Published for SISSA by Springer

Received: *August 24, 2022*
Revised: *October 10, 2022*
Accepted: *October 23, 2022*
Published: *November 2, 2022*

Standard Model predictions for lepton flavour universality ratios of inclusive semileptonic B decays

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Abstract: We present Standard Model predictions for lepton flavour universality ratios of inclusive $B \to X_{(c)}\ell\bar{\nu}_\ell$. For the $\ell = \mu, e$, these ratios are very close to unity as expected. For the τ mode, we update the SM prediction for the branching ratio including power-corrections in the heavy-quark expansion up to $1/m_b^3$. These inclusive ratios serve as an important cross-check of the exclusive $B \to D^{(*)}\ell\bar{\nu}_\ell$ modes, in which tensions exists between the predictions and measurements in those modes.

Keywords: Bottom Quarks, Semi-Leptonic Decays

ArXiv ePrint: 2207.03432

https://doi.org/10.1007/JHEP11(2022)007

JHEP11(2022)007

Contents

1 Introduction

The inclusive $B \to X_c\ell\bar{\nu}_\ell$ decays, with $\ell = \mu, e$, are by now standard candles in the determination of the CKM element $|V_{cb}|$. Employing the heavy quark expansion (HQE), allows the parametrization of these decays in perturbative Wilson coefficients and non-perturbative HQE elements. Thanks to a combined theoretical and experimental effort, these HQE parameters can be extracted from moments of the decay spectrum giving an impressive 2% uncertainty on the inclusive V_{cb} determinations [1, 2].

The experimental measurements of semileptonic $B \to X_c$ usually combine the muon and electron modes (and B^0 and B^+). Recently, the Belle collaboration also provided the first measurement of q^2 moments, separately for the electron and muon modes [3]. No deviations from lepton flavor universality were found. However, given the discrepancies in the rare $b \to s\ell\ell$ modes, it may be worth measuring the ratio

$$R_{\mu/e}(X_c) \equiv \frac{\Gamma(B \to X_c\mu\bar{\nu}_\mu)}{\Gamma(B \to X_c e\bar{\nu}_e)} \,. \tag{1.1}$$

In the Standard Model (SM), this ratio is expected to be close to one, but more elaborate predictions are not available to our knowledge. In this paper, we provide these predictions by taking into account the masses of the leptons, in light of upcoming measurements. These results are interesting due to the recent inconsistencies in the exclusive $B \to D^*$ forward-backward asymmetry measurements (see [4] for more details). Recently, also the final-state radiation effects in the forward-backward asymmetry were studied in detail [5].

In this work, we do not include structure depend nor ultrasoft QED effects as those are challenging to disentangle from the experimental detector efficiencies (for recent works on QED effects in exclusive semileptonic B decays see e.g. [6–9]).

JHEP11(2022)007

While the inclusive light-lepton modes have been studied in depth, the situation is very different for the τ mode. Experimentally, only LEP results [10] and a unpublished Belle analysis [11] of the total rate exists, both having large uncertainties. In addition, the LEP measurement requires assumptions about hadronic effects in order to be interpreted. On the theoretical side, SM predictions for this mode exists using the HQE parameters as input. In this paper, we update these predictions to include HQE parameters up to $1/m_b^3$, which have a relatively large impact. These higher-order terms were first studied in [12], but this reference misses some terms in the ρ_D^3 coefficient. Here we correct these results. We point out that numerically, the difference between our results and [12] is small. In light of the tensions in ratios of the exclusive $B \to D^{(*)}\ell\bar{\nu}_\ell$ versus $B \to D^{(*)}\tau\bar{\nu}_\tau$ (see e.g. [13] for a recent review on semileptonic τ modes), we stress the importance of an independent cross-check in the inclusive channel. For this, the SM predictions derived in this short letter are vital. These predictions can be used in the search for new physics, especially in the tau sector where new measurements are expected soon.

2 Inclusive decay of $b \to c\ell\bar{\nu}_\ell$ with massive leptons

To calculate the inclusive $b \to c$ semileptonic rate, we employ the standard heavy-quark expansion (HQE). This allows us to perform an operator product expansion (OPE) for the triple differential rate in the lepton (neutrino) energy $E_{\ell(\nu)}$ and the dilepton invariant mass q^2 as

$$\frac{d\Gamma}{dE_\ell dq^2 dE_\nu} = \frac{G_F^2|V_{cb}|^2}{16\pi^3} L_{\mu\nu} W^{\mu\nu} \,. \tag{2.1}$$

Here $L_{\mu\nu}$ is the lepton tensor and $W^{\mu\nu}$ the hadronic tensor as defined in e.g. [14]. Expressing the $W^{\mu\nu}$ tensor in Lorentz scalars as usual then gives

