Large N expansion of an integrated correlator in $\mathcal{N} = 4$ SYM

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ABSTRACT: Recently Dorigoni, Green and Wen conjectured a remarkable exact formula for an integrated correlator of four superconformal primary operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In this work, we investigate its large N limit in detail. We show that the formula of Dorigoni, Green and Wen can be recast into the sum over the contributions of (p, q)-strings via the holography in the large N limit. Due to the $SL(2, \mathbb{Z})$ duality, all the contributions are governed by a single function, typically appearing as the fundamental string contribution. The large order behavior for the perturbative genus expansion of this function allows us to reveal the large N non-perturbative corrections. The same result is obtained more systematically by using a Laplace-difference equation for the integrated correlator.

Contents

1 Introduction

The holographic duality between 4d $\mathcal{N} = 4$ SU(N) supersymmetric Yang-Mills theory (SYM) and the type IIB string theory on $AdS_5 \times S^5$ is a prototypical example of the AdS/CFT correspondence [1]. In particular, Montonen-Olive duality of $\mathcal{N} = 4$ SYM [2, 3] corresponds to the SL$(2, \mathbb{Z})$ duality of type IIB string theory [4]. However, implications

of the $\mathrm{SL}(2,\mathbb{Z})$ duality in the context of AdS/CFT correspondence have not been fully explored. This is partly because, in the large N 't Hooft limit

$$N \to \infty, \qquad g_{\mathrm{YM}} \to 0, \qquad \lambda = g_{\mathrm{YM}}^2 N : \text{fixed}, \qquad (1.1)$$

we are focusing on the small g_{YM} regime and the $\mathrm{SL}(2,\mathbb{Z})$ duality is obscured in this limit. However, one can study the holography and the $\mathrm{SL}(2,\mathbb{Z})$ duality at the same time in a regime where

$$N \to \infty, \qquad g_{\mathrm{YM}} : \text{fixed}, \qquad (1.2)$$

which is called "very strong coupling limit" in [5].

In recent remarkable papers [6, 7], Dorigoni, Green and Wen conjectured an exact form of an integrated four-point correlator $\mathcal{G}_N$ of superconformal primary operators in the $\mathbf{20'}$ of the $\mathrm{SU}(4)_R$ symmetry of $\mathcal{N} = 4$ SYM. Such integrated correlators were originally introduced in [8] to analyze the strong coupling limit of usual four-point functions. See [9–15] for the related works on the integrated correlators in $\mathcal{N} = 4$ SYM. The result in [6, 7] is written in terms of the sum over the two-dimensional lattice $m + n\tau$ $((m,n) \in \mathbb{Z}^2)$ where τ is the complexified coupling of $\mathcal{N} = 4$ SYM. It is manifestly invariant under the $\mathrm{SL}(2,\mathbb{Z})$ duality. They computed the worldsheet instanton correction of the form $e^{-2\sqrt{\lambda}}$ in the large N 't Hooft limit.

In [13], Collier and Perlmutter further studied this integrated correlator $\mathcal{G}_N$ using the spectral decomposition in terms of the eigenfunctions of the Laplacian on the upper-half τ-plane with hyperbolic metric. They found that $\mathcal{G}_N$ receives an additional non-perturbative correction of the form $e^{-2\sqrt{\lambda}/g_s}$, where $g_s = g_{\mathrm{YM}}^2/4\pi$ is the string coupling of the bulk type IIB theory; they identified this non-perturbative correction as the "D-string instanton".

In this paper, we will study the large N expansion of the integrated correlator $\mathcal{G}_N$ in detail. As discussed in [13], the lattice sum in [6, 7] can be rewritten as the so-called Poincaré series. We argue that the Poincaré series representation of $\mathcal{G}_N$ can be interpreted as the sum over the contribution of (p,q)-strings via the holography in the large N limit. All of these contributions are symmetric. This is consistent with the expectation from the $\mathrm{SL}(2,\mathbb{Z})$ duality of bulk type IIB string theory. Then we focus on the contribution of the $(1,0)$-string, which can be written as the usual genus expansion $\sum_g N^{2-2g} G_g(\lambda)$ with the fixed 't Hooft coupling λ. However, as noticed by Collier and Perlmutter, the $(1,0)$-string contribution has another representation, which can also be written as the genus-like expansion $\sum_g N^{1-2g} \widetilde{G}_g(\widetilde{\lambda})$ in the dual 't Hooft limit,

$$N \to \infty, \qquad g_{\mathrm{YM}} \to \infty, \qquad \widetilde{\lambda} = \frac{16\pi^2}{g_{\mathrm{YM}}^2} N : \text{fixed}. \qquad (1.3)$$

Both of these genus expansions are determined by a single spectral function $f_g(s)$. From the Laplace-difference equation for $\mathcal{G}_N$ [7], we find the recursion relation for $f_g(s)$ which enables us to compute the genus expansion up to very high genera. Using these data, we find that the genus expansion of $\mathcal{G}_N$ is Borel non-summable. From the large g behavior of $G_g(\lambda)$ and $\widetilde{G}_g(\widetilde{\lambda})$, we find that there are non-perturbative correction of the form $e^{-NA(\pi\lambda^{-1/2})}$ and $e^{-NA(\pi\widetilde{\lambda}^{-1/2})}$, where the instanton action $A(x)$ has the same form with the action of
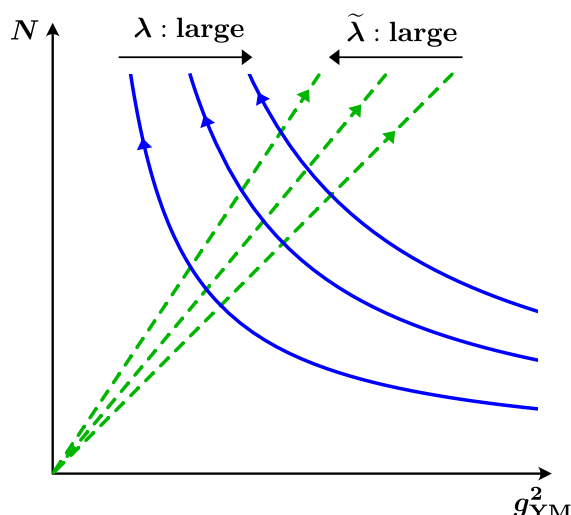
Figure 1. The blue solid lines show slices in the 't Hooft limit, and the green dashed lines show slices in the dual 't Hooft limit. For large but finite N, the strong 't Hooft coupling limit meets the strong dual 't Hooft coupling limit in the very strong coupling regime.

D3-brane which is holographically dual to the so-called "giant Wilson loop" in the large symmetric representation [16–18].

In the usual 't Hooft limit (1.1), the exponential factor $e^{-NA(\pi\lambda^{-1/2})}$ is purely non-perturbative in $1/N$. However, if one considers the dual 't Hooft limit (1.3), this correction can be expressed as $e^{-2\widetilde{\lambda}^{1/2}}$ in terms of $\widetilde{\lambda}$. Such a correction is non-perturbative in $1/\widetilde{\lambda}^{1/2}$, but not in $1/N$, and it is visible at each genus in the dual 't Hooft limit. In fact, we find that the non-perturbative correction $e^{-NA(\pi\lambda^{-1/2})}$ in this limit reproduces the D-string instanton in [13]. We stress that our non-perturbative correction is originally derived in the 't Hooft limit. However, for very large but finite $(N, \lambda, \widetilde{\lambda})$, this can be regarded effectively as the result in an overlapped regime of the 't Hooft limit and the dual 't Hooft limit via the very strong coupling limit. We illustrate it in figure 1. As a consequence, we can extrapolate our result to the dual 't Hooft regime. Also, the non-perturbative correction $e^{-NA(\pi\widetilde{\lambda}^{-1/2})}$ reduces to the worldsheet instanton by taking the 't Hooft limit.

This paper is organized as follows. In section 2, we start with a brief review on the integrated correlator $\mathcal{G}_N$ in $\mathcal{N} = 4$ SYM. We rewrite the integral representation of $\mathcal{G}_N$ in [6, 7] as a sum over (p,q)-string contributions in the large N limit via the holography. The resulting sum is governed by only a single function, the fundamental string contribution. All the (p,q)-string contributions appear democratically with an equal weight. This is a consequence of the $\mathrm{SL}(2, \mathbb{Z})$ duality.

In section 3, we take a deep look at the spectral function $f_g(s)$. This function determines both of the genus functions $G_g(\lambda)$ and $\widetilde{G}_g(\widetilde{\lambda})$ simultaneously. We find a recursion relation for $f_g(s)$. It enables us to compute the large N expansions up to very high genera.

In section 4, we consider the genus expansions of $\mathcal{G}_N$, and see that the functions $G_g(\lambda)$ and $\widetilde{G}_g(\widetilde{\lambda})$ receive non-perturbative corrections in the large (dual) 't Hooft coupling

limit. The former is referred to as the worldsheet instanton, and the latter as the D-string instanton in [13].

In section 5, we explore non-perturbative effects at large N. We show that the genus expansions $\sum_g N^{2-2g} G_g(\lambda)$ and $\sum_g N^{1-2g} \widetilde{G}_g(\widetilde{\lambda})$ receive non-perturbative corrections, and we speculate that they can be interpreted as the D3-brane instanton corrections with magnetic and electric fluxes on its worldvolume, respectively. From the large order behavior, we constrain the form of the non-perturbative corrections. Interestingly, the same corrections are fixed by the Laplace-difference equation that the integrated correlator obeys. We observe that the dual 't Hooft limit of the non-perturbative correction to $\sum_g N^{2-2g} G_g(\lambda)$ agrees with the D-string instanton correction, computed in section 4. Also the 't Hooft limit of the non-perturbative correction to $\sum_g N^{1-2g} \widetilde{G}_g(\widetilde{\lambda})$ coincides with the worldsheet instanton correction.

Finally, we conclude in section 6 with some discussion on the future problems.

In appendix A, we prove the relation between $B_N(t)$ and the matrix model correlator $K_N(x)$ conjectured in [15]. In appendix B, we show that the disconnected and the connected part of $\mathcal{G}_N$ correspond to $G_0(\lambda)$ and $\widetilde{G}_0(\widetilde{\lambda})$ in the large N limit. In appendix C we summarize the method of Richardson extrapolation used in the main text.

2 Sum over (p, q)-strings

We start by reviewing the integrated correlator $\mathcal{G}_N$ in [7], and then rewrite it as a sum over (p, q)-string contributions. $\mathcal{G}_N$ is written as the derivative of the free energy of mass deformed $\mathcal{N} = 2^*$ theory on S^4 [8]

$$\mathcal{G}_N = \frac{1}{4} \Delta_\tau \partial_m^2 \log Z \Big|_{m=0}, \tag{2.1}$$

where Δ_τ is the hyperbolic Laplacian on the upper-half τ-plane

$$\Delta_\tau = \tau_2^2 \left(\frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right), \tag{2.2}$$

and τ is the complexified coupling of the SU(N) $\mathcal{N} = 4$ SYM

$$\tau = \tau_1 + \mathrm{i}\tau_2 = \frac{\theta}{2\pi} + \mathrm{i}\frac{4\pi}{g_{\mathrm{YM}}^2}. \tag{2.3}$$

Via the holographic duality, τ_2 is related to the string coupling g_s of bulk type IIB string theory as

$$\tau_2 = g_s^{-1}. \tag{2.4}$$

The partition function Z of $\mathcal{N} = 2^*$ theory on S^4 is exactly computed by the supersymmetric localization [19] and the Yang-Mills instanton contributions, which correspond to D-instantons in the bulk, are captured by the Nekrasov partition function [20, 21].

After the supersymmetric localization, the partition function Z is reduced to a finite dimensional matrix integral for moduli parameters. However, it is not easy to extract the

full large N expansion from such a matrix integral because its integrand is complicated. Very surprisingly, in [7], a simple exact form of the integrated correlator $\mathcal{G}_N$ is put forward

$$\mathcal{G}_N = \frac{1}{2} \int_0^\infty dt\, B_N(t)\Theta(t,\tau), \tag{2.5}$$

where $\Theta(t,\tau)$ is given by the lattice sum

$$\Theta(t,\tau) = \sum_{(m,n)\in\mathbb{Z}^2} e^{-tM_{m,n}}, \tag{2.6}$$

with

$$M_{m,n} = \frac{\pi}{\tau_2}|m+n\tau|^2, \tag{2.7}$$

and $B_N(t)$ in (2.5) is given by

$$
\begin{aligned}
B_N(t) &= \frac{Q_N(t)}{(1+t)^{2N+1}}, \\
Q_N(t) &= -\frac{1}{2}N(N-1)(1-t)^{N-1}(1+t)^{N+1} \\
&\quad \times \left[(3-6t+3t^2+8Nt)P_N^{(1,-2)}\left(\frac{1+t^2}{1-t^2}\right) + \frac{3t^2-3-8Nt}{1+t}P_N^{(1,-1)}\left(\frac{1+t^2}{1-t^2}\right)\right],
\end{aligned}
\tag{2.8}
$$

where $P_n^{(\alpha,\beta)}(z)$ denotes the Jacobi polynomial. One can show that $\Theta(t,\tau)$ is manifestly $\mathrm{SL}(2,\mathbb{Z})$ invariant

$$\Theta\left(t, \frac{a\tau+b}{c\tau+d}\right) = \Theta(t,\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \tag{2.9}$$

and it satisfies

$$t^{-1}\Theta(t^{-1},\tau) = \Theta(t,\tau). \tag{2.10}$$

It turns out that this formula is very useful both in the exact finite N computation and in the large N analysis.

Fourier expansion of $\mathcal{G}_N$ is obtained by performing the Poisson resummation of $\Theta(t,\tau)$. Using the relation

$$\sum_{m\in\mathbb{Z}} \delta(x-m) = \sum_{\widetilde{m}\in\mathbb{Z}} e^{-2\pi i\widetilde{m}x}, \tag{2.11}$$

$\Theta(t,\tau)$ is written as

$$
\begin{aligned}
\Theta(t,\tau) &= \sum_{(m,n)\in\mathbb{Z}^2} e^{-\frac{\pi t}{\tau_2}\left[(m+n\tau_1)^2+n^2\tau_2^2\right]} \\
&= \int_{-\infty}^{\infty} dx \sum_{(\widetilde{m},n)\in\mathbb{Z}^2} e^{-2\pi i\widetilde{m}x} e^{-\frac{\pi t}{\tau_2}\left[(x+n\tau_1)^2+n^2\tau_2^2\right]} \\
&= \sqrt{\frac{\tau_2}{t}} \sum_{(\widetilde{m},n)\in\mathbb{Z}^2} e^{2\pi i\widetilde{m}n\tau_1-\pi\tau_2\left(\frac{\widetilde{m}^2}{t}+n^2 t\right)} \\
&= \frac{1}{\sqrt{g_s t}} \sum_{(\widetilde{m},n)\in\mathbb{Z}^2} e^{i\widetilde{m}n\theta-\frac{\pi}{g_s}\left(\frac{\widetilde{m}^2}{t}+n^2 t\right)}.
\end{aligned}
\tag{2.12}
$$

Thus the k-instanton contribution is given by

$$\mathcal{G}_N^{(k\text{-inst})} = \frac{1}{2} \sum_{\widetilde{m}n=k} \int_0^\infty \frac{dt}{\sqrt{g_s t}} B_N(t) e^{-\frac{\pi}{g_s}\left(\frac{\widetilde{m}^2}{t} + n^2 t\right)}. \qquad (2.13)$$

In particular, the 0-instanton part ($\widetilde{m} = 0$ or $n = 0$) is exactly given by

$$\mathcal{G}_N^{(0\text{-inst})} = \frac{1}{2} \int_0^\infty \frac{dt}{\sqrt{g_s t}} B_N(t) \left[\vartheta_3(e^{-\frac{2\pi}{g_s t}}) + \vartheta_3(e^{-\frac{2\pi t}{g_s}})\right], \qquad (2.14)$$

where $\vartheta_3(q)$ denotes the Jacobi theta function

$$\vartheta_3(q) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2}. \qquad (2.15)$$

Using the properties of the Jacobi theta function

$$\begin{aligned}
\frac{1}{\sqrt{g_s t}} \vartheta_3(e^{-\frac{2\pi}{g_s t}}) &= \vartheta_3(e^{-2\pi g_s t}), \\
\frac{1}{\sqrt{g_s t}} \vartheta_3(e^{-\frac{2\pi t}{g_s}}) &= \frac{1}{t} \vartheta_3(e^{-\frac{2\pi g_s}{t}}),
\end{aligned} \qquad (2.16)$$

and the symmetry of $B_N(t)$ [7],

$$t^{-1} B_N(t^{-1}) = B_N(t), \qquad (2.17)$$

one can show that the two terms of (2.14) are equal. Finally, we find

$$\begin{aligned}
\mathcal{G}_N^{(0\text{-inst})} &= \int_0^\infty dt\, B_N(t) \vartheta_3(e^{-2\pi g_s t}) \\
&= \int_0^\infty dt\, B_N(t) \sum_{n \in \mathbb{Z}} e^{-n^2 t M_{1,0}}.
\end{aligned} \qquad (2.18)$$

We can naturally interpret this $\mathcal{G}_N^{(0\text{-inst})}$ as the contribution of $(1,0)$-string (or F-string).

