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Perturbation theory for the logarithm of a positive operator

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Abstract: In various contexts in mathematical physics, such as out-of-equilibrium physics and the asymptotic information theory of many-body quantum systems, one needs to compute the logarithm of a positive unbounded operator. Examples include the von Neumann entropy of a density matrix and the flow of operators with the modular Hamiltonian in the Tomita-Takesaki theory. Often, one encounters the situation where the operator under consideration, which we denote by Δ, can be related by a perturbative series to another operator Δ_0, whose logarithm is known. We set up a perturbation theory for the logarithm $\log \Delta$. It turns out that the terms in the series possess a remarkable algebraic structure, which enables us to write them in the form of nested commutators plus some "contact terms".

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1 Introduction

In many different problems in mathematical physics, one needs to compute the logarithm of a positive operator. This is commonplace in asymptotic quantum information theory when one is interested in various quantities constructed from the logarithm of a density matrix. For example, given a reduced density matrix ρ, one needs to compute $\log \rho$ to find the von Neumann entropy and relative entropies. The modular Hamiltonian $K_\rho = -\log \rho$ generalizes the role Hamiltonian plays in a thermal state to an arbitrary state outside the equilibrium ρ. In the same way that equilibrium entropy and energy satisfy laws of thermodynamics, in out-of-equilibrium states ρ, the von Neumann entropy and the expectation value of a modular Hamiltonian satisfy laws of entanglement thermodynamics. Quantum relative entropy generalizes the concept of free energy difference to out-of-equilibrium states. Its calculation also requires calculating the logarithm of a positive operator.

In the past two decades, with growing applications of entanglement entropy in condensed matter and high-energy physics, there have been numerous calculations of such von Neumann and relative entropies in the thermodynamic or continuum limit. The problem is that most of such calculations rely on the analytic continuation method called "*replica* *trick*" whose domain of validity remains physically obscure. The perturbation theory we set up here is a rigorous and unambiguous framework to resolve this problem.

In a general quantum system, in the Tomita-Takesaki theory, given the modular operator Δ_Ω of a state $|\Omega\rangle$ or the relative modular operator of two states $\Delta_{\Psi\Omega}$ one needs to compute their logarithms to obtain the modular flow operator Δ_Ω^{it}, or to calculate relative entropies. In this case, the positive operator in question, Δ_Ω, is unbounded.

Consider an unbounded positive operator Δ. In general, obtaining $\log \Delta$ directly is difficult since it has a simple form only in the spectral decomposition of the operator Δ. We consider the following situation: (i) Δ is related to some other positive operator Δ_0 by a smooth deformation, i.e. there exists a continuous parameter λ and a family of operators $\Delta(\lambda)$ that interpolate between $\Delta(0) = \Delta_0$ and $\Delta(1) = \Delta$; (ii) the logarithm $\log \Delta_0$ is known explicitly. Imagine setting up a perturbative series for $\log \Delta(\lambda)$ in terms of $\log \Delta_0$ for λ small. If the perturbation series converges for $\lambda \leq 1$ one can extend the series to $\lambda = 1$. It is the goal of this paper to set up such a perturbation theory. For a discussion of the fractional powers and the logarithm of bounded operators in the Hilbert space see [1, 2]. For bounded operators that belong to the Lie algebra of a Lie group one often uses the Baker-Campbell-Hausdorff (BCH) expansion to compute the logarithm; see [3]. See also recent discussions [4–9] in the context of quantum field theory.

Our main result is the following series expansion

$$\log \Delta_0 - \log \Delta = \sum_{m=1}^{\infty} Q_m \tag{1.1}$$

where Q_m can be written in the form of nested commutators plus "contact" contributions

$$Q_m = \frac{2\pi}{m} \lim_{\epsilon_i \to 0} \int dt_1 \int dt_2 \dots \int dt_m \, F_{\epsilon_i}(t_1, t_2, \dots, t_m) [\cdots [\boldsymbol{\delta}(t_1), \boldsymbol{\delta}(t_2)], \cdots], \boldsymbol{\delta}(t_m)] + P_m \tag{1.2}$$

and

$$\boldsymbol{\delta}(t) = \Delta_0^{-it} \boldsymbol{\delta} \Delta_0^{it}, \qquad \boldsymbol{\delta} = \frac{\boldsymbol{\alpha}}{1 - \boldsymbol{\alpha}/2}, \qquad \boldsymbol{\alpha} = 1 - \Delta_0^{-\frac{1}{2}} \Delta \Delta_0^{-\frac{1}{2}}, \tag{1.3}$$

$$F_{\epsilon_i}(t_1, t_2, \dots, t_m) = f(t_1) g_{\epsilon_1}(t_2 - t_1) g_{\epsilon_2}(t_3 - t_2) \dots g_{\epsilon_{m-1}}(t_m - t_{m-1}) f(t_m), \tag{1.4}$$

$$f(t) = \frac{1}{2\cosh(\pi t)}, \qquad g_\epsilon(t) = \frac{i}{4} \left[\frac{1}{\sinh(\pi(t - i\epsilon))} + \frac{1}{\sinh(\pi(t + i\epsilon))} \right]. \tag{1.5}$$

In (1.2) P_m are given by terms with two fewer integrals ("contact terms") whose structure are a bit complicated and will be given later. The first few terms of this series are given by

$$Q_1 = \frac{\pi}{2}\int_{-\infty}^{\infty}\frac{dt}{\cosh^2(\pi t)}\boldsymbol{\delta}(t), \tag{1.6}$$

$$Q_2 = \frac{\pi}{4}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{dt_1 dt_2}{\cosh(\pi t_1)\cosh(\pi t_2)}g_\epsilon(t_2-t_1)\left[\boldsymbol{\delta}(t_1),\boldsymbol{\delta}(t_2)\right], \tag{1.7}$$

$$Q_3 = \frac{\pi}{6}\lim_{\epsilon_1,\epsilon_2\to 0}\int\frac{dt_1 dt_2 dt_3}{\cosh(\pi t_1)\cosh(\pi t_3)}g_{\epsilon_1}(t_2-t_1)g_{\epsilon_2}(t_3-t_2)\left[\left[\boldsymbol{\delta}(t_1),\boldsymbol{\delta}(t_2)\right],\boldsymbol{\delta}(t_3)\right] + \frac{\pi}{24}\int\frac{\boldsymbol{\delta}(t)^3}{\cosh^2(\pi t)} \tag{1.8}$$

$$Q_4 = \lim_{\epsilon_1,\epsilon_2,\epsilon_3\to 0}\frac{\pi}{8}\int\frac{dt_1 dt_2 dt_3 dt_4\, g_{\epsilon_1}(t_2-t_1)g_{\epsilon_2}(t_3-t_2)g_{\epsilon_3}(t_4-t_3)}{\cosh(\pi t_1)\cosh(\pi t_4)}\left[\left[\left[\boldsymbol{\delta}(t_1),\boldsymbol{\delta}(t_2)\right],\boldsymbol{\delta}(t_3)\right],\boldsymbol{\delta}(t_4)\right] + \frac{\pi}{32}\lim_{\epsilon\to 0}\int\frac{dt_1 dt_2\, g_\epsilon(t_2-t_1)}{\cosh(\pi t_1)\cosh(\pi t_2)}\left\{\boldsymbol{\delta}(t_2),\left[\boldsymbol{\delta}(t_1),\boldsymbol{\delta}^2(t_2)\right]\right\} \tag{1.9}$$

It is important in (1.2) that performs the integrals keeping ϵ's nonzero and then take the $\epsilon_i \to 0$ limit. We have included quintic contact term P_5 in appendix B.

In the special case the operators Δ and Δ_0 are both bounded one can use the spectral representation of these operators to match our expansion and the BCH expansion order by order in λ.

The plan of the paper is as follows. In section 2, we outline the main steps leading to the proof of (1.2) and give explicit expressions for F_m. In section 3 and section 4 we fill in the details of the proof. In appendix A we give a simple example of harmonic oscillator to illustrate the use of (1.6)–(1.9). In appendix B we present the explicit expression for $P_{m=5}$. Appendices C–E include various fine details for the proof.

2 Outline of the proof

2.1 Setup

We start with the integral representation of the logarithm of an (positive invertible) operator Δ:

$$\log\Delta = \int_0^\infty d\beta\left(\frac{1}{1+\beta} - \frac{1}{\Delta+\beta}\right). \tag{2.1}$$

For unbounded operators Δ, the integral on the right-hand-side should be thought of as a limit of Riemann sums in the strong operator topology induced by the domain of the logarithm of Δ. Thus, we have the operator equality

$$\log\Delta_0 - \log\Delta = \int_0^\infty d\beta\left(\frac{1}{\Delta+\beta} - \frac{1}{\Delta_0+\beta}\right) = \int_0^\infty d\beta\frac{1}{\Delta_0+\beta}(\Delta_0-\Delta)\frac{1}{\Delta+\beta} \tag{2.2}$$

on the common domain of $\log\Delta_0$ and $\log\Delta$.

Introduce the operator

$$\boldsymbol{\alpha} = 1 - \Delta_0^{-\frac{1}{2}}\Delta\Delta_0^{-\frac{1}{2}} \tag{2.3}$$

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which we take to depend on a continuous parameter λ and vanish as $\lambda \to 0$. To lighten the notation, we keep the λ dependence implicit. We stress that $\boldsymbol{\alpha}$ is an unbounded operator despite the fact that it is proportional to a small parameter λ.

Since the function $f(x) = \sqrt{x}(x+\beta)^{-1}$ is bounded for positive β and x the operator $\Delta^{1/2}(\Delta+\beta)^{-1}$ is a bounded operator in the Hilbert space. To make sense of the perturbation theory we assume that there exists a constant c such that $\Delta(\lambda) < c\Delta_0$. As a result, the operator $\Delta_0^{1/2}(\Delta+\beta)^{-1}$ is also bounded. This is part of what we mean by λ being a small perturbation.

