form $A^m - \mathring{A}^m$ is gauge invariant (and so globally defined), and $\mathring{F}^m$ represents a non-trivial cohomology class in the given background. The background dependent extension of (2.59) is given from the Cartan homotopy formula as

$$J^{(2)}_{\rm GZ}(\Lambda) = -\frac{1}{2} \Big(A^m - \mathring{A}^m \Big) \wedge \Big(F^n + \mathring{F}^n \Big) \, \Lambda_m{}^p \Omega_{pn} \ . \tag{2.60}$$

By definition of the Cartan homotopy formula, it follows that the globally defined $E_{7(7)}$ current then suffers from a *classical anomaly*

$$dJ(\Lambda) = \frac{1}{2} \mathring{F}^m {}_\wedge \mathring{F}^n \ \Lambda_m{}^p \Omega_{pn} \ . \tag{2.61}$$

Even without a general classification of the instanton backgrounds that may occur in $\mathcal{N}=8$ supergravity, this result by itself already shows how the continuous $E_{7(7)}$ symmetry can be broken in a non-trivial background. When the gravity background is such that there is a non-trivial cohomology group

$$H^2(\mathbb{Z}) \wedge H^2(\mathbb{Z}) \rightarrow H^4(\mathbb{Z}) \tag{2.62}$$

and both $\mathring{F}^m$ and $\mathring{F}^m {}_\wedge \mathring{F}^n$ define non-trivial cohomology classes in $H^2(\mathbb{Z})$ and $H^4(\mathbb{Z})$, respectively,[12] the $\mathfrak{e}_{7(7)}$ Ward identities will be broken in the background. In this case the

[12]This is *not* the case for dyonic solutions in an asymptotically Minkowskian space-time: even though $\mathring{F}^m$ is non-trivial for such solutions of Maxwell's equations, the product $\mathring{F}^m {}_\wedge \mathring{F}^n$ is trivial.

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1PI generating functional Γ evaluated on $E_{7(7)}$ transformed fields varies as (with appropriate normalisation)

$$\Gamma[g] = \Gamma[\mathbb{1}] + 2\pi\, \Omega_{mp}\, g^p{}_n\, q^m q^n \quad , \tag{2.63}$$

with integer charges $q^m = \frac{1}{2\pi}\int \mathring{F}^m$. This 'classical anomaly' is not affected by the Legendre transform, and the generating functional W of connected diagrams transforms as Γ with respect to $E_{7(7)}$ transformations. As a consequence, the generating functional $Z = \exp[\,iW\,]$ will no longer be invariant under continuous $E_{7(7)}$ transformations, but only with respect to transformations $g \in E_{7(7)}(\mathbb{Z})$. Such backgrounds appear for example in the classification of [30] as $\mathbb{C}P^2$ and $S^2 \times S^2$ type spaces. One might therefore anticipate that $E_{7(7)}$ gets broken to a discrete subgroup when the path integral also includes a sum over such instanton contributions.

However, we should caution readers that the status of 'instanton solutions' in $\mathcal{N} = 8$ supergravity is not clear by any means. Unlike the usual self-duality constraint (which requires a Euclidean metric) the twisted self-duality constraint (2.5) contains an additional 'imaginary unit' J, and any $E_{7(7)}$ invariant Euclidean theory must therefore involve scalar fields parameterising a pseudo-Riemmanian symmetric space $E_{7(7)}/SU^*(8)_{\rm c}$ or $E_{7(7)}/\mathrm{SL}(8)_{\rm c}$ such that the **28** representation is real.[13] It is thus doubtful whether a 'Wick rotation' really makes sense, or whether one should instead look for *real* saddle points in a Lorentzian path integral. The second approach would still require to define the action in a non-trivial non-globally hyperbolic background. It is rather straightforward to modify the classical action similarly as (2.61) such that the equations of motion are not modified, and such that the Lagrangian density is gauge invariant and transforms covariantly with respect to spatial diffeomorphisms. Nonetheless, this Lagrangian density transforms covariantly with respect to $D = 4$ diffeomorphisms only up to terms linear in the equations of motion.

2.5 Transformations in the symmetric gauge

Under the combined action of local SU(8) and rigid $E_{7(7)}$ the 56-bein transforms as

$$\mathcal{V}(x) \;\rightarrow\; \mathcal{V}'(x) = h(x)\mathcal{V}(x)g^{-1} \quad , \qquad h(x) \in \mathrm{SU}(8)\ , \;\; g \in E_{7(7)} \quad . \tag{2.64}$$

For the classical theory, one has the option of either keeping the local SU(8) with linearly realised $E_{7(7)}$, or fixing a gauge for the local SU(8), retaining only the 70 physical scalar fields, whereby the rigid $E_{7(7)}$ becomes realised non-linearly. However, we are here concerned with the *quantised theory*, where the compatibility and mutual consistency of these two descriptions is not immediately evident. Indeed, the SU(8) gauge-invariant formulation of the theory may appear *not* to be well defined at the quantum level because the gauge $\mathfrak{su}(8)$ Ward identity is anomalous at one loop due to the contribution from the

[13] A positive definite 'kinetic term' could then be recovered by decomposing $E_{7(7)}/SU^*(8)_{\rm c} \cong \mathbb{R}^*_+ \times E_{6(6)}/\mathrm{Sp}(4) \times \mathbb{R}^{27}$ (respectively $E_{7(7)}/\mathrm{SL}(8)_{\rm c} \cong \mathrm{SL}(8)/\mathrm{SO}(8) \times \mathbb{R}^{35}$), and dualising 27 axionic scalars (respectively 35) into 2-forms, in analogy with the type IIB D-instantons [31].

spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ fermions [15]. On the other hand, as shown by Marcus [17], the rigid $\mathrm{SU}(8) \subset E_{7(7)}$ left after gauge-fixing is *non-anomalous*, implying the absence of anomalies for the *rigid* $\mathfrak{su}(8)$ current Ward identities in the gauge-fixed formulation of the theory. This is because the rigid $\mathfrak{su}(8)$ symmetry acts linearly on the vector fields, whose chiral nature under SU(8) implies that there is an extra contribution to the anomaly from the vector fields which precisely compensates the contribution from the fermion fields. From the path integral perspective, the main difference between those two kinds of $\mathfrak{su}(8)$ Ward identities can be viewed as resulting from a redefinition of the 56 vector fields as

$$A_{\mathtt{i}}^{IJ} \rightarrow \check{A}_{\mathtt{i}}^{ij} \equiv u^{ij}{}_{IJ} A_{\mathtt{i}}^{IJ} + v^{ij\,IJ} A_{\mathtt{i}\,IJ} \tag{2.65}$$

that is, to the passage between objects transforming under rigid $E_{7(7)}$ and local SU(8), respectively. According to the family's index theorem this change of variables does not leave the path integral measure for the vector fields invariant (because the action of $E_{7(7)}$ on the vector fields is chiral), and thus generates an anomaly. The results of [15] and [17] are therefore perfectly consistent with each other, because the associated sets of Ward identities cannot be both free of anomalies. In the following section we will present an explicit Feynman diagram computation of the vector field contribution to the $\mathfrak{su}(8)$ anomaly. This explicit computation was not given in [17], which relied on the formulation of $\mathcal{N} = 8$ supergravity with only on-shell $E_{7(7)}$ and on arguments based on the family's index theorem.

We emphasize that the $\mathfrak{su}(8)$ anomaly for the local SU(8) gauge invariance is somewhat artificial because it can be compensated by the addition of an appropriate Wess-Zumino term for the $\mathrm{SU}_{\mathrm{c}}(8)$ components of the $E_{7(7)}/\mathrm{SU}_{\mathrm{c}}(8)$ vielbein $\mathcal{V}(x)$ [16]. This procedure replaces the gauge $\mathfrak{su}(8)$ anomaly by a corresponding anomaly of the $\mathfrak{su}(8)$ current Ward identities (with the same coefficient). While restoring local SU(8), the latter by itself would break the rigid $E_{7(7)}$ symmetry, but for $\mathcal{N} = 8$ supergravity this anomaly is cancelled in turn by the contribution from the vector fields! Consequently we anticipate that our results can be re-obtained for the version of $\mathcal{N} = 8$ supergravity with local SU(8) and linearly realised $E_{7(7)}$ such that both descriptions of the quantised theory are consistent, but a detailed verification of this claim remains to be done.

In order to set up the perturbative expansion of the quantised theory, we will nevertheless parameterise the symmetric space $E_{7(7)}/\mathrm{SU}_{\mathrm{c}}(8)$ with explicit coordinates. We will consider as coordinates the scalar fields ϕ^{ijkl} in the **70** of SU(8), which parameterise a representative $\mathcal{V}(x)$ in the *symmetric gauge*, *viz.*

$$\mathcal{V}(x) \equiv \exp \Phi(x) \mathrel{\hat{=}} \exp \begin{pmatrix} 0 & \phi_{ijkl}(x) \\ \phi^{ijkl}(x) & 0 \end{pmatrix} \tag{2.66}$$

with $\Phi \in \mathfrak{e}_{7(7)} \ominus \mathfrak{su}(8)$ and the standard convention $\phi^{ijkl} = (\phi_{ijkl})^*$, (having fixed the SU(8) gauge there is no need any more to distinguish between SU(8) and E_7 indices). After this gauge choice we are left with a *rigid* $E_{7(7)}$ symmetry, whose SU(8) subgroup is realised

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linearly. The remaining rigid E_7 transformations require field dependent compensating SU(8) rotation in order to maintain the chosen gauge (2.66), and are therefore realised *non-linearly* on the 70 scalar fields. In this section, we work out these non-linear transformations in more detail to set the stage for the implementation of the full nonlinear E_7 symmetry at the quantum level. For this purpose we adopt the following notational convention: for any two Lie algebra elements X and Y and any function $f(X)$ that is analytic at $X = 0$, we abbreviate the adjoint action of $f(X)$ on Y by

$$f(X) * Y \equiv f(\mathrm{ad}(X))(Y) \tag{2.67}$$

Here, the right hand side is to be evaluated term by term in the Taylor expansion, where the n-th order term $(\mathrm{ad})^n(X)(Y)$ is the n-fold commutator $[X,[X,...[X,Y]...]]$. It is easy to check that $f(X) * g(X) * Y = (fg)(X) * Y$. For the evaluation of the non-linear transformations the main tool is the Baker-Campbell-Hausdorf formula

$$\begin{aligned}\exp(X)\exp(Y) &= \exp\Big(X + \mathrm{Td}(X) * Y + \mathcal{O}(Y^2)\Big) \\ &= \exp\Big(Y + \mathrm{Td}(-Y) * X + \mathcal{O}(X^2)\Big)\end{aligned} \tag{2.68}$$

with

$$\mathrm{Td}(x) \equiv \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \mathcal{O}(x^2) \ . \tag{2.69}$$

Accordingly we now consider an E_7 transformation with parameter Λ in the **70** of SU(8), *viz.*

$$g \equiv \exp\Lambda \mathrel{\hat{=}} \begin{pmatrix} 0 & \Lambda_{ijkl}(x) \\ \Lambda^{ijkl}(x) & 0 \end{pmatrix} \ . \tag{2.70}$$

Then, by use of (2.68),

$$\exp\Phi\exp(-\Lambda) = \exp\left(\Phi - \frac{\Phi/2}{\tanh(\Phi/2)} * \Lambda - \frac{1}{2}[\Phi,\Lambda] + \mathcal{O}(\Lambda^2)\right) \tag{2.71}$$

where the odd piece is $[\Phi,\Lambda] \equiv \Phi * \Lambda \in \mathfrak{su}(8)$. Now we must choose the compensating SU(8) transformation from the left so as to cancel the third term in the exponential. Using (2.71) and the second line of (2.68), we obtain

$$\begin{aligned}\exp\big(\Phi + \delta\Phi\big) &= \exp\Big(\tanh(\Phi/2) * \Lambda\Big)\ \exp\Phi\ \exp(-\Lambda) \\ &= \exp\left(\Phi - \frac{\Phi}{\tanh\Phi} * \Lambda + \mathcal{O}(\Lambda^2)\right)\end{aligned} \tag{2.72}$$

or

$$\delta^{\mathfrak{e}_{7(7)}}\Phi \equiv \delta^{\mathfrak{e}_{7(7)}}(\Lambda)\Phi = -\frac{\Phi}{\tanh\Phi} * \Lambda \ . \tag{2.73}$$

In the same way one computes the supersymmetry transformation of the scalar fields and the non-linear modifications due to the compensating SU(8) rotations. Infinitesimally, local supersymmetry acts on the scalar fields by a shift along the non-compact directions with parameter

$$\mathcal{X} \;=\; \begin{pmatrix} 0 & \overline{\epsilon}_{[i}\chi_{jkl]} + \frac{1}{24}\varepsilon_{ijklmnpq}\overline{\epsilon}^{m}\chi^{npq} \\ \overline{\epsilon}^{[i}\chi^{jkl]} + \frac{1}{24}\varepsilon^{ijklmnpq}\overline{\epsilon}_{m}\chi_{npq} & 0 \end{pmatrix} \tag{2.74}$$

Observing that this shift acts on $\mathcal{V}$ *from the left* (unlike the E_7 transformation in (2.64) which acts from the right), one computes that, again using (2.68),

$$\exp\Big(\mathcal{X} + \tanh(\Phi/2) * \mathcal{X}\Big)\;\exp\Phi = \exp\left(\Phi + \frac{\Phi}{\sinh\Phi} * \mathcal{X} + \mathcal{O}(\mathcal{X}^2)\right) \quad , \tag{2.75}$$

whence the supersymmetry transformation of Φ is

$$\delta^{Susy}\Phi \equiv \delta^{Susy}(\mathcal{X})\Phi = \frac{\Phi}{\sinh\Phi} * \mathcal{X} \;\; . \tag{2.76}$$

By elementary algebra, this can be re-written in terms of an $E_{7(7)}$ transformation with parameter $\mathcal{X}$,

$$\delta^{Susy}(\mathcal{X})\Phi = -\delta^{\mathfrak{e}_{7(7)}}(\mathcal{X})\Phi - \Phi\tanh(\Phi/2) * \mathcal{X} \;\; . \tag{2.77}$$

We can now check the commutation rules between supersymmetry and $E_{7(7)}$. The expectation is that the commutator of two such transformations gives rise to a supersymmetry transformation whose parameter ϵ' is obtained from the original supersymmetry parameter ϵ by acting on it with the compensating SU(8) transformations induced by the action of $\mathfrak{e}_{7(7)}$ on the fermions, i.e.

$$[\delta^{\mathfrak{e}_{7(7)}}(\Lambda), \delta^{Susy}(\epsilon)] = \delta^{Susy}(\epsilon')\,, \qquad \epsilon' \equiv \delta^{\mathfrak{su}(8)}\left(-\tanh(\Phi/2) * \Lambda\right)\epsilon \;\; . \tag{2.78}$$

At this point it is convenient to modify the supersymmetry variation by requiring the spinor parameter ϵ also to transform with respect to the induced SU(8) transformation as

$$\delta^{\mathfrak{e}_{7(7)}}\epsilon = \delta^{\mathfrak{su}(8)}\Big(\tanh(\Phi/2) * \Lambda\Big)\epsilon \tag{2.79}$$

so ϵ transforms in the same way as the gravitino field ψ under the compensating SU(8). As a consequence, the parameter $\mathcal{X}$ in the adjoint representation simply transforms as

$$\delta^{\mathfrak{e}_{7(7)}}(\Lambda)\,\mathcal{X} = \Big[\tanh(\Phi/2) * \Lambda\,,\,\mathcal{X}\Big] \tag{2.80}$$

which correctly reproduces the corresponding $\mathfrak{su}(8)$ action in the **70**. As we will show below, with this extra compensating transformation we obtain

$$\Big[\delta^{\mathfrak{e}_{7(7)}}, \delta^{Susy}\Big]\,\Phi = 0\,, \tag{2.81}$$

If (2.81) holds on the scalar fields, this commutator will also vanish on functions of Φ as well as on all other fields. Indeed, the only transformation that could still appear is a local

Lorentz transformation which does not act on Φ. To check the absence of the latter we simply evaluate the above commutator on the vierbein field. While $\delta^{\mathfrak{e}_{7(7)}}$ does not act on the vierbein, δ^{Susy} produces a term $\bar{\epsilon}^i\gamma^a\psi_{\mu i} + \bar{\epsilon}_i\gamma^a\psi^i_\mu$. However, with the extra compensating transformation (2.79), this expression becomes a singlet under the induced SU(8) transformation, and therefore the commutator also vanishes on the vierbein. The main advantage of defining the transformation such that (2.81) is satisfied will become apparent when we discuss the quantum theory, because with (2.81) the BRST transformation will commute with $E_{7(7)}$, and this enables us to directly formulate the Ward identities for the full non-linear $E_{7(7)}$ symmetry.

In the remainder of this section we prove the key formula (2.81). It is more convenient to evaluate the commutator on $\exp\Phi$ rather than on Φ itself, because then all the non-linear terms appear via the compensating SU(8) transformation

$$\begin{aligned}\delta^{\mathfrak{e}_{7(7)}}(\Lambda)\exp\Phi &= \big(\tanh(\Phi/2)*\Lambda\big)\exp\Phi - \big(\exp\Phi\big)\,\Lambda \\ \delta^{Susy}(\mathcal{X})\exp\Phi &= \Big(\mathcal{X}+\tanh(\Phi/2)*\mathcal{X}\Big)\exp\Phi\end{aligned} \tag{2.82}$$

from which we read off that

$$\delta^{Susy}(\mathcal{X})\exp\Phi = \delta^{\mathfrak{e}_{7(7)}}(\mathcal{X})\exp\Phi + \mathcal{X}\exp\Phi + \big(\exp\Phi\big)\mathcal{X}\ \ , \tag{2.83}$$

a relation that will be useful below. Using (2.80) we get

$$\begin{aligned}\big[\delta^{\mathfrak{e}_{7(7)}}(\Lambda)\,,\ \delta^{Susy}(\mathcal{X})\big]\exp\Phi &= \\ = \Big(\big(\delta^{\mathfrak{e}_{7(7)}}(\Lambda)\tanh(\Phi/2)\big)*\mathcal{X}\Big)\exp\Phi &- \Big(\big(\delta^{Susy}(\mathcal{X})\tanh(\Phi/2)\big)*\Lambda\Big)\exp\Phi \\ &+ \Big[\tanh(\Phi/2)*\mathcal{X}\,,\ \tanh(\Phi/2)*\Lambda\Big]\exp\Phi \\ &+ \Big(\tanh(\Phi/2)*\Big[\tanh(\Phi/2)*\Lambda\,,\ \mathcal{X}\Big]\Big)\exp\Phi\ \ .\end{aligned} \tag{2.84}$$

To evaluate these terms further we need to make use of the closure property

$$\big[\delta^{\mathfrak{e}_{7(7)}}(\Lambda_1),\delta^{\mathfrak{e}_{7(7)}}(\Lambda_2)\big]\Phi = \Big[\Phi\,,\ \big[\Lambda_1,\Lambda_2\big]\Big]\ \ , \tag{2.85}$$

that is, the fact that the commutator of two compensated $E_{7(7)}$ transformations must close properly into $\mathfrak{su}(8)$. This formula obviously extends to all functions $f(\Phi)$ which are expandable in a power series. Observe that without the compensating $\mathfrak{su}(8)$ transformation, the commutator $[\Lambda_1,\Lambda_2]$ in (2.85) would only act on Φ from the right (corresponding to the uncompensated $E_{7(7)}$ action), while its action from the left is due to the compensating $\mathfrak{su}(8)$. Using (2.82) we now apply this formula to $\exp\Phi$ to obtain

$$\begin{aligned}&\Big(\Big(\delta^{\mathfrak{e}_{7(7)}}(\Lambda_1)\tanh(\Phi/2)\Big)*\Lambda_2\Big)\exp\Phi - \Big(\Big(\delta^{\mathfrak{e}_{7(7)}}(\Lambda_1)\tanh(\Phi/2)\Big)*\Lambda_2\Big)\exp\Phi = \\ &\quad = \Big[\tanh(\Phi/2)*\Lambda_1\,,\ \tanh(\Phi/2)*\Lambda_2\Big]\exp\Phi - \big[\Lambda_1,\Lambda_2\big]\exp\Phi\ \ .\end{aligned} \tag{2.86}$$

Modulo the difference between $\delta^{Susy}(\mathcal{X})$ and $\delta^{\mathfrak{e}_{7(7)}}(\mathcal{X})$, cf. (2.83), this formula allows us to rewrite the right hand side of (2.84) as

$$\left[\mathcal{X}\,,\,\Lambda\right]\exp\Phi + \Big(\tanh(\Phi/2)*\Big[\tanh(\Phi/2)*\Lambda\,,\,\mathcal{X}\Big]\Big)\exp\Phi\ \ . \tag{2.87}$$

Now exploiting (2.83) in the form

$$\begin{aligned}\delta^{Susy}(\mathcal{X})\exp(n\Phi) \,=\, &\delta^{\mathfrak{e}_{7(7)}}(\mathcal{X})\exp(n\Phi) + \mathcal{X}\exp(n\Phi) + \exp(n\Phi)\mathcal{X} + \\ &+ 2\sum_{1\leq m\leq n-1}\exp(m\Phi)\mathcal{X}\exp((n-m)\Phi)\end{aligned} \tag{2.88}$$

and expanding $\tanh(\Phi/2)$ as a formal power series in $\exp\Phi$ we get

$$\delta^{Susy}(\mathcal{X})\tanh(\Phi/2) = \delta^{\mathfrak{e}_{7(7)}}(\mathcal{X})\tanh(\Phi/2) + \mathcal{X} - \tanh(\Phi/2)\,\mathcal{X}\tanh(\Phi/2)\ \ , \tag{2.89}$$

(as can also be checked by expanding the formal power series around $\Phi = 0$). Therefore

$$\begin{aligned}-\,\delta^{Susy}(\mathcal{X})\tanh(\Phi/2)*\Lambda \,=\, &-\delta^{\mathfrak{e}_{7(7)}}(\mathcal{X})\tanh(\Phi/2)*\Lambda\ + \\ &-\left[\mathcal{X}\,,\,\Lambda\right] + \tanh(\Phi/2)*\Big[\mathcal{X}\,,\,\tanh(\Phi/2)*\Lambda\Big]\ \ .\end{aligned} \tag{2.90}$$

Acting with this expression on $\exp\Phi$ we see that these terms cancel the ones in (2.87), which proves the key formula (2.81).

Finally, it is straightforward to see that the remaining gauge symmetries trivially commute with the non-linear action of $\delta^{\mathfrak{e}_{7(7)}}$. Combining all gauge symmetries into a single BRST transformation with generator s in the usual way we therefore see that the relation (2.81) extends to the more general statement[14]

$$[\delta^{\mathfrak{e}_{7(7)}}, s] = 0\ \ . \tag{2.91}$$

Consequently, at the classical level, the BRST (gauge) transformations can be completely disentangled from the non-linear action of $E_{7(7)}$. In the remaining sections it will be our task to elevate this statement to the full quantum theory.

3 The SU(8) anomaly at one loop

As a first application of the formalism developed in the foregoing sections, we now present a Feynman diagram computation of the SU(8) anomaly considered long ago by very different methods. In [17], N. Marcus pointed out the absence of rigid SU(8) anomalies for $\mathcal{N} = 8$ supergravity at one loop; the cancellation is based on the following identity

$$3\times\mathrm{tr}_{\mathbf{8}}X^3 - 2\times\mathrm{tr}_{\mathbf{28}}X^3 + 1\times\mathrm{tr}_{\mathbf{56}}X^3 = (3\times 1 - 2\times 4 + 1\times 5)\,\mathrm{tr}_{\mathbf{8}}X^3 = 0\ , \tag{3.1}$$

[14]For anti-commuting $\mathfrak{e}_{7(7)}$ parameters, this relation becomes an anti-commutator: $\{\delta^{\mathfrak{e}_{7(7)}}, s\} = 0$.

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where X is any $\mathfrak{su}(8)$ generator, and where the first and third contributions are due to the eight gravitinos and the 56 spin-$\frac{1}{2}$ fermions of $\mathcal{N} = 8$ supergravity, while the middle contribution is due to the 28 chiral vectors. In this section we will not consider the fermionic triangle diagrams which can be obtained by standard methods, but concentrate on the vector fields, that is, the middle term in (3.1). The formalism of this paper makes possible (for the first time) a full fledged Feynman diagram calculation because it allows for an off-shell realisation of the chiral properties of the vector fields and their interactions under SU(8). At the end of this section and in the following sections we will extend these considerations to the full $E_{7(7)}$ current, where we will encounter a non-linear variant of the familiar linear anomaly, with three currents and (in principle) any number of scalar field insertions. We will also present arguments showing that this results extends to all loop orders.

