

Constructing Carrollian field theories from null reduction

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ABSTRACT: In this paper, we propose a novel way to construct off-shell actions of d -dimensional Carrollian field theories by considering the null-reduction of the Bargmann invariant actions in $d+1$ dimensions. This is based on the fact that d -dimensional Carrollian symmetry is the restriction of the $(d+1)$ -dimensional Bargmann symmetry to a null hypersurface. We focus on free scalar field theory and electromagnetic field theory, and show that the electric sectors and the magnetic sectors of these theories originate from different Bargmann invariant actions in one higher dimension. In the cases of massless free scalar field and $d=4$ electromagnetic field, we verify the Carrollian conformal invariance of the resulting theories, and find that there appear naturally chain representations and staggered modules of Carrollian conformal algebra.

KEYWORDS: Field Theories in Higher Dimensions, Scale and Conformal Symmetries, Space-Time Symmetries

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1 Introduction

The Carrollian symmetry group was first found by Lévy-Leblond in 1965 [1] (and independently by Sen Gupta [2]) by studying the ultra-relativistic ($c \rightarrow 0$) contraction of the Poincaré group. Later on, the Carroll group was discovered [3] to be one of possible kinematical groups, which means that it could be the spacetime symmetry of a nonrelativistic manifold, the Carrollian manifold. The Carroll group is generated by the Carrollian boost

$$\vec{x}' = \vec{x}, \quad t' = t - \vec{b} \cdot \vec{x}, \quad (1.1)$$

the translations, and the rotations among spacial directions. In the Carrollian limit, the lightcones collapse, and there appears the notion of absolute space. Consequently, the motion of a free Carrollian particle is trivial: it runs without moving [1, 4]. For a massless Carrollian particle, it has infinite dimensional Carrollian conformal symmetry [5]. However, for interacting Carrollian particles, there could have nontrivial dynamics.

Since its discovery, the Carrollian symmetry has been found in various physical systems. The Carrollian boost was discovered in the isometry of plane-gravitational wave [6, 7]. The Carrollian limit was found to control the dynamics of the gravitational field near a spacelike singularity [8, 9]. The Carrollian symmetry appears in the near horizon of black hole as well [10–13]. Recently, it has been applied to study various physical problems in cosmology [14] and condensed matter physics [15, 16]. More importantly, the Carrollian conformal symmetry [17], which is isomorphic to the BMS group [18–21], is essential in the study of $3d$ flat space holography [22–24] and celestial holography [25–27].

The Carrollian invariant field theories can be constructed by taking the ultra-relativistic contraction of a Lorentz invariant field theories [4, 28–33]. One approach is to directly take the limit on the equations of motion. Notably, there are two distinct Carroll contractions, resulting in two different Carrollian field theories, conventionally named as electric sector and magnetic sector respectively [4]. The constructed theories are manifestly on-shell Carrollian invariant, but their off-shell actions need to be constructed separately. Typically, the construction of the electric sector is relatively straightforward, while the construction of the magnetic sector is found to be more challenging [30]. Another way to construct the Carrollian invariant field theories by contraction is based on the Hamiltonian action principle [28]. Even though this approach can yield Carrollian invariant action automatically, after gauging away the extra field and accordingly modifying the transformation rules, it lacks manifest spacetime covariance.¹

In this paper, we propose a novel method to construct off-shell actions of d -dimensional Carrollian field theories by performing null reduction on the actions of $(d+1)$ -dimensional Bargmann theories. The logic behind our construction is similar to the one of the null reduction technique employed in the Galilean case [36]. The key point is that the Carrollian symmetry forms a subgroup of the Bargmann group. Specifically, by disregarding the translation along a null direction, the Bargmann group reduces to the Carroll group. Thus, if we begin with Bargmann-invariant theories and carry out reductions along the null

¹Some examples of Carrollian diffeomorphism invariant theory on general Carrollian manifolds are discussed in [34, 35].

direction, we will obtain the theories that are guaranteed to be Carrollian invariant. Our focus in this work is primarily on the free massless scalar theories and free electromagnetic field theories. Previous discussions in the literature, such as [4, 28], have addressed the existence of two distinct rescalings of the fields when taking the limit $c \rightarrow 0$, leading to the electric sectors and magnetic sectors of the theories. In our approach, these two sectors arise from two distinct Bargmann field theories in one higher dimension. The resulting electric sector is exactly the same as the one found in [28], while the resulting magnetic sector differs slightly from the one in [4, 28]. We demonstrate that our action is off-shell Carrollian invariant,² and subsequently we calculate the correlators using the path-integral formalism.

Another motivation for present study is to find Carrollian conformal invariant theories and study their properties. The Carrollian conformal field theory (CCFT) plays an important role in flat-space holography [22–24] and celestial holography [25–27]. In particular, higher dimensional ($d \geq 3$) CCFT presents some novel features [43]. Firstly, the representations of the higher dimensional Carrollian conformal algebra (CCA) are much more involved. There appear the multiplet structure and staggered modules in the highest-weight representations. Secondly the constraints from the Ward identities on the two-point correlators are less restrictive. It is important to construct concrete CCFTs and study their properties. As will be discussed in section 2.2, the d -dimensional Carrollian conformal group is not a subgroup of the $(d + 1)$ -dimensional Bargmann conformal group. This implies that the null reduction of a Bargmann conformal invariant theory does not automatically yield a CCFT. Therefore we need to check the Carrollian conformal invariance of null-reduced theories case by case. In this work, we demonstrate that the free Carrollian scalar theory and the Carrollian electromagnetic theory in $d = 4$ are really Carrollian conformal invariant, by checking the invariance of the actions and the Ward identities of 2-point correlators of the primary operators. We also discuss the representations of the fields, and find that the staggered modules appear naturally in these theories.

The remaining parts of this paper are organized as follows. In section 2, we give a brief review of the Carrollian symmetry, its conformal extension, and the representations of Carrollian conformal algebra. Then in section 3 we construct the electric sector and the magnetic sector of free Carrollian scalar theory from null reduction of the Bargmann scalar theory. We read the 2-point correlators of the fundamental fields from the path-integral and check the conformal symmetries in both sectors. In section 4, we investigate more subtle and nontrivial models, including the electromagnetic theory and free p -form field theories. For the electromagnetic theory, we discuss in detail the related boost multiplet structures and compute the 2-point correlators of the fundamental fields in both the electric sector and the magnetic sector by using the path-integrals in suitable gauges. In section 5, we discuss the further reduction of the 4d electromagnetic theories from quotient representations. We conclude with some discussions in section 6.

Some technical details are presented in the appendices. In appendix A, we briefly review the construction of staggered module, and discuss the possible staggered modules

²The on-shell reduction from Bargmann theories has been discussed in [4]. For recent studies on BMS invariant theories, refer to [37–42].

involving the scalars in CCFT. In appendix B, we collect the path-integral computations of the 2-point correlators of Carrollian field theories. In appendix C, after briefly reviewing the Ward identities of Carrollian conformal symmetry, we discuss carefully their restrictions on the 2-point correlators of the primary operators in various representations. Different from the discussions in [43], we pay more attention to the correlators with δ -function distribution, which appear in the field theories studied in this paper and in the celestial holography [25–27].

Conventions. In the present work we use the Greek alphabets α, β, \dots as the spacetime indices, and use the Latin alphabets i, j, \dots as spacial indices only. Moreover, we use α, β, γ to be the indices for the Bargmann spacetime, μ, ν, δ for the Carrollian spacetime, and u and v for the null directions.

2 Carrollian symmetry

In this section, we briefly review the Carrollian symmetry and its relation with Bargmann symmetry. Additionally, we introduce the fundamentals of Carrollian conformal symmetry and its representations, which have been thoroughly studied in [43].

2.1 Carrollian symmetry from Bargmann symmetry

In [44], it was shown that Newton-Cartan geometry is associated with the so-called Bargmann group. Then in [4], it was pointed out that the Galilean group and Carroll group and their related geometries could be unified in a relativistic Bargmann space. A Bargmann manifold has three ingredients (\mathcal{B}, G, ξ) : \mathcal{B} is a $(d+1)$ -dimensional manifold, G is a metric of Lorentz signature, and ξ is a nowhere vanishing null vector. In the flat case, it can be described in terms of the coordinates $x^\alpha = (u, \vec{x}, v)$, ($\alpha = 0, 1 \dots d$) as

$$\mathcal{B} = \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}, \quad G = 2dudv + \delta_{ij}dx^i dx^j, \quad \xi = \partial_u, \quad (2.1)$$

where both u, v are the lightcone coordinates and \vec{x} is a $(d-1)$ -vector. The Bargmann group is the isometry group of the flat Bargmann structure (2.1) that keeps the metric G and the null vector ξ invariant, and it is a subgroup of Poincaré group

$$\text{Barg}(d, 1) = \text{ISO}(d, 1) \setminus \{J_d^0, 1/\sqrt{2}(J_0^i - J_d^i)\}. \quad (2.2)$$

The Bargmann generators consist of P_α, J_j^i , and $B_i^{\mathcal{B}}$, where $B_i^{\mathcal{B}}$ are Bargmann boosts. Their realizations as vector fields on the spacetime are shown in table 1. The commutation relations of the generators are

$$\begin{aligned} [B_i^{\mathcal{B}}, P_v] &= -P_i, & [B_i^{\mathcal{B}}, P_j] &= \delta_{ij}P_u, & [B_i^{\mathcal{B}}, P_u] &= 0, \\ [J_j^i, J_l^k] &= \delta^{ik}J_{jl} - \delta_l^i J_j^k + \delta_{jl}J^{ik} - \delta_j^k J_l^i, & & & & (2.3) \\ [J_j^i, P_k] &= \delta_k^i P_j - \delta_{jk}P^i, & [J_j^i, B_k^{\mathcal{B}}] &= \delta_k^i B_j^{\mathcal{B}} - \delta_{jk}B^{\mathcal{B}i}, & \text{others} &= 0. \end{aligned}$$

Geometrically, the Carroll group can be viewed as a subgroup of the Bargmann group that preserves the $v = 0$ null hyper-surface. By restricting to the null hyper-surface $v = 0$, we see immediately that the flat Bargmann structure reduces to the flat

Generators	Vector fields	Finite transformations
P_α	$p_\alpha = \partial_\alpha$	$x^\alpha + x_0^\alpha$
J_j^i	$j_j^i = x^i \partial_j - x_j \partial^i$	$(u, \mathbf{J} \cdot \vec{x}, v)$
$B_i^{\mathcal{B}}$	$b_i^{\mathcal{B}} = v \partial_i - x_i \partial_u$	$(u - \vec{v} \cdot \vec{x} - \frac{1}{2} \vec{v}^2 v, \vec{x} + \vec{v} v, v)$

Table 1. Generators of Bargmann symmetry as vector fields on the spacetime.

Carrollian structure (\mathcal{C}, g, ξ) with the coordinates $x^\mu = (t = u, \vec{x})$, the degenerated metric $g_{\mu\nu} = G_{\mu\nu}|_{v=0} = \delta_\mu^i \delta_\nu^j \delta_{ij}$, and the timelike vector $\xi = \partial_t$. A Bargmann transformation naturally induces a transformations on the $v = 0$ null hyper-surface if it leaves the $v = 0$ null hyper-surface invariant.

The generators of the Carrollian algebra consist of P_μ, J_j^i, B_i , with the following commutation relations

$$\begin{aligned}
 [B_i, P_j] &= \delta_{ij} P_0, & [B_i, P_0] &= 0 \\
 [J_j^i, J_l^k] &= \delta^{ik} J_{jl} - \delta_l^i J_j^k + \delta_{jl} J^{ik} - \delta_j^k J_l^i, & (2.4) \\
 [J_j^i, P_k] &= \delta^i_k P_j - \delta_{jk} P^i, & [J_j^i, B_k] &= \delta^i_k B_j - \delta_{jk} B^i, \quad \text{others} = 0.
 \end{aligned}$$

2.2 Carrollian conformal symmetry

Besides being a subgroup of the Bargmann group, the Carroll group can be obtained from the ultra-relativistic ($c \rightarrow 0$) contraction of the Poincare group as well. Moreover, the Carrollian conformal symmetry naturally arises from the $c \rightarrow 0$ limit of relativistic conformal symmetry. The algebra of the Carrollian conformal group is generated by $\{P_\mu, J_j^i, B_i, D, K_\mu\}$, $\mu = 0, 1, \dots, d-1$, $i, j = 1, \dots, d-1$ with the commutation relations³

$$\begin{aligned}
 [D, P_\mu] &= P_\mu, & [D, K_\mu] &= -K_\mu, & [D, B_i] &= [D, J_j^i] = 0, \\
 [J_j^i, G_k] &= \delta^i_k G_j - \delta_{jk} G^i, & G &\in \{P, K, B\}, \\
 [J_j^i, P_0] &= [J_j^i, K_0] = 0, & & & & (2.5) \\
 [J_j^i, J_l^k] &= \delta^{ik} J_{jl} - \delta_l^i J_j^k + \delta_{jl} J^{ik} - \delta_j^k J_l^i, \\
 [B_i, P_j] &= \delta_{ij} P_0, & [B_i, K_j] &= \delta_{ij} K_0, & [B_i, B_j] &= [B_i, P_0] = [B_i, K_0] = 0, \\
 [K_0, P_0] &= 0, & [K_0, P_i] &= -2B_i, & [K_i, P_0] &= 2B_i, \quad [K_i, P_j] = 2\delta_j^i D + 2J_j^i.
 \end{aligned}$$

This algebra is isomorphic to the Poincaré algebra $\mathfrak{cca}_d \simeq \mathfrak{iso}(d, 1)$, though their homogeneous vector space realizations are different. The actions of the generators of the Carrollian conformal group on space-time point (t, \vec{x}) are shown in table 2.

As shown in the last subsection, the Carroll group appears as the restriction of the Bargmann group on a null hyper-surface of Bargmann manifold. However, this is not true for the conformal case. Recall that the (Lorentzian) conformal group is generated by the diffeomorphisms that transform the metric g^L as

$$a^* g^L = \Omega^2 g^L. \quad (2.6)$$

³The spacial indices are raised (lowered) by δ^{ij} (δ_{ij}).

Generators	Vector fields	Finite transformations
D	$d = t\partial_t + x^i\partial^i$	λx^μ
P_μ	$p_\mu = (\partial_t, \vec{\partial})$	$x^\mu + a^\mu$
K_μ	$k_\mu = (-\vec{x}^2\partial_t, 2\vec{x}x_\mu\partial^\mu - \vec{x}^2\vec{\partial})$	$(\frac{t-a^0\vec{x}^2}{1-2\vec{a}\cdot\vec{x}+\vec{a}^2\vec{x}^2}, \frac{\vec{x}-\vec{a}\vec{x}^2}{1-2\vec{a}\cdot\vec{x}+\vec{a}^2\vec{x}^2})$
B_i	$b_i = x_i\partial_t$	$(t + \vec{v} \cdot \vec{x}, \vec{x})$
J^i_j	$j^i_j = x^i\partial_j - x_j\partial^i$	$(t, \mathbf{J} \cdot \vec{x})$

Table 2. Generators of CCA as the vector fields on the space-time.

In the Bargmann case, there are two geometric notions, the metric G and the invariant null-vector ξ . One can similarly define the conformal transformations of order k as

$$a^*G = \Omega^2G, \quad a^*\xi = \Omega^{-2/k}\xi. \tag{2.7}$$

One may take $k = 2$ to keep the scaling in the ξ direction to be the same as the ones in other directions. It turns out that the conformal version of the Bargmann group is a subgroup of Lorentzian conformal group, keeping the null vector ξ invariant. Indeed, the group is generated by $\{P_\alpha, J^i_j, B_i, D\}$, where D is the dilation generator. But now the Lorentzian special conformal transformations (SCT) do not satisfy the conformal condition (2.7). In other words, the G -preserving and ξ -preserving subgroup of Lorentz conformal group consists of just the Bargmann transformations and a single dilation, but not of special conformal transformations.

However, things become different on the null hyper-surface $v = 0$. Since the metric in the Carrollian manifold is degenerate, the Killing equation $a^*g = \Omega^2g$ is less constrained. Actually the solutions to the Carrollian conformal Killing equations are

$$\begin{aligned} p_i &= \partial_i, & m^i_j &= x^i\partial_j - x_j\partial^i, \\ d &= x^\mu\partial_\mu, & k_i &= 2x_ix^\mu\partial_\mu - x^lx_l\partial_i, & m &= g(x^i)\partial_0, \end{aligned} \tag{2.8}$$

where m 's are the vector fields for infinite-dimensional extensions of the Carrollian conformal transformations and $g(x^i)$ is an arbitrary function of spacial coordinates. Especially, the global transformations in the m 's include the temporal translation, the boosts, and the temporally special conformal translation:

$$p_0 = \partial_0, \quad b_i = x_i\partial_0, \quad k_0 = -x^lx_l\partial_0. \tag{2.9}$$

Thus the (global) Carrollian conformal group⁴ is generated by $\{P_\mu, J^i_j, B_i, D, K_\mu\}$ with $K_0 = (K_{d+1}^L + K_0^L)/\sqrt{2}|_{v=0}$ and $K_i = K_i^L|_{v=0}$. We see that the Carrollian conformal group is not a subgroup of the Bargmann conformal group.

2.3 Representations of Carrollian conformal algebra

The representations of the higher dimensional Carrollian conformal algebra (CCA) are much more involved than the ones of its Lorentzian cousin. The so-called scale-spin representation

⁴In this work, the Carrollian conformal symmetry always means the global one.

was used to study the representation of homogeneous Carrollian conformal group [31]. However, this description is not precise enough to discuss the representations with a complicated boost multiplet structure. In [43], the authors discussed the highest-weight representations (HWR) of the Carrollian conformal group in detail. Here we outline some main results.

A local operator located at generic point is related to the one at the origin by the translation operators as

$$\mathcal{O}^a(x) = \exp\{ix^\mu P_\mu\} \mathcal{O}^a(0) \exp\{-ix^\mu P_\mu\}. \quad (2.10)$$

Hence, it suffices to consider the representation of local operators at the origin $\mathcal{O}^a = \mathcal{O}^a(0)$. The stabilizer algebra of CCA is generated by dilation D , Carrollian rotations $M = \{J, B\}$ and special conformal transformations (SCTs) K . The local operators \mathcal{O}^a can be diagonalized simultaneously into the eigenstates of the dilation and the representations of Carrollian rotations,

$$[D, \mathcal{O}] = \Delta_{\mathcal{O}} \mathcal{O}, \quad [M, \mathcal{O}^a] = \Sigma_b^a \mathcal{O}^b, \quad (2.11)$$

where Δ is the scaling dimension of the operator. The highest-weight operator, which is often called primary operator, in a given representation is defined as the operator with the lowest eigenvalue of dilation generator D , satisfying the primary condition

$$[K, \mathcal{O}] = 0. \quad (2.12)$$

An operator in a highest-weight representation is therefore characterized by the scaling dimension and its representation with respect to the Carrollian rotations.

Taken as an example, the scalar primary operator \mathcal{O}_p at the origin, being the scalar under the Carrollian rotations, has scaling dimension Δ and commutes with all other generators, including the K generators,

$$[D, \mathcal{O}_p] = \Delta \mathcal{O}_p, \quad [J_{ij}, \mathcal{O}_p] = [B_i, \mathcal{O}_p] = 0, \quad [K_\mu, \mathcal{O}_p] = 0. \quad (2.13)$$

The operators generated by acting one generator of P_μ on \mathcal{O}_p are descendants, and all have conformal dimension $(\Delta + 1)$,

$$P_\mu \mathcal{O}_p = [P_\mu, \mathcal{O}_p] = \partial_\mu \mathcal{O}_p, \quad [D, P_\mu \mathcal{O}_p] = (\Delta + 1) P_\mu \mathcal{O}_p. \quad (2.14)$$

Acting the operator K_μ on $P_\mu \mathcal{O}_p$ leads back to \mathcal{O}_p or zero:

$$[K_i, P_j \mathcal{O}_p] = 2\Delta \delta_{ij} \mathcal{O}_p, \quad [K_0, P_i \mathcal{O}_p] = [K_i, P_0 \mathcal{O}_p] = [K_0, P_0 \mathcal{O}_p] = 0. \quad (2.15)$$

Similar to usual CFT, there are higher-order descendant operators of \mathcal{O}_p by acting more momentum operators. The primary operator \mathcal{O}_p together with all its descendants is referred to as the conformal family of \mathcal{O}_p .

One typical feature in the representations of CCA is the staggered structure. The staggered module has emerged in the studies of 2D LogCFT [45–48], BMS free scalar [49] and BMS free fermions [50, 51]. Here we only give a simple example and leave the analysis

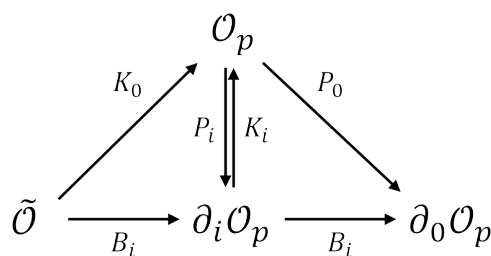


Figure 1. The conformal family of \mathcal{O}_p to the first order descendent level. This is a part of full staggered module (A.14).

of the staggered modules in higher dimensional CCFT to the appendix A. For the above scalar case, there could exist another scalar operator $\tilde{\mathcal{O}}$ with conformal dimension $\Delta + 1$ such that acting K_0 on it gives \mathcal{O}_p . More precisely there are

$$[K_0, \tilde{\mathcal{O}}] = 2\Delta\mathcal{O}_p, \quad [K_i, \tilde{\mathcal{O}}] = 0. \tag{2.16}$$

The relations among \mathcal{O}_p , $\partial_\mu\mathcal{O}_p$ and $\tilde{\mathcal{O}}$ are shown in figure 1. The operators $\{\mathcal{O}_p, \partial_\mu\mathcal{O}_p, \tilde{\mathcal{O}}\}$ form a staggered structure in which $\{\mathcal{O}_p, \partial_\mu\mathcal{O}_p\}$ form a submodule and $\tilde{\mathcal{O}}$ is a quotient. The full staggered module containing the higher-order descendants is shown in (A.14).

In order to study the other primary operators besides the scalar, we need to understand the representations of Carrollian rotations. Because the algebra of Carrollian rotations is not semi-simple, its finite dimensional representations are generically reducible but indecomposable, and are much more complicated than the ones of the usual Lorentzian rotations.

