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Magnetized Baryonic layer and a novel BPS bound in the gauged-non-linear-sigma-model-Maxwell theory in (3+1)-dimensions through Hamilton-Jacobi equation

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Abstract: It is show that one can derive a novel BPS bound for the gauged Non-Linear-Sigma-Model (NLSM) Maxwell theory in (3+1) dimensions which can actually be saturated. Such novel bound is constructed using Hamilton-Jacobi equation from classical mechanics. The configurations saturating the bound represent Hadronic layers possessing both Baryonic charge and magnetic flux. However, unlike what happens in the more common situations, the topological charge which appears naturally in the BPS bound is a non-linear function of the Baryonic charge. This BPS bound can be saturated when the surface area of the layer is quantized. The far-reaching implications of these results are discussed. In particular, we determine the exact relation between the magnetic flux and the Baryonic charge as well as the critical value of the Baryonic chemical potential beyond which these configurations become thermodynamically unstable.

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1 Introduction

The phase diagram of the low energy limit of QCD under extreme conditions (like finite density, low temperatures and when strong magnetic fields are involved) is a very hard nut to crack (see [1–3] and references therein). Not only analytic perturbative methods fail, but also Lattice QCD (LQCD henceforth) is not very effective both due to the sign problem as well as due to the magnetic field (see [4–10] and references therein). In the Infra-Red (IR henceforth) the analysis of the interplay between strong interactions and electromagnetic fields still exhibits difficult unsolved problems. In IR phase, the main role is played by the topological solitons of QCD (see [2, 11–13] and references therein). Topological solitons are characterized by a non-vanishing topological charge which prevents such classical configurations from decaying into the trivial vacuum. There are many relevant physical effects which are genuine features of the fact that a finite amount of topological charge is "forced to live" within a finite spatial region. One of the most important is the appearance of non-homogeneous Baryonic condensates (which would not form in free space). In (1+1)-dimensional models (where the ¨solitons¨ are kinks, such as in the Gross-Neveu model and its variants: see, for instance, [32–40]) it has been possible to show, thanks to integrability, that there is a finite region in the phase diagram (which appears at finite density) where kinks crystals (namely, ordered arrays of kinks) dominate (depending on the value of the isospin chemical potentials). Similar results have been obtained in higher dimensional models (under the assumption that the main fields only depend on one spatial coordinate: see [14–19] and references therein). These results strongly suggest that in the low energy limit of QCD there should appear non-homogeneous condensates similar to the ones appearing in superconductors [20, 21]. Indeed, in (3+1) dimensions there are strong numerical as well as phenomenological evidences (see [22–31] and references therein) suggesting that non-homogeneous Baryonic condensates do appear at low energies and

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temperatures when a finite amount of Baryonic charge is forced into a finite spatial volume. The nice regular shapes of these condensates lead to the name *nuclear pasta phases*. In the references mentioned above, it has been shown that at finite Baryon densities "hadronic tubes" (*nuclear spaghetti*), "Hadronic layers" (*nuclear lasagna*) and so on do appear. The numerical simulations of these systems are very challenging and the situation becomes much worse when the self-consistent electromagnetic interactions of these non-homogeneous Baryonic condensates are taken into account. Consequently, the development of novel analytic tools able to support the numerical simulations is a priority. In the present paper we will analyze the prototype of strongly interacting configurations where the techniques used in [14–19, 32–40] are not especially effective due to the lack of integrability: magnetized non-homogeneous Baryonic condensates in (3+1) dimensions where the fields necessarily depend on all the spatial coordinates.

We will consider here the (3+1)-dimensional gauged non-linear sigma model (gauged NLSM henceforth) Maxwell theory in the case of SU(2) isospin global symmetry which is one of the most relevant effective field theories: see [11, 41] and references therein. One of the main reasons is that such theory is the basic building block of Chiral Perturbation Theory (CPT henceforth, see [42–48] and references therein) being closely related to the low energy limit of QCD in (3+1) dimensions). Moreover, the well known arguments clarifying the interpretation of the topological charge as Baryonic charge (see [49–54] and references therein) do not actually need the explicit presence of the Skyrme term in the gauged NLSM but rather of a mechanism to stabilize the solitons (so that the search for stabilizing mechanism in the gauged NLSM Maxwell theory acquires high priority). The analysis of the conditions allowing the appearance of topologically stable non-homogeneous condensates, besides its intrinsic interest, has important consequences for the thermodynamics of dense nuclear/quark matter (see [1, 3] and references therein). Moreover, when the SU(2) valued scalar field is assumed to be homogeneous and the corresponding spatial fluctuations are neglected, many important physical effects are missed (see [59, 60] and references therein). The more common techniques (which allow both in the Yang-Mills-Higgs and in the Abelian-Higgs models to describe stable topological solitons such as instantons, monopoles and vortices) do not work in the case of the gauged NLSM Maxwell theory. The reason is that, in the case in which the minimal coupling with electromagnetism is not taken into account, there is no obvious BPS bound to saturate for static configurations due to the Derrick scaling argument. On the other hand, when the minimal coupling is turned on, there is some hope (as in the Abelian-Higgs model at critical coupling one can find topologically non-trivial vortices, which would not appear in the spectrum in the absence of gauge field). Unfortunately, no explicit example of BPS bound which can be saturated in the (3+1)-dimensional gauged NLSM Maxwell theory has been found so far. This the main reason why many authors proposed modifications to the NLSM and Skyrme theories (see [61–64] and references therein): the idea is to modify the theory in order to allow the appearance of BPS bound which can actually be saturated. However, the most important question remains open:

> *Is it possible to find novel BPS bounds in the gauged NLSM Maxwell theory (without any modification) in terms of different topological charges which can actually be saturated?*

It is worth emphasizing that the obvious candidate to be the topological charge appearing in the BPS bound (which is the Baryonic charge) does not work very well since the available numerical solutions are always above the bound (see [2, 11–13] and references therein). In the present paper we will construct non-homogeneous magnetized Hadronic condensates depending on all the spatial coordinates in such a way to have non-vanishing Baryonic charge and magnetic flux. This approach also helps to avoid the Landau-Peierls[1] [59] (which cannot be applied when the fields depend in a non-trivial way on the three spatial coordinates). Secondly, a magnetized topologically non-trivial condensates cannot decay into a condensate with vanishing topological charge and magnetic flux. It is worth emphasizing that magnetize Baryonic layers are extremely relevant in many situations such as heavy ions collisions and neutron stars (see [4–10, 26–31] and references therein).

