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Four-loop QCD cusp anomalous dimension at small angle

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Abstract: We calculate the small angle expansion of the four-loop QCD cusp anomalous dimension. As a byproduct of our calculation, we also obtain the four-loop anomalous dimension of the heavy-quark field in HQET. The validity of the calculational setup is cross-checked by the independent calculation of the four-loop QCD beta-function from heavy-quark-gluon vertex renormalization in HQET. We check the obtained results for the cusp anomalous dimension and heavy-quark field anomalous dimension against available analytical and numerical results. Finally, we find that the maximal transcendentality contribution to the QCD Bremsstrahlung function coincides, up to a factor 3/2, with the Bremsstrahlung function in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, at least, through 4 loops.

Keywords: Effective Field Theories of QCD, Higher-Order Perturbative Calculations, Effective Field Theories, Renormalization Group

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1 Introduction

The cusp anomalous dimension Γ_{cusp} was originally introduced in Polyakov's paper [1] to describe the scaling of a cusped Wilson loop with a variation of UV cutoff parameter. The same quantity determines the infrared singularity structure of scattering amplitudes in the chosen QFT theory. The physical meaning of this correspondence is simple. Since the infrared divergencies stem from soft regions of loop integration, the incoming/outgoing particle can be replaced by a cusped Wilson line which IR and UV scaling behavior is governed by the same exponent by dimensional arguments. The notation $\Gamma_{\text{cusp}}(\phi, \alpha_s)$ indicates that this quantity depends on cusp angle ϕ and coupling constant α_s. Of course, the cusp anomalous dimension depends also on the specific variant of quantum field theory. In the case of quantum electrodynamics without massless fermions, the cusp anomalous dimension determines the scaling of quasi-elastic cross sections with the soft-photon energy cut-off thus being a directly observable quantity. Note that in this case result exact in α can be obtained from one-loop calculation, thanks to exponentiation of QED. For non-abelian theories, in particular, for QCD, $\Gamma_{\text{cusp}}(\phi, \alpha_s)$ is a nontrivial series in α_s accessible only via perturbative calculations.

The asymptotics of $\Gamma_{\text{cusp}}(\phi, \alpha_s)$ at large and small angles also provide an important information. The light-like cusp anomalous dimension $K(\alpha_s) = \lim_{\phi\to i\infty} i\Gamma_{\text{cusp}}(\phi, \alpha_s)/\phi$ plays important role for the infrared asymptotics of massless scattering amplitudes and form factors [2–4]. The opposite limit of small angles is also of some interest. In particular, the Bremsstrahlung function $B(\alpha_s) = -\lim_{\phi\to 0} \Gamma_{\text{cusp}}(\phi, \alpha_s)/\phi^2$ determines the energy loss of a charged particle moving along a smooth curved trajectory [5]. In addition, the calculations in the small-angle limit are much more accessible and often precede the full-angle dependence calculations.

The exact angle dependence of $\Gamma_{\text{cusp}}(\phi, \alpha_s)$ is known up to three loops in QCD [6] and supersymmetric Yang-Mills theories [7]. At the four loop order, only partial results are available: fermionic contributions to QCD $\Gamma_{\text{cusp}}(\phi, \alpha_s)$ in small-angle expansion [8–11], abelian part with full angle dependence [8, 10, 12], planar part of angle-dependent cusp anomalous dimension in N=4 SYM [13]. In addition, the Bremsstrahlung function has been calculated at the four-loop level in 3-dimensional ABJM theory, refs. [14, 15].

The goal of the present paper is to provide the small-angle expansion up to ϕ^4 of the QCD four-loop cusp anomalous dimension. As a byproduct, we calculate the anomalous dimension of the heavy quark field in heavy quark effective theory (HQET) [16], extending the partial results of refs. [8–11, 17]. Both calculations require the knowledge of the four-loop HQET propagator-type integrals and provide the first application examples of the results of ref. [18] where a full set of the four-loop HQET propagator master integrals was calculated. As an additional cross-check for all ingredients of the calculation chain we obtain from the renormalization of the heavy-quark-gluon vertex the four-loop QCD beta-function [19, 20] known for a long time and recently extended to five-loop order [21–23].

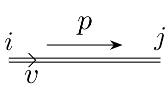
The paper is organized as follows. In section 2 we introduce the HQET framework and in section 3 we present details of our calculation. Section 4 contains four-loop results for the calculated heavy quark field anomalous dimension as well as small angle expansion of QCD cusp anomalous dimension. We conclude in section 5.