For $\beta > 0$, we define the bounded operator

$$A = \frac{\Delta_0^{\frac{1}{2}}}{\Delta_0 + \beta} \tag{2.4}$$

to rewrite the integrand of eq. (2.2) as

$$\begin{aligned} \frac{1}{\Delta+\beta} - \frac{1}{\Delta_0+\beta} &= \frac{1}{\Delta_0+\beta}(\Delta_0 - \Delta)\frac{1}{\Delta+\beta} && (2.5)\\ &= A\boldsymbol{\alpha}\Delta_0^{1/2}(\Delta+\beta)^{-1} = A\boldsymbol{\alpha}(1-\Delta_0^{1/2}A\boldsymbol{\alpha})^{-1}A && (2.6)\\ &= A\boldsymbol{\delta}(1-B\boldsymbol{\delta})^{-1}A, && (2.7)\end{aligned}$$

where we have introduced

$$\boldsymbol{\delta} = \frac{\boldsymbol{\alpha}}{1-\boldsymbol{\alpha}/2}, \qquad B = \frac{\Delta_0 - \beta}{2(\Delta_0+\beta)}\,. \tag{2.8}$$

One might want to naively expand $(1-\Delta_0^{\frac{1}{2}}A\boldsymbol{\alpha})^{-1}$ in the second line of (2.5) in a power series of $\Delta_0^{\frac{1}{2}}A\boldsymbol{\alpha}$. But this is not a good expansion as $\Delta_0^{\frac{1}{2}}A\boldsymbol{\alpha}$ is an unbounded operator. This is similar the approach taken by [9]. It leads to singular integrals which can be sensible only if one provides a prescription to deform the integration Contour. To circumvent this problem, in the third line of (2.5) we introduced the operator $\boldsymbol{\delta}$, which is bounded with a norm $\|\boldsymbol{\delta}\| \le 2$. To see this note that the spectrum of the closure of $\boldsymbol{\alpha}$ is contained in $(-\infty, 1)$ and $\frac{x}{1-x/2}$ is a bounded function in the range $(-2, 2)$. On a dense domain, $\boldsymbol{\delta}$ and its closure agree. Finally, expanding B in the spectral decomposition of Δ_0 we find that the spectrum of B is contained in $\left(0, \frac{1}{2}\right)$. Therefore, $||B|| = 1/2$ and by the Cauchy-Schwarz inequality $||B\boldsymbol{\delta}|| \le 1$.

For $||B\boldsymbol{\delta}|| < 1$ expanding the third line of (2.5) in terms of $B\boldsymbol{\delta}$ gives a convergent series. In general, it is not possible to exclude the $||B\boldsymbol{\delta}|| = 1$ case. To justify our expansion, we will restrict to those vectors $|x\rangle$ in Hilbert space which satisfy

$$||B\boldsymbol{\delta}A|x\rangle|| < ||A|x\rangle||\,. \tag{2.9}$$

On this set, we have the operator equality

$$A\boldsymbol{\delta}(1-B\boldsymbol{\delta})^{-1}A|x\rangle = A\boldsymbol{\delta}\sum_{n=0}^{\infty}(B\boldsymbol{\delta})^n A|x\rangle\,. \tag{2.10}$$

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and the sum on the right-hand-side is pointwise convergent. We will not specify $|x\rangle$ below but its presence should always be kept in mind.

Using (2.5)–(2.10) in (2.2) we then find that

$$\log\Delta_0 - \log\Delta = \int_0^\infty d\beta A\boldsymbol{\delta}\sum_{n=0}^{\infty}(B\boldsymbol{\delta})^n A\,. \tag{2.11}$$

We now further rewrite the above expression using the one-parameter unitary group Δ_0^{it} generated by $\log\Delta_0$. In particular, we use the following integral expressions for A and B

$$A = \frac{1}{\sqrt{\beta}}\int_{-\infty}^{\infty} dt\, f(t)\beta^{it}\Delta_0^{-it}, \qquad B = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt\, g_\epsilon(t)\beta^{it}\Delta_0^{-it} \tag{2.12}$$

with

$$f(t) = \frac{1}{2\cosh(\pi t)}, \qquad g_\epsilon(t) = \frac{i}{4}\left[\frac{1}{\sinh(\pi(t-i\epsilon))} + \frac{1}{\sinh(\pi(t+i\epsilon))}\right]. \tag{2.13}$$

In appendix D, we show that the above $\epsilon\to 0$ limit exists.

Plugging (2.12) into (2.11) we find that

$$\begin{aligned}\log\Delta_0 - \log\Delta = \sum_{m=0}^{\infty}\int_0^\infty \frac{d\beta}{\beta}\int_{-\infty}^{\infty} dt_0\cdots dt_{m+1}\,\beta^{i(t_0+\cdots+t_{m+1})}\\ \times f(t_0)\Delta_0^{-it_0}\boldsymbol{\delta}\left(\prod_{i=1}^{m} g_{\epsilon_i}(t_i)\Delta_0^{-it_i}\boldsymbol{\delta}\right)\Delta_0^{-it_{m+1}}f(t_{m+1})\,.\end{aligned} \tag{2.14}$$

Notice that if we exchange the orders of β and t-integrals (with associated $\epsilon_i\to 0$ limit) the β-integral can be performed explicitly

$$\int_0^\infty \beta^{-1}d\beta\,\beta^{i(t_0+\cdots+t_{m+1})} = 2\pi\delta(t_0+\cdots+t_{m+1})\,. \tag{2.15}$$

Equation (2.14) can then be further written as (shifting the sum of m to start from 1)

$$\log\Delta_0 - \log\Delta = \sum_{m=1}^{\infty} Q_m \tag{2.16}$$

with

$$Q_m = 2\pi\lim_{\epsilon_i\to 0}\int dt_1\int dt_2\ldots\int dt_m\, F_{\epsilon_i}(t_1,t_2,\cdots,t_m)\,\boldsymbol{\delta}(t_1)\cdots\boldsymbol{\delta}(t_m) \tag{2.17}$$

where the kernel F is defined by

$$F_{\epsilon_i}(t_1,t_2,\ldots,t_m) = f(t_1)g_{\epsilon_1}(t_2-t_1)g_{\epsilon_2}(t_3-t_2)\ldots g_{\epsilon_{m-1}}(t_m-t_{m-1})f(t_m) \tag{2.18}$$

and

$$\boldsymbol{\delta}(t) = \Delta_0^{-it}\boldsymbol{\delta}\Delta_0^{it}\,. \tag{2.19}$$

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A variant of (2.16)–(2.18) has appeared previously in [9].[1] The main goal of the paper is to show that the kernel (2.18) has remarkable symmetric properties which enable one to write Q_m in terms of nested commutators of $\boldsymbol{\delta}$'s

$$Q_m = \frac{2\pi}{m}\lim_{\epsilon_i\to 0}\int dt_1\int dt_2\ldots\int dt_m\, F_{\epsilon_i}(t_1,t_2,\ldots,t_m)[\cdots[\boldsymbol{\delta}(t_1),\boldsymbol{\delta}(t_2)],\cdots],\boldsymbol{\delta}(t_m)]+P_m\,. \tag{2.20}$$

The P_m contact term involves terms which contain two fewer integrals. It has the following structure:

$$P_m = \sum_s\int dt_1\cdots dt_{m-3}dt\; J_s(t_1,\cdots t_{m-3};t)M_s(t_1,\cdots t_{m-3};t), \tag{2.21}$$

where s sums over all possible ways in which three t_i's are selected from the set $\{t_1,\cdots t_m\}$ such that at least two of the indices on the chosen t_i are adjacent. The three chosen t_i's are set to be equal to t, with the rest relabeled as $t_1,\cdots t_{m-3}$. J_s is a kernel which can be obtained from F_{ϵ_i} after applying a number of operations which are described in the next subsection. For now, we give some simple examples. Suppose $t_1=t_2=t_3=t$, then

$$M=\delta(t)^3\delta(t_1)\ldots\delta(t_{m-3}),\qquad J=\frac{\pi}{2m}F_\epsilon(t,t_1,\ldots,t_{m-3})\,. \tag{2.22}$$

For $t_1=t_2=t_5=t$, one has $M=\delta(t)^2\delta(t_1)\delta(t_2)\delta(t)\delta(t_3)\ldots\delta(t_{m-3})$ and

$$J=\frac{\pi}{2m}(F_\epsilon(t_2,t_1,t,t_3,t_4\ldots,t_{m-3})-F_\epsilon(t_1,t_2,t,t_3,t_4,\ldots,t_{m-3}))\,. \tag{2.23}$$

As the selected indices become larger, the number of terms in J increases and the terms also become more complicated. For $t_6=t_2=t_3=t$ one has $M=\delta(t_1)\delta(t)^2\delta(t_2)\delta(t_3)\delta(t)\delta(t_4)\ldots\delta(t_{m-3})$, and J has a term of the form $\frac{1}{f(t)^2}F(t_3,t_2,t,t_1)F(t,t_4,t_5,\ldots,t_{m-3})$.

2.2 Basic ideas for the proof

The kernel (2.18) looks complicated, but it satisfies a number of amazing identities under the permutation of its arguments. To explain the basic idea leading to the proof of (2.20), we need to first establish some notation.

Let S_m be the symmetric group of permutations of m-distinct objects. We use the *cycle* notation $(12\cdots m)$ to represent the permutation that sends $1\to 2\to 3\cdots\to(m-1)\to m\to 1$. Any index not listed in the cycle is left untouched. We define the action of an element $\sigma\in S_m$ on a product of operators $\boldsymbol{\delta}(t_{i_1})\boldsymbol{\delta}(t_{i_2})\ldots\boldsymbol{\delta}(t_{i_m})$ and a general function $H(t_{i_1},t_{i_2},\ldots,t_{i_m})$ in the following manner:

$$\sigma(\boldsymbol{\delta}(t_{i_1})\boldsymbol{\delta}(t_{i_2})\ldots\boldsymbol{\delta}(t_{i_m})):=\boldsymbol{\delta}(t_{i_{\sigma(1)}})\boldsymbol{\delta}(t_{i_{\sigma(2)}})\ldots\boldsymbol{\delta}(t_{i_{\sigma(m)}}) \tag{2.24}$$

$$\sigma(H(t_{i_1},t_{i_2},\ldots,t_{i_m})):=H(t_{\sigma(i_1)},t_{\sigma(i_2)},\ldots,t_{\sigma(i_m)}) \tag{2.25}$$

[1]Note that the expansion in [9] is similar to expanding (2.5) in α which is an unbounded operator in our case. To have a convergent series we use the bounded operator $\boldsymbol{\delta}$ as the expansion parameter. Our expressions (2.16)–(2.18) appear to differ from that in [9] in the $i\epsilon$ prescription, the integration contours, and operator orderings.

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Note that in (2.24) σ acts on the left while in (2.25) it acts on the right. See appendix C for further explanations and examples. Let us also introduce the special permutations

$$\mu_j = (j(j-1)(j-2)\dots 4321), \quad \Lambda_j = (1234\dots(j-1)j), \quad \mu_j = \Lambda_j^{-1}. \tag{2.26}$$

One can show that the following statements are true:

1. Introduce the operator

$$T_m = (\mathrm{id} - \mu_m)(\mathrm{id} - \mu_{m-1})\dots(\mathrm{id} - \mu_2) \tag{2.27}$$

where id denotes the identity operation. Then,

$$T_m(\boldsymbol{\delta}(t_1)\dots\boldsymbol{\delta}(t_m)) = [[\cdots[\boldsymbol{\delta}(t_1), \boldsymbol{\delta}(t_2)], \cdots], \boldsymbol{\delta}(t_{m-1})], \boldsymbol{\delta}(t_m)]. \tag{2.28}$$

See [10] for a proof. For completeness, we have included a proof in appendix C.