Accordingly, our first aim will be to compute correlators with the insertion of three SU(8) current operators obtained by restricting the $E_{7(7)}$ current to its SU(8) subgroup. Now it is known (for linearly realised symmetries) that the anomaly involves a trace of the form (with Lie algebra generators X_1, X_2, X_3)

$$\mathrm{Tr}\ \{X_1, X_2\} X_3\ . \tag{3.2}$$

However, there is no invariant symmetric tensor of rank three in the **56** or the adjoint of $E_{7(7)}$, and hence *a priori* also none for its SU(8) subgroup (in these representations) so readers may wonder how one could get an anomaly at all. It is here that the distinction between a linearly realised symmetry and a non-linearly realised one makes all the difference. Namely, as the explicit calculation below will show, the relevant trace involves the complex structure tensor $J^m{}_n$ as an extra factor, so (3.2) is replaced by

$$\mathrm{Tr}\ J\{X_1, X_2\} X_3\ . \tag{3.3}$$

This extra factor (which one might think of as being analogous to the insertion of a γ_5) breaks the manifest symmetry from $E_{7(7)}$ to SU(8), and at the same time allows for the appearance of chirality, and hence a non-vanishing trace (effectively replacing the vector-like $\mathbf{56} = \mathbf{28} \oplus \overline{\mathbf{28}}$ by the chiral **28** in the trace). Nevertheless, the $E_{7(7)}$ symmetry is still present, but necessarily non-linear.

3.1 Feynman rules

With these remarks we can now proceed to the actual computation.[15] We first work out the propagators by starting from the gauge-fixed kinetic term for the vector fields

$$\mathcal{L}_0 = \frac{1}{2}\Omega_{mn}\varepsilon^{\mathtt{ijk}}\partial_0 A^m_{\mathtt{i}}\partial_{\mathtt{j}} A^n_{\mathtt{k}} - \frac{1}{2}G_{mn}(\delta^{\mathtt{ik}}\delta^{\mathtt{jl}} - \delta^{\mathtt{il}}\delta^{\mathtt{jk}})\partial_{\mathtt{i}} A^m_{\mathtt{j}}\partial_{\mathtt{k}} A^n_{\mathtt{l}} + b_m\partial_{\mathtt{i}} A^m_{\mathtt{i}}\ , \tag{3.4}$$

[15] We shall occasionally point out similarities of the present computation with the familiar γ_5 anomaly; readers may therefore find it useful to consult the textbooks [18, 19, 32, 33] for further information on this well known topic.

which is obtained from (2.17) by retaining only the parts quadratic in the fields. Furthermore, in the linearised approximation, we set $h_{\mathtt{ij}} = \delta_{\mathtt{ij}}$ in (3.4) and expand the scalar fields about a given background $\mathring{\Phi}$, so the metric $G_{mn} = G_{mn}(\mathring{\Phi})$[16] also becomes constant (with $\mathring{\Phi} = 0$ we have $G_{mn} = \delta_{mn}$). Going to momentum space, the quadratic operator to be inverted is

$$\Delta^{-1}(p) = \begin{pmatrix} \Omega_{mn}\varepsilon^{\mathtt{ijk}}p_0 p_{\mathtt{k}} + G_{mn}(\delta^{\mathtt{ij}}p^2 - p^{\mathtt{i}}p^{\mathtt{j}}) & ip^{\mathtt{i}}\delta^n_m \\ -ip^{\mathtt{j}}\delta^m_n & 0 \end{pmatrix} , \tag{3.5}$$

with $p^2 \equiv \delta^{\mathtt{ij}}p_{\mathtt{i}}p_{\mathtt{j}}$. The vector propagator is therefore

$$\Delta(p) = \frac{1}{p^2}\begin{pmatrix} \frac{\Omega^{mn}\varepsilon_{\mathtt{ijk}}p_0 p^{\mathtt{k}} - G^{mn}(\delta_{\mathtt{ij}}p^2 - p_{\mathtt{i}}p_{\mathtt{j}})}{p_0{}^2 - p^2 + i\varepsilon} & ip_{\mathtt{i}}\delta^m_n \\ -ip_{\mathtt{j}}\delta^n_m & 0 \end{pmatrix} ; \tag{3.6}$$

it is a (4×56) by (4×56) matrix, with three spatial directions and the fourth component corresponding to the Lagrange multipliers b_m which enforce the condition $\partial_{\mathtt{i}} A^m_{\mathtt{i}} = 0$. The propagating spin-1 degrees of freedom correspond to the residues of the poles of the propagator at $p_0 = \pm|p|$. There is no pole in the off-diagonal components mixing b_m and $A^m_{\mathtt{i}}$, and the residue is given by

$$2|p| \operatorname{res}(\Delta)\big|_{p_0=|p|} = \Omega^{mn}\varepsilon_{\mathtt{ijk}}\hat{p}^{\mathtt{k}} - G^{mn}(\delta_{\mathtt{ij}} - \hat{p}_{\mathtt{i}}\hat{p}_{\mathtt{j}}) , \tag{3.7}$$

where $\hat{p}_{\mathtt{i}} \equiv p_{\mathtt{i}}/|p|$. An important difference between (3.6) and the usual covariant propagator in four dimensions is that (3.6) contains terms which are *odd* under parity (for which $p^{\mathtt{i}} \to -p^{\mathtt{i}}$ and $p_0 \to p_0$). It is these terms, together with the parity odd vertices to be given below, which introduce the extra factor $J^m{}_n$ into the traces, and hence can contribute to chiral anomalies, even if only vector fields circulate in the loop.

We can rephrase these results in canonical language. Consider the free quantum field

$$A^m_{\mathtt{i}}(x) \equiv \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2|p|}}\sum_\sigma \Big(e^{-ix\cdot p}e^{*m}_{\mathtt{i}}(\sigma,p)a(\sigma,p) + e^{ix\cdot p}e^m_{\mathtt{i}}(\sigma,p)a^\dagger(\sigma,p)\Big) , \tag{3.8}$$

where $a^\dagger(\sigma,p)$ and $a(\sigma,p)$ are creation and annihilation operators of asymptotic free particles of momentum p and helicity $h(\sigma) = \pm 1$, and 56 SU(8) quantum numbers σ (we anticipate in this notation that σ determines h by (3.14)),

$$\big[a(\sigma,p), a^\dagger(\sigma',q)\big] = \delta_{\sigma\sigma'}\delta^{(3)}(p-q) . \tag{3.9}$$

[16] In this section we write G_{mn} instead of using the (perhaps more appropriate) notation $\mathring{G}_{mn} \equiv G_{mn}(\mathring{\Phi})$, since $G_{mn}(\Phi)$ does not appear and the notation is therefore unambiguous. Except in (3.16) and (3.18), we refrain from using boldface latters for the spatial components of four-vectors, as it should be clear from the context which is meant.

In order for the operator algebra to reproduce the propagator (3.6)

$$\Big\langle 0\Big|T\Big\{A^m_{\mathtt{i}}(x)A^n_{\mathtt{j}}(y)\Big\}\Big|0\Big\rangle = -i\int\frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{p_0{}^2-p^2+i\varepsilon}\left(\Omega^{mn}\varepsilon_{\mathtt{ijk}}\frac{p_0}{|p|}\hat{p}^{\mathtt{k}}-G^{mn}\left(\delta_{\mathtt{ij}}-\hat{p}_{\mathtt{i}}\hat{p}_{\mathtt{j}}\right)\right), \tag{3.10}$$

the polarisation vectors $e^m_{\mathtt{i}}(\sigma,p)$ and their complex conjugates $e^{*}{}^m_{\mathtt{i}}(\sigma,p)$ must satisfy

$$\sum_\sigma e^m_{\mathtt{i}}(\sigma,p)e^{*}{}^n_{\mathtt{j}}(\sigma,p) = -\Omega^{mn}\varepsilon_{\mathtt{ijk}}\hat{p}^{\mathtt{k}}+G^{mn}(\delta_{\mathtt{ij}}-\hat{p}_{\mathtt{i}}\hat{p}_{\mathtt{j}})\ . \tag{3.11}$$

As usual, the polarisation vectors are transverse

$$\hat{p}^{\mathtt{i}}\, e^m_{\mathtt{i}}(\sigma,p)=0\ . \tag{3.12}$$

With the convention

$$\varepsilon_{\mathtt{i}}{}^{\mathtt{jk}}\hat{p}_{\mathtt{k}}\, e^m_{\mathtt{j}}(\sigma,p) = ih(\sigma)e^m_{\mathtt{i}}(\sigma,p)\ , \tag{3.13}$$

it follows from (3.11) that the polarisation vectors must satisfy in addition

$$J^m{}_n e^n_{\mathtt{i}}(\sigma,p) = ih(\sigma)e^m_{\mathtt{i}}(\sigma,p)\,, \tag{3.14}$$

with the 'complex structure' tensor $J^m{}_n \equiv J^m{}_n(\mathring{\Phi})$, see (2.3). With this extra constraint, there are only 56 independent polarisations, so σ runs from 1 to 56. The linearised equations of motion are then satisfied with a zero gradient $\partial_{\mathtt{i}}A^m_0=0$ in (2.8),

$$\partial_0 A^m_{\mathtt{i}} = \varepsilon_{\mathtt{i}}{}^{\mathtt{jk}}J^m{}_n\partial_{\mathtt{j}}A^n_{\mathtt{k}}\,, \tag{3.15}$$

such that the action of the Lorentz group on $A^m_{\mathtt{i}}$ is the same as in the standard formulation of the free theory in the Coulomb gauge. It follows that the 56 creation operators $a^\dagger(\sigma,p)$ are the same as in the standard formulation of the free theory, and the 28 states of helicity $h=1$ transform in the **28** of SU(8), whereas the 28 states of helicity $h=-1$ transform in the $\overline{\mathbf{28}}$ of SU(8), as required by (3.14).

Note that because of (3.11), the free quantum field $A^m_{\mathtt{i}}(x)$ does not commute with itself at equal time, but satisfies instead

$$\big[A^m_{\mathtt{i}}(x^0,\mathbf{x}),A^n_{\mathtt{j}}(x^0,\mathbf{y})\big] = i\Omega^{mn}\varepsilon_{\mathtt{ijk}}\frac{\partial}{\partial x^{\mathtt{k}}}\,\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\ . \tag{3.16}$$

This equal time commutator could be derived alternatively from the Dirac quantisation of the theory in the Coulomb gauge, with the second class constraints[17]

$$\Pi^{\mathtt{i}}_m-\frac{1}{2}\Omega_{mn}\varepsilon^{\mathtt{ijk}}\partial_{\mathtt{j}}A^n_{\mathtt{k}}\approx 0\,,\qquad \partial^{\mathtt{i}}A^m_{\mathtt{i}}\approx 0\ . \tag{3.17}$$

[17] As in the conventional formulation, the Poisson bracket of the first class Coulomb constraint $\partial_{\mathtt{i}}\Pi^{\mathtt{i}}_m\approx 0$ and the Coulomb gauge constraint $\partial^{\mathtt{i}}A^m_{\mathtt{i}}\approx 0$ is non-degenerate, and they altogether define a set of second class constraints.

The decomposition of the canonical momentum $\Pi^{\mathtt{i}}_m$ in the Darboux basis only coincides with the definition (2.19) of the canonical momentum $\Pi^{\bar{m}\,\mathtt{i}}$ in the conventional formulation of the theory (2.20) up to a factor 2. Although the canonical Poisson brackets therefore differ by a factor 2 in the two formulations, the Dirac brackets are equivalent. The commutation relation (3.16) is consistent with causality, because

$$\left[F^m_{\mathtt{ij}}(x^0,\mathbf{x}),F^n_{\mathtt{kl}}(x^0,\mathbf{y})\right]=2i\Omega^{mn}\varepsilon_{\mathtt{ij[k}}\partial_{\mathtt{l]}}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{3.18}$$

as follows directly from (3.16), and therefore gauge-invariant operators commute at space-like separation $(x^0-y^0)^2<|\mathbf{x}-\mathbf{y}|^2$.

The cubic vertex defining the couplings of the $E_{7(7)}$ current to the vector fields can be obtained from the quadratic action by adding to (3.4) terms with source fields $B_\mu{}^m{}_n$ coupling to the conserved $E_{7(7)}$ current, such that the latter is re-obtained by taking the derivative with respect to the source fields and then setting them equal to zero. Here we will restrict attention to the $\mathfrak{su}(8)$ part of the full $E_{7(7)}$ current, for which the source $B_\mu{}^m{}_n$ leaves the background metric G_{mn} invariant:

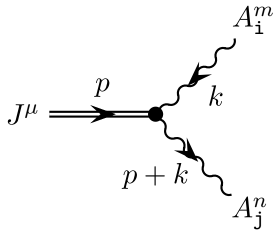
$$B_\mu{}^p{}_m G_{pn}+B_\mu{}^p{}_n G_{pm}=0 \tag{3.19}$$

(for $G_{mn}=\delta_{mn}$ this just means that SU(8) is realised by *anti-symmetric* matrices in the real basis of the **56** representation of $E_{7(7)}$). As is well known, the introduction of such sources corresponds to formally covariantising the action (3.4) with respect to *local* SU(8), such that (3.4) is replaced by the density

$$\begin{aligned}\mathcal{L}_0[B]=&\,\frac{1}{2}\Omega_{mn}\varepsilon^{\mathtt{ijk}}\left(\partial_0 A^m_{\mathtt{i}}+B_0{}^m{}_pA^p_{\mathtt{i}}\right)\left(\partial_{\mathtt{j}}A^n_{\mathtt{k}}+B_{\mathtt{j}}{}^n{}_qA^q_{\mathtt{k}}\right)\\&-\frac{1}{2}G_{mn}(\delta^{\mathtt{ik}}\delta^{\mathtt{jl}}-\delta^{\mathtt{il}}\delta^{\mathtt{jk}})\left(\partial_{\mathtt{i}}A^m_{\mathtt{j}}+B_{\mathtt{i}}{}^m{}_pA^p_{\mathtt{j}}\right)\left(\partial_{\mathtt{k}}A^n_{\mathtt{l}}+B_{\mathtt{k}}{}^n{}_qA^q_{\mathtt{l}}\right)+b_m\left(\partial_{\mathtt{i}}A^m_{\mathtt{i}}+B_{\mathtt{i}}{}^m{}_nA^n_{\mathtt{i}}\right)\end{aligned} \tag{3.20}$$

(in fact, dropping the restriction (3.19) this action becomes covariant with respect to local $E_{7(7)}$, as required for a study of the full $E_{7(7)}$ current, cf. section 5.2). For the fermion fields, the SU(8) tensor structure factorises out, and the vertex associated to one SU(8) current insertion just has the expected structure $\propto(1\pm i\gamma_5)\gamma^\mu$. For vector fields, on the other hand, the Lorentz and $\mathfrak{su}(8)$ tensor structures are *a priori* entangled for the vertices computed from (3.20).[18] Nevertheless, for correlation functions of SU(8) currents only, the two tensor structures can be disentangled by using the property that the only tensors appearing in the trace are the SU(8) invariant tensors δ^m_n and $J^m{}_n$; these can be diagonalised according to the decomposition of the $E_{7(7)}$ representation $\mathbf{56}\cong\mathbf{28}\oplus\overline{\mathbf{28}}$ of SU(8). The calculation shows that all the $\mathfrak{su}(8)$ Lie algebra generators X's can be moved to the left such that the vertex for linking an $\mathfrak{su}(8)$-current J^μ and a chiral boson $A^m_{\mathtt{i}}$ with incoming momenta p and k respectively to a chiral boson $A^n_{\mathtt{j}}$ with outgoing momentum $p+k$

[18] The momentum dependence of the 3-point vertex can be derived in the usual way [32] by writing the corresponding terms from (3.20) in momentum space and symmetrising in the internal legs involving the quantum fields $A^m_{\mathtt{i}}$ (not forgetting the antisymmetry condition (3.19)).



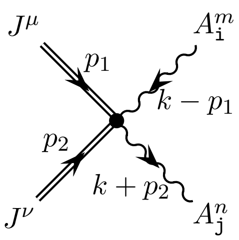
is effectively given by

$$\begin{aligned}
\Upsilon^{0}(k+p,k) &= \begin{pmatrix} i\Omega_{mn}\varepsilon^{\mathtt{ijk}}\left(k_{\mathtt{k}}+\frac{1}{2}p_{\mathtt{k}}\right) & 0 \\ 0 & 0 \end{pmatrix}, \\
\Upsilon^{\mathtt{k}}(k+p,k) &= \\
&\begin{pmatrix} i\Omega_{mn}\varepsilon^{\mathtt{ijk}}\left(k_{0}+\frac{1}{2}p_{0}\right)+iG_{mn}\left((2k^{\mathtt{k}}+p^{\mathtt{k}})\delta^{\mathtt{ij}}-\delta^{\mathtt{ki}}(k^{\mathtt{j}}+p^{\mathtt{j}})-\delta^{\mathtt{kj}}k^{\mathtt{i}}\right) & -\delta^{\mathtt{ki}}\delta^{n}_{m} \\ \delta^{\mathtt{kj}}\delta^{m}_{n} & 0 \end{pmatrix},
\end{aligned} \tag{3.21}$$

where the bottom-left component gets a positive sign because of (3.19). The notation we use here is *formally* very similar to the one used for the familiar fermionic vertices. The vertices Υ^μ are analogous to the $(1 \pm i\gamma_5)\gamma^\mu$ matrices that appear in the corresponding computation of the anomalous fermionic triangle diagram. This analogy is for instance reflected in the identity

$$-\,ip_\mu \Upsilon^\mu(k+p,k) = \Delta^{-1}(k+p) - \Delta^{-1}(k) \ , \tag{3.22}$$

which is analogous to the (trivial) identity $\not{p} = (\not{k} + \not{p}) - \not{k}$, and will be similarly useful to cancel propagators in the diagrams and thereby simplify them. However, in contradistinction to the case of fermion fields which are governed by a first order kinetic term, (3.20) is quadratic in $B_\mu{}^m{}_n$ and thus the insertion of more than one current requires the consideration of contact terms absent in the fermionic triangle. The corresponding vertices $\mathcal{R}^{\mu\nu}$



do not depend on the momenta:

$$\mathcal{R}^{0\mathtt{k}} = \mathcal{R}^{\mathtt{k}0} = \begin{pmatrix} \frac{1}{2}\Omega_{mn}\varepsilon^{\mathtt{ijk}} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{R}^{\mathtt{kl}} = \begin{pmatrix} G_{mn}\left(\delta^{\mathtt{kl}}\delta^{\mathtt{ij}} - \delta^{\mathtt{kj}}\delta^{\mathtt{li}}\right) & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.23}$$

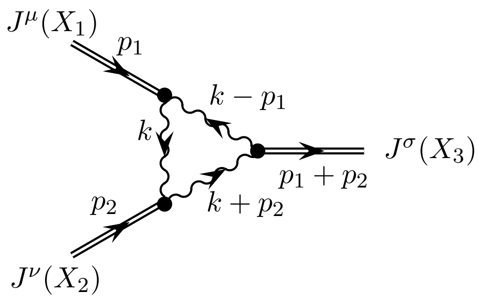
The vertices (3.21) and (3.23) satisfy

$$\Upsilon^\mu(k+p,k)^T = -\Upsilon^\mu(-k,-k-p) \ , \quad \mathcal{R}^{\mu\nu} = \left(\mathcal{R}^{\nu\mu}\right)^T \ , \tag{3.24}$$

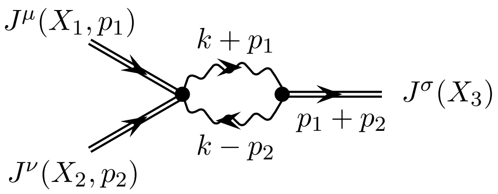
where transposition is defined in the matrix notation, and includes the interchange of the index pairs $(\mathtt{i},m) \leftrightarrow (\mathtt{j},n)$ of the top left component. Furthermore,

$$ip_\nu \mathcal{R}^{\mu\nu} = \Upsilon^\mu(q,l+p) - \Upsilon^\mu(q,l) \ , \tag{3.25}$$

for any choice of momenta l^μ and q^μ. The contribution to the vacuum expectation value of three currents of the one-loop diagrams with vector fields circulating in the loop is encoded in the triangle diagram



and in the one with the orientation of the loop momenta reversed as well as in the six independent permutations of the bubble diagram



Summing all these contributions, we obtain the following expression for the three-point function:

$$\begin{aligned}
\Big\langle J^\mu(X_1,p_1)J^\nu(X_2,p_2)J^\sigma(X_3,-p_1-p_2)\Big\rangle_{\text{vec}} &= i\int \frac{d^4k}{(2\pi)^4}\text{Tr } X_1X_2X_3 \\
&\times\Big(\Upsilon^\mu(k+p_1,k)\Delta(k)\Upsilon^\nu(k,k-p_2)\Delta(k-p_2)\Upsilon^\sigma(k-p_2,k+p_1)\Delta(k+p_1) \\
&+\mathcal{R}^{\mu\nu}\Delta(k-p_2)\Upsilon^\sigma(k-p_2,k+p_1)\Delta(k+p_1)+\Upsilon^\mu(k+p_1,k)\Delta(k)\mathcal{R}^{\nu\sigma}\Delta(k+p_1) \\
&+\Delta(k)\Upsilon^\nu(k,k-p_2)\Delta(k-p_2)\mathcal{R}^{\sigma\mu}\Big) \\
&+i\int\frac{d^4k}{(2\pi)^4}\text{Tr } X_2X_1X_3 \\
&\times\Big(\Upsilon^\nu(k+p_2,k)\Delta(k)\Upsilon^\mu(k,k-p_1)\Delta(k-p_1)\Upsilon^\sigma(k-p_1,k+p_2)\Delta(k+p_2) \\
&+\mathcal{R}^{\nu\mu}\Delta(k-p_1)\Upsilon^\sigma(k-p_1,k+p_2)\Delta(k+p_2)+\Upsilon^\nu(k+p_2,k)\Delta(k)\mathcal{R}^{\mu\sigma}\Delta(k+p_2) \\
&+\Delta(k)\Upsilon^\mu(k,k-p_1)\Delta(k-p_1)\mathcal{R}^{\sigma\nu}\Big) \ .
\end{aligned} \tag{3.26}$$

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Here, X_1, X_2, X_3 are $\mathfrak{su}(8)$ matrices, valued in the $\mathbf{28} \oplus \overline{\mathbf{28}}$ and the trace is to be taken over $(4 \times 56)^2$ matrices corresponding to components of the vector propagator.

Let us compute the divergence of the third current in this expectation value. Using the formulas (3.22), (3.25), one computes that

$$
\begin{aligned}
&i(p_{1\sigma} + p_{2\sigma})\Big\langle J^\mu(X_1, p_1) J^\nu(X_2, p_2) J^\sigma(X_3, -p_1 - p_2)\Big\rangle_{\rm vec} \\
&\quad = i \int \frac{d^4k}{(2\pi)^4} \text{Tr}\ X_1 X_2 X_3 \Big(\Upsilon^\mu(k+p_1, k)\Delta(k)\Upsilon^\nu(k, k+p_1)\Delta(k+p_1) \\
&\qquad - \Upsilon^\mu(k-p_2, k)\Delta(k)\Upsilon^\nu(k, k-p_2)\Delta(k-p_2) + \mathcal{R}^{\mu\nu}\Delta(k+p_1) - \mathcal{R}^{\mu\nu}\Delta(k-p_2)\Big) \\
&\qquad + i \int \frac{d^4k}{(2\pi)^4} \text{Tr}\ X_2 X_1 X_3 \Big(\Upsilon^\mu(k, k+p_2)\Delta(k+p_2)\Upsilon^\nu(k+p_2, k)\Delta(k) \\
&\qquad - \Upsilon^\mu(k, k-p_1)\Delta(k-p_1)\Upsilon^\nu(k-p_1, k)\Delta(k) + \mathcal{R}^{\nu\mu}\Delta(k+p_2) - \mathcal{R}^{\nu\mu}\Delta(k-p_1)\Big).
\end{aligned}
\tag{3.27}
$$

The commutator component $\propto$ Tr $[X_1, X_2]X_3$ of (3.27) gives rise to the vector field contribution to the vacuum expectation value of the insertion of two currents, as required by the current Ward identity

$$
\begin{aligned}
&i(p_{1\sigma} + p_{2\sigma})\Big\langle J^\mu(X_1, p_1) J^\nu(X_2, p_2) J^\sigma(X_3, -p_1 - p_2)\Big\rangle \\
&\qquad = i\Big\langle J^\mu(X_1, p_1) J^\nu([X_2, X_3], -p_1)\Big\rangle + i\Big\langle J^\mu([X_1, X_3], -p_2) J^\nu(X_2, p_2)\Big\rangle \ .
\end{aligned}
\tag{3.28}
$$

By contrast, the anticommutator component of (3.27) is proportional to Tr $J X_1 X_2 X_3$, and reduces to the difference of divergent integrals with respect to a constant shift of the integration variable, as for the one-loop contribution of the fermion fields.[19] The two first lines in (3.27) give rise to the difference of linearly divergent integrals

$$
\begin{aligned}
&\frac{1}{2}\text{Tr}\ \{X_1, X_2\}X_3 \int \frac{d^4k}{(2\pi)^4}\Big(I^{\mu\nu}(k, p_1) - I^{\mu\nu}(k - p_1, p_1) - I^{\mu\nu}(k, -p_1) + I^{\mu\nu}(k + p_2, -p_2)\Big) \\
&\quad = \frac{1}{2}\ \text{Tr}\ \{X_1, X_2\}X_3 \int \frac{d^4k}{(2\pi)^4}\left(p_{1\sigma}\frac{\partial I^{\mu\nu}(k, p_1)}{\partial k_\sigma} + p_{2\sigma}\frac{\partial I^{\mu\nu}(k, -p_2)}{\partial k_\sigma}\right),
\end{aligned}
$$

with

$$
\text{Tr}\ X^3 I^{\mu\nu}(k, p) = \text{Tr}\ X^3\, \Upsilon^\mu(k+p, k)\Delta(k)\Upsilon^\nu(k, k+p)\Delta(k+p) \ , \tag{3.29}
$$

where X can be any SU(8) generator. Because there is no invariant symmetric tensor of rank three in the adjoint of $E_{7(7)}$ by (3.2) the last anticommutator term of (3.27) reduces to a double difference of quadratically divergent integrals

$$
\begin{aligned}
&\frac{1}{2}\text{Tr}\ \{X_1, X_2\}X_3\mathcal{R}^{\mu\nu} \int \frac{d^4k}{(2\pi)^4}\Big(\Delta(k+p_1) - \Delta(k-p_2) - \Delta(k+p_2) + \Delta(k-p_1)\Big) \\
&\quad = \frac{1}{2}\text{Tr}\ \{X_1, X_2\}X_3\mathcal{R}^{\mu\nu}(p_{1\sigma} + p_{2\sigma})(p_{1\rho} - p_{2\rho}) \int \frac{d^4k}{(2\pi)^4}\frac{\partial^2\Delta(k)}{\partial k_\sigma \partial k_\rho}.
\end{aligned}
\tag{3.30}
$$

[19] Because ghosts do not give rise to any term of type (3.3), they do not contribute to the anomaly.