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\ell \mathrm{d}q^2 \mathrm{d}E_\nu} = \frac{G_F^2|V_{cb}|^2}{2\pi^3} \Bigg[& q^2 W_1 + \left(2E_\ell E_\nu - \frac{q^2}{2}\right) W_2 + q^2 \left(E_\ell - E_\nu\right) W_3 \\ & \frac{1}{2} m_\ell^2 \left(-2W_1 + W_2 - 2\left(E_\nu + E_\ell\right) W_3 + q^2 W_4 + 4E_\nu W_5\right) - \frac{1}{2} m_\ell^4 W_4 \Bigg] \,,\end{aligned} \tag{2.2}$$

where we have omitted explicit θ-functions (see [15]).

In general, for $B \to X_c\mu\bar{\nu}_\mu$ and $B \to X_c e\bar{\nu}_e$, lepton masses are neglected. However, for the much heavier decay involving the τ lepton: $B \to X_c\tau\bar{\nu}_\tau$, such an approximation cannot be made. We calculated the total inclusive rate including lepton masses. This calculation differs from the standard case, as now also the structure functions W_4 and W_5 in (2.2) contribute and because the phase space boundaries are affected. We refer to [12, 15] for details.

Considering terms up to $1/m_b^3$, we write the total rate as

$$\Gamma(B \to X_c\ell\bar{\nu}_\ell) = \Gamma_0 \left[C_0^{(0)} + \frac{\alpha_s}{\pi} C_0^{(1)} + C_{\mu_\pi^2}^{\perp} \frac{(\mu_\pi^2)^\perp}{m_b^2} + C_{\mu_G^2}^{\perp} \frac{(\mu_G^2)^\perp}{m_b^2} + C_{\rho_D^3}^{\perp} \frac{(\rho_D^3)^\perp}{m_b^3} + C_{\rho_{LS}^3}^{\perp} \frac{(\rho_{LS}^3)^\perp}{m_b^3} \right], \tag{2.3}$$

where the coefficients depend on

$$\rho \equiv m_c^2/m_b^2\,, \qquad \eta \equiv m_\ell^2/m_b^2\,, \tag{2.4}$$

and

$$\Gamma_0 \equiv \frac{G_F^2|V_{cb}|^2 m_b^5}{192\pi^3}\,(1+A_{\rm ew}), \tag{2.5}$$

which includes the electroweak correction $A_{\rm ew} = 0.014$ [16].

We define the nonperturbative parameters as (see e.g. [17])

$$2m_B\,(\mu_\pi^2)^\perp \equiv -\langle B|\bar{b}_v(iD_\rho)(iD_\sigma)b_v|B\rangle \Pi^{\rho\sigma}\,, \tag{2.6}$$

$$2m_B\,(\mu_G^2)^\perp \equiv \frac{1}{2}\langle B|\bar{b}_v\,[iD_\rho, iD_\lambda]\,(-i\sigma_{\alpha\beta})b_v|B\rangle \Pi^{\alpha\rho}\Pi^{\beta\lambda}, \tag{2.7}$$

$$2m_B\,(\rho_D^3)^\perp \equiv \frac{1}{2}\langle B|\bar{b}_v\,[iD_\rho,[iD_\sigma, iD_\lambda]]\,b_v|B\rangle \Pi^{\rho\lambda}v^\sigma, \tag{2.8}$$

$$2m_B\,(\rho_{LS}^3)^\perp \equiv \frac{1}{2}\langle B|\bar{b}_v\,\{iD_\rho,[iD_\sigma, iD_\lambda]\}\,(-i\sigma_{\alpha\beta})b_v|B\rangle \Pi^{\alpha\rho}\Pi^{\beta\lambda}v^\sigma\,, \tag{2.9}$$

where

$$\Pi_{\mu\nu} = g_{\mu\nu} - v_\mu v_\nu\,. \tag{2.10}$$

The above definitions differ from e.g. [2, 18, 19] where the full covariant derivative was used and not only the spatial component as above, linked via $iD_\mu = v_\mu ivD + D_\perp$. To differentiate, we therefore add a $\perp$ superscript to HQE parameters. The relation between the "perped" and full covariant derivative parameters is

$$(\mu_G^2)^\perp = \mu_G^2 + \frac{\rho_D^3 + \rho_{LS}^3}{m_b}\,, \tag{2.11}$$

while $(\mu_\pi^2)^\perp = \mu_\pi^2, (\rho_{LS}^3)^\perp = \rho_{LS}^3$ and $(\rho_D^3)^\perp = \rho_D^3$ up to terms of order $1/m_b^3$ (see discussion in appendix A of [19]).

