The $AdS_3 \times S^1$ chiral ring

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ABSTRACT: We study $AdS_3 \times S^1 \times Y$ supersymmetric string theory backgrounds with Neveu-Schwarz-Neveu-Schwarz flux that are dual to $\mathcal{N} = 2$ superconformal theories on the boundary. We classify all worldsheet vertex operators that correspond to space-time chiral primaries. We compute space-time chiral ring structure constants for operators in the zero spectral flow sector using the operator product expansion in the worldsheet theory. We find that the structure constants take a universal form that depends only on the topological data of the $\mathcal{N} = 2$ superconformal theory on Y.

Contents

1   Introduction

Holographic duality is likely a generic property of quantum gravity [1, 2]. While convincing examples of the duality are known [3], the duality remains ill-understood. Understanding holographic duality in detail could imply resolving the information paradox, or finding a unitary quantization of gravity in de Sitter space-time. Thus, there is promise in understanding holographic dualities in greater depth. For instance, one can enlarge the class of examples of holographic dualities, or attempt to understand those examples that we know in more detail.

The holographic string theory backgrounds that are most easily exactly quantized in the inverse curvature expansion in the bulk may be the three-dimensional anti-de Sitter backgrounds with Neveu-Schwarz-Neveu-Schwarz flux. The world sheet model is a Wess-Zumino-Witten model on the universal cover of the non-compact group $\mathrm{SL}(2,\mathbb{R})$, and as such may be exactly solvable. Indeed, a lot of progress has been made in explicitly solving the theory (see e.g. [4–6]). Moreover, in the presence of extended supersymmetry, the boundary conformal field theory can be twisted to give rise to a topological conformal field theory. It is a reasonable goal to attempt to prove the holographic duality between the topologically twisted bulk theory and the topological conformal field theory on the boundary in this setting [7].

String theories in three-dimensional anti-de Sitter space with NS-NS flux admit $\mathcal{N}=2$ superconformal symmetry when they are of the form $AdS_3 \times S^1 \times Y$ where Y is a $N=2$ world sheet superconformal field theory [8]. We determine the full spectrum of spacetime chiral primaries in these backgrounds in section 2. We compute structure constants of the space-time chiral ring in section 3. Our technique is to compute the space-time operator products directly, using world sheet techniques. The underlying motivation is to perform the calculation in the bulk in a spirit that will eventually allow for topological twisting. The reward we reap is that the calculation becomes simple and since it is only based on the necessary properties of the $\mathcal{N}=2$ superconformal background, it is generally valid. We thus extend calculations performed in backgrounds with $\mathcal{N}=4$ superconformal symmetry [9, 10] that provided powerful checks of the holographic correspondence. The $\mathcal{N}=2$

superconformal backgrounds are less constrained and could exhibit a richer set of structure constants. We discuss our results and conclude in section 4. In a set of appendices we summarize many details regarding the worldsheet conformal field theory on the $AdS_3 \times S^1 \times Y$ background, and in particular, lay bare some generic properties of $N = 2$ superconformal world sheet theories.

2 The space-time chiral primaries

In this section, we classify the spectrum of space-time chiral primary fields in AdS_3 backgrounds with $\mathcal{N} = 2$ spacetime superconformal symmetry. The classification extends results obtained in various approaches in references [11–14] among others. The chiral primaries form a basis for the chiral ring [15], a topological and non-trivial observable of the extended superconformal field theory.

2.1 The extended superconformal backgrounds

We study a class of string theory backgrounds with spacetime $\mathcal{N} = 2$ superconformal symmetry. When these backgrounds carry only NS-NS flux and can be described by a world sheet conformal field theory, they were argued to take the form $AdS_3 \times S^1 \times Y$ [8] where the factor Y corresponds to a world sheet theory with $N = 2$ world sheet conformal symmetry.[1] The extended spacetime supersymmetry forces the presence of a $\mathrm{U}(1)_R$ symmetry that in turn is geometrically incarnated as a circle factor in the spacetime. These backgrounds provide a large class of theories in which to study holography with extended supersymmetry and thus provide a fertile testing ground for attempting to prove a topologically twisted version of the correspondence between quantum gravity in anti-de Sitter space and conformal field theory [7].

The world sheet central charge of the $AdS_3 \times S^1 \times Y$ theory is critical when it satisfies

$$c_{\text{tot}} = 3 + \frac{6}{k} + 1 + 4 \times \frac{1}{2} + c_Y = 15 \,. \tag{2.1}$$

We have included four world sheet fermion partners for the $AdS_3 \times S^1$ bosons. The level k is equal to the square of the AdS_3 cosmological constant length scale divided by α' (which in turn equals 2π times the fundamental string tension). We conclude that the critical world sheet model contains a factor theory Y with central charge

$$c_Y = 9 - \frac{6}{k} \,. \tag{2.2}$$

We take the radius of the circle equal to $R = \sqrt{k\alpha'}$ which equals the radius of curvature of the AdS_3 space-time.

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[1]It is clear that one can also allow orbifolds thereof and that there may be further wiggle room (e.g. by demanding only chiral $\mathcal{N} = 2$ symmetry). Still, the $AdS_3 \times S^1$ geometry is a natural geometric realization of extended superconformal symmetry.

2.2 The spectrum

We work in the Neveu-Schwarz-Ramond formalism on the world sheet and in a first instance ignore light-cone oscillator degrees of freedom. We consider vertex operators that factorize according to the spacetime factors. The world sheet conformal dimension L_0, the circle left momentum p_L, the space-time energy H and the space-time R-charge $Q_{\rm st}$ are given by the expressions — see e.g. [14] for the detailed background –:

$$L_0 = -\frac{h\,(h-1)}{k} - w\,(h+r) - \frac{(k+2)\,w^2}{4} + \frac{1}{2}\,(f+a)^2 + \frac{p_L^2}{2} + h_Y + N\,, \qquad (2.3)$$

$$p_L = \frac{n'}{R} + \frac{w'R}{\alpha'} = \frac{n'}{\sqrt{2k}} + \sqrt{\frac{k}{2}}w' = \frac{1}{\sqrt{2k}}\,(n'+kw') \qquad (2.4)$$

$$H = h + r + \frac{k+2}{2}w + f + a \qquad (2.5)$$

$$Q_{\rm st} = \sqrt{2k}p_L \qquad (2.6)$$

The quantum numbers appearing in these equations are as follows. First of all, we put $a = 0$ for the world sheet NS sector and $a = -\frac{1}{2}$ in the Ramond sector. The quantum numbers (h, r, w) arise from the $sl(2, \mathbb{R})$ world sheet theory and h is the world sheet $sl(2, \mathbb{R})$ spin, w parameterizes the spectral flow and r measures an elliptic $sl(2, \mathbb{R})$ spin component. We restrict to discrete representations of $sl(2, \mathbb{R})$ and will set $r = 0$ — see [14] for a detailed justification. The super ghosts cancel one of the two pairs of fermions and the quantum number f corresponds to the fermion number arising from the remaining pair of fermions in the $AdS_3 \times S^1$ sector — one takes them to lie in the spatial AdS_3 directions. The quantum numbers (n', w') are the momentum and winding along the circle direction. The conformal dimension of the operator arising from the compact conformal field theory Y is denoted h_Y and N is the oscillator contribution to the conformal dimension. The momentum p_L is determined in terms of the momentum and winding along the circle S^1. The spacetime energy H is the J_0^3 time translation eigenvalue in the (supersymmetric) AdS_3 theory while the spacetime R-charge is proportional to the left-moving momentum along the S^1. Similar formulas hold in the right-moving sector. The space-time spectrum is determined by solving on-shell constraints (among others on L_0) in the Hilbert space of the world sheet conformal field theories.