In the above derivation, we have made use of standard formulas [18, 33] to express the integrals as surface integrals which leads to the final expressions with first and second derivatives on the integrands.

Although finite, these integrals are not absolutely convergent, and they are subject to ambiguities associated to the order of integration of the momentum components k_μ. This ambiguity can be fixed in the conventional case (with fermions in the loop) by requiring Lorentz invariance. However, when photons run in the loop, the integrands are not Lorentz invariant and this prescription cannot be consistently defined.[20] This problem is in fact general in the theory. Indeed, because the Feynman rules are not manifestly Lorentz invariant, and because of the explicit appearance of the Levi-Cività tensor $\varepsilon^{\mathtt{ijk}}$, one cannot regularise the theory with the dimensional regularisation. Nevertheless, we will now explain how one can perform a consistent computation using Pauli-Villars regularisation.

3.2 Pauli-Villars regularisation

The formulation of the theory is defined such that it is formally equivalent to the manifestly diffeomorphism covariant formulation, up to a Gaussian integration of the 28 vector fields $A^{\bar{m}}_{\mathtt{i}}$ as in (2.18), (2.19), (2.20). Therefore, we will require the massive Pauli-Villars vector fields, to be defined through a local formulation after Gaussian integration. This is the case only if the vectors $A^{\bar{m}}_{\mathtt{i}}$ appear in the mass term through $F^{\bar{m}}_{\mathtt{ij}}$ up to a total derivative. The only 'sensible' possibility is therefore to introduce a symmetric tensor Γ_{mn} which is off-diagonal in the Darboux basis (i.e. $\Gamma_{mn} = 0 = \Gamma_{\bar{m}\bar{n}}$)

$$\begin{aligned} \mathcal{L}_0(M) = &\frac{1}{2}\Omega_{mn}\varepsilon^{\mathtt{ijk}}\partial_0 A^m_{\mathtt{i}}\partial_{\mathtt{j}} A^n_{\mathtt{k}} + \frac{i}{2}\Gamma_{mn}\varepsilon^{\mathtt{ijk}} M A^m_{\mathtt{i}}\partial_{\mathtt{j}} A^n_{\mathtt{k}} \\ &-\frac{1}{2}G_{mn}(\delta^{\mathtt{ik}}\delta^{\mathtt{jl}} - \delta^{\mathtt{il}}\delta^{\mathtt{jk}})\partial_{\mathtt{i}} A^m_{\mathtt{j}}\partial_{\mathtt{k}} A^n_{\mathtt{l}} + b_m\partial_{\mathtt{i}} A^m_{\mathtt{i}} \quad . \end{aligned} \tag{3.31}$$

We will show next that this Lagrangian gives rise to the standard equations of motion for the 28 Pauli-Villars vector fields in the Coulomb gauge. Before doing so, note that there is no necessity to modify the interaction terms in the Lagrangian (3.31), and that the tensor Γ_{mn} necessarily breaks SU(8) to (at most) SO(8). In fact, a manifestly SU(8) regularisation would be in contradiction with the possible existence of chiral anomalies. We define Γ_{mn} such that it reads

$$\Gamma_{m\bar{n}} = \Gamma_{\bar{n}m} = \delta_{m\bar{n}} \quad , \tag{3.32}$$

in the Darboux basis. Following the procedure of section 2.2, the manifestly covariant action for the Pauli-Villars vector fields is obtained after a Gaussian integration of the 28 (dual) Pauli-Villars vector fields $A^{\bar{m}}_{\mathtt{i}}$. This amounts to performing the replacement

$$F^m_{0\mathtt{i}} \rightarrow F^m_{0\mathtt{i}} + M A^m_{\mathtt{i}} \quad , \tag{3.33}$$

[20] We are aware that a consistent dimensional regularisation via an SO(3) invariant prescription has been used successfully in other contexts, such as the post-Newtonian approximation in general relativity, where there are no anomalies (T. Damour, private communication). However, this prescription appears to give inconsistent results in the present case.

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in all expressions. In particular, the equations of motion read

$$\partial^{\mathtt{i}}\left(F^{m}_{0\mathtt{i}}+MA^{m}_{\mathtt{i}}\right)=0\ ,\qquad \partial^{\mu}F_{\mathtt{i}\mu}+M\partial_{\mathtt{i}}A_0-M^2A^{m}_{\mathtt{i}}=0\ ,\tag{3.34}$$

and are manifestly gauge invariant with respect to the modified gauge transformations

$$\delta A^{m}_0=\partial_0 c^m+Mc^m\ ,\qquad \delta A^{m}_{\mathtt{i}}=\partial_{\mathtt{i}}c^m\ .\tag{3.35}$$

In the Coulomb gauge $\partial^{\mathtt{i}}A_{\mathtt{i}}=0$, they reduce to

$$A^{m}_0=0\,,\qquad \Box A^{m}_{\mathtt{i}}+M^2A^{m}_{\mathtt{i}}=0\ .\tag{3.36}$$

The substitution (3.33) breaks diffeomorphism invariance manifestly, which can therefore be restored only after the regulator is removed (possibly with a non-Lorentz invariant local counterterm, see below).

With these replacements, the propagator is manifestly massive in the duality invariant formulation. Indeed, one has

$$\Delta^{-1}(p,M)=\begin{pmatrix}\Omega_{mn}\varepsilon^{\mathtt{ijk}}p_0p_{\mathtt{k}}+\Gamma_{mn}\varepsilon^{\mathtt{ijk}}Mp_{\mathtt{k}}+G_{mn}(\delta^{\mathtt{ij}}p^2-p^{\mathtt{i}}p^{\mathtt{j}}) & ip^{\mathtt{i}}\delta^n_m\\ -ip^{\mathtt{j}}\delta^m_n & 0\end{pmatrix}\ ,\tag{3.37}$$

and the propagator is

$$\Delta(p,M)=\frac{1}{p^2}\begin{pmatrix}\frac{\Omega^{mn}\varepsilon_{\mathtt{ijk}}p_0p^{\mathtt{k}}+\Gamma^{mn}\varepsilon_{\mathtt{ijk}}Mp^{\mathtt{k}}-G^{mn}(\delta_{\mathtt{ij}}p^2-p_{\mathtt{i}}p_{\mathtt{j}})}{p_0{}^2-p^2-M^2+i\varepsilon} & ip_{\mathtt{i}}\delta^m_n\\ -ip_{\mathtt{j}}\delta^n_m & 0\end{pmatrix}\ ,\tag{3.38}$$

where Γ^{mn} is the inverse of Γ_{mn} and satisfies

$$\Gamma_{mp}\Omega^{pn}=-\Omega_{mp}\Gamma^{pn}\,,\qquad \Gamma_{mp}G^{pn}=G_{mp}\Gamma^{pn}\,,\qquad \Gamma_{mp}\Gamma^{pn}=\delta^n_m\ .\tag{3.39}$$

Therefore

$$(p_0\Omega_{mp}+M\Gamma_{mp})(p_0\Omega^{pn}+M\Gamma^{pn})=(-p_0{}^2+M^2)\delta^n_m\ ,\tag{3.40}$$

which permits to check (3.38).

To define the associated SU(8)-current vertex one must distinguish the vector and the axial components, respectively, corresponding to the decomposition **63** $\rightarrow$ **28** $\oplus$ **35** of the $\mathfrak{su}(8)$ adjoint under its $\mathfrak{so}(8)$ subalgebra. One can thus consider a manifestly SO(8) invariant regularisation by considering the coupling of the SO(8) current source $B_\mu{}^m{}_n$ to the mass term. So we consider the coupled Lagrangian

$$\begin{aligned}\mathcal{L}_0[B]=&\frac{1}{2}\Omega_{mn}\varepsilon^{\mathtt{ijk}}\left(\partial_0A^m_{\mathtt{i}}+B_0{}^m{}_pA^p_{\mathtt{i}}\right)\left(\partial_{\mathtt{j}}A^n_{\mathtt{k}}+B_{\mathtt{j}}{}^n{}_qA^q_{\mathtt{k}}\right)+\frac{i}{2}\Gamma_{mn}\varepsilon^{\mathtt{ijk}}MA^m_{\mathtt{i}}\left(\partial_{\mathtt{j}}A^n_{\mathtt{k}}+B_{\mathtt{j}}{}^n{}_pA^p_{\mathtt{k}}\right)\\&-\frac{1}{2}G_{mn}(\delta^{\mathtt{ik}}\delta^{\mathtt{jl}}-\delta^{\mathtt{il}}\delta^{\mathtt{jk}})\left(\partial_{\mathtt{i}}A^m_{\mathtt{j}}+B_{\mathtt{i}}{}^m{}_pA^p_{\mathtt{j}}\right)\left(\partial_{\mathtt{k}}A^n_{\mathtt{l}}+B_{\mathtt{k}}{}^n{}_qA^q_{\mathtt{l}}\right)+b_m\left(\partial_{\mathtt{i}}A^m_{\mathtt{i}}+B_{\mathtt{i}}{}^m{}_nA^n_{\mathtt{i}}\right).\end{aligned}\tag{3.41}$$

Note however that the mass term does only couple to the **35** axial component of the source $B_{\mathtt{j}}{}^{n}{}_{p}$, because an axial generators $X_m{}^p$ satisfies

$$X_m{}^p\Gamma_{pn} - X_n{}^p\Gamma_{mp} = 0 \ . \tag{3.42}$$

For simplicity, we will focus on the contribution of the massive Pauli-Villars vector fields to the vacuum expectation value of three axial currents (the three of them in the **35**) for which the vertices Υ^μ are still defined by (3.21). Using (3.42), one obtains that the latter is given by

$$\begin{aligned}
&\Big\langle J^\mu(X_1,p_1)J^\nu(X_2,p_2)J^\sigma(X_3,-p_1-p_2)\Big\rangle_{\rm PV} = i\int\frac{d^4k}{(2\pi)^4}\text{Tr}\ X_1X_2X_3 \qquad (3.43)\\
&\Big(\Delta(k+p_1,M)\Upsilon^\mu(k+p_1,k)\Delta(k,-M)\Upsilon^\nu(k,k-p_2)\Delta(k-p_2,M)\Upsilon^\sigma(k-p_2,k+p_1)\\
&+\Delta(k+p_1,M)R^{\mu\nu}\Delta(k-p_2,M)\Upsilon^\sigma(k-p_2,k+p_1)\\
&+\Delta(k+p_1,M)\Upsilon^\mu(k+p_1,k)\Delta(k,-M)R^{\nu\sigma}+R^{\sigma\mu}\Delta(k,-M)\Upsilon^\nu(k,k-p_2)\Delta(k-p_2,M)\Big)\\
&+i\int\frac{d^4k}{(2\pi)^4}\text{Tr}\ X_2X_1X_3\\
&\Big(\Delta(k+p_2,M)\Upsilon^\nu(k+p_2,k)\Delta(k,-M)\Upsilon^\mu(k,k-p_1)\Delta(k-p_1,M)\Upsilon^\sigma(k-p_1,k+p_2)\\
&+\Delta(k+p_2,M)R^{\nu\mu}\Delta(k-p_1,M)\Upsilon^\sigma(k-p_1,k+p_2)\\
&+\Delta(k+p_2,M)\Upsilon^\nu(k+p_2,k)\Delta(k,-M)R^{\mu\sigma}+R^{\sigma\nu}\Delta(k,-M)\Upsilon^\mu(k,k-p_1)\Delta(k-p_1,M)\Big)
\end{aligned}$$

where the propagator $\Delta(k,-M)$ gets an opposite mass through the commutation with the axial generators (similarly as in the standard fermion triangle). (3.43) is therefore the analogue of (3.26) for $M\neq 0$. In addition to the traces (3.2) and (3.3) there are two more types of traces, both of which give vanishing contribution because

$$\text{Tr}\ (\Omega\Gamma)X_1X_2X_3 = 0 = \text{Tr}\ (G\Gamma)X_1X_2X_3 \ \ , \tag{3.44}$$

Therefore the resulting integral is an *even* function of M. The massive generalisation of (3.22) is

$$-ip_\mu\Upsilon^\mu(k+p,k) = \Delta^{-1}(k+p,M) - \Delta^{-1}(k,-M) - M\Upsilon_5(2k+p) \ \ , \tag{3.45}$$

where

$$\Upsilon_5(p) = \begin{pmatrix} \Gamma_{mn}\varepsilon^{\mathtt{ijk}}p_{\mathtt{k}} & 0 \\ 0 & 0\end{pmatrix} \ \ , \tag{3.46}$$

again indicating the formal similarity of our computation with the usual fermionic triangle diagram. Using the latter identity, one computes that

$$
\begin{aligned}
& i(p_{1\sigma}+p_{2\sigma})\Big\langle J^{\mu}(X_1,p_1)J^{\nu}(X_2,p_2)J^{\sigma}(X_3,-p_1-p_2)\Big\rangle_{\rm PV} \\
& = i\int\frac{d^4k}{(2\pi)^4}{\rm Tr}\ X_1X_2X_3\Big(\Delta(k+p_1,M)\Upsilon^{\mu}(k{+}p_1,k)\Delta(k,-M)\Upsilon^{\nu}(k,k{+}p_1) \\
& \quad -\Upsilon^{\mu}(k{-}p_2,k)\Delta(k,-M)\Upsilon^{\nu}(k,k{-}p_2)\Delta(k{-}p_2,M)+\mathcal{R}^{\mu\nu}\Delta(k{+}p_1)-\mathcal{R}^{\mu\nu}\Delta(k{-}p_2)\Big) \\
& \quad +i\int\frac{d^4k}{(2\pi)^4}{\rm Tr}\ X_2X_1X_3\Big(\Upsilon^{\mu}(k,k+p_2)\Delta(k+p_2,-M)\Upsilon^{\nu}(k+p_2,k)\Delta(k,M) \\
& \quad -\Delta(k,M)\Upsilon^{\mu}(k,k{-}p_1)\Delta(k{-}p_1,-M)\Upsilon^{\nu}(k{-}p_1,k)+\mathcal{R}^{\nu\mu}\Delta(k{+}p_2)-\mathcal{R}^{\nu\mu}\Delta(k-p_1)\Big) \\
& \quad -iM\int\frac{d^4k}{(2\pi)^4}{\rm Tr}\ X_1X_2X_3\Big(\Delta(k{+}p_1,M)\Upsilon^{\mu}(k{+}p_1,k)\Delta(k,-M)\Upsilon^{\nu}(k,k{+}p_1)\Delta(k{+}p_1,M) \\
& \quad \times\Upsilon_5(2k+p_1-p_2)\ +\ \Delta(k+p_1,M)\mathcal{R}^{\mu\nu}\Delta(k-p_2,M)\Upsilon_5(2k+p_1-p_2)\Big) \\
& \quad -iM\int\frac{d^4k}{(2\pi)^4}{\rm Tr}\ X_2X_1X_3\Big(\Delta(k{+}p_2,M)\Upsilon^{\nu}(k{+}p_2,k)\Delta(k,-M)\Upsilon^{\mu}(k,k{-}p_1)\Delta(k{-}p_1,M) \\
& \quad \times\Upsilon_5(2k-p_1+p_2)\ +\ \Delta(k+p_2,M)\mathcal{R}^{\nu\mu}\Delta(k-p_1,M)\Upsilon_5(2k-p_1+p_2)\Big)
\end{aligned}
\tag{3.47}
$$

3.3 Computation of the anomaly coefficient

To compute the anomaly we now follow the standard procedure by subtracting (3.47) (indicated by the subscript "PV") from (3.27) (indicated by the subscript "vec"), and then taking the limit $M\to\infty$. The first two integrals in (3.47) are very similar to the massless case (3.27), and their contribution to the anomaly reduces to a difference of linearly divergent integrals as well. Because such integrals only depend on the leading power in the momentum k, they do not depend on the mass and hence these contributions cancel precisely between (3.27) and (3.47). It follows that the overall contribution of the vector fields to the anomaly is obtained by (minus) the sum of the two last integrals of (3.47) in the limit $M\to\infty$ — just like for the fermionic triangle in the standard computation with Pauli-Villars regulators, see e.g. [33].

For the computation, it is straightforward to see that for the massive propagators and vertices, the relations (3.24) remain unchanged, and that we have in addition,

$$
\Delta(k,M)^T=\Delta(-k,M)\ ,\quad \Upsilon_5(p)^T=\Upsilon_5(-p)\ . \tag{3.48}
$$

These relations permit to prove that the two last integrands in (3.47) are invariant under the substitution

$$
k\leftrightarrow -k\,,\qquad p_1\leftrightarrow p_2\,,\qquad \mu\leftrightarrow\nu\ . \tag{3.49}
$$

It follows that they are equal, and the anomaly being defined as

$$\mathcal{A}^{\mu\nu}_{\rm vec}(p_1,p_2)\,\mathrm{Tr}\ JX_1X_2X_3 \equiv i(p_{1\sigma}+p_{2\sigma})\Big\langle J^\mu(X_1,p_1)J^\nu(X_2,p_2)J^\sigma(X_3,-p_1-p_2)\Big\rangle_{\rm vec+PV}$$
$$-\,i\Big\langle J^\mu(X_1,p_1)J^\nu([X_2,X_3],-p_1)\Big\rangle_{\rm vec+PV}-i\Big\langle J^\mu([X_1,X_3],-p_2)J^\nu(X_2,p_2)\Big\rangle_{\rm vec+PV} \tag{3.50}$$

is given by

$$\mathcal{A}^{\mu\nu}_{\rm vec}(p_1,p_2)=2i\lim_{M\to+\infty}\left[M\int\frac{d^4k}{(2\pi)^4}S^{\mu\nu}(p_1,p_2,k)\right]\quad , \tag{3.51}$$

where the integrand $S^{\mu\nu}(p_1,p_2,k)$ is

$$S^{\mu\nu}(p_1,p_2,k)=-\frac{1}{56}\mathrm{Tr}\ J\Big(\Delta(k+p_1,M)\Upsilon^\mu(k+p_1,k)\Delta(k,-M)\Upsilon^\nu(k,k+p_1)\Delta(k+p_1,M)$$
$$\times\,\Upsilon_5(2k+p_1-p_2)+\Delta(k+p_1,M)\mathcal{R}^{\mu\nu}\Delta(k-p_2,M)\Upsilon_5(2k+p_1-p_2)\Big)\quad . \tag{3.52}$$

$S^{\mu\nu}$ includes three propagators, and takes the form

$$S^{\mu\nu}(p_1,p_2,k)=\frac{MP^{\mu\nu}(p_1,p_2,k)}{(\mathrm{k}{+}\mathrm{p}_1)^2\big((k_0{+}p_{1\,0})^2{-}(\mathrm{k}{+}\mathrm{p}_1)^2{-}M^2{+}i\varepsilon\big)\mathrm{k}^2\big(k_0{}^2{-}\mathrm{k}^2{-}M^2{+}i\varepsilon\big)(\mathrm{k}{-}\mathrm{p}_2)^2\big((k_0{-}p_{2\,0})^2{-}(\mathrm{k}{-}\mathrm{p}_2)^2{-}M^2{+}i\varepsilon\big)} \tag{3.53}$$

where $P^{\mu\nu}$ is a sum of monomials of order eight in p_1, p_2, k and M. One can neglect all terms of order three and higher in p_1 and p_2, because they will not contribute to (3.51). Moreover, for the terms of order two in p_1 and p_2, the denominator can be approximated as well by $\mathrm{k}^6(k_0{}^2-\mathrm{k}^2-M^2+i\varepsilon)^3$ and one can use the usual simplifications

$$\mathrm{k}^{2n}k_0{}^{2m}k_{\mathtt{i}}k_{\mathtt{j}}\sim\frac{1}{3}\delta_{\mathtt{ij}}\mathrm{k}^{2n+2}k_0^{2m}\,,\qquad \mathrm{k}^{2n}k_0{}^{2m+1}k_{\mathtt{i}}\sim 0\,, \tag{3.54}$$

according to the standard integration rules. After a rather tedious computation, we obtain

$$P^{0\mathtt{i}}\sim-\mathrm{k}^4(k_0{}^2-\mathrm{k}^2-M^2)\varepsilon^{\mathtt{ijk}}k_{\mathtt{j}}(2p_{1\mathtt{k}}+6p_{2\mathtt{k}})+\frac{1}{3}\mathrm{k}^4\left(4\mathrm{k}^2-19(k_0{}^2-\mathrm{k}^2-M^2)\right)\varepsilon^{\mathtt{ijk}}p_{1\mathtt{j}}p_{2\mathtt{k}}$$
$$P^{\mathtt{ij}}\sim\mathrm{k}^4\varepsilon^{\mathtt{ijk}}\Big(6(p_{1\,0}+p_{2\,0})(k_0{}^2-\mathrm{k}^2-M^2)k_{\mathtt{k}}-\frac{8}{3}(k_0{}^2-\mathrm{k}^2-M^2)k_0(p_{1\mathtt{k}}+p_{2\mathtt{k}})$$
$$+\frac{1}{3}(k_0{}^2-\mathrm{k}^2-M^2)\left(11(p_{1\,0}+p_{2\,0})(p_{1\mathtt{k}}-p_{2\mathtt{k}})-4(p_{1\,0}p_{2\mathtt{k}}-p_{2\,0}p_{1\mathtt{k}})\right)$$
$$-\frac{8}{3}k_0{}^2(p_{1\,0}p_{1\mathtt{k}}-p_{2\,0}p_{2\mathtt{k}})\Big) \tag{3.55}$$

and $P^{\mathtt{i}0}=-P^{0\mathtt{i}}$ using the symmetries (3.48), (3.49). Using the formula[21]

$$\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{k}^{2n}(-k_0{}^2)^m}{(\mathrm{k}^2-k_0^2+M^2-i\varepsilon)^l}=-i\frac{\Gamma(\frac{1}{2}+m)\Gamma(\frac{3}{2}+n)\Gamma(l-m-n-2)}{(2\pi)^3\Gamma(l)\,M^{2(l-m-n-2)}}\quad , \tag{3.56}$$