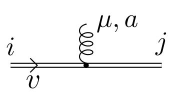
2 Cusp anomalous dimension in HQET framework

Heavy Quark Effective Theory is a well established framework for calculation of both full angle dependent $\Gamma_{\rm cusp}(\phi, \alpha_s)$ [6, 7, 12] and its small-angle expansion [9, 11]. In our work we closely follow the technique employed in refs. [9, 11] for the calculation of $\Gamma_{\rm cusp}(\phi, \alpha_s)$ in small angle expansion. Within the HQET framework the quantity $\Gamma_{\rm cusp}(\phi, \alpha_s)$ is extracted from UV divergences of diagrams with cusped HQET line where the cusp corresponds to an abrupt change of heavy quark velocity.

Being an effective theory of QCD, HQET describes the interaction of heavy quark field h with massless quarks by gluon exchange. The renormalization of HQET theory requires, in addition to QCD renormalization constants, the only new constant Z_h connecting the bare and the renormalized heavy quark fields, which appears in the diagrams with external heavy legs. The QCD renormalization constants up to the four-loop order can be found in ref. [20].

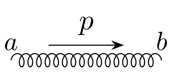
To fix notation we provide Feynman rules for the heavy quark propagator and the heavy-quark-gluon vertex:



$$= \frac{-i\delta_{ij}}{\omega - v\cdot p}\,,$$


$$= igv^\mu T^a_{ij}\,. \tag{2.1}$$

By introducing in eq. (2.1) the residual energy ω we regulate the IR divergencies in the diagrams, then the UV divergences, our main interest, reveal themselves as poles in $\varepsilon = (4-d)/2$. Since one of our goals, h-field anomalous dimension, is known to be gauge dependent quantity, we perform our calculations in a generalized covariant gauge where the gluon propagator has the form



$$= \frac{-i\delta_{ab}}{p^2}\left[g_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right]\,. \tag{2.2}$$

All other QCD Feynman rules are standard and available from ref. [24].

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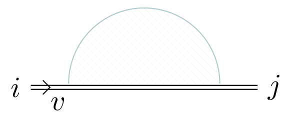
Equipped with the above Feynman rules we are prepared to calculate loop diagrams in HQET. We perform our calculations in two steps. In the first step, we calculate the sum of bare unrenormalized diagrams up to the needed order. In the second step, we carry out renormalization and extract the required renormalization constants.

The first and simplest quantity we need to calculate is the heavy quark field renormalization constant and the corresponding anomalous dimension γ_h. We consider the quantity

$$G_{\rm hh} =$$

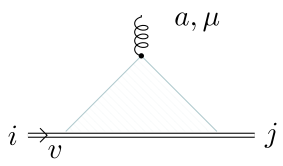


$$\cdot \frac{\delta_{ij}}{\omega N}. \tag{2.3}$$

where N is the number of colors and the dashed blob stands for the sum of the bare 1PI two-point functions up to the four-loop order. For convenience, we contract the color indices to make the expression scalar and choose the normalization factor such that the perturbative expansion of $G_{\rm hh}$ starts with 1.

Another quantity that we want to calculate is the four-loop QCD beta-function. Although this beta-function is known for a long time [19, 20], this calculation provides a crucial check of our setup. We extract this quantity from the heavy-quark-gluon vertex. Similar to the previous case, eq. (2.3), we define the scalar function $G_{\rm ghh}$ accumulating the contributions of 1PI three-point diagrams up to four-loop order:

$$G_{\rm ghh} =$$



$$\cdot \frac{t^a_{ij} v_\mu}{g_s N C_F}. \tag{2.4}$$

To reduce the original problem to the problem of propagator-type diagrams calculation, we apply the IRR trick [25] to diagrams entering in (2.4) and put external gluon momentum to zero.

Finally, we want to consider the cusp anomalous dimension and, in particular, calculate its small angle expansion. We consider the expectation value of an infinite cusped Wilson line depending on two velocities $v^2 = v'^2 = 1, v \cdot v' = \cos\phi$. In momentum representation the perturbative corrections to this expectation value are expressed via HQET diagrams with $[h(v)\bar{h}(v')]$ operator insertion into two-point function with heavy quark velocities v and v' to the left and to the right from the insertion point, respectively. Again, we construct a scalar function $G_{\rm [hh]hh}(\phi)$ corresponding to the sum of bare 1PI diagrams convoluted with

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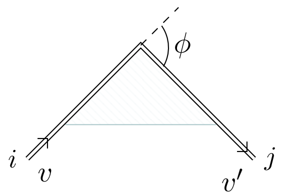
the appropriate tensor:

$$G_{[\mathrm{hh}]\mathrm{hh}}(\phi) =$$



$$\cdot \frac{\delta_{ij}}{N}. \tag{2.5}$$

Using results for bare functions G_X calculated before we proceed with the second step, namely, with the extraction of renormalization constants. The corresponding $\overline{\mathrm{MS}}$ renormalization constants Z_X for each function G_X are determined from the poles cancellation requirements

$$Z_{\mathrm{hh}} \cdot G_{\mathrm{hh}} = \mathcal{O}\left(\varepsilon^0\right), \quad Z_{\mathrm{ghh}} \cdot G_{\mathrm{ghh}} = \mathcal{O}\left(\varepsilon^0\right), \quad Z_{[\mathrm{hh}]\mathrm{hh}}(\phi) \cdot G_{[\mathrm{hh}]\mathrm{hh}}(\phi) = \mathcal{O}\left(\varepsilon^0\right). \tag{2.6}$$