One nontrivial representation for $d = 4$ Carrollian rotation group, which will appear in the following study of this work, is given by primary operators $\mathcal{O}_\alpha = \{\mathcal{O}_v, \mathcal{O}_i, \mathcal{O}_0\}$ with $\alpha = v, 1, 2, 3, 0$, and $i = 1, 2, 3$. The operator \mathcal{O}_α corresponds to a vector representation in $d = 5$ Bargmann space. With respect to three-dimensional spacial rotations J_{ij} , the operators \mathcal{O}_v and \mathcal{O}_0 are scalars, \mathcal{O}_i is a vector, and they are related by the boost generators as follows,

$$\begin{aligned} [J_{kl}, \mathcal{O}_v] &= 0, & [J_{kl}, \mathcal{O}_i] &= \delta_{ik}\mathcal{O}_l - \delta_{il}\mathcal{O}_k, & [J_{kl}, \mathcal{O}_0] &= 0, \\ [B_k, \mathcal{O}_v] &= -\mathcal{O}_k, & [B_k, \mathcal{O}_i] &= \delta_{ik}\mathcal{O}_0, & [B_k, \mathcal{O}_0] &= 0, \end{aligned} \tag{2.17}$$

as illustrated in figure 2. This representation is denoted as $(0) \rightarrow (1) \rightarrow (0)$: 0 and 1 are angular quantum numbers of $\mathfrak{so}(3)$ so that the first (0) stands for \mathcal{O}_v , the last (0) stands for \mathcal{O}_0 , and (1) stands for \mathcal{O}_i operators; the arrows represent the action of boost generator B_i . This representation is reducible because there exist sub-representation such as the $(1) \rightarrow (0)$ part. The boost generators map \mathcal{O}_v to \mathcal{O}_i , thus the representation is not decomposable. There are also descendent operators of \mathcal{O}_α , which together with \mathcal{O}_α form the conformal family. The conformal family structure could be of staggered type, similar to the scalar case.

The generic representation of Carrollian conformal group is more involved. The representation such as $(0) \rightarrow (1) \rightarrow (0)$ is called a multiplet in the sense that the representation contains multiple $\text{SO}(d - 1)$ covariant primary operators which are related by the action of

$$\begin{array}{ccccc}
\mathcal{O}_v & \xrightarrow{B_i} & \mathcal{O}_i & \xrightarrow{B_i} & \mathcal{O}_0 \\
(0) & \longrightarrow & (1) & \longrightarrow & (0)
\end{array}$$

Figure 2. The structure of Bargmann vector operator.

B_i generators. In contrast, the scalar primary operator discussed before is called a singlet. From a theorem by Jakobsen [52], the finite dimensional representations of the Carrollian rotations are all multiplet representations with every sub-sector being the irreducible representation of $\text{SO}(d-1)$. The multiplet representations for $d \geq 3$ have complicated structures, including not only the usual chain representations like $(0) \rightarrow (1) \rightarrow (0)$, which appear in logCFT [53] and $2d$ CCFT [49–51, 54–57], but also novel net representations. In a chain representation, the subsectors are connected in a linear fashion through the boost action, whereas in a net representation the subsectors exhibit a more intricate structure resembling a network. In [43], the possible chain representations have been classified (see (5.1) and (5.2)). Here we would not go into the details, and the interested readers could find them in [43].

It should be stressed that the above discussions about the representations of Carrollian rotation group $M = \{J, B\}$ are independent of the conformal part of the symmetry. Thus it can be applied to the study of general (non-conformal) Carrollian field theories as well.

3 Construction of free Carrollian scalar theories

In section 2.1, we have seen that the Carrollian structure arises as the restriction of the Bargmann structure to the null hyper-surface $v = 0$. This motivates us to construct Carrollian field theories by reducing the Bargmann field theories to the null hyper-surface. Our strategy is straightforward: firstly write an action of the fields using Bargmann geometric invariants; secondly do the null reduction to get the action of Carrollian field theory; moreover check the Carrollian conformal invariance.

In this section, we introduce the procedure of null reduction and illustrate it with the example of a free scalar theory. We defer the applications to electromagnetic theory and general p -form theories to the next section.

3.1 Construction of Carrollian theories

To construct the d -dimensional Carrollian field theories from $(d+1)$ -dimensional Bargmann field theories, we implement the following null reduction procedure. In principle, we may insert a delta-function distribution $\delta(v)$ into a Bargmann action, in order to confine the theory to the $v = 0$ null hyperplane. In practice, we use a taking-limit procedure to reach the delta function and show the Carrollian invariance of the confined theory. Starting from a Bargmann invariant action, we can modify the Bargmann theory by multiplying the Lagrangian \mathcal{L} by an arbitrary function $h(v)$. Such a function is invariant under all the Bargmann transformations listed in table 1 except the translation along v -direction. As a consequence, even though the new Lagrangian $\mathcal{L}_h = h(v)\mathcal{L}$ is not fully Bargmann invariant,

it is still invariant under spacial rotations, Bargmann boosts, and translations along other directions. Finally, if we choose $h(v)$ as a family of functions approaching the delta function, we can confine the integration to the hyper-surface $v = 0$ by taking the limit. The resulting action is then naturally Carrollian invariant.

To illustrate the reduction procedure, let us consider a functional S defined on $d + 1$ dimensions

$$S[\Phi] = \int d^{d+1}x \mathcal{L}(\Phi(x^\alpha), x^\alpha). \tag{3.1}$$

We assume that the integrand \mathcal{L} is well-behaved near $v = 0$. For simplicity, we choose $h(v)$ to be a uniformly distributed function over a small interval of v ,

$$h_\epsilon(v) = \begin{cases} \frac{1}{2\epsilon} & -\epsilon \leq v \leq \epsilon \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Then we can define a smeared functional

$$S_\epsilon[\Phi] \equiv \int d^{d+1}x h_\epsilon(v) \mathcal{L}(\Phi(x^\alpha), x^\alpha), \tag{3.3}$$

and expand \mathcal{L} to the powers of v

$$\begin{aligned} S_\epsilon[\Phi] &= \int dud^{d-1}x \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} dv \mathcal{L}_0 + \mathcal{L}_1 v + \mathcal{O}(v^2) \\ &= \int dud^{d-1}x \mathcal{L}_0 + 0 + \mathcal{O}(\epsilon^2), \end{aligned} \tag{3.4}$$

where $\mathcal{L}_0 = \mathcal{L}|_{v=0, (u, \vec{x}, v = 0)}$ is the restriction of \mathcal{L} to the $v = 0$ surface. Thus taking the $\epsilon \rightarrow 0$ limit singles out the contribution of the fields on $v = 0$ surface

$$S^{\mathcal{C}}[\Phi] \equiv \lim_{\epsilon \rightarrow 0} S_\epsilon[\Phi] = \int dud^{d-1}x \mathcal{L}_0. \tag{3.5}$$

If we start with a Bargmann invariant action $S^{\mathcal{B}}$, this manipulation will yield an Carrollian invariant action $S^{\mathcal{C}}$ on $v = 0$.

However, as have discussed in section 2.2, the Carrollian conformal symmetry is a different story. As shown in [43], we can construct d -dimensional Carrollian conformal theories from $(d + 1)$ -dimensional conformal theories on a null hyper-surface, since the d -dimensional (global) Carrollian conformal group is a subgroup of $(d + 1)$ -dimensional conformal group. Thus using the above procedure we get

$$S^{\text{con}\mathcal{C}}[\Phi] \equiv \lim_{\epsilon \rightarrow 0} S_\epsilon^{\text{con}}[\Phi] = \int dud^{d-1}x \mathcal{L}_0^{\text{con}}, \tag{3.6}$$

the leading expansion in v of $(d + 1)$ -dimensional conformal Lagrangian produces a d -dimensional Carrollian conformal Lagrangian. However, there are subtleties in null reductions, due to geometric invariants. For Bargmann theories, as the geometric invariants are the metric G and the time-like vector ξ , we can construct Carrollian invariant theories in both the electric sector and the magnetic sector. By contrast, the only geometric invariant for a relativistic conformal theory is the metric G , and the same kind of null reduction only

Parent theory	kinds of theories	preserving conformal symmetry
Bargmann theory	electric and magnetic sector	not automatically conformal
Conformal theory	electric sector	automatically conformal

Table 3. Null reduction from different parent theories.

leads to a Carrollian conformal theory in the electric sector. In other words, there could exist Carrollian conformal theories in the magnetic sector, which cannot be obtained by doing null reduction from parent CFTs. We summarise these subtleties in table 3.

In this work, we focus on the construction of both the electric-sector and the magnetic-sector theories from null reduction of Bargmann theories. We will verify the Carrollian conformal symmetry of the resulting theories case by case.

3.2 Carrollian free scalar theories

We aim to construct the Carrollian massless free scalar theories in $d \geq 3$. The building blocks of Bargmann field theories are geometric invariants $G^{\alpha\beta}$ and ξ^α . For a massless free scalar field Φ , there are only two kinds of Bargmann invariant actions:

$$S_E^{\mathbb{B}} = -\frac{1}{2} \int d^{d+1}x \xi^\alpha \xi^\beta \partial_\alpha \Phi \partial_\beta \Phi, \quad S_M^{\mathbb{B}} = -\frac{1}{2} \int d^{d+1}x G^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi. \quad (3.7)$$

The subscript M and E stand for magnetic sector and electric sector, which correspond to magnetic and electric Carrollian field theories [28], respectively. Let us start with the simpler one, i.e., the electric sector first.

Electric sector. In this sector, the Bargmann action is

$$S_E^{\mathbb{B}} = -\frac{1}{2} \int d^{d+1}x \xi^\alpha \xi^\beta \partial_\alpha \Phi \partial_\beta \Phi. \quad (3.8)$$

Expanding Φ near $v = 0$, we have

$$\Phi(u, \vec{x}, v) = \phi(u, \vec{x}) + v\pi(u, \vec{x}) + \mathcal{O}(v^2). \quad (3.9)$$

Inserting this into the action, and choosing $\xi^\alpha = (1, \vec{0}, 0)$, we have

$$S_E^{\mathbb{B}} = -\frac{1}{2} \int d^{d+1}x \partial_u \Phi \partial_u \Phi = -\frac{1}{2} \int d^{d+1}x \partial_u \phi \partial_u \phi + 2v \partial_u \pi \partial_u \phi + \mathcal{O}(v^2). \quad (3.10)$$

Thus we get the Carrollian invariant action

$$S_E^{\mathbb{C}} = \lim_{\epsilon \rightarrow 0} S_{E,\epsilon}^{\mathbb{B}} = -\frac{1}{2} \int d^d x \partial_0 \phi \partial_0 \phi. \quad (3.11)$$

This is actually the electric Carrollian conformal scalar theory with ϕ being the fundamental field. Under an infinitesimal symmetry transformation generated by G , the field changes as $\delta_G \phi(x) = -\xi_G^\mu(x) \partial_\mu \phi(x) + [G, \phi(x)]$, where ξ_G is the vector field corresponding to G in table 2 and the term $[G, \phi(x)]$ is thus the representation of the symmetry on the field.

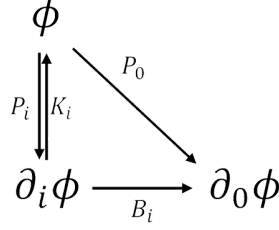


Figure 3. The Carrollian conformal family of ϕ to the first order.

Using the notation $\phi = \phi(0)$ introduced in the section 2.3, we have the actions of the CCA generators on ϕ :

$$\begin{aligned} [D, \phi] &= \Delta_\phi \phi, & [K_\mu, \phi] &= 0, & [P_\mu, \phi] &= \partial_\mu \phi, \\ [J_j^i, \phi] &= 0, & [B_i, \phi] &= 0, & [B_i, \partial_j \phi] &= \delta_{ij} \partial_0 \phi. \end{aligned} \quad (3.12)$$

It can be checked that the action (3.11) is indeed invariant under the Carrollian conformal transformations.

The field ϕ is a primary operator and the relations among its first-order descendants are shown in figure 3. Noticing that $\partial_0 \phi$ is a descendent as well as a primary operator, because under an action of K_μ , we have

$$[K_0, [P_0, \phi]] = [[K_0, P_0], \phi] = 0, \quad [K_i, [P_0, \phi]] = [[K_i, P_0], \phi] = 2[B_i, \phi] = 0. \quad (3.13)$$

The 2-point correlator of ϕ can be calculated via the path-integral

$$\langle \phi(x) \phi(0) \rangle = \frac{i|t|}{2} \delta^{(d-1)}(\vec{x}). \quad (3.14)$$

It satisfies the Ward identities of CCA generators, and this fact confirms again that ϕ is a primary operator. The computation details and the discussions on the 2-pt correlators in a CCFT can be found in appendix C.

Magnetic sector. We now consider the more non-trivial magnetic sector. The Bargmann action is now given by

$$S_M^{\mathbb{B}} = -\frac{1}{2} \int d^{d+1}x G^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi. \quad (3.15)$$

Using the expansion of Φ (3.9), we get

$$S_M^{\mathbb{B}} = -\frac{1}{2} \int d^{d+1}x 2\partial_u \Phi \partial_v \Phi + \partial_i \Phi \partial_i \Phi = -\frac{1}{2} \int d^{d+1}x 2\pi \partial_u \phi + \partial_i \phi \partial_i \phi + \mathcal{O}(v). \quad (3.16)$$

Thus we find the action of magnetic Carrollian scalar theory,

$$S_M^{\mathbb{C}} = -\frac{1}{2} \int d^d x 2\pi \partial_0 \phi + \partial_i \phi \partial_i \phi. \quad (3.17)$$

The fundamental fields in this theory are ϕ and π , which are in the expansion of the field Φ . Classically they are totally independent fields. In this theory, the canonical momentum

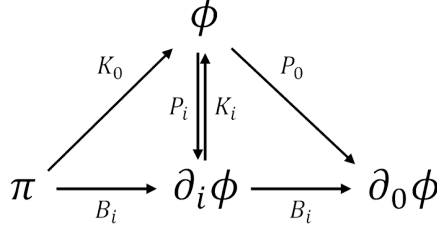


Figure 4. The staggered structure of fields ϕ , $\partial_\mu\phi$ and π .

of scalar field ϕ is $\Pi_\phi = 2\pi$, which matches perfectly with [28]. The Carrollian invariance of this action is quite non-trivial. Under the Bargmann boost $B_i^{\mathcal{B}}$, the Bargmann scalar transforms as

$$\delta_{B_i^{\mathcal{B}}}\Phi = -v\partial_i\Phi + x_i\partial_u\Phi, \quad (3.18)$$

which can be expanded as

$$\delta_{B_i^{\mathcal{B}}}\phi + v\delta_{B_i^{\mathcal{B}}}\pi + \mathcal{O}(v^2) = x_i\partial_u\phi + vx_i\partial_u\pi - v\partial_i\phi + \mathcal{O}(v^2). \quad (3.19)$$

In the leading order of v , we get the infinitesimal transformations of the fields ϕ, π under the Carrollian boost B_i

$$\delta_{B_i}\phi = x_i\partial_0\phi, \quad \delta_{B_i}\pi = x_i\partial_0\pi - \partial_i\phi, \quad (3.20)$$

and furthermore we have

$$\delta_{B_i}\partial_j\phi = x_i\partial_0\partial_j\phi + \delta_{ij}\partial_0\phi. \quad (3.21)$$

This means ϕ transforms as a scalar under Carrollian boost and $(\pi, \partial_i\phi, \partial_0\phi)$ as a $(scalar) \rightarrow (vector)^{(d-1)} \rightarrow (scalar)$ representation, which forms a $(0) \rightarrow (1) \rightarrow (0)$ chain representation. This is somehow expected since ∂_α could be seen as a contravariant Bargmann vector, and thus $(\partial_v) \rightarrow (\partial_i) \rightarrow (\partial_u)$ form a representation of the Carroll group.

Next we would like to check the conformal invariance of this action. Obviously, equipping ϕ with conformal dimension $\Delta_\phi = d/2 - 1$ and π with $\Delta_\pi = d/2$, the action is invariant under the dilation D . The scalar ϕ is still a primary operator as in relativistic CFT, and the field π appears as a part of staggered module of ϕ 's conformal family, as shown in figure 4. More explicitly, the generators act on the fields as

$$\begin{aligned} [D, \phi] &= \Delta_\phi\phi, & [K_\mu, \phi] &= 0, & [P_\mu, \phi] &= \partial_\mu\phi, \\ [J_j^i, \phi] &= 0, & [B_i, \phi] &= 0, & [B_i, \partial_j\phi] &= \delta_{ij}\partial_0\phi, \\ [D, \pi] &= (\Delta_\phi + 1)\pi, & [K_0, \pi] &= 2\Delta_\phi\phi, & [K_i, \pi] &= 0, \\ [J_j^i, \pi] &= 0, & [B_i, \pi] &= -\partial_i\phi. \end{aligned} \quad (3.22)$$

The field π is neither primary nor descendent, as it cannot be generated by acting the generators P on ϕ , while $\partial_0\phi$ is both a primary operator and a descendent of ϕ . Thus we should treat π as an independent field, and the fields $\phi, \partial_i\phi, \partial_0\phi$ and π constitute a staggered module. The emergence of staggered module is a common feature of the magnetic

sector of Carrollian theories, in which the next-to-leading-order field in the expansion of Bargmann field shows up as an independent field in the action.

For the special conformal transformation K_μ , there is

$$\delta_{K_\nu} S_M^{\mathcal{B}} = -\frac{1}{2} \int d^d x \partial_\mu (k_\nu^\mu (2\pi \partial_0 \phi + \partial_i \phi \partial_i \phi)) + \partial_0 (2\Delta_\phi \phi^2), \quad (3.23)$$

where k_ν^μ is the μ component of the generator K_ν . This variation of the action differs from the usual structure for a generator G ,

$$\delta_G S = \int d^d x \partial_\mu (g^\mu \mathcal{L}), \quad (3.24)$$

only by a total derivative $\partial_0 (2\Delta_\phi \phi^2)$. Thus the magnetic theory is Carrollian conformal invariant as well.

We can check the conformal family structure by calculating the correlation functions in the path-integral formalism. The details can be found in appendix B.2. We read the 2-pt correlators of the fundamental fields

$$\begin{aligned} \langle \phi(\vec{x}_1, t_1) \phi(\vec{x}_2, t_2) \rangle &= 0 \\ \langle \phi(\vec{x}_1, t_1) \pi(\vec{x}_2, t_2) \rangle &= -\langle \pi(\vec{x}_1, t_1) \phi(\vec{x}_1, t_2) \rangle = -\frac{i}{2} \text{Sign}(t) \delta^{(d-1)}(\vec{x}) \\ \langle \pi(\vec{x}_1, t_1) \pi(\vec{x}_2, t_2) \rangle &= \frac{i|t|}{2} \vec{\partial}^2 \delta^{(d-1)}(\vec{x}) \end{aligned} \quad (3.25)$$

where $t = t_1 - t_2$ and $\vec{x} = \vec{x}_1 - \vec{x}_2$. The correlator $\langle \phi \phi \rangle = 0$ obviously satisfies the Ward identities. However π is not a primary operator, so the correlators of π do not satisfy the constraints on the primary operators discussed in C. Nevertheless, these correlators indeed satisfy the Ward identities (C.1) with non-vanishing $\langle [K_0, \mathcal{O}_1] \mathcal{O}_2 \rangle$ or $\langle \mathcal{O}_1 [K_0, \mathcal{O}_2] \rangle$ terms.

3.3 Relations between Bargmann correlators and Carrollian correlators

Given the fact that the fundamental fields of Carrollian theories ϕ, π , etc., constitute the components of the Bargmann field Φ , it is reasonable to expect that their correlators form the components of the expanded correlator of Φ . In this section, we demonstrate that this relationship holds true for free scalar theories. Our approach is to convert the derivative operator in (3.3) into an integral kernel, allowing us to derive its “inverse” as the correlator. We show that this correlator is related to the Bargmann correlator by taking the limit $\epsilon \rightarrow 0$, and that it is also related to the Carrollian correlator by first taking the $\epsilon \rightarrow 0$ limit of the generation function and subsequently calculating the correlator. Consequently, we establish the relationship between the Bargmann correlators and the Carrollian correlators. At the end of this subsection, we apply these discussions to the electric and magnetic free scalar theories to reproduce the Carrollian correlators in the last subsection, which were obtained by using the path integral method.