[21] Which itself follows from the standard formula [18]

$$\int_0^\infty\frac{x^{a-1}\,dx}{(x^2+s^2)^b}=s^{a-2b}\,\frac{\Gamma(\frac{a}{2})\Gamma(b-\frac{a}{2})}{2\Gamma(b)}\,.$$

this leads to the integrals

$$\mathcal{A}^{0\mathtt{i}}_{\mathrm{vec}} = 2M^2 i\varepsilon^{\mathtt{ijk}} p_{1\mathtt{j}} p_{2\mathtt{k}} \int \frac{d^4k}{(2\pi)^4} \left(\frac{4}{\left(k^2 - k_0^2 + M^2 - i\varepsilon\right)^3} - \frac{k^{-2}}{\left(k^2 - k_0^2 + M^2 - i\varepsilon\right)^2} \right)$$
$$= 0 \tag{3.57}$$

and

$$\begin{aligned}\mathcal{A}^{\mathtt{ij}}_{\mathrm{vec}} &= 2M^2 i\varepsilon^{\mathtt{ijk}} (p_{1\,0} p_{2\mathtt{k}} - p_{2\,0} p_{1\mathtt{k}}) \int \frac{d^4k}{(2\pi)^4} \left(\frac{8{k_0}^2 k^{-2}}{({k_0}^2 - k^2 - M^2 + i\varepsilon)^3} - \frac{\frac{4}{3} k^{-2}}{(k^2 - k_0^2 + M^2 - i\varepsilon)^2} \right) \\ &\quad + 2M^2 i\varepsilon^{\mathtt{ijk}} (p_{1\,0} + p_{2\,0})(p_{1\mathtt{k}} - p_{2\mathtt{k}}) \int \frac{d^4k}{(2\pi)^4} \left(\frac{\frac{8}{3}{k_0}^2 k^{-2}}{({k_0}^2 - k^2 - M^2 + i\varepsilon)^3} \right. \\ &\quad \left. - \frac{4}{(k^2 - k_0^2 + M^2 - i\varepsilon)^3} - \frac{\frac{1}{3} k^{-2}}{(k^2 - k_0^2 + M^2 - i\varepsilon)^2} \right) \\ &= \frac{1}{6\pi^2} \varepsilon^{\mathtt{ijk}} (p_{1\,0} p_{2\mathtt{k}} - p_{2\,0} p_{1\mathtt{k}}) - \frac{1}{6\pi^2} \varepsilon^{\mathtt{ijk}} (p_{1\,0} + p_{2\,0})(p_{1\mathtt{k}} - p_{2\mathtt{k}}) \quad . \end{aligned} \tag{3.58}$$

The resulting anomaly is not Lorentz invariant, but this is not so surprising since we used a regulator that breaks Lorentz invariance. In order to restore Lorentz invariance, one must renormalise the theory with a finite non-Lorentz invariant counterterm with the appropriate $\mathfrak{su}(8)$ tensor structure. The only such SO(3) invariant density is

$$\delta\mathcal{L} \;\propto\; \varepsilon^{\mathtt{ijk}} J^m{}_n B_0{}^n{}_p B_{\mathtt{i}}{}^p{}_q \partial_{\mathtt{j}} B_{\mathtt{k}}{}^q{}_m \quad . \tag{3.59}$$

It follows that the vacuum expectation value of three current insertions is only defined up to a shift

$$\delta \Big\langle J^{\mathtt{i}}(X_1, p_1) J^{\mathtt{j}}(X_2, p_2) J^0(X_3, -p_1 - p_2) \Big\rangle = -ia\, \varepsilon^{\mathtt{ijk}} (p_{1\mathtt{k}} - p_{2\mathtt{k}}) \mathrm{Tr}\ J X_1 X_2 X_3 \,, \tag{3.60}$$

and permutations. This shift affects the anomaly factor as

$$\delta\mathcal{A}^{0\mathtt{i}}_{\mathrm{vec}} = -\delta\mathcal{A}^{\mathtt{i}0}_{\mathrm{vec}} = a\, \varepsilon^{\mathtt{ijk}} p_{1\mathtt{j}} p_{2\mathtt{k}} \,, \qquad \delta\mathcal{A}^{\mathtt{ij}}_{\mathrm{vec}} = a\, \varepsilon^{\mathtt{ijk}} (p_{1\,0} + p_{2\,0})(p_{1\mathtt{k}} - p_{2\mathtt{k}}) \,, \tag{3.61}$$

and so, choosing $a = \frac{1}{6\pi^2}$, one recovers the *Lorentz invariant anomaly*

$$\mathcal{A}^{\mu\nu}_{\mathrm{vec}}(p_1, p_2) = \frac{1}{6\pi^2} \varepsilon^{\mu\nu\sigma\rho} p_{1\sigma} p_{2\rho} \quad . \tag{3.62}$$

We have thus verified that the anomaly coefficient associated to the vector fields is, as predicted from the family's index theorem, (-2) times the one associated to the Dirac fermion fields. Taking into account the fermionic contributions it follows that the total coefficient of the anomaly vanishes for $\mathcal{N} = 8$ supergravity, in agreement with (3.1).

Within the path-integral formulation of the theory, the variation of the formal integration measure with respect to an infinitesimal $\mathfrak{su}(8)$ local transformation gives rise to a local functional of the fields linear in the $\mathfrak{su}(8)$ parameter $C_{\mathfrak{k}}$ which defines a 1-cocycle over the space of $\mathfrak{su}(8)$ gauge transformations. This factor can be compensated by a redefinition of the local action if this cocycle is trivial in local cohomology. The triviality of this cohomology class is equivalent, via a transgression operation, to the triviality of a 2-cocycle over the moduli space of framed $\mathfrak{su}(8)$-connections B identified modulo $\mathfrak{su}(8)$-gauge transformations in local cohomology. The latter can be computed by means of the family's index theorem [26], as the Chern class of the vector bundle defined over an S^2 two parameters family of $\mathfrak{su}(8)$-gauge orbits of $\mathfrak{su}(8)$ framed connections with fibre the index of the chiral differential operators

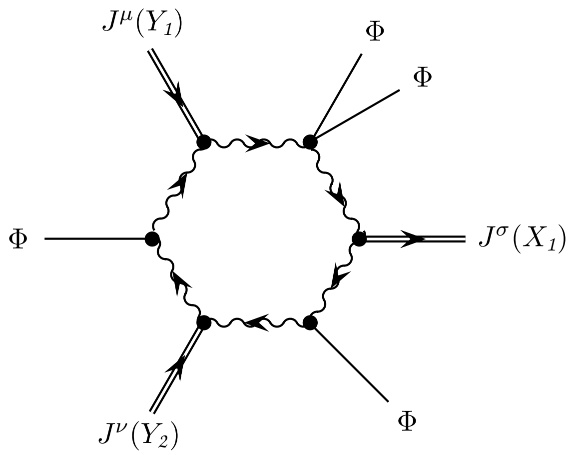
$$(1+i\gamma_5)\not{D}\ , \qquad (1+J\star)d_B\ , \qquad (1+i\gamma_5)\star e^a_\wedge\gamma_a d_B\ , \tag{3.63}$$

acting on the fields of spin 1/2, 1, and 3/2, respectively. A similar construction applies to gravitational and mixed anomalies. According to the family's index theorem, the contribution of the fermion fields and the vectors has been computed in [26], and applied to various supergravity theories in [17], giving for instance the cancelation (3.1) of the $\mathfrak{su}(8)$ anomaly in $\mathcal{N}=8$ supergravity.

Let us now turn to the generalisation of these results to $E_{7(7)}$. Unlike the linear SU(8) anomaly, the full $E_{7(7)}$ current and the non-linearly realised $E_{7(7)}$ symmetry give rise to an *infinite number* of potentially anomalous diagrams. Namely, for the complete $\mathfrak{e}_{7(7)}$ current Ward identities, one must also take into account the potential anomalies associated to the **70** component of the current, as well as diagrams with any number of scalar field insertions. We will write X_1, X_2 for $\mathfrak{su}(8)$ generators, and Y_1, Y_2 for generators in the **70**. Because there is no **63** in the *symmetric* tensor product $(\mathbf{70}\otimes\mathbf{70})_{sym}$,[22] the Ward identity associated to

$$\Big\langle J^\mu(Y_1,p_1)J^\nu(Y_2,p_2)J^\sigma(X_1,-p_1-p_2)\Big\rangle \tag{3.64}$$

cannot be anomalous (we are here using the same notation as in (2.50), with $J^\mu(X)$ denoting the projection of the current J^μ along the Lie algebra element X). However, further anomalies can appear if one includes scalar field insertions, as e.g.



[22] The antisymmetric product just gives the usual contribution to the non-anomalous Ward identity.

This is because the insertion of one scalar field into the diagram does not only add one propagator, but also two derivatives, whence the degree of divergence of the diagram remains the same with any number of external scalar fields (the same is true for fermionic loops, where the insertion of an extra fermionic propagator is accompanied by one derivative, as well as for the current vertex including scalar fields legs, which do not carry derivatives, but do not add propagators either). As a first non-trivial example, consider the vacuum expectation value

$$\Big\langle J^\mu(X_1,p_1)J^\nu(X_2,p_2)J^\sigma(Y_1,p_3)\Phi(Y_2,-p_1-p_2-p_3)\Big\rangle\ . \tag{3.65}$$

It satisfies the $\mathfrak{su}(8)$ Ward identity

$$\begin{aligned}-ip_{2\,\sigma}&\Big\langle J^\mu(X_1,p_1)J^\sigma(X_2,p_2)J^\nu(Y_1,p_3)\Phi(Y_2,-p_1-p_2-p_3)\Big\rangle\\ &=i\Big\langle J^\mu([X_1,X_2],p_1+p_2)J^\sigma(Y_1,p_3)\Phi(Y_2,-p_1-p_2-p_3)\Big\rangle\\ &\quad+i\Big\langle J^\mu(X_1,p_1)J^\sigma([Y_1,X_2],p_2+p_3)\Phi(Y_2,-p_1-p_2-p_3)\Big\rangle\\ &\quad+i\Big\langle J^\mu(X_1,p_1)J^\sigma(Y_1,p_3)\Phi([Y_2,X_2],-p_1-p_3)\Big\rangle\ .\end{aligned} \tag{3.66}$$

By contrast, the $\mathfrak{e}_{7(7)}$ Slavnov-Taylor identity takes a more complicated form because the transformation is non-linear (see (4.17) below for the derivation)

$$\begin{aligned}&-ip_{3\,\sigma}\Big\langle J^\mu(X_1,p_1)J^\nu(X_2,p_2)J^\sigma(Y_1,p_3)\Phi(Y_2,-p_1-p_2-p_3)\Big\rangle\\ &\ =i\Big\langle J^\mu([X_1,Y_1],p_1+p_3)J^\nu(X_2,p_2)\Phi(Y_2,-p_1-p_2-p_3)\Big\rangle\\ &\ +i\Big\langle J^\mu(X_1,p_1)J^\nu([X_2,Y_1],p_2+p_3)\Phi(Y_2,-p_1-p_2-p_3)\Big\rangle\\ &\ +\Big\langle J^\mu(X_1,p_1)J^\nu(X_2,p_2)\Phi^A(p_3)\,\Phi(Y_2,-p_1-p_2-p_3)\Big\rangle\Big\langle\Big[\frac{\Phi}{\tanh(\Phi)}*Y_1\Big]_A\Big\rangle\\ &\ +i\Big\langle J^\mu(X_1,p_1)J^\nu(X_2,p_2)\Big[\frac{\Phi}{\tanh(\Phi)}(-p_1-p_2)*Y_1\Big]_A\Big\rangle\Big\langle\Phi^A(p_1+p_2+p_3)\Phi(Y_2,-p_1-p_2-p_3)\Big\rangle\\ &\ +i\Big\langle J^\mu(X_1,p_1)\Big[\frac{\Phi}{\tanh(\Phi)}(-p_1)*Y_1\Big]_A\Big\rangle\Big\langle J^\nu(X_2,p_2)\Phi^A(p_1+p_3)\Phi(Y_2,-p_1-p_2-p_3)\Big\rangle\\ &\ +i\Big\langle J^\nu(X_2,p_2)\Big[\frac{\Phi}{\tanh(\Phi)}(-p_2)*Y_1\Big]_A\Big\rangle\Big\langle J^\mu(X_1,p_1)\Phi^A(p_2+p_3)\Phi(Y_2,-p_1-p_2-p_3)\Big\rangle.\end{aligned} \tag{3.67}$$

where the index $A=1$ to 70 labels an orthonormal basis of the coset component of the $E_{7(7)}$ Lie algebra.

At one loop, there is a potential anomaly to these Ward identities of the form

$$\propto \varepsilon^{\mu\nu\sigma\rho}p_{1\sigma}p_{2\,\rho}\mathrm{Tr}\ JX_1X_2[Y_1,Y_2]$$

for the $\mathfrak{su}(8)$ Ward identity (3.66), and an anomalous contribution to the Slavnon-Taylor identity (3.67)

$$\propto \varepsilon^{\mu\nu\sigma\rho}\,(3p_{1\sigma}p_{2\,\rho}+(p_{1\sigma}-p_{2\,\sigma})p_{3\,\rho})\ \mathrm{Tr}\ JX_1X_2[Y_1,Y_2]\ . \tag{3.68}$$

Similarly, the Ward identities associated to the vacuum expectations values

$$\begin{aligned}
&\left\langle J^\mu\left(X_1, -\sum_{m=1}^{2N+3} p_m\right) J^\nu(X_2, p_{2N+3}) J^\sigma(Y_1,p_1) \prod_{n=2}^{2+2N} \Phi(Y_n,p_n)\right\rangle \ ,\\
&\left\langle J^\mu\left(X_1, -\sum_{m=1}^{2N+4} p_m\right) J^\nu(Y_1, p_1) J^\sigma(Y_2,p_2) \prod_{n=3}^{4+2N} \Phi(Y_n,p_n)\right\rangle \ ,\\
&\left\langle J^\mu\left(Y_{2N+6}, -\sum_{m=1}^{2N+5} p_m\right) J^\nu(Y_1, p_1) J^\sigma(Y_2,p_2) \prod_{n=3}^{5+2N} \Phi(Y_n,p_n)\right\rangle \ ,
\end{aligned} \tag{3.69}$$

are potentially anomalous for all $N \geq 0$. Computing these anomalies explicitly would involve an infinite number of Feynman diagrams of increasing complexity. Fortunately, as we are going to see in the following section, the coefficients associated to these anomalies are determined by the Wess-Zumino consistency conditions in terms of the $\mathfrak{su}(8)$ anomaly coefficient. It thus follows from the computation of this section that they all vanish.

What about higher loops? Remarkably, for strictly non-renormalisable[23] theories the Adler-Bardeen Theorem is almost trivial in the following sense. By non-renormalisability and power counting higher loop anomalies would have a different form and dimension (involving more derivatives) from the one-loop anomaly studied above. However, such anomalies can be ruled out by the cohomology arguments given in the next section. In conclusion, with the cancellations exhibited above *there are no* $\mathfrak{su}(8)$ *or* $\mathfrak{e}_{7(7)}$ *anomalies in* $\mathcal{N}=8$ *supergravity at any order in perturbation theory.*

4 Wess-Zumino consistency condition

The purpose of this section is to show that 'non-linear' $\mathfrak{e}_{7(7)}$ anomaly is completely determined by the 'linear' $\mathfrak{su}(8)$ anomaly. In this way the determination of an infinite number of potentially anomalous diagrams involving three currents and an arbitrary number of scalar field insertions can be reduced to the single diagram computed in section 3. As already mentioned in the introduction, this result has its differential geometric roots in the homotopy equivalence (1.1).

4.1 The $\mathfrak{e}_{7(7)}$ master equation

The 'non-linear' $\mathfrak{e}_{7(7)}$ Ward identities are Slavnov-Taylor identities, which can be summarised in a master equation for the 1PI generating functional Γ. To simplify the discussion, we will postpone the discussion of the ghost sector and the compatibility with the BRST master equation to the next section.

Because the discussion of this section does not rely on particular properties of $E_{7(7)}$ and applies equally to other supergravity theories coupled to abelian vector fields and scalar

[23] 'Strictly' in the sense that there are no coupling constant of dimension ≥ 0.

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fields parametrising a symmetric space, we keep it general by considering a Lie algebra $\mathfrak{g}$ with decomposition

$$\mathfrak{g} \cong \mathfrak{k} \oplus \mathfrak{p} \quad , \tag{4.1}$$

with maximal 'compact subalgebra' $\mathfrak{k}$ and the 'non-compact' part $\mathfrak{p}$, and the usual commutation relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \ , \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \ , \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k} \ . \tag{4.2}$$

As explained in section 2.5, the transformation $\delta^{\mathfrak{g}}$ in the non-linear realisation acts on the scalar fields as

$$\delta^{\mathfrak{g}}\Phi \equiv \delta^{\mathfrak{k}}\Phi + \delta^{\mathfrak{p}}\Phi = -[C_{\mathfrak{k}}, \Phi] + \frac{\Phi}{\tanh \Phi} * C_{\mathfrak{p}} \quad , \tag{4.3}$$

where the compact subalgebra $\mathfrak{k}$ acts linearly with parameter $C_{\mathfrak{k}}$, while the remaining transformations $\delta^{\mathfrak{p}}$ with parameter $C_{\mathfrak{p}}$ are realised *non-linearly*. With regard to our previous discussion of these transformations in section 2.5, we note two important differences:

1. As we wish to treat the theory within the 'BRST formalism', we will from now on take the transformation parameters $C_{\mathfrak{k}}$ and $C_{\mathfrak{p}}$ as *anti-commuting* (which is the reason why we use the letter C rather than Λ for the transformation parameters).

2. Although the $\mathfrak{g}$ symmetry acts *rigidly*, we will nevertheless take C to be a *local* parameter, i.e. to depend on x. The corresponding source fields $B \equiv B_{\mathfrak{k}} + B_{\mathfrak{p}}$ coupling to the conserved G Noether current consequently transform as (non-abelian) gauge fields with these parameters.

With regard to the second point we emphasise that the introduction of an artificial local G invariance here is merely a formal device (well known to specialists) which will enable us to derive current Ward identities for $\mathfrak{g}$. The sources B are *external fields*, which are not part of any supermultiplet and are not integrated over in the path integral. Hence, the symmetries of the physical degrees of freedom of $\mathcal{N} = 8$ supergravity and their interactions are still the same as before. Similarly, $C_{\mathfrak{k}}$ and $C_{\mathfrak{p}}$, though x-dependent, are not quantum fields. Readers might nevertheless find it convenient to consider them as ghosts for the fictitious local G symmetry, when the G current is coupled to the sources $B_{\mathfrak{p}}$ and $B_{\mathfrak{k}}$. For instance, we will shortly consider a grading that corresponds to the order of the functional in these parameters, and that can be thought of as a ghost number (although it must not be confused with the true ghost number associated to the BRST operator which implements the gauge symmetries of the theory).

With these comments, the action of the transformations (4.3) on the other fields is straightforward to describe. On the fermionic fields (as well as on the supersymmetry ghost or superghosts) the transformations act via an induced $\mathfrak{k}$ transformation with parameter

$$C_{\mathfrak{k}} + \tanh(\Phi/2) * C_{\mathfrak{p}} \tag{4.4}$$

while on the vector fields and their ghosts the variations act linearly with parameter C in the corresponding representation (the **56** of $E_{7(7)}$ for $\mathcal{N}=8$ supergravity). Finally, writing $C \equiv C_{\mathfrak{k}} + C_{\mathfrak{p}}$, we have

$$\delta^{\mathfrak{g}} B = -dC - \{B, C\}\,, \qquad \delta^{\mathfrak{g}} C = -C^2 \tag{4.5}$$

on the current source $B \equiv B_{\mathfrak{k}} + B_{\mathfrak{p}}$ and on the parameter itself, both of which transform in the adjoint of $\mathfrak{g}$ (that is, the **133** of $E_{7(7)}$ for $\mathcal{N}=8$ supergravity). The anticommutator in this formula appears because $\delta^{\mathfrak{g}}$ anticommutes with forms of odd degree.

In summary, on all the fields (but $C_{\mathfrak{k}}$) the differential $\delta^{\mathfrak{g}}$ decomposes into a $\mathfrak{k}$ transformation of parameter $C_{\mathfrak{k}}$, which we will denote $\delta^{\mathfrak{k}}$, and the remaining (coset) transformation $\delta^{\mathfrak{p}}$ with parameter $C_{\mathfrak{p}}$

$$\delta^{\mathfrak{g}} = \delta^{\mathfrak{k}} + \delta^{\mathfrak{p}}\,, \qquad \delta^{\mathfrak{k}} \equiv \delta^{\mathfrak{k}}(C_{\mathfrak{k}})\,,\ \delta^{\mathfrak{p}} \equiv \delta^{\mathfrak{p}}(C_{\mathfrak{p}}) \tag{4.6}$$

For instance, and as a consequence, (4.5) splits as follows

$$\begin{aligned}
\delta^{\mathfrak{k}} B_{\mathfrak{k}} &= -dC_{\mathfrak{k}} - \{B_{\mathfrak{k}}, C_{\mathfrak{k}}\}, \quad & \delta^{\mathfrak{k}} B_{\mathfrak{p}} &= -\{B_{\mathfrak{p}}, C_{\mathfrak{k}}\},\\
\delta^{\mathfrak{p}} B_{\mathfrak{k}} &= -\{B_{\mathfrak{p}}, C_{\mathfrak{p}}\}, & \delta^{\mathfrak{p}} B_{\mathfrak{p}} &= -dC_{\mathfrak{p}} - \{B_{\mathfrak{k}}, C_{\mathfrak{k}}\},\\
\delta^{\mathfrak{k}} C_{\mathfrak{p}} &= -\{C_{\mathfrak{k}}, C_{\mathfrak{p}}\}, & \delta^{\mathfrak{p}} C_{\mathfrak{p}} &= 0,
\end{aligned} \tag{4.7}$$

$\delta^{\mathfrak{k}}$ is a nilpotent differential defined on all the fields, including $C_{\mathfrak{k}}$ with

$$\delta^{\mathfrak{k}} C_{\mathfrak{k}} = -{C_{\mathfrak{k}}}^2\ . \tag{4.8}$$

By contrast, $\delta^{\mathfrak{p}}$ makes sense only on expressions which do not depend on $C_{\mathfrak{k}}$. If such expressions are moreover $\mathfrak{k}$-invariant, the operator $\delta^{\mathfrak{p}}$ is nilpotent as a consequence of (4.2), i.e.

$$\delta^{\mathfrak{p}}(C_{\mathfrak{p}}) \circ \delta^{\mathfrak{p}}(C_{\mathfrak{p}}) = \delta^{\mathfrak{k}}(C_{\mathfrak{p}}^2) \approx 0\ . \tag{4.9}$$

We will refer to such $\mathfrak{k}$-invariant expressions which do not depend of $C_{\mathfrak{k}}$ as '$\mathfrak{k}$-*basic*'; and the cohomology of the nonlinear operator $\delta^{\mathfrak{p}}$ on the complex of $\mathfrak{k}$-basic expressions, as the *equivariant cohomology* $\mathcal{H}^{\bullet}_K(\delta^{\mathfrak{p}})$ (see for example [34] for a mathematical definition).

We will write $S[\varphi, B]$ for the classical action coupled to G-current sources B, where by φ^a we designate *all* the fields of the theory including ghosts. For each field φ^a we introduce a source $\varphi^{\mathfrak{g}}_a$ for the non-linear $\mathfrak{g}$ transformation $\delta^{\mathfrak{p}}(C_{\mathfrak{p}})$ of the field φ^a of anti-commuting parameter $C_{\mathfrak{p}}$. We define the action coupled to sources by

$$\Sigma[\varphi, \varphi^{\mathfrak{g}}, B, C] \equiv \frac{1}{\kappa^2} S[\varphi, B] - \int d^4x \sum_a (-1)^a \varphi^{\mathfrak{g}}_a \delta^{\mathfrak{p}}(C_{\mathfrak{p}}) \varphi^a\ \ , \tag{4.10}$$

where $(-1)^a$ is ± 1 depending of the Grassmann parity of the field φ^a, and the letter a labels *all* the fields of the theory. Of course, the parity of the antifields is the reverse of the corresponding fields, such that the action Σ is bosonic and of zero ghost number. $\Sigma[\varphi, \varphi^{\mathfrak{g}}, B, C]$ satisfies the linear functional identity

$$\delta^{\mathfrak{k}}\Sigma = \int d^4x \Bigg(\sum_a \delta^{\mathfrak{k}}(C_{\mathfrak{k}})\varphi^a \frac{\delta^L\Sigma}{\delta\varphi^a} + \sum_a \delta^{\mathfrak{k}}(C_{\mathfrak{k}})\varphi^{\mathfrak{g}}_a \frac{\delta^L\Sigma}{\delta\varphi^{\mathfrak{g}}_a} - (dC_{\mathfrak{k}} + \{B_{\mathfrak{k}}, C_{\mathfrak{k}}\}) \cdot \frac{\delta^L\Sigma}{\delta B_{\mathfrak{k}}} - \{C_{\mathfrak{k}}, B_{\mathfrak{p}}\} \cdot \frac{\delta^L\Sigma}{\delta B_{\mathfrak{p}}} - \{C_{\mathfrak{k}}, C_{\mathfrak{p}}\} \cdot \frac{\delta^L\Sigma}{\delta C_{\mathfrak{p}}} \Bigg) = 0\,, \quad (4.11)$$

associated to the $\mathfrak{k}$-current Ward identities, and the bilinear functional identity

$$\int d^4x \Bigg(\sum_a \frac{\delta^R\Sigma}{\delta\varphi^{\mathfrak{g}}_a}\frac{\delta^L\Sigma}{\delta\varphi^a} - (dC_{\mathfrak{p}} + \{B_{\mathfrak{k}}, C_{\mathfrak{p}}\}) \cdot \frac{\delta^L\Sigma}{\delta B_{\mathfrak{p}}} - \{C_{\mathfrak{p}}, B_{\mathfrak{p}}\} \cdot \frac{\delta^L\Sigma}{\delta B_{\mathfrak{k}}} - \sum_a \varphi^{\mathfrak{g}}_a \delta^{\mathfrak{k}}({C_{\mathfrak{p}}}^2)\varphi^a \Bigg) = 0\,, \quad (4.12)$$

associated to the $\mathfrak{p}$-current Slavnov-Taylor identities, where the dots stand for the appropriately normalised K-invariant scalar products.