Noticing that $M_{m,n}$ in (2.7) has the property

$$M_{np,nq} = n^2 M_{p,q}, \qquad (2.19)$$

we can rewrite (2.5) as[1]

$$\begin{aligned}
\mathcal{G}_N &= \frac{1}{2} \int_0^\infty dt\, B_N(t) \sum_{(m,n) \in \mathbb{Z}^2} e^{-t M_{m,n}} \\
&= \frac{1}{2} \int_0^\infty dt\, B_N(t) \sum_{\substack{p,q \\ \gcd(p,q)=1}} \sum_{n \in \mathbb{Z}} e^{-t M_{np,nq}} \\
&= \frac{1}{2} \int_0^\infty dt\, B_N(t) \sum_{\substack{p,q \\ \gcd(p,q)=1}} \sum_{n \in \mathbb{Z}} e^{-n^2 t M_{p,q}} \\
&= \frac{1}{2} \sum_{\substack{p,q \\ \gcd(p,q)=1}} \mathcal{G}_N^{(0\text{-inst})}(M_{p,q}),
\end{aligned} \qquad (2.20)$$

---

[1]This rewriting is equivalent to the Poincaré sum discussed in [13] (see e.g. [22]). We would like to thank Eric Perlmutter for pointing this out.

where we defined

$$\mathcal{G}_N^{(0\text{-inst})}(M) := \int_0^\infty dt\, B_N(t) \sum_{n\in\mathbb{Z}} e^{-n^2 tM} = \int_0^\infty dt\, B_N(t)\vartheta_3(e^{-2Mt}), \qquad (2.21)$$

which, of course, reduces to the 0-instanton part (2.18) when $M = M_{1,0} = \pi g_s$. Now, we interpret (2.20) as the sum over the contribution of (p,q)-strings with relatively prime (p,q). In fact, $M_{p,q}$ has a physical meaning as the square of the (p,q)-string tension [23]. As expected from the SL$(2,\mathbb{Z})$ duality invariance of $\mathcal{G}_N$, all contributions of the (p,q)-strings have the same functional form $\mathcal{G}_N^{(0\text{-inst})}(M)$ as a function of $M = M_{p,q}$. In other words, all (p,q)-strings appear democratically in $\mathcal{G}_N$ with an equal weight.

Let us further elaborate on our interpretation of (p,q)-strings. The Poincare series representation (2.20) is just a mathematical statement of the decomposition of SL$(2,\mathbb{Z})$ invariant function of τ and $\overline{\tau}$. In general, it has nothing to do with (p,q)-strings if we pick an arbitrary SL$(2,\mathbb{Z})$ invariant function. However, we claim that for our particular case of the integrated correlator $\mathcal{G}_N$, the decomposition (2.20) has a physical interpretation as a sum over (p,q)-string contributions in the large N limit via the AdS/CFT correspondence between $\mathcal{N} = 4$ SYM and type IIB string on $AdS_5 \times S^5$. As we will see in section 4, $\mathcal{G}_N^{(0\text{-inst})}(M_{1,0})$ receives the so-called worldsheet instanton correction which scales as

$$\mathcal{G}_N^{(0\text{-inst})}(M_{1,0}) \sim e^{-\sqrt{M_{1,0}N}}. \qquad (2.22)$$

Here we ignored the numerical factors such as 2 and π for simplicity. The worldsheet instanton action $\sqrt{M_{1,0}N}$ is written in terms of α' of the bulk type IIB string and the curvature radius R_{AdS} of AdS_5 as

$$\sqrt{M_{1,0}N} \sim \frac{R_{\text{AdS}}^2}{\alpha'}. \qquad (2.23)$$

Since $\mathcal{G}_N^{(0\text{-inst})}(M_{1,0})$ and $\mathcal{G}_N^{(0\text{-inst})}(M_{p,q})$ have exactly the same functional form, (2.22) implies that $\mathcal{G}_N^{(0\text{-inst})}(M_{p,q})$ receives the correction

$$\mathcal{G}_N^{(0\text{-inst})}(M_{p,q}) \sim e^{-\sqrt{M_{p,q}N}}. \qquad (2.24)$$

Using the dictionary (2.23), the instanton action appearing in (2.24) is written as

$$\sqrt{M_{p,q}N} \sim \sqrt{\frac{M_{p,q}}{M_{1,0}}} \frac{R_{\text{AdS}}^2}{\alpha'} = \frac{|p+q\tau|}{\alpha'} R_{\text{AdS}}^2, \qquad (2.25)$$

where we used the definition of $M_{p,q}$ in (2.7). One can see that the first factor $|p+q\tau|/\alpha'$ in (2.25) is equal to the tension of (p,q)-string and $\sqrt{M_{p,q}N}$ has the expected form of the action of (p,q)-string instanton in bulk type IIB string theory

$$\sqrt{M_{p,q}N} = (\text{tension of } (p,q)\text{-string}) \times (\text{area}). \qquad (2.26)$$

This serves as strong evidence supporting our interpretation of the decomposition (2.20) of $\mathcal{G}_N$ as a sum over (p,q)-string contributions.

In the following sections, we will focus on the zero-instanton term $\mathcal{G}_N^{(0\text{-inst})}(M)$. This is the only building block of the full function $\mathcal{G}_N$. It turns out that the non-perturbative corrections in the large N expansion of $\mathcal{G}_N^{(0\text{-inst})}(M)$ do not correspond to the D-instanton. The D-instanton effect arises only after we sum over all (p,q) in (2.20) and perform the Poisson resummation (2.12). In other words, the D-instanton cannot be seen in $\mathcal{G}_N^{(0\text{-inst})}(M_{p,q})$ with fixed (p,q).

In the large N limit, only a single term in the sum (2.20) is dominant. Which (p,q) becomes dominant is determined by the condition

$$\min_{(p,q)} M_{p,q}. \tag{2.27}$$

This condition divides the upper-half τ-plane into several domains. For instance, the $(1,0)$-string (or F-string) is dominant in the fundamental domain of $\text{SL}(2,\mathbb{Z})$. In general, the condition (2.27) leads to a tessellation of the upper-half τ-plane, which is similar to the phase diagram of 3d gravity (see e.g. figure 3 in [24]).

3 Spectral representation

The N-dependence of the integrated correlator (2.5) is encoded in the function $B_N(t)$ in the integrand. As discussed in [7, 13], it is more convenient to consider the coefficient $c(s,N)$ appearing in the small t expansion of $B_N(t)$

$$B_N(t) = \sum_{s=2}^{\infty} (-1)^s c(s,N) t^{s-1}. \tag{3.1}$$

Although $c(s,N)$ is only defined for positive integer s in this expression, one can naturally define the analytic continuation of $c(s,N)$ to the complex s-plane, as emphasized in [7, 13].

In this section, we first consider the large N limit of $c(s,N)$. We will see that the usual resummation technique allows us to re-construct the finite N result of $c(s,N)$ from the large N expansion. In the next section, we will perform the similar analysis for $\mathcal{G}_N^{(0\text{-inst})}(M_{1,0})$, and it turns out that the large N expansion of $\mathcal{G}_N^{(0\text{-inst})}(M_{1,0})$ receives non-perturbative corrections.

3.1 Genus expansion

Let us consider $c(s,N)$ defined in (3.1). For small N, we can compute the coefficient $c(s,N)$ explicitly, as in (A.30) for $N = 2, 3, 4$. It is interesting to notice that one can write down an exact closed form of $c(s,N)$ in (A.29) for finite N.

In this paper, we are rather interested in the large N limit. Since we already have the exact form for finite N, we can see its large N behavior numerically. It turns out that $c(s,N)$ has different leading large N behaviors for $s > 1/2$ and $s < 1/2$ (see figure 2).

$$c(s,N) \approx \begin{cases} p(s)N^{1+s}, & (s > 1/2), \\ p(1-s)N^{2-s}, & (s < 1/2), \end{cases} \tag{3.2}$$
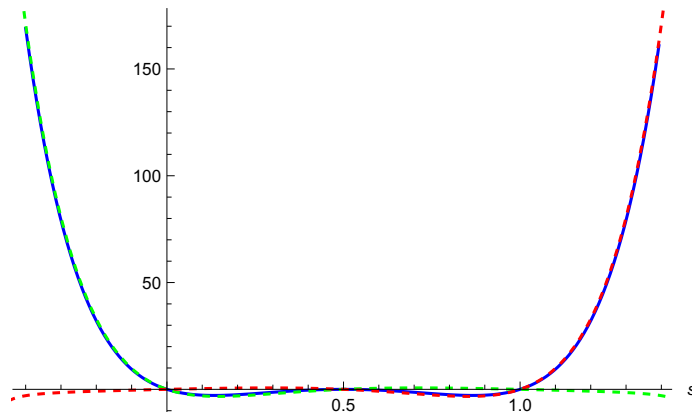
Figure 2. Plot of $c(s, N)$. The blue solid curve is the exact result of $c(s, N)$ while the red and green dashed curves represent $p(s)N^{1+s}$ and $p(1-s)N^{2-s}$, respectively. In this figure we set $N = 15$.

where

$$p(s) := \frac{4^{s-1}(s-1)(2s-1)\Gamma(s+1/2)}{\sqrt{\pi}\Gamma(s+2)\Gamma(s)}. \tag{3.3}$$

Recall that $c(s, N)$ is exactly symmetric under $s \leftrightarrow 1 - s$. This symmetry suggests that the large N asymptotic expansion of $c(s, N)$ has the following symmetric form:[2]

$$c(s, N) = p(s)N^{1+s}\sum_{g=0}^{\infty}\frac{f_g(s)}{N^{2g}} + p(1-s)N^{2-s}\sum_{g=0}^{\infty}\frac{f_g(1-s)}{N^{2g}}. \tag{3.4}$$

We will check the validity of this conjectural expansion later. The first few terms of $f_g(s)$ were obtained in [7],

$$
\begin{aligned}
f_0(s) &= 1, \\
f_1(s) &= \frac{(s+1)s(s-1)(s-6)}{24(2s-3)}, \\
f_2(s) &= \frac{(s+1)s(s-1)(s-2)(s-3)(s-4)(5s^2 - 47s + 30)}{5760(2s-3)(2s-5)}.
\end{aligned} \tag{3.5}
$$

Here we show a recursion relation to determine $f_g(s)$ very efficiently. The key relation is the following difference equation for $c(s, N)$:

$$N(N-1)c(s, N+1) - \Big[2(N^2 - 1) + s(s-1)\Big]c(s, N) + N(N+1)c(s, N-1) = 0, \tag{3.6}$$

as argued in [7]. Since this is a second order difference equation for N, it has two independent solutions. This equation is also invariant under $s \leftrightarrow 1 - s$. These properties imply that the two asymptotic series on the right hand side in (3.4) correspond to two independent solutions to the difference equation (3.6).

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[2]This large N expansion is slightly different from the one in [7], in which the authors seemed to assume $s > 1/2$ implicitly.

Let us introduce $p(s,N)$ as an asymptotic expansion:

$$p(s,N) := p(s)N^{1+s} \sum_{g=0}^{\infty} \frac{f_g(s)}{N^{2g}}. \tag{3.7}$$

Then, the large N expansion (3.4) is simply written as

$$c(s,N) = p(s,N) + p(1-s,N). \tag{3.8}$$

As discussed above, $p(s,N)$ should be a solution to (3.6). Therefore, we substitute the asymptotic series (3.7) into the difference equation (3.6). To do so, we rewrite $p(s,N\pm 1)$ as

$$\begin{aligned}
p(s,N\pm 1) &= p(s) \sum_{g=0}^{\infty} f_g(s)(N\pm 1)^{1+s-2g} \\
&= p(s)N^{1+s} \sum_{g=0}^{\infty} \frac{f_g(s)}{N^{2g}} \sum_{j=0}^{\infty} \binom{1+s-2g}{j} \left(\pm\frac{1}{N}\right)^j.
\end{aligned} \tag{3.9}$$

In the computation, we need the sum/difference of them, and we have more convenient forms

$$\begin{aligned}
p(s,N+1) + p(s,N-1) &= 2p(s)N^{1+s} \sum_{g=0}^{\infty} N^{-2g} \sum_{h=0}^{g} \binom{1+s-2h}{2g-2h} f_h(s), \\
p(s,N+1) - p(s,N-1) &= 2p(s)N^{s} \sum_{g=0}^{\infty} N^{-2g} \sum_{h=0}^{g} \binom{1+s-2h}{1+2g-2h} f_h(s).
\end{aligned} \tag{3.10}$$

After some lengthy computations, we find the following recursion relation:

$$f_g(s) = \sum_{h=0}^{g-1} \frac{(2+4g-2h-s)\Gamma(2+s-2h)}{g(1+2g-2s)\Gamma(3+2g-2h)\Gamma(1+s-2g)} f_h(s). \tag{3.11}$$

This equation determines $f_g(s)$ from $f_h(s)$ ($0 \le h \le g-1$). Therefore, starting with $f_0(s) = 1$, all the functions $f_g(s)$ are uniquely fixed. One can quickly compute $f_g(s)$ up to very high genera.

3.2 Pole cancellation

We show a simple consistency test for the large N expansion (3.4). The function $c(s,N)$ is defined for arbitrary s. However, the coefficients $f_g(s)$ have poles at $s = 3/2, 5/2, \ldots$, and $p(s)$ has poles at $s = -1/2, -3/2, \ldots$. This means that $p(s,N)$ itself is ill-defined for these values of s. We observe that the combination $p(s,N)+p(1-s,N)$ is well-defined for any s. All the singular terms in $p(s,N)$ are canceled by those in $p(1-s,N)$! As an example, let us check this property at $s = 3/2$. At order $\mathcal{O}(N^{1/2})$, we find the following singular behaviors:

$$\begin{aligned}
\lim_{s\to 3/2} p(s)N^{1+s}\frac{f_1(s)}{N^2} &= -\frac{3\sqrt{N}}{8\pi^{3/2}(s-3/2)} + \cdots, \\
\lim_{s\to 3/2} p(1-s)N^{2-s}\frac{f_0(1-s)}{N^0} &= +\frac{3\sqrt{N}}{8\pi^{3/2}(s-3/2)} + \cdots.
\end{aligned} \tag{3.12}$$

Thus the poles at $s = 3/2$ cancel to each other at this order. In general we expect that the pole at $s = k + 1/2$ ($k = 1, 2, \cdots$) is canceled order by order in the genus expansion

$$\lim_{s \to k+\frac{1}{2}} \left[p(s) N^{1+s} \frac{f_{g+k}(s)}{N^{2(g+k)}} + p(1-s) N^{2-s} \frac{f_g(1-s)}{N^{2g}} \right] = \text{(regular)}. \qquad (3.13)$$

We have checked this relation for various k and g. It would be interesting to find a general proof of this relation.