First, let us show how to convert the derivative operator into an integral kernel by considering some $(d + 1)$ -dimensional discrete toy models. Usually, an quadratic action

containing a derivative operator $\hat{D}(\partial_\alpha)$ can be expressed as an integral with the kernel $D(x_1 - x_2)$

$$\int d^{d+1}x \Phi(x) \hat{D}(\partial_\alpha) \Phi(x) = \int d^{d+1}x_1 d^{d+1}x_2 \Phi(x_1) D(x_1^\alpha - x_2^\alpha) \Phi(x_2). \quad (3.26)$$

Here $\hat{D}(\partial_\alpha)$ is a function of ∂_α , which can be expanded to finite powers of ∂_α . In our specific case, the action (3.3) considered in this paper is slightly different from the above one by an insertion of the function $h_\epsilon(v)$. Consequently, the corresponding kernel is different from the conventional one, which will be denoted as D_ϵ where the subscript ϵ indicates the presence of $h_\epsilon(v)$,

$$\int d^{d+1}x h_\epsilon(v) \Phi(x) \hat{D}(\partial_\alpha) \Phi(x) = \int d^{d+1}x_1 d^{d+1}x_2 \Phi(x_1) D_\epsilon(x_1^\alpha - x_2^\alpha) \Phi(x_2). \quad (3.27)$$

Let us start from the following discrete example, whose continuous limit is $\int d^{(d+1)}x h_\epsilon(v) \Phi \partial_\mu \Phi$,

$$\begin{aligned} & \sum_{\{n\}} h_{n_v}^\epsilon \Phi_{\{n\}} (\Phi_{\{n\}} - \Phi_{\{n_\mu-1\}}) \\ &= \sum_{\{n\}\{m\}} \Phi_{\{n\}} \delta_{n_0, m_0} \cdots (\delta_{n_\mu, m_\mu} - \delta_{n_\mu, m_\mu-1}) \cdots \delta_{n_{d-1}, m_{d-1}} h_{n_v}^\epsilon h_{m_v}^\epsilon \Phi_{\{m\}} \\ &= \sum_{\{n\}\{m\}} \Phi_{\{n\}} \delta_{n_0, m_0} \cdots (-\delta_{n_\mu, m_\mu} + \delta_{n_\mu-1, m_\mu}) \cdots \delta_{n_{d-1}, m_{d-1}} h_{n_v}^\epsilon h_{m_v}^\epsilon \Phi_{\{m\}}. \end{aligned} \quad (3.28)$$

Here $\{n\}$ stands for the point at $(n_0, \dots, n_\mu, \dots, n_{d-1})$, $\{n_\mu - 1\}$ is short for $(n_0, \dots, n_\mu - 1, \dots, n_{d-1}, n_v)$, $\Phi_{\{n\}}$ is the field defined at $\{n\}$, whose continuous limit is $\Phi(x^\alpha)$, and $h_{n_v}^\epsilon$ is defined such that

$$h_{n_v}^\epsilon = \sum_{|l_v| \leq \epsilon} \frac{1}{2\epsilon} \delta_{n_v, l_v} h_{n_v}^\epsilon \xrightarrow{\text{continuous limit}} h_\epsilon(v). \quad (3.29)$$

The function $h_{n_v}^\epsilon$ restricts the action by only counting the interaction in $|n_v| < \epsilon$. Taking the continuous limit on both sides, the equation (3.28) becomes

$$\begin{aligned} & \int d^{d+1}x h_\epsilon(v) \Phi(x) \partial_\mu \Phi(x) \\ &= \int d^{d+1}x_1 d^{d+1}x_2 \Phi(x_1) \left(\frac{1}{2} (-\partial_{x_1^\mu} + \partial_{x_2^\mu}) (\delta(x_1^\mu - x_2^\mu) h_\epsilon(v_1) h_\epsilon(v_2)) \right) \Phi(x_2). \end{aligned} \quad (3.30)$$

In other words, the modified integral can be rewritten in terms of the integral kernel $D_{1,\epsilon}(x_1^\mu - x_2^\mu, v_1, v_2)$:

$$\int d^{d+1}x h_\epsilon(v) \Phi(x) \partial_\mu \Phi(x) = \int d^{d+1}x_1 d^{d+1}x_2 \Phi(x_1) D_{1,\epsilon}(x_1^\mu - x_2^\mu, v_1, v_2) \Phi(x_2), \quad (3.31)$$

with

$$D_{1,\epsilon}(x_1^\mu - x_2^\mu, v_1, v_2) = -\frac{1}{2} (\partial_{x_1^\mu} - \partial_{x_2^\mu}) (\delta(x_1^\mu - x_2^\mu) h_\epsilon(v_1) h_\epsilon(v_2)). \quad (3.32)$$

Similarly, we may consider the following example, whose continuous limit is $\int d^{(d+1)}x h_\epsilon(v)\Phi\partial_v\Phi$,

$$\begin{aligned}
 & \sum_{\{n\}} h_{n_v}^\epsilon \Phi_{\{n\}} (\Phi_{\{n\}} - \Phi_{\{n_v-1\}}) \\
 &= \sum_{\{n\}\{m\}} \Phi_{\{n\}} \delta_{n_0, m_0} \cdots \delta_{n_{d-1}, m_{d-1}} h_{n_v}^\epsilon (h_{m_v}^\epsilon - h_{m_v-1}^\epsilon) \Phi_{\{m\}} \\
 &= \sum_{\{n\}\{m\}} \Phi_{\{n\}} \delta_{n_0, m_0} \cdots \delta_{n_{d-1}, m_{d-1}} (-h_{n_v}^\epsilon + h_{n_v-1}^\epsilon) h_{m_v}^\epsilon \Phi_{\{m\}}.
 \end{aligned} \tag{3.33}$$

In this case, the integral in the continuous limit could be rewritten as

$$\int d^{d+1}x h_\epsilon(v)\Phi(x)\partial_v\Phi(x) = \int d^{d+1}x_1 d^{d+1}x_2 \Phi(x_1) D_{2,\epsilon}(x_1^\mu - x_2^\mu, v_1, v_2) \Phi(x_2), \tag{3.34}$$

with

$$D_{2,\epsilon}(x_1^\mu - x_2^\mu, v_1, v_2) = -\frac{1}{2}(\partial_{v_1} - \partial_{v_2})(\delta(x_1^\mu - x_2^\mu)h_\epsilon(v_1)h_\epsilon(v_2)). \tag{3.35}$$

The examples (3.28) and (3.33) pertain solely to first-order derivatives. They correspond to the integrals containing uncontracted derivatives, thus result in the integrals of vectors rather than scalars. From this viewpoint, they should be regarded as formal expressions, distinct from Bargmann invariant actions. Nonetheless, they illustrate the corresponding kernels of first-order derivative operators.

The above argument can be generalized to the higher-derivative operators. The reason behind is that higher-derivative operators correspond to longer range interactions in the lattice model, and the corresponding kernel can be derived in the same way. Now let us consider a quadratic local action with a generic derivative operator $\int d^{d+1}x h_\epsilon(v)\Phi(x)\hat{D}(\partial_\alpha)\Phi(x)$. The action could be written in terms of the corresponding integral kernel $D_\epsilon(x_1^\mu - x_2^\mu, v_1, v_2)$

$$\int d^{d+1}x h_\epsilon(v)\Phi(x)\hat{D}(\partial_\alpha)\Phi(x) = \int d^{d+1}x_1 d^{d+1}x_2 \Phi(x_1) D_\epsilon(x_1^\mu - x_2^\mu, v_1, v_2) \Phi(x_2), \tag{3.36}$$

where

$$D_\epsilon(x_1^\mu - x_2^\mu, v_1, v_2) = \hat{D}\left(-\frac{1}{2}(\partial_{x_1^\alpha} - \partial_{x_2^\alpha})\right)(\delta(x_1^\mu - x_2^\mu)h_\epsilon(v_1)h_\epsilon(v_2)). \tag{3.37}$$

It is illuminating to write the kernel in the momentum space

$$\begin{aligned}
 & D_\epsilon(x_1^\mu - x_2^\mu, v_1, v_2) \\
 &= \int \frac{d^d p_\mu dp_{1v} dp_{2v}}{(2\pi)^d (2\pi)^2} \hat{D}\left(-ip_\mu, -\frac{i}{2}(p_{1v} - p_{2v})\right) \frac{\sin \epsilon p_{1v} \sin \epsilon p_{2v}}{\epsilon^2 p_{1v} p_{2v}} e^{ip_\mu(x_1-x_2)^\mu} e^{ip_{1v}v_1} e^{ip_{2v}v_2} \\
 &= \int \frac{d^{d+1} p_\alpha dp_{v+}}{(2\pi)^{d+1} (2\pi)} \hat{D}(-ip_\alpha) \frac{\cos 2\epsilon p_v - \cos 2\epsilon p_{v+}}{\epsilon^2 (p_{v+}^2 - p_v^2)} e^{ip_\alpha(x_1-x_2)^\alpha} e^{ip_{v+}(v_1+v_2)},
 \end{aligned} \tag{3.38}$$

where $p_v \equiv (p_{1v} - p_{2v})/2$ and $p_{v+} \equiv (p_{1v} + p_{2v})/2$.

As the action is not invariant under the translation in v direction, there is no mathematically rigorous definition for the inverse of D_ϵ , namely there does not exist D_ϵ^{-1} such that

$$\int d^{(d+1)}x_2 D_\epsilon(x_1, x_2) D_\epsilon^{-1}(x_2, x_3) = \delta(x_1^\mu - x_3^\mu) \delta(v_1 - v_3). \tag{3.39}$$

The kernel $D_\epsilon(x_1, x_2)$ only counts the interaction in $|v| < \epsilon$, thus the translation symmetry along v is broken, and the inverse can not be properly defined for $|v| > \epsilon$. However, we may impose a looser inverse condition

$$\int d^{(d+1)}x_2 D_\epsilon(x_1^\alpha, x_2^\alpha) D_\epsilon^{-1}(x_2^\alpha, x_3^\alpha) = \delta(x_1^\mu - x_3^\mu) \delta(v_1) \delta(v_3), \quad (3.40)$$

such that we can find the “inverse” $D_\epsilon^{-1}(x_1^\mu - x_2^\mu, v_1, v_2)$. Noticing that this $D_\epsilon^{-1}(x_1^\mu - x_2^\mu, v_1, v_2)$ is not the rigorous inverse of $D_\epsilon(x_1, x_2)$, but rather an inverse on $v = 0$ hypersurface. It takes the following form in the momentum space

$$\begin{aligned} & D_\epsilon^{-1}(x_1^\mu - x_2^\mu, v_1, v_2) \\ &= \int \frac{d^{d+1}p_\alpha}{(2\pi)^{d+1}} \frac{dp_{v+}}{(2\pi)} \frac{1}{\hat{D}(-ip_\alpha)} \frac{\epsilon^2 (p_{v+}^2 - p_v^2)}{\cos 2\epsilon p_v - \cos 2\epsilon p_{v+}} e^{ip_\alpha(x_1-x_2)^\alpha} e^{ip_{v+}(v_1+v_2)}. \end{aligned} \quad (3.41)$$

It turns out that this form of inverse D_ϵ^{-1} is useful in our study. The reason we use the modified normalization condition is that the Carrollian action is simply defined on the null hyper-surface and we need not require the translation symmetry along v .

With the integral kernel and its inverse, we can discuss the relation between the Bargmann correlators and Carrollian correlators. In general, the modified quadratic Bargmann action can be written as

$$S_\epsilon^{\mathcal{B}} = \int d^{(d+1)}x h_\epsilon(v) \Phi \hat{D} \Phi = \int d^{d+1}x_1 d^{d+1}x_2 \Phi(x_1^\alpha) D_\epsilon(x_1^\mu - x_2^\mu, v_1, v_2) \Phi(x_2^\alpha), \quad (3.42)$$

with $D_\epsilon(x_1^\mu - x_2^\mu, v_1, v_2)$ being defined in (3.37). The corresponding modified generating functional, labeled by the subscript ϵ , is

$$\begin{aligned} \mathcal{Z}_\epsilon^{\mathcal{B}}[J] &= \int \mathcal{D}\Phi \exp\left(i S_\epsilon^{\mathcal{B}} + i \int d^{d+1}x \delta(v) J(x^\alpha) \Phi(x^\alpha)\right) \\ &= \int \mathcal{D}\Phi \exp\left(i \int d^{d+1}x_1 d^{d+1}x_2 \Phi(x_1^\alpha) D_\epsilon(x_1^\mu - x_2^\mu, v_1, v_2) \Phi(x_2^\alpha) \right. \\ &\quad \left. + i \int d^{d+1}x_1 d^{d+1}x_2 J(x_1^\alpha) \delta(v_1) \delta(x_1^\mu - x_2^\mu) \delta(v_2) \Phi(x_2^\alpha)\right) \\ &= N \exp\left(-\frac{i}{4} \int d^{d+1}x_1 d^{d+1}x_2 J(x_1^\alpha) D_\epsilon^{-1}(x_1^\mu - x_2^\mu, v_1, v_2) J(x_2^\alpha)\right), \end{aligned} \quad (3.43)$$

and the modified correlator is

$$\langle \Phi(x_1^\mu, v_1) \Phi(x_2^\mu, v_2) \rangle_\epsilon = \frac{1}{Z_\epsilon^{\mathcal{B}}[0]} \frac{(-i)^2 \delta^2}{\delta J(x_1^\alpha) \delta J(x_2^\alpha)} \mathcal{Z}_\epsilon^{\mathcal{B}}[J] \Big|_{J=0} = -\frac{1}{2} D_\epsilon^{-1}(x_1^\mu - x_2^\mu, v_1, v_2). \quad (3.44)$$

In the limit $\epsilon \rightarrow 0$, this correlator reduces to the Bargmann correlator $\langle \Phi(x_1^\mu, v_1) \Phi(x_2^\mu, v_2) \rangle$,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \langle \Phi(x_1^\mu, v_1) \Phi(x_2^\mu, v_2) \rangle_\epsilon &= \lim_{\epsilon \rightarrow 0} \frac{(-i)^2 \delta^2}{\delta J(x_1^\alpha) \delta J(x_2^\alpha)} \frac{\mathcal{Z}_\epsilon^{\mathcal{B}}[J]}{Z_\epsilon^{\mathcal{B}}[0]} \Big|_{J=0} \\ &= \int \frac{d^{d+1}p_\alpha}{(2\pi)^{d+1}} -\frac{1}{2} \hat{D}^{-1}(-ip_\alpha) e^{ip_\alpha(x_1-x_2)^\alpha} \int \frac{dp_{v+}}{2\pi} e^{ip_{v+}(v_1+v_2)} \\ &= 2\delta(v_1 + v_2) \langle \Phi(x_1^\alpha) \Phi(x_2^\alpha) \rangle, \end{aligned} \quad (3.45)$$

where $\langle \Phi(x_1^\alpha) \Phi(x_2^\alpha) \rangle$ is the normal Bargmann correlator without inserting $h_\epsilon(v)$ in the action. On the other way, we may interchange the order of taking the $\epsilon \rightarrow 0$ limit and taking the functional derivatives so that we can find the relation between modified correlator (3.44) and the Carrollian correlators. Taking the $\epsilon \rightarrow 0$ limit first, we see that the modified Bargmann generating functional (3.43) becomes the Carrollian generating functional,

$$Z_\epsilon^{\mathcal{B}}[J] \xrightarrow{\epsilon \rightarrow 0} Z^{\mathcal{C}}[J] = \int \mathcal{D}\phi \mathcal{D}\pi \cdots \exp\left(iS^{\mathcal{C}} + i \int d^{d+1}x J(x) \delta(v) (\phi(x^\mu) + \pi(x^\mu)v + \cdots)\right). \quad (3.46)$$

Here we have expanded the Bargmann field $\Phi(x) = \phi(x^\mu) + \pi(x^\mu)v + \cdots$. Note that the source terms corresponding to the component fields are $J_\phi = \int dv J(x) \delta(v)$, $J_\pi = \int dv v J(x) \delta(v)$, etc. Next, we take functional derivatives to read the correlation functions,

$$\begin{aligned} & \frac{-i\delta}{\delta J(x_1^\alpha)} \frac{-i\delta}{\delta J(x_2^\alpha)} \lim_{\epsilon \rightarrow 0} Z_\epsilon^{\mathcal{B}}[J]|_{J=0} \\ &= \int \mathcal{D}\phi \mathcal{D}\pi \cdots \delta(v_1) (\phi(x_1^\mu) + \pi(x_1^\mu)v_1 + \cdots) \delta(v_2) (\phi(x_2^\mu) + \pi(x_2^\mu)v_2 + \cdots) \exp iS^{\mathcal{C}} \\ &= \delta(v_1) \delta(v_2) (\langle \phi(x_1^\mu) \phi(x_2^\mu) \rangle + v_1 \langle \pi(x_1^\mu) \phi(x_2^\mu) \rangle \\ & \quad + v_2 \langle \phi(x_1^\mu) \pi(x_2^\mu) \rangle + v_1 v_2 \langle \pi(x_1^\mu) \pi(x_2^\mu) \rangle + \cdots) \end{aligned} \quad (3.47)$$

Matching the last line in (3.45) with the one of (3.47), we find

$$\begin{aligned} & 2\delta(v_1 + v_2) \langle \Phi(x_1^\mu, v_1) \Phi(x_2^\mu, v_2) \rangle \\ &= \delta(v_1) \delta(v_2) (\langle \phi(x_1^\mu) \phi(x_2^\mu) \rangle + v_1 \langle \pi(x_1^\mu) \phi(x_2^\mu) \rangle + v_2 \langle \phi(x_1^\mu) \pi(x_2^\mu) \rangle + v_1 v_2 \langle \pi(x_1^\mu) \pi(x_2^\mu) \rangle + \cdots). \end{aligned} \quad (3.48)$$

It shows that the Carrollian correlators are given by the expansion of the Bargmann correlator, as expected. This matching is feasible only if assuming that taking $\epsilon \rightarrow 0$ limit and taking the functional derivatives are commutative, and we believe this assumption is generally correct.

Now we show the relation (3.48) explicitly in the electric sector and the magnetic sector of the Carrollian scalar theories. For the electric scalar sector (labeled by superscript E) (3.8), the 2-point correlators of the Bargmann field Φ and the Carrollian field ϕ are respectively

$$\begin{aligned} \langle \Phi(x_1^\mu, v_1) \Phi(x_2^\mu, v_2) \rangle^E &= i \int \frac{d^{(d+1)}p}{(2\pi)^{(d+1)}} e^{ip_\alpha x^\alpha} \frac{-1}{p_0^2} = \frac{i|t_1 - t_2|}{2} \delta^{(d-1)}(\vec{x}_1 - \vec{x}_2) \delta(v_1 - v_2), \\ \langle \phi(x_1^\mu) \phi(x_2^\mu) \rangle^E &= \frac{i|t_1 - t_2|}{2} \delta^{(d-1)}(\vec{x}_1 - \vec{x}_2). \end{aligned} \quad (3.49)$$

Noticing the fact that

$$2\delta(v_1 + v_2) \delta(v_1 - v_2) = \delta(v_1) \delta(v_2), \quad (3.50)$$

we find that the relation (3.48) indeed holds.

For the magnetic sector (labeled by superscript M) of scalar (3.15), the 2-point Bargmann correlator of Φ is

$$\langle \Phi(x_1^\mu, v_1) \Phi(x_2^\mu, v_2) \rangle^M = \int \frac{d^{(d+1)}p}{(2\pi)^{(d+1)}} e^{ip_\alpha(x_1 - x_2)^\alpha} \frac{-1}{\vec{p}^2 + 2p_0 p_v}. \quad (3.51)$$

Integrating p_v and expanding in v_i , we find

$$\begin{aligned}
 \langle \Phi(x_1^\mu, v_1) \Phi(x_2^\mu, v_2) \rangle^M &= \int \frac{d^d p}{(2\pi)^d} \frac{-i}{2p_0} e^{-\frac{i\vec{p}^2(v_1-v_2)}{2p_0}} \text{Sign}(v_1-v_2) \\
 &= \frac{\text{Sign}(v_1-v_2)}{v_1-v_2} (v_1-v_2) \int \frac{d^d p}{(2\pi)^d} \left(\frac{-i}{2p_0} - \frac{\vec{p}^2}{4p_0^2} (v_1-v_2) + \dots \right) \\
 &= |v_1-v_2|^{-1} \left(0 + v_1 \int \frac{d^d p}{(2\pi)^d} \frac{-i}{2p_0} + v_2 \int \frac{d^d p}{(2\pi)^d} \frac{i}{2p_0} + v_1 v_2 \int \frac{d^d p}{(2\pi)^d} \frac{\vec{p}^2}{2p_0^2} + \dots \right).
 \end{aligned} \tag{3.52}$$

Viewing $|v_1-v_2|^{-1}$ as a generalized function, it is proportional to $\delta(v_1-v_2)$ by the canonical regularization [58]

$$\frac{1}{\Gamma(0)|x|} = \delta(x), \tag{3.53}$$

thus the equation (3.48) matches the Bargmann correlator with the correlators of Carrollian fields (B.7),

$$\begin{aligned}
 &2\delta(v_1+v_2) \langle \Phi(x_1^\mu, v_1) \Phi(x_2^\mu, v_2) \rangle^M \\
 &\propto 2\delta(v_1+v_2) \delta(v_1-v_2) \left(0 + v_1 \int \frac{d^d p}{(2\pi)^d} \frac{-i}{2p_0} + v_2 \int \frac{d^d p}{(2\pi)^d} \frac{i}{2p_0} + v_1 v_2 \int \frac{d^d p}{(2\pi)^d} \frac{\vec{p}^2}{2p_0^2} + \dots \right) \\
 &= \delta(v_1) \delta(v_2) \left(\langle \phi(x_1) \phi(x_2) \rangle^M + v_1 \langle \pi(x_1) \phi(x_2) \rangle^M \right. \\
 &\quad \left. + v_2 \langle \phi(x_1) \pi(x_2) \rangle^M + v_1 v_2 \langle \pi(x_1) \pi(x_2) \rangle^M + \dots \right).
 \end{aligned} \tag{3.54}$$

Strictly speaking, the relation (3.48) for the magnetic sector of scalar theory is not exact since there is a divergent overall factor $\Gamma(0)$. Despite of the overall divergent factors, the Bargmann correlators could be related to the Carrollian correlators with correct patterns and relative coefficients.

It is not hard to find that the above discussions also apply to the massive case. Since turning on the mass term amounts to adding a constant term in the derivative operator $\hat{D}(\partial_\alpha) \rightarrow \hat{D}(\partial_\alpha) + m^2$, it does not spoil the discussions.

4 Carrollian p -form field theories

The above construction of Carrollian scalar field theories from Bargmann field theories can be extended to other kinds of field theories. In this section, we study the construction of Carrollian p -form field theories, which has been discussed in [28]. The p -form field theories appear in the low energy effective action of superstring theory, and could be important in studying the physics near singularity. We start from Carrollian 1-form field, which gives rise to Carrollian electromagnetic theories. These theories were first studied in [4].

4.1 Electromagnetic theories

In this subsection, we construct Carrollian electromagnetic theories from electromagnetic theories in Bargmann space. There are two kinds of Bargmann invariant actions for a vector

field in general⁵

$$S_E^{\mathbb{B}} = -\frac{1}{4} \int d^{d+1}x G^{\alpha\gamma} \xi^\beta \xi^\delta F_{\alpha\beta} F_{\gamma\delta}, \quad S_M^{\mathbb{B}} = -\frac{1}{4} \int d^{d+1}x G^{\alpha\gamma} G^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta}. \quad (4.1)$$

Here $F_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha$ is the usual field-strength tensor, and a_α is the gauge potential, which is a vector in the Bargmann space. As we have done in the scalar case, we first expand the fundamental fields to the powers of v near $v = 0$,

$$a_\alpha(u, \vec{x}, v) = A_\alpha(u, \vec{x}) + v \pi_\alpha(u, \vec{x}) + \mathcal{O}(v^2), \quad (4.2)$$

and insert this expansion into the action to read the Carrollian theories. Consequently, we obtain the actions of electric sector and magnetic sector of Carrollian electromagnetic theory, from $S_E^{\mathbb{B}}$ and $S_M^{\mathbb{B}}$, respectively.