2. For any function $H(t_1, t_2, \dots, t_m)$ we have

$$\int H(t_1, t_2, \dots, t_m)\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\dots\boldsymbol{\delta}(t_m) = \int H(t_1, t_2, \dots, t_m)\, T_m(\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\dots\boldsymbol{\delta}(t_m)) + \int \boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\dots\boldsymbol{\delta}(t_m)\, \Sigma_m H(t_1, t_2, \dots, t_m) \tag{2.29}$$

where the operation Σ_m is defined as

$$\Sigma_m H(t_1, t_2, \dots, t_m) = \sum_{k=1}^{m-1} (-1)^{k-1}\, \Xi_k^m H(t_1, t_2, \dots, t_m) \tag{2.30}$$

with

$$\Xi_k^m H(t_1, t_2, \dots, t_m) \equiv \sum_{i_1=k+1}^{m} \sum_{i_2=k}^{i_1-1} \sum_{i_3=k-1}^{i_2-1} \cdots \sum_{i_k=2}^{i_{k-1}-1} \Lambda_{i_1}\Lambda_{i_2}\dots\Lambda_{i_k} H(t_1, t_2, \dots, t_m). \tag{2.31}$$

3. For a general operator $O(t_1, \dots, t_m)$ and function F_ϵ of (2.18) we have

$$\lim_{\epsilon_i\to 0}\int O(t_1,\dots,t_m)\Xi_k^m F_\epsilon(t_1,\dots,t_m) = (-1)^k \lim_{\epsilon_i\to 0}\int O(t_1,\dots,t_m) F_\epsilon(t_1,\dots,t_m) + \sum_{j=k+2}^{m} N_j[O] \tag{2.32}$$

where $N_j[O]$ are "contact terms" containing only $m-2$ integrals. They will be given explicitly at the end.

4. If we set $H = F_\epsilon$ in (2.29) and use (2.32) in (2.30) we find that

$$Q_m = \frac{2\pi}{m} \lim_{\epsilon_i\to 0} \int F_\epsilon(t_1, t_2, \dots, t_m) T_m\Big(\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\dots\boldsymbol{\delta}(t_m)\Big) + \frac{2\pi}{m}\sum_{k=1}^{m-1}(-1)^{k-1}\sum_{j=k+2}^{m} N_j \tag{2.33}$$

which leads to (2.20) upon using (2.28) together with the identification

$$P_m = \frac{2\pi}{m}\sum_{k=1}^{m-1}(-1)^{k-1}\sum_{j=k+2}^{m} N_j\,. \tag{2.34}$$

We now give the explicit expressions for N_j. First, let us introduce some definitions. For integers $q_2 < q_1$ we define

$$L^{\epsilon}_{q_1,q_2}(t) \equiv \frac{\Lambda_{q_2} F_\epsilon(t_1,\dots,t_{q_1})}{g_\epsilon(t_1-t_{q_2})g_\epsilon(t_{q_2+1}-t_1)}\Big|_{t_1=t_{q_2}=t_{q_2+1}=t}, \tag{2.35}$$

$$R^{(q_1)}_{q_2}[M] \equiv \lim_{\epsilon_i\to 0}\int L^{\epsilon}_{q_1,q_2}(t) M(t_1,t_2,\dots,t_{q_1})|_{t_1=t_{q_2}=t_{q_2+1}=t} \tag{2.36}$$

where M is some operator and on the right hand side of (2.36) the integrations are over all distinct t_i's (i.e. q_1-2 integrations). Given an operator $O(t_1,t_2,\dots,t_m)$, for each $p\le k$ and $k+3\le j\le m$ where k is the lower index in the object Ξ^m_k, we define the operator W as

$$W(t_1,t_2,\dots,t_p,t_{j-1},t_j) = \frac{1}{f(t_j)f(t_{j-1})}\sum_{i_{p+1}=k-p+1}^{j-p-2}\sum_{i_{p+2}=k-p}^{i_{p+1}-1}\dots\sum_{i_k=2}^{i_{k-1}-1} \tag{2.37}$$
$$\times\int\left(\Pi^{j-2}_{n=p+1}dt_n\Pi^{m}_{l=j+1}dt_l\right)\chi_{pj}O(t_1,\dots t_m)g_{j,j+1,\epsilon_j}g_{j+1,j+2,\epsilon_{j+1}}\cdots g_{m-1,m,\epsilon_{m-1}}f_m$$

where χ_{pj} is given by

$$\chi_{pj}(t_{p+1},t_{p+2},\dots,t_{j-2},t_{j-1}) = \Pi^{j-p}_{n=j-1}\Lambda_n\Pi^{k}_{l=p+1}\Lambda_{i_l}\left(f_1 g_{1,2,\epsilon_1}g_{2,3,\epsilon_2}\cdots g_{j-p-2,j-p-1,\epsilon_{j-p-2}}\right), \tag{2.38}$$

and we have used the following shorthand notations

$$f_m \equiv f(t_m),\quad g_{m-1,m,\epsilon_{m-1}} \equiv g_{\epsilon_{m-1}}(t_m-t_{m-1})\,. \tag{2.39}$$

We also introduce

$$\tilde{O}_j(y_1,y_2,\dots,y_{p+2}) = W(y_{p+2},y_{p+1},\dots,y_3,y_2,y_1) \tag{2.40}$$

which amounts to the relabelling

$$y_1 = t_j,\quad y_2 = t_{j-1},\quad y_{3+q} = t_{p-q},\qquad 0\le q\le p-1 \tag{2.41}$$

Finally, the expression for $N_j[O]$ is given by

$$N_j[O] = \begin{cases} \frac{1}{4}\sum_{p=1}^{k}\sum_{l=2}^{p+1} R^{(p+2)}_l[\tilde{O}_j] & j\ge k+3 \\ \frac{1}{4}\sum_{l=2}^{k+1} R^{(k+2)}_l[\tilde{O}_{k+2}] & j=k+2\end{cases} \tag{2.42}$$

for $m\ge 2$ and $1\le k\le m-1$.

In summary, to obtain the kernel J of (2.21) corresponding to t_j, t_{r+1}, t_r for some $1\le r< j-1$, we use the following algorithm:

1. Choose an integer k such that $1\le k\le j-2$. If $k=j-2$ then set $p=k$, otherwise choose p in the range $r+1\le p\le k$.

2. For each such k and p, apply the string of permutations on the right-hand-side of eq. (2.38) to the first $j-p-2$ arguments of F_ϵ to obtain χ_{pj}. Sum over these permutations.

3. Keep all arguments above j unpermuted, and delete the arguments between $j-p-2$ and j. Divide the result by $\frac{1}{f(t_j)f(t_{j-1})}$.

4. Define the variables y_i as in (2.41). Apply the permutation Λ_{p+2-r} to $F(y_1,\ldots,y_{p+2})$ and divide by $4g_\epsilon(y_1-y_{p+2-r})g_\epsilon(y_{p+3-r}-y_1)$. Now set $t_j=t_{r+1}=t_r$

5. Multiply this by the result obtained in Step 3 after reexpressing y_i in terms of t_i. Perform the sum over p. Sum over k after multipying by $(-1)^{k-1}$. Multiply the whole result by $\frac{2\pi}{m}$.

To obtain J corresponding to the choice $t_j=t_{j-1}=t_r$ when $1<r<j-2$, the previous steps are followed with the choice $p=r$ and no sum over p in the last step. To obtain J corresponding to the choice $t_j=t_{j-1}=t_1$ follow the previous steps after setting $p=1$ and to the result, add the term $\frac{\pi\Lambda_{j-1}F(t_1,\ldots,t_m)}{2m(g_\epsilon(t_1-t_{j-1})g_\epsilon(t_j-t_1))}$. This exhausts all the cases.

The rest of the paper is devoted to establishing (2.28)–(2.32) and justifying the existence of $\epsilon_i\to 0$ limit. In section 3 we prove (2.28) and (2.29). In section 4 we prove (2.32). Appendix D discusses in detail the $\epsilon_i\to 0$ limit. In appendix E we examine more carefully the interchange of β-integral and $\epsilon_i\to 0$ limit used in (2.14).

3 Permutation identities (I)

In this section, we present a proof of eq. (2.29). Consider the integral

$$I_m=\int H(t_1,t_2,\ldots,t_m)\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\ldots\boldsymbol{\delta}(t_m) \tag{3.1}$$

for some function $H(t_1,t_2,\ldots,t_m)$. Then, we have

$$\begin{aligned}I_m&=\int H(t_1,t_2,\ldots,t_m)(1-(12)+(12))\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\ldots\boldsymbol{\delta}(t_m)\\&=\int H(t_1,t_2,\ldots,t_m)\Big[(1-(321))(1-(12))+(321)(1-(12))+(12)\Big]\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\ldots\boldsymbol{\delta}(t_m)\\&=\int H(t_1,t_2,\ldots,t_m)\Big[(1-(4321))(1-(321))(1-(12))\\&\qquad+(4321)(1-(321))(1-(12))+(321)(1-(12))+(12)\Big]\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\ldots\boldsymbol{\delta}(t_m)\,.\end{aligned} \tag{3.2}$$

Note that the third line is simply $T_4[\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_3)\boldsymbol{\delta}(t_4)]\boldsymbol{\delta}(t_5)\cdots\boldsymbol{\delta}(t_m)$, while the fourth line when expanded gives

$$\sum_{k=1}^{3}(-1)^{k-1}\sum_{i_1=k+1}^{4}\sum_{i_2=k}^{i_1-1}\sum_{i_3=k-1}^{i_2-1}\cdots\sum_{i_k=2}^{i_{k-1}-1}\mu_{i_1}\mu_{i_2}\ldots\mu_{i_k}\Big(\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_3)\boldsymbol{\delta}(t_4)\Big)\boldsymbol{\delta}(t_5)\cdots\boldsymbol{\delta}(t_m)\,. \tag{3.3}$$

Continuing this process repeatedly, we get

$$\begin{aligned}I_m=&\int H(t_1,t_2,\ldots,t_m)\,T_m\Big[\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\ldots\boldsymbol{\delta}(t_m)\Big]+\int H(t_1,t_2,\ldots,t_m)\\&\times\sum_{k=1}^{m-1}(-1)^{k-1}\sum_{i_1=k+1}^{m}\sum_{i_2=k}^{i_1-1}\sum_{i_3=k-1}^{i_2-1}\cdots\sum_{i_k=2}^{i_{k-1}-1}\mu_{i_1}\mu_{i_2}\ldots\mu_{i_k}\Big(\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\ldots\boldsymbol{\delta}(t_m)\Big)\,.\end{aligned} \tag{3.4}$$