Here we replace bare parameters $a_{s,B}$ and $a_{\xi,B} = 1 - \xi$ entering G_i with its renormalized counterparts:

$$a_{s,B} = \mu^{2\varepsilon} Z_{a_s} a_s, \qquad a_{\xi,B} = Z_{a_\xi} a_\xi. \tag{2.7}$$

From vertex renormalization constants (2.6), dividing by external legs Z-factors we determine gauge parameter independent combinations:

$$Z_{a_s} = \frac{Z_{\mathrm{ghh}}^2}{Z_{\mathrm{hh}}^2 Z_A}, \qquad Z_{\mathrm{cusp}}(\phi) = \frac{Z_{[\mathrm{hh}]\mathrm{hh}}(\phi)}{Z_{\mathrm{hh}}}, \tag{2.8}$$

where Z_{hh} found before and gluon field renormalization constant Z_A is known from refs. [19, 20]. Gauge parameter independence of Z_{a_s} and $Z_{\mathrm{cusp}}(\phi)$ allows us to calculate G_{ghh} and $G_{[\mathrm{hh}]\mathrm{hh}}(\phi)$ as expansion around $\xi = 0$, keeping only the first term of expansion to verify its cancellation in (2.8) as additional test on the validity of the obtained results. Another test comes from the HQET Ward identity implying that:

$$Z_{\mathrm{hh}} = \lim_{\phi \to 0} Z_{[\mathrm{hh}]\mathrm{hh}}(\phi). \tag{2.9}$$

From the four-loop result for Z_{a_s} in (2.8) we can derive a well known expression for the four-loop QCD beta-function within the $\overline{\mathrm{MS}}$ renormalization scheme:

$$\beta_{a_s} = \frac{d a_s}{d \log \mu^2} = \frac{-\varepsilon a_s}{1 + a_s \partial_{a_s} \log Z_{a_s}} = -\varepsilon a_s - \sum_{n=0}^{\infty} b_n a_s^{n+2}, \tag{2.10}$$

with $b_0 = \frac{11}{3} C_A - \frac{4}{3} n_f T_F$. The agreement of the obtained coefficients b_{0-3} with the results of refs. [19, 20] provides a strong check of our calculation setup.

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3 Calculation details

To calculate bare Green functions introduced in section 2 we have developed a highly automatized setup. Its workflow starts with the generation of diagrams with `DIANA` [26], which internally calls `QGRAF` [27]. We generate the propagator-type (two-point) diagrams for the calculation of $G_{\rm hh}$ and vertex-type (three-point) diagrams for the calculation of $G_{\rm ghh}$. The former diagrams have been reused in $G_{\rm [hh]hh}$ calculation since the diagrams with the cusp on the Wilson line are in one-to-one correspondence with the diagrams obtained by an auxiliary leg insertion in all possible ways on the heavy-quark line in two-point diagrams. The insertion point corresponds to a cusp, so we replace $v \to v'$ in all h-propagators to the right of this point. After that, all propagators dependent on v' are expanded in the vicinity of $\phi = 0$ with a recursive application of the identity

$$\frac{1}{1-2\,k\cdot v'} = \underbrace{\frac{1}{1-2\,k\cdot v}}_{\mathcal{O}(\phi^0)} + \underbrace{\frac{1}{1-2\,k\cdot v}\,\frac{2k\cdot(v'-v)}{1-2\,k\cdot v'}}_{\mathcal{O}(\phi)} \tag{3.1}$$

After the decomposition of v' in numerator with $v' = v\cos\phi + n_\perp \sin\phi$, where $n_\perp^2 = 1$, and $v\cdot n_\perp = 0$ we are left with scalar products of $n_\perp$ with loop momenta. Since the result of the loop integration is independent of the $n_\perp$ direction it is possible to replace $n_\perp^{\mu_1}\dots n_\perp^{\mu_n} \to \langle n_\perp^{\mu_1}\dots n_\perp^{\mu_n}\rangle$, where $\langle\bullet\rangle$ denotes averaging over perpendicular directions. For reference, we present explicit formulae for this averaging

$$\langle n_\perp^{\mu_1}\dots n_\perp^{\mu_{2s-1}}\rangle = 0, \quad \langle n_\perp^{\mu_1}\dots n_\perp^{\mu_{2s}}\rangle = \frac{(1/2)_s}{(3/2-\varepsilon)_s}\mathcal{S}\prod_{k=1}^{s} g_\perp^{\mu_{2k-1}\mu_{2k}}\,, \tag{3.2}$$