Here we disentangled the linear and the non-linear Ward identities, however, in order to discuss possible anomalies it will be more convenient to combine both of them into a single bilinear G *master equation*

$$\big(\Sigma, \Sigma\big)_{\mathfrak{g}} = 0\,, \quad (4.13)$$

which can be obtained by introducing sources for the sources B and the parameter C [14]. In the absence of anomalies, the above master equation can be elevated to a G master equation for the full effective action, i.e. the 1PI generating functional Γ

$$\big(\Gamma, \Gamma\big)_{\mathfrak{g}} = 0\,. \quad (4.14)$$

This, then, is the equation which encapsulates the $\mathfrak{g}$ invariance of the theory up to any given order in perturbation theory.

Before discussing the anomalies, let us give an example of Slavnov-Taylor identities that can be obtained from the (to be proved to be) non-anomalous master equation (4.14). For example, one can consider correlation functions involving scalar fields only, with

$$\left(\prod_{n\in I} X_n \cdot \frac{\delta}{\delta\Phi(x_n)} X \cdot \frac{\delta}{\delta C(x)} \big(\Gamma, \Gamma\big)_{\mathfrak{g}} \right)\Bigg|_0 = 0\,, \quad (4.15)$$

where the notation $|_0$ means that we set all the classical field φ^a and sources to zero after differentiation. This gives the Ward identity

$$\frac{\partial}{\partial x^\mu}\Big\langle J^\mu(X, x) \prod_{n\in I} \Phi(X_n, x_n)\Big\rangle = \sum_{J\subset I}\Big\langle \Phi^A(x) \prod_{m\in J} \Phi(X_m, x_m)\Big\rangle\Big\langle \Big[\frac{\Phi}{\tanh(\Phi)}(x) * X\Big]_A \prod_{n\in I\setminus J} \Phi(X_n, x_n)\Big\rangle\,, \quad (4.16)$$

where the sum over $J \subset I$ is the sum over all odd subsets of indices J inside the odd set of indices I. In the same way (3.67) is the Fourier transform of

$$\left(X_1 \cdot \frac{\delta}{\delta B_{\mathfrak{k}\,\mu}(x_1)} \right) \left(X_2 \cdot \frac{\delta}{\delta B_{\mathfrak{k}\,\nu}(x_2)} \right) \left(Y_1 \cdot \frac{\delta}{\delta C(y_1)} \right) \left(Y_2 \cdot \frac{\delta}{\delta \Phi(y_2)} \right) \left(\Gamma , \Gamma \right)_{\mathfrak{g}} \Big|_0 = 0 \ . \quad (4.17)$$

Let us first briefly recall why the existence of anomalies is equivalent to a cohomology problem. It is well known that the master equation (4.14) can in principle be broken by the renormalisation process at each order n in perturbation theory, such that

$$\left(\Gamma_n , \Gamma_n \right)_{\mathfrak{g}} = \hbar^n \mathcal{A}_n + \mathcal{O}(\hbar^{n+1}) \,, \quad (4.18)$$

where $\Gamma_n \equiv \sum_{p\le n} \hbar^p \, \Gamma^{(p)}$ is the n-loop renormalised 1PI generating functional, and $\mathcal{A}_n$ is a local functional of the fields and antifields linear in C. Because of the 'anti-Jacobi' functional identity

$$\left(\Gamma , \left(\Gamma , \Gamma \right)_{\mathfrak{g}} \right)_{\mathfrak{g}} = 0 \quad (4.19)$$

the anomaly nevertheless satisfies the Wess-Zumino consistency condition

$$\left(\Gamma_n , \mathcal{A}_n \right)_{\mathfrak{g}} = \mathcal{O}(\hbar) \quad (4.20)$$

and therefore

$$\left(\Sigma , \mathcal{A}_n \right)_{\mathfrak{g}} = 0 \ \ , \quad (4.21)$$

where Σ is the classical action. If $\mathcal{A}_n$ satisfies

$$\mathcal{A}_n = \left(\Sigma , \Sigma^{(\flat\, n)} \right)_{\mathfrak{g}} \ \ , \quad (4.22)$$

for a local functional of the fields $\Sigma^{(\flat\, n)}$, the anomaly is trivial, because one can simply add it to the bare action in order to define a 1PI generating functional which is not anomalous at this order (as we did for example in the last section with the counterterm (3.59) in order to restore Lorentz invariance). The existence of an anomaly therefore requires that the cohomology of the linearised Slavnov-Taylor operator $\mathcal{F} \rightarrow (\Sigma, \mathcal{F})$ on the set of local functionals $\{\mathcal{F}\}$ of the fields is non-trivial. This cohomology is equivalent to the cohomology $\mathcal{H}^1(\delta^{\mathfrak{g}})$ of the differential operator $\delta^{\mathfrak{g}}$ which generates the non-linear $\mathfrak{e}_{7(7)}$ action on the set of local functionals of the fields identified modulo the equations of motion [35].

As we already pointed out, the property that $\mathcal{A}$ is a local functional is known as the quantum action principle [14]. This principle holds true generally for any well defined regularisation scheme. Because of the rather non-standard character of the duality invariant formulation of the theory we are using, it is important to show that such consistent regularisation scheme exists for the theory. Although a fully rigorous proof of the validity of the quantum action principle within the Pauli-Villars regularisation scheme defined in the preceding section is beyond the scope of the present paper, the one-loop computation of the preceding section provides a strong indication for its validity.

4.2 The SU(8)-equivariant cohomology of $\mathfrak{e}_{7(7)}$

To investigate the general structure of anomalies, we need a basis of local functionals. For this purpose it is convenient to consider functions of the fields and their covariant derivatives, defined as

$$d_B\Phi \equiv d\Phi + [\, B_{\mathfrak{k}}, \Phi\,] - \frac{\Phi}{\tanh\Phi} * B_{\mathfrak{p}} \tag{4.23}$$

for the scalars, and similarly for the other fields. Keeping in mind that $\delta^{\mathfrak{g}}$ and the exterior derivative *anti*-commute, we then have

$$\delta^{\mathfrak{g}}\ (d_B\Phi) = -\{\, C_{\mathfrak{k}},\, d_B\Phi\,\} - d_B\left(\frac{\Phi}{\tanh\Phi}\right) * C_{\mathfrak{p}}\ . \tag{4.24}$$

In deriving this formula, we make use of the closure property (2.85) in the form

$$\delta^{\mathfrak{p}}(C_{\mathfrak{p}})\left(\frac{\Phi}{\tanh\Phi}\right) * B_{\mathfrak{p}} + \delta^{\mathfrak{p}}(B_{\mathfrak{p}})\left(\frac{\Phi}{\tanh\Phi}\right) * C_{\mathfrak{p}} = -\big\{B_{\mathfrak{p}}, C_{\mathfrak{p}}\big\} * \Phi\,, \tag{4.25}$$

which allows us to trade one expression (the variation of $\Phi/\tanh\Phi$) which we cannot write in closed-form in terms of another (the $B_{\mathfrak{p}}$ covariantisation of the last term in (4.24)) which we also cannot write in closed-form.

Given a basis of local functionals, a potential anomaly $\mathcal{A}$ decomposes into a term linear in $C_{\mathfrak{k}}$ and a term linear in $C_{\mathfrak{p}}$

$$\mathcal{A} = \int (\mathcal{F}\cdot C_{\mathfrak{k}} + \mathcal{G}\cdot C_{\mathfrak{p}})\ , \tag{4.26}$$

with two local functionals $\mathcal{F}$ and $\mathcal{G}$ of the fields, the current sources and their derivatives. $\mathcal{F}$ and $\mathcal{G}$ take values in $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Accordingly, the Wess-Zumino consistency condition $\delta^{\mathfrak{g}}\mathcal{A} = 0$ decomposes into three components

$$\begin{aligned}\int \delta^{\mathfrak{k}}\,(\mathcal{F}\cdot C_{\mathfrak{k}}) &= 0\ \ ,\\ \int \Big(\delta^{\mathfrak{p}}\mathcal{F}\cdot C_{\mathfrak{k}} + \delta^{\mathfrak{k}}\,(\mathcal{G}\cdot C_{\mathfrak{p}})\Big) &= 0\ \ ,\\ \int \left(-\mathcal{F}\cdot {C_{\mathfrak{p}}}^2 + \delta^{\mathfrak{p}}\mathcal{G}\cdot C_{\mathfrak{p}}\right) &= 0\ \ ,\end{aligned} \tag{4.27}$$

corresponding to the coefficients of $C_{\mathfrak{k}}^2, C_{\mathfrak{k}}C_{\mathfrak{p}}$ and $C_{\mathfrak{p}}^2$, respectively. The first equation is the condition that $\int \mathcal{F}\cdot C_{\mathfrak{k}}$ defines a consistent anomaly for the $\mathfrak{k}$ current Ward identity. *A priori*, there are therefore two kinds of anomalies, the ones associated to the linearly realised subgroup K and determined by $\int \mathcal{F}\cdot C_{\mathfrak{k}}$ in $\mathcal{H}^1(\delta^{\mathfrak{k}})$,[24] which would have to be extended to the non-linear representation by an appropriate $\int \mathcal{G}\cdot C_{\mathfrak{p}}$; and *'genuinely non-linear anomalies'*, with $\int \mathcal{F}\cdot C_{\mathfrak{k}} = 0$, associated to the non-linear transformations only and given by $\int \mathcal{G}\cdot C_{\mathfrak{p}}$. The latter expression is then a $\mathfrak{k}$-invariant functional of the fields and

[24]The superscript on $\mathcal{H}$ here refers to the 'ghost number'.

the current sources which is $\delta^{\mathfrak{p}}$ closed by (4.27). If it can be written as the $\delta^{\mathfrak{g}}$ variation of a functional of the fields, the latter must be K invariant, and the action of $\delta^{\mathfrak{g}}$ and $\delta^{\mathfrak{p}}$ on it are identical. Such a functional $\int \mathcal{G}\cdot C_{\mathfrak{p}}$, if non-trivial, defines by definition a cocycle representative of the equivariant cohomology $\mathcal{H}^1_K(\delta^{\mathfrak{p}})$ of $\delta^{\mathfrak{g}}$ with respect to K. This property can be summarised in the following exact sequence

$$0 \hookrightarrow \mathcal{H}^1_K(\delta^{\mathfrak{p}}) \xrightarrow{\iota} \mathcal{H}^1(\delta^{\mathfrak{g}}) \xrightarrow{\pi} \mathcal{H}^1(\delta^{\mathfrak{k}})\,, \tag{4.28}$$

which states that to each element of $\mathcal{H}^1_K(\delta^{\mathfrak{p}})$ there corresponds one element of $\mathcal{H}^1(\delta^{\mathfrak{g}})$, and that all the other elements of $\mathcal{H}^1(\delta^{\mathfrak{g}})$ correspond to elements of $\mathcal{H}^1(\delta^{\mathfrak{k}})$ (although π is not necessarily surjective *a priori*).

Let us first consider the non-trivial anomalies associated to $\mathfrak{k}$ current anomalies. The anomalies associated to a linearly realised group are well known, and are classified by symmetric Casimirs. A nice way to derive such anomalies is by means of the 'Russian formula' [36–39]

$$(d+\delta^{\mathfrak{k}})\,(B_{\mathfrak{k}}+C_{\mathfrak{k}}) + (B_{\mathfrak{k}}+C_{\mathfrak{k}})^2 = F^{(0)}_{\mathfrak{k}} \equiv dB_{\mathfrak{k}} + B_{\mathfrak{k}}{}^2 \;\;, \tag{4.29}$$

to derive a $(d+\delta^{\mathfrak{k}})$-cocycle from any symmetric Casimirs by use of the Cartan homotopy formula. In four dimensions, the relevant Casimir is the symmetric tensor of rank three, and the Cartan homotopy formula gives

$$(d+\delta^{\mathfrak{k}})\mathrm{Tr}\,\left(\tilde{B}_{\mathfrak{k}}F^{(0)\,2}_{\mathfrak{k}} - \frac{1}{2}\tilde{B}_{\mathfrak{k}}{}^3F^{(0)}_{\mathfrak{k}} + \frac{1}{10}\tilde{B}_{\mathfrak{k}}{}^5\right) = \mathrm{Tr}\;F^{(0)\,3}_{\mathfrak{k}} = 0\,, \tag{4.30}$$

where we define the *extended connection* (always indicated by a tilde) as

$$\tilde{B}_{\mathfrak{k}} \equiv B_{\mathfrak{k}} + C_{\mathfrak{k}}\,, \tag{4.31}$$

and the trace is taken in the complex representation of $\mathfrak{k}$. Picking the component of the Chern-Simons function of form degree four, one obtains from this equation the conventional non-abelian Adler-Bardeen anomaly

$$\mathcal{A}_{\mathfrak{k}} = \int \mathrm{Tr}\;dC_{\mathfrak{k}}\left(B_{\mathfrak{k}}F^{(0)}_{\mathfrak{k}} - \frac{1}{2}B_{\mathfrak{k}}{}^3\right)\,, \tag{4.32}$$

which satisfies

$$\delta^{\mathfrak{k}}\mathcal{A}_{\mathfrak{k}} = 0\;. \tag{4.33}$$

Here we are specifically interested in the case when the rigid symmetry group G does *not* admit an invariant tensor of rank three, as for $G = E_{7(7)}$. In this case the trace must be taken in a *complex representation* of the subgroup K, and (4.30) cannot be defined from the straightforward extension of the 'linear' Russian formula (4.29) to the linear formula

$$(d+\delta^{\mathfrak{g}})\,(B+C) + (B+C)^2 = F_{\mathfrak{g}} = dB + B^2\,, \tag{4.34}$$

for the full Lie algebra $\mathfrak{g}$, because this formula would only make sense in a linear representation of $E_{7(7)}$. Instead we must now look for a *non-linear variant of the Russian formula.* To this aim, we first observe that the closure of the non-linear representation of $\mathfrak{g}$ on the fermion fields implies

$$\delta^{\mathfrak{g}}\left(C_{\mathfrak{k}}+\tanh(\Phi/2)*C_{\mathfrak{p}}\right)+\left(C_{\mathfrak{k}}+\tanh(\Phi/2)*C_{\mathfrak{p}}\right)^2=0\ . \tag{4.35}$$

in the given complex representation of $\mathfrak{k}$. This formula (which is a non-linear analogue of the usual BRST variation, cf. second formula in (4.5)) suggests that one can define a non-linear Russian formula for the $\mathfrak{g}$ symmetry in the fundamental representation of $\mathfrak{k}$. The most natural guess for the (extended) 'non-linear $\mathfrak{g}$ connection' is

$$\tilde{B}\equiv B_{\mathfrak{k}}+C_{\mathfrak{k}}+\tanh(\Phi/2)*\left(B_{\mathfrak{p}}+C_{\mathfrak{p}}\right), \tag{4.36}$$

which is indeed valued in the Lie subalgebra $\mathfrak{k}$. In turn, this motivates the following definition of the (extended) $\mathfrak{g}$ field strength, *viz.*

$$\tilde{F}_{\mathfrak{g}}\equiv(d+\delta^{\mathfrak{g}})\tilde{B}+\tilde{B}^2\ , \tag{4.37}$$

which one then computes (using the extended version of (4.35) to $B+C$) to be

$$\tilde{F}_{\mathfrak{g}}=F_{\mathfrak{k}}+\tanh(\Phi/2)*F_{\mathfrak{p}}+d_B\left(\tanh(\Phi/2)\right)*\left(B_{\mathfrak{p}}+C_{\mathfrak{p}}\right), \tag{4.38}$$

with

$$F_{\mathfrak{k}}\equiv dB_{\mathfrak{k}}+{B_{\mathfrak{k}}}^2+{B_{\mathfrak{p}}}^2\,,\qquad F_{\mathfrak{p}}\equiv dB_{\mathfrak{p}}+\{B_{\mathfrak{k}},B_{\mathfrak{p}}\}\,, \tag{4.39}$$

and $d_B\tanh(\Phi/2)$ defined in terms of the covariant derivative (4.23) similarly as in (4.24). In contradistinction to the conventional Russian formula, the extended field-strength (4.38) is not only the 'horizontal' two-form curvature, but has an extra component linear in the parameter $C_{\mathfrak{p}}$. Nevertheless, it does not depend on $C_{\mathfrak{k}}$, and transforms covariantly with respect to $\mathfrak{k}$ in the adjoint representation $\mathfrak{k}$,

$$\delta^{\mathfrak{k}}\tilde{F}_{\mathfrak{g}}=-\left[\,C_{\mathfrak{k}}\,,\,\tilde{F}_{\mathfrak{g}}\,\right]\ . \tag{4.40}$$

With these definitions, the Cartan homotopy formula

$$(d+\delta^{\mathfrak{g}})\mathrm{Tr}\,\left(\tilde{B}_{\mathfrak{g}}{\tilde{F}_{\mathfrak{g}}}^2-\frac{1}{2}{\tilde{B}_{\mathfrak{g}}}^3\tilde{F}_{\mathfrak{g}}+\frac{1}{10}{\tilde{B}_{\mathfrak{g}}}^5\right)=\mathrm{Tr}\ {\tilde{F}_{\mathfrak{g}}}^3 \tag{4.41}$$

therefore admits a non-vanishing right-hand-side (whereas for a linear representation of $E_{7(7)}$, the right hand side of (4.41) would simply vanish). But because it is independent of $C_{\mathfrak{k}}$ and $\mathfrak{k}$-invariant, and hence $\mathfrak{k}$-basic, it defines a cocycle of the equivariant cohomology $\mathcal{H}^2_K(\delta^{\mathfrak{p}})$. Note that it is a cocycle of 'ghost number' 2, because the associated 4-form component is of 'ghost number' 2. We here tacitly use the corollary of the algebraic Poincaré lemma, i.e. all d-closed functions of the fields and their derivatives of form-degree $p\leq 3$

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are d-exact, whence the cohomology of a differential $\delta^{\mathfrak{p}}$ in the complex of local functionals of 'ghost number' n is isomorphic to the cohomology of the extended differential $d+\delta^{\mathfrak{p}}$ in the complex of functions of the fields of form-degree plus 'ghost number' $4+n$ [40].

If this cocycle is trivial in $\mathcal{H}^2_K(\delta^{\mathfrak{p}})$, i.e. if there exists a $\mathfrak{k}$-basic function $\tilde{M}$ such that

$$(d+\delta^{\mathfrak{p}})\tilde{M} = \mathrm{Tr}\ \tilde{F}_{\mathfrak{g}}{}^3\,, \tag{4.42}$$

one can extend the $\mathfrak{k}$ Adler-Bardeen anomaly to a $\mathfrak{g}$ anomaly by considering the integral of the 4-form component

$$\mathcal{A} = \int\left(\mathrm{Tr}\ \left(\tilde{B}_{\mathfrak{g}}\tilde{F}_{\mathfrak{g}}{}^2 - \frac{1}{2}\tilde{B}_{\mathfrak{g}}{}^3\tilde{F}_{\mathfrak{g}} + \frac{1}{10}\tilde{B}_{\mathfrak{g}}{}^5\right)\Big|_{(4,1)} - M_{(4,1)}\right) \tag{4.43}$$

It follows that the only possible obstruction to extend a $\mathfrak{k}$ anomaly in $\mathcal{H}^1(\delta^{\mathfrak{k}})$ to a full $\mathfrak{g}$ anomaly in $\mathcal{H}^1(\delta^{\mathfrak{g}})$ are defined by cohomology classes of the second equivariant cohomology group $\mathcal{H}^2_K(\delta^{\mathfrak{p}})$.

One can summarise these properties into the exact sequence

$$0 \hookrightarrow \mathcal{H}^1_K(\delta^{\mathfrak{p}}) \xrightarrow{\ \iota\ } \mathcal{H}^1(\delta^{\mathfrak{g}}) \xrightarrow{\ \pi\ } \mathcal{H}^1(\delta^{\mathfrak{k}}) \longrightarrow \mathcal{H}^2_K(\delta^{\mathfrak{p}}) \tag{4.44}$$

which states that the $\mathcal{H}^1(\delta^{\mathfrak{g}})$ and $\mathcal{H}^1(\delta^{\mathfrak{k}})$ only differ by cocycles associated to equivariant cohomology classes. The last arrow is the map which associates to any K consistent anomaly, the corresponding invariant polynomial in $\tilde{F}_{\mathfrak{g}}$.

Now that we have motivated our interest in the equivariant cohomology, we are going to prove that it is trivial. The intuitive idea is the following, the equivariant cohomology on the set of local functional of the fields is closely related to the equivariant cohomology on the set of functions of the scalars only, and the latter is homomorphic to the De Rham cohomology of the coset space $G/K \cong \mathbb{R}^n$ which is trivial [34].

In order to carry out this program, it will turn out to be useful to introduce a filtration in terms of the order of the functional in naked scalar fields Φ, (considering $d_B\Phi$ as independent). The expansion of the variation of Φ and its covariant derivative are

$$\begin{aligned}
\delta^{\mathfrak{p}}\Phi &= C_{\mathfrak{p}} + \frac{1}{3}[\Phi,[\Phi,C_{\mathfrak{p}}]] - \frac{1}{45}[\Phi,[\Phi,[\Phi,[\Phi,C_{\mathfrak{p}}]]]] + \mathcal{O}(\Phi^6)\ , \\
d_B\Phi &= -B_{\mathfrak{p}} + d_{B_{\mathfrak{k}}}\Phi - \frac{1}{3}[\Phi,[\Phi,B_{\mathfrak{p}}]] + \frac{1}{45}[\Phi,[\Phi,[\Phi,[\Phi,B_{\mathfrak{p}}]]]] + \mathcal{O}(\Phi^6)\ , \\
\delta^{\mathfrak{g}}_{C_{\mathfrak{k}}}(d_{B_{\mathfrak{k}}}\Phi) &= -d_{B_{\mathfrak{k}}}C_{\mathfrak{p}} + [\Phi,\{B_{\mathfrak{p}},C_{\mathfrak{p}}\}] - \frac{1}{3}d_{B_{\mathfrak{k}}}[\Phi,[\Phi,C_{\mathfrak{p}}]] + \mathcal{O}(\Phi^4)\ .
\end{aligned} \tag{4.45}$$

The first order in Φ of the equivariant differential only acts on Φ itself as $\delta^{\mathfrak{g}\,(-1)}_{C_{\mathfrak{k}}}\Phi = C_{\mathfrak{p}}$. Any SU(8)-invariant local function of the fields admits an expansion

$$X = \sum_{k\in\mathbb{N}} X^{(n+k)} \tag{4.46}$$

and

$$\delta^{\mathfrak{p}}X = 0 \quad \Rightarrow \quad \delta^{\mathfrak{p}\,(-1)}X^{(n)} = 0\ . \tag{4.47}$$

If $X^{(n)}$ depends non-trivially on $C_{\mathfrak{p}}$ or Φ, then there exist a function $Y^{(n+1)}$ such that $X^{(n)} = \delta^{\mathfrak{p}\,(-1)} Y^{(n+1)}$ [40]. To see this, let us define the trivialising homotopy σ, which acts trivially on all fields, but $C_{\mathfrak{p}}$

$$\sigma C_{\mathfrak{p}} = \Phi \quad , \qquad \{\sigma, \delta^{\mathfrak{p}\,(-1)}\} X = N X \equiv \int d^4 x \left(C_{\mathfrak{p}} \frac{\delta}{\delta C_{\mathfrak{p}}} + \Phi \frac{\delta}{\delta \Phi} \right) X \quad , \tag{4.48}$$

and

$$[N, \delta^{\mathfrak{p}\,(-1)}] = [N, \sigma] = 0 \quad . \tag{4.49}$$

If $X^{(n)}$ depends non-trivially on $C_{\mathfrak{p}}$ or Φ, $N^{-1} X^{(n)}$ exists and

$$X^{(n)} = \{\sigma, \delta^{\mathfrak{p}\,(-1)}\} N^{-1} X^{(n)} = \delta^{\mathfrak{p}\,(-1)}\, \sigma N^{-1} X^{(n)} \tag{4.50}$$

Now with $Y^{(n+1)} = \sigma N^{-1} X^{(n)}$,[25]

$$X - \delta^{\mathfrak{p}} Y^{(n+1)} = X^{(n+1)} - \delta^{\mathfrak{p}\,(0)} Y^{(n+1)} + \mathcal{O}(\Phi^{n+2}) \quad . \tag{4.51}$$

Using the trivialising homotopy, one proves in the same way that $Y^{(n+2)}$ exists such that

$$X - \delta^{\mathfrak{p}} \left(Y^{(n+1)} + Y^{(n+2)} \right) = X^{(n+2)} - \delta^{\mathfrak{p}\,(1)} Y^{(n+1)} - \delta^{\mathfrak{p}\,(0)} Y^{(n+2)} + \mathcal{O}(\Phi^{n+3}) \quad . \tag{4.52}$$

Iteratively one proves that there exist a formal power series

$$Y = \sum_{k \in \mathbb{N}} Y^{(n+1+k)} \tag{4.53}$$

in Φ that trivialises X,

$$X = \delta^{\mathfrak{p}} Y \quad . \tag{4.54}$$

This proof extends trivially to functionals [40], and therefore

$$\mathcal{H}^n_K(\delta^{\mathfrak{p}}) \cong 0 \qquad \text{for } n \geq 1 \quad . \tag{4.55}$$

As a direct consequence, the exact sequence (4.44) implies the isomorphism

$$\mathcal{H}^1(\delta^{\mathfrak{g}}) \cong \mathcal{H}^1(\delta^{\mathfrak{k}}) \quad . \tag{4.56}$$

The equivalence of these two cohomology groups is a main result of this paper: it states that the $\mathfrak{e}_{7(7)}$ consistent anomalies are in one-to-one correspondence with the $\mathfrak{su}(8)$ consistent anomalies. In particular, it follows that their coefficients are the same, establishing as a corollary that the absence of anomalies for the $\mathfrak{su}(8)$ current Ward identities implies the absence of anomalies for the non-linear $\mathfrak{e}_{7(7)}$ Ward identities. This statement completes our proof that the rigid $E_{7(7)}$ symmetry of $d = 4$ $\mathcal{N} = 8$ supergravity is not anomalous in perturbation theory.