The techniques introduced in [65–79] allowed the construction of analytical solutions that describe multi-solitons at finite density for both, in the non-linear sigma model as well as in the Skyrme model minimally coupled to the Maxwell theory. However, in this approach the electromagnetic field self-consistently generated by the non-homogeneous Baryonic condensates has necessarily the electric components of the same order as the magnetic components. Therefore, in all the situations in which the magnetic field dominates, this framework must be modified. This is the aim of the present paper in which the theory of Hamilton-Jacobi equation will be used to derive a novel BPS bound which can be saturated for magnetized configurations.

This paper is organized as follows: in the second section the gauged NLSM-Maxwell theory will be introduced. In the third section the ansatz to describe magnetized Baryonic layers will be described. In the fourth section the technique to derive novel BPS bound using the Hamilton-Jacobi equation in classical mechanics will be explained. In the fifth section such technique will be applied to construct analytically BPS magnetized Baryonic configurations. In the sixth section, the main physical characteristics of these non-homogeneous condensates will be presented. In the final section some conclusions will be drawn.

2 The gauged non-linear sigma model

At finite density, a fundamental question which has been only partially addressed is whether or not the gauged non-linear sigma model admits topologically non-trivial non-homogeneous condensates with crystal-like structures. The Chiral Soliton Lattice (CSL henceforth: see [14, 80–82] and references therein) is a very interesting (3+1)-dimensional example in which the Pionic field only depends on one spatial coordinate. This model can be supported by strong external fields (as explained in the above references). However, the question of the magnetic field generated self-consistently by topologically non-trivial non-homogeneous condensates depending in a non-trivial way on all the spatial coordinates remains open. One would like to know the magnetic field generated by these condensates, whether or not such magnetic field grows with the Baryonic charge and so on. With the available numerical

[1]They showed that in an isotropic system in three or fewer dimensions, thermal fluctuations destroy condensates depending on only one spatial coordinate.

methods, to answer to such a question cannot be achieved (yet). In the present paper we will construct non-homogeneous condensates depending on all the spatial coordinates in such a way to have non-vanishing topological charge. Therefore, the classical no-go argument by Landau and Peierls [59] is avoided directly and, moreover, the issue of the stability (which, usually, is a quite difficult problem to analyze) in the present case will be clarified by the novel BPS bound which will be derived in the next sections.

As it can be shown using, for instance, Chiral Perturbation Theory (χPT henceforth, see [45–48]), the low energy QCD action[2] (to order $O(p^2)$) reads

$$S=\int\frac{d^4v}{4}\left\{-\widehat{K}\ \mathrm{Tr}\left[\Sigma^{\mu}\Sigma_{\mu}\right]-\frac{1}{e^2}F_{\mu\nu}F^{\mu\nu}\right\}\,, \tag{2.1}$$

where e is the electric charge and $\widehat{K}$ is the coupling constant of the gauged NLSM. The constant $\widehat{K}$ is related with the Pions decay constant f_π as follows (see [54]):

$$\widehat{K}=\frac{(f_\pi)^2}{4}\,,\quad f_\pi\approx 130\quad MeV\,. \tag{2.2}$$

The above action in eq. (2.1) can be rewritten as follows:

$$S=\frac{1}{4e^2}\int\frac{d^4v}{4}\left\{-K\ \mathrm{Tr}\left[\Sigma^{\mu}\Sigma_{\mu}\right]-F_{\mu\nu}F^{\mu\nu}\right\}\,,\quad K=e^2\widehat{K}\,. \tag{2.3}$$

The above form of the action is very convenient since the field equations will only depend on the "rescaled" Pions decay constant K defined here above while the factor $1/e^2$ will enter only as global factor in the definition of the energy density and total energy. The pions mass could also be included, however in the present case can be safely neglected as the mass of the solitons which will be constructed here is much bigger than the Pions mass. Moreover

$$\Sigma_\mu=\Sigma^{-1}D_\mu\Sigma=\Sigma_\mu^j t_j,\ F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,,\ t_j=i\sigma_j\,,$$

where σ_i are the Pauli matrices and the SU(2) valued Σ field can be parametrized using Euler angles (as any element of SU(2) can be written in a unique way in Euler parametrization):

$$\Sigma=\exp\left(t_3G_1\right)\exp\left(t_2G_2\right)\exp\left(t_3G_3\right)\,, \tag{2.4}$$

where $G_j\left(x^\mu\right)$ (with $j=1,2,3$) are the three scalar degrees of freedom of SU(2). The covariant derivative is defined as

$$D_\mu\Sigma=\partial_\mu\Sigma+A_\mu\left[t_3,\Sigma\right]\,, \tag{2.5}$$

where ∂_μ is the usual partial derivative. The field equations for the gauged NLSM read

$$D_\mu\Sigma^\mu=0\,, \tag{2.6}$$
$$\partial_\mu F^{\mu\nu}=J^\nu\,, \tag{2.7}$$

[2]We have chosen the units in such a way that both the speed of light and the Planck divided by 2π are set to 1: $c=1$, $\hbar=1$.

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where the current J^{μ} is given by

$$J^{\mu}=\frac{K}{2}\mathrm{Tr}\left[\widehat{O}\Sigma^{\mu}\right]\,,\qquad \widehat{O}=\Sigma^{-1}t_3\Sigma-t_3\,. \tag{2.8}$$

The energy-momentum tensor is given by

$$T_{\mu\nu}=\frac{1}{e^2}\left\{-\frac{K}{2}\mathrm{Tr}\left[\Sigma_{\mu}\Sigma_{\nu}-\frac{1}{2}g_{\mu\nu}\Sigma^{\alpha}\Sigma_{\alpha}\right]+\bar{T}_{\mu\nu}\right\}\,,$$

with

$$\bar{T}_{\mu\nu}=-F_{\mu\alpha}F_{\nu}^{\ \alpha}+\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}\,. \tag{2.9}$$