2.3 The spacetime chiral primaries

Our goal is to classify the single fundamental string states that are spacetime chiral primaries. These states satisfy the equality [15]:

$$2H = Q_{\rm st}\,. \qquad (2.7)$$

As physical states, they are subject to the on-shell condition $L_0 = \frac{1}{2}$. In a first step, we show that the space-time chiral primary equation (2.7) and the world sheet on-shell condition can be rendered independent of the spectral flow parameter w by shifting the world sheet fermion number $f = f' - w$, or in other words, by applying spectral flow to the fermion sector as well. Firstly, the chiral primary condition (2.7) allows us to write the left

circle momentum p_L entirely in terms of the AdS_3 quantum numbers. Substituting this into the equation for the world sheet conformal dimension we obtain

$$L_0 = \frac{h(1 + 2a + 2f')}{k} + \frac{(k+2)}{2k}(f' + a)^2 + h_Y + N \, . \tag{2.8}$$

We see that indeed, the condition is independent of the spectral flow parameter w.

Spacetime chiral primaries can arise from both the world sheet Ramond and NS sectors and we discuss each of these in turn. For simplicity, we work chirally — the physical spectrum is obtained by suitably combining left and right moving world sheet excitations in a modular invariant, GSO projected theory.

2.3.1 The Ramond sector

In the Ramond sector we set the parameter $a = -1/2$. Substituting this into the world sheet conformal dimension L_0 (2.8) we find that it is minimized for fermion number $f' = 0$. We attain the minimum:

$$L_0 = \frac{k+2}{8k} + h_Y + N \, . \tag{2.9}$$

In the Ramond sector of the theory Y, the conformal dimension is bounded from below:

$$h_Y \geq \frac{c_Y}{24} = \frac{3}{8} - \frac{1}{4k} \, . \tag{2.10}$$

For a space-time chiral primary state, we therefore find that the world sheet conformal dimension satisfies $L_0 \geq 1/2$. The physical state condition $L_0 = 1/2$ is satisfied when there are no oscillator excitations and *only* for Ramond ground states of the theory Y.

There are two more constraints to be satisfied involving the spacetime and world sheet R-charges. The first is that the spacetime R-charge is given by the momentum and winding on the circle. Combining this with the spacetime chiral primary condition we find:

$$2H = 2h - 1 + kw = Q_{\text{st}} = n' + kw' \, , \tag{2.11}$$

where we recall that w is the spectral flow number and (n', w') are the momentum and winding on the circle. The constraint can be rewritten in the form

$$2h - 1 = n' + k(w' - w) \, , \tag{2.12}$$

which states that the combination of spin $2h-1$ is an integer plus an integer multiple of k. For discrete states in AdS_3 the spins are constrained to lie in the interval $2h-1 \in (0, k]$. We conclude that there are only a finite number of spins h allowed. The last constraint in the classification problem of space-time chiral primaries in the world sheet Ramond sector is to implement the world sheet GSO projection. We discuss this constraint in subsection 2.4.2.

2.3.2 The Neveu-Schwarz sector

In the NS sector we set $a = 0$ and the world sheet conformal dimension takes the form

$$L_0 = \frac{(2f' + 1)h}{k} + \frac{k+2}{2k}(f')^2 + h_Y + N \, . \tag{2.13}$$

Extremizing with respect to the fermion number f' (while ignoring the mild implicit GSO dependence of the dimension h_Y on f'), we find that the fermion number f' takes the values 0 or -1.

Case 1. For the case $f' = 0$ we obtain the world sheet operator dimension $L_0 = \frac{h}{k} + h_Y$. The on-shell value is still $L_0 = 1/2$. Using the constraint on the allowed range of the spin h we obtain a bound on the conformal dimension h_Y of the operator factor in the theory Y:[2]

$$0 \le h_Y \le \frac{1}{2} - \frac{1}{2k} \,. \tag{2.14}$$

Furthermore, we need to implement the GSO constraint:

$$q_{\text{ws}} = 1 - 2h_Y + q_Y \in \text{odd integers} \,. \tag{2.15}$$

By combining the GSO constraint with the bound on the dimension h_Y we find that for a fixed world sheet R-charge q_Y only one dimension h_Y can solve the GSO constraint (because the range of $2h_Y$ is smaller than one). The unitarity of the $N = 2$ superconformal theory on Y implies the inequality $2h_Y \ge |q_Y|$. For positive world sheet R-charge, $q_Y \ge 0$, there are only the primaries satisfying $2h_Y = q_Y + 2\mathbb{Z}$ that can therefore solve the GSO constraint. Moreover, if the charge q_Y is positive, then by the constraint on $2h_Y$, which is less than one, we must have $2h_Y = q_Y$. We conclude that world sheet chiral primaries are the only states that satisfy the constraint in case the world sheet R-charge in the theory Y is positive.

For negative R-charge, we have $2h_Y \ge -q_Y$, or $-2h_Y \le q_Y$. We conclude that: $-q_Y \le 1 - 1/k$ in order for $2h_Y$ to exist. We therefore have the constraint

$$0 \ge q_Y \ge -1 + 1/k \,. \tag{2.16}$$

We add the bound on $1 - 2h_Y$ and on q_Y to find that:

$$-1 + 2/k \le 1 - 2h_Y + q_Y < 1 + 1/k \,. \tag{2.17}$$

For this to be an odd integer, we need $2h_Y = q_Y$ which implies that both are zero in a unitary theory Y. Thus, this is the identity operator in the unitary conformal field theory Y. It also falls into the previous category, with positive R-charge. Therefore, there are no extra space-time chiral primaries in this category. The only solutions to all the constraints that can give rise to a spacetime chiral primary is a world sheet chiral primary with conformal dimension h_Y in the range (2.14).

Case 2. For the case $f' = -1$ we find $L_0 = \frac{1}{2} + \frac{1-h}{k} + h_Y = \frac{1}{2}$ for the on-shell constraint. Solving this constraint and using the range of spins h, we find the same range (2.14) for the conformal dimension:

$$0 \le h_Y \le \frac{1}{2} - \frac{1}{2k} \tag{2.18}$$

The GSO constraint in this case is given by

$$q_{\text{ws}} = -1 + 2h_Y + q_Y \in \text{odd integers} \,. \tag{2.19}$$

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[2]The bound on the spin h gives a weaker lower bound. Unitarity of the Y-theory implies that $h_Y \ge 0$.

We proceed as before and observe that for a given charge q_Y there is only a single value of h_Y that can solve the GSO constraint. Using the unitarity bound of the theory Y, let us discuss the two cases of positive and negative R-charge. For $q_Y \leq 0$ only primaries satisfying $2h_Y = -q_Y + 2\mathbb{Z}$ can solve the GSO constraint. Given the constraint on $2h_Y$, we must therefore have a primary with $2h_Y = -q_Y$. Thus only world sheet anti-chiral primaries satisfy all the constraints in this case. For $q_Y \geq 0$ we have the unitarity constraint $2h_Y \geq q_Y$. The bound on h_Y thus implies a bound on q_Y:

$$0 \leq q_Y \leq 1 - \frac{1}{k} \, . \tag{2.20}$$

Adding this to the bound on $2h_Y$ we obtain

$$-1 \leq -1 + 2h_Y + q_Y \leq 1 - \frac{2}{k} \, . \tag{2.21}$$

This is the same combination that occurs in the expression of the world sheet R-charge and for this to be an odd integer we must have $2h_Y + q_Y = 0$. In conjunction with the unitarity bound for $q_Y \geq 0$, the only possibility is that both the dimension and the R-charge vanish. Thus, in the second case, only the anti-chiral primaries in the world sheet theory Y give rise to space-time chiral primaries.