[25] Note that although $\delta^{\mathfrak{p}\,(0)}$ vanishes on Φ and the fermion fields, it acts non-trivially on the electromagnetic fields and their ghost, and so $\delta^{\mathfrak{p}\,(0)} Y^{(n+1)}$ does not vanish in general.

In the remaining part of this section, we want to illustrate in some more detail how a potential $\mathfrak{su}(8)$ Adler-Bardeen anomaly would generalise to an $\mathfrak{e}_{7(7)}$ anomaly. The three-form component $\mathrm{Tr}\ \tilde{F}_{\mathfrak{k}}{}^3|_{(3,3)}$ is cubic in $C_{\mathfrak{p}}$, and being $\delta^{\mathfrak{p}}$-closed by construction, there exists an SU(8)-invariant function $M_{(3,2)}$ of the fields quadratic in $C_{\mathfrak{p}}$ such that

$$\mathrm{Tr}\ \tilde{F}_{\mathfrak{k}}{}^3|_{(3,3)} = \delta^{\mathfrak{g}} M(F_{\mathfrak{k}}, F_{\mathfrak{p}}, B_{\mathfrak{p}}, d_B\Phi, \Phi, C_{\mathfrak{p}})_{(3,2)} \ . \tag{4.57}$$

Then $\mathrm{Tr}\ \tilde{F}_{\mathfrak{k}}{}^3|_{(4,2)} - dM_{(3,2)}$ is itself $\delta^{\mathfrak{p}}$-closed because of the Bianchi identity,

$$\delta^{\mathfrak{p}} \Bigl(\mathrm{Tr}\ \tilde{F}_{\mathfrak{k}}{}^3|_{(4,2)} - dM_{(3,2)}\Bigr) = -d\,\mathrm{Tr}\ \tilde{F}_{\mathfrak{k}}{}^3|_{(3,3)} + d\,\delta^{\mathfrak{p}} M_{(3,2)} = 0 \ , \tag{4.58}$$

and being quadratic in $C_{\mathfrak{p}}$, there exists a K-invariant function $M_{(4,1)}$ of the fields linear in $C_{\mathfrak{p}}$ such that

$$\mathrm{Tr}\ \tilde{F}_{\mathfrak{k}}{}^3|_{(4,2)} = \delta^{\mathfrak{g}} M(F_{\mathfrak{k}}, F_{\mathfrak{p}}, B_{\mathfrak{p}}, d_B\Phi, \Phi, C_{\mathfrak{p}})_{(4,1)} + dM(F_{\mathfrak{k}}, F_{\mathfrak{p}}, B_{\mathfrak{p}}, d_B\Phi, \Phi, C_{\mathfrak{p}})_{(3,2)} \ . \tag{4.59}$$

The consistent $E_{7(7)}$ anomaly is defined as

$$\int \biggl(\mathrm{Tr}\ \Bigl(\tilde{B}_{\mathfrak{k}}\tilde{F}_{\mathfrak{k}}{}^2 - \frac{1}{2}\tilde{B}_{\mathfrak{k}}{}^3\tilde{F}_{\mathfrak{k}} + \frac{1}{10}\tilde{B}_{\mathfrak{k}}{}^5\Bigr)\Big|_{(4,1)} - M(F_{\mathfrak{k}}, F_{\mathfrak{p}}, B_{\mathfrak{p}}, d_B\Phi, \Phi, C_{\mathfrak{p}})_{(4,1)}\biggr) \tag{4.60}$$

where

$$\begin{aligned}
&\mathrm{Tr}\ \Bigl(\tilde{B}_{\mathfrak{k}}\tilde{F}_{\mathfrak{k}}{}^2 - \frac{1}{2}\tilde{B}_{\mathfrak{k}}{}^3\tilde{F}_{\mathfrak{k}} + \frac{1}{10}\tilde{B}_{\mathfrak{k}}{}^5\Bigr)\Big|_{(4,1)} \qquad\qquad (4.61)\\
&\quad = \mathrm{Tr}\ (C_{\mathfrak{k}} + \tanh(\Phi/2) * C_{\mathfrak{p}})\, d\Bigl((B_{\mathfrak{k}} + \tanh(\Phi/2) * B_{\mathfrak{p}})\, d\,(B_{\mathfrak{k}} + \tanh(\Phi/2) * B_{\mathfrak{p}})\\
&\qquad + \frac{1}{2}(B_{\mathfrak{k}} + \tanh(\Phi/2) * B_{\mathfrak{p}})^3\Bigr)\\
&\qquad + \mathrm{Tr}\ d_B\,(\tanh(\Phi/2)) * C_{\mathfrak{p}}\Bigl((B_{\mathfrak{k}} + \tanh(\Phi/2) * B_{\mathfrak{p}})\, d\,(B_{\mathfrak{k}} + \tanh(\Phi/2) * B_{\mathfrak{p}})\\
&\qquad + d\,(B_{\mathfrak{k}} + \tanh(\Phi/2) * B_{\mathfrak{p}})\ (B_{\mathfrak{k}} + \tanh(\Phi/2) * B_{\mathfrak{p}}) + \frac{3}{2}(B_{\mathfrak{k}} + \tanh(\Phi/2) * B_{\mathfrak{p}})^3\Bigr).
\end{aligned}$$

One computes perturbatively that

$$\begin{aligned}
M_{(3,2)} = \frac{1}{8}\mathrm{Tr}\ \Bigl(-[\Phi, B_{\mathfrak{p}}]\{C_{\mathfrak{p}}, B_{\mathfrak{p}}\}^2 + \frac{1}{2}\bigl(3\{C_{\mathfrak{p}}, d_{B_{\mathfrak{k}}}\Phi\} - [\Phi, d_{B_{\mathfrak{k}}}C_{\mathfrak{p}}]\bigr)\Bigl\{[\Phi, B_{\mathfrak{p}}], \{C_{\mathfrak{p}}, B_{\mathfrak{p}}\}\Bigr\}\Bigr)&\\
+\ \mathcal{O}(\Phi^3) \ ,&
\end{aligned} \tag{4.62}$$

and

$$\begin{aligned}
M_{(4,1)} = \frac{3}{8}\mathrm{Tr}\ &\Bigl\{\{C_{\mathfrak{p}}, B_{\mathfrak{p}}\} - 2\{C_{\mathfrak{p}}, d_{B_{\mathfrak{k}}}\Phi\} + [\Phi, d_{B_{\mathfrak{k}}}C_{\mathfrak{p}}], [\Phi, B_{\mathfrak{p}}]\Bigr\}\bigl(F_{\mathfrak{k}} - B_{\mathfrak{p}}{}^2\bigr)\\
&+ \frac{1}{4}\mathrm{Tr}\ \Bigl\{[\Phi, B_{\mathfrak{p}}], \{C_{\mathfrak{p}}, B_{\mathfrak{p}}\}\Bigr\}\bigl([\Phi, F_{\mathfrak{p}}] + \{B_{\mathfrak{p}}, d_{B_{\mathfrak{k}}}\Phi\}\bigr) + \mathcal{O}(\Phi^3) \ .
\end{aligned} \tag{4.63}$$

There is no difficulty in computing higher order terms in Φ, but the complete solution is not obvious. Anyway, the important property is that it exists, at least as a formal power series in Φ (the issue of convergence being irrelevant in perturbative theory).

The results of this section extend straightforwardly to any consistent K anomaly for any supergravity theory in arbitrary dimensions. For example, the solution for $\operatorname{Tr} \tilde{F}_{\mathfrak{k}}$ is rather trivial when K admits a U(1) factor, as for lower $\mathcal{N}$-extended supergravity theories. In that case, $\operatorname{Tr} \tilde{F}_{\mathfrak{g}\,\wedge} R^{ab}{}_{\wedge} R_{ab} = 0$ and one has the anomaly

$$\mathcal{A}_{\mathfrak{u}(1)} = \int \operatorname{Tr} \, \left(C_{\mathfrak{k}} + \tanh(\Phi/2) * C_{\mathfrak{p}} \right) R^{ab}{}_{\wedge} R_{ab} \ . \tag{4.64}$$

In particular, this anomaly does not vanish when $C_{\mathfrak{p}}$ is constant, and the current sources are set to zero. It follows that the rigid Ward identities are anomalous at one-loop if the coefficient does not vanish, as is the case for minimal $\mathcal{N} = 4$ supergravity with duality group $\mathrm{SL}(2,\mathbb{R})$, and more generally for $\mathcal{N} \leq 4$ supergravities.

More specifically, for $G = \mathrm{SL}(2,\mathbb{R})$, we can spell out the above formulas in explicit detail. In this case, the second relation in (4.5) becomes

$$\delta^{\mathfrak{sl}_2} \alpha = i w \bar{w} \ , \qquad \delta^{\mathfrak{sl}_2} w = -2 i \alpha w \ , \tag{4.65}$$

where $\alpha \equiv C_{\mathfrak{k}}$ and $w \equiv C_{\mathfrak{p}}$ are real and complex anticommuting numbers, respectively. If we denote by ϕ the complex scalar parametrising the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ (the analogue of Φ) the formula (4.3) can be worked out as

$$\delta^{\mathfrak{sl}_2} \phi = \left(\frac{1}{2} + \frac{|\phi|}{\tanh 2|\phi|} \right) w - 2 i \alpha \phi + \left(\frac{1}{2|\phi|^2} - \frac{1}{|\phi| \tanh 2|\phi|} \right) \bar{w} \phi^2 \tag{4.66}$$

and the anomaly (4.64) reads

$$\mathcal{A}_{\mathfrak{u}(1)} = \int \left(\alpha - \frac{i \tanh |\phi|}{2|\phi|} \left(\bar{\phi} w - \phi \bar{w} \right) \right) R^{ab}{}_{\wedge} R_{ab} \tag{4.67}$$

Equivalently, within the triangular gauge parametrisation of $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ by the complex modulus $\tau = \tau_1 + i\tau_2$, and the (hopefully self explanatory) notation

$$C = \begin{pmatrix} C_h & C_e \\ C_f & -C_h \end{pmatrix} \ , \tag{4.68}$$

the algebra reads

$$\begin{aligned} \delta^{\mathfrak{sl}_2} C_h &= -C_e C_f \ , \qquad \delta^{\mathfrak{sl}_2} C_e = -2 C_h C_e \ , \qquad \delta^{\mathfrak{sl}_2} C_f = 2 C_h C_f \ , \\ \delta^{\mathfrak{sl}_2} \tau &= -C_e - 2 C_h \tau + C_f \tau^2 \ . \end{aligned} \tag{4.69}$$

The consistent anomaly (4.64) then becomes ($C_f \tau_2$ being the parameter of the compensating $\mathfrak{u}(1)$ transformation in the triangular gauge)

$$\mathcal{A}_{\mathfrak{u}(1)} = \int C_f \tau_2 \, R^{ab}{}_{\wedge} R_{ab} \ . \tag{4.70}$$

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Indeed, explicit computation shows that

$$\delta^{\mathfrak{sl}_2}(C_f \tau_2) = 0 \ , \tag{4.71}$$

but $C_f \tau_2$ itself cannot be written as $\delta^{\mathfrak{sl}_2}\mathcal{F}(\tau)$: indeed, the vanishing of the C_e component of $\delta^{\mathfrak{sl}_2}\mathcal{F}(\tau)$ implies that $\mathcal{F}$ is a function of τ_2 only, and the vanishing of the C_h component then entails that $\mathcal{F}$ must be constant.

In the conventional formulation of $\mathcal{N} = 4$ supergravity, and similarly in $\mathcal{N} = 2$ supergravity with a semi-simple duality group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(2,n)$, we see that the non-linearly realised generator $\mathbf{f}$ of $\mathfrak{sl}_2$ is anomalous at one-loop. More generally, in $\mathcal{N} = 2$ supergravity theories with vector multiplets scalar fields parametrising a symmetric special Kähler manifold, the duality group will be anomalous at one-loop. Indeed, one computes similarly as in [17] that the addition of matter multiplets does not permit to cancel the U(1) gravitational anomaly, the anomaly coefficient of (4.64) being proportional to $24 + 12n_{\mathrm{V}}$ in $\mathcal{N} = 4$ supergravity coupled to n_{V} vector multiplets, and to $102 + 10n_{\mathrm{V}} + 3n_{\mathrm{H}}$ in $\mathcal{N} = 2$ supergravity coupled to n_{V} vector multiplets and n_{H} hypermultiplets.

5 Compatibility of $E_{7(7)}$ with gauge invariance

Up to this point we have discussed the properties of the $E_{7(7)}$ symmetry and its possible anomalies, irrespectively of its compatibility with gauge invariance. We now extend this discussion to the full quantum theory, with the aim of deriving the Ward identities associated to the conservation of the $E_{7(7)}$ Noether current, thereby corroborating our main claim that the non-linear $E_{7(7)}$ symmetry is compatible with all gauge symmetries of the theory, in the sense that it can be implemented order by order in a loop expansion of the full effective action. To this aim, we have to make use of the BRST formalism (see e.g. [19, 40]). Because the algebra of gauge transformations is 'open',[26] we have to go one step further by including higher order ghost interactions [41], and ultimately bring in the full machinery of the Batalin-Vilkovisky formalism [42]. In addition to the usual ghosts and antighosts (anti-commuting for the bosonic transformations, and commuting for the supersymmetry transformations) this requires introducing 'antifields' for all fields and ghost fields of the theory. The compatibility of the of $E_{7(7)}$ with the BRST symmetry is then encoded into two corresponding mutually compatible 'master equations'.

Now, a complete treatment of our $E_{7(7)}$ invariant formulation of maximal supergravity along these lines would be very involved and cumbersome, and certainly unsuitable for practical computations of the type performed in [5, 6]. Instead, we here focus on the specific features of the duality invariant formulation in comparison with the conventional formulation of the theory, and the fact that the cancellation of $E_{7(7)}$ anomalies, together with the well admitted absence of diffeomorphism and supersymmetry anomalies in four

[26]That is, the gauge algebra closes only modulo the equations of motion.

space-time dimensions, eliminates any obstruction towards implementing the BRST and $E_{7(7)}$ master equations at any order in perturbation theory. We emphasise again that these results do not preclude the appearance of divergent counterterms, but they ensure that potential divergences must respect the full $E_{7(7)}$ symmetry of the theory.

5.1 Batalin-Vilkovisky formalism

Following [19] we will designate by $e^{\ddagger\,\mu}_{a}$, $A^{\ddagger\,\mathrm{i}}_{m}$, $\psi^{\ddagger\,\mu}_{i}$ and $\chi^{\ddagger}_{ijk}$ the antifields associated to the vierbein, the vector fields, the gravitino and the Dirac fields, respectively, and by ξ^μ, $\Omega^a{}_b$ and c^m the anticommuting ghost fields associated to diffeomorphism invariance, Lorentz invariance, and abelian gauge invariance, respectively; the commuting supersymmetry ghost is ϵ^i. In addition we also need antifields $\xi^\ddagger, \Omega^{\ddagger\,ab}, c^{\ddagger\,m}$ and $\epsilon^{\ddagger\,i}$ for these ghost fields.

Regarding gauge-fixing, the $\mathfrak{e}_{7(7)}$ Ward identities can be implemented without further ado as long as the gauge-fixing manifestly preserves $E_{7(7)}$ invariance. Of course, this is trivially the case for any sensible gauge choice for diffeomorphism and Lorentz invariance, and it is also true for the Coulomb gauge we are using. An $E_{7(7)}$-invariant gauge choice for local supersymmetry can be achieved in terms of the SU(8)-covariant derivative

$$D_\mu \psi^i_\nu = \partial_\mu \psi^i - \frac{1}{3}\left(u_{jk}{}^{IJ}\partial_\mu u^{ik}{}_{IJ} - v_{jkIJ}\partial_\mu v^{ikIJ}\right)\psi^j_\nu \; . \tag{5.1}$$

(for instance by setting $D^\mu \psi^i_\mu = 0$), with the extra proviso that the supersymmetry antighost and the Nielsen-Kallosh field transform in the non-linear representation of $E_{7(7)}$ conjugate to the one of ψ^i and ϵ^i.

In the conventional formulation of the theory, the supersymmetry algebra closes on the fermionic fields only modulo terms linear in the fermionic equations of motion. Within the Batalin-Vilkovisky approach, this problem is cured by introducing terms quadratic in the fermion antifields in the action. The functional form of the BRST operator then includes these terms in the BRST transformation of the fermions. Let us briefly recall how this works. Collectively designating the fields and ghosts as $\varphi^{\mathtt{a}}$, and their Grassmann parity as $(-1)^{\mathtt{a}}$, we have

$$s^2\varphi^{\mathtt{a}} = \sum_{\mathtt{b}} K(\varphi)^{\mathtt{a}\mathtt{b}}\frac{\delta^L \Sigma}{\delta\varphi^{\mathtt{b}}} \; . \tag{5.2}$$

This equation simply expresses the fact that algebra closes (that is, $s^2 \approx 0$) only if the equations of motion are imposed. Introducing antifields $\varphi^\ddagger_{\mathtt{a}}$, the action $\Sigma[\varphi, \varphi^\ddagger]$ reads

$$\Sigma(\varphi^{\mathtt{a}}, \varphi^\ddagger_{\mathtt{a}}) = \frac{1}{\kappa^2}S[\varphi] - \int d^4x \left(\sum_{\mathtt{a}}(-1)^{\mathtt{a}}\varphi^\ddagger_{\mathtt{a}} s\varphi^{\mathtt{a}} + \frac{\kappa^2}{2}\sum_{\mathtt{a}\mathtt{b}}\varphi^\ddagger_{\mathtt{a}} K^{\mathtt{a}\mathtt{b}}(\varphi)\varphi^\ddagger_{\mathtt{b}}\right) , \tag{5.3}$$

The symmetry of the action and the closure of the algebra can then be combined into a single *BRST master equation*[27] (indexed by a ‡ to distinguish it from the $E_{7(7)}$ master

[27] See e.g. [40] for further information.

equation to be introduced below)

$$(\Sigma, \Sigma)_{\ddagger} \equiv \sum_{a} \int d^4x \frac{\delta^R \Sigma}{\delta \varphi^{\ddagger}_a} \frac{\delta^L \Sigma}{\delta \varphi^a} = 0 \ . \tag{5.4}$$

This equation requires in addition that

$$sK^{ab} + \frac{1}{2} \sum_c \left(K^{ac} \frac{\partial^L s\varphi^b}{\partial \varphi^c} + (-1)^{(a+1)(b+1)} K^{bc} \frac{\partial^L s\varphi^a}{\partial \varphi^c} \right) = 0 \,, \tag{5.5}$$

$$\sum_d \left(K^{cd} \frac{\partial^L K^{ab}}{\partial \varphi^d} + (-1)^{c(a+b)} K^{ad} \frac{\partial^L K^{bc}}{\partial \varphi^d} + (-1)^{b(c+a)} K^{bd} \frac{\partial^L K^{ca}}{\partial \varphi^d} \right) = 0 \ . \tag{5.6}$$

These identities are automatically satisfied modulo the equations of motion by integrability of definition (5.2). For $\mathcal{N} = 8$ supergravity in the conventional formulation, the term quadratic in the antifields in (5.3) only involves the fermionic antifields $\psi^{*\,i}_\mu$ and $\chi^{*\,ijk}$ in a first approximation (i.e. neglecting the antifield dependent terms in (5.2)). In supergravity, such K^{ab} components associated to the fermions are bilinear in the superghosts ϵ^i (and depend as well on the vierbeine and the scalar fields), and the validity of the identity (5.5) is ensured by certain cubic Fierz identities in ϵ^i. (5.6) is trivially satisfied because K^{ab} only depends on fields on which the algebra is satisfied off-shell. Nevertheless, this modification of the BRST transformation of the fermions also affects the closure of the algebra on the bosons, such that the BRST transformation of the Lorentz ghost $\Omega^a{}_b$ must include terms linear in the fermion antifields as well. This entails terms quadratic in the antifields involving the Lorentz ghost antifield $\Omega^{\ddagger}_{ab}$ as well. Considering these terms in the nilpotency of the linearised Slavnov-Taylor operator $(\Sigma, \cdot\,)$ on the vierbeine, i.e.

$$\left(\Sigma, \left(\Sigma, e^a_\mu\right)_{\ddagger}\right)_{\ddagger} = \kappa^2 \sum_a \Big(\bar\epsilon_i \gamma^a K^{i\,a}_\mu(\varphi) + \bar\epsilon^i \gamma^a K_{\mu i}{}^a(\varphi) + K^a{}_b{}^a(\varphi) e^b_\mu \Big) \varphi^{\ddagger}_a \ \ , \tag{5.7}$$

where $K^{i\,a}_\mu$, $K_{\mu i}{}^a$ and $K^a{}_b{}^a$ are the components of K^{ab} to be contracted with $\bar\psi^{\ddagger\mu}_i$, $\bar\psi^{\ddagger\mu i}$ and $\Omega^{\ddagger\,b}_a$, respectively, one observes that $K^a{}_b{}^a$ is determined in function of $K^{i\,a}_\mu$, $K_{\nu i}{}^a$, such that the modification of the action to be carried out amounts to replacing the gravitino antifields appearing in the term quadratic in the fermion antifields by

$$\psi^{\ddagger\,\mu}_i \to \psi^{\ddagger\,\mu}_i - e^\mu_a \gamma_b \Omega^{\ddagger\,ab} \epsilon_i \ . \tag{5.8}$$

In our manifestly $E_{7(7)}$-invariant formulation, the situation is the same with regard to the fermion fields, but now the vector fields are also governed by a first order Lagrangian (first order in the time derivative), and hence the algebra of gauge transformations on the vectors likewise involves the equations of motion. It is important that the equation of motion of the vector fields here appears as (2.6), and not in its integrated form (2.8), as required for the consistency of the Batalin-Vilkovisky formalism. We have checked that the diffeomorphism transformations do close among themselves, and that local supersymmetry closes on the vector fields. However, their commutator on the vector fields close modulo the equations of motion of the fermion fields, *viz.*

$$s^2 A^{IJ}_{\mathtt{i}} = \frac{N\xi^0}{\sqrt{h}} e^{0a} e^b_{\mathtt{i}} \Bigg[u_{ij}{}^{IJ} \left(\bar{\epsilon}^i \gamma_\mu \gamma_{ab} \frac{\delta^L S}{\delta \bar{\psi}_{\mu\, j}} + 12 \bar{\epsilon}_k \gamma_{ab} \frac{\delta^L S}{\delta \bar{\chi}_{ijk}} \right) - v^{ijIJ} \left(\bar{\epsilon}_i \gamma_\mu \gamma_{ab} \frac{\delta^L S}{\delta \bar{\psi}^j_\mu} + 12 \bar{\epsilon}^k \gamma_{ab} \frac{\delta^L S}{\delta \bar{\chi}^{ijk}} \right) \Bigg] . \tag{5.9}$$

To remain consistent with the basic symmetry property $K^{ab} = -(-1)^{ab} K^{ba}$, the closure of diffeomorphisms with local supersymmetry on the fermion fields then requires correspondingly the equations of motion of the vector fields. We checked that this is indeed the case, and that the fermion equations of motion are not involved (as follows trivially from Lorentz invariance). The quadratic terms in the fermion antifields are also modified by non-manifestly diffeomorphism invariant terms, such that they are manifestly duality invariant (and so do not depend on the scalar fields).