3.3 Borel resummation

The systematic computation of $f_g(s)$ allows us to resum the large N expansion (3.4). Here we numerically show that the large N resummation correctly reproduces the exact result for finite N. We use the so-called Borel-Padé resummation. Its main advantage is that one can apply it to factorially divergent series.

Let us briefly review the Borel-Padé resummation technique. Let us consider a formal $1/N$ expansion

$$F(N) = \sum_{g=0}^{\infty} \frac{F_g}{N^{2g}}. \qquad (3.14)$$

We first define the Borel transform by

$$\mathcal{B}_a[F](\zeta) := \sum_{g=0}^{\infty} \frac{F_g}{\Gamma(2g + a)} \zeta^{2g}, \qquad (3.15)$$

with a parameter a. Then, the Borel resummation of (3.14) is given by

$$F_a^{\text{Borel}}(N) = \int_0^{\infty} d\zeta \, e^{-\zeta} \zeta^{a-1} \mathcal{B}_a[F] \left(\frac{\zeta}{N} \right) \qquad (3.16)$$

Typically, the parameter a is set as unity, but we can choose it whatever as we want. In practice, we have F_g up to a finite order. In this case, we replace the Borel transform (3.15) by its Padé approximant. We refer to this procedure as the Borel-Padé resummation.

We apply this procedure to $p(s, N)$. We repeat the computation for various s and N. The result is shown in table 1. We confirmed that the naive Borel-Padé resummation of the combination $p(s, N) + p(1-s, N)$ correctly reproduces the exact value of $c(s, N)$. This means that there are no non-perturbative corrections to $c(s, N)$ in the large N limit. On the other hand, one can also consider the large N expansion of $\mathcal{G}_N^{(0\text{-inst})}(M_{1,0})$. We find that this large N expansion receives non-perturbative corrections. This difference comes from an exchange of the summation and the integration, as we will see in the next section.

4 Worldsheet instanton and D-string instanton

As we have shown in (2.20), the full integrated correlator $\mathcal{G}_N$ is constructed by the single function $\mathcal{G}_N^{(0\text{-inst})}(M)$, the zero instanton part. In this section, we consider the large N expansion of the F-string contribution $\mathcal{G}_N^{(0\text{-inst})}(M_{1,0})$. It turns out that there are non-perturbative corrections to this expansion.

s	N	$p^{\text{Borel}}(s, N)$	$p^{\text{Borel}}(1 - s, N)$	$1 - \dfrac{p^{\text{Borel}}(s, N) + p^{\text{Borel}}(1 - s, N)}{c(s, N)}$
$2/3$	2	-0.0563855378210	0.0440398588087	6.32×10^{-46}
	3	-0.111597704486	0.0759324095617	1.00×10^{-56}
	4	-0.180689554674	0.111594396415	1.05×10^{-65}
$5/4$	2	0.605741506110	-0.254179006110	6.78×10^{-45}
	3	1.45334429406	-0.343725153435	1.02×10^{-55}
	4	2.73979678726	-0.426144748682	1.03×10^{-64}
$\sqrt{3}$	2	3.52020101650	0.329163888886	3.51×10^{-44}
	3	13.6225376249	0.365956652856	4.91×10^{-55}
	4	32.1651458968	0.394892370253	4.68×10^{-64}

Table 1. The Borel-Padé resummation of the asymptotic $1/N$ expansion (3.4). The individual resummation $p^{\text{Borel}}(s, N)$ or $p^{\text{Borel}}(1 - s, N)$ does not reproduce the exact result of $c(s, N)$. The sum of them does. We used $f_g(s)$ up to $g = 120$ and the diagonal Padé approximant of the Borel transform. The numerical values of the large N resummation show a remarkable agreement even for $N = 2$. We also set $a = 1$ in the numerical computation.

4.1 Genus expansion

Let us derive the genus expansion of $\mathcal{G}_N^{(\text{0-inst})}(M_{1,0})$. We start with the following integral representation of the full correlator $\mathcal{G}_N$ [13]:

$$\mathcal{G}_N = \frac{N(N-1)}{4} - \frac{1}{2} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{ds}{2\pi i} \frac{\pi}{\sin \pi s} c(s, N) E_s^* \tag{4.1}$$

where E_s^* denotes the non-holomorphic Eisenstein series

$$E_s^* = \Gamma(s) \sum_{(m,n) \neq (0,0)} (M_{m,n})^{-s} = \int_0^\infty dt\, t^{s-1} \Big[\Theta(t, \tau) - 1 \Big]. \tag{4.2}$$

To proceed, we further use the Fourier expansion of E_s^*,

$$E_s^* = \sum_{k \in \mathbb{Z}} e^{ik\theta} \mathcal{F}_k \tag{4.3}$$

where

$$\mathcal{F}_k = \begin{cases} 2\Gamma(s)\zeta(2s)(M_{1,0})^{-s} + 2\Gamma(1-s)\zeta(2-2s)(M_{1,0})^{s-1}, & (k = 0), \\ 4|k|^{s-\frac{1}{2}}\sigma_{1-2s}(|k|)\sqrt{\tau_2} K_{s-\frac{1}{2}}(2\pi|k|\tau_2), & (k \neq 0). \end{cases} \tag{4.4}$$

Above, $K_\nu(x)$ denotes the modified Bessel function of the second kind and $\sigma_s(n)$ is the divisor function

$$\sigma_s(n) = \sum_{d|n} d^s. \tag{4.5}$$

See e.g. [7] for a derivation of (4.4).

Our interest is the zero-instanton sector, which is originally given by (2.18). With the help of the integral representation (4.1), we also obtain another integral representation,

$$\mathcal{G}_N^{(\text{0-inst})}(M_{1,0}) = \frac{N(N-1)}{4} - \frac{1}{2}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{ds}{2\pi i}\frac{\pi}{\sin \pi s}c(s,N)\mathcal{F}_0. \tag{4.6}$$

Plugging (3.8) and (4.4) into this integral, we have four contributions. Since there are simple symmetric relations

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{ds}{2\pi i}\frac{\pi}{\sin \pi s}p(1-s,N)\Gamma(1-s)\zeta(2-2s)(M_{1,0})^{s-1}$$
$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{ds}{2\pi i}\frac{\pi}{\sin \pi s}p(s,N)\Gamma(s)\zeta(2s)(M_{1,0})^{-s},$$
$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{ds}{2\pi i}\frac{\pi}{\sin \pi s}p(s,N)\Gamma(1-s)\zeta(2-2s)(M_{1,0})^{s-1} \tag{4.7}$$
$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{ds}{2\pi i}\frac{\pi}{\sin \pi s}p(1-s,N)\Gamma(s)\zeta(2s)(M_{1,0})^{-s},$$

we obtain

$$\mathcal{G}_N^{(\text{0-inst})}(M_{1,0}) = \frac{N(N-1)}{4} + I_N + \widetilde{I}_N, \tag{4.8}$$

where

$$I_N := -\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{ds}{2\pi i}\frac{2\pi}{\sin \pi s}p(s,N)\Gamma(1-s)\zeta(2-2s)(M_{1,0})^{s-1},$$
$$\widetilde{I}_N := -\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{ds}{2\pi i}\frac{2\pi}{\sin \pi s}p(1-s,N)\Gamma(1-s)\zeta(2-2s)(M_{1,0})^{s-1}. \tag{4.9}$$

Using the large N expansion (3.7) and exchanging the sum and the integral, these functions admit the genus expansions,

$$I_N = \sum_{g=0}^{\infty}N^{2-2g}\left[-\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{ds}{2\pi i}\frac{2\pi}{\sin \pi s}p(s)f_g(s)\Gamma(1-s)\zeta(2-2s)(M_{1,0}N)^{s-1}\right], \tag{4.10}$$
$$\widetilde{I}_N = \sum_{g=0}^{\infty}N^{1-2g}\left[-\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{ds}{2\pi i}\frac{2\pi}{\sin \pi s}p(1-s)f_g(1-s)\Gamma(1-s)\zeta(2-2s)(M_{1,0}^{-1}N)^{1-s}\right],$$

where $M_{1,0}N$ and $M_{1,0}^{-1}N$ are related to the 't Hooft coupling λ and the dual 't Hooft coupling $\widetilde{\lambda}$ by

$$\lambda = 4\pi g_s N = 4M_{1,0}N,$$
$$\widetilde{\lambda} = \frac{\lambda}{g_s^2} = 4\pi g_s^{-1}N = 4\pi^2 M_{1,0}^{-1}N. \tag{4.11}$$

We should emphasize that the dual 't Hooft coupling $\widetilde{\lambda}$ is not the S-dual of λ

$$\widetilde{\lambda} \neq 4M_{0,1}N. \tag{4.12}$$

This is not the end of the story. Each of these functions turns out to give unusual genus expansions. For instance, in the weak coupling regime $\lambda \ll 1$, one can close the

integration contour in I_N by adding the infinite right semi-circle. Then one obtains the following weak coupling expansion of I_N,

$$
\begin{aligned}
I_N = N^2 & \left(-\frac{1}{4} + \frac{3\zeta(3)}{8\pi^2}\lambda - \frac{75\zeta(5)}{128\pi^4}\lambda^2 + \mathcal{O}(\lambda^3) \right) \\
& + N^0 \left(\frac{\lambda^{1/2}}{16} - \frac{3\zeta(3)}{8\pi^2}\lambda + \frac{75\zeta(5)}{128\pi^4}\lambda^2 + \mathcal{O}(\lambda^3) \right) \\
& + N^{-2} \left(\frac{13\lambda^{1/2}}{8192} - \frac{\lambda^{3/2}}{1536} + \frac{945\zeta(9)}{4096\pi^8}\lambda^4 + \mathcal{O}(\lambda^5) \right) + \mathcal{O}(N^{-4}).
\end{aligned}
\tag{4.13}
$$

The weak coupling expansion has unusual fractional power terms. Such terms are however canceled by taking into account the counterpart $\widetilde{I}_N$, as discussed in [13]. To see it, we rewrite $\widetilde{I}_N$ as

$$
\widetilde{I}_N = \sum_{g=0}^{\infty} N^{1-2g} \left[-\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{ds}{2\pi i} \frac{2\pi}{\sin \pi s} p(1-s) f_g(1-s)\Gamma(1-s)\zeta(2-2s) \left(\frac{\lambda}{4N}\right)^{s-1} \right]
\tag{4.14}
$$

Since in the 't Hooft limit, we first take $N \to \infty$ by keeping λ finite, $\lambda/(4N)$ is less than unity for any λ in this limit. Therefore, we should close the integration contour of $\widetilde{I}_N$ in the same way as I_N by adding the infinite right semi-circle. Then we find

$$
\widetilde{I}_N = \frac{N}{4} - N^0 \frac{\lambda^{1/2}}{16} + N^{-2} \left(-\frac{13\lambda^{1/2}}{8192} + \frac{\lambda^{3/2}}{1536} \right) + \mathcal{O}(N^{-4}),
\tag{4.15}
$$

which cancels the fractional power terms of λ in (4.13). Therefore we conclude that the combination $I_N + \widetilde{I}_N$ gives the usual genus expansion without the fractional power terms of λ.[3] Let us denote

$$
\mathcal{G}_N^{(0\text{-inst})}(M_{1,0}) \sim \sum_{g=0}^{\infty} N^{2-2g} G_g(\lambda),
\tag{4.16}
$$

where

$$
G_g(\lambda) := -\int_{c_g-i\infty}^{c_g+i\infty} \frac{ds}{2\pi i} \frac{2\pi}{\sin \pi s} p(s) f_g(s)\Gamma(1-s)\zeta(2-2s) \left(\frac{\lambda}{4}\right)^{s-1}.
\tag{4.17}
$$

We have used $\sim$ in (4.16) because these genus expansions receive non-perturbative corrections, as we will see later. Note also that the integration contour in this expression should be chosen so that the weak coupling expansion in λ does not include fractional power terms. It is not hard to find them. For $g = 0, 1$, we find

$$
1 < c_0 < 2, \qquad \frac{3}{2} < c_1 < 2.
\tag{4.18}
$$

For $g \geq 2$, we have

$$
g + \frac{1}{2} < c_g < 2g + 1 \qquad (g \geq 2).
\tag{4.19}
$$

---

[3]In [13], the condition (3.13) was obtained by requiring the absence of the fractional power terms of λ in the small λ expansion. We should stress that our logic of the derivation of (3.13) is different from [13]: we derived the condition (3.13) from the regularity of the combination $p(s, N) + p(1-s, N)$ in (3.8) and the absence of the fractional power terms of λ is a consequence of (3.13), not the other way around.

We can also derive another genus expansion in terms of the dual coupling $\widetilde{\lambda}$. For $\widetilde{\lambda} \ll 1$, $\widetilde{I}_N$ admits the following weak coupling expansion:

$$
\begin{aligned}
\widetilde{I}_N = {} & N\left(\frac{\widetilde{\lambda}^2}{480} - \frac{5\widetilde{\lambda}^3}{12096} + \frac{7\widetilde{\lambda}^4}{115200} + \mathcal{O}(\widetilde{\lambda}^5)\right) \\
& + N^{-1}\left(\frac{3\zeta(3)\widetilde{\lambda}^{3/2}}{64\pi^3} - \frac{\widetilde{\lambda}^2}{480} + \frac{5\widetilde{\lambda}^3}{12096} + \mathcal{O}(\widetilde{\lambda}^4)\right) \\
& + N^{-3}\left(\frac{39\zeta(3)\widetilde{\lambda}^{3/2}}{32768\pi^3} - \frac{45\zeta(5)\widetilde{\lambda}^{5/2}}{4096\pi^5} + \frac{\widetilde{\lambda}^5}{443520} + \mathcal{O}(\widetilde{\lambda}^6)\right) + \mathcal{O}(N^{-5}).
\end{aligned}
\tag{4.20}
$$

The fractional powers are canceled by I_N:

$$
I_N = -N^{-1}\frac{3\zeta(3)\widetilde{\lambda}^{3/2}}{64\pi^3} + N^{-3}\left(-\frac{39\zeta(3)\widetilde{\lambda}^{3/2}}{32768\pi^3} + \frac{45\zeta(5)\widetilde{\lambda}^{5/2}}{4096\pi^5}\right) + \mathcal{O}(N^{-5}).
\tag{4.21}
$$

Therefore in this case, we obtain

$$
\mathcal{G}_N^{(\text{0-inst})}(M_{1,0}) \sim \frac{N(N-1)}{4} + \sum_{g=0}^{\infty} N^{1-2g}\widetilde{G}_g(\widetilde{\lambda}),
\tag{4.22}
$$

where

$$
\widetilde{G}_g(\widetilde{\lambda}) := -\int_{\widetilde{c}_g-\mathrm{i}\infty}^{\widetilde{c}_g+\mathrm{i}\infty} \frac{ds}{2\pi\mathrm{i}} \frac{2\pi}{\sin \pi s} p(1-s) f_g(1-s) \Gamma(1-s)\zeta(2-2s)\left(\frac{\widetilde{\lambda}}{4\pi^2}\right)^{1-s}.
\tag{4.23}
$$

The real number $\widetilde{c}_g$ should be chosen so that the weak coupling expansion in $\widetilde{\lambda}$ does not include any fractional powers. It is easy to check that we can set $\widetilde{c}_g = 1 - c_g$. Therefore the same result is obtained by using the same contour in $G_g(\lambda)$,

$$
\widetilde{G}_g(\widetilde{\lambda}) = -\int_{c_g-\mathrm{i}\infty}^{c_g+\mathrm{i}\infty} \frac{ds}{2\pi\mathrm{i}} \frac{2\pi}{\sin \pi s} p(s) f_g(s) \Gamma(s)\zeta(2s)\left(\frac{\widetilde{\lambda}}{4\pi^2}\right)^s.
\tag{4.24}
$$

4.2 Worldsheet instanton

After identifying the integration contours, we can discuss the strong coupling expansions. Let us first consider $G_g(\lambda)$. Our logic in this subsection is as follows. The large λ expansion of $G_g(\lambda)$ is Borel non-summable. This means that the Borel resummation of the large λ expansion has a discontinuity due to an ambiguity of the resummation. It is well-known that this discontinuity is related to non-perturbative corrections, which are interpreted as worldsheet instanton corrections in the present case.