3 Magnetized BPS Baryonic layers

As it happens with the superconductors described by the Abelian-Higgs model, the physical properties are largely determined by the magnetic field. This is the reason why it is so important to analyze the physics of magnetized Baryonic layers. Unfortunately, in this situation, the techniques developed in [65–77] cannot be applied as these require the presence also of an electric field. Therefore, in order to answer to the question "how can we describe Baryonic layers generating in a self-consistent way only a magnetic field without electric component", we need a different approach. In other words, we need to solve the coupled system of three non-linear PDEs arising from the gauged NLSM minimally coupled to a U(1) gauge potential together with the corresponding four Maxwell equations (where the U(1) current arising from the NLSM in eq. (2.8)) in the case in which the layers generate a pure magnetic field (no electric components) to be as close as possible to the solitonic solutions of the Abelian-Higgs model. Since the main physical motivation of the present work is to study the appearance of topologically non-trivial non-homogeneous condensates at finite density, finite volume effects are of crucial importance. The easiest way to take them into account is to use the following flat metric

$$ds^2=-dt^2+L_r^2dr^2+L^2\left(dx^2+dy^2\right)\,, \tag{3.1}$$

where $4\pi^3L_rL^2$ is the volume of the box in which the gauged solitons are living. The adimensional Cartesian coordinates r, x and y have the ranges

$$0\leq r\leq 2\pi\,,\quad 0\leq x\leq\pi\,,\quad 0\leq y\leq 2\pi\,. \tag{3.2}$$

As it will be clear when analyzing the magnetized Baryonic layers in the next sections, the coordinates x and y are coordinates tangent to the layers while the coordinate r is orthogonal to the layers. The reason is that such layers are localized in the r direction as the corresponding energy and Baryon densities only depend on r. Thus, (once r is fixed at the position of the layer) one moves along the layers by moving x and y. On the other hand, one moves away from the layer by moving r away from the position of the layer (which is the value of r at which the energy-density has its maximum). Consequently, the area A of the layer is

$$A=2\pi^2L^2\,. \tag{3.3}$$

Following the strategy of [74, 75] and [76], let us consider the following static ansatz for the SU(2) valued scalar field and for the U(1) gauge field

$$\Sigma(x^\mu) = \exp\left(py\ t_3\right)\exp\left(H(r)\ t_2\right)\exp\left(px\ t_3\right)\ , \tag{3.4}$$
$$A_\mu = \left(0, 0, \frac{p}{2} - u\left(r\right), -\frac{p}{2} + u\left(r\right)\right)\ ,\ p \neq 0\,,\ p \in \mathbb{N}$$

where p must be an integer according to the theory of Euler angles for SU(N) (see, for instance, [87–89] and references therein). Thus, in the above ansatz (which depend in a non-trivial way on the three spatial coordinates), only two scalar degrees of freedom (namely, the two profiles $H(r)$ and $u(r)$) are turned on (in a similar way as it happens with the vortex in the superconductors where the field equations reduce to two coupled equations for the Higgs profile and for one component of the U(1) gauge potential). The first relevant observation is that the full system of seven coupled non-linear field equations is reduced to the following two coupled ODEs:

$$H'' + 4\left(\frac{L_r}{L}\right)^2 \sin\left(2H\right)\left(\left(\frac{p}{2}\right)^2 - u^2\right) = 0 \tag{3.5}$$

$$u'' - 4KL_r^2 \sin^2\left(H\right) u = 0\,, \tag{3.6}$$

where, of course,

$$X' = \frac{d}{dr}X\ ,\ \ X'' = \frac{d^2}{dr^2}X$$

and so on. Moreover, with the above ansatz, the energy-density T_{00} reduces to

$$T_{00} = \frac{1}{e^2}\left\{\frac{K}{L^2}\left[p^2\cos^2\left(H\right) + 4\sin^2\left(H\right)u^2\right] + \frac{K\left(H'\right)^2}{2L_r^2} + \frac{\left(u'\right)^2}{\left(L_r L\right)^2}\right\}\ , \tag{3.7}$$

and the field equations corresponding to the ansatz in eq. (3.4) can be derived using the above energy-density as starting point for a variational principle.

The topological (Baryonic) charge of the gauged non-linear sigma model [49–58] is given by

$$B = \frac{1}{24\pi^2}\int_S \rho_B\ ,\quad \rho_B = \rho_{B1} + \rho_{B2}\,, \tag{3.8}$$

where

$$\rho_{B1} = \epsilon^{ijk}\mathrm{Tr}\left\{\left(\Sigma^{-1}\partial_i\Sigma\right)\left(\Sigma^{-1}\partial_j\Sigma\right)\left(\Sigma^{-1}\partial_k\Sigma\right)\right\}\ , \tag{3.9}$$

$$\rho_{B2} = -3\epsilon^{ijk}\mathrm{Tr}\left\{\partial_i\left[A_j t_3\left(\Sigma^{-1}\partial_k\Sigma + \left(\partial_k\Sigma\right)\Sigma^{-1}\right)\right]\right\}\ , \tag{3.10}$$

are the two topological density contributions (A_j being the spatial components of the gauge potential). With the above ansatz we get

$$\rho_B = J_0^B = -12p\left[u(1+\cos(2H))\right]'\ . \tag{3.11}$$

4 The BPS bound: a novel strategy based on the Hamilton-Jacobi equation

Before going back to the case of interest in this paper, let us outline a general strategy to derive a novel BPS bound using Hamilton-Jacobi equation. To the best of author's knowledge, this strategy to construct BPS bounds is new. Suppose one has the positive definite energy-density T_{00} of a static configurations of two (or more) interacting degrees of freedom, J_1 and J_2 (in the present case, J_1 and J_2 are related to H and u) which depend on one spatial coordinate (let´s say, r):

$$T_{00} = \frac{(\partial_r J_1)^2}{2} + \frac{(\partial_r J_2)^2}{2} + V(J_1, J_2) \ , \tag{4.1}$$

where $V(J_1, J_2)$ is the interaction term between the two degrees of freedom (which is supposed to be known). Being the configuration static (by hypothesis) the field equations for J_1 and J_2 can be derived using T_{00} as action density. In order to get a BPS bound for J_1 and J_2 we would like to sum two quantities (let us call them Γ_1 and Γ_2) to the gradients of J_1 and J_2 with the following properties:

$$\frac{(\partial_r J_1 \pm \Gamma_1)^2}{2} + \frac{(\partial_r J_2 \pm \Gamma_2)^2}{2} = T_{00} + \text{total derivative} \, .$$