Electric sector. The action of electric Carrollian U(1) gauge theory is

$$S_E = -\frac{1}{2} \int d^d x F_{0i} F_{0i}, \quad (4.3)$$

where $F_{0i} = \partial_0 A_i - \partial_i A_0$, and the fundamental fields are $A_\mu = (A_0, A_i)$. In this theory, the field A_v and the second-order field π_α get decoupled, and only the leading-order gauge symmetry survives. From the perspective of the representations of Carrollian rotations, A_μ is in the sub-representation $(1) \rightarrow (0)$ of the full A_α representation $(0) \rightarrow (1) \rightarrow (0)$.

It is well known that $4d$ Lorentzian electromagnetic theory is conformal invariant with the field-strength tensor $F_{\mu\nu}$ (other than the vector potential fields A_μ) being the primary operators. However, the story is different in the Carrollian case. As will be shown later, both the electric sector and the magnetic sector are Carrollian conformal invariant. Moreover, the gauge potential A_μ itself is now the primary operators with conformal dimension $\Delta_A = 1$ in $d = 4$. The actions of the symmetry generators on A_μ are

$$\begin{aligned} [D, A_\mu] &= \Delta_A A_\mu, & [K_\mu, A_\nu] &= 0, & \Delta_A &= \frac{d-2}{2}, \\ [J^i_j, A_k] &= \delta_k^i A_j - \delta_{jk} A^i, & [J^i_j, A_0] &= 0, & & \\ [B_k, A_i] &= \delta_{ik} A_0, & [B_k, A_0] &= 0. & & \end{aligned} \quad (4.4)$$

Remarkably, the field-strength tensor F_{0i} is also a primary operator since it satisfies $[K_\mu, F_{0i}] = 0$. Thus in the electric sector of Carrollian electromagnetic theory given by (4.3), both A_μ and F_{0i} are primary operators. This is supported by the fact that both the

⁵For $d = 3, 4$ there are extra topological terms in one higher dimension constructed by the contraction with the Levi-Civita tensor:

$$\begin{aligned} S_{\text{top}}^{\mathbb{B}} &= -\frac{1}{4} \int d^4 x \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}, & (d=3) \\ S_{\text{top}}^{\mathbb{B}} &= -\frac{1}{4} \int d^5 x \xi_\lambda \epsilon^{\lambda\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}, & (d=4). \end{aligned}$$

In this paper, we do not discuss these topological terms. For higher dimensions, there could be the actions involved more than two field-strength tensors, e.g., $d = 5$ with $\mathcal{L} = \epsilon^{\alpha\beta\gamma\delta\sigma\rho} F_{\alpha\beta} F_{\gamma\delta} F_{\sigma\rho}$.

correlators of A_μ and the correlators of F_{0i} satisfy the Ward identities of $d = 4$ Carrollian conformal symmetry.

The action (4.3) has a gauge symmetry which transforms the potential A_μ as

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \omega_0(x). \quad (4.5)$$

We may select a gauge $\partial_0 A_0 = 0$ which is Carrollian conformal invariant to compute the path-integral. When we take this gauge and select the Landau gauge $\xi = 0$, the correlators are simply proportional to the Dirac δ -functions (see appendix B.3 for details)

$$\langle A_i(x) A_j(0) \rangle = \frac{i}{2} \delta_{ij} |t| \delta^{(3)}(\vec{x}), \quad \text{others} = 0, \quad (4.6)$$

which obey the Ward identities for the Carrollian conformal symmetries.

In fact, $\xi^\mu \xi^\nu \partial_\mu a_\nu = \partial_u a_u$ is an appropriate Bargmann gauge-fixing term whose leading order in v is exactly $\partial_0 A_0 = 0$. Using this gauge-fixing term and taking the gauge $\xi = 0$, we find that the path-integral gives the Bargmann correlator

$$\langle a_i(x) a_j(0) \rangle = \frac{i}{2} \delta_{ij} |u| \delta^{(3)}(\vec{x}) \delta(v), \quad (4.7)$$

whose leading order in v gives the Carrollian correlator

$$\langle a_i(x) a_j(0) \rangle = \langle A_i(x) A_j(0) \rangle + \mathcal{O}(v). \quad (4.8)$$

Finally, let us consider the Carrollian Maxwell equations. We denote the magnetic and electric field as

$$\mathbf{B}_k = \frac{1}{2} \epsilon^{ijk} F_{ij}, \quad \mathbf{E}_k = F_{0k}. \quad (4.9)$$

Although not appearing in the action, $F_{ij} = \partial_i A_j - \partial_j A_i$ can be verified to be a primary operator. It is clear that the on-shell equations in the electric sector give the first line of the electric Carroll contraction of the Maxwell equations given in [4],

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, & \frac{\partial \mathbf{E}}{\partial t} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0. \end{aligned} \quad (4.10)$$

The other two equations in the second line are indeed automatically satisfied, and in this sense we say this action is a realization of the electric Carrollian electromagnetism and call it the electric sector.

Magnetic sector. The action of the magnetic Carrollian U(1) gauge theory is

$$\begin{aligned} S_M &= -\frac{1}{4} \int d^d x \delta^{ik} \delta^{jl} F_{ij} F_{kl} + 4\delta^{ij} F_{0i} F_{vj} - 2F_{0v} F_{0v} \\ &= -\frac{1}{4} \int d^d x \delta^{ik} \delta^{jl} (\partial_i A_j - \partial_j A_i)(\partial_k A_l - \partial_l A_k) \\ &\quad + 4\delta^{ij} (\partial_0 A_i - \partial_i A_0)(\pi_j - \partial_j A_v) - 2(\pi_0 - \partial_0 A_v)^2. \end{aligned} \quad (4.11)$$

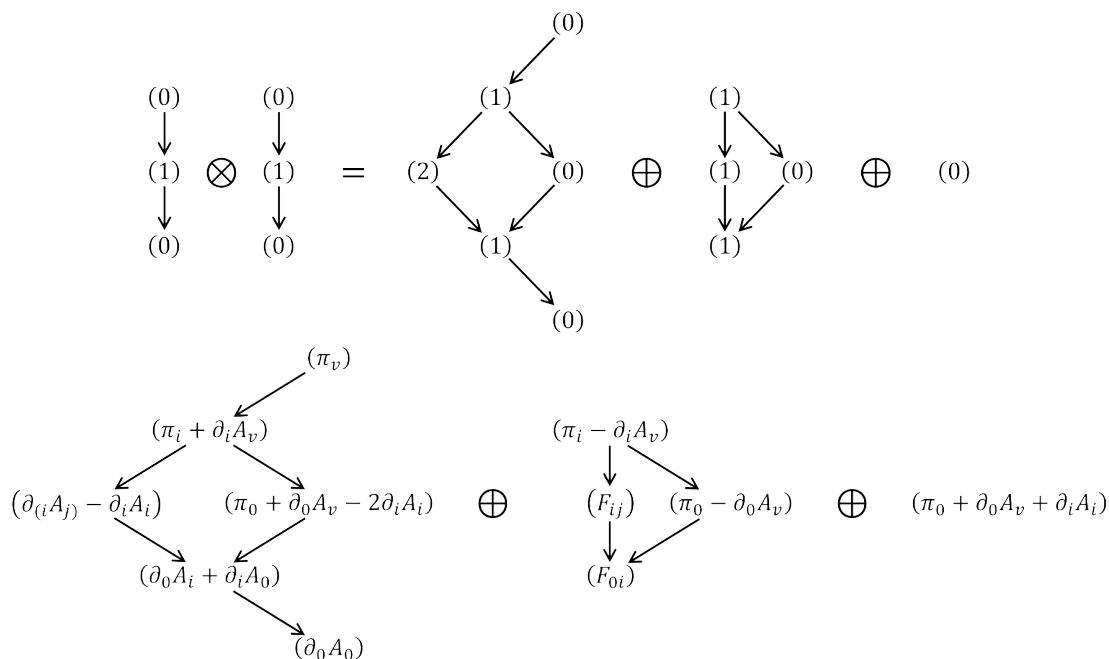


Figure 5. The representation structure of field strength tensors under Carrollian boost B_i (arrows) for $d = 4$.

The fundamental fields appearing in the action are $A_\alpha = (A_0, A_i, A_v)$ and $\pi_\mu = (\pi_0, \pi_i)$, with π_v being decoupled.

Under the Carrollian rotations, the potential fields A_α form a $(0) \rightarrow (1) \rightarrow (0)$ representation, but the representation of π fields is more complex. Actually, under the Carrollian rotations, and especially under the boost generators, the π fields together with the first-order derivatives of A_α , i.e., $\partial_\mu A_\alpha$, form the tensor product of two $(0) \rightarrow (1) \rightarrow (0)$ representations, as shown in figure 5. The upper part of figure 5 shows the decomposition of the tensor product, while the lower part shows the field realizations. The actions of the Carrollian rotations on the fields are:

$$\begin{aligned}
 [J_{kl}, A_v] &= 0, & [J_{kl}, A_i] &= \delta_{ik} A_l - \delta_{il} A_k, & [J_{kl}, A_0] &= 0, \\
 [J_{kl}, \pi_i] &= \delta_{ik} \pi_l - \delta_{il} \pi_k, & [J_{kl}, \pi_0] &= 0, & [J_{kl}, \pi_v] &= 0, \\
 [B_k, A_v] &= -A_k, & [B_k, A_i] &= \delta_{ik} A_0, & [B_k, A_0] &= 0, \\
 [B_k, \pi_i] &= \delta_{ik} \pi_0 - \partial_k A_i, & [B_k, \pi_0] &= -\partial_k A_0, & [B_k, \pi_v] &= -\pi_k - \partial_k A_v,
 \end{aligned} \tag{4.12}$$

which can be read off from the expansion of Bargmann field a_α , similar to the equation (3.19). It is easy to see that only the middle part in figure 5, the anti-symmetric part, appears in the action of the magnetic sector. It is straightforward to verify that the action (4.11) is invariant under these actions and thus represents a Carrollian field theory.

Now we consider the Carrollian conformal symmetries. Unlike the free scalar theories that are Carrollian conformal in generic dimension, the magnetic sector of Carrollian electromagnetic theory is conformal only in $d = 4$. The potential A_μ is a primary operator with conformal dimension $\Delta_A = 1$, while the combinations of π_μ fields and $\partial_\mu A_\nu$ are

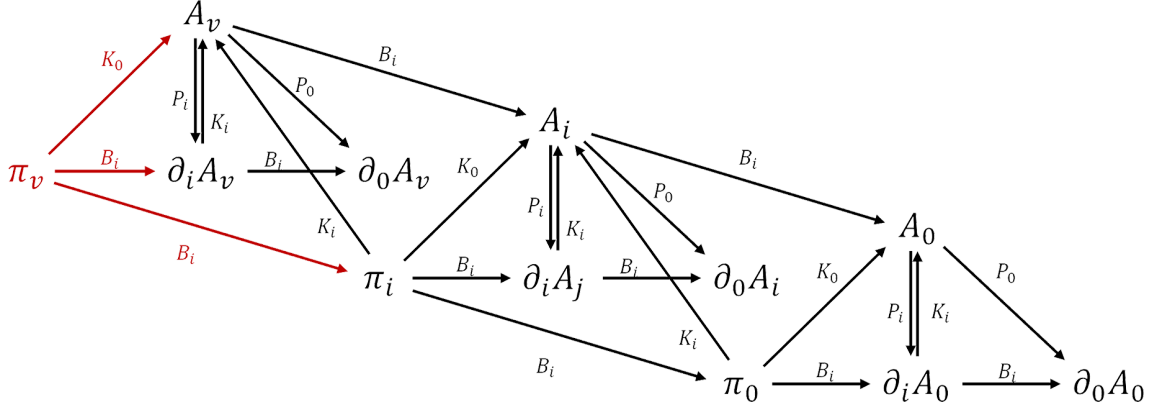


Figure 6. The full structure of the fields $\{A_\alpha, \pi_\alpha, \partial_\mu A_\alpha\}$. The arrows labels the action of some Carrollian conformal generators, whose explicit actions are shown in (4.12) and (4.13). The black part shows the relations among the fields in the magnetic sector of electromagnetic theory, while the red part involving π_v does not appear in the theory.

descendent operators with dimension $\Delta_\pi = \Delta_A + 1 = 2$. The explicit actions of the CCA generators on the fields are given by

$$\begin{aligned}
 [D, A_\alpha] &= \Delta_A A_\alpha = A_\alpha, & [D, \pi_\alpha] &= \Delta_\pi \pi_\alpha = 2\pi_\alpha, & [K_\mu, A_\alpha] &= 0, \\
 [K_i, \pi_j] &= 2\delta_{ij} A_v, & [K_i, \pi_0] &= -2A_i, & [K_i, \pi_v] &= 0, \\
 [K_0, \pi_i] &= 2\Delta_A A_i, & [K_0, \pi_0] &= 2(\Delta_A - 1)A_0 = 0, & [K_0, \pi_v] &= 2(\Delta_A + 1)A_v = 4A_v,
 \end{aligned} \tag{4.13}$$

where $\alpha = 0, i, v$, and $\mu = 0, i$. It can be further checked that the field-strength tensors $F_{vi}, F_{v0}, F_{ij}, F_{0i}$ are all primary operators as well.

Different from the electric sector, the fields in the magnetic sector do not form a staggered structure. The relations between the fields are illustrated in figure 6. The representations of the operators $\{A_\alpha, \pi_\mu, \partial_\mu A_\alpha\}$ do not have a staggered structure. The fields π_i which potentially lead to a staggered structure are combinations of the descendent operators $\partial_i A_v$ and the primary operator F_{vi} , in the form of $\pi_i = F_{vi} + \partial_i A_v$. Similarly, $\pi_0 = F_{v0} + \partial_0 A_v$ does not cause a staggered structure. As shown in figure 6, the field π_v may lead to a staggered module, but it does not appear in the action. The actions of the CCA generators on π_v are listed in (4.12) and (4.13), as well as in figure 6. Thus we can safely say that the magnetic sector of electromagnetic theory does not contain staggered module.

In this case, the gauge fixing should be treated carefully. The gauge transformation in Bargmann U(1) gauge theory is $a_\mu(u, \vec{x}, v) \rightarrow a_\mu(u, \vec{x}, v) + \partial_\mu \Omega(u, \vec{x}, v)$, where $\Omega(x^\alpha)$ is the gauge parameter. Considering the expansion to the powers of v and keeping the leading-order term in v , we find that there are two sets of gauge transformations for the magnetic Carrollian U(1) theory

$$\begin{aligned}
 A_i(x) &\rightarrow A_i(x) + \partial_i \omega_0(x), & A_0(x) &\rightarrow A_0(x) + \partial_0 \omega_0(x), \\
 A_v(x) &\rightarrow A_v(x) + \omega_1(x), & \pi_i(x) &\rightarrow \pi_i(x) + \partial_i \omega_1(x), & \pi_0(x) &\rightarrow \pi_0(x) + \partial_0 \omega_1(x),
 \end{aligned} \tag{4.14}$$

where $\omega_0(x), \omega_1(x)$ appearing in the expansion of the Bargmann gauge parameters $\Omega(x^\alpha) = \omega_0(x^\mu) + \omega_1(x^\mu)v + \dots$ are referred to as the first-order and the second-order gauge parameters, respectively. We find that the following gauge-fixing term is manifestly Carrollian conformal invariant

$$\mathcal{L}_{gf} = -\frac{1}{2\xi_1}(\partial_0 A_0)^2 - \frac{1}{2\xi_2}(3\partial_i A_0 \partial_i A_0 + 2\partial_0 A_0(2\pi_0 - \partial_0 A_v - \partial_i A_i)). \quad (4.15)$$

In this expression we have summed over repeated indices. With the help of this gauge-fixing term, the correlators are now computable in the path-integral formalism. In the Landau-type gauge $\xi_2 = 0$, the correlators can be organized in a relatively compact form

$$\begin{aligned} \langle A_v(x)A_v(0) \rangle &= -2i|t|\delta^{(3)}(\vec{x}), & \langle A_v(x)\pi_i(0) \rangle &= -\langle \pi_i(x)A_v(0) \rangle = \frac{3i}{2}|t|\partial_i\delta^{(3)}(\vec{x}), \\ \langle A_v(x)\pi_0(0) \rangle &= -\langle \pi_0(x)A_v(0) \rangle = i\text{Sign}(t)\delta^{(3)}(\vec{x}), \\ \langle A_i(x)\pi_j(0) \rangle &= -\langle \pi_j(x)A_i(0) \rangle = -\frac{i}{2}\delta_{ij}\text{Sign}(t)\delta^{(3)}(\vec{x}), \\ \langle \pi_i(x)\pi_j(0) \rangle &= \langle \pi_j(x)\pi_i(0) \rangle = \frac{i}{2}|t|\left(\partial_i\partial_j\delta^{(3)}(\vec{x}) + \delta_{ij}\vec{\partial}^2\delta^{(3)}(\vec{x})\right), \\ \langle \pi_i(x)\pi_0(0) \rangle &= \langle \pi_0(x)\pi_i(0) \rangle = \frac{i}{2}\text{Sign}(t)\partial_i\delta^{(3)}(\vec{x}), & \langle \pi_0(x)\pi_0(0) \rangle &= i\delta(t)\delta^{(3)}(\vec{x}), \end{aligned} \quad (4.16)$$

with all other correlators being vanishing. It should be noted that even if we use the gauge-fixing Lagrangian without the ξ_1 -term, the path-integral is still well-defined and the final result is the same. The details of calculations can be found in appendix B.4.

Different from the scalar case, the above correlators cannot be reduced directly from those in the Bargmann theory. This is because the complicated gauge choice lacks a corresponding consistent choice in the Bargmann theory.

With all field components reduced from the Bargmann fields being unmodified, the approach adopted above preserves the Carrollian symmetry and gauge symmetry as much as possible. However, it is still possible to perform the following field redefinition

$$\Pi_i = \pi_i - \partial_i A_v, \quad \Pi_0 = \pi_0 - \partial_0 A_v, \quad (4.17)$$

to simplify the action

$$S'_M[A_i, A_0, \Pi_i, \Pi_0] = -\frac{1}{4}\int d^d x (\partial_i A_j - \partial_j A_i)^2 + 4\Pi_i(\partial_0 A_i - \partial_i A_0) - 2\Pi_0^2. \quad (4.18)$$

After this field transformation, the A_v field is implicit as it is absorbed into the Π -fields. The Π -fields and thus the action (4.18) are invariant under gauge transformation generated by ω_1 in (4.14). Actually, the Π -fields lose the interpretation as the sub-leading-order term of the Bargmann fields a_μ , and they match the leading-order term of Bargmann field strength tensors $F_{v\mu}$, even though they are not fundamental fields in Bargmann U(1) gauge theory. Owing to this fact, the second-order gauge symmetry is hidden because it leaves all the remaining fields invariant, while the first-order gauge symmetry and the Carrollian symmetry continue to be manifest. The action (4.18) suggests that the Π_0 field is decoupled from other fields, and appears only as a mass term without dynamics. Nevertheless, it is

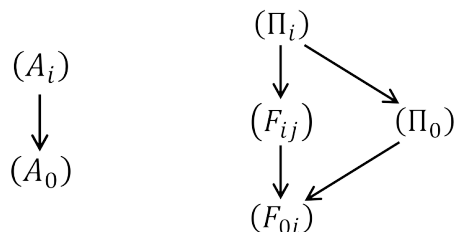


Figure 7. The representations of redefined fields in the magnetic sector of electromagnetic theory. The A_μ fields are in (1) \rightarrow (0) representation, while the Π fields as well as the field strength tensors are in a net representation.

helpful to keep it in the action to show the structure of the representation and to explicitly exhibit the Carrollian (conformal) symmetry without using the equations of motion. The transformation rules for the modified fields under the Carrollian boosts and SCTs are

$$\begin{aligned}
 [B_k, A_i] &= \delta_{ik} A_0, & [B_k, A_0] &= 0, \\
 [B_k, \Pi_i] &= \delta_{ik} \Pi_0 + (\partial_i A_k - \partial_k A_i) = \delta_{ik} \Pi_0 + F_{ik}, \\
 [B_k, \Pi_0] &= \partial_0 A_k - \partial_k A_0 = F_{0k}, \\
 [K_\mu, A_\nu] &= [K_\mu, \Pi_\nu] = 0, & \mu, \nu &= 0, i.
 \end{aligned}
 \tag{4.19}$$

These relations can be illustrated in figure 7.

The modified fields are all Carrollian primaries, and Π_i are the conjugate momenta of A_i . The correlation functions can be derived both from the combination of the previous correlators in (4.16), or directly from the path-integral by adding the gauge fixing term $-\frac{1}{2\xi}(\partial_0 A_0)^2$ to (4.18) and taking $\xi = 0$ in the end. The two approaches turn out to be consistent, and lead to the correlators of the following forms

$$\begin{aligned}
 \langle \Pi_0(x) \Pi_0(0) \rangle &= i \delta(t) \delta^{(3)}(\vec{x}), \\
 \langle A_i(x) \Pi_j(0) \rangle &= -\langle \Pi_j(x) A_i(0) \rangle = -\frac{i}{2} \delta_{ij} \text{Sign}(t) \delta^{(3)}(\vec{x}), \\
 \langle \Pi_i(x) \Pi_j(0) \rangle &= \langle \Pi_i(x) \Pi_j(0) \rangle = \frac{i}{2} |t| (\delta_{ij} \partial^2 - \partial_i \partial_j) \delta^{(3)}(\vec{x}).
 \end{aligned}
 \tag{4.20}$$

As expected, the Π_0 field behaves as a non-kinematic field. Nevertheless, we should still keep it in the theory if we want to write down the Ward identities associated to Carrollian boost symmetries, since $[B_k, \Pi_i] = \delta_{ik} \Pi_0 + F_{ik}$.