The large λ expansion is obtained by closing the contour along the infinite left semicircle. In the following, we will often omit the contour of the s-integrals in $G_g(\lambda)$ and $\widetilde{G}_g(\widetilde{\lambda})$ if it is the path from $c_g - \mathrm{i}\infty$ to $c_g + \mathrm{i}\infty$. We first rewrite $G_g(\lambda)$ in (4.17) as

$$
\begin{aligned}
G_g(\lambda) = {} & -\int \frac{ds}{2\pi\mathrm{i}} \frac{\pi}{\sin \pi s} p(s) f_g(s) \Gamma(1-s) \sum_{n=1}^{\infty} 2(n^2 M_{1,0} N)^{s-1} \\
= {} & -\sum_{n=1}^{\infty} 2n T_F \int_0^{\infty} dw\, e^{-n T_F w} \int \frac{ds}{2\pi\mathrm{i}} \frac{\pi}{\sin \pi s} p(s) f_g(s) \Gamma(1-s) \frac{(4w)^{2-2s}}{\Gamma(3-2s)}
\end{aligned}
\tag{4.25}
$$

where we used $\zeta(2-2s) = \sum_{n=1}^{\infty} n^{2s-2}$ and T_F is defined by

$$T_F = 4\sqrt{M_{1,0}N} = 2\sqrt{\lambda}. \tag{4.26}$$

At strong coupling, by picking up the poles, this can be written as

$$G_g(\lambda) = \sum_{n=1}^{\infty} 2nT_F \int_0^{\infty} dw \, e^{-nT_F w} [\phi_g^+(w) + \phi_g^-(w)], \tag{4.27}$$

where $\phi_g^+(w)$, $\phi_g^-(w)$ are residue contributions from poles on the right/left half s-plane, respectively. There are finite number of poles on the strip $0 < \operatorname{Re} s < c_g$, while there are infinite number of poles on the left half s-plane.

Since we are interested in the discontinuity, the negative part $\phi_g^-(w)$ is important, which is given by

$$\phi_g^-(w) = \sum_{\ell=0}^{\infty} \beta_\ell f_g(-\ell - 1/2) \frac{(4w)^{2\ell+3}}{\Gamma(2\ell+4)} \tag{4.28}$$

with β_ℓ being

$$\beta_\ell = \operatorname{Res}_{s=-\ell-\frac{1}{2}} \left[-\frac{\pi}{\sin \pi s} p(s)\Gamma(1-s) \right] = \frac{\Gamma\left(\ell - \frac{1}{2}\right)\Gamma\left(\ell + \frac{5}{2}\right)\Gamma(2\ell+3)}{2^{4\ell+4}\pi\Gamma^2(\ell+1)}. \tag{4.29}$$

As we will see in the next section, the large order behavior of the genus sum is governed by the positive part $\phi_g^+(w)$.

For the genus-zero part we find the following analytic expression:

$$\phi_0^-(w) = \sum_{\ell=0}^{\infty} \beta_\ell \frac{(4w)^{2\ell+3}}{\Gamma(2\ell+4)} = -2w^3 \, {}_2F_1\left(-\frac{1}{2}, \frac{3}{2}, 1, w^2\right). \tag{4.30}$$

This agrees with the result in [7]. $\phi_0^-(w)$ has a branch cut along the positive real w-axis running from $w = 1$ to $w = \infty$. To see the discontinuity of $\phi_0^-(w)$ across the cut, it is convenient to rewrite it in terms of the complete elliptic integrals

$$\phi_0^-(w) = \frac{4w^3}{\pi} \Big[K(w^2) - 2E(w^2) \Big], \tag{4.31}$$

where $K(w^2)$ and $E(w^2)$ denote the complete elliptic integral of the first and the second kind, respectively. From the known discontinuity of $K(z)$ and $E(z)$

$$\begin{aligned}
K(z+\mathrm{i}0) - K(z-\mathrm{i}0) &= 2\mathrm{i}K(1-z), \\
E(z+\mathrm{i}0) - E(z-\mathrm{i}0) &= 2\mathrm{i}K(1-z) - 2\mathrm{i}E(1-z),
\end{aligned} \tag{4.32}$$

we find the discontinuity of $\phi_0^-(w)$

$$\begin{aligned}
\operatorname{Disc} \phi_0^-(w) &:= \phi_0^-(w+\mathrm{i}0) - \phi_0^-(w-\mathrm{i}0) \\
&= -\frac{8\mathrm{i}w^3}{\pi} \Big[K(1-w^2) - 2E(1-w^2) \Big] \\
&= 4\mathrm{i}w^3 \, {}_2F_1\left(-\frac{1}{2}, \frac{3}{2}, 1, 1-w^2\right).
\end{aligned} \tag{4.33}$$

This reproduces eq. (5.35) in [7]. Then the worldsheet instanton corrections to $G_0(\lambda)$ is computed by expanding $\text{Disc}\, \phi_0^-(w)$ around $w = 1$ and integrate it term by term

$$\Delta G_0 = 8\mathrm{i} \int_1^\infty dw \sum_{n=1}^\infty n T_F e^{-n T_F w} w^3 \, {}_2F_1\left(-\frac{1}{2}, \frac{3}{2}, 1, 1 - w^2\right)$$

$$= 8\mathrm{i} \sum_{n=1}^\infty e^{-n T_F}\left[1 + \frac{9}{2 n T_F} + \frac{117}{8 (n T_F)^2} + \frac{489}{16 (n T_F)^3} + \cdots\right] \tag{4.34}$$

$$= 8\mathrm{i}\left[\mathcal{L}_0(T_F) + \frac{9}{2}\mathcal{L}_1(T_F) + \frac{117}{8}\mathcal{L}_2(T_F) + \frac{489}{16}\mathcal{L}_3(T_F) + \cdots\right],$$

where we defined

$$\mathcal{L}_m(T_F) = \sum_{n=1}^\infty e^{-n T_F}(n T_F)^{-m} = (T_F)^{-m}\text{Li}_m(e^{-T_F}), \tag{4.35}$$

and $\text{Li}_m(e^{-T_F})$ denotes the polylogarithm. Note that T_F corresponds to the action of the worldsheet one-instanton and the sum over n in (4.35) is interpreted as the multi-covering of the worldsheet instantons. The appearance of the polylog is reminiscent of the worldsheet instanton corrections in the topological string.

In general, ΔG_0 is expanded as

$$\Delta G_0 = 8\mathrm{i} \sum_{m=0}^\infty a_m \mathcal{L}_m(T_F) = 8\mathrm{i} \sum_{n=1}^\infty e^{-n T_F} \sum_{m=0}^\infty \frac{a_m}{(n T_F)^m}. \tag{4.36}$$

As discussed in [7], a_m obeys the recursion relation

$$a_m = -\frac{3\left(2m^4 - 16m^3 + 31m^2 + 13m - 15\right)}{2m\left(2m^2 - 10m + 3\right)} a_{m-1}$$
$$- \frac{(m-6)(m-2)\left(2m^2 - 6m - 5\right)}{2\left(2m^2 - 10m + 3\right)} a_{m-2}. \tag{4.37}$$

By using this recursion relation and the Richardson extrapolation (see appendix C), we find the large m behavior,

$$a_m \sim \frac{12}{\pi}(-1)^m \Gamma(m - 3), \qquad (m \gg 1). \tag{4.38}$$

Since a_m has an alternating sign at large m, the summation over m in (4.36) is now Borel summable. This implies that the median resummation of the large λ expansion of $G_0(\lambda)$ is a resurgent function [7].

Next consider the genus-one part. For $g = 1$ we find

$$\phi_1^-(w) = \frac{\left(-41 w^4 + 81 w^2 - 36\right) K(w^2)}{96\pi w \left(w^2 - 1\right)^2} + \frac{\left(10 w^6 - 65 w^4 + 99 w^2 - 36\right) E(w^2)}{96\pi w \left(w^2 - 1\right)^3} \tag{4.39}$$

$$= -\frac{1}{64 w}\left[(10 w^4 + 81 w^2 + 36)\, {}_2F_1\left(\frac{3}{2}, \frac{5}{2}, 1, w^2\right) + 9(3 w^4 + w^2 - 4)\, {}_2F_1\left(\frac{5}{2}, \frac{5}{2}, 1, w^2\right)\right].$$

Using (4.32), we find the discontinuity of $\phi_1^-(w)$

$$
\begin{aligned}
\mathrm{Disc}\,&\phi_1^-(w) \\
&= -\frac{\mathrm{i}w\left(31w^4-57w^2+18\right)K\left(1-w^2\right)}{48\pi\left(w^2-1\right)^3} - \frac{\mathrm{i}\left(10w^6-65w^4+99w^2-36\right)E\left(1-w^2\right)}{24\pi w\left(w^2-1\right)^3} \\
&= -\frac{\mathrm{i}}{2^{11}w}\left[8(10w^4+81w^2+36)\,{}_2F_1\!\left(\frac{3}{2},\frac{5}{2},4,1-w^2\right)+27(3w^4+w^2-4)\,{}_2F_1\!\left(\frac{5}{2},\frac{5}{2},5,1-w^2\right)\right].
\end{aligned}
\tag{4.40}
$$

Then (4.27) gives rise to the genus-one worldsheet instanton correction

$$
\begin{aligned}
\sum_{n=1}^{\infty} 2nT_F \int_1^{\infty} dw\, e^{-nT_F w}\mathrm{Disc}\,\phi_1^-(w) \\
&= \sum_{n=1}^{\infty} \mathrm{i}e^{-nT_F}\left[-\frac{127}{256}+\frac{927}{2048}(nT_F)^{-1}-\frac{3897}{4096}(nT_F)^{-2}+\cdots\right] \\
&= \mathrm{i}\left[-\frac{127}{256}\mathcal{L}_0(T_F)+\frac{927}{2048}\mathcal{L}_1(T_F)-\frac{3897}{4096}\mathcal{L}_2(T_F)+\cdots\right],
\end{aligned}
\tag{4.41}
$$

which agrees with eq. (5.45) in [7]. However, it turns out that this is not the end of the story. Since $\phi_1^-(w)$ has a pole at $w=1$, there are additional contributions to the imaginary part of $G_1(\lambda)$. The Laurent expansion of $\phi_1^-(w)$ around $w=1$ reads

$$
\phi_1^-(w) = \frac{1}{4\pi}\left[\frac{1}{24(w-1)^3}-\frac{3}{32(w-1)^2}-\frac{77}{128(w-1)}+\mathcal{O}((w-1)^0)\right],
\tag{4.42}
$$

which also contributes to the imaginary part of $G_1(\lambda)$

$$
\begin{aligned}
-2\pi\mathrm{i}\sum_{n=1}^{\infty}\mathrm{Res}_{w=1}&\left[2nT_F e^{-nT_F w}\phi_1^-(w)\right] \\
&= \mathrm{i}\sum_{n=1}^{\infty} e^{-nT_F}\left[-\frac{1}{48}(nT_F)^3-\frac{3}{32}(nT_F)^2+\frac{77}{128}nT_F\right] \\
&= \mathrm{i}\left[-\frac{1}{48}\mathcal{L}_{-3}(T_F)-\frac{3}{32}\mathcal{L}_{-2}(T_F)+\frac{77}{128}\mathcal{L}_{-1}(T_F)\right].
\end{aligned}
\tag{4.43}
$$

Adding (4.41) and (4.43), the imaginary part of $G_1(\lambda)$ becomes

$$
\begin{aligned}
\Delta G_1 = \mathrm{i}\Bigg[&-\frac{1}{48}\mathcal{L}_{-3}(T_F)-\frac{3}{32}\mathcal{L}_{-2}(T_F)+\frac{77}{128}\mathcal{L}_{-1}(T_F) \\
&-\frac{127}{256}\mathcal{L}_0(T_F)+\frac{927}{2048}\mathcal{L}_1(T_F)-\frac{3897}{4096}\mathcal{L}_2(T_F)+\cdots\Bigg].
\end{aligned}
\tag{4.44}
$$

We stress that the first three terms are missing in [7]. These terms should be included to re-construct the analytic function from the large λ expansion of $G_1(\lambda)$.

In a similar manner, we find ΔG_g for $g = 2, 3$

$$\Delta G_2 = \mathrm{i}\left[\frac{1}{36864}\mathcal{L}_{-6}(T_F) + \frac{7}{40960}\mathcal{L}_{-5}(T_F) - \frac{857}{491520}\mathcal{L}_{-4}(T_F) - \frac{1681}{327680}\mathcal{L}_{-3}(T_F)\right.$$

$$\left. + \frac{28087}{2621440}\mathcal{L}_{-2}(T_F) + \frac{24825}{1048576}\mathcal{L}_{-1}(T_F) - \frac{153027}{4194304}\mathcal{L}_0(T_F) + \frac{726327}{8388608}\mathcal{L}_1(T_F) + \cdots\right],$$

$$\Delta G_3 = \mathrm{i}\left[-\frac{1}{42467328}\mathcal{L}_{-9}(T_F) - \frac{11}{47185920}\mathcal{L}_{-8}(T_F) + \frac{2489}{1321205760}\mathcal{L}_{-7}(T_F)\right.$$

$$+ \frac{147661}{7927234560}\mathcal{L}_{-6}(T_F) + \frac{94303}{2348810240}\mathcal{L}_{-5}(T_F) - \frac{3293543}{42278584320}\mathcal{L}_{-4}(T_F)$$

$$\left. - \frac{33749753}{56371445760}\mathcal{L}_{-3}(T_F) + \cdots\right]. \tag{4.45}$$

In general, the discontinuity of $G_g(\lambda)$ takes the form

$$\Delta G_g = \mathrm{i}\sum_{m=0}^{\infty} a_{g,m}\mathcal{L}_{-3g+m}(T_F) = \mathrm{i}\sum_{n=1}^{\infty} e^{-nT_F}\sum_{m=0}^{\infty} a_{g,m}(nT_F)^{3g-m}, \tag{4.46}$$

where $a_{0,m} = 8a_m$.