We list all coefficients, except $C_0^{(1)}$ in appendix A, for completeness. Setting $\eta \to 0$, reproduces the well-known rate [18, 20, 21]

The coefficients agree with [12] (and previous results in [22, 23] for $C_0, C_{\mu_\pi^2}$ and $C_{\mu_G^2}$) up to a difference in the $C_{\rho_D^3}$. The discrepancy with [12] arises due to the more involved integrations which now contain additional delta functions. For the total rate, where no cut on lepton energy is required, it is easiest to first perform the integration over the lepton energy E_ℓ analytically (as the structure functions W do not depend on E_ℓ.). In the limit $\rho = \eta$, our calculation can be checked and agrees with [24]. We have also contacted the authors of [12], who now agree with our results.[1] Finally, after finalizing this paper, also [26] appeared in which the α_s corrections to the ρ_D^3 coefficient for massive leptons was calculated. At LO, this paper reproduces our results.

We recalculated the perturbative corrections for the partonic rate $C_0^{(1)}$ which agree with [27, 28]. Our analysis does not include α_s^2 corrections, which are known [29] but only

[1]After finalizing our paper, we were made aware of [25] which agrees with the calculation in [12]. We assume that the difference with our result arises from the same reasons outlined before. We note that also the recent [26] agrees with our result.

m_b^{kin}	(4.573 ± 0.012) GeV
$\overline{m}_c(2\,\text{GeV})$	(1.092 ± 0.008) GeV
$\big(\mu_\pi^2(\mu)\big)_{\text{kin}}$	(0.477 ± 0.056) GeV^2
$\big(\mu_G^2(\mu)\big)_{\text{kin}}$	(0.306 ± 0.050) GeV^2
$\big(\rho_D^3(\mu)\big)_{\text{kin}}$	(0.185 ± 0.031) GeV^3
$\big(\rho_{LS}^3(\mu)\big)_{\text{kin}}$	(-0.130 ± 0.092) GeV^3
V_{cb}	$(42.16 \pm 0.51) \cdot 10^{-3}$

Table 1. Numerical inputs taken from [1], where the HQE parameters are defined in the perp basis. For the charm mass, we use the $\overline{\text{MS}}$ scheme at 2 GeV. All other hadronic parameters are in the kinetic scheme at $\mu = 1$ GeV.

available for fixed m_b/m_c. To fully include such effects in a state-of-the-art manner, a new analysis is required. We briefly discuss these corrections in the following. We note that for $\eta = 0$, these corrections are even known up to α_s^3 [30].

3 SM predictions for inclusive rates including masses

With the coefficients C_i for the total rate, we can now in principle predict the branching ratios for semileptonic $b \to c$ decays. However, the light lepton decays and their moments are used to determined the HQE parameters and V_{cb}. Therefore, such predictions are not very instructive for light mesons. For those, we therefore restrict ourselves to ratios of semileptonic modes. For the tau modes, we also discuss the total branching ratio.

As is customary, we work in the kinetic mass scheme, which can be related to the pole mass via a perturbative series [31–33]. For our numerical analysis, we use the input values listed in table 1 obtained from [1] obtained from leptonic energy and hadronic invariant mass moments. These values can be compared with those obtained from a recent analysis using q^2 moments [2]. In the default analysis of [2], also $1/m_b^4$ terms were included, such that the HQE elements cannot directly be compared. However, table 6 provides the HQE parameters up to $1/m_b^3$ in the full covariant derivative basis, which can be directly compared. We note that the values for m_b and m_c are similar, as these are used as constraints on the fit in [2]. However, as was discussed in [2], especially the value of $\rho_D^3 = 0.03 \pm 0.02\,\text{GeV}^3$ differs from the value of [1] quoted in table 1. The difference between the two should be clarified, possibly by performing a combined analysis of lepton energy, hadronic invariant mass and q^2 moments. However, for our current analysis, because the uncertainties on the HQE parameters from the q^2 analysis are somewhat larger than those from [1], we use the latter here as inputs. In the next section, we comment on how our numerical results would differ if the values of [2] were used instead.

3.1 Lepton flavour universality ratios

We define the ratios $R_{\mu/e}$ as in (1.1) and define equivalently $R_{\tau/\mu}$ and $R_{\tau/e}$. In such ratios, V_{cb} drops out, but the HQE parameters do not completely, due to different mass effects.