Finally, let us point out that there is a strong upper bound on the conformal dimension of an operator in the theory Y that can feature in a space-time chiral primary. The upper bound on its conformal dimension is

$$h_Y \leq \frac{1}{2} - \frac{1}{2k} \, , \tag{2.22}$$

and the upper bound on the absolute value of its world sheet R-charge is therefore $|q_Y| \leq 1 - 1/k$. Note that it is smaller than one. Yet, the central charge of the theory Y is $c_Y = 9 - 6/k$ and therefore the bound is considerably stronger than the unitarity bound within the world sheet theory Y.

A corollary. We observe a distinction between the range in which the cosmological length scale is larger than the string scale, $k \geq 1$, and in which it is smaller, $k < 1$. In the latter case, we cannot obtain spacetime chiral primaries from the world sheet NS sector for a unitary theory Y because of the bound (2.14). An application of this corollary is that the space-time vacuum which lies in the NS sector is not among the normalizable spacetime chiral primaries for $k < 1$. This is the telltale signature of the phase transition from a thermal phase dominated by black holes at high energy to a phase governed by long strings [16]. See also [11, 17–19] for further discussion of the physics at and beyond the phase transition.

2.4 The covariant chiral primary vertex operators

So far we worked out the spectrum of spacetime chiral primaries in the absence of light cone oscillator excitations. That method provides a rather direct access to the spectrum of physical excitations. For the purpose of computing the spacetime operator product expansions, it is however useful to write the vertex operators for these states in a covariant form.

2.4.1 The Neveu-Schwarz sector

The covariant vertex operators in the left moving Neveu-Schwarz sector in picture number (-1) take the form [20, 21]:

$$O^{(-1)} = c \, e^{-\phi} \, O_{\text{matter}} \,. \tag{2.23}$$

The world sheet conformal dimension has the same expression as before, but we allow for light cone bosonic and fermionic oscillator excitations. The conformal dimension of the c ghost is -1, and the $e^{-\phi}$ ghost excitation has dimension $1/2$. Therefore the matter operator of the unintegrated vertex operator remains of dimension $1/2$. Light cone oscillator excitations in the matter operator would only raise the world sheet conformal dimension and are therefore incompatible with the space-time chiral primary constraint. We still need to check whether our excitation is BRST closed, and whether there are excitations that are BRST trivial. Given the light-cone analysis of spacetime chiral primaries, we suspect that the BRST cohomology comprises of all the states we found previously and none other.

In order to check this, we need to compute the action of $Q_{BRST} = 1/(2\pi i) \oint (cT + \gamma G)$ on these states. The first term gives zero because of the on-shell constraint $L_0 = \frac{1}{2}$ (and the absence of oscillators). The second term will only act when we have a non-trivial fermion excitation, and from our results, this can only be the fermion ψ^- in the transverse AdS_3 directions that lowers the space-time conformal dimension. If such a fermion is present, the condition for BRST closedness will be that the $sl(2,\mathbb{R})$ lowering operator j_0^- is zero on the bosonic state, i.e. that it is a j_0^3 lowest weight state. Our choice $r = 0$ for the j_0^3 momentum of the discrete $sl(2,\mathbb{R})$ representations guarantees that this is indeed the case — it precisely cements the choice of a lowest weight state in a discrete D^+ representation [14]. Moreover, in the compact theory Y, (anti-)chiral primaries are automatically annihilated by $G_{+1/2}$ and therefore, indeed, all the covariant states that we identified are BRST closed.

To prove that none of these states is BRST exact, it is sufficient to recall that the BRST charge does not change the space-time conformal dimension or charge, and therefore any relevant seed would need to be a space-time chiral primary. We have just proven that those are all BRST closed and therefore they cannot give rise to non-trivial BRST exact states.

We conclude that we have two classes of covariant NS sector states. The first class has $H = h + \frac{kw}{2}$ and in the left-moving sector it is given by the vertex operator:

$$O_{\text{NS}_1}^{a\,(-1)} = c \, e^{-\phi} \, \Phi_h^w \, \psi_w \, e^{i\sqrt{\frac{2}{k}}(h+\frac{kw}{2})X} \, V_{c.p.}^a \,. \tag{2.24}$$

The left-moving circle momentum is $p_L = \sqrt{\frac{2}{k}}(h + \frac{kw}{2})$ and the conformal dimension of the chiral primary in the Y sector is determined from the on-shell condition to be

$$h_Y^a = \frac{1}{2} - \frac{h}{k} \,. \tag{2.25}$$

We denote the chiral ring elements by $V_{c.p.}^a$ where $a = 1, \ldots \dim(\mathcal{R}_Y)$, where $\mathcal{R}_Y$ is the chiral ring of the $\mathcal{N} = 2$ superconformal world sheet conformal field theory Y.

Finally, we make a few remarks about the spectrally flowed operators we have defined. The spectral flow in the bosonic $sl(2,\mathbb{R})$ sector is analyzed in [22]. The spacetime conformal dimension associated to the flowed states is listed in equation (2.3). The spectral flow in

the fermionic sector is described in detail in [23]. We have denoted the vacuum in this spectrally flowed fermionic sector $|\tilde{0}\rangle$ as being created by the operator ψ_w. It corresponds to the state

$$\psi_w \longleftrightarrow |\tilde{0}\rangle = \frac{1}{k^{w/2}}\psi^-_{-w+\frac{1}{2}}\psi^-_{-w+\frac{3}{2}}\cdots\psi^-_{-\frac{1}{2}}|0\rangle\,. \tag{2.26}$$

It has world sheet fermion number $-w$ and world sheet conformal dimension $\frac{w^2}{2}$. It has zero spacetime R-charge and spacetime conformal dimension equal to $-w$.

The second class of NS sector states consists of states with space-time conformal dimension $H = h + \frac{kw}{2} - 1$ and the left-moving vertex operator takes the form

$$O^{\bar{a}\,(-1)}_{\mathrm{NS}_2} = c\ e^{-\phi}\ \Phi^w_h\ \psi_{w+1}\ e^{i\sqrt{\frac{2}{k}}(h+\frac{kw}{2}-1)X}\ V^{\bar{a}}_{a.c.p.}\,. \tag{2.27}$$

The momentum along the circle is determined in terms of the spacetime R-charge: $p_L = \sqrt{\frac{2}{k}}(h + \frac{kw}{2} - 1)$. The world sheet conformal dimension of the anti-chiral primary state in the sector Y is

$$h^{\bar{a}}_Y = \frac{h-1}{k}\,. \tag{2.28}$$

We have denoted the anti-chiral primaries by $V^{\bar{a}}_{a.c.p.}$, where the index $\bar{a}$ again labels world sheet ring elements. We will assume that we have a left world sheet $\mathrm{U}(1)_R$ conjugation symmetry and that the left chiral ring is isomorphic to the left anti-chiral ring.

We have denoted by ψ_{w+1} the single fermionic excitation on top of the ground state (2.26):

$$\psi_{w+1} \longleftrightarrow \frac{1}{\sqrt{k}}\psi^-_{-w-\frac{1}{2}}|\tilde{0}\rangle = \frac{1}{k^{(w+1)/2}}\psi^-_{-w-\frac{1}{2}}\psi^-_{-w+\frac{1}{2}}\cdots\psi^-_{-\frac{1}{2}}|0\rangle\,. \tag{2.29}$$

The world sheet and spacetime quantum numbers shift appropriately by the addition of this extra fermion.

2.4.2 The Ramond sector

We perform a covariant analysis in the left-moving Ramond sector as well — it is to be combined with a right-moving Ramond or Neveu-Schwarz sector. We consider the covariant vertex operators in the $(-\frac{1}{2})$ picture:

$$O^{(-\frac{1}{2})}_{\mathrm{R}} = c\ e^{-\frac{\phi}{2}}\ O_{\mathrm{matter}} \tag{2.30}$$

where O_{matter} is of conformal dimension $5/8$. The matter vertex operators that we identified previously are again the pertinent ones, but we now take into account the contribution of the fermionic zero modes in the light-cone directions to the world sheet conformal dimension. We can for instance set the light cone fermion number to $s_0 = +1/2$ (as in flat space superstring theory [21]). Effectively, we are then looking for solutions of $L_0 = 5/8 - 1/8 = 1/2$, and our problem reduces to the one that we solved previously.