4.3 D-string instanton

As argued in [13], the large $\widetilde{\lambda}$ expansion of $\widetilde{G}_g(\widetilde{\lambda})$ is Borel non-summable and hence there are non-perturbative ambiguity in the large $\widetilde{\lambda}$ expansion of $\widetilde{G}_g(\widetilde{\lambda})$. To see this, we rewrite $\widetilde{G}_g(\widetilde{\lambda})$ in (4.24) into the multi-covering form using the functional identity of the Riemann zeta function

$$\Gamma(s)\zeta(2s) = \pi^{2s-\frac{1}{2}}\Gamma\left(\frac{1}{2} - s\right)\zeta(1 - 2s). \tag{4.47}$$

Then $\widetilde{G}_g(\widetilde{\lambda})$ in (4.24) is written as

$$\widetilde{G}_g(\widetilde{\lambda}) = -\int \frac{ds}{2\pi\mathrm{i}}\frac{\pi}{\sin\pi s}p(s)f_g(s)\sum_{n=1}^{\infty}\frac{2}{n\sqrt{\pi}}\Gamma(1/2 - s)(\pi^2 n^2 M_{1,0}^{-1}N)^s \tag{4.48}$$

$$= -T_D\int \frac{ds}{2\pi\mathrm{i}}\frac{\pi}{\sin\pi s}p(s)f_g(s)\sum_{n=1}^{\infty}\int_0^{\infty}dw\, e^{-nT_D w}(nT_D)^{1+3g}\frac{w^{1+3g-2s}\Gamma(1/2 - s)}{2^{4s-1}\sqrt{\pi}\Gamma(2 + 3g - 2s)},$$

where

$$T_D = 4\pi\sqrt{M_{1,0}^{-1}N} = 2\sqrt{\widetilde{\lambda}} = \frac{1}{g_s}T_F. \tag{4.49}$$

Closing the contour along the infinite left semi-circle and picking up the residues, we find

$$\widetilde{G}_g(\widetilde{\lambda}) = T_D\sum_{n=1}^{\infty}\int_0^{\infty}dw\, e^{-nT_D w}(nT_D)^{1+3g}[\psi_g^+(w) + \psi_g^-(w)], \tag{4.50}$$

where $\psi_g^+(w)$, $\psi_g^-(w)$ are the contributions of poles from the right/left half s-plane. In particular the negative pole part is

$$\psi_g^-(w) = \sum_{\ell=0}^{\infty}\beta_\ell f_g(-\ell - 1/2)\frac{2^{4\ell+3}w^{2+3g+2\ell}\Gamma(\ell + 1)}{\sqrt{\pi}\Gamma(\ell + 3/2)\Gamma(3 + 3g + 2\ell)}. \tag{4.51}$$

The first few terms of $\psi_g^-(w)$ read

$$\psi_0^-(w) = \frac{w^2(4w^2-3)}{2\pi\sqrt{1-w^2}},$$

$$\psi_1^-(w) = \frac{1}{768\pi}\left[-\frac{w}{\sqrt{1-w^2}} - 24\arcsin w + 25w - 18\log\frac{1+\sqrt{1-w^2}}{2}\right],$$

$$\psi_2^-(w) = \frac{1}{23592960\pi}\left[-45\left(13w^2+120\right)w^2\log\frac{1+\sqrt{1-w^2}}{2} - \frac{5047w^4-4501w^2-586}{\sqrt{1-w^2}}\right.$$
$$\left. + \frac{4907w^4}{4} - 4794w^2 - 586\right]. \tag{4.52}$$

Some comments are in order here. In the definition of $\psi_g^-(w)$ in (4.51), we have divided the summand by the factor $\Gamma(3+3g+2\ell)$ which depends on g. Instead, we could have divided by the g-independent factor, e.g. $\Gamma(3+2\ell)$. However, in that case $\psi_g^-(w)$ would have the singularity at $w=1$ of the form $(1-w^2)^{-m-\frac{1}{2}}$ with $m \geq 1$. In order to avoid such higher order singularity at $w=1$, we have divided in (4.51) by the factor $\Gamma(3+3g+2\ell)$. Then, as we can see from (4.52), the singularity of $\psi_g^-(w)$ at $w=1$ is at most $(1-w^2)^{-\frac{1}{2}}$ whose integral is convergent near $w=1$.[4]

Taking the discontinuity of $\psi_g^-(w)$ across the cut running from $w=1$ to $w=\infty$, we find the imaginary part of $\widetilde{G}_g(\widetilde{\lambda})$

$$\Delta\widetilde{G}_0 = \frac{\mathrm{i}T_D}{\sqrt{2\pi}}\left[\mathcal{L}_{-\frac{1}{2}}(T_D) + \frac{39}{8}\mathcal{L}_{\frac{1}{2}}(T_D) + \frac{1785}{128}\mathcal{L}_{\frac{3}{2}}(T_D) + \frac{22365}{1024}\mathcal{L}_{\frac{5}{2}}(T_D)\cdots\right],$$

$$\Delta\widetilde{G}_1 = \frac{\mathrm{i}T_D}{\sqrt{2\pi}}\left[-\frac{1}{384}\mathcal{L}_{-\frac{7}{2}}(T_D) - \frac{17}{1024}\mathcal{L}_{-\frac{5}{2}}(T_D) + \frac{805}{16384}\mathcal{L}_{-\frac{3}{2}}(T_D)\right.$$
$$\left. - \frac{4019}{131072}\mathcal{L}_{-\frac{1}{2}}(T_D) + \cdots\right], \tag{4.53}$$

$$\Delta\widetilde{G}_2 = \frac{\mathrm{i}T_D}{\sqrt{2\pi}}\left[\frac{1}{294912}\mathcal{L}_{-\frac{13}{2}}(T_D) + \frac{43}{1310720}\mathcal{L}_{-\frac{11}{2}}(T_D)\right.$$
$$\left. - \frac{2251}{20971520}\mathcal{L}_{-\frac{9}{2}}(T_D) - \frac{25099}{33554432}\mathcal{L}_{-\frac{7}{2}}(T_D) + \cdots\right].$$

In general, we have

$$\Delta\widetilde{G}_g = \frac{\mathrm{i}T_D}{\sqrt{2\pi}}\sum_{m=0}^{\infty}\widetilde{a}_{g,m}\mathcal{L}_{-1/2-3g+m}(T_D)$$
$$= \frac{\mathrm{i}T_D}{\sqrt{2\pi}}\sum_{n=1}^{\infty}e^{-nT_D}\sum_{m=0}^{\infty}\widetilde{a}_{g,m}(nT_D)^{1/2+3g-m}. \tag{4.54}$$

Note that the discontinuity of $\psi_0^-(w)$ has a simple form

$$\mathrm{Disc}\,\psi_0^-(w) = 2\mathrm{i}\frac{w^2(4w^2-3)}{2\pi\sqrt{w^2-1}}, \tag{4.55}$$

---

[4]Similarly, if we replace $\Gamma(2\ell+4)$ by $\Gamma(2\ell+4+3g)$ in the definition of $\phi_g^-(w)$ in (4.28), $\phi_g^-(w)$ does not have a pole at $w=1$. Then we do not have to treat the contribution of the residue at $w=1$ separately, as we did in (4.43).

and hence we can write down the expansion coefficients of $\Delta \widetilde{G}_0$ in a closed form

$$\widetilde{a}_{0,m} = \frac{(-1)^m 2^{-m-2}(4m^2+35)\Gamma\left(m-\frac{7}{2}\right)\Gamma\left(m+\frac{5}{2}\right)}{\pi\Gamma(m+1)}. \tag{4.56}$$

Since $\widetilde{a}_{0,m}$ has an alternating sign, the summation over m in (4.54) is Borel summable. This is similar to the behavior of a_m in (4.38).

In [13], the imaginary part $\Delta \widetilde{G}_g$ is interpreted as the contribution of D-string instanton. However, we should stress that $\Delta \widetilde{G}_g$ is not the S-dual of the worldsheet instanton correction ΔG_g. In particular, they have a different functional form. On the other hand, the sum over the (p,q)-strings in (2.20) is an exact statement and the (p,q)-string contribution $\mathcal{G}_N^{(\text{0-inst})}(M_{p,q})$ has exactly the same form as the contribution of $(1,0)$-sting (or F-string) $\mathcal{G}_N^{(\text{0-inst})}(M_{1,0})$.

5 Large N non-perturbative corrections

In this section, we study the large genus behavior of $G_g(\lambda)$ and $\widetilde{G}_g(\widetilde{\lambda})$. It turns out that both of the genus expansions $\sum_g N^{2-2g}G_g(\lambda)$ and $\sum_g N^{1-2g}\widetilde{G}_g(\widetilde{\lambda})$ are Borel non-summable. This means that the genus expansions receive non-perturbative corrections in $1/N$. In fact, we observe that the Borel non-summability leads to the following discontinuities

$$\text{Disc}\left[\sum_{g=0}^{\infty} N^{2-2g}G_g(\lambda)\right]^{\text{Borel}} = \sum_{n=1}^{\infty}\frac{1}{n}G_{\text{np}}(N,nx),$$

$$\text{Disc}\left[\sum_{g=0}^{\infty} N^{1-2g}\widetilde{G}_g(\widetilde{\lambda})\right]^{\text{Borel}} = \sum_{n=1}^{\infty}\widetilde{G}_{\text{np}}(N,ny), \tag{5.1}$$

where $G_{\text{np}}(N,x)$ and $\widetilde{G}_{\text{np}}(N,y)$ are given by

$$G_{\text{np}}(N,x) = \text{i}N\frac{(8Nx)^{\frac{3}{2}}}{\sqrt{2\pi}}(1+x^2)^{-\frac{1}{4}}e^{-NA(x)}\sum_{k=0}^{\infty}\frac{g_k(x)}{\left[8Nx(1+x^2)^{\frac{3}{2}}\right]^k},$$

$$\widetilde{G}_{\text{np}}(N,y) = 8\text{i}N^2 e^{-NA(y)}\sum_{k=0}^{\infty}\frac{h_k(y)}{\left[8Ny(1+y^2)^{\frac{3}{2}}\right]^k}, \tag{5.2}$$

with $x=\pi/\sqrt{\lambda}, y=\pi/\sqrt{\widetilde{\lambda}}$ and

$$A(z) = 4\left[\text{arcsinh}\,(z) + z\sqrt{1+z^2}\right]. \tag{5.3}$$

$g_k(z)$ and $h_k(z)$ in (5.2) are polynomials of z.

Curiously, the dual 't Hooft limit of $G_{\text{np}}(N,x)$ formally reduces to the D-string instanton, which are computed in the previous section. Similarly, $\widetilde{G}_{\text{np}}(N,y)$ reduces to the worldsheet instanton in the 't Hooft limit. That is to say, we observe that

$$\sum_{n=1}^{\infty}\frac{1}{n}G_{\text{np}}(N,nx) \xrightarrow{\text{dual 't Hooft limit}} \sum_{g=0}^{\infty}N^{1-2g}\Delta\widetilde{G}_g(\widetilde{\lambda}),$$

$$\sum_{n=1}^{\infty}\widetilde{G}_{\text{np}}(N,ny) \xrightarrow{\text{'t Hooft limit}} \sum_{g=0}^{\infty}N^{2-2g}\Delta G_g(\lambda). \tag{5.4}$$

Note that $G_{\mathrm{np}}(N, x)$ is originally derived in the 't Hooft limit, not in the dual 't Hooft limit. One should understand it as an extrapolation from the 't Hooft limit to the dual 't Hooft limit via the very strong coupling limit, as shown in figure 1. It is not obvious for us why the extrapolation to the dual 't Hooft limit of $G_{\mathrm{np}}(N, x)$ reproduces the D-string instanton. However, as a result, the first line of (5.4) agrees with the result in [13].

In the bulk type IIB string picture, we speculate that the non-perturbative corrections $G_{\mathrm{np}}(N, x)$ and $\widetilde{G}_{\mathrm{np}}(N, y)$ correspond to the D3-brane instanton corrections with magnetic and electic fluxes on its worldvolume, respectively. We will give some discussion on this interpretation in subsection 5.4.

5.1 $G_{\mathbf{np}}(N, x)$

Let us first consider the resummation of $G_g(\lambda)$ in the large λ regime. In this regime, $G_g(\lambda)$ is expanded as

$$G_g(\lambda) \sim \sum_{m=0}^{\infty} c_{g,m} \lambda^{g-\frac{1}{2}-m} \tag{5.5}$$

where $c_{g,m}$ are some coefficients. To see the large g behavior in the large λ regime, the terms with positive powers of λ in (5.5) are relevant. These terms come from the poles at positive half-integer s in (4.17). Then we find

$$G_g^+ = \sum_{j=1}^{g} \mathrm{Res}_{s=j+\frac{1}{2}} \left[-\frac{2\pi}{\sin \pi s} p(s) f_g(s) \Gamma(1-s) \zeta(2-2s) \left(\frac{\lambda}{4}\right)^{s-1} \right], \tag{5.6}$$

where the superscript of G_g^+ refers to the positive power of λ. The first few terms read

$$
\begin{aligned}
G_1^+ &= -\frac{\sqrt{\lambda}}{16}, \\
G_2^+ &= \frac{\lambda^{3/2}}{1536} - \frac{13\sqrt{\lambda}}{8192}, \\
G_3^+ &= \frac{\lambda^{5/2}}{98304} - \frac{25\lambda^{3/2}}{393216} + \frac{1533\sqrt{\lambda}}{4194304}, \\
G_4^+ &= \frac{3\lambda^{7/2}}{5242880} - \frac{595\lambda^{5/2}}{100663296} + \frac{11473\lambda^{3/2}}{268435456} - \frac{1203917\sqrt{\lambda}}{4294967296}.
\end{aligned}
\tag{5.7}
$$

We have computed G_g^+ up to $g = 126$[5] and determined the large g behavior of $G_g(\lambda)$ numerically using the Richardson extrapolation (see appendix C for the details of this method). We find that the large g asymptotic behavior of $G_g(\lambda)$ is given by

$$G_g(\lambda) \sim 16 \left(\frac{x}{\pi}\right)^{\frac{3}{2}} (1+x^2)^{-\frac{1}{4}} A(x)^{-2g-\frac{1}{2}} \sum_{k=0}^{\infty} g_k(x) \left[\frac{A(x)}{8x(1+x^2)^{\frac{3}{2}}}\right]^k \Gamma\left(2g + \frac{1}{2} - k\right), \tag{5.8}$$

where x is defined by

$$x = \frac{\pi}{\sqrt{\lambda}} = \sqrt{\frac{\pi}{4g_s N}}. \tag{5.9}$$

---

[5]The data of G_g^+ are available upon request to the authors.

We find the first few terms of $g_k(x)$

$$
\begin{aligned}
g_0(x) &= 1, \\
g_1(x) &= \frac{28x^4}{3} + 14x^2 + \frac{39}{8}, \\
g_2(x) &= \frac{104x^8}{9} + \frac{104x^6}{3} + \frac{113x^4}{2} + 47x^2 + \frac{1785}{128}, \\
g_3(x) &= \frac{1888x^{12}}{405} + \frac{944x^{10}}{45} + \frac{683x^8}{5} + \frac{772x^6}{3} + \frac{7195x^4}{32} + \frac{6951x^2}{64} + \frac{22365}{1024},
\end{aligned}
\tag{5.10}
$$

and $A(x)$ in (5.8) is given by (5.3). As reviewed in appendix C, the asymptotic behavior (5.8) leads to (5.2), the discontinuity of the resummed genus expansion or twice of the non-perturbative correction in $1/N$. Although the result (5.2) is a very complicated function of x, we are quite confident of our numerics since we found more than 20 digits of agreement in (5.8) by applying the Richardson extrapolation to our data $\{G_g^+\}_{g \le g_{\max}}$ with $g_{\max} = 126$.

As a demonstration of the Richardson extrapolation, in figure 3 we show the plot of the series

$$
A_g = 2g\sqrt{\frac{G_g^+}{G_{g+1}^+}},
\tag{5.11}
$$

and its 30-th Richardson transform $A_g^{(30)}$ for $\lambda = 100$ (see (C.2) for the definition of the k-th Richardson transform). From the discussion in appendix C, this series approaches $A(x)$ in the large g limit. Indeed, we find that the last term of this series $\{A_g^{(30)}\}_{g \le g_{\max}-30}$ agrees with $A(x)$ in (5.3) within 39 digits accuracy

$$
\frac{A_{g_{\max}-30}^{(30)}}{A(x)} - 1 = 3.5904 \times 10^{-40}.
\tag{5.12}
$$

We have also performed a similar analysis for $g_k(x)$ with various values of λ and found a good numerical agreement using the method of Richardson extrapolation.

We observe that the constant term of $g_k(x)$ is equal to the coefficients appearing in $\Delta\widetilde{G}_0$ in (4.54)

$$
g_k(0) = \widetilde{a}_{0,k}.
\tag{5.13}
$$

This is not an accident. In fact, we find that the D-string instanton correction $\Delta\widetilde{G}_g$ is reproduced from $G_{\mathrm{np}}(N, x)$ in (5.2) by taking the scaling limit

$$
N \to \infty, \qquad x \to 0, \qquad T_D = 8Nx : \text{fixed}.
\tag{5.14}
$$

This is nothing but the dual 't Hooft limit. More precisely, we further assume the strong coupling $T_D \gg 1$. Therefore, we consider the regime where $\sqrt{\lambda} \sim N \to \infty$ and $\sqrt{\widetilde{\lambda}} \gg 1$.
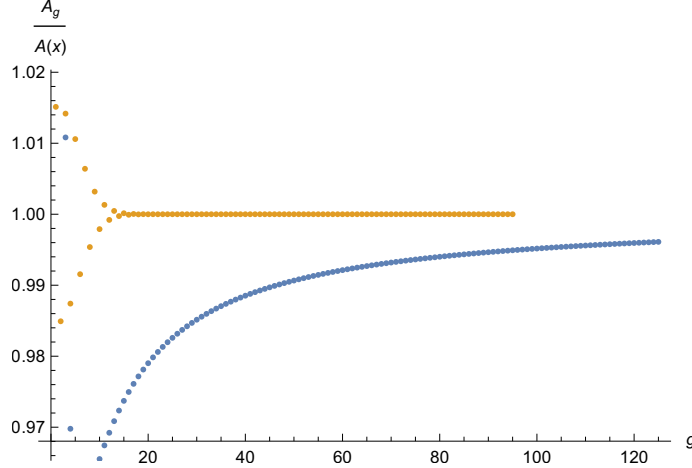
Figure 3. Plot of $A_g/A(x)$ for $\lambda = 100$. The blue dots represent $A_g/A(x)$ defined in (5.11) while the orange dots represent its 30-th Richardson transform $A_g^{(30)}/A(x)$.