where $g_\perp^{\alpha\beta} = g^{\alpha\beta} - v^\alpha v^\beta$, $c_s = c\cdot(c+1)\cdot\ldots\cdot(c+s-1)$ is the Pocchammer symbol, and $\mathcal{S}$ is the normalized (i.e., $\mathcal{S}1 = 1$) symmetrization operator with respect to permutations of $\mu_1,\dots,\mu_{2k}$. From eq. (3.1) it is obvious that the calculation of higher orders of expansion in ϕ requires the reduction of integrals with higher powers of denominators and scalar products in the numerator. In our work we consider expansion to ϕ^4, corresponding to two first non-trivial orders in the small-angle expansion of $\Gamma_{\rm cusp}(\phi)$.

Next, we calculate the Dirac traces and simplify expressions with `FORM` [28] and perform the color algebra in terms of color invariants with `COLOR` [29] ending up with a set of scalar integrals.

Due to the presence of linear propagators (2.1), we need to perform partial fraction decomposition of linear dependent propagators. With the implementation based on the package `TopoID` [30] and the private version of the `LiteRed` package, we obtain expressions containing integrals with an independent set of scalar products only which can be mapped on the set of 19 auxiliary topologies considered in [18]. For reduction to master integrals calculated in [18], we use `FIRE6` [31] in combination with `LiteRed` [32, 33].

All diagrams up to the three-loop order as well as the four-loop diagrams needed for $G_{\rm hh}$ are calculated keeping the full dependence on gauge-fixing parameter ξ, but to reduce required calculation time, we perform expansion in ξ to leading order in four-loop diagrams $G_{\rm ghh}$ and $G_{\rm [hh]hh}$, since corresponding renormalization constants (2.8) extracted from these functions are gauge-parameter independent and cancelation of the ξ dependence in the expanded form is a sufficient check on the validity of the obtained result.

4 Results and discussion

From the results for renormalization constants obtained in the previous section, we derive anomalous dimensions by taking logarithmic derivatives in the renormalization scale:[1]

$$\gamma_h = \frac{d\log Z_{\rm hh}}{d\log\mu} = 2\beta_{a_s}\frac{\partial\log Z_{\rm hh}}{\partial a_s} + 2\beta_{a_\xi}\frac{\partial\log Z_{\rm hh}}{\partial a_\xi}, \tag{4.1}$$

$$\Gamma_{\rm cusp}(\phi) = -\frac{d\log Z_{\rm cusp}(\phi)}{d\log\mu} = -2\beta_{a_s}\frac{d\log Z_{\rm cusp}(\phi)}{da_s}. \tag{4.2}$$

Here $a_s = \alpha_s/(4\pi)$ and $a_\xi = 1-\xi$, the strong coupling beta-function β_{a_s} was introduced in (2.10) and the beta-function of the gauge fixing parameter a_ξ is defined as follows:

$$\beta_{a_\xi} = \frac{da_\xi}{d\log\mu^2} = -\beta_{a_s}\frac{a_\xi\partial_{a_s}\log Z_{a_\xi}}{1+a_\xi\partial_{a_\xi}\log Z_{a_\xi}}\,. \tag{4.3}$$

The complete result for the HQET field anomalous dimension up to four-loop order is