	$R_{\tau/\mu}(X_c)\cdot 10^{-2}$	$R_{\tau/e}(X_c)\cdot 10^{-2}$	$R_{\mu/e}(X_c)\cdot 10^{-2}$
$\xi_{\rm LO}$	23.557	23.429	99.458
$\xi_{\rm NLO}$	5.446	5.451	0.144
$\xi_{\mu_G^2}$	-2.165	-2.161	-0.0315
$\xi_{\mu_\pi^2}$	0	0	0
$\xi_{\rho_{LS}^3}$	0.4735	0.4726	0.0068
$\xi_{\rho_D^3}$	-6.785	-6.765	-0.0709

Table 2. SM predictions for the inclusive LFU ratios. We list the different contributions independently of the value of the HQE parameters and separated according to (3.1).

We split the contributions $R(X_c)$ according to

$$R\left(X_c\right) = \xi_{\rm LO} + \xi_{\rm NLO}\left(\frac{\alpha_s}{\pi}\right) + \xi_{\mu_G^2}\left(\mu_G^2\right)^{\perp} + \xi_{\mu_\pi^2}\left(\mu_\pi^2\right)^{\perp} + \xi_{\rho_{LS}^3}\left(\rho_{LS}^3\right)^{\perp} + \xi_{\rho_D^3}\left(\rho_D^3\right)^{\perp}. \tag{3.1}$$

Using then the input for m_b and m_c as in table 1, we find the SM predictions as listed in table 2. The coefficients ξ can then be used to obtain $R(X_c)$ for any set of HQE parameters. We note that μ_π^2 completely drops out in such ratios, because it can be absorbed into the partonic rate because of reparametrization invariance (see e.g. [18]). The effect of ρ_D^3 is relatively large even though this is a $1/m_b^3$ contribution. We do not include an additional uncertainty for missed higher-order terms of order $1/m_b^4$ and beyond.

Using the HQE parameters as listed in table 1, we find for the light leptons

$$R_{\mu/e}(X_c)|_{{\rm NLO}+1/m_b^2+1/m_b^3} = 0.9945 \pm 0.0001\,. \tag{3.2}$$

The uncertainty is obtained by combining all uncertainties of the input parameters in quadrature. In addition, we vary the scale of $\alpha_s(\mu)$ from $m_b/2 < \mu < 2m_b$.

For the τ modes, we find

$$\begin{aligned} R_{\tau/\mu}(X_c)|_{{\rm NLO}+1/m_b^2+1/m_b^3} &= 0.220 \pm 0.004 \\ R_{\tau/e}(X_c)|_{{\rm NLO}+1/m_b^2+1/m_b^3} &= 0.218 \pm 0.004\,. \end{aligned} \tag{3.3}$$

This is in agreement with previous determination in [34], which includes terms up to $1/m_b^2$ in the 1S-scheme:

$$R(X_c)_{\rm FLR} = 0.223 \pm 0.004\,. \tag{3.4}$$

In this case, the uncertainty is dominated by m_b and λ_1 (i.e. the HQE element in the infinite mass limit) and includes an additional uncertainty of half of the α_s^2 term. It however does not include an additional uncertainty due the missed $1/m_b^3$ terms.

Finally, also a calculation of only the partonic rates at $\mathcal{O}(\alpha_s^2)$ exists [29]

$$R(X_c)_{\rm BM} = 0.237 \pm 0.031\,, \tag{3.5}$$

JHEP11(2022)007

which is based on the on-shell scheme. It was found that α_s^2 effects in the $R_{\tau/\ell}(X_c)$ ratio are very small. While the ratio of leading order decay rates is a rapidly changing function of m_b, m_c and m_τ, radiative corrections to $\mathcal{B}(B \to X_c \tau \nu)$ and $\mathcal{B}(B \to X_c \ell \nu)$ are correlated, so they cancel out in the ratio that is largely independent of the quark masses. Here we do not include these α_s^2 effects as [29] only provides them at fixed m_c/m_b. However, we have verified that the α_s^2 corrections are only 2−3 % of the NLO order contribution. Therefore, our uncertainty estimate obtained by varying α_s accounts for these effects. We also note that our α_s corrections are half of those in [29], due to the switch to the kinetic scheme. In addition, there are α_s corrections to HQE parameters that are not written in (3.1) and not take into account. These corrections are known for massless leptons [35, 36]. For massive leptons they became available very recently [26], i.e. after finalizing our paper. The corrections of the α_s corrections to the chromomagnetic μ_G^2 and ρ_D^3 coefficients were found to be a the sub-percent level, and thus well within our uncertainty. Given the large dependence on the value of ρ_D^3 discussed below, which first has to be clarified, we leave update these theoretical predictions to a future study.