In eq. (3.4), the permutations act on the operators. But for subsequent applications, we need the permutations to act on functions. This is achieved by the obervations

$$\int H(t_1,t_2,\dots,t_m)\mu_{i_1}\dots\mu_{i_k}\Big(\boldsymbol{\delta}(t_1)\dots\boldsymbol{\delta}(t_m)\Big) \tag{3.5}$$

$$= \int H(t_1,t_2,\dots,t_m)\boldsymbol{\delta}(t_{\tau(1)})\boldsymbol{\delta}(t_{\tau(2)})\dots\boldsymbol{\delta}(t_{\tau(m)}) \qquad \tau = \mu_{i_1}\star\mu_{i_2}\star\dots\star\mu_{i_k} \tag{3.6}$$

$$= \int H(t_{\sigma(1)},t_{\sigma(2)}\dots t_{\sigma(m)})\boldsymbol{\delta}(t_{\tau\sigma(1)})\boldsymbol{\delta}(t_{\tau\sigma(2)})\dots\boldsymbol{\delta}(t_{\tau\sigma(m)}) \tag{3.7}$$

$$= \int \tau^{-1}H(t_1,t_2,\dots,t_m)\boldsymbol{\delta}(t_1)\dots\boldsymbol{\delta}(t_m) \tag{3.8}$$

$$= \int (\mu^{-1}_{i_k}\star\mu^{-1}_{i_{k-1}}\dots\mu^{-1}_{i_2}\star\mu^{-1}_{i_1})H(t_1,t_2,\dots,t_m)\boldsymbol{\delta}(t_1)\dots\boldsymbol{\delta}(t_m) \tag{3.9}$$

$$= \int \Big[\Lambda_{i_1}\Lambda_{i_2}\dots\Lambda_{i_k}H(t_1,t_2,\dots,t_m)\Big]\boldsymbol{\delta}(t_1)\dots\boldsymbol{\delta}(t_m) \tag{3.10}$$

where in (3.6), we used (C.5), in (3.7), used the permutation invariance of n-dimensional integrals, in (3.8), chose $\sigma = \tau^{-1}$, and finally in (3.10), used (C.6) and that μ_j and Λ_j are inverse of each other. Using (3.10) in (3.4) we then find (2.29).

4 Permutation identities (II)

In this section, we prove (2.32) which is the most nontrivial step in the proof of (2.20). Let us first note the identity

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt\, \frac{\sinh(\pi t)}{\sinh(\pi(t\pm i\epsilon))}h(t) = \int_{-\infty}^{\infty} dt\, h(t) \tag{4.1}$$

where $h(t)$ is a regular function at $t = 0$. In subsequent manipulations, we will abbreviate identities of this type by dropping the integral and the limit as

$$\frac{\sinh(\pi t)}{\sinh(\pi(t\pm i\epsilon))} = 1 \tag{4.2}$$

which should (hopefully) cause no confusion. We also remind the reader of the short-hand notation introduced in (2.39). For instance, from (2.18) we have

$$F^{\epsilon}_m \equiv F_{\epsilon_i}(t_1,t_2,\cdots t_m) = f_1 g_{1,2,\epsilon_1} g_{2,3,\epsilon_2}\cdots g_{m-1,m,\epsilon_{m-1}} f_m \tag{4.3}$$

and

$$F^{\epsilon}_m = \left(\frac{g_{m-1,m,\epsilon_{m-1}}f_m}{f_{m-1}}\right)F^{\epsilon}_{m-1}\,. \tag{4.4}$$

4.1 Preparation

Before proving (2.32) we first prove a lemma.

Lemma. Consider the operator

$$I_m[O] \equiv \lim_{\epsilon_i\to 0}\int F^\epsilon_m O(t_1,t_2,\ldots,t_m), \tag{4.5}$$

where $O(t_1,\cdots t_m)$ is a general operator and F^ϵ_m is defined in (4.3). For all $m\geq 2$, we have

$$\sum_{l=2}^{m}\Lambda_l I_m = -I_m + \frac{1}{4}\sum_{l=2}^{m-1} R_l^{(m)}[O] \tag{4.6}$$

where $R_q^{(m)}[O]$ is defined in eq. (2.36). We remind the reader that Λ_l only acts on the kernel function F.

Proof. We prove this by induction on m. Taking $m=2$, $\Lambda_2=(12)$, we get

$$\Lambda_2 I_2 = -I_2 \tag{4.7}$$

which is trivially true by the antisymmetry of the function g. So the correct base case is $m=3$. Performing a few relabelings of the t_i, we get

$$\Lambda_2 I_3+\Lambda_3 I_3=\lim_{\epsilon_1\to 0}\int dt_2\int dt_1 \lim_{\epsilon_2\to 0}\int dt_3\Big(F_{\epsilon_1,\epsilon_2}(t_2,t_3,t_1)+F_{\epsilon_1,\epsilon_2}(t_2,t_1,t_3)\Big)O(t_1,t_2,t_3) \tag{4.8}$$

Using the identity

$$\begin{aligned}&4\cosh(\pi t_2)\sinh(\pi(t_1-t_2))\sinh(\pi(t_2-t_3))\\ &=-\cosh(\pi(t_1-t_2+t_3))-\cosh(\pi(3t_2-t_1-t_3))+\cosh(\pi(t_3+t_2-t_1))+\cosh(\pi(t_1-t_3+t_2))\end{aligned} \tag{4.9}$$

and some algebraic manipulations we can write the term in the parentheses of (4.8) as

$$\begin{aligned}&F_{\epsilon_1,\epsilon_2}(t_2,t_3,t_1)+F_{\epsilon_1,\epsilon_2}(t_2,t_1,t_3)=\\ &-4\left(\frac{i}{4}\right)\frac{f(t_1)f(t_2)f(t_3)\cosh(\pi t_2)\sinh(\pi(t_1-t_2))\sinh(\pi(t_2-t_3))}{|\sinh(\pi(t_3-t_2+i\epsilon_1))|^2|\sinh(\pi(t_1-t_2+i\epsilon_1))|^2}g_{\epsilon_2}(t_1-t_3)\sinh(\pi(t_1-t_3))\\ &+\frac{i}{4}\frac{f(t_1)f(t_2)f(t_3)g_{\epsilon_2}(t_1-t_3)}{|\sinh(\pi(t_3-t_2+i\epsilon_1))|^2|\sinh(\pi(t_1-t_2+i\epsilon_1))|^2}\sinh(\pi(t_1-t_3))\\ &\times\Big[\cosh(\pi(t_1-t_2+t_3))(\cosh(3\pi i\epsilon_1)-1)-\cosh(\pi(t_3+t_2-t_1))(\cosh(i\pi\epsilon_1)-1)\\ &\quad-\cosh(\pi(t_1-t_3+t_2))(\cosh(i\pi\epsilon_1)-1)+\cosh(\pi(3t_2-t_1-t_3))(\cosh(i\pi\epsilon_1)-1)\Big].\end{aligned} \tag{4.10}$$

Integrating against $O(t_1,t_2,t_3)$ and taking $\epsilon_i\to 0$, we get

$$\begin{aligned}\Lambda_2 I_3+\Lambda_3 I_3 &= -\lim_{\epsilon_1\to 0}\int F_{\epsilon_1,\epsilon_1}(t_1,t_2,t_3)O(t_1,t_2,t_3)\\ &-\frac{1}{8}\lim_{\epsilon_1\to 0}\int O(t_1,t_2,t_3)\,\frac{f(t_1)f(t_2)f(t_3)}{|\sinh(\pi(t_3-t_2+i\epsilon_1))|^2|\sinh(\pi(t_1-t_2+i\epsilon_1))|^2}\\ &\times\Big[\cosh(\pi(t_1-t_2+t_3))(\cosh(3\pi i\epsilon_1)-1)-\cosh(\pi(t_3+t_2-t_1))(\cosh(i\pi\epsilon_1)-1)\\ &\quad-\cosh(\pi(t_1-t_3+t_2))(\cosh(i\pi\epsilon_1)-1)+\cosh(\pi(3t_2-t_1-t_3))(\cosh(i\pi\epsilon_1)-1)\Big]\end{aligned} \tag{4.11}$$

The second term of (4.11) can be simplified by noting the identity

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dx \frac{\epsilon}{\epsilon^2+(x-a)^2} f(x) = \pi f(a)\,. \tag{4.12}$$

for f continuous in a, bounded and integrable. Thus, we obtain

$$\Lambda_2 I_3 + \Lambda_3 I_3 = -\lim_{\epsilon_1\to 0}\int F_{\epsilon_1,\epsilon_1}(t_1,t_2,t_3)O(t_1,t_2,t_3) + \frac{1}{4}\int_{-\infty}^{\infty} f(t)^2 O(t,t,t)dt\,. \tag{4.13}$$

From the definitions (2.35)–(2.36) we note that

$$R_2^{(3)}[O(t_1,t_2,t_3)] = \int dt f(t)^2 O(t,t,t) \tag{4.14}$$

and thus

$$\sum_{j=2}^{3} \Lambda_j I_3[O] = -I_3[O] + \frac{1}{4}R_2^{(3)}[O], \tag{4.15}$$

which establishes the lemma for the case $m=3$.

Suppose the lemma holds for some $m \geq 4$, i.e,

$$\sum_{l=2}^{m-1} \Lambda_l I_{m-1}[O] = -I_{m-1}[O] + \frac{1}{4}\sum_{l=2}^{m-2} R_l^{(m-1)}[O] \tag{4.16}$$

Then, we have

$$\begin{aligned}\sum_{l=2}^{m} \Lambda_l I_m[O] &= \lim_{\epsilon\to 0}\sum_{j=2}^{m-2}\int \Lambda_l F_{m-1}^{\epsilon}\left(\frac{g_{m-1,m,\epsilon_{m-1}} f_m}{f_{m-1}}\right) O(t_1,\ldots,t_m)\\ &+ \lim_{\epsilon\to 0}\int \Lambda_m F_m^{\epsilon} O(t_1,\ldots,t_m) + \lim_{\epsilon\to 0}\int \frac{g_{1,m,\epsilon_{m-1}} f_m}{f_1}\Lambda_{m-1}F_{m-1}^{\epsilon} O(t_1,t_2,\ldots,t_m)\,.\end{aligned} \tag{4.17}$$

Rearranging the terms above gives

$$\begin{aligned}\sum_{l=2}^{m}\Lambda_l I_m[O] &= \lim_{\epsilon\to 0}\sum_{l=2}^{m-1}\int \Lambda_l F_{m-1}^{\epsilon}\left(\frac{g_{m-1,m,\epsilon_{m-1}} f_m}{f_{m-1}}\right) O(t_1,\ldots,t_m) + \lim_{\epsilon\to 0}\int \Lambda_m F_m^{\epsilon} O(t_1,\ldots,t_m)\\ &+ \lim_{\epsilon_{m-1}\to 0}\int dt_m \lim_{\epsilon\to 0}\int f_m\left(\frac{g_{1,m,\epsilon_{m-1}}}{f_1} - \frac{g_{m-1,m,\epsilon_{m-1}}}{f_{m-1}}\right)\Lambda_{m-1}F_{m-1}^{\epsilon} O(t_1,t_2,\ldots,t_m).\end{aligned} \tag{4.18}$$

With a computation very similar to the case $m=3$ we find the relation

$$\lim_{\epsilon\to 0}\int \Lambda_m F_m^{\epsilon} O(t_1,\ldots,t_m) \tag{4.19}$$

$$+\lim_{\epsilon_{m-1}\to 0}\int dt_m \lim_{\epsilon\to 0}\int f_m\left(\frac{g_{1,m,\epsilon_{m-1}}}{f_1} - \frac{g_{m-1,m,\epsilon_{m-1}}}{f_{m-1}}\right)\Lambda_{m-1}F_{m-1}^{\epsilon} O(t_1,t_2,\ldots,t_m) \tag{4.20}$$

$$= \frac{1}{4}R_{m-1}^{(m)}[O]. \tag{4.21}$$

Finally, we perform the integration over t_m in the first term on right hand side of (4.18) and use the induction hypothesis to obtain

$$\sum_{l=2}^{m}\Lambda_l I_m[O] = -I_m[O] + \frac{1}{4}\sum_{l=2}^{m-1} R_l^{(m)}[O] \tag{4.22}$$

which proves the lemma.