$$\begin{aligned}
\gamma_h =& -2a_sC_F(3-a_\xi)+a_s^2C_F\left\{\frac{32}{3}n_fT_F-C_A\left(\frac{179}{6}-4a_\xi-\frac{1}{2}a_\xi^2\right)\right\}\\
&+a_s^3\left\{C_F^2n_fT_F(102-96\zeta_3)+C_F\left(\frac{160}{27}(n_fT_F)^2+C_An_fT_F\left(\frac{782}{27}+96\zeta_3-\frac{17}{2}a_\xi\right)\right.\right.\\
&\left.\left.-C_A^2\left(\frac{23815}{216}+\frac{123}{4}\zeta_3+\frac{4}{15}\pi^4-\left(\frac{271}{16}-\frac{4}{45}\pi^4+6\zeta_3\right)a_\xi-\left(\frac{39}{16}+\frac{3}{4}\zeta_3\right)a_\xi^2-\frac{5}{8}a_\xi^3\right)\right)\right\}\\
&+a_s^4\left\{C_F^2\left[(n_fT_F)^2\left(-\frac{3296}{27}-\frac{32}{15}\pi^4+384\zeta_3\right)\right.\right.\\
&\left.+C_An_fT_F\left(\frac{21703}{27}+\frac{88}{15}\pi^4-928\zeta_3-480\zeta_5-\left(\frac{767}{6}-\frac{4}{15}\pi^4-88\zeta_3\right)a_\xi\right)\right]\\
&-\frac{d_F^{abcd}d_F^{abcd}}{N}n_f\left(\frac{512}{3}\pi^2-256\zeta_3-\frac{512}{3}\pi^2\zeta_3+320\zeta_5\right)-C_F^3n_fT_F\left(\frac{560}{3}+592\zeta_3-960\zeta_5\right)\\
&+C_F\left[(n_fT_F)^3\left(\frac{256}{27}-\frac{256}{9}\zeta_3\right)-C_A(n_fT_F)^2\left(\frac{2054}{81}-\frac{32}{15}\pi^4+384\zeta_3+\left(\frac{2152}{243}-\frac{32}{3}\zeta_3\right)a_\xi\right)\right.\\
&+C_A^2n_fT_F\left(\frac{30617}{81}-\frac{16}{3}\pi^2-\frac{3097}{540}\pi^4+\frac{5506}{3}\zeta_3+\frac{104}{9}\pi^2\zeta_3-96\zeta_3^2-\frac{1534}{3}\zeta_5\right.\\
&\left.-\left(\frac{37957}{1944}-\frac{1}{15}\pi^4+82\zeta_3+\frac{16}{27}\pi^2\zeta_3-\frac{4}{9}\zeta_5\right)a_\xi-\left(\frac{109}{36}-\frac{1}{180}\pi^4+\frac{7}{3}\zeta_3\right)a_\xi^2\right)\\
&-C_A^3\left(\frac{471001}{648}-\frac{781}{36}\pi^2+\frac{10501}{2160}\pi^4-\frac{850}{1701}\pi^6+\frac{212237}{288}\zeta_3+\frac{709}{36}\pi^2\zeta_3-\frac{451}{4}\zeta_3^2-\frac{3859}{12}\zeta_5\right.\\
&-\left(\frac{1690475}{15552}-\frac{2}{9}\pi^2-\frac{9109}{4320}\pi^4+\frac{472}{8505}\pi^6+\frac{3839}{48}\zeta_3+\frac{164}{27}\pi^2\zeta_3+\frac{11}{4}\zeta_3^2-\frac{272}{9}\zeta_5\right)a_\xi\\
&-\left(\frac{6707}{576}+\frac{1}{12}\pi^2-\frac{121}{1080}\pi^4+\frac{653}{48}\zeta_3+\frac{13}{36}\pi^2\zeta_3-\frac{169}{48}\zeta_5\right)a_\xi^2-\left(\frac{149}{48}-\frac{1}{480}\pi^4+\frac{21}{16}\zeta_3\right)a_\xi^3\\
&\left.\left.-\left(\frac{19}{32}+\frac{1}{96}\zeta_3+\frac{5}{48}\zeta_5\right)a_\xi^4\right)\right]
\end{aligned}$$

[1] For historical reasons, we take derivatives in $\log\mu$ rather than in $\log\mu^2$ for γ_h and $\Gamma_{\rm cusp}(\phi)$.

$$
-\frac{d_F^{abcd}d_A^{abcd}}{N}\left[\frac{16}{3}\pi^2-\frac{128}{15}\pi^4-\frac{224}{405}\pi^6-\frac{569}{4}\zeta_3-\frac{320}{3}\pi^2\zeta_3+384\zeta_3^2+\frac{4815}{4}\zeta_5\right.
-\left(\frac{16}{3}\pi^2-\frac{884}{2835}\pi^6-11\zeta_3-\frac{128}{3}\pi^2\zeta_3+24\zeta_3^2+540\zeta_5\right)a_\xi-\left(\frac{3}{2}\zeta_3-\frac{75}{2}\zeta_5\right)a_\xi^2
\left.+3\zeta_3a_\xi^3+\left(\frac{7}{4}\zeta_3-\frac{5}{4}\zeta_5\right)a_\xi^4\right]\Bigg\}+\mathcal{O}\left(a_s^5\right) \tag{4.4}
$$

The terms up to α_s^3 agree with the results of the three-loop HQET calculation [34] and results obtained as a byproduct of the three-loop QCD on-shell renormalization [35]. The analytical result for the four-loop part is new and in full agreement with the results of numerical calculation [17] and partial four-loop results for fermionic contributions calculated in [9, 11]. We note also that the terms proportional to a_ξ^{L-k} with $L \geqslant 2$ and $k = 0, 1$ coincide, including the new terms for $L = 4$, with the corresponding terms in quark anomalous dimension γ_q, [23].