The above requirement implies that

$$\frac{(\Gamma_1)^2}{2} + \frac{(\Gamma_2)^2}{2} = V(J_1, J_2) \ , \tag{4.2}$$

$$\Gamma_1 \partial_r J_1 + \Gamma_2 \partial_r J_2 = \text{total derivative} \, . \tag{4.3}$$

Moreover, eq. (4.3) can be satisfied by requiring that

$$\Gamma_1 = \frac{\partial W}{\partial J_1} \ , \ \Gamma_2 = \frac{\partial W}{\partial J_2} \ , \quad W = W(J_1, J_2) \, , \tag{4.4}$$

since (if the above condition is satisfied) then

$$\Gamma_1 \partial_r J_1 + \Gamma_2 \partial_r J_2 = \partial_r \left(W(J_1, J_2) \right) \ .$$

Hence, putting together eqs. (4.4) and (4.2) we arrive at the main requirement able to provide a BPS bound:

$$\left(\frac{\partial W}{\partial J_1} \right)^2 + \left(\frac{\partial W}{\partial J_2} \right)^2 = 2V(J_1, J_2) \ \Rightarrow \tag{4.5}$$

$$\text{total derivative} = \partial_r W \, .$$

In other words, if one is able to solve the above Hamilton-Jacobi equation eq. (4.5) for the function $W(J_1, J_2)$ where the role of the potential is played by $V(J_1, J_2)$, then one get the following bound:

$$E \geq |W(2\pi) - W(0)| \ .$$

The corresponding BPS equation are

$$\partial_r J_k \pm \frac{\partial W}{\partial J_k} = 0\,, \quad k = 1, 2\,.$$

Due to the huge amount of literature on the Hamilton-Jacobi equation, this strategy is extremely convenient and, as we will now show, it allows to derive new topological bound where the corresponding topological charge is a non-linear function of the "obvious topological charge" which one would consider, at a first glance, as candidate to derive BPS equations. Although the relations between the Hamilton-Jacobi equation and SUSY are well known (see [90, 91] and references therein), such approach has not been used so far to derive novel BPS bounds in (3+1)-dimensional field theories which are not supersymmetric (at least, not in the usual sense) such as the (3+1)-dimensional gauged NLSM Maxwell theory.

5 Application: magnetized BPS Baryonic layers

A beautiful application of the above strategy is the derivation of a BPS bound for magnetized Baryonic layers. As it has been already mentioned, it is not possible to solve the system in eqs. (3.5) and (3.6) using the techniques in [65–77]. The reason is that in those references, the non-homogeneous Baryonic condensates produce electromagnetic field in which the electric component is of the same order as the magnetic one. In those cases, the complete set of field equations can be solved analytically since the electromagnetic field is of force-free type. In the case of non-homogeneous condensates generating pure magnetic fields (as vortices in superconductors) one must use different techniques. Despite the fact that in the (3+1)-dimensional gauged NLSM — Maxwell theory, usually the search for BPS bound has not produced explicit analytic solutions in topologically non-trivial sectors, the technique introduced here above open a new unexpected window.

Using the ideas described in the previous section together with the definitions

$$\Gamma_1 = \frac{L_r}{K^{1/2}} \frac{\partial W}{\partial H}\,, \quad \Gamma_2 = \frac{L_r L}{\sqrt{2}} \frac{\partial W}{\partial u}\,,$$

one obtains the following Hamilton-Jacobi equation associated to the energy-density in eq. (3.7):

$$\frac{L_r^2}{2K}\left(\frac{\partial W}{\partial H}\right)^2 + \left(\frac{L_r L}{2}\right)^2 \left(\frac{\partial W}{\partial u}\right)^2 = \frac{K}{L^2}\left[p^2 \cos^2(H) + 4\sin^2(H)\, u^2\right]\,. \tag{5.1}$$

Such Hamilton-Jacobi equation can be solved analytically provided the following relation holds

$$L = \frac{p}{\sqrt{2K}} \;\Leftrightarrow\; A = \pi^2 \frac{p^2}{K} \;\Leftrightarrow\; A = \left(\frac{2\pi p}{e f_\pi}\right)^2\,, \tag{5.2}$$

where we have expressed explicitly the quantization condition in terms of the Pions decay constant using eqs. (2.1) and (2.2). Hence, the BPS bound (to be defined here below) can be saturated when the surface area of the layers is quantized in terms of the Pions decay constant. When the above equation is satisfied, the solution of the Hamilton-Jacobi equation (5.1) is

$$W = \frac{4K^{3/2}}{pL_r} u \cos H \;\Rightarrow \tag{5.3}$$
$$\frac{\partial W}{\partial H} = -\frac{4K^{3/2}}{pL_r} u \sin H\,, \quad \frac{\partial W}{\partial u} = \frac{4K^{3/2}}{pL_r} \cos H\,.$$

The above solution of the Hamilton-Jacobi equation associated to the present theory allows to rewrite the energy-density in a BPS style as follows:

$$T_{00} = \frac{1}{e^2}\left\{\frac{K}{2\,(pL_r)^2}\left[\left(pH' \pm 4K^{1/2}L_r u \sin H\right)^2 + 4\left(u' \mp pK^{1/2}L_r \cos H\right)^2\right] \pm \frac{dW}{dr}\right\} \tag{5.4}$$

where W in eq. (5.3) is the solution the Hamilton-Jacobi equation (5.1) (in the following we will consider the upper signs in the above equation, the analysis with the lower signs is analogous). Consequently, it is possible to derive the following bound:

$$E = \int \sqrt{-g} d^3x T_{00} = AL_r \int_0^{2\pi} T_{00} dr \geq |Q|\ , \quad Q = \frac{AL_r}{e^2}\,|W(2\pi) - W(0)|\ , \tag{5.5}$$

where the area A of the Baryonic layers is quantized according to eq. (5.2).

One could naively think that such BPS trick should not work in the present case since (in the case without Maxwell) the NLSM does not allow the derivation of a BPS bound (and this is the reason why Skyrme introduced the Skyrme term). In fact, as it happens in the case of the Abelian Higgs model, the presence of the minimal coupling with the Maxwell gauge theory changes considerably the situation. Therefore, the two main questions to answer are:

1) Does the saturation of the bound implies the second order field equations?

2) Can the first order BPS equation be actually solved?