4.2 Final proof

We now prove (2.32) which using the definition (4.5) can be written as

$$\Xi_k^m I_m[O] = (-1)^k I_m[O] + \sum_{j=k+2}^{m} N_j[O] \tag{4.23}$$

with $N_j[O]$ given in eq. (2.42).

Our strategy is to use induction on m with a fixed k. First, consider $m = k+1$ for which we have

$$\Xi_k^{k+1} = \Lambda_m \Lambda_{m-1} \Lambda_{m-2} \dots \Lambda_2 \,. \tag{4.24}$$

It can be checked by an explicit computation

$$\Xi_k^{k+1} F_m = F(t_m, t_{m-1}, t_{m-2}, \dots, t_3, t_2, t_1) = (-1)^{m-1} F_m \tag{4.25}$$

following from $g_\epsilon(-t) = -g_\epsilon(t)$. This completes the proof for the case $m = k+1$. Next, consider $m = k+2$, which is the base case for our induction argument. By explicit computation one can show that

$$\Xi_k^{k+2} I_m[O] = \sum_{i_1=m-1}^{m} \sum_{i_2=m-2}^{i_1-1} \dots \sum_{i_{m-2}=2}^{i_{m-3}-1} \Lambda_{i_1} \dots \Lambda_{i_{m-2}} I_m[O] = \sum_{l=2}^{m} (\Lambda_l I_m)[O'] \tag{4.26}$$

where O' is defined as

$$O'(y_1, \dots y_m) \equiv O(y_m, y_{m-1}, \dots, y_1) \,. \tag{4.27}$$

Now applying Lemma 1 (4.6) to the right hand side of (4.26) we find

$$\Xi_k^{k+2} I_m[O] = -I_m[O'] + \frac{1}{4} \sum_{j=1}^{k+1} R_j[O'] = (-1)^k I_m[O] + \frac{1}{4} \sum_{l=2}^{k+1} R_l^{(k+2)}[O'] \tag{4.28}$$

where in the last step we have used the fact that $g_\epsilon(t)$ is an odd function. This completes the proof for $m = k+2$.

Now, suppose (2.32) holds for $m \geq k+3$. Remember the definition of Ξ_k^m from (2.31). It can be checked explicitly that $\Xi_k^m = \Xi_k^{m-1} + \Lambda_m \Xi_{k-1}^{m-1}$. Then, we have

$$\Xi_k^m F_m^\epsilon = \sum_{i_1=k+1}^{m} \sum_{i_2=k}^{i_1-1} \sum_{i_3=k-1}^{i_2-1} \dots \sum_{i_k=2}^{i_{k-1}-1} \Lambda_{i_1} \Lambda_{i_2} \dots \Lambda_{i_k} F_m^\epsilon \tag{4.29}$$

$$= \Xi_k^{m-2} \left(F_{m-1}^\epsilon \frac{g_{m-1,m} f_m}{f_{m-1}} \right) + \Lambda_{m-1} \Xi_{k-1}^{m-2} \left(F_{m-1}^\epsilon \frac{g_{m-1,m} f_m}{f_{m-1}} \right) + \Lambda_m \Xi_{k-1}^{m-1} F_m^\epsilon \tag{4.30}$$

$$= \frac{g_{m-1,m} f_m}{f_{m-1}} \Xi_k^{m-1} F_{m-1}^\epsilon + \Lambda_m \Xi_{k-1}^{m-1} F_m^\epsilon + C \Lambda_{m-1} \Xi_{k-1}^{m-2} F_{m-1}^\epsilon \tag{4.31}$$

Here,

$$C = \frac{g_{1,m} f_m}{f_1} - \frac{g_{m-1,m} f_m}{f_{m-1}} \,. \tag{4.32}$$

If we integrate against $O(t_1, t_2, \ldots, t_m)$ and take $\epsilon_i \to 0$, the first term in (4.31) goes to

$$(-1)^k I_m[O] + \sum_{j=k+2}^{m-1} N_j[O] \tag{4.33}$$

by the induction hypothesis. To finish the proof we need to show that the second and the third term add up to $N_m[O]$.

Let us denote the sum of the second and the third terms in (4.31) as V. First, split V into

$$V = S_1 + V_1 \tag{4.34}$$
$$S_1 = \Lambda_m \Xi_{k-1}^{m-3} F_m^\epsilon + C\Lambda_{m-1} \Xi_{k-1}^{m-3} F_{m-1}^\epsilon \tag{4.35}$$
$$V_1 = \Lambda_m \Lambda_{m-1} \Xi_{k-2}^{m-2} F_m^\epsilon + \Lambda_m \Lambda_{m-2} \Xi_{k-2}^{m-3} F_m^\epsilon + C\Lambda_{m-1} \Lambda_{m-2} \Xi_{k-2}^{m-3} F_{m-1}^\epsilon \tag{4.36}$$

where we have repeatedly used $\Xi_k^m = \Lambda_m \Xi_{k-1}^{m-1} + \Xi_k^{m-1}$. Let us rename $y_1 = t_m$, $y_2 = t_{m-1}$ and $y_3 = t_1$ for the moment. We find

$$S_1 = \frac{\chi_{1m}(t)'}{f_{m-1}} \sum_{l=1}^{3} \Lambda_l F_3^\epsilon(y_1, y_2, y_3) \tag{4.37}$$

$$\chi_{1m}(t)' = \Lambda_{m-1} \sum_{i_2=k}^{m-3} \sum_{i_3=k-1}^{i_2-1} \cdots \sum_{i_k=2}^{i_{k-1}-1} \Lambda_{i_2} \ldots \Lambda_{i_k} (f_1 g_{1,2,\epsilon_1} g_{2,3,\epsilon_2} \ldots g_{m-3,m-2,\epsilon_{m-3}}) \,. \tag{4.38}$$

Integrating S_1 against $O(t_1, \ldots, t_m)$ and taking $\epsilon \to 0$, we find this is precisely the $p = 1, j = m$ term in $N_j[O]$ that we are looking for, after applying the Lemma, in eq. (2.42).

Now, V_1 can be further split into

$$V_1 = S_2 + V_2 \tag{4.39}$$
$$S_2 = \Lambda_m \Theta_2^m \Xi_{k-2}^{m-4} F_m^\epsilon + C\Lambda_{m-1} \Lambda_{m-2} \Xi_{k-2}^{m-4} F_{m-1}^\epsilon \tag{4.40}$$
$$V_2 = \Lambda_m \Theta_3^m \Xi_{k-3}^{m-4} F_m^\epsilon + \Lambda_m \Lambda_{m-1} \Lambda_{m-2} \Lambda_{m-3} \Xi_{k-4}^{m-4} F_m^\epsilon + C\Lambda_{m-1} \Lambda_{m-2} \Lambda_{m-3} \Xi_{k-3}^{m-4} F_{m-1}^\epsilon \tag{4.41}$$

where

$$\Theta_2^m \equiv \Lambda_{m-1} + \Lambda_{m-2} \tag{4.42}$$

$$\Theta_3^m \equiv \Lambda_{m-1}\Lambda_{m-2} + \Lambda_{m-1}\Lambda_{m-3} + \Lambda_{m-2}\Lambda_{m-3} = \sum_{i_1=m-2}^{m-1} \sum_{i_2=m-3}^{i_1-1} \Lambda_{i_1} \Lambda_{i_2} \,. \tag{4.43}$$

It is convenient to rename $y_1 = t_m$, $y_2 = t_{m-1}$, $y_3 = t_2$ and $y_4 = t_1$. Now, using the equalities

$$\Lambda_m \Lambda_{m-1} \Xi_{k-1}^{m-4} F_m^\epsilon = \chi'_{2m} g_{m,m-1} g_{2,m} g_{1,2} f_1 \tag{4.44}$$
$$\Lambda_m \Lambda_{m-2} \Xi_{k-1}^{m-4} F_m^\epsilon = \chi'_{2m} g_{2,m-1} g_{m,2} g_{1,m} f_1 \tag{4.45}$$
$$\Lambda_{m-1} \Lambda_{m-2} \Xi_{k-1}^{m-4} F_{m-1}^\epsilon = \chi'_{2m} g_{2,m-1} g_{1,2} f_1 \tag{4.46}$$

we can write S_2 as

$$S_2 = \frac{\chi'_{2m}}{f_{m-1}} \left(\sum_{j=1}^{4} \Lambda_j F_4(y_1, y_2, y_3, y_4) \right) \,. \tag{4.47}$$

Note that since

$$\chi'_{2m}(t) = \Lambda_{m-1}\Lambda_{m-2}\sum_{i_3=k-1}^{m-4}\sum_{i_4=k-2}^{i_2-1}\cdots\sum_{i_k=2}^{i_{k-1}-1}\Lambda_{i_2}\ldots\Lambda_{i_k}(f_1 g_{1,2,\epsilon_1} g_{2,3,\epsilon_2}\ldots g_{m-4,m-3,\epsilon_{m-4}}) \tag{4.48}$$

if we integrate S_2 in (4.47) against $O(t_1,\ldots,t_m)$ and take all the $\epsilon \to 0$ we find the $p=2, j=m$ term in $N_j[O]$ in eq. (2.42).