We can recover the magnetic Carrollian electromagnetism found in [4], where the Maxwell equations are

$$\begin{aligned}
 \nabla \cdot \mathbf{E} &= 0, & \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= 0, \\
 \nabla \cdot \mathbf{B} &= 0, & \frac{\partial \mathbf{B}}{\partial t} &= 0.
 \end{aligned}
 \tag{4.21}$$

To reveal it, we need to define

$$\mathbf{B}_k = \frac{1}{2} \epsilon^{ijk} F_{ij}, \quad \mathbf{E}_k = F_{vk} = \Pi_k
 \tag{4.22}$$

The equations of motions for (4.18) are given by:

$$\begin{aligned}
\delta A_i &: \quad \partial_j F_{ij} = \partial_0 \Pi_i, \\
\delta A_0 &: \quad \partial_i \Pi_i = 0, \\
\delta \Pi_i &: \quad F_{0i} = 0, \\
\delta \Pi_0 &: \quad \Pi_0 = 0.
\end{aligned}
\tag{4.23}$$

The on-shell equations with regard to $\pi_i, \pi_0/\Pi_i, \Pi_0$ provide us with $F_{0i} = \Pi_0 = 0$, and the equations with regard to A_0, A_i give the first line in (4.21), after using $(\nabla \times \mathbf{B})_i = \partial_j F_{ij}$.

A simple calculation shows that $\nabla \cdot \mathbf{B} = \epsilon^{ijk} \partial_k F_{ij} = 0$. Finally, from $F_{0i} = 0$ we have

$$\partial_0 F_{ij} = \partial_i F_{0j} - \partial_j F_{0i} = 0,$$

which implies the last Maxwell equation $\partial_0 \mathbf{B} = 0$. Under $F_{0i} = \Pi_0 = 0$, the on-shell boost transformation rule becomes

$$[B_i, \mathbf{E}_j] = \epsilon^{ijk} \mathbf{B}_k = -(\mathbf{b}_i \times \mathbf{B})_j. \tag{4.24}$$

Besides, it is well worth mentioning that in the case $d = 3$, the corresponding $(d + 1) = 4$ Bargmann theory possesses the electromagnetic duality $F \leftrightarrow *F$. After restriction to the null hyperplane, the magnetic U(1) Lagrangian (4.11) is invariant under the transformations

$$\pi_0 - \partial_0 A_v \leftrightarrow F_{12}, \quad \pi_1 - \partial_1 A_v \leftrightarrow F_{02}, \quad \pi_2 - \partial_2 A_v \leftrightarrow F_{01}. \tag{4.25}$$

Note that this kind of duality is intrinsic in the magnetic sector.

4.2 p -form theories

It is straightforward to extend the above construction to the p -form free theory. Here we only briefly introduce the construction of the Carrollian action, without discussing the Carrollian conformal symmetry. For a general p -form field a , the field strength F is a $(p + 1)$ -form. Similar to the electromagnetic case, there are only two kinds of Bargmann invariant actions:

$$\begin{aligned}
S_E^{\mathcal{B}} &= -\frac{1}{2(p+1)!} \int d^{d+1}x \xi^{\alpha_1} \xi^{\beta_1} G^{\alpha_2 \beta_2} \dots G^{\alpha_{p+1} \beta_{p+1}} F_{\alpha_1 \dots \alpha_{p+1}} F_{\beta_1 \dots \beta_{p+1}}, \\
S_M^{\mathcal{B}} &= -\frac{1}{2(p+1)!} \int d^{d+1}x G^{\alpha_1 \beta_1} \dots G^{\alpha_{p+1} \beta_{p+1}} F_{\alpha_1 \dots \alpha_{p+1}} F_{\beta_1 \dots \beta_{p+1}}.
\end{aligned}
\tag{4.26}$$

Other combinations of G, ξ and F would be vanishing since the field strength is anti-symmetric. In the above actions, $F_{\alpha_1 \dots \alpha_{p+1}} = (p + 1) \partial_{[\alpha_1} a_{\alpha_2 \dots \alpha_{p+1}]}$ is the field strength tensor, and $a_{\alpha_1 \dots \alpha_p}$ is a p -form gauge potential in Bargmann space. The expansion of the field a to the powers of v near $v = 0$ is

$$a_{\alpha_1 \dots \alpha_p}(u, \vec{x}, v) = A_{\alpha_1 \dots \alpha_p}(u, \vec{x}) + v \pi_{\alpha_1 \dots \alpha_p}(u, \vec{x}) + \mathcal{O}(v^2). \tag{4.27}$$

For the p -form gauge theory, the action of the electric sector is

$$S_E = -\frac{1}{2(p+1)!} \int d^d x F_{0i_2 \dots i_{p+1}} F_{0i_2 \dots i_{p+1}}, \tag{4.28}$$

and the action of the magnetic sector is

$$\begin{aligned}
 S_M &= -\frac{1}{2(p+1)!} \int d^d x F_{i_1 \dots i_{p+1}} F_{i_1 \dots i_{p+1}} \\
 &\quad + 2(p+1) F_{0i_2 \dots i_{p+1}} F_{vi_2 \dots i_{p+1}} - p(p+1) F_{0vi_3 \dots i_{p+1}} F_{0vi_3 \dots i_{p+1}} \\
 &= -\frac{1}{2(p+1)!} \int d^d x (p+1)^2 (\partial_{[i_1} A_{i_2 \dots i_{p+1]}})^2 \\
 &\quad + 2(p+1)^2 \partial_{[0} A_{i_2 \dots i_{p+1]}} (\pi_{i_2 \dots i_{p+1}} + \partial_{[i_2} A_{i_3 \dots i_{p+1]}v}) \\
 &\quad - p(p+1) (\pi_{0i_3 \dots i_{p+1}} + \partial_{[0} A_{i_3 \dots i_{p+1]}v})^2.
 \end{aligned} \tag{4.29}$$

After integrating out $\pi_{ui_2 \dots i_p}$ and $A_{0i_2 \dots i_p}$, we find the following action

$$S_M = \int d^d x \frac{(p+1)}{p!} \Pi_{i_2 \dots i_{p+1}} \partial_{[0} A_{i_2 \dots i_{p+1]}} - \frac{1}{2(p-1)!} (\partial_{[i_1} A_{i_2 \dots i_{p+1]}})^2, \tag{4.30}$$

which is the same as the one in [28]. The fundamental fields in this action are $A_{\alpha_1 \dots \alpha_p}$ and $\pi_{\mu_1 \dots \mu_p}$, where $\Pi_{i_1 \dots i_p}$ are canonical momenta of the fields $A_{i_1 \dots i_p}$. The representation of A is the totally anti-symmetric part of $[(0) \rightarrow (1) \rightarrow (0)]^{\otimes p}$, and the representation of $(\pi, \partial A)$ is the totally anti-symmetric part of $[(0) \rightarrow (1) \rightarrow (0)] \otimes [A]$ in $d = 4$.

5 Carrollian field theories from further reduction

In the last few sections, we have constructed Carrollian invariant field theories from null reduction of Bargmann field theories. Although we managed to obtain a bunch of Carrollian field theories including scalar theories, $U(1)$ theories and p -form theories, we can take a further step. In this section, we will show that some of these theories can be modified by removing some of its field components, and the resulting theories are still Carrollian invariant. The essential point is that both the removed fields and the remaining fields still form *bona fide* sub-representations of Carrollian rotations. Such modifications result in intrinsically different theories. In this way, we are able to discuss Carrollian field theories which can not be directly reduced from Bargmann field theories.

As reviewed in section 2.3, a multiplet representation of Carrollian rotation group is reducible but indecomposable. It could be organized as either chain representation or net representation. An interesting property is that we can always find sub-representations in a multiplet representation. The presence of such sub-representations allows us to remove them and construct another shorter but well-defined representation of the Carrollian rotations using the remaining fields. We have seen in [43] that the possible chain representations of at least rank 2 are in one of the following patterns:

Rank 2:

$$\begin{aligned}
 (j) &\rightarrow (j), \quad j \neq 0 \\
 (j) &\rightarrow (j+1), \\
 (j) &\rightarrow (j-1).
 \end{aligned} \tag{5.1}$$

\geq Rank 3:

$$\begin{aligned}
 & (0) \rightarrow (1) \rightarrow (0), \\
 & \cdots \rightarrow (j) \rightarrow (j+1) \rightarrow (j+2) \rightarrow \cdots, \\
 & \cdots \rightarrow (j) \rightarrow (j-1) \rightarrow (j-2) \rightarrow \cdots.
 \end{aligned} \tag{5.2}$$

For $(0) \rightarrow (1) \rightarrow (0)$ representation, both $(1) \rightarrow (0)$ and (0) are the sub-representations. After removing one of these sub-representations from the chain, the resulting quotient representation (0) or $(0) \rightarrow (1)$ still belongs to the allowed patterns. Similar results hold for the increasing and decreasing chains. Let us consider the chain $(j) \rightarrow (j \pm 1) \rightarrow (j \pm 2) \rightarrow \cdots \rightarrow (j \pm n)$ of length n . For any integer $m < n$ the sub-chain starting at $(j \pm (m + 1))$ and ending at $(j \pm n)$ will give a sub-representation. Upon removing this sub-representation from the full representation we get a shorter chain $(j) \rightarrow (j \pm 1) \rightarrow (j \pm 2) \rightarrow \cdots \rightarrow (j \pm m)$ as a quotient representation. Moreover, we can generalize the above operation to net representations and notice that removing a sub-net from the whole net will give another legal net representation. Although this seems not to be the unique way to get *bona fide* representation of the Carrollian rotations by removing fields, because for example we can also remove the first few terms and the last few terms of a chain representation to get another representation, it is necessary to remove just sub-representations to get quotient representations in order that the reduced action is Carrollian invariant.

Consequently, we are able to reduce the field components from already established Carrollian field theories so as to obtain new Carrollian field theories of the remaining field components. The only requirement is that the removed fields should consist of only fundamental fields with no derivatives and they form a sub-representation. Here we use $\Phi = (\Phi_i)_{i \in \mathcal{I}}$ to stand for the original Carrollian fields and $S[\Phi]$ for the original action, Ψ for the fields making up the sub-representation, $\bar{\Phi} = (\bar{\Phi}_j)_{j \in \mathcal{I}' \subset \mathcal{I}}$ for the remaining fields after removing Ψ from Φ , and $\bar{S}[\bar{\Phi}]$ for the modified action. For any generator g of Carrollian (conformal) group, the invariance of modified action under the transformation can be seen from the variation

$$\bar{\delta}_g \bar{S}[\bar{\Phi}] = \sum_{j \in \mathcal{I}'} \frac{\delta S}{\delta \bar{\Phi}_j} \Big|_{\Psi=0} \bar{\delta}_g \bar{\Phi}_j = \left(\sum_{i \in \mathcal{I}} \frac{\delta S}{\delta \Phi_i} \delta_g \Phi_i \right) \Big|_{\Psi=0} = \delta_g S[\Phi] \Big|_{\Psi=0} = 0, \tag{5.3}$$

where $\bar{\delta}_g \bar{\Phi}_j = \delta_g \Phi_j \Big|_{\Psi=0}$. The removed fields must fit into a sub-representation in this argument, because $\delta_g \Psi$ should only depend on Ψ and becomes vanishing after the reduction. This does not only make sense for Carrollian symmetry, but also for Carrollian conformal symmetry.

Now we take the $U(1)$ magnetic sector as a non-trivial example. In this case, as analyzed in previous sections, the potential fields $A_\alpha = (A_v, A_i, A_0)$ form a $(0) \rightarrow (1) \rightarrow (0)$ representation. As discussed above, (A_0) and $(A_i) \rightarrow (A_0)$ are two sub-representations, which can be removed from the theory. Let us consider the resulting Lagrangians case by case.

$A_0 = 0$. The simplest choice is $A_0 = 0$, and the reduced action is given by

$$\bar{S}_M[A_v, A_i, \pi_i, \pi_0] = -\frac{1}{4} \int d^d x (\partial_i A_j - \partial_j A_i)^2 + 4(\partial_0 A_i)(\pi_i - \partial_i A_v) - 2(\pi_0 - \partial_0 A_v)^2. \tag{5.4}$$

The actions of the boost generators and SCTs in $d = 4$ are modified to be

$$\begin{aligned} [B_k, A_v] &= -A_k, & [B_k, A_i] &= 0, & [B_k, \pi_i] &= \delta_{ik}\pi_0 - \partial_k A_i, & [B_k, \pi_0] &= 0, \\ [K_0, \pi_i] &= 2A_i, & [K_i, \pi_0] &= -2A_i, & [K_i, \pi_j] &= 2\delta_{ij}A_v. \end{aligned} \quad (5.5)$$

The action is still invariant under all the Carrollian conformal transformations. However, the gauge transformation on A_0 is no longer a symmetry, and the action is now only invariant under the second order gauge transformation

$$A_v(x) \rightarrow A_v(x) + \omega_1(x), \quad \pi_i(x) \rightarrow \pi_i(x) + \partial_i \omega_1(x), \quad \pi_0(x) \rightarrow \pi_0(x) + \partial_0 \omega_1(x). \quad (5.6)$$

It is not difficult to work out the 2-point correlators via the path-integral, if we carefully choose the gauge-fixing term to be $-\frac{1}{2\xi}(2\pi_0 - \partial_0 A_v - \partial_i A_i)^2$,

$$\begin{aligned} \langle A_v(x) A_v(0) \rangle &= -i \left(2 - \frac{\xi}{2} \right) |t| \delta^{(3)}(\vec{x}), & \langle A_v(x) \pi_i(0) \rangle &= -\langle \pi_i(x) A_v(0) \rangle = \frac{i}{2} (3 - \xi) |t| \partial_i \delta^{(3)}(\vec{x}), \\ \langle A_v(x) \pi_0(0) \rangle &= -\langle \pi_0(x) A_v(0) \rangle = i \left(1 - \frac{\xi}{2} \right) \text{Sign}(t) \delta^{(3)}(\vec{x}), \\ \langle A_i(x) \pi_j(0) \rangle &= -\langle \pi_j(x) A_i(0) \rangle = -\frac{i}{2} \delta_{ij} \text{Sign}(t) \delta^{(3)}(\vec{x}), \\ \langle \pi_i(x) \pi_j(0) \rangle &= \langle \pi_j(x) \pi_i(0) \rangle = \frac{i}{2} |t| \left((1 - \xi) \partial_i \partial_j \delta^{(3)}(\vec{x}) + \delta_{ij} \vec{\partial}^2 \delta^{(3)}(\vec{x}) \right), \\ \langle \pi_i(x) \pi_0(0) \rangle &= \langle \pi_0(x) \pi_i(0) \rangle = \frac{i}{2} (1 - \xi) \text{Sign}(t) \partial_i \delta^{(3)}(\vec{x}), & \langle \pi_0(x) \pi_0(0) \rangle &= i(1 - \xi) \delta(t) \delta^{(3)}(\vec{x}). \end{aligned} \quad (5.7)$$

Similarly, after redefining Π -fields as in (4.17) to absorb A_v , we get the action

$$\bar{S}_M[A_i, \Pi_i, \Pi_0] = -\frac{1}{4} \int d^d x (\partial_i A_j - \partial_j A_i)^2 + 4\Pi_i \partial_0 A_i - 2\Pi_0^2. \quad (5.8)$$

The corresponding actions under the symmetry generators are

$$\begin{aligned} [B_k, A_i] &= 0, \\ [B_k, \Pi_i] &= \delta_{ik} \Pi_0 + (\partial_i A_k - \partial_k A_i), & [B_k, \Pi_0] &= \partial_0 A_k, \\ [K_\mu, A_\nu] &= [K_\mu, \Pi_\nu] = 0, & \mu, \nu &= 0, i. \end{aligned} \quad (5.9)$$

In this formulation, we are free of gauge redundancy, and do not need to impose gauge fixing. The correlators are now

$$\begin{aligned} \langle \Pi_0(x) \Pi_0(0) \rangle &= i \delta(t) \delta^{(3)}(\vec{x}), \\ \langle A_i(x) \Pi_j(0) \rangle &= -\langle \Pi_j(x) A_i(0) \rangle = -\frac{i}{2} \delta_{ij} \text{Sign}(t) \delta^{(3)}(\vec{x}), \\ \langle \Pi_i(x) \Pi_j(0) \rangle &= \langle \Pi_i(x) \Pi_j(0) \rangle = \frac{i}{2} |t| \left(\delta_{ij} \vec{\partial}^2 - \partial_i \partial_j \right) \delta^{(3)}(\vec{x}), \end{aligned} \quad (5.10)$$

which are the same as the $\xi = 0$ correlators in (4.20) if neglecting the ones involving A_0 . In fact, the resulting theory is intrinsically different from the original theory. This can be seen

from the equations of motion

$$\begin{aligned}
 \delta A_i &: \quad \partial_i F_{ij} = \partial_0 \Pi_i, \\
 \delta \Pi_i &: \quad \partial_0 A_i = 0, \\
 \delta \Pi_0 &: \quad \Pi_0 = 0.
 \end{aligned}
 \tag{5.11}$$

Here is no equation with respect to A_0 , which leads to $\partial_i \Pi_i = 0$.

As it stands, there are four apparently different Lagrangians starting from the Bargmann U(1) theory, including (4.11), (4.18), (5.4), and (5.8), each having different number of fundamental fields and gauge symmetries.

$\mathbf{A}_i = \mathbf{A}_0 = \mathbf{0}$. If the field components A_i, A_0 in the $(0) \rightarrow (1) \rightarrow (0)$ representation are removed, the action will reduce to a simpler one,

$$\bar{S}_M[A_v, \pi_0] = \frac{1}{2} \int d^d x (\pi_0 - \partial_0 A_v)^2.
 \tag{5.12}$$

Not only do A_i, A_0 vanish, but π_i also decouples. The actions of the boost generators are

$$[B_k, A_v] = 0, \quad [B_k, \pi_0] = 0.
 \tag{5.13}$$

Since we can redefine a new scalar field $\chi = \pi_0 - \partial_0 A_v$ to absorb A_v , the Lagrangian includes purely a quadratic term $\frac{1}{2}\chi^2$ without dynamics and thus the theory is somehow trivial.⁶

From the above examples, we see that we can always obtain new Carrollian invariant actions by starting from a Carrollian field theory, setting some field components vanishing, but ensuring the remaining field components to be in a sub-representation. These resulting theories cannot be read from the Bargmann action directly. This paves a new way to find more Carrollian theories. It is more effective, if the original representation is complicated, as there are more choices to get sub-representations.

6 Discussions

In this work, we tried to construct the Carrollian invariant field theories by restricting the parent Bargmann invariant theories to a null hyper-surface. Such a null reduction guarantees the resulting theories to be Carrollian invariant. We mainly focused on the free massless scalar and U(1) electromagnetic theories, and managed to reproduce the known electric-sector and magnetic-sector theories in the literature. The theories we constructed are manifest off-shell Carrollian invariant.

Another focus in this work is on the Carrollian conformal invariance. We found that for both the free scalar and U(1) electromagnetic theories in $d = 4$, their electric-sector and magnetic-sector theories are all Carrollian conformal invariant. We computed the 2-point correlators by using the path-integral formalism, and found the correlators to be consistent with the Ward identities. One remarkable point is that even in the simple theories

⁶Due to the constraint from the equation of motion, the system with a Lagrangian $L = (q_1 - \frac{d}{dt} q_2)^2$ has no physical degrees of freedom and is therefore trivial. For further details, please refers to chapter 1.6 of [59].

studied in this work, the operators have constituted the generic representations of Carrollian conformal algebra [43]. In the magnetic sector of free scalar, there appears the staggered structure, similar to the case in the BMS free scalar theory [49] and the BMS free fermion theory [50, 51]. More interestingly, in the magnetic sector of Carrollian electromagnetic theory, the gauge potentials form a chain representation, and the restricted field strength form a net representation of CCA. Another distinct feature in Carrollian electromagnetic theory is that both the gauge potential and the field strength are primary operators in $d = 4$.

The null-reduction method can be applied to other constructions. There are several directions worthy of pursuing:

- *Non-conformal massive & interacting Carrollian theory.* Although the discussions in the present work cover only the free massless scalar theories and the free massless vector theories, there are no obstacles for the construction to be generalized to other cases, including the massive scalar theories and the interacting theories, say Yang-Mills theory and (scalar) QED [60]. This is due to the fact that if we do not require the conformal symmetry, there are more options on the Bargmann actions, for example the massive theory or the theory with general interaction terms.
- *Fermionic theory.* Recently, in [61] the Carrollian Clifford algebra were studied and the actions for the Carroll fermions were constructed. It would be interesting to reconsider the fermionic theory from the reduction of the Bargmann fermion. Moreover, one may study the supersymmetric Carrollian field theory.
- *Higher-order derivative theory.* We expect that the null-reduction method can be applied to the construction of higher-derivative Carrollian field theory as well. In this case, the higher-order components in the expansion of the Bargmann field will become relevant in the construction.
- *Carrollian gravity.* There have already existed two approaches to Carrollian gravity. One is by gauging the Carrollian algebra [62–64], and the other is by using the contraction of Lorentz theory [28]. A comparison between the magnetic theory from Carrollian contraction and the construction from gauge procedure were made in [65], while the electric sector is missing in the gauging description. In [35], the authors claimed that the two sectors correspond to the leading and next-to-leading order theory in the expansion of general relativity. The Bargmann reduction may provide another viewpoint of this issue, as it could in principle lead to both the electric sector and the magnetic sector of Carrollian gravity at the same time.

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A Staggered modules in higher dimensional Carrollian CFT

The staggered modules (i.e. representations) appear in 2d Logarithmic CFTs, e.g. [45–48], and 2d Carrollian CFTs [49–51]. In this appendix we briefly review the features of this type of modules and discuss its analog in higher dimensional Carrollian CFTs. We adopt a condensed notation without explicit indices on the generators of the conformal algebra or on the operators. The generators of the Carrollian conformal algebra contain dilatation D , generalized rotation M , translation P and SCT K , with the following commutation relations

$$[D, P] = p_D P, \quad [D, K] = k_D K, \quad (\text{A.1})$$

$$[M, P] = p_M P, \quad [M, K] = k_M P, \quad [P, K] = dD + mM \quad (\text{A.2})$$

The actions on primary operators \mathcal{O}_i are $D\mathcal{O}_i = \Delta_i \mathcal{O}_i$ and $M\mathcal{O}_i = \xi_i \mathcal{O}_i$.

Mathematically, the staggered modules come from non-trivial module extensions. To understand the structure of a generic module of an algebra \mathfrak{g} , we can try breaking it into the simplest pieces — irreducible modules, $V_N \stackrel{?}{=} \bigoplus_{i=1}^N W_i$. But this direct sum decomposition cannot be achieved for non-semisimple Lie algebras, since it loses the track of the relations between different W_i -s. For this reason we need a method of sewing W_i -s back to V_N , and this leads to the problem of module extensions, see e.g. [66, 67].