First of all, the first order BPS equations

$$H' + \frac{4K^{1/2}L_r}{p} u \sin H = 0 \tag{5.6}$$
$$u' - pK^{1/2}L_r \cos H = 0 \tag{5.7}$$

actually imply the second order field equations in eqs. (3.5) and (3.6) as it can be verified directly by deriving eqs. (5.6) and (5.7) with respect to r and comparing the result with eqs. (3.5) and (3.6). The above BPS system has a very clear meaning: u' (which represent the intensity of the magnetic field) is maximal when $|\cos H| \sim 1$ which implies that $H' \sim 0$ (since $|\sin H| \sim 0$), on the other hand, when H' is large the magnetic field is small: the physical interpretation of this fact will be discussed in a moment.

One of the most important physical implications of the present exact results is that these allow to derive analytically the non-linear relation between the Baryonic charge and the magnetic flux. In order to achieve this goal, instead of solving directly eqs. (5.6) and (5.7) it is better to use the BPS condition to derive the analytic relation between the SU(2) profile H and the gauge field profile u as follows:

$$\begin{aligned} &\frac{dH}{du} = -\frac{4}{p^2} u \tan H \;\Rightarrow \\ &H(r) = \arcsin\left[\exp\left(-\frac{2\,(u(r))^2}{p^2} + I_0\right)\right] , \end{aligned} \tag{5.8}$$

while I_0 is an integration constant (once again one can check that if one plugs the expression for $H(r)$ in the second order field equations together with eqs. (5.6) and (5.7) the second order field equations are identically satisfied).

In conclusion, using eq. (5.8), the complete set of field equations is reduced to the simple quadrature:

$$u' = pK^{1/2} L_r \sqrt{1 - \exp\left(-\frac{4\,(u(r))^2}{p^2} + 2I_0\right)} . \tag{5.9}$$

As it will be discussed in the next sections, the integration constant I_0 can be chosen in order to satisfy the required boundary conditions.

In conclusion, the answers to the above 2 questions are affirmative in both cases. As it will be explained in more details in the next section, these BPS configurations represent magnetized Baryonic layers possessing a non-trivial Baryonic charge as well as a non-vanishing magnetic flux. To the best of author‘s knowledge, this is the first example of a BPS bound which can be saturated with an explicit analytic configuration in the gauged NLSM-Maxwell theory in (3+1) dimensions which is static, magnetized and with non-vanishing Baryonic charge. Such a surprising result has very intriguing phenomenological consequences on the relations between magnetic flux, Baryonic charge and Baryonic chemical potential (which will be discussed in the following sections). Obviously, there are many very nice papers on solitons in the gauged NLSM Maxwell theory (see [93–103] and references therein), however, most of them employ numerical methods which prevent, for instance, to disclose the explicit relation between the Baryonic charge and the magnetic flux. On the more theoretical side, the fact that, in the gauged NLSM-Maxwell theory it is possible to construct non-trivial BPS bound which can be saturated opens an intriguing possibility related to supersymmetry (SUSY henceforth) as well as to the findings in [104, 105]. The first obvious observation is that whenever a non-trivial BPS bound which can be saturated is found, it is natural to ask whether such a bound is a manifestation of some hidden SUSY. In [104, 105] the authors discussed a supersymmetric extension of the Skyrme model: due to the auxiliary field equations, such a model lacks a kinetic term. On the other hand, in the present paper it has been shown that the gauged NLSM-Maxwell theory could possess a non-linear realization of SUSY, which manifests itself in the fact that the natural charge appearing in the BPS bound is a non-linear function of the "obvious topological charge". Since the NLSM is the natural kinetic term for Skyrme theory, one may wonder whether the inclusion of the minimal coupling with Maxwell theory could allow a supersymmetric

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version of the Skyrme model with a standard kinetic term: I hope to come back on this issue in a future publication.

5.1 Topological and electric currents and magnetic field

The behavior of the magnetic field and of the SU(2) profile can be very easily described thanks to availability of the BPS equations (5.6), (5.7) and (5.8). First of all, the magnetic field $\mathbf{B}_i = \varepsilon_{ijk}F_{jk}$ (whose intensity is proportional to u') only possesses components tangent to the Baryonic layer ($|\mathbf{B}_\theta| = |\mathbf{B}_\varphi| \neq 0$) while the component orthogonal to the layer vanishes: $\mathbf{B}_r = 0$. To see this, it is enough to remind that the energy and Baryon densities of this non-homogeneous condensate only depend on r (so that its position can be identified with the local maximum of the Baryon density). Moreover, from eq. (5.6) and (5.7) one can see that the magnetic field u' is maximal where $|\cos H| \sim 1$ (so that $\sin H \sim 0$ which implies $H' \sim 0$). Consequently, the magnetic field reaches its highest intensity where H' vanishes. This is very interesting since the places where H' vanishes correspond to the regions where the SU(2) chiral field does not contribute to the Baryon density (since the contribution to the Baryon density of the SU(2) valued chiral field is proportional to H'). This implies that the SU(2) chiral field acts as a superconductor in the sense that it tends to suppress the magnetic field in its interior and, moreover, the magnetic field is tangent to the chiral soliton.

In this respect, an important issue must be discussed: the magnetic flux inside the layer should be supplemented by the condensation of the electric charges outside. In other words, the question is: *which is the structure of layers between the magnetic-flux carrying ones?* The exact answer to this question is very complicated at microscopic level as it involves the analysis of quarks dynamics strongly coupled with the present BPS chiral magnetized solitons (I will come back on this issue in a future publication). On the other hand, at a qualitative level, the structure of *layers between the magnetic-flux carrying ones* can be described well in terms of the gauged NLSM itself. Indeed, in order to achieve this goal, one needs exact solutions of the gauged NLSM with the shape of a layer (so that these can fit between the magnetic-flux carrying ones) and with non-vanishing electric charge density (the property to be topologically stable would be extremely welcome as well). This type of exact solutions would be very good candidates to be the *layers between the magnetic-flux carrying ones*. In the references [78, 79] it has been possible to construct analytic solutions of the gauged NLSM and Skyrme model representing Baryonic layers with non-vanishing electric charge density which could not be deformed continuously to the trivial configuration due to the non-triviality of the topological charge. These configurations in [78, 79] are good candidates to be the sought *layers between the magnetic-flux carrying ones*. The details of this interpretation will be discussed in a future publication.