Now, the pattern is clear: we split the remainder V_j into S_{j+1} and V_{j+1} until $j=k$. Since $V_k = 0$ we have

$$V = \sum_{p=1}^{k} S_p, \quad S_p = \Lambda_m \Theta_p^m \Xi_{k-p}^{m-p-2} F_m^\epsilon + C\Lambda_{m-1}\Lambda_{m-2}\cdots\Lambda_{m-p}\Xi_{k-p}^{m-p-2}F_{m-1}^\epsilon, \tag{4.49}$$

where

$$\Theta_p^m = \sum_{i_1=m-2}^{m-1}\sum_{i_2=m-3}^{i_1-1}\cdots\sum_{i_{p-1}=m-p}^{i_{p-2}-1}\Lambda_{i_1}\cdots\Lambda_{i_{p-1}}\,. \tag{4.50}$$

Note that Θ_p^m is a direct generalization of (4.43)) and $\Xi_0^j = 1$.

The basic idea to compute S_p is the same as that of S_2. One has to show that S_p can be written as

$$S_p = \frac{\chi'_{pm}}{f_{m-1}}\left(\sum_{j=1}^{p+2}\Lambda_j F_{p+2}(y_1,y_2,\cdots y_{p+2})\right) \tag{4.51}$$

with

$$y_1 = t_m, \quad y_2 = t_{m-1}, \quad y_{3+q} = t_{p-q} \ \text{ for } \ 0 \le q \le p-1\,. \tag{4.52}$$

It is important to remember that χ'_{pm} does not depend on any of the y variables. It then follows from the Lemma that after integrating against $O(t_1,\ldots,t_m)$ and sending $\epsilon \to 0$, S_p has the form required in eq. (2.42).

To finish the proof, we need to demonstrate (4.51).[2] For this purpose, let us look at the first term of S_p in (4.49):

$$\Lambda_m \Theta_p^m \Xi_{k-p}^{m-p-2} F_m^\epsilon\,. \tag{4.53}$$

Expanding Ξ explicitly we have

$$\Xi_{k-p}^{m-p-2}F_m^\epsilon = \sum_{i_{p+1}=k-p+1}^{m-p-2}\sum_{i_{p+2}=k-p}^{i_{p+1}-1}\cdots\sum_{i_k=2}^{i_{k-1}-1}\Lambda_{i_{p+1}}\ldots\Lambda_{i_k}F_m^\epsilon \tag{4.54}$$

and

$$\Lambda_{i_{p+1}}\Lambda_{i_{p+2}}\ldots\Lambda_{i_k}F_m^\epsilon = \chi_1 g_{m-p-1,m-p}\cdots g_{m-1,m} f_m \tag{4.55}$$

$$\chi_1 = f_{\tau(1)} g_{\tau(1)\tau(2)} g_{\tau(2)\tau(3)}\cdots g_{\tau(m-p-3),\tau(m-p-2)} g_{\tau(m-p-2),m-p-1} \tag{4.56}$$

where the permutation τ and the function χ_1 depend on the multi-index $i_{p+1},\cdots,i_k$. It is crucial that the permutation τ does not touch any of the indices above $m-p-2$. To

[2] It is enough to take $k \ge 3$ as the previous computations of $S_{1,2}$ are sufficent to cover $k=2$.

lighten the notation, we suppress the i dependence of χ_1. Equation (4.54)–(4.55) further imply

$$\Xi^{m-p-2}_{k-p} F^{\epsilon}_m = \chi_2 g_{m-p-1,m-p} \dots g_{m-1,m} f_m \tag{4.57}$$

where we have introduced

$$\chi_2 = \sum_{i_{p+1}=k-p+1}^{m-p-2} \sum_{i_{p+2}=k-p}^{i_{p+1}-1} \cdots \sum_{i_k=2}^{i_{k-1}-1} \chi_1 \,. \tag{4.58}$$

Next, note that

$$\Lambda_m \Theta^m_p = \Lambda_m \sum_{j=0}^{p-1} K_j, \quad K_j = \left(\Pi^j_{l=1}\Lambda_{m-l}\right)\left(\Pi^{p-1}_{n=j+1}\Lambda_{m-n-1}\right). \tag{4.59}$$

Every cyclic permutation appearing in (4.59), has the exact same action on χ_2, therefore

$$\Lambda_m K_j \chi_2 = \chi'_{pm} = \Lambda_{m-1} \dots \Lambda_{m-p} \chi_2 \,. \tag{4.60}$$

The first equality of (4.60) says that the action of $\Lambda_m K_j$ on χ_2 is independent of j and we have named the resulting function χ'_{pm} in (4.51). The second equality says χ'_{pm} can also be obtained from the permutation $\Lambda_{m-1} \cdots \Lambda_{m-p}$ acting on χ_2. The reasons behind (4.60) are: (i) the number of cycles in each permutation string acting on χ_2 is the same; (ii) the length of each cycle is larger than the highest index occurring in χ_2. As a simple example, consider the following permutation on some function $G(t_1, t_2, t_3)$

$$\Lambda_{10}\Lambda_9\Lambda_8 G(t_1,t_2,t_3) = \Lambda_6\Lambda_5\Lambda_4 G(t_1,t_2,t_3) \,. \tag{4.61}$$

The χ'_{pm} in eq. (4.60) has the precise expression we saw for the $j = m$ contact contribution in eq. (2.42) which is

$$\chi'_{pm} = \Lambda_{m-1} \cdots \Lambda_{m-p} \sum_{i_{p+1}=k-p+1}^{m-p-2} \sum_{i_{p+2}=k-p}^{i_{p+1}-1} \cdots \\ \times \sum_{i_k=2}^{i_{k-1}-1} \Lambda_{i_{p+1}}\Lambda_{i_{p+2}} \dots \Lambda_{i_k}(f_1 g_{1,2,\epsilon_1} g_{2,3,\epsilon_2} \dots g_{m-p-2,m-p-1,\epsilon_{m-p-2}}) \tag{4.62}$$

Now, let us look at how K_j act on the product of the functions in (4.57) other than χ_2. To this end, we define

$$G(u_1, u_2, \dots, u_l) \equiv g_{\epsilon_1}(u_2 - u_1) g_{\epsilon_2}(u_3 - u_2) \dots g_{\epsilon_{l-1}}(u_l - u_{l-1}) \,. \tag{4.63}$$

We then find

$$\Lambda_m \sum_{j=0}^{p-1} K_j G(t_{m-p-1}, t_{m-p}, \dots t_{m-1}, t_m) = \sum_{j=0}^{p-1} G(t_{m-1}, t_p, t_{p-1} \dots, t_{j+2}, t_m, t_{j+1}, t_j, \dots, t_2, t_1) \,. \tag{4.64}$$

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Collecting everything together we thus find (4.53) can be written as

$$\Lambda_m \Theta_p^m \Xi_{k-p}^{m-p-2} F_m^\epsilon = \chi'_{pm} f_1 \sum_{j=0}^{p-1} G(t_{m-1}, t_p, t_{p-1}, \ldots, t_{j+2}, t_m, t_{j+1}, t_j \ldots, t_2, t_1)\,. \tag{4.65}$$

The second term of S_p in (4.49) can be treated exactly in parallel, and we find

$$C\Lambda_{m-1}\Lambda_{m-2}\ldots\Lambda_{m-p}\Xi_{k-p}^{m-p-2}F_{m-1}^\epsilon = \chi'_{pm} f_1 C G(t_{m-1}, t_p, t_{p-1}, \ldots, t_2, t_1) \tag{4.66}$$

Plugging the explicit expression for C, combining (4.65) and (4.66), cancelling some terms, and relabeling the arguments as in (4.52), we then obtain (4.51). This completes the proof.

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A Harmonic oscillator example

As an illustration of the perturbation series (1.1) here we consider a very simple example.

Consider Δ_0 given by the (unnormalized) thermal density matrix for a harmonic oscillator

$$\Delta_0 = \exp(-\beta H), \qquad H = \hat{a}^\dagger \hat{a} \tag{A.1}$$

and Δ by a "squeezed transform" of Δ_0

$$\Delta = S(r)^\dagger \exp(-\beta H) S(r), \quad S(r) = \exp\left(\frac{1}{2} r \hat{a}^2 - \frac{1}{2} r \hat{a}^{\dagger 2}\right), \qquad r > 0\,. \tag{A.2}$$

We will compute $\log\Delta_0 - \log\Delta$ as a perturbative expansion in the parameter r two ways, firstly using the BCH (which in this example is extremely simple) and then using the formula (1.2).

From BCH, it is straightforward to see that

$$\Delta = \exp(-\beta H'), \quad H' = S^\dagger H S = \sigma^\dagger \sigma, \quad \sigma = S^\dagger \hat{a} S = \hat{a}\cosh r - \hat{a}^\dagger \sinh r, \tag{A.3}$$

and thus

$$\log\Delta_0 - \log\Delta = \beta(H'-H) = (2\hat{a}^\dagger\hat{a}+1)\left(\beta r^2 + \frac{\beta r^4}{3}\right) - (\hat{a}^2 + \hat{a}^{\dagger 2})\left(r\beta + \frac{2\beta r^3}{3}\right) + \mathcal{O}(r^5)\,. \tag{A.4}$$

Now let us consider the perturbation series (1.2). Note that

$$\Delta_0^{-it}\hat{a}\Delta_0^{it} = \hat{a}e^{-i\beta t}, \quad \Delta_0^{-1/2}\hat{a}\Delta_0^{1/2} = \hat{a}e^{-\beta/2}, \quad \Delta_0^{1/2}\hat{a}\Delta_0^{-1/2} = \hat{a}e^{\beta/2} \tag{A.5}$$

Here, the domain one should consider is the set of finitely excited Harmonic oscillator states. Throughout, we will be working on this set. Then, we get

$$\boldsymbol{\alpha}=1-\Delta_0^{-1/2}\Delta_1\Delta_0^{-1/2} \tag{A.6}$$

$$=\frac{1}{2}r\Delta_0^{-1/2}[\hat{a}^2-\hat{a}^{\dagger 2},\Delta_0]\Delta_0^{-1/2}-\frac{1}{8}r^2\Delta_0^{-1/2}[\hat{a}^2-\hat{a}^{\dagger 2},[\hat{a}^2-\hat{a}^{\dagger 2},\Delta_0]]\Delta_0^{-1/2} \tag{A.7}$$

$$+\frac{r^3}{3!2^3}\Delta_0^{-1/2}[\hat{a}^2-\hat{a}^{\dagger 2},[\hat{a}^2-\hat{a}^{\dagger 2},[\hat{a}^2-\hat{a}^{\dagger 2},\Delta_0]]\Delta_0^{-1/2} \tag{A.8}$$

$$-\frac{r^4}{2^4 4!}\Delta_0^{-1/2}[\hat{a}^2-\hat{a}^{\dagger 2},[\hat{a}^2-\hat{a}^{\dagger 2},[\hat{a}^2-\hat{a}^{\dagger 2},[\hat{a}^2-\hat{a}^{\dagger 2},\Delta_0]\Delta_0^{-1/2}, \tag{A.9}$$

$$\boldsymbol{\delta}=\frac{\boldsymbol{\alpha}}{1-\boldsymbol{\alpha}/2} \tag{A.10}$$