The standard way of decomposing V_N is to choose a maximal submodule $V_{N-1} \subset V_N$ and to take the quotient $V_N/V_{N-1} =: W_N$, then by the maximality of V_{N-1} , W_N is irreducible. Repeatedly we get a series of submodules,

$$0 = V_0 \subset V_1 \subset \dots \subset V_{N-1} \subset V_N, \quad (\text{A.3})$$

which is called the Jordan-Holder composition series of V_N . The irreducible modules $W_i = V_i/V_{i-1}$ are called the factors and N is called the length of V_N . The composition series is not unique, but the length and the factors are invariants of V_N itself. If relaxing the condition of maximality, the resulting composition series will be shorter than the Jordan-Holder one, and W_i can be reducible modules.

On the other way, we can compose W_i -s into some bigger V_N . Let us start from $N = 2$. More broadly, dropping the condition of irreducibility and considering two arbitrary modules W_1 and W_2 , we find that the composed V_2 must satisfy the quotient condition $W_2 = V_2/W_1$, or written in a short exact sequence,

$$0 \longrightarrow W_1 =: V_1 \xrightarrow{\iota} V_2 \xrightarrow{\pi} W_2 \longrightarrow 0. \quad (\text{A.4})$$

In this notation the intertwining map ι is injective and π is surjective, and they characterize the ways of W_1 being embedded into V_2 and W_2 being projected from V_2 . For the same pair (W_1, W_2) there can be inequivalent (ι, π) -s corresponding to not necessarily isomorphic V_2 -s, and each triplet (V_2, ι, π) is called an extension of W_1 by W_2 . For $N > 2$, we can introduce new irreducible modules W_i and repeat the preceding step recursively,

$$0 \longrightarrow V_{i-1} \xrightarrow{\iota_i} V_i \xrightarrow{\pi_i} W_i \longrightarrow 0. \quad (\text{A.5})$$

In this decomposition-composition procedure, we find that there can be other solutions of the constraint (A.4) providing new modules of the algebra, besides the original module V_N . This phenomenon happens for the representations of non-semisimple Lie algebras and infinite dimensional representations of semi-simple or affine Lie algebras and Virasoro algebras. The former case appears in the Carrollian, Galilean and Schrodinger⁷ (conformal) field theories, and the latter one appears in relativistic CFTs.

Now the remaining problem is to solve the module extensions of (A.4). It turns out that the equivalence classes of different extensions constitute a basis of the Ext vector space $\text{Ext}_{\mathfrak{g}}(W_2, W_1)$, and the trivial extension $W_1 \oplus W_2$ corresponds to the zero element, see e.g. chapter 3 & 7 of [67]. The vector space Ext is hard to compute, but in practice we only need to construct certain extensions according to the physical problem by constraining the undetermined coefficients in (ι, π) .

Actually the module extension problem has already been encountered in the construction of finite dimensional modules of Carrollian rotation algebra [43]. For example, the $4d$ electric vector $V = (0) \rightarrow (1)$ is an extension of (1) by (0) . Here (1) is the submodule W_1 , and the morphisms (ι, π) are given by the actions of generators from $W_2 = (0)$ to $\{(1), (0)\}$, which can be constrained by the commutation relations using the Wigner-Eckart theorem. In this simple example, the Ext vector space can be computed as $\text{Ext}_{\mathfrak{g}}((0), (1)) = \mathbb{C}$, hence the electric vector $(0) \rightarrow (1)$ is the only nontrivial extension. We leave the technical computation of $\text{Ext}_{\mathfrak{g}}((0), (1)) = \mathbb{C}$ to the end of this appendix.

After introducing module extensions, we give a sketchy analysis of the staggered modules in the Logarithmic and Carrollian CFTs. For simplicity we focus on $N = 2$ and assume that W_1, W_2 are singlet highest-weight modules, i.e. the corresponding primary operators $\mathcal{O}_1, \mathcal{O}_2$ (indices omitted) are irreducible representations of the generalized rotation subalgebra⁸ $\{D, M\}$. By the surjectivity of π , we can choose the pre-images of \mathcal{O}_2 in the extended module V , $\tilde{\mathcal{O}}_2 \in V$, $\pi(\tilde{\mathcal{O}}_2) = \mathcal{O}_2$, and denote the conformal dimensions of $\mathcal{O}_1, \tilde{\mathcal{O}}_2$ as Δ_1, Δ_2 respectively.⁹ To preserve the grading of the dilatation D , the difference $l = \Delta_2 - \Delta_1 \in \mathbb{Z}$ must be an integer. All the possible nontrivial extended modules V can be cast into three types:

- $l = 0$. It can be shown in this case that for some generators $g \in \{D, M\}$ of the dilatation or the generalized rotation algebra, $\mathcal{O}_1 = g\tilde{\mathcal{O}}_2$. The resulting module corresponds to a rank-2 logarithmic multiplet in LogCFT, a boost multiplet in $2d$ Carrollian CFT, or a chain multiplet in higher dimensional Carrollian CFT.
- $l < 0$. In this case \mathcal{O}_1 and $\tilde{\mathcal{O}}_2$ can be related by some lowering operators: $\mathcal{O}_1 = P^1 \dots P^{|l|}\tilde{\mathcal{O}}_2$. Then V itself is a reducible highest-weight module, and \mathcal{O}_1 are singular in the sense that it is simultaneously primary and descendent.

⁷The “alien operators” introduced in [68] belong to a class of neutral operators in Schrodinger CFT, and could enter into the story of staggered modules.

⁸For reducible but indecomposable $\mathcal{O}_1, \mathcal{O}_2$, this corresponds to the case $N > 2$ and is more sophisticated than $N = 2$ case.

⁹The choice of $\tilde{\mathcal{O}}_2$ admits gauge redundancies: for each $\mathcal{O} \in W_1$ we have $\pi(\mathcal{O}) = 0$ and $\pi(\tilde{\mathcal{O}}_2 + \mathcal{O}) = \mathcal{O}_2$.

- $l > 0$. This type is called (strictly) staggered module and reveals new features comparing with the above two types. There can exist singular vectors in V , and $\tilde{\mathcal{O}}_2$ can be neither primary nor descendent.

To obtain this result we need to consider how W_2 is “staggered” with W_1 . The action of conformal algebra on W_2 is determined by the operators $A = (D - \Delta_2)\tilde{\mathcal{O}}_2$, $A' = (M - \xi_2)\tilde{\mathcal{O}}_2$, $B = K\tilde{\mathcal{O}}_2$, $C = P\tilde{\mathcal{O}}_2$. The operators C generate descendents and is irrelevant to the discussion. Noticing that the intertwining map π commutes with the action, we have $\pi(A) = (D - \Delta_2)\pi(\tilde{\mathcal{O}}_2) = (D - \Delta_2)\mathcal{O}_2 = 0$, hence by the exactness of (A.4) $A \in \text{im}(\pi) = \ker(\iota) = W_1$. Similarly $A', B \in W_1$. Then the extended module is illustrated by the following diagram:

$$\begin{array}{ccccc}
 \Delta_2 - 1 & & B & & \\
 & & \swarrow K & & \\
 \Delta_2 & & A' \xleftarrow{M} \mathcal{O}_2 \xrightarrow{D} A & &
 \end{array} \tag{A.6}$$

For $l = 0$ there is no operator with conformal dimension less than Δ_2 , hence $B = 0$ and $\tilde{\mathcal{O}}_2$ are primary operators in V . There are no other primary operators besides \mathcal{O}_1 , hence $\{A, A'\}$ are linear combinations of \mathcal{O}_1 . On the contrary, due to the irreducibility of \mathcal{O}_1 under $\{D, M\}$, either $\mathcal{O}_1 \cap \{A, A'\} = 0$ or $\mathcal{O}_1 \subset \{A, A'\}$. The former case implies $A = A' = 0$, hence $V = W_1 \oplus W_2$ is trivial. The latter case implies \mathcal{O}_1 and $\{A, A'\}$ are linear combinations of each other, hence $\mathcal{O}_1 = g\tilde{\mathcal{O}}_2$ for some generator $g \in \{D, M\}$. For $l < 0$ the analysis is similar: if V is non-trivial then \mathcal{O}_1 are in the descendents of $\tilde{\mathcal{O}}_2$, hence are singular in V .

Now we consider the interesting case $l > 0$ and assume the extension V is non-trivial. At the level Δ_2 the operator number equals $\#(\mathcal{O}_2) + \#(P^1 \dots P^{|l|}\mathcal{O}_1)$, hence there must be operators not coming from the descendents of \mathcal{O}_1 . By the gauge redundancy $\tilde{\mathcal{O}}_2 \rightarrow \tilde{\mathcal{O}}_2 + \mathcal{O}$, $\mathcal{O} \in W_1$ we can choose the extra operators to be $\tilde{\mathcal{O}}_2$. If $B \neq 0$ then $\tilde{\mathcal{O}}_2$ are neither primary nor descendent. There can be singular vectors from $\{A, A'\}$. Firstly, $KA = (D - \Delta_2 + 1)K\tilde{\mathcal{O}}_2 = (D - \Delta_2 + 1)B = 0$, hence KA is singular in V if $A \neq 0$. Secondly, from $KA' = (M - \xi_2 + k_M)B$, the matrix equation $(M - \xi_2 + k_M)B = 0$ can have non-vanishing solutions B_0 , which further provides singular vectors $A'_0 \subset \{A'\}$ with $KA'_0 = (M - \xi + b)B_0 = 0$. For example, in $2d$ Carrollian CFT, supposing that the two singlet primaries are related by $\mathcal{O}_1 = L_1\tilde{\mathcal{O}}_2$, we have $M_1A' = (M_0 - \xi)M_1\tilde{\mathcal{O}}_2 = 0$ and $L_1A' = (M_0 - \xi)\mathcal{O}_1 + M_1\tilde{\mathcal{O}}_2 = 0$, hence $A' = M_0\tilde{\mathcal{O}}_2$ is a singular vector if not vanishing.

Finally let us consider a special case $\mathcal{O}_1 = K\tilde{\mathcal{O}}_2$ and $K\mathcal{O}_1 = 0$, i.e. $l = \Delta[K]$. This case is still broad enough to include the known staggered modules in the present work, in $2d$ Carrollian CFT and even in Schrodinger CFT. Writing A, A' as linear combinations of $P\mathcal{O}_1$, we have

$$D\tilde{\mathcal{O}}_2 = aP\mathcal{O}_1, \quad M\tilde{\mathcal{O}}_2 = bP\mathcal{O}_1, \tag{A.7}$$

subject to the matrix equation $a(p_M + \xi_1)P\mathcal{O}_1 = b(p_D + \Delta_1)P\mathcal{O}_1$ from $[D, M] = 0$. To exploit the additional information $K\mathcal{O}_1 = 0$, applying K on both sides we get

$$(\Delta_1 - k)\mathcal{O}_1 = a(d\Delta_1 + m\xi_1)\mathcal{O}_1, \quad (\xi_1 - k)\mathcal{O}_1 = b(d\Delta_1 + m\xi_1)\mathcal{O}_1. \quad (\text{A.8})$$

These three matrix equations provide strong constraints on the undetermined parameter matrices a, b .

A.1 Staggered (scalar, scalar) modules

In this subsection we construct the staggered modules from two scalars ϕ, π with $l = 1$. The Carrollian magnetic-sector scalar falls into this class.

The two highest-weight modules W_1, W_2 are generated by ϕ, π respectively, and we can make the most general ansatz respecting the conformal dimension and spin of $\text{SO}(3)$ without Levi-Civita tensor,

$$\begin{aligned} [D, \phi] &= \Delta_\phi \phi, & [J^i_j, \phi] &= 0, & [B_i, \phi] &= 0, & [K_\mu, \phi] &= 0, \\ [D, \pi] &= \Delta_\pi \pi + c_0 \partial_0 \phi, & [J^i_j, \pi] &= 0, & [B_i, \pi] &= c_1 \partial_i \phi, \\ [K_0, \pi] &= c_2 \phi, & [K_i, \pi] &= 0. \end{aligned} \quad (\text{A.9})$$

As a consistency check we unfreeze the condition $\Delta_\pi - \Delta_\phi = l = 1$ and treat Δ_π as a free parameter. From the definition of the representation, the nontrivial constraints come from the commutators $[B_i, K_i], [D, B_i]$. The relation $[[B_i, K_i], \pi] = [B_i, [K_i, \pi]] - [K_i, [B_i, \pi]]$ gives $(c_2 + 2c_1 \Delta_\phi)\phi = 0$, and from $[D, B_i]$ we have $c_1(1 + \Delta_\phi - \Delta_\pi)\partial_i \phi = 0$. Solving them with respect to Δ_ϕ , we find the solutions can be classified as,

$$\text{trivial:} \quad c_0 = c_1 = c_2 = 0, \quad \Delta_\pi \text{ is arbitrary.} \quad (\text{A.10})$$

$$B\text{-staggered:} \quad c_0 = 0, c_2 + 2c_1 \Delta_\phi = 0, \quad \Delta_\pi = \Delta_\phi + 1 \quad (\text{A.11})$$

$$D\text{-staggered:} \quad c_0 \neq 0, c_1 = c_2 = 0, \quad \Delta_\pi \text{ is arbitrary.} \quad (\text{A.12})$$

$$\{B, D\}\text{-staggered:} \quad c_0 \neq 0, c_2 + 2c_1 \Delta_\phi = 0, \quad \Delta_\pi = \Delta_\phi + 1 \quad (\text{A.13})$$

For the first solution the modules generated by ϕ, π are decoupled and the extension is trivial. For the second solution, the independent parameter c_1 can be absorbed by the field renormalization $\phi \rightarrow a_1 \phi, \pi \rightarrow a_2 \pi$, hence the magnetic scalar is the only non-trivial extension. For the third and fourth solutions with $c_0 \neq 0$, the free parameter c_0 indicates the dilatation D contains Jordan blocks, which is similar to the staggered modules in Logarithmic CFTs. Besides, $[D - \Delta_\pi, \pi] = c_0 \partial_0 \phi$ is a singular vector, consistent with the previous analysis that $A = (D - \Delta_2)\tilde{\mathcal{O}}_2$ is singular.

The structure of the extended module V is illustrated as below. Notice that due to $[K_0, P_0] = 0$, the operator $\partial_0 \phi$ is singular in W_1 and V , and $W_{1,2} = W_1/W_{1,1}$ is an additional

quotient module.

$$\begin{array}{ccccccc}
 \Delta_\phi & & & & \phi & & \\
 & & P_0 & & \swarrow & & K_{0,c_2} \\
 \Delta_\phi + 1 & & \partial_0 \phi & \xleftarrow{B_i} & \partial_i \phi & \xleftarrow{B_{i,c_1}} & \pi \\
 & & \vdots & & \vdots & & \vdots \\
 \Delta_\phi + n + 1 & & \partial^n \partial_0 \phi & \xleftarrow{\dots} & \partial^n \partial_i \phi & \xleftarrow{\dots} & \partial^n \pi \\
 & & \vdots & & \vdots & & \vdots \\
 & & W_{1,1} & & W_{1,2} & & W_2
 \end{array}
 \tag{A.14}$$

Finally if allowing the spacial Levi-Civita tensor, the ansatz should be modified by

$$[J^i_j, \pi] = e_1 \epsilon_j^i \partial_0 \phi + d_1 \epsilon_j^{ik} \partial_k \phi, \quad [B_i, \pi] = c_1 \partial_i \phi + e_2 \epsilon^{ij} \partial_j \phi, \tag{A.15}$$

where the d -term and e -terms only appear in $d = 3, 4$ respectively. Similar computation shows that for $d = 4$ the d -term must vanish, while for $d = 3$, only e_2 is forced to vanish, and e_1 is a free parameter. The extra staggered module in $d = 3$ is

$$\begin{aligned}
 [D, \phi] &= \Delta_\phi \phi, & [J^i_j, \phi] &= 0, & [B_i, \phi] &= 0, & [K_\mu, \phi] &= 0, \\
 [D, \pi] &= (\Delta_\phi + 1)\pi + c_0 \partial_0 \phi, & [J^i_j, \pi] &= e_1 \epsilon_j^i \partial_0 \phi, & [B_i, \pi] &= c_1 \partial_i \phi, \\
 [K_0, \pi] &= -2c_1 \Delta_\phi \phi, & [K_i, \pi] &= 0.
 \end{aligned}
 \tag{A.16}$$

A.2 Computation of $\text{Ext}_{\mathfrak{g}}((0), (j))$

The traditional way of computing the vector space $\text{Ext}(W_2, W_1)$ is through the projective or injective resolution of modules. But unlike the case of semisimple Lie algebras, we find that the category of finite dimensional modules of $\mathfrak{iso}(3)$ does not have enough projective/injective modules, and infinite dimensional modules must enter into the game.

Instead, we can utilize the relation between Ext and Lie algebra cohomology, $\text{Ext}_{\mathfrak{g}}(\mathbb{C}, M) \simeq H^1(\mathfrak{g}, M)$, see e.g. [67]. For $W_2 = (0) = \mathbb{C}$ and $W_1 = (1) = M$, the problem gets reduced to the computation of the first cohomology $H^1(\mathfrak{g}, M)$ of $\mathfrak{iso}(3)$. The cohomology of semi-direct product of Lie algebras can be calculated via the Hochschild-Serre spectral sequence [69, 70]. We need only theorem 13 in [70]: supposing Lie algebra \mathfrak{g} and its ideal subalgebra \mathfrak{l} such that $\mathfrak{g}/\mathfrak{l}$ is semisimple, for the module M of \mathfrak{g} , the cohomology is

$$H^n(\mathfrak{g}, M) \simeq \bigoplus_{i+j=n} H^i(\mathfrak{g}/\mathfrak{l}, \mathbb{C}) \otimes H^j(\mathfrak{l}, M)^{\mathfrak{g}}, \tag{A.17}$$

where $M^{\mathfrak{g}} \equiv H^0(\mathfrak{g}, M)$ means the invariant vectors in M : $M^{\mathfrak{g}} := \{x \in M : gx = 0, \forall g \in \mathfrak{g}\}$.

In our case, $\mathfrak{g} = \mathfrak{iso}(3)$, $\mathfrak{l} = \mathbb{C}^3$, $\mathfrak{g}/\mathfrak{l} = \mathfrak{so}(3)$ and M is the singlet module (1) of $\mathfrak{iso}(3)$. By the Wighthead lemma the first cohomology of semisimple Lie algebra vanishes $H^1(\mathfrak{so}(3), \mathbb{C}) = 0$, hence there is only one term at the right-hand side of (A.17)

$$H^1(\mathfrak{iso}(3), (1)) = H^0(\mathfrak{so}(3), \mathbb{C}) \otimes H^1(\mathbb{C}^3, (1))^{\mathfrak{g}}. \quad (\text{A.18})$$

The H^1 is just Hom, hence $H^1(\mathbb{C}^3, (1)) = \text{Mat}_3(\mathbb{C})$. Then $\text{Mat}_3(\mathbb{C})^{\mathfrak{g}}$ contains a constant diagonal matrix $c \cdot I$, which is invariant under the rotations. Finally with $H^0(\mathfrak{so}(3), \mathbb{C}) = \mathbb{C}$ we get $H^1(\mathfrak{iso}(3), (1)) = \mathbb{C}$. Hence the electric vector is the only nontrivial extension of (1) by (0). With the same method we can show $\text{Ext}_{\mathfrak{g}}((0), (j)) = 0$ for $j \neq 1$, and this agrees with the result of the chain representations.

B Path-integral formalism for Carrollian theories

In this section we present the path-integral formalism for free Carrollian field theories, including the scalar and electromagnetic field theories. In each theory, we discuss the electric sector and magnetic sector separately.

B.1 Electric sector of scalar

The electric sector of Carrollian scalar theory has the action

$$S_E^{\mathbb{C}} = -\frac{1}{2} \int d^d x \partial_t \phi \partial_t \phi = -\frac{1}{2} \int d^d x \phi (-\partial_t^2) \phi \quad (\text{B.1})$$

up to a boundary term. Its corresponding generating functional is

$$\begin{aligned} \mathcal{Z}_E^{\mathbb{C}}[j_\phi] &= N \int \mathcal{D}\phi \mathcal{D}\pi \exp\left(i\left(S_E^{\mathbb{C}} + \int d^d x j_\phi \phi\right)\right) \\ &= N' \exp\left(i \int d^d x \frac{1}{2} j_\phi(x) \Pi_E(x-y) j_\phi(y)\right) \end{aligned} \quad (\text{B.2})$$

where $\Pi_E(x-y) = \int \frac{d^d p}{(2\pi)^d} e^{i(\vec{p}\cdot\vec{x} + \omega t)} \cdot \frac{1}{\omega^2} = -\frac{|t|}{2} \delta^{(d-1)}(\vec{x})$. Thus the 2-pt correlator is simply

$$\begin{aligned} \langle \phi(x) \phi(y) \rangle &= (-i)^2 \frac{1}{\mathcal{Z}[0]} \frac{\delta^2}{\delta j_\phi(x) \delta j_\phi(y)} \mathcal{Z}_E^{\mathbb{C}}[J] \Big|_{j_\phi=0} \\ &= -i \Pi_E(x-y) = \frac{i|t_x - t_y|}{2} \delta^{(d-1)}(\vec{x} - \vec{y}) \end{aligned} \quad (\text{B.3})$$

B.2 Magnetic sector of scalar

In order to perform the path-integral for the magnetic sector action, we need to rewrite the action in a standard balanced quadratic form:

$$S_M^{\mathbb{C}} = -\frac{1}{2} \int d^d x 2\pi \partial_t \phi + \partial_i \phi \partial_i \phi = -\frac{1}{2} \int d^d x \Phi^\dagger \hat{A} \Phi \quad (\text{B.4})$$

where $\Phi = (\phi, \pi)$ and the operator $\hat{A} = \begin{pmatrix} -\partial^2 & -\partial_t \\ \partial_t & 0 \end{pmatrix}$.