The main result of the present paper, namely the small angle expansion of the cusp anomalous dimension has the following form:

$$
\Gamma_{\rm cusp}(\phi) = \Gamma^{(2)}\phi^2 + \Gamma^{(4)}\phi^4 + \mathcal{O}(\phi^6) \tag{4.5}
$$

and the complete results for the first two terms of the small-angle expansion of the cusp anomalous dimension up to four-loop order derived from (4.2) read

$$
\Gamma^{(2)}=-\frac{4}{3}a_sC_F-a_s^2C_F\left\{C_A\left(\frac{376}{27}-\frac{8}{9}\pi^2\right)-\frac{80}{27}n_fT_F\right\}
+a_s^3\left\{C_F^2n_fT_F\left(\frac{220}{9}-\frac{64}{3}\zeta_3\right)+\frac{64}{81}C_F(n_fT_F)^2\right.
-C_FC_A^2\left(\frac{946}{9}-\frac{1360}{81}\pi^2+\frac{8}{9}\pi^4+\frac{40}{9}\zeta_3\right)
\left.+C_FC_An_fT_F\left(\frac{3112}{81}-\frac{320}{81}\pi^2+\frac{224}{9}\zeta_3\right)\right\}
+a_s^4\left\{\frac{d_F^{abcd}d_F^{abcd}}{N}n_f\left(\frac{640}{27}\pi^2+\frac{320}{27}\pi^4-\frac{1024}{9}\pi^2\zeta_3\right)\right.
+\frac{d_F^{abcd}d_A^{abcd}}{N}\left(\frac{64}{27}\pi^2-\frac{512}{27}\pi^4-\frac{128}{135}\pi^6+\frac{2176}{9}\pi^2\zeta_3\right)
-C_F^2(n_fT_F)^2\left(\frac{9568}{243}+\frac{64}{135}\pi^4-\frac{2560}{27}\zeta_3\right)+C_F(n_fT_F)^3\left(\frac{256}{243}-\frac{512}{81}\zeta_3\right)
+C_F^2C_An_fT_F\left(\frac{103772}{243}-\frac{880}{27}\pi^2+\frac{176}{135}\pi^4-\frac{10880}{27}\zeta_3+\frac{256}{9}\pi^2\zeta_3-\frac{320}{3}\zeta_5\right)
-C_FC_A(n_fT_F)^2\left(\frac{7340}{243}-\frac{2432}{729}\pi^2-\frac{224}{405}\pi^4+\frac{8960}{81}\zeta_3\right)
+C_FC_A^2n_fT_F\left(\frac{96322}{243}-\frac{59072}{729}\pi^2+\frac{352}{81}\pi^4+\frac{57776}{81}\zeta_3-\frac{896}{27}\pi^2\zeta_3-\frac{1760}{9}\zeta_5\right)
-C_FC_A^3\left(\frac{178022}{243}-\frac{143624}{729}\pi^2+\frac{9682}{405}\pi^4-\frac{80}{81}\pi^6+\frac{19024}{81}\zeta_3-\frac{256}{27}\pi^2\zeta_3-\frac{1240}{9}\zeta_5\right)
\left.-C_F^3n_fT_F\left(\frac{1144}{27}+\frac{1184}{9}\zeta_3-\frac{640}{3}\zeta_5\right)\right\}+\mathcal{O}\left(a_s^5\right), \tag{4.6}
$$

$$
\begin{aligned}
\Gamma^{(4)} =& -\frac{4}{45}a_s C_F - a_s^2 C_F\left\{C_A\left(\frac{364}{405}-\frac{8}{135}\pi^2\right)-\frac{16}{81}n_f T_F\right\}\\
&+a_s^3\left\{C_F^2 n_f T_F\left(\frac{44}{27}-\frac{64}{45}\zeta_3\right)+\frac{64}{1215}C_F(n_f T_F)^2\right.\\
&-C_F C_A^2\left(\frac{18074}{6075}-\frac{320}{243}\pi^2+\frac{8}{135}\pi^4+\frac{3512}{675}\zeta_3\right)\\
&\left.+C_F C_A n_f T_F\left(\frac{328}{135}-\frac{64}{243}\pi^2+\frac{224}{135}\zeta_3\right)\right\}\\
&+a_s^4\left\{\frac{d_F^{abcd}d_F^{abcd}}{N}n_f\left(-\frac{1472}{225}-\frac{10048}{2025}\pi^2+\frac{3136}{2025}\pi^4+\frac{18176}{225}\zeta_3-\frac{4096}{675}\pi^2\zeta_3-\frac{1024}{9}\zeta_5\right)\right.\\
&+\frac{d_F^{abcd}d_A^{abcd}}{N}\left(\frac{512}{243}-\frac{53696}{3645}\pi^2-\frac{1984}{675}\pi^4-\frac{128}{2025}\pi^6+\frac{36224}{405}\zeta_3+\frac{9344}{225}\pi^2\zeta_3-\frac{896}{9}\zeta_5\right)\\
&-C_F^2(n_f T_F)^2\left(\frac{9568}{3645}+\frac{64}{2025}\pi^4-\frac{512}{81}\zeta_3\right)+C_F(n_f T_F)^3\left(\frac{256}{3645}-\frac{512}{1215}\zeta_3\right)\\
&+C_F^2 C_A n_f T_F\left(\frac{99812}{3645}-\frac{176}{81}\pi^2+\frac{176}{2025}\pi^4-\frac{10496}{405}\zeta_3+\frac{256}{135}\pi^2\zeta_3-\frac{64}{9}\zeta_5\right)\\
&-C_F C_A(n_f T_F)^2\left(\frac{20804}{10935}-\frac{2432}{10935}\pi^2-\frac{224}{6075}\pi^4+\frac{1792}{243}\zeta_3\right)\\
&+C_F C_A^2 n_f T_F\left(\frac{224414}{273375}-\frac{286696}{54675}\pi^2+\frac{4688}{30375}\pi^4+\frac{2389232}{30375}\zeta_3-\frac{1664}{675}\pi^2\zeta_3-\frac{1504}{135}\zeta_5\right)\\
&+C_F C_A^3\left(\frac{9434794}{273375}+\frac{216896}{18225}\pi^2-\frac{5782}{3375}\pi^4+\frac{16}{243}\pi^6-\frac{5252768}{30375}\zeta_3+\frac{2912}{675}\pi^2\zeta_3+\frac{4168}{45}\zeta_5\right)\\
&\left.-C_F^3 n_f T_F\left(\frac{1144}{405}+\frac{1184}{135}\zeta_3-\frac{128}{9}\zeta_5\right)\right\}+\mathcal{O}\left(a_s^5\right)
\end{aligned}
\tag{4.7}
$$