In this limit $G_{\rm np}(N, x)$ in (5.2) is expanded as

$$
G_{\rm np}\!\left(N, \frac{T_D}{8N}\right) = \frac{iT_D}{\sqrt{2\pi}} e^{-T_D} \sum_{g=0}^{\infty} N^{1-2g} \sum_{m=0}^{\infty} \widetilde{a}_{g,m} T_D^{1/2+3g-m}
$$

$$
= \frac{iT_D}{\sqrt{2\pi}} e^{-T_D} \Bigg[N\!\left(T_D^{1/2} + \frac{39}{8} T_D^{-1/2} + \frac{1785}{128} T_D^{-3/2} + \frac{22365}{1024} T_D^{-5/2} + \cdots \right)
$$

$$
+ N^{-1}\!\left(-\frac{1}{384} T_D^{7/2} - \frac{17}{1024} T_D^{5/2} + \frac{805}{16384} T_D^{3/2} + \cdots \right) \tag{5.15}
$$

$$
+ N^{-3}\!\left(\frac{1}{294912} T_D^{13/2} + \frac{43}{1310720} T_D^{11/2} - \frac{2251}{20971520} T_D^{9/2} + \cdots \right) + \mathcal{O}(N^{-5}) \Bigg].
$$

This reproduces the single covering ($n = 1$) part of the D-string instanton in (4.53)! Conversely, if we use (4.54) we can go to the multi-covering structure:

$$
\sum_{g=0}^{\infty} N^{1-2g} \Delta \widetilde{G}_g(\widetilde{\lambda}) = \sum_{g=0}^{\infty} N^{1-2g} \frac{iT_D}{\sqrt{2\pi}} \sum_{n=1}^{\infty} e^{-nT_D} \sum_{m=0}^{\infty} \widetilde{a}_{g,m} (nT_D)^{1/2+3g-m}
$$

$$
= \sum_{n=1}^{\infty} \frac{1}{n} G_{\rm np}(N, nx). \tag{5.16}
$$

We conjecture that this gives the complete discontinuity of the genus resummation of $G_g(\lambda)$, and it implies the first line of (5.1).

5.2 $\widetilde{G}_{\rm np}(N, y)$

We can repeat the similar analysis for the second line of (5.4). To see this, let us consider the large g asymptotics of $\widetilde{G}_g(\widetilde{\lambda})$. In the large $\widetilde{\lambda}$ regime, $\widetilde{G}_g(\widetilde{\lambda})$ is expanded as

$$
\widetilde{G}_g(\widetilde{\lambda}) \sim \sum_{m=0}^{\infty} \widetilde{c}_{g,m} \widetilde{\lambda}^{g+\frac{1}{2}-m}. \tag{5.17}
$$

Note that the coefficients $\widetilde{c}_{g,m}$ can be computed by the integral representation (4.24). Interestingly, these coefficients are simply related to the coefficients $c_{g,m}$ in $G_g(\lambda)$,

$$\widetilde{c}_{g,m} = \frac{c_{m,g}}{(4\pi)^{1+2g-2m}}. \tag{5.18}$$

We found it by formally rewriting the genus expansion $\sum_g N^{2-2g}G_g(\lambda)$ at strong coupling in terms of $\widetilde{\lambda}$.

As in the case of $G_g(\lambda)$, $\widetilde{G}_g(\widetilde{\lambda})$ in the large $\widetilde{\lambda}$ regime is also approximated by

$$\widetilde{G}_g^+ = \sum_{j=1}^{g} \operatorname{Res}_{s=j+\frac{1}{2}} \left[-\frac{2\pi}{\sin \pi s} p(s) f_g(s) \Gamma(s) \zeta(2s) \left(\frac{\widetilde{\lambda}}{4\pi^2} \right)^s \right], \tag{5.19}$$

where the superscript of $\widetilde{G}_g^+$ refers to the positive power of $\widetilde{\lambda}$. The first few terms of $\widetilde{G}_g^+$ read

$$
\begin{aligned}
\widetilde{G}_1^+ &= -\frac{3\zeta(3)\widetilde{\lambda}^{3/2}}{64\pi^3}, \\
\widetilde{G}_2^+ &= \frac{45\zeta(5)\widetilde{\lambda}^{5/2}}{4096\pi^5} - \frac{39\zeta(3)\widetilde{\lambda}^{3/2}}{32768\pi^3}, \\
\widetilde{G}_3^+ &= \frac{4725\zeta(7)\widetilde{\lambda}^{7/2}}{2097152\pi^7} - \frac{1125\zeta(5)\widetilde{\lambda}^{5/2}}{1048576\pi^5} + \frac{4599\zeta(3)\widetilde{\lambda}^{3/2}}{16777216\pi^3}, \\
\widetilde{G}_4^+ &= \frac{99225\zeta(9)\widetilde{\lambda}^{9/2}}{67108864\pi^9} - \frac{2811375\zeta(7)\widetilde{\lambda}^{7/2}}{2147483648\pi^7} + \frac{1548855\zeta(5)\widetilde{\lambda}^{5/2}}{2147483648\pi^5} - \frac{3611751\zeta(3)\widetilde{\lambda}^{3/2}}{171798691184\pi^3}.
\end{aligned} \tag{5.20}
$$

We have computed $\widetilde{G}_g^+$ up to $g = 131$.[6] As in the previous subsection, we can determine the large g behavior of $\widetilde{G}_g^+$ numerically by the method of Richardson extrapolation using our data of $\{\widetilde{G}_g^+\}_{g \leq 131}$. The resulting non-perturbative correction is given by (5.2), where the action $A(y)$ is the same as (5.3) but the argument y is replaced by

$$y = \frac{\pi}{\sqrt{\widetilde{\lambda}}} = \sqrt{\frac{\pi g_s}{4N}}. \tag{5.21}$$

The first few terms of the coefficients $h_k(y)$ in (5.2) are given by

$$
\begin{aligned}
h_0(y) &= 1, \\
h_1(y) &= \frac{28y^4}{3} + 14y^2 + \frac{9}{2}, \\
h_2(y) &= \frac{104y^8}{9} + \frac{104y^6}{3} + 53y^4 + 45y^2 + \frac{117}{8}, \\
h_3(y) &= \frac{1888y^{12}}{405} + \frac{944y^{10}}{45} + \frac{1984y^8}{15} + \frac{878y^6}{3} + 290y^4 + \frac{585y^2}{4} + \frac{489}{16}.
\end{aligned} \tag{5.22}
$$

Again, we observe that the constant term of $h_k(y)$ is the same as the coefficients of ΔG_0 in (4.34)

$$h_k(0) = a_k. \tag{5.23}$$

---

[6]The data of $\widetilde{G}_g^+$ are available upon request to the authors.

This suggests that $\widetilde{G}_{\rm np}(N, y)$ is related to the worldsheet instanton studied in the previous section. To see this, we take the scaling limit

$$N \to \infty, \qquad y \to 0, \qquad T_F = 8Ny : \text{fixed}. \tag{5.24}$$

Then $\widetilde{G}_{\rm np}(N, y)$ is expanded as

$$\begin{aligned}
\widetilde{G}_{\rm np}\left(N, \frac{T_F}{8N}\right) = {\rm i}e^{-T_F}&\Bigg[8N^2\left(1 + \frac{9}{2}T_F^{-1} + \frac{117}{8}T_F^{-2} + \frac{489}{16}T_F^{-3} + \cdots\right) \\
&+ N^0\left(-\frac{1}{48}T_F^3 - \frac{3}{32}T_F^2 + \frac{77}{128}T_F + \cdots\right) \\
&+ N^{-2}\left(\frac{1}{36864}T_F^6 + \frac{7}{40960}T_F^5 - \frac{857}{491520}T_F^4 + \cdots\right) + \mathcal{O}(N^{-4})\Bigg].
\end{aligned} \tag{5.25}$$

This reproduces the single covering ($n = 1$) part of the worldsheet instanton corrections (4.34), (4.44), and (4.45). The multi-covering structure is given by

$$\begin{aligned}
\sum_{g=0}^{\infty} N^{2-2g}\Delta G_g(\lambda) &= \sum_{g=0}^{\infty} N^{2-2g} \cdot {\rm i}\sum_{n=1}^{\infty} e^{-nT_F}\sum_{m=0}^{\infty} a_{g,m}(nT_F)^{3g-m} \\
&= \sum_{n=1}^{\infty} \widetilde{G}_{\rm np}(N, ny),
\end{aligned} \tag{5.26}$$

which is almost parallel to (5.16). Again, we conjecture that this gives the complete discontinuity of the genus resummation of $\widetilde{G}_g(\widetilde{\lambda})$, and it implies the second line of (5.1).

5.3 Laplace-difference equation

Following [25], one can compute $G_{\rm np}(N, x)$ and $\widetilde{G}_{\rm np}(N, y)$ as trans-series solutions of the Laplace-difference equation obeyed by the exact $\mathcal{G}_N$ [7]

$$(\Delta_\tau - 2)\mathcal{G}_N = N^2(\mathcal{G}_{N+1} + \mathcal{G}_{N-1} - 2\mathcal{G}_N) - N(\mathcal{G}_{N+1} - \mathcal{G}_{N-1}), \tag{5.27}$$

where Δ_τ is the Laplacian defined in (2.2). Here we show that the Laplace-difference equation fixes both $G_{\rm np}(N, x)$ and $\widetilde{G}_{\rm np}(N, y)$ systematically.

Let us first consider $G_{\rm np}(N, x)$. Since $G_{\rm np}(N, x)$ is independent of τ_1, when acting on $G_{\rm np}(N, x)$ the Laplacian Δ_τ is replaced by

$$\tau_2^2\frac{\partial^2}{\partial\tau_2^2}G_{\rm np}(N, x) = \frac{1}{4}\left(x^2\partial_x^2 - x\partial_x\right)G_{\rm np}(N, x), \tag{5.28}$$

where x is related to τ_2 by

$$x = \sqrt{\frac{\pi\tau_2}{4N}}. \tag{5.29}$$

Thus $G_{\rm np}(N, x)$ should satisfy

$$\begin{aligned}
\left(\frac{x^2\partial_x^2 - x\partial_x}{4} - 2\right)&G_{\rm np}(N, x) \\
&= N^2\Big[G_{\rm np}(N + 1, x_+) + G_{\rm np}(N - 1, x_-) - 2G_{\rm np}(N, x)\Big] \\
&\quad - N\Big[G_{\rm np}(N + 1, x_+) - G_{\rm np}(N - 1, x_-)\Big],
\end{aligned} \tag{5.30}$$

where

$$x_\pm = x(1 \pm N^{-1})^{-\frac{1}{2}}. \tag{5.31}$$

At the leading order in the large N expansion of (5.30), the action $A(x)$ should satisfy

$$\frac{1}{2}x\partial_x A(x) = 2\sinh\left(\frac{1}{2}A(x) - \frac{1}{4}x\partial_x A(x)\right). \tag{5.32}$$

One can check that $A(x)$ in (5.3) indeed satisfies the above equation (5.32). One can also check that $G_{\rm np}(N,x)$ in (5.2) satisfies (5.32) order by order in the $1/N$ expansion. Alternatively, we can compute $g_k(x)$ in (5.2) recursively from the Laplace-difference equation (5.32). However, the constant term of $g_k(x)$ is not fixed by the equation (5.32) alone; we have to supply the information of $g_k(0) = \widetilde{a}_{0,k}$ as an input. With this input, we can compute $g_k(x)$ recursively up to any desired order. Using this result of $g_k(x)$, we observed that the highest and the next highest degree terms of $g_k(x)$ have the structure

$$g_k(x) = c_k\left(x^{4k} + \frac{3k}{2}x^{4k-2}\right) + \mathcal{O}(x^{4k-4}), \tag{5.33}$$

where c_k is determined by the generating function

$$\sum_{k=0}^{\infty} \frac{c_k}{(8N)^k} = \frac{1+N}{N}\left(\frac{\Gamma(N+1)}{\sqrt{2\pi N}N^N e^{-N}}\right)^2. \tag{5.34}$$

Similarly, $\widetilde{G}_{\rm np}(N,y)$ can be characterized as a trans-series solution of the following Laplace-difference equation

$$\begin{aligned}
\left(\frac{y^2\partial_y^2 + 3y\partial_y}{4} - 2\right) &\widetilde{G}_{\rm np}(N,y) \\
&= N^2\left[\widetilde{G}_{\rm np}(N+1,y_+) + \widetilde{G}_{\rm np}(N-1,y_-) - 2\widetilde{G}_{\rm np}(N,y)\right] \\
&\quad - N\left[\widetilde{G}_{\rm np}(N+1,y_+) - \widetilde{G}_{\rm np}(N-1,y_-)\right],
\end{aligned} \tag{5.35}$$

where $y_\pm = y(1 \pm N^{-1})^{-\frac{1}{2}}$. Again, one can check that $\widetilde{G}_{\rm np}(N,y)$ in (5.2) satisfies (5.35) order by order in the $1/N$ expansion. Alternatively, one can compute the coefficient $h_k(y)$ recursively from the Laplace-difference equation (5.35). However, as in the case of $g_k(x)$, the constant term $h_k(0)$ is not determined by the equation (5.35) alone, and we should supply the information of $h_k(0) = a_k$ as input. With this input, we can compute $h_k(y)$ up to any desired order from the equation (5.35).

5.4 Interpretation as D3-brane instantons

In this subsection, we discuss possible interpretations of $G_{\rm np}(N,x)$ and $\widetilde{G}_{\rm np}(N,y)$ as D3-brane instanton corrections in bulk type IIB string picture. This interpretation comes from the observation that $A(z)$ in (5.3) is reminiscent of the action of the so-called "giant Wilson loop" in the k-th symmetric representation with large k [16–18]. In the bulk IIB string theory on $AdS_5 \times S^5$, this giant Wilson loop corresponds to a D3-brane with k units of

electric flux along its worldvolume. As shown in [16], the action of the giant Wilson loop is given by $\frac{1}{2}A(\kappa)$ with the same function $A(z)$ in (5.3) and the argument κ is given by

$$\kappa = \frac{k\sqrt{\lambda}}{4N} = k\sqrt{\frac{\pi g_s}{4N}}. \tag{5.36}$$

We can see that κ is related to y in (5.21) as

$$\kappa = ky. \tag{5.37}$$

Thus it is natural to interpret $\widetilde{G}_{\mathrm{np}}(N, y)$ as the D3-brane instanton correction and the sum over n in (5.26) as the sum over the electric fluxes on the D3-brane. Similarly, $G_{\mathrm{np}}(N, x)$ can be interpreted as the magnetic analogue of the D3-brane in [16–18] since x in (5.9) is related to y in (5.21) by the replacement $g_s \to g_s^{-1}$, and the sum over n in (5.16) corresponds to the sum over the magnetic fluxes on the D3-brane.