Using the explicit expression of H in terms of u in eq. (5.8) one can compute explicitly the Baryonic charge, the new topological charge in eq. (5.5) and the magnetic flux. This allows to answer to the following questions: once the magnetic flux is given, which is the allowed value of the Baryonic charge B (and viceversa)? Which is the critical value of the Baryonic chemical potential beyond which these solitons become thermodynamically unstable? Within neutron stars as well as in Heavy Ions Collisions [9, 10], where such non-homogeneous Baryonic condensates are expected, such questions are of utmost importance. In order to proceed, one observes that from the first order BPS equations one arrives at

$$\begin{aligned}
u' &= pK^{1/2}L_r\sqrt{1-\exp\left(-\frac{4\,(u(r))^2}{p^2}+2I_0\right)} \Rightarrow \\
U' &= K^{1/2}L_r\sqrt{1-\exp\left(-4U^2+2I_0\right)}\,, \quad U=\frac{u(r)}{p} \qquad (5.10)\\
dr &= \frac{du}{pK^{1/2}L_r\sqrt{1-\exp\left(-\frac{4u^2}{p^2}+2I_0\right)}} \Rightarrow \\
2\pi &= \frac{1}{pK^{1/2}L_r}\int_{u(0)}^{u(2\pi)}\frac{du}{\sqrt{1-\exp\left(-\frac{4u^2}{p^2}+2I_0\right)}}\,. \qquad (5.11)
\end{aligned}$$

First of all, from eq. (5.10), one can define a "rescaled" variable $U=u/p$ in such a way that the corresponding first order BPS equation for the "rescaled" function U does not depend on p at all so that, in particular, U does not depend on p locally. At a qualitative level, this tells that $u(r)$ (and consequently $u(2\pi)$) scales linearly with p. The above equation (5.11) links the values of $u(r)$ at 0 and 2π to the integration constant I_0. Although it is not the most general option, it is convenient to assume $u(0)=0$ since this choice (which can be achieved with a gauge transformation) clarifies the relations between the magnetic flux, the Baryon charge and the topological charge appearing in the BPS bound. Hence, eq. (5.11) becomes

$$2\pi = \frac{1}{K^{1/2}L_r}\int_0^{\frac{u(2\pi)}{p}}\frac{d\tau}{\sqrt{1-\exp\left(-4\tau^2+2I_0\right)}}\,,\quad u(0)=0\,. \qquad (5.12)$$

In order to disclose the physical meaning of $u(2\pi)$ let us compute the total magnetic flux $\mathbf{\Phi}$ in the φ-direction (which is proportional to the magnetic flux along the θ direction):

$$\mathbf{\Phi} = LL_r\int dr d\theta F_{r\theta} = \frac{p\pi L_r}{\sqrt{2K}}\left(u(2\pi)-u(0)\right) \Longrightarrow \qquad (5.13)$$

$$u\left(2\pi\right) = \frac{\sqrt{2K}\mathbf{\Phi}}{p\pi L_r}\,. \qquad (5.14)$$

Therefore, $u(2\pi)$ is proportional to the magnetic flux along the $\varphi-$direction. With the above choice of the sign in the BPS equations the magnetic flux is positive while the Baryonic charge is negative (see eq. (5.22) here below) while with the other choice one would obtain a negative magnetic flux and a positive Baryonic charge. Since u scales with p, then the total magnetic flux $\mathbf{\Phi}$ scales with p^2:

$$\frac{\mathbf{\Phi}}{p^2} \sim \Psi_0\,, \qquad (5.15)$$

where Ψ_0 plays the role[3] of elementary magnetic flux (so that it does not depend on p). The above result is reassuring since eq. (5.15) says that the magnetic flux is proportional to the surface area of the hadronic layers.

[3] More precisely: Ψ_0 is proportional to the elementary magnetic flux times an adimesnional number which is fixed by requiring the physical boundary conditions described here below.

The dependence of I_0 on $u(2\pi)$ can be determined by solving the equation

$$2\pi = \frac{1}{K^{1/2}L_r}\int_0^{\frac{u(2\pi)}{p}} \frac{d\tau}{\sqrt{1-\exp\left(-4\tau^2+2I_0\right)}}\,,$$

for I_0 in terms of the other parameters (which can always be done numerically). On the other hand, it is possible to determine the qualitative behavior of such dependence by observing that (either for not too large values of $u(2\pi)/p$ or for large values of $K^{1/2}L_r$) the integration variable τ is small and for small τ one gets the estimate

$$\begin{aligned}2\pi &= \frac{1}{K^{1/2}L_r}\int_0^{\frac{u(2\pi)}{p}} \frac{d\tau}{\sqrt{1-\exp\left(-4\tau^2+2I_0\right)}} \sim \frac{1}{K^{1/2}L_r}\int_0^{\frac{u(2\pi)}{p}} \frac{d\tau}{\sqrt{1-\exp\left(2I_0\right)}}\\ &\sim \frac{1}{pK^{1/2}L_r}\frac{u(2\pi)}{\sqrt{1-\exp\left(2I_0\right)}}\end{aligned} \tag{5.16}$$

Thus, taking into account eq. (5.14), one obtains

$$\left(1-\frac{(u(2\pi))^2}{(2\pi pL_r)^2\,K}\right) \sim \exp\left(2I_0\right) \ \Rightarrow \tag{5.17}$$

$$\exp\left(2I_0\right) \sim \left(1-\frac{\mathbf{\Phi}^2}{2\left(p\pi L_r\right)^4}\right) \tag{5.18}$$

Since $u(2\pi)$ fixes uniquely I_0 through eq. (5.16), and $u(2\pi)$ is in one-to-one correspondence with the magnetic flux through eq. (5.14), the problem to fix the integration constant I_0 is equivalent to the problem to find a suitable physical condition to fix the total magnetic flux $\mathbf{\Phi}$. In order to find such physical condition, let us consider the Baryonic charge which is proportional to the integral of ρ_B in eq. (3.11) using eq. (5.8):

$$\rho_B = 24p\frac{d}{dr}\left(\Theta(r)\right)\,, \tag{5.19}$$

$$\Theta\left(r\right) = u\left(r\right)\left(\exp\left(-\frac{4\left(u(r)\right)^2}{p^2}+2I_0\right)-1\right)\,, \tag{5.20}$$

$$u(0)=0 \Rightarrow \ \Theta\left(0\right) = u\left(0\right)\,.$$

Consequently, the Baryonic charge is

$$B = \frac{1}{24\pi^2}\int_S \rho_B = 4p\Theta\left(2\pi\right) \Rightarrow \tag{5.21}$$

$$B = 4\frac{\sqrt{2K}\mathbf{\Phi}}{\pi L_r}\left(\exp\left(-\frac{8K\mathbf{\Phi}^2}{p^4\left(\pi L_r\right)^2}+2I_0\right)-1\right)\,. \tag{5.22}$$