The terms up to α_s^3 agree with the full angle-dependent results [6, 7] expanded in ϕ^2. Four-loop part is new and its fermionic contributions are in agreement with the partial four-loop results from [9, 11]. Results (4.6) and (4.7) are obtained for the QCD-like case, where heavy quark and massless quarks are in the same representation. For the case of different representation R of the Wilson line, the result can be easily modified by exchanging the single power of C_F with C_R and by replacements $\frac{d_F^{abcd}d_F^{abcd}}{N} \to \frac{d_R^{abcd}d_F^{abcd}}{N_R}$ and $\frac{d_F^{abcd}d_A^{abcd}}{N} \to \frac{d_R^{abcd}d_A^{abcd}}{N_R}$ for quartic Casimir invariants.

It is interesting to compare our result for QCD Γ_{cusp} small-angle expansion with available results in $\mathcal{N}=4$ SYM. In particular, we compare the Bremsstrahlung function which is known in $\mathcal{N}=4$ SYM as an all-order expression [5]. By retaining in front of a_s^L only the terms of highest transcendental weight $2L-2$ in QCD Bremsstrahlung function $B^{\text{QCD}} = -\Gamma^{(2)}$ from eq. (4.6) we obtain

$$
B_{\text{MT}}^{\text{QCD}} = \frac{4}{3}C_F a_s - \frac{8}{9}C_F C_A \pi^2 a_s^2 + \frac{8}{9}C_F C_A^2 \pi^4 a_s^3 - \left\{\frac{80}{81}C_F C_A^3 - \frac{128}{135}\frac{d_F^{abcd}d_A^{abcd}}{N}\right\}\pi^6 a_s^4 \,. \tag{4.8}
$$

Note that all color factors entering $B_{\text{MT}}^{\text{QCD}}$ are of the maximal non-abelian nature.

All-order expression for the $B^{\mathcal{N}=4}$ from ref. [5] reads[2]

$$B^{\mathcal{N}=4} = \frac{a_s}{2\pi^2}\partial_{a_s} \log\left[L_{N-1}^{(1)}\left(-4\pi^2 a_s\right) e^{2\pi^2 a_s(1-1/N)}\right], \tag{4.9}$$

where $L_n^{(\alpha)}$ is the generalized Laguerre polynomial. This Bremsstrahlung function, in addition to the contribution of gluons, also involves the contribution of auxiliary scalar fields which leads to different results for B^{QCD} and $B^{\mathcal{N}=4}$ already at one loop: they differ by a factor of 3/2. Remarkably, the same relation holds at least to the four loops when we replace B^{QCD} by its maximal transcendentality part:

$$B^{\mathcal{N}=4} = \frac{3}{2}B^{\text{QCD}}_{\text{MT}} + \mathcal{O}\left(a_s^5\right). \tag{4.10}$$

Moreover, we have checked that the above relation also holds for arbitrary representation of Wilson line once we substitute $C_F \to C_R$ and $d_F^{abcd}d_A^{abcd} \to d_R^{abcd}d_A^{abcd}$ in eq. (4.8) and use the perturbative result of ref. [36] for $B^{\mathcal{N}=4}$ in representation R. The relation (4.10) can be interpreted as the manifestation of the maximal transcendentality principle [37, 38].