The quadratic terms in the antifields of the gauge fields are responsible for the quartic terms in the ghosts that appear in supergravity [42, 43]. It follows that in the duality invariant formulation, we will also have quartic terms depending on the diffeomorphism ghost ξ^0, the supersymmetry ghost ϵ^i, the abelian antighost $\bar{c}_m$ and the supersymmetry antighost η_i, which in a flat Landau-type gauge for local supersymmetry like $D^\mu \psi^i_\mu \approx 0$, would for example be of the form

$$\frac{N\xi^0}{\sqrt{h}} e^{0a} e^b_{\mathtt{i}} \partial_{\mathtt{i}} \bar{c}_{IJ} \left(u_{ij}{}^{IJ} \bar{\epsilon}^i \gamma^\mu \gamma_{ab} D_\mu \eta^j - v^{ijIJ} \bar{\epsilon}_i \gamma^\mu \gamma_{ab} D_\mu \eta_j \right) + \text{c.c.} \, . \tag{5.10}$$

In general, such vertices do not contribute to amplitudes of physical fields. However, the renormalisation of the theory in the absence of regularisation preserving all gauge symmetries requires the renormalisation of the composite BRST transformations. In consequence, the correlation functions involving the insertion of the BRST transformation of the vector fields do involve such vertices.

Note that one can obtain the solution Σ of the master equation in the covariant formulation form the duality invariant one, by carrying out the Gaussian integration of the momentum variable $\Pi^{m\,\mathtt{i}}$ for the complete action Σ with antifields, similarly as in the second section. Considering for example the terms

$$S^{\text{ghost}}_{\text{vec}} = \int d^4x \Bigg(A^{*\,\mathtt{i}}_m \left(- \left(\xi^{\mathtt{j}} + N^{\mathtt{j}} \xi^0 \right) F^m_{\mathtt{ij}} + \frac{N}{2\sqrt{h}} \xi^0 h_{\mathtt{ij}} \varepsilon^{\mathtt{jkl}} J^m{}_n F^n_{\mathtt{kl}} \right) - c^*_m \left(\frac{1}{2} \xi^{\mathtt{i}} \xi^{\mathtt{j}} F^m_{\mathtt{ij}} + \xi^0 \xi^{\mathtt{i}} \left(-N^{\mathtt{j}} F^m_{\mathtt{ij}} + \frac{N}{2\sqrt{h}} h_{\mathtt{ij}} \varepsilon^{\mathtt{jkl}} J^m{}_n F^n_{\mathtt{kl}} \right) \right) + \cdots \Bigg) \tag{5.11}$$

and the $F^m_{\mathtt{ij}}$ dependent terms that appear in the supersymmetry transformations of the fermions, as well as the gauge-fixing terms, one sees that the vector fields only appear through their field strength $F^m_{\mathtt{ij}}$. It is therefore important (and true!) that the whole

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quantum action can be treated in the way described in the second section. The $\mathrm{Re}[F^{IJ}_{\mathtt{ij}}]$ dependent terms in the supersymmetry variation of the fermions and of the $\mathfrak{sl}_2(\mathbb{C})$ ghost $\Omega^a{}_b$, as well as the ones in the diffeomorphism variation of the vector fields and their ghost c^{IJ}, are replaced upon Gaussian integration of the momentum variables $\Pi^{IJ\,\mathtt{i}}$ by the solution of $\Pi^{IJ\,\mathtt{i}}$ according to their equation of motion. This step will restore manifest diffeomorphism invariance. All these terms will also produce quadratic terms in the sources, which define the required equations of motion in order to close the gauge algebra in the $E_{7(7)}$ invariant formulation of the theory. This way one obtains that the only terms quadratic in $\mathrm{Im}[A^{*\,\mathtt{i}}_{IJ}]$ involve the diffeomorphism ghost ξ^μ, and that they vanish once one puts the source $\mathrm{Re}[A^{*\,\mathtt{i}}_{IJ}]$ equal to zero, in agreement with the explicit computation in the formalism.

5.2 BRST extended current

Having set up the BRST transformations and the Batalin-Vilkovisky framework, the next task is to define the $\mathfrak{e}_{7(7)}$ current Ward identities in such a way that their mutual consistency is preserved also in perturbative quantisation. To this aim, one must in principle couple the whole chain of operators associated to the current via the BRST descent equations, that is, extend the current constructed in section 2.4 by appropriate ghost and antifeld terms. Because the classical current defines a physical Noether charge which is BRST invariant, one has

$$sJ_{(3,0)}(\Lambda) = -dJ_{(2,1)}(\Lambda) \quad , \tag{5.12}$$

where we now write $J_{(3,0)} \equiv J$, indicating the form degree and ghost number. Considering the functional BRST operator s acting on both fields and antifields, the conservation of the current reads

$$dJ_{(3,0)}(\Lambda) = -sJ_{(4,-1)}(\Lambda) \quad , \tag{5.13}$$

where $J_{(4,-1)}$ is the composite operator linear in the antifields

$$J_{(4,-1)}(\Lambda) \equiv \sum_a \varphi^\ddagger_a \delta^{\mathfrak{e}_{7(7)}}(\Lambda)\varphi^a \quad . \tag{5.14}$$

Then

$$dJ_{(3,0)}(\Lambda) = \sum_a \left(\delta^{\mathfrak{e}_{7(7)}}(\Lambda)\varphi^a \frac{\delta^L \Sigma}{\delta \varphi^a} + \delta^{\mathfrak{e}_{7(7)}}(\Lambda)\varphi^\ddagger_a \frac{\delta^L \Sigma}{\delta \varphi^\ddagger_a} \right) \quad , \tag{5.15}$$

as defined by the Noether procedure on the complete gauge fixed action $\Sigma[\varphi, \varphi^\ddagger]$, and where $\varphi^\ddagger_a$ transforms with respect to $E_{7(7)}$ in the representation conjugate to φ^a.

The whole chain of operators appearing in the descent equations defines an extended form $\tilde{J}$ which is a cocycle of the extended differential $d + s$ [35],

$$(d+s)\left(J_{(4,-1)} + J_{(3,0)} + J_{(2,1)} + J_{(1,2)} + J_{(0,3)}\right) = 0 \quad . \tag{5.16}$$

The complete form of the extended current $\tilde{J}$ which now also depends on the ghosts and antifields is again very complicated, and its explicit form would not be very illuminating.

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Let us nevertheless discuss some salient features of this extended current, neglecting terms depending on the antifields and terms linear in the equations of motion. With these assumptions we can take $J_{(4,-1)}$ to vanish, and $J_{(3,0)}$ can be identified with the Gaillard-Zumino current constructed in section 2.4, where we also disregard the 'curl component' leading to a trivial cocycle. Let us first rewrite the components of $J_{(3,0)}$ in terms of differential forms, cf. (2.51), (2.53)

$$\begin{aligned} R^i{}_j &= -2ie^a_\wedge\left(\bar\psi^i_\wedge\gamma_a\psi_j - \frac{1}{8}\delta^i_j\bar\psi^k_\wedge\gamma_a\psi_k\right) - \frac{1}{48}\varepsilon_{abcd}e^b_\wedge e^c_\wedge e^d\left(\bar\chi^{ikl}\gamma^a\chi_{jkl} - \frac{1}{8}\delta^i_j\bar\chi^{klp}\gamma^a\chi_{klp}\right), \\ R_{ijkl} &= -\star\hat{\mathcal{A}}_{ijkl} + \frac{i}{2}e^a_\wedge e^b_\wedge\left(\bar\chi_{[ijk}\gamma_{ab}\psi_{l]} + \frac{1}{4!}\varepsilon_{ijklpmnpq}\bar\chi^{mnp}\gamma_{ab}\psi^q\right) \quad , \end{aligned} \tag{5.17}$$

and

$$J^{(2)}_{\text{GZ}}(\Lambda) = -\frac{1}{2}\,A^m_\wedge F^n\,\Lambda_m{}^p\Omega_{pn}\ . \tag{5.18}$$

Conveniently, the extended current $\tilde J$ takes a form similar to the current constructed in section 2.4

$$\tilde J(\Lambda) = -\frac{1}{24}e^{i_\xi}\text{tr}\left(\mathcal{V}^{-1}\tilde R\mathcal{V}\Lambda\right) + \tilde J^{(2)}_{\text{GZ}}(\Lambda)\ . \tag{5.19}$$

so we only need to explain how to obtain the 'tilded' version of the above currents. The operator i_ξ is the (commuting) Cartan contraction with respect to the anti-commuting vector ξ^μ; its exponentiated action takes care automatically of all modifications involving the diffeomorphism ghost fields ξ^μ [44]. In order to understand how to extend the remaining piece $J^{(2)}_{\text{GZ}}(\Lambda)$ to $\tilde J^{(2)}_{\text{GZ}}(\Lambda)$, it is again convenient to write a Russian formula

$$(d+s)\,(A^m + c^m) = e^{i_\xi}\tilde F^m\,, \tag{5.20}$$

where the extended curvature $\tilde F$ is defined as

$$\begin{aligned} \tilde F^{IJ} &\equiv F^{IJ} + u_{ij}{}^{IJ}\left(\frac{1}{4}\bar\epsilon_k e^a\gamma_a\chi^{ijk} + 2\bar\epsilon^i\psi^j + \bar\epsilon^i\epsilon^j\right) - v^{ijIJ}\left(\frac{1}{4}\bar\epsilon^k e^a\gamma_a\chi_{ijk} + 2\bar\epsilon_i\psi_j + \bar\epsilon_i\epsilon_j\right) \\ &= \hat F^{IJ} + u_{ij}{}^{IJ}\left(\frac{1}{4}\overline{[\psi+\epsilon]}_k e^a\gamma_a\chi^{ijk} + \overline{[\psi+\epsilon]}^i[\psi+\epsilon]^j\right) \\ &\quad - v^{ijIJ}\left(\frac{1}{4}\overline{[\psi+\epsilon]}^k e^a\gamma_a\chi_{ijk} + \overline{[\psi+\epsilon]}_i[\psi+\epsilon]_j\right). \end{aligned} \tag{5.21}$$

The gravitinos here appear only in the supercovariantisation of F^{IJ} or through the combination $\psi^i + \epsilon^i$. In addition we need the nilpotent extended differential [44]

$$\tilde d \equiv e^{-i_\xi}(d+s)e^{i_\xi} = d + s - \mathcal{L}_\xi + i_{(\bar\epsilon\gamma\epsilon)}\ , \tag{5.22}$$

where $i_{(\bar\epsilon\gamma\epsilon)}$ is the Cartan contraction with respect to the vector $\bar\epsilon_i\gamma^\mu\epsilon^i$. Defining

$$\tilde A^m \equiv A^m + c^m - i_\xi A^m\,, \tag{5.23}$$

it is obvious that

$$(d+s)\,e^{i_\xi}\left(\tilde A^m_\wedge\tilde F^n\,\Lambda_m{}^p\Omega_{pn}\right) = e^{i_\xi}\left(\tilde F^m_\wedge\tilde F^n\,\Lambda_m{}^p\Omega_{pn}\right)\ . \tag{5.24}$$

The right-hand-side being gauge-invariant, the extended form $\tilde{R}$ can be obtained from the equation

$$\tilde{d}\left(\frac{1}{24}\mathrm{tr}\left(\mathcal{V}^{-1}\tilde{R}\mathcal{V}\Lambda\right)\right)=\frac{1}{4}\tilde{F}^m_{\wedge}\tilde{F}^n\,\Lambda_m{}^p\Omega_{pn}\ . \tag{5.25}$$

which is an extended version of the Gaillard-Zumino construction. For any gauge invariant extended form, such as $\tilde{R}$ or $\tilde{F}^m$, supersymmetry covariance implies that the gravitino field ψ^i only appears in supercovariant forms, or 'naked', through the wedge product of $\psi^i+\epsilon^i$ with supercovariant forms. It follows that $\tilde{R}$ is simply obtained from R by performing the replacement $\psi^i\rightarrow\psi^i+\epsilon^i$ everywhere inside (5.17).

The $(4,0)$ component of (5.25) is simply the current conservation. To see that (5.25) is indeed satisfied for the other components, let us consider the $(0,4)$ component of this equation. From (5.21) we see that the right hand side is the $\mathfrak{e}_{7(7)}$ component of the square of $u_{ij}{}^{IJ}\bar{\epsilon}^i\epsilon^j-v^{ijIJ}\bar{\epsilon}_i\epsilon_j$. By $E_{7(7)}$ covariance, the scalar fields dependence then reduces to a similarity transformation with respect to $\mathcal{V}$ (as the left hand side), and one can concentrate on the $\mathfrak{e}_{7(7)}$ element quadratic in $\bar{\epsilon}^i\epsilon^j$. Because (for commuting spinors)

$$\bar{\epsilon}^{[i}\epsilon^j\bar{\epsilon}^k\epsilon^{l]}=0\,, \tag{5.26}$$

this term only contributes in the $\mathfrak{su}(8)$ component $i(\bar{\epsilon}^i\epsilon^k)(\bar{\epsilon}_j\epsilon_k)-\frac{i}{8}\delta^i_j(\bar{\epsilon}^k\epsilon^l)(\bar{\epsilon}_k\epsilon_l)$. Because $\tilde{R}$ has a vanishing $(0,3)$ component, the left hand side is the Cartan contraction of its $(1,2)$ component $-2ie^a\left(\bar{\epsilon}^i\gamma_a\epsilon_j-\frac{1}{8}\delta^i_j\bar{\epsilon}^l\gamma_a\epsilon_l\right)$ with the vector $\epsilon_i\gamma^\mu\epsilon^i$. Using the Fierz identity

$$(\bar{\epsilon}^i\epsilon^k)(\bar{\epsilon}_j\epsilon_k)-\frac{1}{8}\delta^i_j(\bar{\epsilon}^k\epsilon^l)(\bar{\epsilon}_k\epsilon_l)=-\frac{1}{2}(\bar{\epsilon}_k\gamma^a\epsilon^k)\left(\bar{\epsilon}^i\gamma_a\epsilon_j-\frac{1}{8}\delta^i_j\bar{\epsilon}^l\gamma_a\epsilon_l\right)\,, \tag{5.27}$$

one obtains the validity of the $(0,4)$ component of (5.17).

Considering the complete antifield dependent extended current $\tilde{J}$,[28] one can couple the $E_{7(7)}$ current to the action in a way fully consistent with BRST invariance. Indeed, considering sources $B_{(p,1-p)}$ for each component of the current $\tilde{J}$, one obtains that

$$\Sigma[B]=\Sigma+\int\tilde{B}_{\wedge}\tilde{J}\,, \tag{5.28}$$

where we use the Berezin notation

$$\int\mathrm{Tr}\;\tilde{B}_{\wedge}\tilde{J}=\int\mathrm{Tr}\;\left(B_{(0,1)}J_{(4,-1)}+B_{(1,0)\,\wedge}J_{(3,0)}+B_{(2,-1)\,\wedge}J_{(2,1)}+B_{(3,-2)\,\wedge}J_{(1,2)}+B_{(4,-3)}J_{(0,3)}\right), \tag{5.29}$$

satisfies the master equation

$$(\Sigma,\Sigma)_{\ddagger}-\int d\tilde{B}_{\wedge}\frac{\delta\Sigma}{\delta\tilde{B}}=0\ . \tag{5.30}$$

[28]We have computed the complete ξ^μ dependent part of $\tilde{J}$ including the antifields to check that the non-manifest Lorentz invariance does not give rise to extra difficulties. Nevertheless, its exhibition would not shed much light in this discussion. However we have not computed explicitly the ϵ^i dependent terms that would involve the quadratic terms in the antifields of the solution Σ of the master equation.

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This formal notation means that

$$(d+s)\tilde{B} = 0 \ . \tag{5.31}$$

This would be enough for insertions of one single current in a BRST invariant way, but consistency with $E_{7(7)}$ will require the consideration of higher order terms in $\tilde{B}$ in Σ, such that these equations are then only valid up to quadratic terms in the sources $B_{(p,1-p)}$.

Introducing a source for the $E_{7(7)}$ current, the rigid $\mathfrak{e}_{7(7)}$ Ward identity is promoted to a local $\mathfrak{e}_{7(7)}$ Ward identity expressing the conservation of the $E_{7(7)}$ current, such that

$$\delta^{\mathfrak{e}_{7(7)}} \tilde{B} = -dC - \{\tilde{B}, C\} \ . \tag{5.32}$$

All the components of $\tilde{B}$ thus transform in the adjoint representation, and $B_{(1,0)}$ transforms as an $\mathfrak{e}_{7(7)}$ gauge field. In order for the current Ward identity to be satisfied, each derivative in the action must be replaced by an $\mathfrak{e}_{7(7)}$ covariant derivative with respect to the gauge field $B_{(1,0)}$. It follows that the linear component is defined as $\int \mathrm{Tr}\, B_{(1,0)} {}_\wedge J_{(3,0)}$, by definition of the Noether current. The kinetic terms of the scalar fields, the Maxwell fields, their ghosts, and the supersymmetry ghost being quadratic in derivatives, they give rise to bilinear terms in $B_{(1,0)}$ in the action. The compatibility with BRST invariance therefore requires to also add quadratic terms in the other sources defining $\tilde{B}$.

In order to ensure that $\delta^{\mathfrak{g}}$ anticommutes with s, one must then define the BRST transformation of $\tilde{B}$ such that

$$(d+s)\tilde{B} + \tilde{B}^2 = 0 \ . \tag{5.33}$$

In this way one has the consistent 'very extended' Russian formula

$$(d+s+\delta^{\mathfrak{g}})\left(\tilde{B}+C\right) + \left(\tilde{B}+C\right)^2 = 0 \, , \tag{5.34}$$

and

$$sB_{(0,1)} = -{B_{(0,1)}}^2 \qquad sB_{(1,0)} = -dB_{(0,1)} - [B_{(1,0)}, B_{(0,1)}] \ . \tag{5.35}$$

The master equation for the completed $\Sigma[\tilde{B}]$ (including quadratic couplings in $\tilde{B}$ is therefore

$$(\Sigma, \Sigma)_\ddagger - \int (d\tilde{B} + \tilde{B}^2)_\wedge \cdot \frac{\delta \Sigma}{\delta \tilde{B}} = 0 \, , \tag{5.36}$$

It is straightforward to compute the solution $\Sigma[\tilde{B}]$ for a non-linear sigma model coupled to gravity, but the derivation of the complete solution in the case of $\mathcal{N} = 8$ supergravity is beyond the scope of this paper. Nevertheless, one can say that this solution can be written as

$$\Sigma[\tilde{B}] = \frac{1}{\kappa^2} S[\varphi, \tilde{B}] - \int d^4x \left(\sum_{\mathfrak{a}} (-1)^{\mathfrak{a}} \varphi^\ddagger_{\mathfrak{a}} s_{\tilde{B}} \varphi^{\mathfrak{a}} + \frac{\kappa^2}{2} \sum_{\mathfrak{a}\mathfrak{b}} \varphi^\ddagger_{\mathfrak{a}} K^{\mathfrak{a}\mathfrak{b}}(\varphi) \varphi^\ddagger_{\mathfrak{b}} \right) \, , \tag{5.37}$$

such that $s_{\tilde{B}}$ defines a differential operator which is nilpotent modulo the equations of motion of $S[\varphi, \tilde{B}]$ satisfying

$$s_{\tilde{B}} S[\varphi, \tilde{B}] = 0 \ , \tag{5.38}$$

and which anti-commutes with $\delta^{\mathfrak{g}}(C)$ for a x dependent parameter C.[29] We emphasise that this is not equivalent to gauging the theory with respect to a local $E_{7(7)}$ symmetry, because the components of $\tilde{B}$ are classical sources and do not constitute part of a supermultiplet in the conventional sense.

In order to arrive at a consistent definition of the BRST master equation (5.4) and the $\mathfrak{e}_{7(7)}$ master equation (4.14), one has to introduce sources $\varphi^{\mathfrak{g}}_a$ for the non-linear symmetry, sources (or antifields) for the BRST transformations, as well as sources $\varphi^{\ddagger\mathfrak{g}}_a$ for the non-linear transformations of the BRST transformations [45], which all transform with respect to $E_{7(7)}$ in the representation conjugate to the one of the corresponding fields. Given the $E_{7(7)}$ invariant solution (5.37) to the BRST master equation, one computes that the complete action[30]

$$\begin{aligned}\Sigma = \frac{1}{\kappa^2} S[\varphi,\tilde{B}] - \int d^4x \sum_a (-1)^a \Big(\varphi^\ddagger_a s_{\tilde{B}} \varphi^a + \varphi^{\mathfrak{g}}_a \delta^{\mathfrak{p}}(C_{\mathfrak{p}}) \varphi^a + \varphi^{\ddagger\mathfrak{g}}_a \delta^{\mathfrak{p}}(C_{\mathfrak{p}}) s_{\tilde{B}} \varphi^a \Big) \\ - \frac{\kappa^2}{2} \int d^4x \sum_{a\mathfrak{b}} \Big(\varphi^\ddagger_a - (-1)^a \delta^{\mathfrak{p}}(C_{\mathfrak{p}}) \varphi^{*\mathfrak{g}}_a \Big) K^{a\mathfrak{b}}(\varphi) \Big(\varphi^\ddagger_a - (-1)^{\mathfrak{b}} \delta^{\mathfrak{p}}(C_{\mathfrak{p}}) \varphi^{\ddagger\mathfrak{g}}_{\mathfrak{b}} \Big) \ , \quad (5.39)\end{aligned}$$

yields a consistent solution of the BRST master equation

$$\int d^4x \sum_a \left(\frac{\delta^R \Sigma}{\delta \varphi^\ddagger_a} \frac{\delta^L \Sigma}{\delta \varphi^a} - (-1)^a \varphi^{\mathfrak{g}}_a \frac{\delta^L \Sigma}{\delta \varphi^{\ddagger\mathfrak{g}}_a} \right) - \int (d\tilde{B} + \tilde{B}^2)_\wedge \cdot \frac{\delta \Sigma}{\delta \tilde{B}} = 0 \,, \quad (5.40)$$

the linear $\mathfrak{su}(8)$ Ward identity

$$\begin{aligned}\int d^4x \sum_a \left(\delta^{\mathfrak{k}}(C_{\mathfrak{k}}) \varphi^a \frac{\delta^L \Sigma}{\delta \varphi^a} + \delta^{\mathfrak{k}}(C_{\mathfrak{k}}) \varphi^{\mathfrak{g}}_a \frac{\delta^L \Sigma}{\delta \varphi^{\mathfrak{g}}_a} + \delta^{\mathfrak{k}}(C_{\mathfrak{k}}) \varphi^\ddagger_a \frac{\delta^L \Sigma}{\delta \varphi^\ddagger_a} + \delta^{\mathfrak{k}}(C_{\mathfrak{k}}) \varphi^{\ddagger\mathfrak{g}}_a \frac{\delta^L \Sigma}{\delta \varphi^{*\mathfrak{g}}_a} \right) \\ - \int \left(\Big(dC_{\mathfrak{k}} + \{\tilde{B}_{\mathfrak{k}}, C_{\mathfrak{k}}\} \Big)_\wedge \cdot \frac{\delta^L \Sigma}{\delta \tilde{B}_{\mathfrak{k}}} + \{C_{\mathfrak{k}}, \tilde{B}_{\mathfrak{p}}\}_\wedge \cdot \frac{\delta^L \Sigma}{\delta \tilde{B}_{\mathfrak{p}}} + \{C_{\mathfrak{k}}, C_{\mathfrak{p}}\} \cdot \frac{\delta^L \Sigma}{\delta C_{\mathfrak{p}}} \right) = 0 \,, \quad (5.41)\end{aligned}$$

and the $E_{7(7)}$ master equation

$$\begin{aligned}\int d^4x \sum_a \left(\frac{\delta^R \Sigma}{\delta \varphi^{\mathfrak{g}}_a} \frac{\delta^L \Sigma}{\delta \varphi^a} + (-1)^a \varphi^\ddagger_a \frac{\delta^L \Sigma}{\delta \varphi^{\ddagger\mathfrak{g}}_a} + (-1)^a \delta^{\mathfrak{k}}({C_{\mathfrak{p}}}^2) \varphi^{\ddagger\mathfrak{g}}_a \frac{\delta^L \Sigma}{\delta \varphi^\ddagger_a} - \varphi^{\mathfrak{g}}_a \delta^{\mathfrak{k}}({C_{\mathfrak{p}}}^2) \varphi^a \right) \\ - \int \left(\Big(dC_{\mathfrak{p}} + \{\tilde{B}_{\mathfrak{k}}, C_{\mathfrak{p}}\} \Big)_\wedge \cdot \frac{\delta^L \Sigma}{\delta \tilde{B}_{\mathfrak{p}}} + \{C_{\mathfrak{p}}, \tilde{B}_{\mathfrak{p}}\}_\wedge \cdot \frac{\delta^L \Sigma}{\delta \tilde{B}_{\mathfrak{k}}} \right) = 0 \ . \quad (5.42)\end{aligned}$$

According to the quantum action principle [14], these functional identities are satisfied by the n-loop 1PI generating functional Γ_n, modulo possible anomalies defined by local functionals $\mathcal{A}^{\mathfrak{g}}_n$ and $\mathcal{A}^\ddagger_n$. We have established in this paper that there is no non-trivial

[29]Whereas the BRST operator s anti-commutes with $\delta^{\mathfrak{g}}(C)$ only for constant parameter C.