Hence, the integration constant I_0 must be fixed in such a way that B is an integer:

$$|B| = 4\frac{\sqrt{2K}\mathbf{\Phi}}{\pi L_r}\left(1-\exp\left(-\frac{8K\mathbf{\Phi}^2}{p^4\left(\pi L_r\right)^2}+2I_0\right)\right) = n\,,\ \ n\in\mathbb{N}_+\,. \tag{5.23}$$

The above equation is a trascendental equation for I_0 which cannot be solved in closed form. Nevertheless, the qualitative behavior can be described as follows. For large magnetic fluxes,

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the exponential terms is small so that B and $\mathbf{\Phi}$ are proportional and both of them are quantized in terms of n on the right hand side of eq. (5.23). Moreover, since the magnetic flux is of order of p^2, the integer n on the right hand side of eq. (5.23) must be of order of p^2:

$$n \sim p^2 .$$

Therefore, for large values of the magnetic flux, the Baryonic charge also grows with p^2.

In this limit in which eqs. (5.15) and (5.18) are valid, one can rewrite eq. (5.23) as a trascendental equation for the elementary flux Ψ_0:

$$\begin{aligned} |B| = p^2 = 4p^2 \frac{\sqrt{2K}\Psi_0}{\pi L_r}\left(1 - \left(1 - \frac{\Psi_0^2}{2\left(\pi L_r\right)^4}\right)\exp\left(-\frac{8K\Psi_0^2}{\left(\pi L_r\right)^2}\right)\right) \Rightarrow \\ 1 \approx 4\frac{\sqrt{2K}\Psi_0}{\pi L_r}\left(1 - \left(1 - \frac{\Psi_0^2}{2\left(\pi L_r\right)^4}\right)\exp\left(-\frac{8K\Psi_0^2}{\left(\pi L_r\right)^2}\right)\right) \Rightarrow \end{aligned} \tag{5.24}$$

$$\frac{\pi L_r}{4\sqrt{2K}\Psi_0} \approx \left(1 - \left(1 - \frac{\Psi_0^2}{2\left(\pi L_r\right)^4}\right)\exp\left(-\frac{8K\Psi_0^2}{\left(\pi L_r\right)^2}\right)\right) . \tag{5.25}$$

When the magnetic flux is not large numerical methods must be used. On the other hand, in the applications in neutron stars and heavy ions collisions (see [9, 10]) one expects large values of the magnetic flux, which is the situation which will be mostly analyzed in the following.

It is interesting to compare the Baryonic charge with the topological charge Q appearing in the BPS bound:

$$\begin{aligned} Q &= \frac{AL_r}{e^2}\left[W(2\pi) - W(0)\right] , \quad u(0) = 0 \Rightarrow W(0) = 0 \Rightarrow \\ Q &= \frac{4\pi^2 p K^{1/2}}{e^2} u(2\pi)\sqrt{1 - \exp\left(-\frac{4\left(u(2\pi)\right)^2}{p^2} + 2I_0\right)} \Rightarrow \\ Q &= 4\sqrt{2}\pi \frac{K}{e^2 L_r}\mathbf{\Phi}\sqrt{1 - \exp\left(-\frac{8K\mathbf{\Phi}^2}{p^4\left(\pi L_r\right)^2} + 2I_0\right)} \end{aligned}$$

It is easy to see that the ratio Q/B of the topological charge which appears naturally in the BPS bound over the Baryon charge is not constant:

$$\left|\frac{Q}{B}\right| = \frac{\pi^2\sqrt{K}}{e^2}\frac{1}{\sqrt{1 - \exp\left(-\frac{8K\mathbf{\Phi}^2}{p^4\left(\pi L_r\right)^2} + 2I_0\right)}} . \tag{5.26}$$

Therefore, when the magnetic flux is large, the two topological charges are proportional. On the other hand, when the magnetic flux is small, the above ratio is very large and the topological charge which appears in the bound is much larger than the Baryonic charge. An important conclusion from the above relation is that it may happen that the topological charge suitable to derive a saturable BPS bound is a non-linear function of the "obvious" BPS charge (the Baryonic charge in the present case) which may not be suitable to achieve a BPS bound which can be saturated.

6 Thermodynamics

The direct computations of the classical grand canonical partition function and the corresponding free energy of this family of BPS Magnetized Baryonic layers presents some difficulties: it cannot be computed in a closed form due to the fact that the dependence on the discrete labels (such as p in eq. (3.4)) of the energy of these BPS configurations is determined by trascendental equations. The best method to analyze the above partition function is the saddle point analysis with the refined technique of resurgence (for detailed reviews see [106–108] and references therein). I hope to come back soon on the analysis of the resurgent behavior of the partition function in eq. (6.1). On the other hand, in the limit of large magnetic flux (when eqs. (5.15) and (5.18) are valid) one can use the proportionality of the magnetic flux with p^2: $\mathbf{\Phi} \approx \Psi_0 p^2$ to estimate the critical Baryonic chemical potential μ^* beyond which one expects a change in the partition function. As it has been already discussed, this is an excellent approximation in many situations of physical interest (such as neutron stars and Heavy Ions Collisions [9, 10]). The classical grand canonical partition function is the sum over the discrete label p of the factor $\exp\left[-\beta\left(E\left(p\right)+\mathbb{P}V-\mu_B B\right)\right]$ (where β is the inverse temperature, $\mathbb{P}$ is the pressure, V is the volume and μ_B is the Baryonic chemical potential and $E(p)$ is in eq. (5.5)). Both the pressure and the volume (due to the quantization condition in eq. (5.2)) depend on the discrete integer label p:

$$
\begin{aligned}
V &= V(p) = 2\pi^3 L_r \frac{p^2}{K} \\
\mathbb{P} &= \mathbb{P}(p) = -\frac{\partial E\left(p\right)}{\partial V} = -\frac{\partial L_r}{\partial V}\frac{\partial E\left(p\right)}{\partial L_r} = -\frac{K}{2\pi^3 p^2}\frac{\partial E\left(p\right)}{\partial L_r} \Rightarrow \\
\mathbb{P}V &= -L_r \frac{\partial E\left(p\right)}{\partial L_r}\,,
\end{aligned}
$$