Finally, let us write the small-angle expansion of Γ_{cusp} in a slightly modified form

$$\Gamma_{\text{cusp}}(\phi) = -3\Gamma^{(2)}A(x) + \widetilde{\Gamma}^{(4)}\phi^4 + \mathcal{O}\left(\phi^6\right) = \Gamma^{(2)}\cdot\left(\phi^2 + \frac{\phi^4}{15}\right) + \widetilde{\Gamma}^{(4)}\phi^4 + \mathcal{O}\left(\phi^6\right),$$

where $x = e^{i\phi}$ and $A(x) = \phi\cot\phi - 1$ is simply the angular dependence of one-loop cusp anomalous dimension in QCD. The modified coefficient $\widetilde{\Gamma}^{(4)} = \Gamma^{(4)} - \frac{1}{15}\Gamma^{(2)}$ has a substantially simpler form than $\Gamma^{(4)}$ in eq. (4.7):

$$\begin{aligned}
\widetilde{\Gamma}^{(4)} = {} & \frac{4}{135}C_A C_F a_s^2 + a_s^3 C_F\left\{C_A^2\left(\frac{24496}{6075} + \frac{16}{81}\pi^2 - \frac{368}{75}\zeta_3\right) - \frac{32}{243}C_A n_f T_F\right\} \\
& + a_s^4\left\{C_F^2 C_A n_f T_F\left(-\frac{88}{81} + \frac{128}{135}\zeta_3\right) + C_F C_A (n_f T_F)^2\frac{1216}{10935}\right. \\
& + C_F C_A^2 n_f T_F\left(-\frac{6999736}{273375} + \frac{2888}{18225}\pi^2 - \frac{4112}{30375}\pi^4 + \frac{314944}{10125}\zeta_3 - \frac{512}{2025}\pi^2\zeta_3 + \frac{256}{135}\zeta_5\right) \\
& + C_F C_A^3\left(\frac{22786444}{273375} - \frac{67432}{54675}\pi^2 - \frac{3628}{30375}\pi^4 - \frac{4777168}{30375}\zeta_3 + \frac{7456}{2025}\pi^2\zeta_3 + \frac{11264}{135}\zeta_5\right) \\
& + \frac{d_F^{abcd}d_F^{abcd}}{N}n_f\left(-\frac{1472}{225} - \frac{1472}{225}\pi^2 + \frac{512}{675}\pi^4 + \frac{18176}{225}\zeta_3 + \frac{1024}{675}\pi^2\zeta_3 - \frac{1024}{9}\zeta_5\right) \\
& \left. + \frac{d_F^{abcd}d_A^{abcd}}{N}\left(\frac{512}{243} - \frac{54272}{3645}\pi^2 - \frac{3392}{2025}\pi^4 + \frac{36224}{405}\zeta_3 + \frac{17152}{675}\pi^2\zeta_3 - \frac{896}{9}\zeta_5\right)\right\}
\end{aligned} \tag{4.11}$$

First, we see that $\widetilde{\Gamma}^{(4)}$ contains only one color factor $\frac{d_F^{abcd}d_F^{abcd}}{N}$ which survives in the QED limit. This fact is quite anticipated agreeing with the results of refs. [10, 12], where $\Gamma_{\text{cusp}}^{QED}(\phi)$ was represented as $\Gamma_{\text{cusp}}^{QED}(\phi) = \gamma(\alpha)A(x) + (\alpha/\pi)^4 n_f B(x)$. Perhaps a less anticipated observation is that the coefficients of all remaining color structures in $\widetilde{\Gamma}^{(4)}$ are now free from the highest transcendental weight contribution.

[2] Note that scalar field contribution depends on auxiliary angle θ in “inner” space, and eq. (4.9) corresponds to the case $\theta = 0$.

5 Conclusion

In the present paper, we have calculated the small angle expansion of the QCD cusp anomalous dimension and the four-loop anomalous dimension of the heavy quark field in HQET. The obtained results agree with partial analytical and numerical results available in the literature. We have also performed a stringent test of our calculational setup by independent four-loop QCD beta-function derivation from the HQET vertex renormalization. The obtained results are the first application of the HQET propagator-type master integrals calculated in ref. [18]. The highly automated setup developed in the course of this work allows one to obtain yet higher terms in the small-angle expansion of $\Gamma_{\rm cusp}$ (once they are needed) as well as to calculate similar quantities in the HQET framework. The obtained missing parts of the full QCD result for $\Gamma_{\rm cusp}$ allowed us to compare the QCD result with $\mathcal{N}=4$ SYM predictions for the Bremsstrahlung function and observe the applicability of the maximal transcendentality principle up to four loops.

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