Although we do not have strong evidence of the above interpretation other than the agreement of the instanton action $A(z)$ in (5.3) and $A(\kappa)$ in [16], at the technical level, appearance of the D3-brane as the non-perturbative correction is anticipated from the matrix model representation of $\mathcal{G}_N^{(0\text{-inst})}$. As reviewed in appendix A, $\mathcal{G}_N^{(0\text{-inst})}$ is written as

$$\mathcal{G}_N^{(0\text{-inst})} = -\int_0^\infty \frac{dw\,w}{2\sinh^2 w}\partial_{g_s}g_s^2\partial_{g_s}\langle \mathrm{Tr}\,e^{iwM}\,\mathrm{Tr}\,e^{-iwM}\rangle, \tag{5.38}$$

where the expectation value is defined by the Gaussian integral (A.4) over the hermitian matrix M. The non-perturbative correction comes from the pole of the w-integral in (5.38)

$$w = i\pi n \quad (n \in \mathbb{Z}), \tag{5.39}$$

and the sum over the electric/magnetic fluxes corresponds to the sum over the poles in (5.39). On the other hand, the expectation value of the 1/2 BPS circular Wilson loop with winding number k is also given by the same Gaussian matrix model due to supersymmetric localization [19]

$$W(C_k) = \langle \mathrm{Tr}\,e^{\pi k M}\rangle. \tag{5.40}$$

Recall that the expectation values of the 1/2 BPS Wilson loop with winding number k and the 1/2 BPS Wilson loop in the k-th symmetric representation behave in the same way at the leading order in the large k limit with fixed κ in (5.36). Thus, one can argue that the technical reason of the appearance of the D3-brane instanton corrections in the integrated correlator is that $\mathcal{G}_N^{(0\text{-inst})}$ in (5.38) and the 1/2 BPS Wilson loop (5.40) share the same underlying Gaussian matrix model due to supersymmetric localization.

It would be interesting to find more evidence for our interpretation of D3-brane instantons and understand the D3-brane configurations which contribute to the non-perturbative corrections $G_{\mathrm{np}}(N, x)$ and $\widetilde{G}_{\mathrm{np}}(N, y)$ more directly in the bulk type IIB string theory. We leave this as an interesting future problem.

6 Conclusions and outlook

In this paper, we studied the large N limit of the integrated correlator $\mathcal{G}_N$. We argued that the Poincaré series representation of $\mathcal{G}_N$ can be interpreted as a sum over (p,q)-string contributions in the large N limit via the holography. Due to the $SL(2,\mathbb{Z})$ duality, it is natural that all the (p,q)-strings contribute democratically. We found the recursion relation for $f_g(s)$ to compute the large N expansions of $\mathcal{G}_N^{(0\text{-inst})}(M_{1,0})$ up to very high genera.

We showed that the genus expansions $\sum_g N^{2-2g} G_g(\lambda)$ and $\sum_g N^{1-2g} \widetilde{G}_g(\widetilde{\lambda})$ receive non-perturbative corrections $G_{\text{np}}(N,x)$ and $\widetilde{G}_{\text{np}}(N,y)$, respectively. We speculate that these nonperturbative corrections correspond to D3-brane instanton corrections in the bulk type IIB string theory. The forms of these non-perturbative corrections are constrained by the large order behaviors of $G_g(\lambda)$ and $\widetilde{G}_g(\widetilde{\lambda})$, and given by (5.2). Interestingly, $G_{\text{np}}(N,x)$ reproduces the D-string instanton correction in the dual 't Hooft limit while $\widetilde{G}_{\text{np}}(N,y)$ reproduces the worldsheet instanton correction in the 't Hooft limit. We confirmed that the same corrections are fixed by the Laplace-difference equation for the integrated correlator.

There are many open questions. In this paper, we have not studied the D-instanton corrections in detail (see appendix A.3 for the exact result of one-instanton contribution at finite N). As we emphasized at the end of section 2, D-instanton corrections cannot be seen in the (p,q)-string contribution $\mathcal{G}_N^{(0\text{-inst})}(M_{p,q})$ with fixed (p,q). We have seen that the $(1,0)$-string contribution $\mathcal{G}_N^{(0\text{-inst})}(M_{1,0})$ is naturally organized into $G(\lambda)$ and $\widetilde{G}(\widetilde{\lambda})$, whose non-perturbative corrections $e^{-2\sqrt{\lambda}}$ and $e^{-2\sqrt{\widetilde{\lambda}}}$ are interpreted as the worldsheet instanton and the D-string instanton, respectively. However, we do not see the mixed contribution of these instantons. On the other hand, in the grand partition function of ABJM theory on S^3, it is known that there are "bound states" of worldsheet instantons and membrane instantons [26]. It turns out that these bound states can be absorbed into the worldsheet instantons by a certain redefinition of the chemical potential, which corresponds to the quantum mirror map of the topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$ [27]. It would be interesting to see if such a structure exists in the integrated correlator of $\mathcal{N} = 4$ SYM as well.

In a recent paper [14], the lattice sum representation of the integrated correlator (2.1) is generalized to the $\mathcal{N} = 4$ SYM with gauge groups $SO(N)$ and $Sp(N)$. It would be interesting to repeat our analysis with these gauge groups and see how the unoriented worldsheet contributions fit into the whole picture. It would also be interesting to generalize our analysis to another type of integrated correlator $\partial_m^4 \log Z|_{m=0}$. As discussed in [12, 13], in this correlator there appear certain generalized Eisenstein series, which satisfy inhomogeneous Laplace equations on the upper-half τ-plane. We leave these generalization of our analysis for future works.

Acknowledgments

A Exact results at finite N

A.1 Relation between $B_N(t)$ and $K_N(x)$

The zero instanton part $\mathcal{G}_N^{(0\text{-inst})}$ is written as

$$\mathcal{G}_N^{(0\text{-inst})} = \int_0^\infty dt B_N(t) \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{g_s t}} e^{-\frac{\pi n^2}{g_s t}}. \tag{A.1}$$

On the other hand, $\mathcal{G}_N^{(0\text{-inst})}$ is also written as [28]

$$\mathcal{G}_N^{(0\text{-inst})} = \int_0^\infty dw \frac{w K_N(x)}{\sinh^2 w}, \quad x = \frac{g_s w^2}{\pi}. \tag{A.2}$$

Here $K_N(x)$ is given by a certain correlator in the Gaussian matrix model

$$K_N(x) = -\frac{1}{2} \partial_x x^2 \partial_x \text{SFF}(x), \tag{A.3}$$

where

$$\text{SFF}(x) = \left\langle \text{Tr}\, e^{iwM} \text{Tr}\, e^{-iwM} \right\rangle = \frac{\int dM e^{-\frac{2\pi}{g_s} \text{Tr}\, M^2} \text{Tr}\, e^{iwM} \text{Tr}\, e^{-iwM}}{\int dM e^{-\frac{2\pi}{g_s} \text{Tr}\, M^2}}. \tag{A.4}$$

$\text{SFF}(x)$ in (A.4) is known as the spectral form factor if we regard w as "time" and M as a random Hamiltonian. The matrix model correlator (A.4) is naturally decomposed into the disconnected part and the connected part

$$\begin{aligned}
\text{SFF}_{\text{dis}}(x) &= \left\langle \text{Tr}\, e^{iwM} \right\rangle \left\langle \text{Tr}\, e^{-iwM} \right\rangle, \\
\text{SFF}_{\text{conn}}(x) &= \left\langle \text{Tr}\, e^{iwM} \text{Tr}\, e^{-iwM} \right\rangle_{\text{conn}}.
\end{aligned} \tag{A.5}$$

It is known that $\text{SFF}_{\text{dis}}(x)$ and $\partial_x \text{SFF}_{\text{conn}}(x)$ can be written in a closed form in terms of the Laguerre polynomials [29–31]

$$\begin{aligned}
\text{SFF}_{\text{dis}}(x) &= e^{-x} L_{N-1}^{(1)}(x)^2, \\
\partial_x \text{SFF}_{\text{conn}}(x) &= N e^{-x} \left[L_{N-1}^{(1)}(x) L_{N-1}(x) - L_{N-2}^{(1)}(x) L_N(x) \right].
\end{aligned} \tag{A.6}$$

It turns out that $\text{SFF}_{\text{dis}}(x)$ and $\text{SFF}_{\text{conn}}(x)$ are not independent; they are related by [29]

$$\partial_x^2 \text{SFF}_{\text{conn}}(x) = -\text{SFF}_{\text{dis}}(x). \tag{A.7}$$

As discussed in [14], $B_N(t)$ and $K_N(x)$ are related by the Laplace transformation

$$B_N(t) = -\left(t\partial_t + \frac{1}{2} \right) \int_0^\infty dx e^{-tx} K_N(x) = \int_0^\infty dx e^{-tx} \left(tx - \frac{1}{2} \right) K_N(x). \tag{A.8}$$

Indeed, we can easily check that plugging (A.8) into (A.1) we recover (A.2)

$$\begin{aligned}
\mathcal{G}_N^{(0\text{-inst})} &= \int_0^\infty \frac{2 g_s w dw}{\pi} K_N(x) \int_0^\infty dt \left(t\frac{g_s w^2}{\pi} - \frac{1}{2} \right) e^{-t\frac{g_s w^2}{\pi}} \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{g_s t}} e^{-\frac{\pi n^2}{g_s t}} \\
&= \int_0^\infty dw w K_N(x) \sum_{n \in \mathbb{Z}} 2|n| e^{-2|n|w} \\
&= \int_0^\infty dw \frac{w K_N(x)}{\sinh^2 w}.
\end{aligned} \tag{A.9}$$

Using the relation between $B_N(t)$ and $K_N(x)$ in (A.8), one can easily show the basic properties of $B_N(t)$ found in [7]. The first property is

$$\int_0^\infty \frac{dt}{\sqrt{t}} B_N(t) = 0, \tag{A.10}$$

which follows from

$$\int_0^\infty \frac{dt}{\sqrt{t}} e^{-tx} \left(tx - \frac{1}{2} \right) = -\int_0^\infty dt \partial_t(\sqrt{t} e^{-tx}) = 0. \tag{A.11}$$

The second property is

$$\int_0^\infty dt B_N(t) = \frac{1}{4} N(N-1). \tag{A.12}$$

This can be shown as

$$\begin{aligned}
\int_0^\infty dt \int_0^\infty dx e^{-tx} &\left(tx - \frac{1}{2} \right) K_N(x) \\
&= \int_0^\infty dx \frac{1}{2x} K_N(x) \\
&= -\frac{1}{4} \int_0^\infty dx \frac{1}{x} \partial_x x^2 \partial_x \mathrm{SFF}(x) \\
&= \frac{\mathrm{SFF}(0) - \mathrm{SFF}(\infty)}{4},
\end{aligned} \tag{A.13}$$

where we performed the integration by parts in the last equality. It is well known that the early time and the late time value of the spectral form factor are given by

$$\mathrm{SFF}(0) = N^2, \quad \mathrm{SFF}(\infty) = N, \tag{A.14}$$

which reproduces (A.12).

Using the explicit form of $\mathrm{SFF}(x)$ in (A.6), one can prove the relation (A.8) by directly computing the Laplace transform of $K_N(x)$.[7] To this end, it is natural to define

$$\begin{aligned}
B_{\mathrm{dis}}(t) &= \int_0^\infty dx e^{-tx} \left(tx - \frac{1}{2} \right) \left(-\frac{1}{2} \partial_x x^2 \partial_x \mathrm{SFF}_{\mathrm{dis}}(x) \right), \\
B_{\mathrm{conn}}(t) &= \int_0^\infty dx e^{-tx} \left(tx - \frac{1}{2} \right) \left(-\frac{1}{2} \partial_x x^2 \partial_x \mathrm{SFF}_{\mathrm{conn}}(x) \right).
\end{aligned} \tag{A.15}$$

Let us first consider $B_{\mathrm{dis}}(t)$

$$\begin{aligned}
B_{\mathrm{dis}}(t) &= \frac{1}{2} (t\partial_t + 1/2) \int_0^\infty dx e^{-tx} \partial_x x^2 \partial_x e^{-x} L_{N-1}^{(1)}(x)^2 \\
&= \frac{1}{2} (t\partial_t + 1/2) \int_0^\infty dx e^{-(1+t)x} (-2tx + t^2 x^2) L_{N-1}^{(1)}(x)^2 \\
&= -\frac{1}{2} (t\partial_t + 1/2)(2t + t^2 \partial_t) I
\end{aligned} \tag{A.16}$$

where I is given by

$$I = \int_0^\infty dx e^{-(1+t)x} x L_{N-1}^{(1)}(x)^2. \tag{A.17}$$

Using the formula in [32], this is evaluated as

$$I = N^2 \frac{t^{2N-2}}{(1+t)^{2N}} {}_2F_1(1-N, 1-N, 2; t^{-2}). \tag{A.18}$$

$B_{\text{conn}}(t)$ can be computed in a similar manner. After some algebra, we find

$$
\begin{aligned}
B_{\text{dis}}(t) &= \frac{3Nt^{2N-2}(1-t^2-2Nt+2N^2t^2)}{4(t-1)(1+t)^{2N+1}} X \\
&\quad + \frac{Nt^{2N-2}\left[-3(1-t^2)^2+12Nt-24N^2t^2-4Nt^3+16N^3t^3\right]}{4(t-1)^2(1+t)^{2N+2}} Y, \\
B_{\text{conn}}(t) &= \frac{3Nt^{2N-2}(1-t^2+2Nt-2N^2)}{4(t-1)(1+t)^{2N+1}} X \\
&\quad - \frac{Nt^{2N-2}\left[-3(1-t^2)^2+12Nt^3-24N^2t^2-4Nt+16N^3t\right]}{4(t-1)^2(1+t)^{2N+2}} Y,
\end{aligned}
\tag{A.19}
$$

where

$$
\begin{aligned}
X &= t^{3-2N} {}_2F_1(-N, 2-N, 2; t^2) + {}_2F_1(-N, 2-N, 2; t^{-2}), \\
Y &= t^{3-2N} {}_2F_1(-N, 2-N, 2; t^2) - {}_2F_1(-N, 2-N, 2; t^{-2}).
\end{aligned}
\tag{A.20}
$$

We observe that $B_{\text{dis}}(t)$ and $B_{\text{conn}}(t)$ are related by

$$B_{\text{conn}}(t) = t^{-1} B_{\text{dis}}(t^{-1}). \tag{A.21}$$

Adding $B_{\text{dis}}(t)$ and $B_{\text{conn}}(t)$ in (A.19), we find a simple expression of $B_N(t)$

$$B_N(t) = B_{\text{dis}}(t) + B_{\text{conn}}(t) = \frac{N(N^2-1)t^{2N-2}}{(1+t)^{2N+1}} \left[\frac{3}{2}(1+t)X + \frac{4Nt}{t-1}Y\right]. \tag{A.22}$$

One can show that X and Y in (A.20) are related to the Jacobi polynomials as

$$
\begin{aligned}
(N+1)t^{2N-2}X &= (1-t^2)^N \left[(t-1)P_N^{(1,-2)}\left(\frac{1+t^2}{1-t^2}\right) + P_N^{(1,-1)}\left(\frac{1+t^2}{1-t^2}\right)\right], \\
(N+1)t^{2N-2}Y &= (1-t^2)^N \left[(t+1)P_N^{(1,-2)}\left(\frac{1+t^2}{1-t^2}\right) - P_N^{(1,-1)}\left(\frac{1+t^2}{1-t^2}\right)\right],
\end{aligned}
\tag{A.23}
$$

which implies the equivalence of (A.22) and (2.8).