Again, we need to check how the BRST cohomology interfaces with our solution set. The L_0 constraint is once more taken care of while the fermionic part of the BRST operator also annihilates the state because of the fact that we have solved the Dirac equation by

picking signs of light cone momenta as well as $s_0 = 1/2$ — see [21] as well as below. There are no BRST exact states by the reasoning we described above.

We summarize our results for the covariant Ramond sector vertex operator in picture $(-\frac{1}{2})$. The left-moving part of the matter vertex operator has conformal dimension $H = h + \frac{kw}{2} - \frac{1}{2}$ and takes the factorized form:

$$O_{\text{matter}} = \Phi_h^w \, e^{i\sqrt{\frac{2}{k}}\left(h + \frac{kw}{2} - \frac{1}{2}\right)X} \, P_{\text{matter}} \,. \tag{2.31}$$

The momentum along the S^1 direction is given by $p_L = \sqrt{\frac{2}{k}}(h + \frac{kw}{2} - \frac{1}{2})$. Let us be a bit more precise about the spin and internal fields that make up the operator P_{matter}. We refer to the discussion in [8] and appendix A for further details. The four fermions that are the world sheet superpartners of the $AdS_3 \times S^1$ directions are paired up and bosonized in order to write the vertex operators in the Ramond sector. We have

$$\partial \hat{H}_1 = \frac{2}{k}\psi^1\psi^2 \,, \qquad \partial \hat{H}_0 = -i\sqrt{\frac{2}{k}}\psi^0\psi^3 \,. \tag{2.32}$$

The hatted bosons have been defined in equation (A.25) — appropriate cocycle factors have been included. The remaining factor in the matter operator can then be written as:

$$P_{\text{matter}} = e^{-i\left(\frac{1}{2} + w\right)\hat{H}_1} e^{\pm \frac{i}{2}\hat{H}_0} \, \Sigma^{\bar{a}} \,, \tag{2.33}$$

where $\Sigma^{\bar{a}}$ are the Ramond ground state operators of the $\mathcal{N} = 2$ superconformal theory Y. The label $\bar{a}$ runs over the Ramond ground states of the theory. As explained in [15, 24] it is possible to realize the Ramond ground states either by acting with the chiral ring elements on the Ramond ground state with the lowest R-charge or equivalently by acting with the anti-chiral ring elements on the Ramond ground state with the highest R-charge. Thus, the ground states can be labelled by either a or $\bar{a}$. Our choice of labelling the states by the anti-chiral primaries anticipates calculations to come.

The on-shell condition requires us to impose that the world sheet $N = 1$ super current mode G_0 acting on the state is zero. The super current splits into three terms corresponding to the factored $AdS_3 \times S^1 \times Y$ model:

$$G = \frac{2}{k}\psi^A J_A + i\,\psi^0 \partial X + G^Y \,. \tag{2.34}$$

When we act with the supercurrent on the vertex operator (2.31), the coefficient of the simple pole is proportional to:

$$p_L\gamma^0 + \frac{2}{k}\gamma^3\left(h + \frac{kw}{2} - \frac{1}{2}\right) = \sqrt{\frac{2}{k}}\left(h + \frac{kw}{2} - \frac{1}{2}\right)\gamma^0\left(1 + \sqrt{\frac{2}{k}}\gamma^0\gamma^3\right) \tag{2.35}$$

where the γ^μ matrices represent the fermion zero modes. The right hand side vanishes only if the coefficient of the field $\hat{H}_0$ in the exponent of the vertex operator is $\frac{1}{2}$, as we assumed previously. Finally we impose a GSO projection on the vertex operator. We write the Ramond sector ground state as (see appendix B for details):

$$\Sigma^{\bar{a}} = e^{i\sqrt{\frac{3}{c_Y}}q_R^{\bar{a}}Z} \Pi^{\bar{a}} \,, \tag{2.36}$$

where $q_R^{\bar{a}}$ is the R-charge of the Ramond ground state in the Y-theory. Here Z bosonizes the $U(1)_R$ charge of the Y theory and we have separated out the factor that carries the world sheet R-charge. The chiral GSO projection is equivalent to the condition of locality with respect to the spacetime supersymmetry generators. Following [8] we choose a set of space-time supersymmetry generators (see equation (C.5) in appendix C), whose world sheet operators are given by:

$$e^{-\frac{\phi}{2}} S_r^{\pm} = e^{-\frac{\phi}{2}} \; e^{-ir(\hat{H}_1 \mp \hat{H}_0)} \; e^{\pm i \frac{X}{\sqrt{2k}}} \; e^{\pm i \sqrt{\frac{c_Y}{12}} Z} \,, \qquad (2.37)$$

where $r = \pm \frac{1}{2}$. We use the standard free field operator product expansions

$$
\begin{aligned}
e^{-\frac{\phi}{2}} \cdot e^{-\frac{\phi}{2}} &\approx \frac{e^{-\phi}}{z^{\frac{1}{4}}} \\
e^{-ir\hat{H}_1} \cdot e^{-i\left(\frac{1}{2}+w\right)\hat{H}_1} &\approx z^{r\left(w+\frac{1}{2}\right)} e^{-i\left(r+\frac{1}{2}+w\right)\hat{H}_1} \\
e^{\pm ir\hat{H}_0} \cdot e^{\frac{i}{2}\hat{H}_0} &\approx z^{\pm\frac{r}{2}} e^{i\left(\pm r+\frac{1}{2}\right)\hat{H}_0} \\
e^{\pm i \frac{X}{\sqrt{2k}}} \cdot e^{\frac{i}{\sqrt{2k}}(2h-1+kw)X} &\approx z^{\pm \frac{2h-1+kw}{2k}} e^{\frac{i}{\sqrt{2k}}(2h-1+kw\pm 1)X} \\
e^{\pm i \sqrt{\frac{c_Y}{12}} Z} \cdot e^{i\sqrt{\frac{3}{c_Y}} q_R^{\bar{a}} Z} &\approx z^{\pm \frac{q_R^{\bar{a}}}{2}} e^{i\sqrt{\frac{3}{c_Y}}\left(q_R^{\bar{a}} \pm \frac{c_Y}{6}\right)Z} \,,
\end{aligned}
\qquad (2.38)
$$

to show that the locality of the operator product expansion amounts to the constraint:

$$-\frac{1}{2} + r\left(1 \pm 1\right) + 2w\left(r \pm \frac{1}{2}\right) \pm \left(\frac{2h-1}{k} + q_R^{\bar{a}}\right) \in \text{ odd integers.} \qquad (2.39)$$

Since $r \pm \frac{1}{2}$ is always an integer, multiplying by $2w$ always equals an even integer. Thus the w-dependent term drops out from the oddness constraint. For both choices of signs the dependence on the index r on the space-time supercharges trivializes as well and we must satisfy:

$$\frac{1}{2} + \frac{2h-1}{k} + q_R^{\bar{a}} \in \text{ odd integers} \qquad (2.40)$$

Thus locality with all four supercharges is ensured by satisfying the charge constraint (2.40).