[30]The only sources $\varphi^{\ddagger\mathfrak{g}}_a$ that are involved quadratically in the action are $\psi^{\ddagger\mathfrak{g}\,\mu}_i$, $\chi^{\ddagger\mathfrak{g}}_{ijk}$, $A^{\ddagger\mathfrak{g}\,\mathtt{i}}_m$, and $-\delta^{\mathfrak{p}}(C_{\mathfrak{p}})$ is defined as a linear $\mathfrak{e}_{7(7)}$ transformation on $A^{\ddagger\mathfrak{g}\,\mathtt{i}}_m$, and as an $\mathfrak{su}(8)$ transformation of parameter $\tanh(\Phi/2){*}C_{\mathfrak{p}}$ on $\psi^{\ddagger\mathfrak{g}\,\mu}_i$ and $\chi^{\ddagger\mathfrak{g}}_{ijk}$.

anomaly for the non-linear $E_{7(7)}$ master equation. It is commonly admitted (although no general proof exists to our knowledge) that there is no non-trivial anomaly to the BRST master equation in four dimensions (that is, diffeomorphisms and local supersymmetry are non-anomalous in four space-time dimensions). Once one has enforced the $E_{7(7)}$ master equation, the cohomology of the BRST operator of ghost number one associated to the possible anomalies to the BRST symmetry must be defined on the complex of $E_{7(7)}$ invariant functionals. Nevertheless, it rather obvious that the a BRST antecedent of an $E_{7(7)}$ invariant solution to the BRST Wess-Zumino consistency condition can always be chosen to be $E_{7(7)}$ invariant. We therefore conclude that there exists a renormalisation scheme such that these three functional identities are satisfied by the 1PI generating functional Γ to all orders in perturbation theory.

The Pauli-Villars regularisation employed in this paper breaks all these Ward identities, and so the determination of the non-invariant finite counterterms would require checking their validity in each order of perturbation theory. In principle, preserving $E_{7(7)}$ invariance requires testing the $\mathfrak{e}_{7(7)}$ Ward identities separately, and local supersymmetry will not be enough. As an example, the three-loop supersymmetry invariant starting as the square of the Bel-Robinson tensor does not preserve $E_{7(7)}$ invariance [22, 23]. Therefore, the supersymmetry master equation does not determine its coefficient in the bare action, independently of the property that there is no logarithmic divergence at this order, and one must use the $E_{7(7)}$ master equation to determine its value. $L = 3$ is therefore the first loop order at which a renormalisation prescription may fail to preserve $E_{7(7)}$ invariance. One would expect that the prescription used in [5, 6] to compute $\mathcal{N} = 8$ on-shell amplitudes should satisfy the $\mathfrak{e}_{7(7)}$ Slavnov-Taylor identities, but this needs to be checked.

The BRST master equation and the $E_{7(7)}$ master equation are more constraining than the requirement of local supersymmetry and rigid $E_{7(7)}$ invariance. For this reason it would be interesting to see if the prospective divergent counterterms at 7 and 8 loop could possibly be ruled out by these master equations.

5.3 Energy Coulomb divergences

There is still one subtlety concerning the Coulomb gauge which we have not yet addressed. It is well known that the Coulomb gauge in non-abelian gauge theories gives rise to energy divergences which are not easily dealt with in the renormalisation program [46, 47]. Because the ghost 'kinetic' term does not involve a time derivative, any ghost loop contribution is the energy integral of a function independent of the energy k_0, which diverges linearly. However, in the flat Coulomb gauge we use the ghost field c^m only appear in its free 'kinetic' term

$$-\bar{c}_m \partial_{\mathtt{i}} \partial_{\mathtt{i}} c^m \ . \tag{5.43}$$

Therefore, although the antighost $\bar{c}_m$ couples to the other fields via the diffeomorphism ghosts ξ^μ and the supersymmetry ghosts ϵ^i, and so 'ghost particles' can decay, they cannot be created, and there is no closed loops involving the ghost c^m. It follows that the Coulomb energy divergences do not appear in the loop corrections to amplitudes. It is in fact very important that the Coulomb gauge we use is field independent for this property to be true. For instance, a metric dependent gauge such as $\partial_{\mathtt{i}}(\sqrt{h}h^{\mathtt{ij}}A_{\mathtt{j}})$ would give rise to the ghost Lagrangian

$$-\bar{c}_m\partial_{\mathtt{i}}\left(\sqrt{h}h^{\mathtt{ij}}\partial_{\mathtt{j}}c^m\right)\,, \tag{5.44}$$

whence perturbation theory would involve energy divergences through the couplings to the metric. Although BRST invariance in principle guarantees that these energy divergences should cancel with the energy divergences involving vector fields, the compensating process might be difficult to exhibit.

Even within the 'free Coulomb gauge', the energy divergences do not disappear when one considers insertions of non-gauge-invariant composite operators, and in particular when one considers insertions of the $E_{7(7)}$ current, since the latter couples to the ghosts in a way very similar as in non-abelian gauge theory, in such a way that (5.43) is replaced by

$$-\bar{c}_m D_{\mathtt{i}}D_{\mathtt{i}}c^m \tag{5.45}$$

with the $E_{7(7)}$ covariant derivative $D_{\mathtt{i}}c^m \equiv \partial_{\mathtt{i}}c^m + B_{\mathtt{i}}^m{}_n c^n$. For all (and only for) the correlation functions of N $E_{7(7)}$ currents, there is one one-loop diagram associated to a 'ghost particle' interacting with each of the currents for each ordering of the currents, which gives an integral of the form

$$\begin{aligned}\Big\langle\prod_{a=1}^N J^{\mathtt{i}_a}(X_a,p_a)\Big\rangle_{\text{ghost}} &= -2\sum_\varsigma \operatorname{Tr}\left(\prod_{\varsigma(a)=1}^N X_{\varsigma(a)}\right)\times\\ &\int\frac{d^4k}{(2\pi)^4}\frac{\left(2k^{\mathtt{i}_N}-\sum_{c=1}^{N-1}p_c^{\mathtt{i}_N}\right)\prod_{a=1}^{N-1}\left(2\left(k^{\mathtt{i}_a}+\sum_{b=1}^{a-1}p_b^{\mathtt{i}_a}\right)+p_a^{\mathtt{i}_a}\right)}{\prod_{a=1}^N\left(k+\sum_{b=1}^{a-1}p_b\right)^2}+\text{C.T.}\,,\end{aligned} \tag{5.46}$$

where the sum over ς is the sum over non-cyclic permutations, (i.e. the permutations identified modulo cyclic ones), and C.T. correspond to the diagrams involving contact terms.

The contributions of the vector fields to such insertion is given at one-loop by

$$\begin{aligned}\Big\langle\prod_{a=1}^N J^{\mathtt{i}_a}(X_a,p_a)\Big\rangle_{\text{vec}} &= (-i)^N\sum_\varsigma \operatorname{Tr}\prod_{a=1}^N X_{\varsigma(a)}\int\frac{d^4k}{(2\pi)^4}\prod_{b=1}^N\Big(\Delta(k_{\varsigma,b})\Upsilon^{\mathtt{i}_{\varsigma(b)}}(k_{\varsigma,b},k_{\varsigma,b}+p_{\varsigma(b)})\Big)\\ &\quad+\text{C.T.},\end{aligned} \tag{5.47}$$

where $k_{\varsigma,a}=k+\sum_{\varsigma(b)=1}^{\varsigma(a)-1}p_{\varsigma(b)}$ and the sum over ς is the sum over non-cyclic permutations. The leading order in $\mathtt{k}_0$ in the limit ${k_0}^2\rightarrow+\infty$ of the product

$$\begin{aligned}&\Delta(k)\Upsilon^{\mathtt{k}}(k,k+p) =\\ &\frac{1}{k^2}\begin{pmatrix} i\delta^m_n\left(\delta^{\mathtt{j}}_{\mathtt{i}}k^{\mathtt{k}}-\delta^{\mathtt{k}}_{\mathtt{i}}k^{\mathtt{j}}+k_{\mathtt{i}}\delta^{\mathtt{kj}}\right)+\mathcal{O}({k_0}^{-1}) & \Omega^{mn}\varepsilon^{\mathtt{k}}{}_{\mathtt{il}}k^{\mathtt{l}}{k_0}^{-1}+\mathcal{O}({k_0}^{-2})\\ \Omega_{mn}\varepsilon^{\mathtt{jkl}}k_{\mathtt{l}}k_0+\mathcal{O}(1) & ik^{\mathtt{k}}\delta^m_n\end{pmatrix}\end{aligned} \tag{5.48}$$

is such that

$$\left\langle \prod_{a=1}^{N} J^{\mathtt{i}_a}(X_a,p_a)\right\rangle_{\rm vec} = \sum_{\varsigma} \operatorname{tr} \prod_{a=1}^{N} X_{\varsigma(a)} \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left(\prod_{b=1}^{N} K^{\mathtt{i}_{\varsigma(b)}}(k_{\varsigma,b}) + \mathcal{O}({k_0}^{-1})\right) , \qquad (5.49)$$

with

$$K^{\mathtt{k}}(k) = \frac{1}{k^2}\begin{pmatrix} \delta^{\mathtt{j}}_{\mathtt{i}} k^{\mathtt{k}} - \delta^{\mathtt{k}}_{\mathtt{i}} k^{\mathtt{j}} + k_{\mathtt{i}}\delta^{\mathtt{kj}} & \varepsilon^{\mathtt{k}}{}_{\mathtt{il}} k^{\mathtt{l}} \\ \varepsilon^{\mathtt{jkl}} k_{\mathtt{l}} & k^{\mathtt{k}} \end{pmatrix} , \qquad (5.50)$$

where we used the property that the trace is invariant with respect to inverse rescalings of the two off-diagonal components,[31] and the property that the contact terms are subleading in k_0 because

$$\Delta(k)\mathcal{R}^{\mathtt{ij}} = \begin{pmatrix} \mathcal{O}({k_0}^{-1}) & 0 \\ \mathcal{O}(1) & 0 \end{pmatrix} . \qquad (5.51)$$

We observe that this matrix can be written

$$K^{\mathtt{k}}(k) = \frac{k_{\mathtt{i}}\sigma^{\mathtt{i}}}{k^2}\sigma^{\mathtt{k}} , \qquad (5.52)$$

where the $\sigma^{\mathtt{i}}$ are the 4×4 pure imaginary Pauli matrices,

$$\sigma^{\mathtt{k}} \equiv i\begin{pmatrix} \varepsilon_{\mathtt{i}}{}^{\mathtt{jk}} & \delta^{\mathtt{k}}_{\mathtt{i}} \\ -\delta^{\mathtt{kj}} & 0 \end{pmatrix} , \qquad (5.53)$$

satisfying

$$\sigma^{\mathtt{i}}\sigma^{\mathtt{j}} = \delta^{\mathtt{ij}} - i\varepsilon^{\mathtt{ijk}}\sigma^{\mathtt{k}} . \qquad (5.54)$$

Rewriting the 'leading' vector field contribution to the N $\mathfrak{su}(8)$ currents insertion in this way,

$$\left\langle \prod_{a=1}^{N} J^{\mathtt{i}_a}(X_a,p_a)\right\rangle_{\rm vec} = \int \frac{dk_0}{2\pi} \sum_{\varsigma} \operatorname{tr} \prod_{a=1}^{N} X_{\varsigma(a)} \int \frac{d^3k}{(2\pi)^3} \operatorname{tr}\left(\prod_{b=1}^{N} \frac{1}{\not{k}_{\varsigma,b}}\sigma^{\mathtt{i}_{\varsigma(b)}} + \mathcal{O}({k_0}^{-1})\right) , \qquad (5.55)$$

one recognises that the integrand

$$\sum_{\varsigma} \operatorname{tr} \prod_{a=1}^{N} X_{\varsigma(a)} \int \frac{d^3k}{(2\pi)^3} \operatorname{tr} \prod_{b=1}^{N} \frac{1}{\not{k}_{\varsigma,b}}\sigma^{\mathtt{i}_{\varsigma(b)}} , \qquad (5.56)$$

is the one-loop N $\mathfrak{su}(8)$-currents insertion in a three-dimensional theory of free bosonic spinor fields.

[31] This can easily be proved using a similarity transformation of the form $K \rightarrow S^{-1}KS$ with

$$S = \begin{pmatrix} \delta^{\mathtt{j}}_{\mathtt{i}} {k_0}^{\frac{1}{2}} & 0 \\ 0 & {k_0}^{-\frac{1}{2}} \end{pmatrix}$$

It follows that the contribution to the N $\mathfrak{su}(8)$-current insertions responsible for energy divergences can be computed in an Euclidean three-dimensional effective theory, with 56 doublets of anti-commuting scalar fields $\bar{c}_m$, c^m and 56 Dirac spinor fields λ^m, understood as SU(2) $\mathbf{2} \oplus \bar{\mathbf{2}}$ real spinors with

$$\bar{\lambda}_m = \lambda^{n\,T} G_{nm} \,, \tag{5.57}$$

coupled to an external $\mathfrak{su}(8)$-current as

$$S^{\rm 3D} = \int d^3x \left(\frac{1}{2} \bar{\lambda}_m \not{D} \lambda^m - \bar{c}_m D_{\tt i} D^{\tt i} c^m \right) \,. \tag{5.58}$$

The corresponding contributions to the N $\mathfrak{su}(8)$-currents insertions are

$$\exp(-\Gamma[B]) = \frac{{\rm Det}[D_{\tt i} D^{\tt i}]}{{\rm Det}[\not{D}]^{\frac{1}{2}}} \,. \tag{5.59}$$

and therefore do not vanish. Nevertheless, they can be compensated by the contribution of a trivial free-theory. Consider the fermionic fields $\theta^m_{\tt i}$, $\bar{\theta}_m$ and the bosonic fields L^m, $\bar{L}_m$, with BRST transformations

$$s\theta^m_{\tt i} = \partial_{\tt i} L^m \,, \qquad sL^m = 0 \,, \qquad s\bar{L}_m = \bar{\theta}_m \,, \qquad s\bar{\theta}_m = 0 \,. \tag{5.60}$$

The BRST invariant Lagrangian

$$\frac{1}{2}\Omega_{mn}\varepsilon^{\tt ijk}\theta^m_{\tt i}\partial_{\tt j}\theta^n_{\tt k} + s\left(\bar{L}_m\partial_{\tt i}\theta^m_{\tt i}\right) = \frac{1}{2}\Omega_{mn}\varepsilon^{\tt ijk}\theta^m_{\tt i}\partial_{\tt j}\theta^n_{\tt k} + \bar{\theta}_m\partial_{\tt i}\theta^m_{\tt i} + \bar{L}_m\partial_{\tt i}\partial_{\tt i}L^m \,, \tag{5.61}$$

is a fermionic equivalent of the abelian Chern-Simons Lagrangian. The coupling of this theory to the current gives rise to a contribution to the N $\mathfrak{su}(8)$-current insertions which cancels the ratio of determinants (5.59). One can therefore disregard the energy divergences without affecting the BRST symmetry, although the extended current (5.19) is modified by a non-trivial BRST cocycle

$$\tilde{J}(\Lambda)_{\rm C} \approx \frac{1}{2} dt_{\wedge} \left(dx^{\tt i}\theta^m_{\tt i} + L^m\right)_{\wedge}\left(dx^{\tt j}\theta^n_{\tt j} + L^n\right)\Omega_{np}\Lambda_m{}^p \,. \tag{5.62}$$

Nevertheless, this term vanishes when the equations of motion are imposed with the appropriate boundary conditions,

$$\partial_{[\tt i}\theta^m_{\tt j]} = 0 \,, \quad \partial_{\tt i}\theta^m_{\tt i} = 0 \quad \Rightarrow \quad \theta^m_{\tt i} = 0 \,. \tag{5.63}$$

This contribution to the energy divergences is reproduced by the Pauli-Villars fields, within the prescription for the vector fields defined in section 3.2, and the prescription for the ghosts that their Pauli-Villars Lagrangian is mass-independent. For the ghosts, this implies that their contribution is entirely eliminated by their Pauli-Villars 'partners', and one simply omits them at one-loop. This prescription is rather natural, since it preserves the BRST symmetry associated to the abelian gauge invariance of the Pauli-Villars vector fields (the mass term in (3.31) being $M\Gamma_{mn}\varepsilon^{\mathtt{ijk}}A^m_{\mathtt{i}}F^n_{\mathtt{jk}}$). The leading k_0 independent integrand in (5.47) is mass-independent for the Pauli-Villars vector field Feynman rules as well, and that is why their contribution cancel precisely the vector fields energy Coulomb divergences.

By property of the Pauli-Villars regularisation, the regularised divergences in M can be computed by expending the integrant in powers of the external momenta (since $p^2 \ll M^2$ and $p_0^2 \ll M^2$), and no non-local divergent contribution can be produced. The energy divergences are therefore consistently eliminated within the Pauli-Villars regularisation. One computes indeed that the divergent contribution to the regularised two-points function is

$$\Bigl\langle J^{\mathtt{i}}(X_1,p)J^{\mathtt{j}}(X_2,-p)\Bigr\rangle_{\text{vec+PV}} \sim \frac{i}{48\pi^2}\mathrm{Tr}\ (X_1X_2)\left(aM^2 - \left(\delta^{\mathtt{ij}}\left(p^2-p_0^2\right) - p^{\mathtt{i}}p^{\mathtt{j}}\right)\ln M\right), \tag{5.64}$$

similarly as for the Dirac fermion contribution. In particular, we see that the Coulomb energy divergence

$$\Bigl\langle J^{\mathtt{i}}(X_1,p)J^{\mathtt{j}}(X_2,-p)\Bigr\rangle_{\text{ghost}+\lambda\bar\lambda} = \int\frac{dk_0}{4\pi}\ \mathrm{Tr}\ (X_1X_2)\ \frac{1}{|p|}\left(\delta^{\mathtt{ij}}p^2 - p^{\mathtt{i}}p^{\mathtt{j}}\right)\ , \tag{5.65}$$

does not require a 'catastrophic' non-local renormalisation

$$\propto \int\frac{d^4p}{(2\pi)^4}\frac{M}{|p|}\left(\delta^{\mathtt{ij}}p^2 - p^{\mathtt{i}}p^{\mathtt{j}}\right)\mathrm{Tr}\ B_{\mathtt{i}}(p)B_{\mathtt{j}}(-p)\ , \tag{5.66}$$

within the prescription. The coefficient a depends on the axial / vector character of the elements X_1 and X_2, and is not unambiguously determined within the prescription, because it diverges logarithmically in the UV (i.e. at $\alpha \rightarrow 0$)

$$\begin{aligned} a_{\text{A}} &= \int_0^\infty d\alpha\left(\frac{5}{3}M^{-2}\alpha^{-2}\left(e^{-\alpha M^2}-1\right) + \left(3\alpha^{-1}+2M^2\right)e^{-\alpha M^2}\right) \\ a_{\text{V}} &= \int_0^\infty d\alpha\left(\frac{5}{3}M^{-2}\alpha^{-2}\left(e^{-\alpha M^2}-1\right) + \frac{1}{3}\alpha^{-1}e^{-\alpha M^2}\right)\ . \end{aligned} \tag{5.67}$$

This difficulty is not associated to the Coulomb divergences, but to the general property that the Pauli-Villars regularisation does not permit to regularise divergences behaving like $\sim M^2\ln M$. For example, the same problem appears in the Dirac fermion contribution to the two-point function when X_1 and X_2 are axial. These divergences are irrelevant anyway, since they do not affect the renormalised correlation functions at higher orders.

6 Conclusions

We have exhibited in this paper the consistency of the duality invariant formulation of $\mathcal{N} = 8$ supergravity in perturbation theory. The non-standard non-manifestly Lorentz invariant Feynman rules turn out to satisfy the quantum action principle, and diffeomorphism invariance can therefore be maintained through appropriate renormalisations. The theory can be gauge-fixed within the Batalin-Vilkovisky formalism, and although the abelian ghosts exhibit Coulomb energy divergences in insertions of the $E_{7(7)}$ current, these divergences are consistently removed within the Pauli-Villars regularisation.

Furthermore, we have solved the Wess-Zumino consistency conditions for the anomaly associated to the non-linear $\mathfrak{e}_{7(7)}$-current Ward idendities, and shown that these solutions are uniquely determined in terms of the corresponding solutions to the Wess-Zumino consistency condition associated to the linear $\mathfrak{su}(8)$-current Ward identity. It follows that any non-linear $E_{7(7)}$ anomaly in perturbation theory is entirely determined by the one-loop coefficient of the linear $\mathfrak{su}(8)$ anomaly. In particular, we have explicitly computed the one-loop contribution of the vector fields to the anomaly, establishing the validity of the family's index prediction, and therefore the vanishing of the anomaly at one-loop.

The main result of the paper is that the non-linear Slavnov-Taylor identities associated to the $\mathfrak{e}_{7(7)}$ Ward identities are maintained at all orders in perturbation theory, if one renormalises the theory appropriately. Although we proved this theorem within the symmetric gauge, it remains in principle valid within the SU(8) gauge invariant formulation [16].

What are the implications of the non-linear $E_{7(7)}$ symmetry for possible logarithmic divergences of the theory? Regarding the definition of BPS supersymmetric invariants which cannot be written as full superspace integrals (but as integrals over subspaces of superspace classified by their BPS degree), the linear approximation suggests that they cannot be duality invariant. Indeed, the BPS invariants are defined in the linearised approximation as partial superspace integrals of functions of the scalar superfield $W_{ijkl}(x,\theta) = \phi_{ijkl} + \mathcal{O}(\theta)$, but there is no $E_{7(7)}$ invariant function that can be built out of these scalar fields in any SU(8) representation. It is therefore hard to see how such supersymmetric invariants (i.e. the supersymmetrisations of the Bel-Robinson square R^4, $\partial^4 R^4$ and $\partial^6 R^4$) could be made invariant under the full non-linear duality symmetry. Nevertheless, this argument may not be entirely 'watertight', as a similar argument appears to fail in higher dimensions, where, however, the duality groups are non-exceptional. For instance, the logarithmic divergences found in dimensions ≥ 6 imply that there must exist an SO$(5,5)$ invariant 1/8 BPS counterterm in six dimensions, an SL$(5,\mathbb{R})$ invariant 1/4 BPS counterterm in seven dimensions, and an SL$(2,\mathbb{R})\times$SL$(3,\mathbb{R})$ invariant 1/2 BPS counterterm in eight dimensions. Nevertheless, [22, 23] exhibited that the 1/2 BPS invariant is not $E_{7(7)}$ invariant, which implies that the absence of logarithmic divergence at 3-loop is a consequence of the $\mathfrak{e}_{7(7)}$ Ward identities.

The duality invariance may therefore entail various non-renormalisation theorems, which might explain the absence of logarithmic divergences in maximal supergravity in five dimensions at four loops [6], and in maximal supergravity in four dimensions at three, five and six loops. A similar argument would lead to the conclusion that $\mathcal{N} = 6$ supergravity admits its first logarithmic divergence at five loops, and $\mathcal{N} = 5$ supergravity at four loops. However, establishing such non-renormalisation theorems will require further investigation of BPS invariants in supergravity.

As another application, the $\mathfrak{e}_{7(7)}$ Slavnov-Taylor identities such as (4.16) may imply special identities among the on-shell amplitudes in the 'multi-soft-momenta limit', generalising the ones derived in [48] at all orders in perturbation theory.

As shown by several examples (see e.g. [7]), the study of supersymmetric counterterms is not enough to reach definite conclusions regarding the appearance of certain logarithmic divergences in supersymmetric theories. The non-linear $\mathfrak{e}_{7(7)}$ Ward identities may therefore imply more stringent restrictions than one would deduce from the existence of $E_{7(7)}$ invariant supersymmetric counterterms.

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