Finally, we comment on the numerical differences for our predictions if we would have used the inputs [2]. We note that for the q^2 analysis, the extracted μ_π^2 has a large uncertainty [2]. However, as this contribution drops out in the ratios this does not affect our predictions. For the τ modes, we find that the predictions shift by around 1σ. Specifically, we find

$$
\begin{aligned}
R_{\tau/\mu}(X_c)|^{q^2}_{\mathrm{NLO}+1/m_b^2+1/m_b^3} &= 0.225 \pm 0.004 \\
R_{\tau/e}(X_c)|^{q^2}_{\mathrm{NLO}+1/m_b^2+1/m_b^3} &= 0.224 \pm 0.004 \,,
\end{aligned}
\tag{3.6}
$$

where we have added a subscript indicating that these predictions use the HQE parameters from the q^2 moments in [2].

3.2 Ratios for semileptonic $B \to X$

Experimentally, in order to obtain the semileptonic $B \to X_c$, the $B \to X_u$ background has to be dealt with. On the other hand, as pointed out in [14], this V_{ub}^2/V_{cb}^2 suppressed contribution can also be calculated in the local OPE. Naively taking the $B \to X_c$ rate and setting $\rho \to 0$ works up to $1/m_b^2$, but at order $1/m_b^3$ additional four-quark operators (weak annihilation) have to be introduced that cure the divergence arising in the ρ_D^3 term (see e.g. [37] for references and discussions). For charm, such effects were studied in [38] using semileptonic D meson data from CLEO [39]. For $B \to X_u$, this issue will be discussed specifically in an upcoming publication [40]. However, at the moment, we can make a reliable estimate for the $R(X)$ ratio by calculating the $B \to X_u$ effects by setting $\rho_D^3 \to 0$. We then have

$$
\Gamma(B \to X \ell \bar{\nu}_\ell) = \Gamma(B \to X_c \ell \bar{\nu}_\ell) + \left(\frac{|V_{ub}|}{|V_{cb}|}\right)^2 \Gamma(B \to X_c \ell \bar{\nu}_\ell)|_{\rho \to 0, \rho_D^3 \to 0} \,. \tag{3.7}
$$

To derive ratios of the $B \to X$ semileptonic rates, we use the exclusive V_{ub} determination from [41]:

$$
V_{ub}|_{\mathrm{excl.}} = (3.77 \pm 0.15) \cdot 10^{-3} \,, \tag{3.8}
$$

JHEP11(2022)007

which is in agreement at the $1-2\sigma$ level with the recent inclusive determinations [42]. For V_{cb}, we take the recent inclusive determination in $V_{cb} = (42.16 \pm 0.51) \cdot 10^{-3}$ [1].

We then find

$$R_{\tau/\mu}(X) = 0.221 \pm 0.004\,, \tag{3.9}$$
$$R_{\tau/e}(X) = 0.220 \pm 0.004\,, \tag{3.10}$$
$$R_{\mu/e}(X) = 0.994 \pm 0.001\,. \tag{3.11}$$

We do not quote the $R(X_u)$ as there we do not have the V_{ub}^2 suppression. As such, weak annihilation and ρ_D^3 effects may play a bigger role.

Finally, we note that experimentally, usually a lower cut on the lepton energy E_ℓ employed. Alternatively, also a q^2 cut can be imposed, as suggested first in [19], where q^2 moments of the spectrum are advertised. A q^2 cut is easier to implement for the α_s corrections, therefore we also quote ratios with such a cut. Here we take $q^2_{\rm cut} = 3\,{\rm GeV}^2$ as a default cut. The full expression with an arbitrary $q^2_{\rm cut}$ can be provide by the authors. We find using the inputs in table 1 of [1]

$$R_{\tau/\mu}(X)_{q^2_{\rm cut}} = 0.352 \pm 0.004\,, \tag{3.12}$$
$$R_{\tau/e}(X)_{q^2_{\rm cut}} = 0.352 \pm 0.004\,, \tag{3.13}$$
$$R_{\mu/e}(X)_{q^2_{\rm cut}} = 0.999 \pm 0.001\,. \tag{3.14}$$

We also explicitly provide the ratios using the masses and HQE parameters listed in table 6 of [2] and $V_{cb} = (42.69 \pm 0.63) \cdot 10^{-3}$ [2]. We obtain

$$R_{\tau/\mu}(X)_{q^2_{\rm cut}} = 0.359 \pm 0.005\,, \tag{3.15}$$
$$R_{\tau/e}(X)_{q^2_{\rm cut}} = 0.358 \pm 0.005\,, \tag{3.16}$$
$$R_{\mu/e}(X)_{q^2_{\rm cut}} = 0.998 \pm 0.002\,, \tag{3.17}$$

which agrees with the determinations above at the $1-2\sigma$ level as expected, but with larger uncertainties. This is due to the larger relative uncertainty on ρ_D^3.