The classification problem has been reduced to one that depends solely on the level k and the nature of the world sheet conformal field theory Y. The number of solutions we find will depend on the spectrum of world sheet R-charges q_Y. A general constraint on that spectrum is that it is bounded by the central charge of the theory Y [15]:

$$-\frac{3}{2} + \frac{1}{k} \le q_R^{\bar{a}} \le \frac{3}{2} - \frac{1}{k} \,. \qquad (2.41)$$

Given the equations (2.40), (2.41) and the bounds on the spin h, it becomes clear that the world sheet R-charge q_{ws} can only take the value 1. For that value, we have:

$$q_R^{\bar{a}} = \frac{1}{2} - \frac{2h-1}{k} \qquad (2.42)$$

and we find the bound:

$$|q_R^{\bar{a}}| \le \frac{1}{2} \qquad (2.43)$$

on the world sheet Ramond sector R-charge.

To summarize, the left moving Ramond vertex operator for physical states in the covariant formalism, in which the charges are constrained by (2.40) is:

$$O_{\mathrm{R}}^{\bar{a}\,(-\frac{1}{2})} = c\; e^{-\frac{\phi}{2}}\; \Phi_h^w\; e^{i\sqrt{\frac{2}{k}}\left(h+\frac{kw}{2}-\frac{1}{2}\right)X}\; e^{-i\left(\frac{1}{2}+w\right)\hat{H}_1}\; e^{\frac{i}{2}\hat{H}_0}\; \Sigma^{\bar{a}}\,. \tag{2.44}$$

Thus we determined the covariant vertex operators corresponding to the space-time chiral primaries.

2.4.3 Zero winding vertex operators at a generic boundary point

In this subsection, we include the spacetime chiral primary operators into global $sl(2,\mathbb{R})$ multiplets. We use a variable x to keep track of the different states in one multiplet — the variable has the intuitive interpretation of designating a point on the boundary of the AdS_3 manifold [4]. For simplicity, we concentrate on the operators that have spectral flow number $w = 0$ from now on. To put the operators Φ_h of spin h at a generic point x, it is useful to temporarily combine a discrete lowest weight representation with a discrete highest weight representation plus a finite dimensional representation, and define the $sl(2)$ multiplet combination $\Phi_h(x)$

$$\Phi_h(x;z) = \sum_{m\in\mathbb{Z}} \Phi_{h,m,\bar{m}}(z)x^{-m-h}\bar{x}^{-\bar{m}-h}\,\Phi_{h,m,\bar{m}}(z,\bar{z})\,. \tag{2.45}$$

Similarly, we combine the $sl(2,\mathbb{R})$ adjoint fermions [4]

$$\psi(x;z) = -\psi^+(z) + 2x\psi^3(z) - x^2\psi^-(z)\,. \tag{2.46}$$

A primary field of spin s in the $sl(2,\mathbb{R})_{k+2}$ model satisfies the operator product expansion with the currents $j^A(z)$:

$$j^A(z)\mathcal{O}_s(x;z) \approx -\frac{D_x^A \mathcal{O}_s(x;z)}{z-w}\,, \tag{2.47}$$

where the differential operators representing $sl(2,\mathbb{R})$ read

$$D_x^- = \partial_x\,, \qquad D_x^3 = x\frac{\partial}{\partial x} + s\,, \qquad D_x^+ = x^2\partial_x + 2sx\,. \tag{2.48}$$

The fields $\Phi_h(x;z)$ and $\psi(x;z)$ have spins equal to h and -1 respectively. The affine $sl(2,\mathbb{R})$ currents also transform in the adjoint:

$$j(x;z) = -j^+(z) + 2xj^3(z) - x^2j^-(z)\,. \tag{2.49}$$

We can form analogous combinations for the fermionic current $\hat{j}(x;z)$ (see appendix A for definitions and conventions) and the supersymmetric current $J(x;z) = j(x;z)+\hat{j}(x;z)$. The standard operator product expansions between the world sheet fields that we have summarized in appendix A can then be compactly encoded in the operator product expansions:

$$j(x_1;z)\Phi_h(x_2;w) \approx \frac{1}{z-w}\left((x_1-x_2)^2\partial_{x_2} - 2h(x_1-x_2)\right)\Phi_h(x_2;w) \tag{2.50}$$

$$\hat{j}(x_1;z)\psi(x_2;w) \approx \frac{1}{z-w}\left((x_1-x_2)^2\partial_{x_2} + 2(x_1-x_2)\right)\psi(x_2;w) \tag{2.51}$$

$$\psi(x_1;z_1)\psi(x_2;z_2) \approx k\frac{(x_{12})^2}{z_{12}}\,. \tag{2.52}$$

In terms of these fields, one can write down the zero spectral flow left-moving vertex operators in the NS sector. We choose a particular normalization whose utility will become apparent:

$$O_{\mathrm{NS_1}}^{a\,(-1)}(x;z) = \frac{1}{\sqrt{k}(2h-1)}\, c\, e^{-\phi}\, \Phi_h(x;z)\, e^{i\sqrt{\frac{2}{k}}hX}\, V_{c.p.}^a\,, \qquad h_Y^a = \frac{1}{2} - \frac{h}{k}\,, \quad (2.53)$$

$$O_{\mathrm{NS_2}}^{\bar{a}\,(-1)}(x;z) = \frac{1}{\sqrt{k}}\, c\, e^{-\phi}\, \Phi_h(x;z)\, \psi(x;z)\, e^{i\sqrt{\frac{2}{k}}(h-1)X}\, V_{a.c.p.}^{\bar{a}}\,, \qquad h_Y^{\bar{a}} = \frac{h-1}{k}\,. \quad (2.54)$$

In the Ramond-sector one analogously combines spin fields to form a primary field $S(x)$ such that it is of spin $s = -\frac{1}{2}$:

$$S(x;z) = x\, e^{-\frac{i}{2}\hat{H}_1 + \frac{i}{2}\hat{H}_0} + e^{\frac{i}{2}\hat{H}_1 - \frac{i}{2}\hat{H}_0}\,. \quad (2.55)$$

To check that the vertex operator is BRST closed and transforms as claimed, it is useful to use the $\hat{j}^A$ currents expressed in terms of the bosons. Finally, we have the left-moving Ramond sector vertex operator at a generic point x on the boundary:

$$O_{\mathrm{R}}^{\bar{a}\,(-\frac{1}{2})}(x;z) = \frac{1}{\sqrt{k}}\, c\, e^{-\frac{\phi}{2}}\, \Phi_h(x;z)\, e^{i\sqrt{\frac{2}{k}}\left(h-\frac{1}{2}\right)X}\, S(x)\, \Sigma^{\bar{a}}\,. \quad (2.56)$$

2.5 A change of picture

It will likewise be useful to have the covariant vertex operators available in multiple pictures. We derive the operators in this subsection. To find the zero picture operators in the NS sector, we apply picture changing to the (-1) picture operators we have found. The latter are schematically of the form:

$$O^{(-1)} = c\, e^{-\phi}\, O_{\mathrm{matter}}\,, \quad (2.57)$$

where both the matter vertex operator O_{matter} and $e^{-\phi}$ have dimension equal to $1/2$. The vertex operator O_{matter} is a superconformal primary of the $N = 1$ world sheet algebra. The picture changing operator acts on the matter operator as [20, 21]:

$$O_{\mathrm{matter}}^{(0)}(0) = G_{-1/2} \cdot O_{\mathrm{matter}}(0) = \oint_0 dz\, G(z)\, O_{\mathrm{matter}}(0)\,, \quad (2.58)$$

where $G_{-1/2}$ is a mode of the $N = 1$ world sheet supercurrent G. The calculation can again be split into three terms corresponding to the factors $AdS_3 \times S^1 \times Y$ — see equation (2.34). It is useful to recall that we suppose $N = 2$ superconformal symmetry in the world sheet theory Y and that we have the generic embedding of the $N = 1$ super current G into the $N = 2$ superconformal algebra [21]:

$$G^Y = \frac{1}{\sqrt{2}}\left(G^{+,Y} + G^{-,Y}\right)\,. \quad (2.59)$$

Let's implement these statements case by case.