A.2 Closed form of $c(s, N)$

The spectral function $c(s, N)$ is defined as the expansion coefficient of $B_N(t)$ in (3.1). From (A.8), one can show that $c(s, N)$ is given by the Mellin transform of $K_N(x)$

$$c(s, N) = \left(s - \frac{1}{2}\right) \int_0^\infty dx \frac{x^{s-1}}{\Gamma(s)} K_N(x). \tag{A.24}$$

From (A.6) and (A.7), $K_N(x)$ is written as

$$K_N(x) = -\frac{1}{2}\partial_x \left(x^2 \partial_x \left[e^{-x} L_{N-1}^{(1)}(x)^2\right]\right) + \frac{1}{2}\partial_x \left(x^2 \int_\infty^x dy\, e^{-y} L_{N-1}^{(1)}(y)^2\right). \tag{A.25}$$

Then, by the partial integration, the Mellin transform of (A.25) becomes

$$\int_0^\infty dx\, x^{s-1} K_N(x) = \frac{s-1}{2} \int_0^\infty dx \left(-s x^{s-1} + \frac{x^{s+1}}{s+1} \right) e^{-x} L_{N-1}^{(1)}(x)^2. \qquad (A.26)$$

Using the relation

$$\int_0^\infty dx\, x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)} \delta_{n,m},$$

$$L_n^{(\alpha)}(x) = \sum_{i=0}^n \binom{\alpha - \beta + n - i - 1}{n - i} L_i^{(\beta)}(x), \qquad (A.27)$$

we find

$$\int_0^\infty dx\, x^{s-1} K_N(x)$$
$$= \frac{s-1}{2} \sum_{i=0}^{N-1} \left[-s \binom{N-s-i}{N-1-i}^2 \frac{\Gamma(i+s)}{\Gamma(i+1)} + \frac{1}{s+1} \binom{N-s-i-2}{N-1-i}^2 \frac{\Gamma(i+s+2)}{\Gamma(i+1)} \right]. \qquad (A.28)$$

This summation can be written in terms of the hypergeometric function $_3F_2$. Finally, we find the exact form of $c(s, N)$ at finite N

$$c(s,N) = \frac{s(s-1/2)}{2\Gamma(N)^2 \Gamma(2-s)\Gamma(1-s)} \left[\Gamma(N+1-s)^2 {}_3F_2(\{1-N,1-N,s\},\{s-N,s-N\},1) \right.$$

$$\left. - (s-1)^2 s^2 \Gamma(N-1-s)^2 {}_3F_2(\{1-N,1-N,s+2\},\{s+2-N,s+2-N\},1) \right]. \qquad (A.29)$$

At finite N, $c(s, N)$ is a polynomial in s. For instance,

$$c(s,2) = \frac{1}{2}(1-2s)^2 s(s-1),$$
$$c(s,3) = \frac{1}{4}(1-2s)^2 s(s-1)(6-s+s^2), \qquad (A.30)$$
$$c(s,4) = \frac{1}{24}(1-2s)^2 s(s-1)(4-s+s^2)(18-s+s^2).$$

Note also that $c(s, N)$ has a symmetry

$$c(1-s, N) = c(s, N). \qquad (A.31)$$

A.3 Exact one-instanton contribution at finite N

From (2.13), the one-instanton contribution is written as

$$\mathcal{G}_N^{(\text{1-inst})} = \int_0^\infty \frac{dt}{\sqrt{g_s t}} B_N(t) e^{-\frac{\pi}{g_s}(t+t^{-1})}. \qquad (A.32)$$

In order to evaluate this integral, it is convenient to perform the change of integration variable

$$t = e^{2x}. \qquad (A.33)$$

Then (A.32) becomes

$$\mathcal{G}_N^{(\text{1-inst})} = \frac{2}{\sqrt{g_s}} \int_{-\infty}^{\infty} dx\, e^x B_N(e^{2x}) e^{-\frac{2\pi}{g_s}\cosh 2x}. \tag{A.34}$$

Using (A.22), one can show that $e^x B_N(e^{2x})$ is expanded as

$$e^x B_N(e^{2x}) = \frac{N(N^2-1)}{\sqrt{\pi}(\cosh x)^{2N+1}} \sum_{j=0}^{N-1} (\cosh x)^{2j}(-1)^{N+j+1}\Gamma(N+1/2-j)\Gamma(N-1)$$
$$\times \left[\frac{9}{4\Gamma(j)\Gamma(N-j)\Gamma(N+3-j)} + \frac{N}{\Gamma(j+1)\Gamma(N-1-j)\Gamma(N+2-j)} \right]. \tag{A.35}$$

Plugging this into (A.32), we find

$$\mathcal{G}_N^{(\text{1-inst})} = \frac{2\Gamma(N+2)}{\sqrt{g_s}} e^{-\frac{2\pi}{g_s}} \sum_{j=0}^{N-1}(-1)^{N+j+1} U\!\left(\frac{1}{2},\frac{1}{2}+j-N,\frac{4\pi}{g_s}\right)\Gamma\!\left(N+\frac{1}{2}-j\right)$$
$$\times \left[\frac{9}{4\Gamma(j)\Gamma(N-j)\Gamma(N+3-j)} + \frac{N}{\Gamma(j+1)\Gamma(N-1-j)\Gamma(N+2-j)} \right], \tag{A.36}$$

where $U(a,b,z)$ denotes the Toriconi's confluent hypergeometric function. For instance, when $N=2$ and $N=3$ (A.36) becomes

$$\mathcal{G}_2^{(\text{1-inst})} = \frac{12\pi^2}{g_s^2} e^{-\frac{2\pi}{g_s}} - \frac{3\pi^2}{g_s^{3/2}}\left(1+\frac{8\pi}{g_s}\right)e^{\frac{2\pi}{g_s}}\text{Erfc}\left(2\sqrt{\frac{\pi}{g_s}}\right), \tag{A.37}$$

$$\mathcal{G}_3^{(\text{1-inst})} = \frac{3\pi^2}{g_s^2}\left(37+\frac{40\pi}{g_s}\right)e^{-\frac{2\pi}{g_s}} - \frac{3\pi^2}{8g_s^{3/2}}\left(27+\frac{336\pi}{g_s}+\frac{320\pi^2}{g_s^2}\right)e^{\frac{2\pi}{g_s}}\text{Erfc}\left(2\sqrt{\frac{\pi}{g_s}}\right),$$

where $\text{Erfc}(z)$ denotes the complementary error function. This agrees with the result in [7].

From the exact result (A.36), we find the small g_s expansion of $\mathcal{G}_N^{(\text{1-inst})}$

$$\mathcal{G}_N^{(\text{1-inst})} = -\frac{3\Gamma(N-1/2)}{4\sqrt{\pi}\Gamma(N-1)} e^{-\frac{2\pi}{g_s}} \left[1 + \frac{3(N-4)}{8(2N-3)}\frac{g_s}{\pi} - \frac{15N(N+1)}{128(2N-3)(2N-5)}\frac{g_s^2}{\pi^2} \right.$$
$$\left. + \frac{105N(N+1)(N-6)}{1024(2N-3)(2N-5)(2N-7)}\frac{g_s^3}{\pi^3} - \frac{4725N(N+1)(N^2-11N+38)}{32768(2N-3)(2N-5)(2N-7)(2N-9)}\frac{g_s^4}{\pi^4} + \cdots \right]. \tag{A.38}$$

B Relation between $B_{\text{dis}}, B_{\text{conn}}$ and $G_0(\lambda), \tilde{G}_0(\tilde{\lambda})$

From the matrix model picture, there is a natural decomposition of the zero-instanton part into the disconnected and the connected part

$$\mathcal{G}_{\text{dis}}^{(\text{0-inst})} = \int_0^{\infty} dt B_{\text{dis}}(t)\vartheta_3(e^{-2\pi g_s t}),$$
$$\mathcal{G}_{\text{conn}}^{(\text{0-inst})} = \int_0^{\infty} dt B_{\text{conn}}(t)\vartheta_3(e^{-2\pi g_s t}), \tag{B.1}$$

where $B_{\text{dis}}(t)$ and $B_{\text{conn}}(t)$ are given by (A.19). We observe that $\mathcal{G}_{\text{dis}}^{(0\text{-inst})}$ approaches $N^2 G_0(\lambda)$ in the 't Hooft limit, and $\mathcal{G}_{\text{conn}}^{(0\text{-inst})}$ approaches $N\widetilde{G}_0(\widetilde{\lambda})$ in the dual 't Hooft limit

$$
\begin{aligned}
\lim_{N\to\infty} N^{-2}\mathcal{G}_{\text{dis}}^{(0\text{-inst})}\Big|_{g_s=\frac{\lambda}{4\pi N}} &= G_0(\lambda), \\
\lim_{N\to\infty} N^{-1}\mathcal{G}_{\text{conn}}^{(0\text{-inst})}\Big|_{g_s=\frac{4\pi N}{\widetilde{\lambda}}} &= \widetilde{G}_0(\widetilde{\lambda}),
\end{aligned}
\tag{B.2}
$$

where $G_0(\lambda)$ and $\widetilde{G}_0(\widetilde{\lambda})$ are given by [8, 13, 28]

$$
\begin{aligned}
G_0(\lambda) &= \int_0^\infty dw\, w \frac{J_1(w\sqrt{\lambda}/\pi)^2 - J_2(w\sqrt{\lambda}/\pi)^2}{\sinh^2 w}, \\
\widetilde{G}_0(\widetilde{\lambda}) &= \frac{\widetilde{\lambda}}{2\pi}\int_0^1 dw \frac{1}{\sinh^2(w\sqrt{\widetilde{\lambda}})}\frac{w^2(4w^2-3)}{\sqrt{1-w^2}}.
\end{aligned}
\tag{B.3}
$$

As we can see from figure 4, (B.2) is a good approximation at large N.

The relation between $\mathcal{G}_{\text{dis}}^{(0\text{-inst})}$ and $G_0(\lambda)$ can be understood as follows. In the large N limit, $\langle \text{Tr}\, e^{\pm iwM}\rangle$ becomes

$$
\langle \text{Tr}\, e^{\pm iwM}\rangle = N\frac{2J_1(\sqrt{4Nx})}{\sqrt{4Nx}},
\tag{B.4}
$$

and the disconnected part of $K_N(x)$ is given by

$$
-\frac{1}{2}\partial_x x^2 \partial_x \left(N\frac{2J_1(\sqrt{4Nx})}{\sqrt{4Nx}}\right)^2 = N^2\Big[J_1(\sqrt{4Nx})^2 - J_1(\sqrt{4Nx})^2\Big],
\tag{B.5}
$$

which reproduces $G_0(\lambda)$ in (B.3).

On the other hand, we do not have a clear understanding of the reason why the connected part $\mathcal{G}_{\text{conn}}^{(0\text{-inst})}$ corresponds to $\widetilde{G}_0(\widetilde{\lambda})$. However, there is some indication of this correspondence from the behavior of the spectral form factor. It is known that at late times the connected part is dominant in $\text{SFF}(x)$ and it becomes constant, which is usually called plateau, beyond some critical value of x

$$
\text{SFF}_{\text{conn}}(x) = N, \quad (x \geq x_c),
\tag{B.6}
$$

where x_c is given by [29, 30]
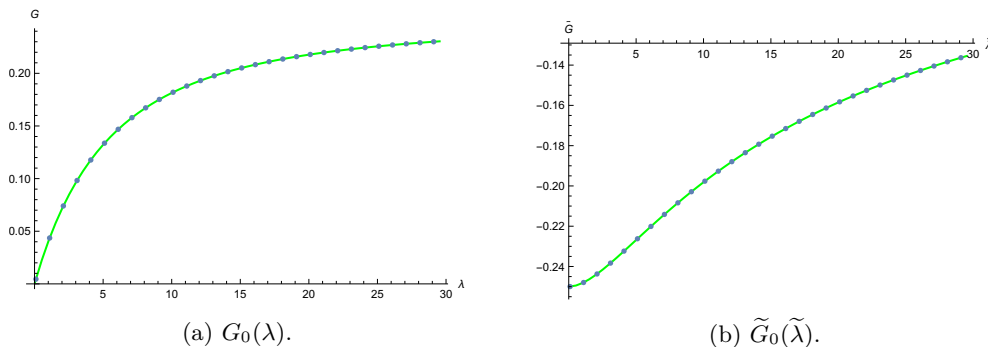
$$
x_c = 4N.
\tag{B.7}
$$

The corresponding critical time w_H, so-called Heisenberg time, is given by

$$
\frac{g_s w_H^2}{\pi} = x_c \quad \Rightarrow \quad w_H = \sqrt{\widetilde{\lambda}}.
\tag{B.8}
$$

From (A.2), one can estimate the order of the non-perturbative correction

$$
\sinh^{-2} w_H \sim e^{-2w_H} = e^{-2\sqrt{\widetilde{\lambda}}},
\tag{B.9}
$$

which is exactly the order of D-string instanton. This suggests the relation between $\mathcal{G}_{\text{conn}}^{(0\text{-inst})}$ and $\widetilde{G}_0(\widetilde{\lambda})$. It would be interesting to understand this relation better.

(a) $G_0(\lambda)$.

(b) $\widetilde{G}_0(\widetilde{\lambda})$.

Figure 4. Plot of a $G_0(\lambda)$ and b $\widetilde{G}_0(\widetilde{\lambda})$. The solid green curves represent the left hand side of (B.2) for $N = 20$, while the blue dots represent the genus-zero result in (B.3).

C Richardson extrapolation

In the numerical study of the asymptotic behavior of a sequence, one can use the standard technique of Richardson extrapolation which accelerates the convergence of sequence towards the leading asymptotics. Given a sequence $\{S_m\}_{m=1,2,\dots}$

$$S_m = s_0 + \frac{s_1}{m} + \frac{s_2}{m^2} + \cdots, \qquad \lim_{m \to \infty} S_m = s_0, \tag{C.1}$$

its k-th Richardson transform is defined as

$$S_m^{(k)} := \sum_{n=0}^{k} \frac{(-1)^{k+n}(m+n)^n S_{m+n}}{n!(k-n)!}. \tag{C.2}$$

After this transformation the subleading terms in S_m are canceled up to m^{-k}, i.e. $S_m^{(k)} = s_0 + \mathcal{O}(m^{-k-1})$ and hence the sequence $S_m^{(k)}$ has a much faster convergence to s_0 compared to the original S_m. However, we lose some data in this transformation as a cost of the faster convergence: if we know the original sequence S_m up to $m = m_{\max}$, the data of k-th Richardson transform $S_m^{(k)}$ in (C.2) are available only up to $m = m_{\max} - k$.

One can apply the Richardson extrapolation to estimate the large order behavior of the perturbative series

$$G \sim \sum_{m=0}^{\infty} a_m z^m. \tag{C.3}$$

When this series is Borel non-summable, there is an ambiguity of the form

$$\Delta G = \pi \mathrm{i} z^{-\frac{b}{2}} e^{-\frac{A}{\sqrt{z}}} \sum_{n=0}^{\infty} f_n z^n. \tag{C.4}$$

Then the large m behavior of a_m is related to ΔG as

$$a_m \sim \int_0^\infty \frac{dz}{2\pi \mathrm{i} z^{m+1}} \Delta G = \sum_{n=0}^{\infty} f_n A^{-2m-b+n} \Gamma(2m+b-n). \tag{C.5}$$

As explained in [33], one can extract A, b, f_n by constructing some sequences which converge to them. First, A is extracted from the sequence

$$A_m = 2m\sqrt{\frac{a_m}{a_{m+1}}}. \tag{C.6}$$

From (C.5) one can see that A_m approaches A as $m \to \infty$. In practice, A is estimated numerically by applying the Richardson extrapolation to the sequence $\{A_m\}_{m=1,2,\dots}$. Once we find A, then we can extract b from the sequence

$$b_m = m\left(A^2 \frac{a_{m+1}}{4m^2 a_m} - 1\right) - \frac{1}{2}, \tag{C.7}$$

which converges to b in the limit $m \to \infty$. Finally, f_n can be found from the sequence

$$f_{n,m} = \frac{1}{A^{-2m-b+n}\Gamma(2m+b-n)}\left[a_m - \sum_{k=0}^{n-1} f_k A^{-2m-b+k}\Gamma(2m+b-k)\right], \tag{C.8}$$

which converges to f_n as $m \to \infty$. For instance, f_0 can be obtained from the large m limit of $f_{0,m}$ given by

$$f_{0,m} = \frac{a_m}{A^{-2m-b}\Gamma(2m+b)}. \tag{C.9}$$

Once we know f_k ($0 \le k \le n-1$), one can extract f_n numerically from the large m asymptotics of the sequence $\{f_{n,m}\}_{m=1,2,\dots}$ using the Richardson extrapolation.

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