where, in the above formulas, we have used the fact that for fixed p the volume is proportional to L_r. It is worth to note here that the product $\mu_B B\left(p\right)$ should be considered as positive with both choices of the sign in the BPS equations. Indeed, when B is negative (which is the case when the flux is positive, as it has been already mentioned) μ_B also must be negative (since the chemical potential of the anti-Baryons is opposite to the chemical potential of the Baryon which is positive: see [2] and references therein). Hence, in this approximation, the grand canonical partition function Z reads:

$$
Z = \sum_{p=-\infty\,,\ p\neq 0}^{+\infty} \exp\left[-\beta\left(E\left(p\right) - L_r\frac{\partial E\left(p\right)}{\partial L_r} - \mu_B B\left(p\right)\right)\right] = Z\left(\beta, \mu_B, L_r\right) \tag{6.1}
$$

$$
\begin{aligned}
E\left(p\right) &= 4\sqrt{2}\pi\frac{K}{e^2 L_r}\left|\Psi_0\right| p^2\sqrt{1-\exp\left(-\frac{8K\Psi_0^2}{\left(\pi L_r\right)^2}+2I_0\right)}\,, \\
\mu_B B\left(p\right) &= \mu_B p^2\,, \quad \mu_B \geq 0\,.
\end{aligned}
$$

Consequently, in this limit one can express the partition function Z in terms of the elliptic theta function

$$Z = \sum_{p \in \mathbb{Z}} \left(q^{1/2}\right)^{p^2} - 1\,, \quad q^{1/2} = \exp\left[-\beta\left(a\left(\Psi_0\right) - L_r \frac{\partial a\left(\Psi_0\right)}{\partial L_r} - \mu_B\right)\right]\,, \tag{6.2}$$

$$a\left(\Psi_0\right) = 4\sqrt{2}\pi \frac{K}{e^2 L_r} \left|\Psi_0\right| \sqrt{1 - \exp\left(-\frac{8K\Psi_0^2}{\left(\pi L_r\right)^2} + 2I_0\right)} \approx 2^{7/4}\pi^{3/2} \frac{K^{3/4}}{e^2} \left|\frac{\Psi_0}{L_r}\right|^{1/2} \tag{6.3}$$

where K (which is a "rescaled" version of the Pions decay constant) is defined in eqs. (2.1) and (2.2) while eq. (5.24) has been taken into account and the term "-1" in eq. (6.2) is due to the fact that the $p = 0$ terms is absent in the present family since, when $p = 0$ the topological charge vanishes. Thus, one expects a change in the behavior of the grand canonical partition function when the chemical potential μ_B becomes of the same order of the term $a\left(\Psi_0\right) - L_r \frac{\partial a(\Psi_0)}{\partial L_r}$:

$$\mu_B \sim a\left(\Psi_0\right) - L_r \frac{\partial a\left(\Psi_0\right)}{\partial L_r}$$

which (using eq. (6.3)) gives the following estimate

$$\mu^* \approx 3\left(2^{3/4}\pi^{3/2} \frac{K^{3/4}}{e^2}\right) \left|\frac{\Psi_0}{L_r}\right|^{1/2} = 3\left(2^{3/4}\pi^{3/2}\right) \left(\frac{f_\pi}{2}\right)^{3/2} \left|\frac{\Psi_0}{eL_r}\right|^{1/2}\,, \tag{6.4}$$

where we have expressed explicitly the critical chemical potential in terms of the Pions decay constant using eqs. (2.1) and (2.2). Obviously, a more careful analysis of the partition function is needed as, for instance, in order to justify the fact that the chemical potential for configurations with negative B is the opposite of the chemical potential of the configurations with positive B a proper semi-classical treatment of these solitons would be necessary.[4] Nevertheless, the above estimate in eq. (6.4) provides with the correct qualitative behavior in the classical limit which is not spoiled in the regime in which the semi-classical corrections are small. Therefore, the critical Baryonic chemical potential grows with $\left|\frac{\Psi_0}{L_r}\right|^{1/2}$. This means, in a sense, that the bigger is $\left|\frac{\Psi_0}{L_r}\right|$, the more difficult is to destabilize these hadronic layers. The above results also imply that in order for these configurations to survive in a large box (large L_r), Ψ_0 must increase accordingly. In view of the difficulties that LQCD faces when dealing with Baryonic chemical potential, the present formalism can prove very useful for phenomenological applications at finite Baryon density with non-vanishing magnetic fluxes. I will come back on these relevant issue in a future publication.

7 Conclusions and perspectives

Using the equation of Hamilton-Jacobi from classical mechanics, in the present paper the first analytic examples of static non-homogeneous condensates possessing both Baryonic

[4] It is worth emphasizing that, in systems with dominant magnetic interactions, the fact that the chemical potential of the particles is opposite to the chemical potential of the anti-particles extends to the world of quasi-particles as well. For instance, in [109] it has been shown that the two magnon eigenmodes of the system are described by an equal and opposite chemical potential, in analogy with a particle-antiparticle pair.

charge and magnetic flux have been constructed. Such configurations saturate a novel BPS bound (provided the surface area of the Baryonic layer is quantized) in which the topological charge Q is a non-linear function of the Baryonic charge (which is the "obvious" topological charge which one would consider at first in a BPS bound). The present formalism allows to compute the Baryonic charge as a function of the magnetic flux as well as the dependence of Q on the Baryonic charge. The computation of the classical grand canonical partition function associated to this family of BPS configurations allows to estimate the Baryonic chemical potential beyond which such configurations become thermodynamically unstable. The present formalism can have a huge impact on the analysis of chiral perturbation theory and, more generically, of QCD under extreme conditions of non-vanishing Baryon density, strong magnetic fields and low temperatures (where LQCD experiences problems and analytic perturbative methods are ineffective). For instance, it allows to determine how much magnetic flux is needed to increase the Baryonic charge of the condensate (a computation which would be difficult to do with other methods). Another interesting application of the present results is the analysis of the coupling of the quarks to such gauged magnetized BPS Baryonic configurations (using the well known Dirac-like equation which describes the interactions of quarks with SU(2) chiral fields: see [100, 110–113] and references therein). I hope to come back soon on these interesting topics in a future publication.

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