2.5.1 The NS$_1$ case: no fermion in the AdS factor

If there is no fermion $\psi(x)$ present in the vertex operator, then picture changing leads to:

$$O_{\mathrm{NS_1}}^{a\,(0)} = \frac{1}{\sqrt{k}\,(2h-1)}\, \left(\frac{2}{k}\psi_A D_x^A + G_{-1/2}^{S^1+Y}\cdot\right) \Phi_h(x)\, e^{i\sqrt{\frac{2}{k}}hX}\, V_{c.p.}^a\,. \quad (2.60)$$

2.5.2 The NS₂ case: a fermion in the AdS factor

We next consider the NS_2 case in which the vertex operator (2.54) has left-moving dimension $H = h - 1$. Acting with the picture changing operator we find:

$$O_{\mathrm{NS}_2}^{\bar{a}\,(0)} = \frac{1}{\sqrt{k}}\big(J(x) + \frac{2}{k}\psi(x)\psi_A D_x^A + \psi(x)\, G^{S^1+Y} \cdot \big)\Phi_h(x)\, e^{i\sqrt{\frac{2}{k}}(h-1)X}\, V_{a.c.p.}^{\bar{a}}. \qquad (2.61)$$

The second term in the parentheses can be simplified:

$$
\begin{aligned}
\psi(x)&\left(\psi_A D_x^A\right)\Phi_h(x) \\
&= \left(-\psi^+ + 2x\psi^3 - x^2\psi^-\right)\left(\frac{1}{2}\psi^- D_x^+ + \frac{1}{2}\psi^+ D_x^- - \psi^3 D_x^3\right)\Phi_h(x) \\
&= \left[\frac{\psi^+\psi^-}{2}\left(x^2 D_x^- - D_x^+\right) + \psi^3\psi^+\left(xD_x^- - D_x^3\right) + \psi^3\psi^-\left(xD_x^+ - x^2 D_x^3\right)\right]\Phi_h(x) \\
&= -\frac{kh}{2}\hat{j}(x)\,\Phi_h(x)\,.
\end{aligned}
\qquad (2.62)
$$

Substituting this into the expression (2.61) we find

$$O_{\mathrm{NS}_2}^{\bar{a}\,(0)} = \frac{1}{\sqrt{k}}\,\left(j(x) + (1-h)\,\hat{j}(x) + \psi(x)\, G_{-1/2}^{S^1+Y} \cdot\right)\Phi_h(x)\, e^{i\sqrt{\frac{2}{k}}(h-1)X}\, V_{a.c.p.}^{\bar{a}}. \qquad (2.63)$$

2.5.3 Ramond sector vertex operators

It will be useful to know the Ramond vertex operators in the $(-\frac{3}{2})$ picture as well:

$$O_{\mathrm{R}}^{\bar{a}\,(-\frac{3}{2})}(x;z) = \frac{2}{(2h-1)}\, c\, e^{-\frac{3\phi}{2}}\, \Phi_h(x;z)\, e^{i\sqrt{\frac{2}{k}}\left(h-\frac{1}{2}\right)X}\, \widetilde{S}(x)\, \Sigma^{\bar{a}}, \qquad (2.64)$$

where

$$\widetilde{S}(x;z) = -x\, e^{-\frac{i}{2}\hat{H}_1 - \frac{i}{2}\hat{H}_0} + e^{\frac{i}{2}\hat{H}_1 + \frac{i}{2}\hat{H}_0}\,. \qquad (2.65)$$

Acting with the picture changing operator on this operator, we indeed obtain the Ramond operator (2.56) in the $(-\frac{1}{2})$ picture. The operator $\tilde{S}$ has opposite chirality from the operator S.

In summary, we have determined the full spectrum of chiral primaries in space-time. We have described all the relevant covariant vertex operators. For those that are in the zero spectral flow sector, we have provided additional detail. A summary of the latter vertex operators is provided in table 1.

3 Chiral ring structure constants

Our goal in this section is to compute structure constants of the spacetime $\mathcal{N}=2$ superconformal chiral ring, involving operators which are in the world sheet sector of zero spectral flow. We will derive a space-time operator product expansion from a world sheet operator product expansion. The advantage of this method is that it only involves data that survives a topological twist of the theory. In particular, we bypass the calculation of the physical three-point functions and two-point functions both in the space-time and on the world sheet. Our method simplifies calculations in the literature for $\mathcal{N}=4$ superconformal theories. We also extend them to the class of $\mathcal{N}=2$ theories which exhibit qualitatively new features.

	Vertex Operator	Dimension H
$O_{\text{NS}_1}^{a\,(-1)}$	$\frac{1}{\sqrt{k}(2h-1)}\ c\ e^{-\phi}\ \Phi_h(x)\ e^{i\sqrt{\frac{2}{k}}hX}\ V_{c.p.}^a$	h
$O_{\text{NS}_1}^{a\,(0)}$	$\frac{1}{\sqrt{k}(2h-1)}\ (\frac{2}{k}\psi_A D_x^A + G_{-1/2}^{S^1+Y}\cdot)\Phi_h(x)\ e^{i\sqrt{\frac{2}{k}}hX}\ V_{c.p.}^a$	h
$O_{\text{NS}_2}^{\bar{a}\,(-1)}$	$\frac{1}{\sqrt{k}}\ c\ e^{-\phi}\ \Phi_h(x)\ \psi(x)\ e^{i\sqrt{\frac{2}{k}}(h-1)X}\ V_{a.c.p.}^{\bar{a}}$	$h-1$
$O_{\text{NS}_2}^{\bar{a}\,(0)}$	$\frac{1}{\sqrt{k}}\ (j(x) + (1-h)\hat{j}(x) + \psi(x)G_{-1/2}^{S^1+Y}\cdot)\Phi_h(x)\ e^{i\sqrt{\frac{2}{k}}(h-1)X}\ V_{a.c.p.}^{\bar{a}}$	$h-1$
$O_{\text{R}}^{\bar{a}\,(-\frac{1}{2})}$	$\frac{1}{\sqrt{k}}\ c\ e^{-\frac{\phi}{2}}\Phi_h(x)\ e^{i\sqrt{\frac{2}{k}}(h-\frac{1}{2})X}S(x)\Sigma^{\bar{a}}$	$h-\frac{1}{2}$
$O_{\text{R}}^{\bar{a}\,(-\frac{3}{2})}$	$\frac{2}{(2h-1)}\ c\ e^{-\frac{3\phi}{2}}\Phi_h(x)\ e^{i\sqrt{\frac{2}{k}}(h-\frac{1}{2})X}\widetilde{S}(x)\Sigma^{\bar{a}}$	$h-\frac{1}{2}$

Table 1. The world sheet vertex operators representing space-time chiral primaries and their space-time left-moving conformal dimension, in the subsector with spectral flow number equal to zero.

3.1 Pictures and products

In the NS sector we have determined covariant vertex operators in the (-1) as well as the (0) pictures. We wish to calculate the space-time chiral ring. Since the picture number adds in the product of operators, we can multiply an integrated zero picture vertex operator into a fixed (-1) picture vertex operator and obtain once more a (-1) picture operator. Multiplying two vertex operators in the Ramond sector with picture number $(-\frac{1}{2})$ leads to an NS operator with picture number minus one. In the R-NS mixed operator product expansions we take the Ramond and NS operators in their canonical pictures to generate a Ramond sector operator in the $(-\frac{3}{2})$ picture.