3.3 Inclusive decay of $b \to c\tau\bar{\nu}_\tau$

Using (2.3), we update the SM predictions for the branching ratio of the τ-mode. Splitting the branching ratio as in (3.1) and taking $V_{cb} = (42.16 \pm 0.51) \cdot 10^{-3}$ [1], and m_b and m_c from table 1, we find the contributions ξ given in table 3. These results use the averaged decay rate $\tau_B = 1.579$ ps [10], which can be adjusted for the $B^{+,0}$ by multiplying with $\tau_{B^{+,0}}/\tau_B$. As the branching ratio depends V_{cb}^2, predictions using $V_{cb} = (41.69 \pm 0.63) \cdot 10^{-3}$ [2] can be easily obtained by a re-scaling.

Using the inputs for the HQE parameters in table 1, we can calculating the branching ratio directly from the OPE:

$$\begin{aligned}
&\mathcal{B}(B \to X_c\tau\bar{\nu}_\tau)_{\rm OPE} \\
&= \Big(2.34 \pm 0.07|_{m_b} \pm 0.03|_{m_c} \pm 0.02|_{\mu_G^2} + 0.01|_{\rho_{LS}^3} + 0.04|_{\rho_D^3} + 0.06|_{\alpha_s} + 0.05|_{V_{cb}}\Big)\,\% \\
&= (2.34 \pm 0.13)\%\ ,
\end{aligned} \tag{3.18}$$

	$\mathcal{B}(B \to X_c \tau \bar{\nu}_\tau)$ [%]	$\mathcal{B}(B \to X \tau \bar{\nu}_\tau)$ [%]
$\xi_{\rm LO}$	3.042	3.095
$\xi_{\rm NLO}$	-3.064	-3.020
$\xi_{\mu_G^2}$	-0.557	-0.564
$\xi_{\mu_\pi^2}$	-0.0727	-0.074
$\xi_{\rho_{LS}^3}$	0.122	0.123
$\xi_{\rho_D^3}$	-1.408	-1.408

Table 3. Predictions for the branching ratio within the local OPE, using $V_{cb} = (42.16 \pm 0.51) \cdot 10^{-3}$ [1] and split according to (3.1). We quote the flavour-averaged rate. Predictions for the charged or neutral B decay can be obtained by multiplying with $\tau_{B^{+,0}}/\tau_B$.

where we specify the different contributions to the uncertainty and in the last line we summed these in quadrature. Again, we do not include an additional uncertainty due to missed higher-order terms. For completeness we also quote the B^+ and B^0 rates separately

$$\mathcal{B}(B^+ \to X_c^+ \tau \bar{\nu}_\tau) = (2.43 \pm 0.13)\% \,.$$
$$\mathcal{B}(B^0 \to X_c^0 \tau \bar{\nu}_\tau) = (2.25 \pm 0.13)\% \,. \tag{3.19}$$

Our value agrees with [12], despite a missed ρ_D^3 contribution in that paper. Finally, following the procedure outlined in section 3.2, we find the $B \to X$ rate as

$$\mathcal{B}(B \to X \tau \bar{\nu}_\tau) = (2.39 \pm 0.13)\,\% \,. \tag{3.20}$$

These determinations are in agreement with the LEP measurement of the inclusive branching fraction of the admixture of bottom baryons [10]

$$\mathcal{B}(b\text{-admix} \to X \tau \bar{\nu}_\tau) = (2.41 \pm 0.23)\% \,, \tag{3.21}$$

which only to leading order in the HQE can be interpreted as the individual hadron rates.

In addition, there exists an unpublished Belle measurement of the $R_{\tau/(e,\mu)}(X)$ [11]:

$$R(X) \equiv \frac{\mathcal{B}(B \to X \tau \bar{\nu}_\tau)}{\mathcal{B}(B \to X \ell \bar{\nu}_\ell)} = 0.298 \pm 0.022 \,, \tag{3.22}$$

where $\ell = \mu, e$. Comparing this with our estimate in (3.9), we observe a slight tension. Alternatively, we may also estimate the relation between $R(X)$ and $R(X_c)$, by subtracting the theoretically calculated rate. We find

$$R(X) = \begin{cases} R_{\tau/\mu}(X_c) \left(1 + 1.012 \dfrac{|V_{ub}|^2}{|V_{cb}|^2}\right) & \text{for } \ell = \mu \,, \\[2ex] R_{\tau/e}(X_c) \left(1 + 1.014 \dfrac{|V_{ub}|^2}{|V_{cb}|^2}\right) & \text{for } \ell = e \,. \end{cases} \tag{3.23}$$

JHEP11(2022)007

Therefore, we will interpret $R(X) = R(X_c)$. Comparing then (3.22) with our predictions in (3.3), we again observe a slight tension.