$$=-r\sinh\beta(\hat{a}^2+\hat{a}^{\dagger 2})+\frac{r^2}{4}[\hat{a}^2,\hat{a}^{\dagger 2}]\sinh(2\beta)+\frac{r^3}{12}\sinh^3\beta(\hat{a}^2+\hat{a}^{\dagger 2})^3 \tag{A.11}$$

$$-\frac{r^3}{24}\cosh\beta\sinh(2\beta)[\hat{a}^2-\hat{a}^{\dagger 2},[\hat{a}^2,\hat{a}^{\dagger 2}]] \tag{A.12}$$

$$-\frac{r^4}{3.2^4}\sinh^2\beta\sinh2\beta\Big((\hat{a}^2+\hat{a}^{\dagger 2})^2[\hat{a}^2,\hat{a}^{\dagger 2}]+[\hat{a}^2,\hat{a}^{\dagger 2}](\hat{a}^2+\hat{a}^{\dagger 2})^2+(\hat{a}^2+\hat{a}^{\dagger 2})[\hat{a}^2,\hat{a}^{\dagger 2}](\hat{a}^2+\hat{a}^{\dagger 2})\Big) \tag{A.13}$$

$$+\sinh2\beta\frac{r^4}{4!2^3}\Big(\cosh^2\beta[[[\hat{a}^2,\hat{a}^{\dagger 2}],\hat{a}^2-\hat{a}^{\dagger 2}],\hat{a}^2-\hat{a}^{\dagger 2}]-\sinh^2\beta[[[\hat{a}^2,\hat{a}^{\dagger 2}],\hat{a}^2+\hat{a}^{\dagger 2}],\hat{a}^2+\hat{a}^{\dagger 2}]\Big). \tag{A.14}$$

We define

$$s(t)=\hat{a}^2e^{-2i\beta t}+\hat{a}^{\dagger 2}e^{2i\beta t}, \qquad d(t)=\hat{a}^2e^{-2i\beta t}-\hat{a}^{\dagger 2}e^{2i\beta t} \tag{A.15}$$

in terms of which we obtain

$$\boldsymbol{\delta}(t)=-r\sinh\beta s(t)+\frac{r^2}{4}[\hat{a}^2,\hat{a}^{\dagger 2}]\sinh2\beta \tag{A.16}$$

$$+\frac{r^3}{12}\sinh^3\beta s(t)^3-\frac{r^3}{24}\cosh\beta\sinh(2\beta)[d(t),[\hat{a}^2,\hat{a}^{\dagger 2}]] \tag{A.17}$$

$$-\frac{r^4}{3.2^4}\sinh^2\beta\sinh2\beta\Big(s(t)^2[\hat{a}^2,\hat{a}^{\dagger 2}]+[\hat{a}^2,\hat{a}^{\dagger 2}]s(t)^2+s(t)[\hat{a}^2,\hat{a}^{\dagger 2}]s(t)\Big) \tag{A.18}$$

$$+\sinh2\beta\frac{r^4}{4!2^3}\Big(\cosh^2\beta[[[\hat{a}^2,\hat{a}^{\dagger 2}],d(t)],d(t)]-\sinh^2\beta[[[\hat{a}^2,\hat{a}^{\dagger 2}],s(t)],s(t)]\Big). \tag{A.19}$$

We would need the expressions

$$\int dt\frac{e^{ixt}}{\cosh^2(\pi t)}=\frac{x}{\pi}\frac{1}{\sinh(x/2)}, \tag{A.20}$$

$$\lim_{\epsilon\to0}\int\frac{dt}{\cosh(\pi t)}\frac{e^{ix(t-a)}}{\sinh(\pi(t-a+i\epsilon))}=2i\frac{e^{-x/2}-e^{-iax}}{1-e^{-x}}\frac{e^{-x/2}}{\cosh(\pi a)}, \tag{A.21}$$

$$\lim_{\epsilon\to0}\int\frac{dt}{\cosh(\pi t)}\frac{e^{ix(t-a)}}{\sinh(\pi(t-a-i\epsilon))}=2i\frac{e^{x/2}-e^{-iax}}{1-e^{-x}}\frac{e^{-x/2}}{\cosh(\pi a)}. \tag{A.22}$$

Plugging all this in the expressions in (1.6)–(1.9), and adding all the contributions, we get, upto $\mathcal{O}(r^4)$,

$$\sum_i Q_i = -r\beta(\hat{a}^2 + \hat{a}^{\dagger 2}) + \frac{1}{2}\beta r^2[\hat{a}^2, \hat{a}^{\dagger 2}] - \frac{\beta r^3}{12}[\hat{a}^2 - \hat{a}^{\dagger 2}, [\hat{a}^2, \hat{a}^{\dagger 2}]] \tag{A.23}$$

$$- \frac{\beta r^4}{96}([[[\hat{a}^2, \hat{a}^{\dagger 2}], \hat{a}^2], \hat{a}^{\dagger 2}] + [[[\hat{a}^2, \hat{a}^{\dagger 2}], \hat{a}^{\dagger 2}], \hat{a}^2]) \tag{A.24}$$

$$= (2\hat{a}^\dagger \hat{a} + 1)\left(\beta r^2 + \frac{\beta r^4}{3}\right) - (\hat{a}^2 + \hat{a}^{\dagger 2})\left(r\beta + \frac{2\beta r^3}{3}\right). \tag{A.25}$$

where we have combined terms using the identity

$$[s_1, s_2^3] - 3s_2[s_1, s_2]s_2 = [[[s_1, s_2], s_2], s_2]\,. \tag{A.26}$$

Eq. (A.25) and eq. (A.4) agree precisely.

B Contact term at quintic order

For completeness, we evaluate the formula given for the contact terms P_m for the case $m = 5$. We get

$$P_5 = \frac{\pi}{40}\lim_{\epsilon_1,\epsilon_2\to 0} \int \frac{g_{\epsilon_1}(t_2 - t_1)g_{\epsilon_2}(t_3 - t_2)}{\cosh(\pi t_1)\cosh(\pi t_3)}\Big(\boldsymbol{\delta}(t_1)^3\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_3) + \boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)^2\boldsymbol{\delta}(t_3)$$
$$+ \boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_3)\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_1)^2 + \boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_3)^3 + \boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)^2\boldsymbol{\delta}(t_3)\boldsymbol{\delta}(t_2)$$
$$+ \boldsymbol{\delta}(t_1)^2\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_3)\boldsymbol{\delta}(t_1) - \boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_3)\boldsymbol{\delta}(t_1)^2 - \boldsymbol{\delta}(t_2)^2\boldsymbol{\delta}(t_3)\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_1)$$
$$- \boldsymbol{\delta}(t_1)^2\boldsymbol{\delta}(t_3)\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_1) - \boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_3)\boldsymbol{\delta}(t_2)^2 + \boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)^3\boldsymbol{\delta}(t_3)\Big). \tag{B.1}$$

The first three terms above come from the object Ξ_1^5 applied to the kernel. The next three terms come from Ξ_3^5. The last five terms come from Ξ_2^5.

This can be further simplified as

$$P_5 = \frac{\pi}{40}\lim_{\epsilon_1,\epsilon_2\to 0} \int \frac{g_{\epsilon_1}(t_2 - t_1)g_{\epsilon_2}(t_3 - t_2)}{\cosh(\pi t_1)\cosh(\pi t_3)}\Big([\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_1), \boldsymbol{\delta}(t_2)^2\boldsymbol{\delta}(t_3)]$$
$$+ \boldsymbol{\delta}(t_1)[\boldsymbol{\delta}(t_3), \boldsymbol{\delta}(t_2)]\boldsymbol{\delta}(t_1)^2 + \boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)[\boldsymbol{\delta}(t_2), \boldsymbol{\delta}(t_3)]\boldsymbol{\delta}(t_2) + \boldsymbol{\delta}(t_1)^2[\boldsymbol{\delta}(t_2), \boldsymbol{\delta}(t_3)]\boldsymbol{\delta}(t_1)$$
$$+ \boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)^3\boldsymbol{\delta}(t_3) + \boldsymbol{\delta}(t_1)^3\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_3) + \boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_3)^3\Big). \tag{B.2}$$

C Action of the permutation group on operators and functions

In this appendix, we elaborate more on the action of the permutation group on operators and functions introduced in (2.24)–(2.25).

S_m denotes the symmetric group of permutations on m-distinct objects and we use $(1, 2, \ldots, m)$ to denote the permutation that sends $1 \to 2 \to 3 \cdots \to (m-1) \to m \to 1$. We will follow the convention where group composition, denoted by $\star$ in S_m is from left to right, e.g.,

$$(1234) \star (124) = (1423) \tag{C.1}$$

$$(12345) \star (342) = (13245) \tag{C.2}$$

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We define

$$\sigma(\boldsymbol{\delta}(t_{i_1})\boldsymbol{\delta}(t_{i_2})\cdots\boldsymbol{\delta}(t_{i_m})) = \boldsymbol{\delta}(t_{i_{\sigma(1)}})\boldsymbol{\delta}(t_{i_{\sigma(2)}})\cdots\boldsymbol{\delta}(t_{i_{\sigma(m)}}) \tag{C.3}$$

$$\sigma(F(t_{i_1}, t_{i_2}, \cdots, t_{i_m})) = F(t_{\sigma(i_1)}, t_{\sigma(i_2)}, \cdots, t_{\sigma(i_m)})\,. \tag{C.4}$$

Note that in (C.3) the permutations act to the left, while in (C.4), the permutations act to the right. More explicitly,

$$\sigma(\tau(\boldsymbol{\delta}(t_{i_1})\boldsymbol{\delta}(t_{i_2})\cdots\boldsymbol{\delta}(t_{i_m}))) = \sigma\star\tau(\boldsymbol{\delta}(t_{i_1})\boldsymbol{\delta}(t_{i_2})\cdots\boldsymbol{\delta}(t_{i_m})), \tag{C.5}$$

$$\sigma(\tau(F(t_{i_1}, t_{i_2}, \cdots, t_{i_m}))) = \tau\star\sigma(F(t_{i_1}, t_{i_2}, \cdots, t_{i_m})))\,. \tag{C.6}$$

Now consider some examples

$$(123)F(t_2,t_1,t_3) = F(t_3,t_2,t_1), \quad (123)\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_3) = \boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_3)\boldsymbol{\delta}(t_2), \tag{C.7}$$

$$(12)((123)F(t_2,t_1,t_3)) = F(t_3,t_1,t_2) = (23)F(t_2,t_1,t_3), \tag{C.8}$$

$$(12)((123)\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_3)) = \boldsymbol{\delta}(t_3)\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_2) = (13)\boldsymbol{\delta}(t_2)\boldsymbol{\delta}(t_1)\boldsymbol{\delta}(t_3), \tag{C.9}$$

with $(12)\star(123) = (13)$ and $(123)\star(12) = (23)$.