The product of two space-time chiral primary operators inserted at boundary points x_1 and x_2 is regular as a function of the space-time distance $x_1 - x_2$ between the operators. The structure constants of the chiral ring arise from the zeroth order term in $x_1 - x_2$. For example, starting with two operators of the type NS$_2$ we obtain

$$\lim_{x_1 \to x_2} \int d^2 z_1 O_{\text{NS}_2}^{\bar{a}\,(0)}(x_1, \bar{x}_1; z_1, \bar{z}_1) O_{\text{NS}_2}^{\bar{b}\,(-1)}(x_2, \bar{x}_2; z_2, \bar{z}_2) = \mathcal{A}^{2\bar{a}\,2\bar{b}}{}_{2\bar{c}} O_{\text{NS}_2}^{\bar{c}\,(-1)}(x_2, \bar{x}_2; z_2, \bar{z}_2)$$
$$+ O(x_1 - x_2, \bar{x}_1 - \bar{x}_2), \quad (3.1)$$

where $\mathcal{A}$ denotes the space-time structure constant for the product under consideration and we have for now only written down one (out of two possible) term(s) on the right hand side. We imagine that this calculation takes place inside a perturbative string correlation function. We assume throughout that the operator product in space-time arises from the region in which the world sheet vertex operators are close to each other. See e.g. [4, 25] for discussions of these points in more generic contexts.

We make one more preliminary remark. We note that we have a graded ring. The Neveu-Schwarz sector vertex operators are even while Ramond sector vertex operators are odd. This leads to a space-time fermion number grading of the space-time ring.

3.2 The NS₂-NS₂ to NS₂ structure constants

In the following subsections, we write out our laundry list of structure constants. We start out with the structure constants obtained by colliding the NS$_2$ operators and giving rise to another NS$_2$ operator as in equation (3.1). We have $H_i = h_i - 1$ for all three operators involved in the operator product expansion. For a while, we focus our attention on the world sheet holomorphic sector. An important ingredient in our calculation is the operator product expansion between the zero and minus one picture operators:

$$\frac{1}{\sqrt{k}} \int d^2z \left(j(x_1;z) + (1-h_1)\hat{j}(x_1;z) + \psi(x_1;z) G_{-1/2}^{S^1+Y} \cdot \right) \Phi_{h_1}(x_1;z) e^{i\sqrt{\frac{2}{k}}(h_1-1)X(z)} V_{a.c.p.}^{\bar{a}}(z)$$

$$\times \frac{1}{\sqrt{k}} \, c \, e^{-\phi} \, \psi(x_2;w) \, \Phi_{h_2}(x_2;w) \, e^{i\sqrt{\frac{2}{k}}(h_2-1)X(w)} \, V_{a.c.p.}^{\bar{b}}(w). \tag{3.2}$$

The first operator is inserted at the world sheet point z while the second operator is inserted at w. Before we proceed, let us make a remark about world sheet operator dimensions. Consider the NS-sector operators at ghost number -1. They are of the form $O_{\text{NS}}^{(-1)} = c \, e^{-\phi} \, O_{\text{matter}}^{(-1)}$. The total operator has dimension zero. The operator $O_{\text{matter}}^{(-1)}$ has dimension one half. Taking into account the dimension of the fermion operator this leads to the conclusion that the combination

$$W_i^{\bar{a}}(x_i;z) = \Phi_{h_i}(x_i;z) \, e^{i\sqrt{\frac{2}{k}}(h_i-1)X} \, V_{a.c.p.}^{\bar{a}}(z), \tag{3.3}$$

has dimension zero. Similarly, in $O_{\text{matter}}^{(0)}$, the combination $W_i^{\bar{a}}$ has dimension zero as well since the total operator has dimension one which equals the dimension of the current.[3] This will play a role in simplifying our analysis. Moreover, world sheet fermion number conservation will imply important selection rules on our calculations — we will exploit them in due course.

We have only shown the left moving contribution to the operator product and we need to specify the choice of vertex operators in the right moving sector as well. For simplicity, we concentrate on the chiral-chiral ring in space-time, such that we can work with the same type of vertex operators for the space-time right-movers. One can still have either of the classes of states NS$_i$ with $i = 1, 2$ for the right moving sector. We choose the NS$_2$ states also in the right moving sector. From the form of the ghost number zero operator in (3.2) we see that there are three independent calculations to be done; schematically the complete contribution to the operator product expansion therefore includes a sum of terms that arises from the product of left and right moving contributions:

$$(T_1 + T_2 + T_3)\left(\widetilde{T}_1 + \widetilde{T}_2 + \widetilde{T}_3\right). \tag{3.4}$$

We shall evaluate each of these three terms in turn working in the left moving sector first.

3.2.1 The first term

We begin with the first term that includes the bosonic AdS_3 current $j(x;z)$. We need to evaluate the operator product:

$$\frac{1}{k} \, j(x_1;z) W_1^{\bar{a}}(x_1;z) \, c(w) \, e^{-\phi(w)} \, \psi(x_2;w) \, W_2^{\bar{b}}(x_2;w). \tag{3.5}$$

---

[3]A similar reasoning holds in the R-sector.

The fermion, the ghost and the superghosts go along for the ride and we see that if any space-time chiral primary appears in this operator product at all, it would be an NS_2 vertex operator. The only non-trivial factor is the (holomorphic) operator product expansion of the operator $(j\,W_1^{\bar{a}})(x_1; z)$ with $W_2(x_2; w)$, which we denote by t_1. We deal with an operator product expansion of composite operators of the form $(BC)(z)$ with $A(w)$, where we identify $A(w) = W_2^{\bar{b}}(x_2; w)$, $(BC)(z) = (j\,W_1^{\bar{a}})(x_1; z)$. What we shall rather do is calculate the operator product expansion of $A(z)$ with $(BC)(w)$ given by the generalized Wick theorem for interacting fields [26]:

$$A(z)(BC)(w) \approx \frac{1}{2\pi i} \oint_w \frac{dy}{y-w} \left(\overline{A(z)B(y)}\; C(w) + B(y)\; \overline{A(z)C(w)} \right). \qquad (3.6)$$

To obtain the required operator product expansion we then interchange $z \leftrightarrow w$, and Taylor expand the fields that are evaluated at z around the point w. We thus first compute:

$$t_1 = \frac{1}{2\pi i\,k} \oint_w \frac{dy}{y-w} \left(\overline{W_2^{\bar{b}}(x_2; z)j(x_1; y)}\; W_1^{\bar{a}}(x_1; w) + j(x_1; y)\; \overline{W_2^{\bar{b}}(x_2; z)W_1^{\bar{a}}(x_1; w)} \right), \quad (3.7)$$

We use the basic $\Phi\Phi$ as well as the $j\Phi$ operator product expansions, which are given in the appendix (equations (A.47) and (A.37)). Combining these with the operator product expansion in the circle direction and using the definition of the world sheet chiral ring of the Y-theory,

$$V_{a.c.p.}^{\bar{a}}(z)V_{a.c.p.}^{\bar{b}}(w) \approx R^{\bar{a}\bar{b}}{}_{\bar{c}}V_{a.c.p.}^{\bar{c}}(w)\,, \qquad (3.8)$$

one can write down the relevant WW and Wj operator product expansions:

$$W_2^{\bar{b}}(x_2; z)\,W_1^{\bar{a}}(x_1; w) \approx \int dh\, (w-z)^{\frac{-h(h-1)}{k}+\frac{(h_1+h_2-2)^2}{k}+h_c^Y}\,(x_1-x_2)^{h-h_1-h_2}\,C_{h_1,h_2}{}^h\,R^{\bar{a}\bar{b}}{}_{\bar{c}}$$

$$\Phi_h(x_2; w)\,e^{i\sqrt{\frac{2}{k}}(h_1+h_2-2)X}\,V_{a.c.p.}^{\bar{c}}(w) \qquad (3.9)$$

$$W_2^{\bar{b}}(x_2; z)\,j(x_1; y) \approx -\frac{1}{z-y}\left((x_2-x_1)^2\,\partial_{x_1} - 2h_2\,(x_1-x_2)\right)W_2^{\bar{b}}(x_2; z)$$

$$+ \left((x_2-x_1)^2\,\partial_{x_1} - 2h_2\,(x_1-x_2)\right)\partial_z W_2^{\bar{b}}(x_2; z) \qquad (3.10)$$