Besides calculating the rate directly from the OPE as in (3.18), we may also give predictions of the branching ratio by multiplying them with the measured flavor-averaged light-meson branching ratio. Following the detailed discussion in [13], we take

$$\mathcal{B}(B \to X_c \ell \bar{\nu}_\ell) = (10.48 \pm 0.13)\%, \tag{3.24}$$

which differs slightly from those quoted by [1] and [10]. Averaging our predictions for the muon and electron ratios in (3.3), and multiplying with (3.24), we find

$$\mathcal{B}(B \to X_c \tau \bar{\nu}_\tau)_{\text{Exp+OPE}} \equiv \mathcal{B}(B \to X_c \ell \bar{\nu}_\ell)\, R_{\tau/\ell}(X_c) = (2.30 \pm 0.05)\%\,, \tag{3.25}$$

which is in perfect agreement with, but has a much smaller uncertainty than our direct calculation in (3.18). Using the inputs in [2], and the ratios in (3.6), we find

$$\mathcal{B}(B \to X_c \tau \bar{\nu}_\tau)^{q^2}_{\text{Exp+OPE}} \equiv \mathcal{B}(B \to X_c \ell \bar{\nu}_\ell)\, R_{\tau/\ell}(X_c)|^{q^2} = (2.35 \pm 0.05)\%\,, \tag{3.26}$$

which agrees at 1σ level.

Similarly, we can convert the unpublished Belle measurement in (3.22). In [11], this is multiplied with the measured isospin-average branching fraction $\mathcal{B}(B \to X \ell \bar{\nu}_\ell) = (10.86 \pm 0.16)\%$ to obtain $\mathcal{B}(B \to X \tau \bar{\nu}_\tau) = (3.23 \pm 0.25)\%$. This is in tension with the value we find from the direct OPE calculation in (3.20). Using (3.22), we multiply with (3.24) to find

$$\mathcal{B}(B \to X_c \tau \bar{\nu}_\tau)_{\text{Belle}} = (3.12 \pm 0.23)\,\%. \tag{3.27}$$

Multiplying the previous theoretical determination of $R(X_c)$ in (3.4) [34] with (3.24) gives

$$\mathcal{B}(B \to X_c \tau \bar{\nu}_\tau)_{\text{FLR}} = (2.34 \pm 0.05)\%, \tag{3.28}$$

which is in agreement with the value reported in [43].

Finally, multiplying (3.5) with the branching ratio in (3.24) gives

$$\mathcal{B}(B \to X_c \tau \bar{\nu}_\tau)_{\text{BM}} = (2.47 \pm 0.04)\%. \tag{3.29}$$

It is also interesting to compare our inclusive predictions with a sum over exclusive. To this extend, we follow the recent [43]. Using the HFLAV-averaged SM predictions for $R(D)$ and $R(D^*)$ and the measured rates for the light-modes, combined with the prediction for $\mathcal{B}(B \to D^{**} \ell \bar{\nu}_\ell)$ [44], they find [43]

$$\sum_{X_c \in D^{(*,**)}} \mathcal{B}(B \to X_c \tau \bar{\nu}_\tau) = (2.14 \pm 0.06)\,\%\,. \tag{3.30}$$

Interestingly, this sum over exclusive modes does not saturate our calculated fully inclusive rate. In fact, using the HQE inputs from the q^2 moment analysis predict a larger branching ratio in (3.27) leaving more room for additional states to saturate the rate. We summarize and visualize our findings in figure 1.