We now give a proof eq. (2.28) [10]. Notice that

$$(m, m-1\ldots 4321)X\boldsymbol{\delta}(t_m) = \boldsymbol{\delta}(t_m)X, \tag{C.10}$$

for *any* product of operators X not involving $\boldsymbol{\delta}(t_m)$, and thus

$$(\mathrm{id} - (m, m-1, \ldots 4321))X\boldsymbol{\delta}(t_m) = [X, \boldsymbol{\delta}(t_m)]\,. \tag{C.11}$$

Now (2.28) follows by induction. For $m = 2$ it is obviously true. Assume its true for $m-1$. Then we have

$$\begin{aligned} T_m(\boldsymbol{\delta}(t_1)\ldots\boldsymbol{\delta}(t_m)) &= (\mathrm{id} - (m, m-1, \ldots 4321))\Big[T_{m-1}(\boldsymbol{\delta}(t_1)\ldots\boldsymbol{\delta}(t_{m-1}))\Big]\boldsymbol{\delta}(t_m) \\ &= [T_{m-1}(\boldsymbol{\delta}(t_1)\ldots\boldsymbol{\delta}(t_{m-1})), \boldsymbol{\delta}(t_m)] \end{aligned} \tag{C.12}$$

which completes the proof.

D The $\epsilon \to 0$ limit

In this appendix, we include a proof of the existence of the $\epsilon \to 0$ limit in (1.2). In other words, we establish the identity in (2.12) treating the $\epsilon \to 0$ limit carefully.

Using the spectral decomposition of Δ in (E.9) for any $\epsilon, \beta > 0$ and vectors $|x\rangle$ and $|y\rangle$ we have

$$\lim_{\epsilon\to 0}\langle x|\frac{\Delta^{1-\epsilon}}{\Delta+\beta}|y\rangle = \frac{-i}{2}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{dt}{\sinh(\pi(t-i\epsilon))}\beta^{-it}\langle y|\Delta^{-it}|x\rangle \tag{D.1}$$

Note that we can freely interchange the λ and t integrals because the integrand above is an absolutely convergent function. Our goal is to show the limit above gives $\langle y|\frac{\Delta}{\Delta+\beta}|x\rangle$.

The following inequalities

$$\frac{e^{(1-\epsilon)\lambda}}{e^{\lambda}+\beta} < \frac{e^{\lambda}}{e^{\lambda}+\beta} < 1 \qquad \forall\lambda > 0, \tag{D.2}$$

$$\frac{e^{(1-\epsilon)\lambda}}{e^{\lambda}+\beta} < \frac{1}{e^{\lambda}+\beta} < \frac{1}{\beta} \qquad \forall\lambda < 0. \tag{D.3}$$

imply that $\frac{e^{\lambda(1-\epsilon)}}{e^{\lambda}+\beta}$ is dominated by an integrable function. Then, the Lebesgue dominated convergence theorem [11] implies that

$$\lim_{\epsilon\to 0} \int \frac{e^{\lambda(1-\epsilon)}}{e^{\lambda}+\beta} \langle x|P(d\lambda)|y\rangle = \int \frac{e^{\lambda}}{e^{\lambda}+\beta} \langle x|P(d\lambda)|y\rangle \,. \tag{D.4}$$

This is sufficient to guarantee

$$\frac{-i}{2}\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{dt}{\sinh(\pi(t-i\epsilon))}\beta^{-it}\langle y|e^{itK}|x\rangle = \langle y|\frac{\Delta}{\Delta+\beta}|x\rangle. \tag{D.5}$$

E Interchange of β integral with $\epsilon \to 0$ limit

Finally, we consider the question of interchange of the order of the β integral and the $\epsilon \to 0$ limit in (2.14). For concreteness, consider the $m = 2$ in (2.14):

$$Q_2^{(\epsilon)} = \int_0^{\infty} \frac{d\beta}{\beta}\lim_{\epsilon\to 0}\int dt_0 dt_1 dt_2 \beta^{i(t_0+t_1+t_2)} f(t_0) g_\epsilon(t_1) f(t_2)\langle x|\Delta^{-it_0}\boldsymbol{\delta}\Delta^{-it_1}\boldsymbol{\delta}\Delta^{-it_2}|y\rangle, \tag{E.1}$$

$$f(t) = \frac{1}{2\cosh(\pi t)}, \tag{E.2}$$

$$g_\epsilon(t) = \frac{i}{4}\left[\frac{1}{\sinh(\pi(t-i\epsilon))} + \frac{1}{\sinh(\pi(t+i\epsilon))}\right]. \tag{E.3}$$

Our aim is to justify bringing the limit $\epsilon \to 0$ out of the integral. The integrals over the t_i can be done explicitly to get

$$Q_2^{(\epsilon)} = \int_0^{\infty} d\beta\Big(\beta^{\epsilon} X_\epsilon(\beta) - \beta^{1-\epsilon} Y_\epsilon(\beta)\Big), \tag{E.4}$$

$$X_\epsilon(\beta) = \langle x|A\boldsymbol{\delta}\left(\frac{\Delta^{1-\epsilon}}{\Delta+\beta}\right)\boldsymbol{\delta}A|y\rangle, \tag{E.5}$$

$$Y_\epsilon(\beta) = \langle x|A\boldsymbol{\delta}\left(\frac{\Delta^{\epsilon}}{\Delta+\beta}\right)\boldsymbol{\delta}A|y\rangle. \tag{E.6}$$

In appendix D we show that

$$\lim_{\epsilon\to 0} X_\epsilon(\beta) = \langle x|A\boldsymbol{\delta}\left(\frac{\Delta}{\Delta+\beta}\right)\boldsymbol{\delta}A|y\rangle = X(\beta), \tag{E.7}$$

$$\lim_{\epsilon\to 0} Y_\epsilon(\beta) = \langle x|A\boldsymbol{\delta}\left(\frac{1}{\Delta+\beta}\right)\boldsymbol{\delta}A|y\rangle = Y(\beta). \tag{E.8}$$

It is instructive to think of the spectral decomposition of the positive operator Δ:

$$\Delta = \int_{\lambda\in\mathcal{R}} e^{-\lambda} P(d\lambda), \tag{E.9}$$

where $P(d\lambda)$ is a positive-operator valued measure. Then, for any $0 < \epsilon < 1$, $\beta > 0$ and all vectors $|x\rangle$ we have

$$\langle x|\frac{\Delta^{1-\epsilon}}{\Delta+\beta}|x\rangle \leq \langle x|H|x\rangle \tag{E.10}$$

where

$$H = \int_{0<\lambda<1} \frac{1}{\lambda+\beta} P(d\lambda) + \int_{\lambda>1} \frac{\lambda}{\lambda+\beta} P(d\lambda)\,. \tag{E.11}$$

By the Cauchy-Schwarz inequality we have the estimates

$$\left|\beta^\epsilon X_\epsilon(\beta) - \beta^{1-\epsilon} Y_\epsilon(\beta)\right| < 2F(\beta), \qquad \forall 0 < \beta < 1, \tag{E.12}$$

$$\left|\beta^\epsilon X_\epsilon(\beta) - \beta^{1-\epsilon} Y_\epsilon(\beta)\right| < 2\beta F(\beta), \qquad \forall 1 < \beta, \tag{E.13}$$

$$F(\beta) = \langle x|A\boldsymbol{\delta} H\boldsymbol{\delta} A|x\rangle + \langle y|A\boldsymbol{\delta} H\boldsymbol{\delta} A|y\rangle. \tag{E.14}$$

Since $\int_0^1 F(\beta) < \infty$ and $\int_1^\infty \beta F(\beta) < \infty$ the dominated convergence theorem guarantees

$$\lim_{\epsilon\to 0} Q_2^{(\epsilon)} = \int_0^\infty d\beta (X_\beta - \beta Y(\beta)), \tag{E.15}$$

which implies

$$\int_0^\infty \frac{d\beta}{\beta} \lim_{\epsilon\to 0} \int dt_0 dt_1 dt_2 \beta^{i(t_0+t_1+t_2)} f(t_0) g_\epsilon(t_1) f(t_2) \langle x|\Delta^{-it_0}\boldsymbol{\delta}\Delta^{-it_1}\boldsymbol{\delta}\Delta^{-it_2}|y\rangle \tag{E.16}$$

$$= \lim_{\epsilon\to 0} \int_0^\infty \frac{d\beta}{\beta} \int dt_0 dt_1 dt_2 \beta^{i(t_0+t_1+t_2)} f(t_0) g_\epsilon(t_1) f(t_2) \langle x|\Delta^{-it_0}\boldsymbol{\delta}\Delta^{-it_1}\boldsymbol{\delta}\Delta^{-it_2}|y\rangle. \tag{E.17}$$

The argument above generalizes to the m^{th} term in (2.14) and justifies the interchange of the β and $\epsilon \to 0$ limits. This generalization makes it clear that the order of limits of the $\epsilon_i \to 0$ does not matter in (2.14).

References

[1] H. Komatsu, *Fractional powers of operators*, *Pacific J. Math.* **19** (1966) 285.

[2] J.B. Conway and B.B. Morrel, *Roots and logarithms of bounded operators on Hilbert space*, *J. Funct. Anal.* **70** (1987) 171.

[3] V.S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Springer, New York, NY, U.S.A. (2013) [DOI:10.1007/978-1-4612-1126-6].

[4] T. Faulkner, *Bulk emergence and the RG flow of entanglement entropy*, *JHEP* **05** (2015) 033 [arXiv:1412.5648] [INSPIRE].

[5] S. Balakrishnan, T. Faulkner, Z.U. Khandker and H. Wang, *A general proof of the quantum null energy condition*, *JHEP* **09** (2019) 020 [arXiv:1706.09432] [INSPIRE].

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[6] T. Faulkner, R.G. Leigh, O. Parrikar and H. Wang, *Modular Hamiltonians for deformed half-spaces and the averaged null energy condition*, *JHEP* **09** (2016) 038 [arXiv:1605.08072] [INSPIRE].

[7] T. Faulkner, R.G. Leigh and O. Parrikar, *Shape dependence of entanglement entropy in conformal field theories*, *JHEP* **04** (2016) 088 [arXiv:1511.05179] [INSPIRE].

[8] T. Faulkner et al., *Nonlinear gravity from entanglement in conformal field theories*, *JHEP* **08** (2017) 057 [arXiv:1705.03026] [INSPIRE].

[9] G. Sárosi and T. Ugajin, *Modular Hamiltonians of excited states, OPE blocks and emergent bulk fields*, *JHEP* **01** (2018) 012 [arXiv:1705.01486] [INSPIRE].

[10] D. Blessenohl and H. Laue, *Generalized Jacobi identities*, *Note Matem.* **08** (1988) 111.

[11] M. Reed and B. Simon, *Methods of modern mathematical physics I: functional analysis*, Academic Press (1980).

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