By adding a source term and then performing the Gaussian integral, we easily get the related generating functional $\mathcal{Z}[J]$ in the following form

$$\begin{aligned}
 \mathcal{Z}_M^c[J] &= N \int \mathcal{D}\phi \mathcal{D}\pi \exp\left(i\left(S_M^c + \int d^d x J\Phi\right)\right) \\
 &= N \int \mathcal{D}\phi \mathcal{D}\pi \exp\left(i \int d^d x \left(-\frac{1}{2}\Phi^\dagger \hat{A}\Phi + J\Phi\right)\right) \\
 &= N' \exp\left(i \int d^d x \frac{1}{2} J^\dagger \hat{A}^{-1} J\right) \\
 &= N' \exp\left(i \int d^d x \int d^d y \frac{1}{2} J^\dagger(x) \Pi_M(x-y) J(y)\right)
 \end{aligned} \tag{B.5}$$

where $J = (j_\phi, j_\pi)$, and $\Pi_M(x-y)$ is the Green function of \hat{A} , satisfying $\hat{A}\Pi(x-y) = \delta^{(d-1)}(\vec{x})$. The Green function can be calculated by inverting \hat{A} in the momentum space,

$$\begin{aligned}
 \Pi_M(x-y) &= \int \frac{d^d p}{(2\pi)^d} e^{i(\vec{p}\cdot\vec{x}+\omega t)} \cdot \begin{pmatrix} \vec{p}^2 & -i\omega \\ i\omega & 0 \end{pmatrix}^{-1} \\
 &= \int \frac{d^d p}{(2\pi)^d} e^{i(\vec{p}\cdot\vec{x}+\omega t)} \cdot \begin{pmatrix} 0 & -i \\ i & -\vec{p}^2 \\ \omega & \omega^2 \end{pmatrix}.
 \end{aligned} \tag{B.6}$$

Then the 2-pt correlator can be read from the partition function,

$$\begin{aligned}
 \langle \Phi_i(x) \Phi_j(y) \rangle &= (-i)^2 \frac{1}{Z[0]} \frac{\delta^2}{\delta J_i(x) \delta J_j(y)} \mathcal{Z}_M^c[J] \Big|_{J=0} \\
 &= -\frac{i}{2} (\Pi_{ij}(x-y) + \Pi_{ji}(y-x)) \\
 &= -\frac{i}{2} \int \frac{d^d p}{(2\pi)^d} e^{i(\vec{p}\cdot\vec{x}+\omega t)} \cdot (\Pi_{ij}(p) + \Pi_{ji}(-p)) \\
 &= \int \frac{d^d p}{(2\pi)^d} e^{i(\vec{p}\cdot\vec{x}+\omega t)} \cdot \begin{pmatrix} 0 & -\frac{1}{\omega} \\ \frac{1}{\omega} & i\frac{\vec{p}^2}{\omega^2} \end{pmatrix} \\
 &= \frac{i}{2} \begin{pmatrix} 0 & -\text{Sign}(t_x - t_y) \delta^{(d-1)}(\vec{x} - \vec{y}) \\ \text{Sign}(t_x - t_y) \delta^{(d-1)}(\vec{x} - \vec{y}) & |t_x - t_y| \vec{\partial}^2 \delta^{(d-1)}(\vec{x} - \vec{y}) \end{pmatrix}.
 \end{aligned} \tag{B.7}$$

This gives us the correlators in (3.25).

B.3 Electric sector of U(1) theory

For the electric sector of U(1), it is very similar to the magnetic sector of scalar. The action is

$$S_E^c = -\frac{1}{2} \int d^d x F_{0i} F_{0i} = -\frac{1}{2} \int d^d x \Phi^\dagger \hat{B} \Phi \tag{B.8}$$

with $\Phi = (A_i, A_0)$ and

$$\hat{B} = \begin{pmatrix} -\partial_0^2 \delta_{ij} & \partial_0 \partial_i \\ \partial_0 \partial_j & -\vec{\partial}^2 \end{pmatrix}. \tag{B.9}$$

Then with the temporal gauge-fixing term $\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_0 A_0)^2$, we may read the correlators. In $d = 4$, the correlators are

$$\begin{aligned} \langle A_i(x)A_j(0) \rangle &= \langle A_j(x)A_i(0) \rangle = \frac{i}{2}\delta_{ij}|t|\delta^{(3)}(\vec{x}) + \frac{i\xi}{12}t^3\text{Sign}(t)\partial_i\partial_j\delta^{(3)}(\vec{x}), \\ \langle A_i(x)A_0(0) \rangle &= \langle A_0(x)A_i(0) \rangle = \frac{i\xi}{4}t^2\text{Sign}(t)\partial_i\delta^{(3)}(\vec{x}), \quad \langle A_0(x)A_0(0) \rangle = \frac{i\xi}{2}|t|\delta^{(3)}(\vec{x}). \end{aligned} \quad (\text{B.10})$$

Moreover, in the Landau gauge $\xi = 0$, the correlators containing A_0 vanish and the remaining ones are

$$\langle A_i(x)A_j(0) \rangle = \frac{i}{2}\delta_{ij}|t|\delta^{(3)}(\vec{x}). \quad (\text{B.11})$$

B.4 Magnetic sector of U(1) theory

For the U(1) magnetic sector, we can rewrite the action as

$$\begin{aligned} S_M^{\mathcal{C}} &= -\frac{1}{4}\int d^d x F_{ij}F_{ij} + 4F_{0i}F_{vi} - 2F_{0v}^2 \\ &= -\frac{1}{4}\int d^d x (\partial_i A_j - \partial_j A_i)^2 + 4(\partial_0 A_i - \partial_i A_0)(\pi_i - \partial_i A_v) - 2(\pi_0 - \partial_0 A_v)^2 \\ &= -\frac{1}{2}\int d^d x \Phi^\dagger \hat{B} \Phi \end{aligned} \quad (\text{B.12})$$

with $\Phi = (A_v, A_i, A_0, \pi_i, \pi_0)$ and

$$\hat{B} = \begin{pmatrix} \partial_0^2 & \partial_0\partial_j & -\vec{\partial}^2 & 0 & -\partial_0 \\ \partial_0\partial_i & -\vec{\partial}^2\delta_{ij} + \partial_i\partial_j & 0 & -\partial_0\delta_{ij} & 0 \\ -\vec{\partial}^2 & 0 & 0 & \partial_j & 0 \\ 0 & \partial_0\delta_{ij} & -\partial_i & 0 & 0 \\ \partial_0 & 0 & 0 & 0 & -1 \end{pmatrix}. \quad (\text{B.13})$$

Since there are abundant gauge symmetries, this matrix is not invertible in the momentum space. In order to properly inverse it and perform the path-integral, we ought to include the gauge-fixing terms.

The gauge fixing of magnetic U(1) sector is tricky, especially if we want to keep the Carrollian conformal invariance. We actually need two gauge-fixing terms for the first-order and second-order gauge transformations in (4.14). Just as we prefer to choose Lorentz invariant gauge in the relativistic gauge theory, the ideal gauge-fixing term $\mathcal{L}_{gf} d^d x$ should be a Carrollian conformal invariant one in $d = 4$. The choice of the first-order gauge fixing is simple by setting the temporal derivative of the time component of A_μ to zero, $\partial_0 A_0 = 0$, with the help of a R_ξ -type auxiliary field, leading to the gauge-fixing term $\mathcal{L}_1 = -\frac{1}{2\xi_1}(\partial_0 A_0)^2$. However, it turns out to be impossible to find a usual quadratic gauge-fixing term for the second-order gauge transformation, which retains the invariances under both the boost and special conformal transformations simultaneously. Nevertheless, it is possible to find a Carrollian conformal invariant gauge-fixing term, if we allow for more general terms. By direct calculation, we find that the gauge-fixing term (4.15) turns up to be Carrollian conformal invariant up to total derivatives.

Unlike the usual gauge-fixing term of a quadratic form, which can be implemented into the path-integral by the standard Faddeev-Popov procedure, the gauge-fixing term (4.15) here contains a part which is obviously not quadratic, and one may question whether such an exotic term will function properly as a gauge fixing term for the second-order gauge transformation. To show that (4.15) can play the role of gauge fixing, we need to use a generalized “Stueckelberg trick” as in chapter 14.5 of [71]. Shortly speaking, this trick works as follows. One may formally multiply and then divide the path-integral of the generating function by a suitable infinite function $f(\xi)$ which is an integration of auxiliary field, then in the multiplication one exchanges the orders of doing the integrations, and shift the auxiliary field to get the gauge-fixing term R_ξ . To apply this trick to the case at hand, we need to select $f(\xi)$ appropriately to make it consistent with our non-quadratic gauge-fixing term. We can consider the following function of ξ_1, ξ_2 , which is an integration over two auxiliary fields $\alpha(x)$ and $\beta(x)$,

$$\begin{aligned} f(\xi_1, \xi_2) &= \int \mathcal{D}\alpha(x) \mathcal{D}\beta(x) \exp \left\{ -i \int d^d x \left(\frac{1}{2\xi_1} (\partial_0^2 \alpha)^2 + \frac{1}{2\xi_2} \left(3(\partial_0 \partial_i \alpha)^2 + 2\partial_0^2 \alpha \partial_0 \beta \right) \right) \right\} \\ &= \int \mathcal{D}\alpha(x) \mathcal{D}\beta(x) \exp \left\{ -i \int d^d x \left(\frac{1}{2\xi_1} \left(\partial_0^2 \alpha + \frac{\xi_1}{\xi_2} \partial_0 \beta \right)^2 + \frac{3}{2\xi_2} (\partial_0 \partial_i \alpha)^2 - \frac{\xi_1}{2\xi_2^2} (\partial_0 \beta)^2 \right) \right\}. \end{aligned}$$

This integral is actually divergent, but is in a form of Gaussian integral if we treat $\partial_0 \alpha(x)$, $\partial_i \alpha(x)$, and $\partial_0 \beta$ as independent integration variables. Then, we shift the auxiliary fields one by one

$$\alpha(x) \rightarrow \alpha(x) - \frac{1}{\partial_0} A_0(x), \quad \beta(x) \rightarrow \beta(x) - \frac{1}{\partial_0} \left((2\pi_0(x) - \partial_0 A_v(x) - \partial_i A_i(x)) + \vec{\partial}^2 \alpha(x) \right),$$

where we have used the notation that $h(x) = \frac{1}{\partial_0} g(x)$ is the solution of $\partial_0 h(x) = g(x)$. As the shift does not change the integral, we have

$$\begin{aligned} f(\xi_1, \xi_2) &= \int \mathcal{D}\alpha(x) \mathcal{D}\beta(x) \exp \left\{ -i \int d^d x \left(\frac{1}{2\xi_1} (\partial_0^2 \alpha - \partial_0 A_0)^2 \right. \right. \\ &\quad \left. \left. + \frac{1}{2\xi_2} \left(3(\partial_0 \partial_i \alpha - \partial_i A_0)^2 + 2(\partial_0^2 \alpha - \partial_0 A_0)(\partial_0 \beta - (2\pi_0 - \partial_0 A_v - \partial_i A_i) - \vec{\partial}^2 \alpha) \right) \right) \right\}. \end{aligned} \tag{B.14}$$

We can multiply and divide (B.14) when doing the path-integral, and use the “Stueckelberg trick” to perform the gauge transformation shift, with $\alpha(x), \beta(x)$ being the gauge parameters,

$$\begin{aligned} A_i(x) &\rightarrow A_i(x) + \partial_i \alpha(x), & A_0(x) &\rightarrow A_0(x) + \partial_0 \alpha(x), \\ A_v(x) &\rightarrow A_v(x) + \beta(x), & \pi_i(x) &\rightarrow \pi_i(x) + \partial_i \beta(x), & \pi_0(x) &\rightarrow \pi_0(x) + \partial_0 \beta(x). \end{aligned} \tag{B.15}$$

Since the measures $\mathcal{D}\alpha(x) \mathcal{D}\beta(x) \mathcal{D}\Phi(x)$, the action $S_M^c[\Phi(x)]$, and gauge-invariant operators \mathcal{O}_i are all invariant under the transformations, we have

$$\begin{aligned} &\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \\ &= \frac{1}{Z[0]} \left(\frac{1}{f(\xi_1, \xi_2)} \int \mathcal{D}\alpha \mathcal{D}\beta \right) \times \int \mathcal{D}\Phi \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \\ &\quad \times \exp \left\{ i \left(S_M^c - \frac{1}{2\xi_1} (\partial_0 A_0)^2 - \frac{1}{2\xi_2} (3\partial_i A_0 \partial_i A_0 + 2\partial_0 A_0 (2\pi_0 - \partial_0 A_v - \partial_i A_i)) \right) \right\}. \end{aligned} \tag{B.16}$$

Therefore, we see that the term (4.15) appears in the action naturally. In other words, (4.15) can be taken as the gauge-fixing term. With this term, it is not difficult to perform the standard inversion in the momentum space, and then use Fourier transform to calculate the correlators. The 2-point correlator of the field can be written as

$$\langle \Phi_i(x) \Phi_j(y) \rangle = -\frac{i}{2} (\Pi_{ij}(x-y) + \Pi_{ji}(y-x)), \quad (\text{B.17})$$

where $\Pi_{ij}(x-y)$ is the position space Green function, as in (B.6). It needs to be pointed out that even if we drop the ξ_1 term and only keep the ξ_2 term in \mathcal{L}_{gf} , the inversion is also possible and well-defined.

In the end, we find the 2-point correlators in the magnetic sector of U(1) theory

$$\begin{aligned} \langle A_v(x) A_v(0) \rangle &= -2i \left(1 + \frac{\xi_2^2}{4\xi_1} \right) |t| \delta^{(3)}(\vec{x}) - \frac{i\xi_2}{12} t^3 \text{Sign}(t) \vec{\partial}^2 \delta^{(3)}(\vec{x}), \\ \langle A_v(x) A_i(0) \rangle &= \langle A_i(x) A_v(0) \rangle = \frac{i\xi_2}{4} t^2 \text{Sign}(t) \partial_i \delta^{(3)}(\vec{x}), \\ \langle A_v(x) A_0(0) \rangle &= \langle A_0(x) A_v(0) \rangle = \frac{i\xi_2}{2} |t| \delta^{(3)}(\vec{x}), \\ \langle A_v(x) \pi_i(0) \rangle &= -\langle \pi_i(x) A_v(0) \rangle = \frac{3i}{2} \left(1 + \frac{\xi_2^2}{3\xi_1} \right) |t| \partial_i \delta^{(3)}(\vec{x}) + \frac{i\xi_2}{12} t^3 \text{Sign}(t) \partial_i \vec{\partial}^2 \delta^{(3)}(\vec{x}), \\ \langle A_v(x) \pi_0(0) \rangle &= -\langle \pi_0(x) A_v(0) \rangle = i \left(1 + \frac{\xi_2^2}{2\xi_1} \right) \text{Sign}(t) \delta^{(3)}(\vec{x}) + \frac{i\xi_2}{4} t^2 \text{Sign}(t) \vec{\partial}^2 \delta^{(3)}(\vec{x}), \\ \langle A_i(x) \pi_j(0) \rangle &= -\langle \pi_j(x) A_i(0) \rangle = -\frac{i}{2} \delta_{ij} \text{Sign}(t) \delta^{(3)}(\vec{x}) - \frac{i\xi_2}{4} t^2 \text{Sign}(t) \partial_i \partial_j \delta^{(3)}(\vec{x}), \\ \langle A_i(x) \pi_0(0) \rangle &= -\langle \pi_0(x) A_i(0) \rangle = \langle A_0(x) \pi_i(0) \rangle = -\langle \pi_i(x) A_0(0) \rangle = -\frac{i\xi_2}{2} |t| \partial_i \delta^{(3)}(\vec{x}), \\ \langle A_0(x) \pi_0(0) \rangle &= -\langle \pi_0(x) A_0(0) \rangle = -\frac{i\xi_2}{2} \text{Sign}(t) \delta^{(3)}(\vec{x}), \\ \langle \pi_i(x) \pi_j(0) \rangle &= \langle \pi_j(x) \pi_i(0) \rangle = \frac{i}{2} \left(1 + \frac{\xi_2^2}{\xi_1} \right) |t| \left(\partial_i \partial_j \delta^{(3)}(\vec{x}) + \delta_{ij} \vec{\partial}^2 \delta^{(3)}(\vec{x}) \right) \\ &\quad + \frac{i\xi_2}{12} t^3 \text{Sign}(t) \partial_i \partial_j \vec{\partial}^2 \delta^{(3)}(\vec{x}), \\ \langle \pi_i(x) \pi_0(0) \rangle &= \langle \pi_0(x) \pi_i(0) \rangle = \frac{i}{2} \left(1 + \frac{\xi_2^2}{\xi_1} \right) \text{Sign}(t) \partial_i \delta^{(3)}(\vec{x}) + \frac{i\xi_2}{4} t^2 \text{Sign}(t) \partial_i \vec{\partial}^2 \delta^{(3)}(\vec{x}), \\ \langle \pi_0(x) \pi_0(0) \rangle &= i \left(1 + \frac{\xi_2^2}{\xi_1} \right) \delta(t) \delta^{(3)}(\vec{x}) + \frac{i\xi_2}{2} t^2 |t| \vec{\partial}^2 \delta^{(3)}(\vec{x}). \end{aligned} \quad (\text{B.18})$$

It can be checked directly that for every choice of ξ_1, ξ_2 , the Ward Identities are all satisfied, which reveals the Carrollian conformal invariance of the theory. Though these expressions look complicated, we can select the Landau-type gauge $\xi_2 = 0$ to simply them and obtain

the nonvanishing correlators listed in (4.16). Here we list them again for completeness.

$$\begin{aligned}
\langle A_v(x)A_v(0) \rangle &= -2i|t|\delta^{(3)}(\vec{x}), & \langle A_v(x)\pi_i(0) \rangle &= -\langle \pi_i(x)A_v(0) \rangle = \frac{3i}{2}|t|\partial_i\delta^{(3)}(\vec{x}), \\
\langle A_v(x)\pi_0(0) \rangle &= -\langle \pi_0(x)A_v(0) \rangle = i\text{Sign}(t)\delta^{(3)}(\vec{x}), \\
\langle A_i(x)\pi_j(0) \rangle &= -\langle \pi_j(x)A_i(0) \rangle = -\frac{i}{2}\delta_{ij}\text{Sign}(t)\delta^{(3)}(\vec{x}), \\
\langle \pi_i(x)\pi_j(0) \rangle &= \langle \pi_j(x)\pi_i(0) \rangle = \frac{i}{2}|t|\left(\partial_i\partial_j\delta^{(3)}(\vec{x}) + \delta_{ij}\partial^2\delta^{(3)}(\vec{x})\right), \\
\langle \pi_i(x)\pi_0(0) \rangle &= \langle \pi_0(x)\pi_i(0) \rangle = \frac{i}{2}\text{Sign}(t)\partial_i\delta^{(3)}(\vec{x}), & \langle \pi_0(x)\pi_0(0) \rangle &= i\delta(t)\delta^{(3)}(\vec{x}).
\end{aligned}
\tag{B.19}$$

C Ward identities and 2-point correlation functions

In this appendix, we review the constraints on the 2-point correlation functions of the primary operators from the Ward identities of Carrollian conformal symmetries. There could be four classes of the correlators with different structures, which will be labeled by Case 1.1, Case 1.2, Case 2.1, and Case 2.2. It turns out that the correlators discussed in the main text belong to Case 2.1.

Similar to the case in CFT, the structure of 2-point correlation functions in CCFT is very much constrained by the Ward identities of the symmetries. For the Carrollian conformal symmetries, the corresponding Ward identities are listed in (C.1),

$$\begin{aligned}
P_\mu : & \quad (\partial_1^\mu + \partial_2^\mu)\langle \mathcal{O}_1\mathcal{O}_2 \rangle = 0, \\
D : & \quad x^\mu\partial^\mu\langle \mathcal{O}_1\mathcal{O}_2 \rangle + \Delta_1\langle \mathcal{O}_1\mathcal{O}_2 \rangle + \Delta_2\langle \mathcal{O}_1\mathcal{O}_2 \rangle = 0, \\
J_{ij} : & \quad (x^i\partial^j - x^j\partial^i)\langle \mathcal{O}_1\mathcal{O}_2 \rangle + \langle (J^{ij}\mathcal{O}_1)\mathcal{O}_2 \rangle + \langle \mathcal{O}_1(J^{ij}\mathcal{O}_2) \rangle = 0, \\
B_i : & \quad x^i\partial_t\langle \mathcal{O}_1\mathcal{O}_2 \rangle + \langle [B_i, \mathcal{O}_1]\mathcal{O}_2 \rangle + \langle \mathcal{O}_1[B_i, \mathcal{O}_2] \rangle = 0, \\
K_0 : & \quad (\langle [K_0, \mathcal{O}_1]\mathcal{O}_2 \rangle + \langle \mathcal{O}_1[K_0, \mathcal{O}_2] \rangle) - x^i(\langle [B_i, \mathcal{O}_1]\mathcal{O}_2 \rangle - \langle \mathcal{O}_1[B_i, \mathcal{O}_2] \rangle) = 0, \\
K_i : & \quad (\langle [K_i, \mathcal{O}_1]\mathcal{O}_2 \rangle + \langle \mathcal{O}_1[K_i, \mathcal{O}_2] \rangle) + x^i(\Delta_1 - \Delta_2)\langle \mathcal{O}_1\mathcal{O}_2 \rangle \\
& \quad + x^j(\langle [J^i_j, \mathcal{O}_1]\mathcal{O}_2 \rangle - \langle \mathcal{O}_1[J^i_j, \mathcal{O}_2] \rangle) + t(\langle [B_i, \mathcal{O}_1]\mathcal{O}_2 \rangle - \langle \mathcal{O}_1[B_i, \mathcal{O}_2] \rangle) = 0.
\end{aligned}
\tag{C.1}$$

It should be mentioned that we have used the techniques explained in the appendix of [43] to simplify the expression for Carrollian special conformal transformation generators K_0, K_i . These identities hold for all of the correlators appearing in this article. The one from the translational generator P_μ requires that

$$\langle \mathcal{O}_1\mathcal{O}_2 \rangle = f(x^\mu),
\tag{C.2}$$

where $x^\mu = x_1^\mu - x_2^\mu$.