JHEP11(2022)007

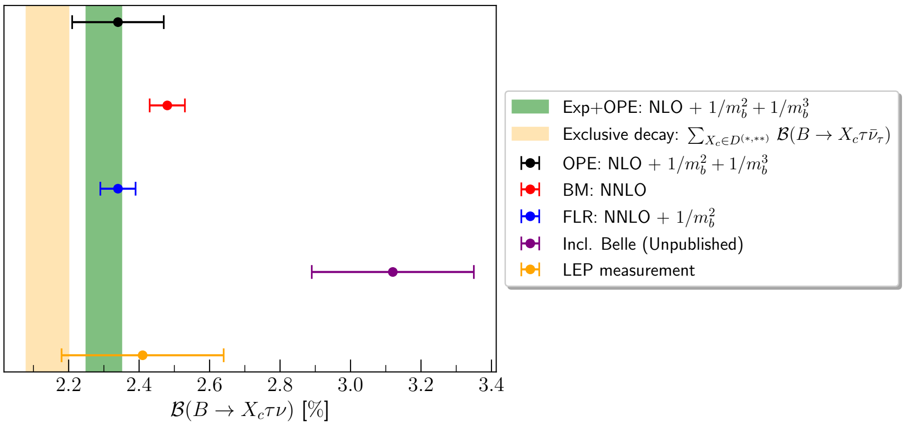

Figure 1. Comparison of our predictions for the branching ratio $\mathcal{B}(B \to X_c\tau\nu)$ with previous determinations and with the sum over exclusives from [43]. We also quote the measurements of LEP and the unpublished Belle measurement (see text for details).

4 Conclusion

We calculated the SM predictions for the lepton flavour universality ratios of semileptonic inclusive B decays. In these predictions, we only considered the mass effects, and included HQE parameters up to $1/m_b^3$. We corrected a previous calculation in [12], which missed some terms in the ρ_D^3 contribution.

In addition, we present updated results of the Standard Model for the branching ratio of the $B \to X_c\tau\bar\nu_\tau$ decay. Experimentally, for this rate only a LEP measurement and an unpublished Belle analysis are available. In light of the discrepancies between data and experiment in the universality ratios of exclusive semileptonic $B \to D^{(*)}$ update measurements of this observable are highly wanted. A detailed analysis of the effect of new physics operators on inclusive semitauonic decays is in progress [45].

Acknowledgments

We thank Florian Bernlochner for suggesting this project in light of upcoming measurements. Additionally, we thank him, Thomas Mannel and Matteo Fael for their comments. We thank A. Rusov for discussions about [12]. This research was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under grant 396021762 - TRR 257.

A Total rate

In this appendix, we explicitly give the coefficients of the rate in (2.3).

We note that all these coefficients except $C_{\rho_D^3}$ agree with [12] when transforming basis from the spatial derivative "perped" basis used here to the full covariant derivative basis via (2.11). Explicitly this means that

$$C_{\mu_\pi^2} = C^\perp_{\mu_\pi^2}\,, \quad C_{\mu_G^2} = C^\perp_{\mu_G^2}\,, \quad C_{\rho_D^3} = C^\perp_{\rho_D^3} + C^\perp_{\mu_G^2}\,, \quad C_{\rho_{LS}^3} = 0\,. \tag{A.1}$$

We find

$$C_0^{(0)} = R\left[1-7\rho-7\rho^2+\rho^3-\left(7-12\rho+7\rho^2\right)\eta-7\left(1+\rho\right)\eta^2+\eta^3\right] \qquad (A.2)$$
$$-12\left[\rho^2\ \ln\frac{(1+\rho-\eta-R)^2}{4\rho}-\eta^2\ln\frac{(1+\eta-\rho+R)^2}{4\eta}-\rho^2\eta^2\ \ln\frac{(1-\rho-\eta-R)^2}{4\rho\eta}\right],$$

$$C^{\perp}_{\mu_G^2} = \frac{R}{2}\left[-3+5\rho-19\rho^2+5\rho^3+\left(5+28\rho-35\rho^2\right)\eta-\left(19+35\rho\right)\eta^2+5\eta^3\right] \qquad (A.3)$$
$$-6\left[\rho^2\ \ln\frac{(1+\rho-\eta-R)^2}{4\rho}-\eta^2\ln\frac{(1+\eta-\rho+R)^2}{4\eta}-5\rho^2\eta^2\ \ln\frac{(1-\rho-\eta-R)^2}{4\rho\eta}\right],$$

In addition, we have

$$C^{\perp}_{\mu_\pi^2} = -\frac{C_0}{2}\,, \qquad C^{\perp}_{\rho_{LS}^3} = -C^{\perp}_{\mu_G^2} \qquad (A.4)$$

$$C^{\perp}_{\rho_D^3} = \frac{R}{6}\left\{77+5\rho^3+\rho^2\left(13-35\eta\right)+13\eta-59\eta^2+5\eta^3-\rho\left(11+12\eta+35\eta^2\right)\right\} \qquad (A.5)$$
$$+\left\{\eta^2\left(10\rho^2+8\eta-2\right)\ \ln\left[\frac{(1-\rho-\eta-R)^2}{4\eta\rho}\right]\right.$$
$$\left.+\left(8+6\rho^2-8\eta-6\eta^2\right)\ \ln\left[\frac{(1+\rho-\eta-R)^2}{4\rho}\right]\right\},$$

where $R=\sqrt{\rho^2+(-1+\eta)^2-2\rho\left(1+\eta\right)}$.

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JHEP11(2022)007