As shown in [43], by solving the Ward identities, the 2-point correlators of the operators in a CCFT is generically composed of two independent types, one being of the power-law form, the other being proportional to the Dirac δ -function. In [43], the authors have discussed the one of the power-law form in detail. In this appendix, we mainly focus on the 2-point correlators for the primary operators in chain representations, and pay

more attention to the correlators which appear as the generalized functions¹⁰ in general d dimensions, including the Dirac δ -functions. The techniques used here is similar to the ones in [43], and we strongly recommend the reader to find more details there.

It should also be stressed that here we only consider the correlators of the primary operators. Some operators in the staggered modules, like π in the magnetic scalar theory, are special in the sense that they are neither primary ($KO \neq 0$) nor descendent, and their correlators can not be constrained by the discussions here. Even though these operators do obey some Ward identities from their transformation laws, which help us to determine their correlators, there is short of general rules on the correlators of these operators.

For the primary operators $\mathcal{O}_1, \mathcal{O}_2$, their 2-point correlation function $f = \langle \mathcal{O}_1 \mathcal{O}_2 \rangle$ is a homogeneous function by using the Ward identity of D ,

$$D : \quad (t\partial_t + x^i \partial_i) f(t, \vec{x}) + (\Delta_1 + \Delta_2) f(t, \vec{x}) = 0. \quad (\text{C.3})$$

The solution to this equation is a combination of two independent solutions, the power-law functions and the generalized functions like the (derivatives of) Dirac δ -distribution. For example, the one-dimensional version of this differential equation is

$$x\partial f(x) + \lambda f(x) = 0, \quad (\text{C.4})$$

with the solution being

$$f(x) = c_1 x^{-\lambda} + c_2 \partial^{(\lambda-1)} \delta(x), \quad (\text{C.5})$$

where c_i are constants, and $c_2 \neq 0$ for $\lambda = 1, 2, \dots$. In the Carrollian case, t direction and x_i directions could be considered separately, and thus the solution to (C.3) is simply

$$f(t, \vec{x}) = g(t)g(\vec{x}), \quad (\text{C.6})$$

where $g(t)$ and $g(\vec{x})$ are the homogeneous generalized functions of the form (C.5).

Another important constraint is from the Ward identity of B_i on the lowest-level correlators $f = \langle \mathcal{O}_1 \mathcal{O}_2 \rangle$:

$$B_i : \quad x^i \partial_t f(t, \vec{x}) = 0. \quad (\text{C.7})$$

By the ‘‘lowest-level’’, we mean $[B_i, \mathcal{O}_1] = [B_i, \mathcal{O}_2] = 0$. Considering the fact $x\delta(x) = 0$, we find four independent solutions,

$$\partial_t f = 0 : \quad \begin{cases} f(t, \vec{x}) \propto P(\vec{x}), & (\text{Case 1.1}) \\ f(t, \vec{x}) \propto \prod_i \partial_i^{n_i} (\vec{\partial}^2)^n \delta^{(d-1)}(\vec{x}), \quad \Delta_1 + \Delta_2 = d-1 + \sum_i n_i + 2n, & (\text{Case 1.2}) \end{cases}$$

$$x^i f = 0 : \quad \begin{cases} f(t, \vec{x}) \propto P(t) \delta^{(d-1)}(\vec{x}), & (\text{Case 2.1}) \\ f(t, \vec{x}) \propto \partial_t^{n_t} \delta(t) \delta^{(d-1)}(\vec{x}), \quad \Delta_1 + \Delta_2 = d + n_t, & (\text{Case 2.2}) \end{cases}$$

where both $P(t)$ and $P(\vec{x})$ are the power-law functions, and Case 1.2 appears for $\Delta_1 + \Delta_2 = d-1, d, d+1, \dots$ and Case 2.2 appears for $\Delta_1 + \Delta_2 = d, d+1, d+2, \dots$. In fact, the correlators of the primary operators in this paper belong to Case 2.1.

¹⁰A nice introduction to the generalized functions can be found in [58].

The Case 1.1 with $f(t, \vec{x}) \propto P(\vec{x})$ being the power-law function has been discussed in [43]. In the rest of this section, we first repeat the constraints in Case 1.1 and then discuss the other situations.

C.1 Case 1.1 and Case 1.2

As shown in [43], the chain representations can have the following forms

$$\begin{aligned}
 & (j) \\
 & (j) \rightarrow (j), \quad j \neq 0 \\
 & (0) \rightarrow (1) \rightarrow (0), \\
 & \dots \rightarrow (j) \rightarrow (j+1) \rightarrow (j+2) \rightarrow \dots, \\
 & \dots \rightarrow (j) \rightarrow (j-1) \rightarrow (j-2) \rightarrow \dots.
 \end{aligned} \tag{C.8}$$

For Case 1.1, the correlators could be the power-law functions of x^μ , and the non-vanishing 2-point correlators only appear in the case that $\mathcal{O}_1, \mathcal{O}_2$ have (partially) inverse structure, and the selection rule is $\Delta_1 = \Delta_2$. The correlator takes the form

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \frac{C (t/|\vec{x}|)^r I_{j_1, j_2}^{m_1, m_2}}{|\vec{x}|^{(\Delta_1 + \Delta_2)}} \delta_{\Delta_1, \Delta_2}, \tag{C.9}$$

where I is a rank-0 homogeneous function of x_i representing the tensor structure of O_i .

For Case 1.2, $\Delta_1 + \Delta_2 \geq d - 1 \in \mathbb{Z}$, there exists another solution for the lowest-level 2-point correlators,

$$f(t, \vec{x}) \propto \prod_i \partial_i^{n_i} (\vec{\partial}^2)^n \delta^{(d-1)}(\vec{x}), \quad \sum_i n_i = \Delta_1 + \Delta_2 - (d-1) - 2n, \quad n_i \in \mathbf{N}^+. \tag{C.10}$$

For the higher-level correlators, the solutions are of the form $f'(t, \vec{x}) \propto t^r \prod_i \partial_i^{n'_i} (\vec{\partial}^2)^n \delta^{(d-1)}(\vec{x})$ with $\sum_i n'_i - 2n - r = \Delta_1 + \Delta_2 - (d-1), n'_i \in \mathbf{N}^+$. The full restriction on the 2-point correlators in Case 1.2 is similar to Case 1.1, except the case that one of the operators is a scalar, which will be discussed separately later. The reason that the selection rule is (almost) the same is that the power laws are proportional to (derivatives of) Dirac δ -functions under canonical regularization [58]:

$$\left. \frac{2}{\Omega_{(d-1)}} \frac{r^\lambda}{\Gamma\left(\frac{\lambda+d-1}{2}\right)} \right|_{\lambda=-(d-1)-2k} = \frac{(-1)^k (d-2)!}{2^k k! (d-1+2k-2)!} (\vec{\partial}^2)^k \delta^{(d-1)}(\vec{x}) \tag{C.11}$$

for $k = 0, 1, 2, \dots$, with $r^2 = \sum_i x_i^2$. As a result, most of the constraints from the Ward identities are the same as the ones in Case 1.1. Thus if $\Delta_1 + \Delta_2 \geq d - 1 \in \mathbb{Z}$ and $\Delta_1 = \Delta_2$, the correlators are non-vanishing for \mathcal{O}_1 and \mathcal{O}_2 in partially inverse representations, and the structures of the correlators are of the form

$$\left\langle \mathcal{O}_{1, l_1}^{\{s_1\}} \mathcal{O}_{2, l_2}^{\{s_2\}} \right\rangle = C t^r (D_{s_1} D_{s_2} (\vec{\partial}^2)^n \delta^{(d-1)}(\vec{x}) - \text{traces}), \quad \text{with } D_{s_i} = \partial_{s_{i,1}} \cdots \partial_{s_{i, l_i}} \tag{C.12}$$

The explicit selection rule is rather tedious, and we do not repeat them here. The interested readers may refer [43] for detailed discussions.

The exceptional situation in Case 1.2 is when one of the primary operators is in scalar representation (0). In this case, there is one additional set of the selection rules, due to the special property of Dirac δ -function. In the following, we explain how this additional selection rule emerges and show the structure of the correlators in this situation. Firstly, for the simplest case that both \mathcal{O}_1 and \mathcal{O}_2 are scalars with $\Delta_1 + \Delta_2 = d - 1$, the correlator is $f = \langle \mathcal{O}_1 \mathcal{O}_2 \rangle \propto \delta^{(d-1)}(\vec{x})$ in Case 1.2. It is known that

$$\begin{aligned} \text{Case 1.1: } & x_i f \propto \frac{x_i}{r^{(d-1)}} \neq 0, \\ \text{Case 1.2: } & x_i f \propto x_i \delta^{(d-1)}(\vec{x}) = 0, \end{aligned} \quad (\text{C.13})$$

which makes the constraints from the Ward identities of K_i on f for Case 1.1 and 1.2 different,

$$\begin{aligned} \text{Case 1.1: } & \text{solution: } f = \frac{C_1}{r^{(d-1)}}, \quad \text{constraint: } \Delta_1 = \Delta_2 = \frac{d-1}{2}, \\ \text{Case 1.2: } & \text{solution: } f = C_2 \delta^{(d-1)}(\vec{x}), \quad \text{constraint: } \Delta_1 + \Delta_2 = d-1. \end{aligned} \quad (\text{C.14})$$

Thus for Case 1.2, we have the selection rule

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = C \delta^{(d-1)}(\vec{x}), \quad \mathcal{O}_1, \mathcal{O}_2 \in (0), \quad \Delta_1 + \Delta_2 = d-1. \quad (\text{C.15})$$

Next, we consider the case that \mathcal{O}_1 is in more complicated chain representation. In the case that $\mathcal{O}_1 \in (j)$ is a symmetric traceless tensor (STT) with spin j , $\mathcal{O}_2 \in (0)$ is a scalar. Using the fact $x_i \partial_i \delta^{(d-1)}(\vec{x}) = -\delta^{(d-1)}(\vec{x})$, we find that the restrictions from the Ward identities of K_i are

$$\langle \mathcal{O}_1^{\{s_1, \dots, s_j\}} \mathcal{O}_2 \rangle = C (\partial_{s_1} \dots \partial_{s_j} \delta^{(d-1)}(\vec{x}) - \text{traces}), \quad \Delta_1 = 1, \quad \Delta_2 = d-2+j. \quad (\text{C.16})$$

The ‘‘traces’’ term is the trace of $\partial_{s_1} \dots \partial_{s_j} \delta^{(d-1)}(\vec{x})$, and subtracting this term makes the correlators respect the traceless condition of \mathcal{O}_1 . Moreover for $\mathcal{O}_1 \in (j)_2 \rightarrow (j)_1$ and $\mathcal{O}_2 \in (0)$, we have¹¹

$$\begin{aligned} \langle \mathcal{O}_{1,(j)_2}^{\{s_1, \dots, s_j\}} \mathcal{O}_2 \rangle &= C (\partial_{s_1} \dots \partial_{s_j} \delta^{(d-1)}(\vec{x}) - \text{traces}), \quad \langle \mathcal{O}_{1,\text{others}} \mathcal{O}_2 \rangle = 0, \\ & \Delta_1 = 1, \quad \Delta_2 = d-2+j, \end{aligned} \quad (\text{C.17})$$

For \mathcal{O}_1 being a decreasing chain, $\mathcal{O}_1 \in (j+n) \rightarrow (j+n-1) \dots \rightarrow (j+1) \rightarrow (j)$ and $\mathcal{O}_2 \in (0)$, we have

$$\langle \mathcal{O}_{1,l_1=j+r}^{\{s_1, \dots, s_{l_1}\}} \mathcal{O}_2 \rangle = \frac{C t^r}{r!} (\partial_{s_1} \dots \partial_{s_{l_1}} \delta^{(d-1)}(\vec{x}) - \text{traces}), \quad \Delta_1 = 1, \Delta_2 = d-2+j, \quad (\text{C.18})$$

where \mathcal{O}_{1,l_1} is the spin- l_1 part of \mathcal{O}_1 . For \mathcal{O}_1 in an increasing chain representation, $\mathcal{O}_1 \in (j) \rightarrow (j+1) \dots \rightarrow (j+n-1) \rightarrow (j+n)$, and $\mathcal{O}_2 \in (0)$, the correlators vanish except for the highest-rank sector in \mathcal{O}_1 . Namely, we have

$$\langle \mathcal{O}_{1,(j)}^{\{s_1, \dots, s_j\}} \mathcal{O}_2 \rangle = C (\partial_{s_1} \dots \partial_{s_j} \delta^{(d-1)}(\vec{x}) - \text{traces}), \quad \langle \mathcal{O}_{1,\text{others}} \mathcal{O}_2 \rangle = 0, \quad \Delta_1 + \Delta_2 = d-1+j. \quad (\text{C.19})$$

¹¹Here we use subscripts to distinguish different sectors of $(j)_2 \rightarrow (j)_1$ with the same spin j . Similar notation for $(0)_3 \rightarrow (1)_2 \rightarrow (0)_1$ will appear below.

And finally, for $\mathcal{O}_1 \in (0)_3 \rightarrow (1)_2 \rightarrow (0)_1$ and $\mathcal{O}_2 \in (0)$, we have

$$\langle \mathcal{O}_{1,(0)_3} \mathcal{O}_2 \rangle = C \delta^{(d-1)}(\vec{x}), \quad \langle \mathcal{O}_{1,\text{others}} \mathcal{O}_2 \rangle = 0, \quad \Delta_1 + \Delta_2 = d - 1. \quad (\text{C.20})$$

We have presented all the exceptional cases involving a scalar primary operator. Here we only discuss the case that the other operator belong to a chain representation, and we do not discuss the case that the other operator is in a net-like representation.

C.2 Case 2.1 and 2.2

Case 2.1 and Case 2.2 come from the fact that $x^i \delta^{(d-1)}(\vec{x}) = 0$ solves the equation of the Ward identities of B_i . The selection rules for these two cases are very different from the ones in Case 1.1 and Case 1.2.

First we consider Case 2.1 with the operators being the symmetric traceless tensors (STTs) and in the singlet representations (j). Since the spacial dependence in the correlators is always $\delta^{(d-1)}(\vec{x})$, the only possible non-vanishing lowest-level correlator is from the case that \mathcal{O}_1 and \mathcal{O}_2 have the same spin, $l_1 = l_2$. It can be checked that the Ward identities of K_i are manifestly satisfied using the fact that $x^i \delta^{(d-1)}(\vec{x}) = 0$, and there is no selection rule on Δ_1 and Δ_2 . Therefore we have

$$\begin{aligned} \langle \mathcal{O}_1 \mathcal{O}_2 \rangle &= C t^{(d-1-\Delta_1-\Delta_2)} \delta^{(d-1)}(\vec{x}), & l_1 = l_2 = 0, \\ \langle \mathcal{O}_1^{i_1} \mathcal{O}_2^{j_1} \rangle &= C \delta_{j_1}^{i_1} t^{(d-1-\Delta_1-\Delta_2)} \delta^{(d-1)}(\vec{x}), & l_1 = l_2 = 1, \\ \langle \mathcal{O}_1^{i_1 i_2} \mathcal{O}_2^{j_1 j_2} \rangle &= C \left(\delta_{j_1}^{i_1} \delta_{j_2}^{i_2} + \delta_{j_2}^{i_1} \delta_{j_1}^{i_2} - \frac{2}{d-1} \delta^{i_1 i_2} \delta_{j_1 j_2} \right) t^{(d-1-\Delta_1-\Delta_2)} \delta^{(d-1)}(\vec{x}), & l_1 = l_2 = 2, \\ & \vdots \\ \langle \mathcal{O}_1^{i_1 \dots i_s} \mathcal{O}_2^{j_1 \dots j_s} \rangle &= C \left(\delta_{j_1}^{i_1} \dots \delta_{j_s}^{i_s} - \text{trace} \right) t^{(d-1-\Delta_1-\Delta_2)} \delta^{(d-1)}(\vec{x}), & l_1 = l_2 = s. \end{aligned} \quad (\text{C.21})$$

The ‘‘trace’’ term is to cancel the trace of \mathcal{O}_1 indices and the trace of \mathcal{O}_2 indices, as both \mathcal{O}_1 and \mathcal{O}_2 are STTs. The coefficient C 's are undetermined constants.

For the chain representations, there are very limited restrictions for the correlators being non-vanishing. The calculations show that if two chain representations have the same sub-sector, the correlators of the operators in these sub-sectors and in the higher levels are non-vanishing. In other words, if

$$\begin{aligned} \mathcal{O}_1 &\in \dots \rightarrow (j_{n+1}) \rightarrow (j_n) \rightarrow (j_{n-1}) \rightarrow \dots, \\ \mathcal{O}_2 &\in \dots \rightarrow (j_{m+1}) \rightarrow (j_m) \rightarrow (j_{m-1}) \rightarrow \dots, \end{aligned} \quad \text{with } j_n = j_m \quad (\text{C.22})$$

then

$$\langle \mathcal{O}_{1,l_1=j_{\geq n}} \mathcal{O}_{2,l_2=j_{\geq m}} \rangle \neq 0. \quad (\text{C.23})$$

For the chains, there are the selection rules on Δ_1 and Δ_2 , but the specific selection rule must be discussed case by case. For examples, we have

$$\begin{aligned} & \left\langle \mathcal{O}_{1,l_1}^{\{s_1, \dots, s_{l_1}\}} \mathcal{O}_{2,l_2}^{\{r_1, \dots, r_{l_2}\}} \right\rangle \\ &= C \frac{(d-1-\Delta_1-\Delta_2)! t^{(d-1-\Delta_1-\Delta_2+l_1+l_2)}}{(d-1-\Delta_1-\Delta_2+l_1+l_2)!} (\partial_{s_1} \dots \partial_{s_{l_1}} \partial_{r_1} \dots \partial_{r_{l_2}} \delta^{(d-1)}(\vec{x}) - \text{traces}) \quad (\text{C.24}) \\ & \quad \text{for } \mathcal{O}_1, \mathcal{O}_2 \in \dots \rightarrow (2) \rightarrow (1) \rightarrow (0), \quad \text{with } \Delta_1 = \Delta_2 = 1. \end{aligned}$$

$$\begin{aligned} & \left\langle \mathcal{O}_{1,l_1}^{\{s_1, \dots, s_{l_1}\}} \mathcal{O}_{2,l_2}^{\{r_1, \dots, r_{l_2}\}} \right\rangle \\ &= C \frac{(d-1-\Delta_1-\Delta_2)! t^{(d-1-\Delta_1-\Delta_2+l_1+l_2-2)}}{(d-1-\Delta_1-\Delta_2+l_1+l_2)!} \left(\delta_{(s_1}^{(r_1)} \partial_{s_2} \dots \partial_{s_{l_1})} \partial^{r_2} \dots \partial^{r_{l_2}} \right) \delta^{(d-1)}(\vec{x}) - \text{traces} \\ & \quad \text{for } \mathcal{O}_1, \mathcal{O}_2 \in \dots \rightarrow (3) \rightarrow (2) \rightarrow (1), \quad \text{with } \Delta_1 = \Delta_2 = 0. \quad (\text{C.25}) \end{aligned}$$

Especially, we have

$$\begin{aligned} \left\langle \mathcal{O}_{1,(0)_3} \mathcal{O}_{2,(0)_3} \right\rangle &= C \frac{t^{(d-1-2\Delta+2)}}{(5-2\Delta)(\Delta-3)} \partial^2 \delta^{(d-1)}(\vec{x}) \\ \left\langle \mathcal{O}_{1,(0)_3} \mathcal{O}_{2,(1)_2}^r \right\rangle &= C \frac{t^{(d-1-2\Delta+1)}}{(\Delta-3)} \partial_r \delta^{(d-1)}(\vec{x}) \\ \left\langle \mathcal{O}_{1,(1)_2}^s \mathcal{O}_{2,(0)_3} \right\rangle &= C \frac{t^{(d-1-2\Delta+1)}}{(\Delta-3)} \partial_s \delta^{(d-1)}(\vec{x}) \quad (\text{C.26}) \\ \left\langle \mathcal{O}_{1,(0)_3} \mathcal{O}_{2,(0)_1} \right\rangle &= C t^{(d-1-2\Delta)} \delta^{(d-1)}(\vec{x}) \\ \left\langle \mathcal{O}_{1,(1)_2}^s \mathcal{O}_{2,(1)_2}^r \right\rangle &= C \frac{1-\Delta}{\Delta-3} t^{(d-1-2\Delta)} \delta_{sr} \delta^{(d-1)}(\vec{x}) \\ \left\langle \mathcal{O}_{1,(0)_1} \mathcal{O}_{2,(0)_3} \right\rangle &= C t^{(d-1-2\Delta)} \delta^{(d-1)}(\vec{x}) \quad \langle \text{others} \rangle = 0 \\ & \quad \text{for } \mathcal{O}_1, \mathcal{O}_2 \in (0)_3 \rightarrow (1)_2 \rightarrow (0)_1, \quad \text{with } \Delta_1 = \Delta_2 = \Delta \end{aligned}$$

The selection rule for Case 2.2 is the same as the ones for Case 2.1. Different from the relation between Case 1.1 and 1.2, there is no exceptional situation. The analog of the exceptional case in Case 1.2 is when $\Delta_1 + \Delta_2 = d$ with the correlator $f \propto \delta(t) \delta^{(d-1)}(\vec{x})$, but the constraint from the Ward identities of K_i gives similar selection rules for Case 2.1 and 2.2.

The correlators appeared in the main text are all of Case 2.1. The primary operator in the electric scalar theory is the field ϕ , and the correlator is $\langle \phi(x) \phi(0) \rangle = \frac{i}{2} |t| \delta^{(d-1)}(\vec{x})$. It can be checked that the correlator satisfies the Ward identities, no matter if the temporal part is in power of t or $|t|$, and this correlator matches the form of (C.21). Similar to the electric scalar theory, the magnetic scalar theory have the primary operator ϕ with $\langle \phi(x) \phi(0) \rangle = 0$, which obviously matches the form of (C.21). The primary operators in the electric sector of electromagnetic theory are $A_\mu = (A_0, A_i)$. They are in $(1) \rightarrow (0)$ representation and the corresponding correlators are (B.10). These correlators have the same form with (C.24). Finally, the fundamental operators $A_\alpha = (A_v, A_i, A_0)$ in the magnetic

sector of electromagnetic theory are primary operators, which are in $(0) \rightarrow (1) \rightarrow (0)$ representation. Their correlators are in (B.18) which match the ones in (C.26) with $\Delta_1 = \Delta_2 = 1$.

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