In the second operator product we have expanded the W_2 operator at z. We omit the term proportional to the derivative of the operator as that is a world sheet descendant and we know that it cannot lead to a spacetime chiral primary. Substituting into the generalized Wick theorem (3.6) leads to the result:

$$t_1 \approx \frac{1}{2\pi i\,k} \oint_w \frac{dy}{y-w}\left[-\frac{1}{z-y}\left((x_1-x_2)^2\partial_{x_2} - 2h_2(x_1-x_2)\right)\overline{W_2^{\bar{b}}(x_2; z)W_1^{\bar{a}}(x_1; w)} \right.$$

$$\left. + \int dh \frac{(x_1-x_2)^{h-h_1-h_2}C_{h_1h_2}{}^h\,R^{\bar{a}\bar{b}}{}_{\bar{c}}}{(w-z)^{\frac{h(h-1)}{k}-\frac{(h_1+h_2-2)^2}{k}-h_c^Y}}e^{i\sqrt{\frac{2}{k}}(h_1+h_2-2)X}\,V_{a.c.p.}^{\bar{c}}(w)\overline{j(x_1; y)\Phi_h(x_2; w)} \right].$$

$$(3.11)$$

Substituting the basic operator product expansions once again we find

$$t_1 \approx \frac{1}{2\pi i\,k} \oint_w \frac{dy}{y-w} \Bigg[-\frac{1}{z-y}\left((x_1-x_2)^2\partial_{x_2} - 2h_2(x_1-x_2)\right) \tag{3.12}$$

$$\int dh \frac{(x_1-x_2)^{h-h_1-h_2} C_{h_1h_2}{}^h R^{\bar a \bar b}{}_{\bar c}}{(z-w)^{\frac{h(h-1)}{k}-\frac{(h_1+h_2-2)^2}{k}-h_c^Y}}\, e^{i\sqrt{\frac{2}{k}}(h_1+h_2-2)X} V^{\bar c}_{a.c.p.}(w)\Phi_h(x_2;w)$$

$$+ \int dh \frac{(x_1-x_2)^{h-h_1-h_2} C_{h_1h_2}{}^h R^{\bar a \bar b}{}_{\bar c}}{(w-z)^{\frac{h(h-1)}{k}-\frac{(h_1+h_2-2)^2}{k}-h_c^Y}(y-w)}\, e^{i\sqrt{\frac{2}{k}}(h_1+h_2-2)X}\, V^{\bar c}_{a.c.p.}(w)$$

$$\times \left((x_1-x_2)^2\partial_{x_2} - 2h(x_1-x_2)\right)\Phi_h(x_2;w)$$

$$+ \int dh \frac{(x_1-x_2)^{h-h_1-h_2} C_{h_1h_2}{}^h R^{\bar a \bar b}{}_{\bar c}}{(w-z)^{\frac{h(h-1)}{k}-\frac{(h_1+h_2-2)^2}{k}-h_c^Y}}\, e^{i\sqrt{\frac{2}{k}}(h_1+h_2-2)X} V^{\bar c}_{a.c.p.}(w)\ :j_1(w)\Phi_h(x_2;w):\ \Bigg].$$

By evaluating the integral over y, we see that the second term vanishes. The last term does not give rise to a space-time chiral primary, and so we are left with the first term only:

$$t_1 \approx \frac{1}{2\pi i\,k} \oint_w \frac{dy}{y-w} \Bigg[-\frac{1}{z-y}\left((x_1-x_2)^2\partial_{x_2} - 2h_2(x_1-x_2)\right)$$

$$\int dh \frac{(x_1-x_2)^{h-h_1-h_2} C_{h_1h_2}{}^j R^{\bar a \bar b}{}_{\bar c}}{(z-w)^{\frac{h(h-1)}{k}-\frac{(h_1+h_2-2)^2}{k}-h_c^Y}}\, e^{i\sqrt{\frac{2}{k}}(h_1+h_2-2)X} V^{\bar c}_{a.c.p.}(w)\Phi_h(x_2;w) \Bigg] \tag{3.13}$$

The y-integral simply sets $y = w$. To proceed further we must include the contribution from the right-movers. An important point is that there is only a single, common integral over the $sl(2,\mathbb{R})$ spin h. Reverting to our notation for the operator product in (3.4) we obtain

$$T_1 \widetilde{T}_1 \approx \frac{1}{k^2} \int \frac{dh}{|z-w|^{2+2\frac{h(h-1)}{k}-2\frac{(h_1+h_2-2)^2}{k}-2h_c^Y}} C_{h_1h_2}{}^h R^{\bar a \bar b}{}_{\bar c} e^{i\sqrt{\frac{2}{k}}(h_1+h_2-2)X}\, V^{\bar c}_{a.c.p.}(w,\bar w)$$

$$\left|\left((x_1-x_2)^2\partial_{x_2} - 2h_2(x_1-x_2)\right)\right|^2 |x_1-x_2|^{2h-2h_1-2h_2}\Phi_h(x_2;w,\bar w) \tag{3.14}$$

The structure constants $C_{h_1h_2}{}^h$ and $R^{\bar a \bar b}{}_{\bar c}$ now include the contribution from the right movers as well.[4] We use once more the fact that we are calculating the spacetime chiral ring. Spacetime R-charge conservation and unitarity imply [15] that the right hand side of equation (3.14) should be independent of the boundary distance $x_1 - x_2$. This picks out the value $h = h_3 = h_1 + h_2 - 1$ as the one on which to concentrate. The world sheet conformal dimension h_Y^c of the operator V^c is fixed by world sheet R-charge conservation to be

$$h_Y^c = h_Y^a + h_Y^b = \frac{h_1+h_2-2}{k} = \frac{h_3-1}{k}\,. \tag{3.15}$$

In the second equality we have used the relation between the dimension h_Y and the $sl(2,\mathbb{R})$ spin for the NS$_{2,a}$ vertex operators. Combining this with the momentum along the S^1 direction we see that this leads to the propagation of an on-shell state that again corresponds to

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[4]To avoid clutter, we do not write out the in principle necessary zoo of indices.

a spacetime chiral primary. This is consistent with the general discussion of deriving space-time operator product expansions from world sheet operator product expansions in [25] in which the dominant contribution to the z-integral arises from on-shell states. To recapitulate, precisely at the value of $h = h_3 = h_1 + h_2 - 1$, the state (whose vertex operator includes the $V^{\bar{c}}_{a.c.p.}$, Φ_h and the S^1 vertex operator with momentum $p_L = h_3 - 1$) turns out to be on-shell and this leads to a pole in the h-integral. The contribution of this state is extracted by taking the residue in the h-integral[5]

$$\text{Res}_{h=h_3} \frac{1}{|z|^{2+f(h)}} = \frac{1}{f'(h_3)} \delta^2(z, \bar{z}) \,, \tag{3.16}$$

where, in our case we have $f'(h_3) = (2h_3 - 1)/k = (2h_1 + 2h_2 - 3)/k$. Performing the h-integral we obtain

$$T_1 \widetilde{T}_1 \approx \frac{C_{h_2, h_1}{}^{h_1 + h_2 - 1} R^{\bar{a}\bar{b}}{}_{\bar{c}}}{k(2h_1 + 2h_2 - 3)} \left| (x_1 - x_2)^2 \partial_{x_2} - 2h_2(x_1 - x_2) \right|^2 |x_1 - x_2|^{-2} \, \Phi_{h_1 + h_2 - 1}(w, \bar{w})$$

$$e^{i\sqrt{\frac{2}{k}}(h_1 + h_2 - 2)X} \, V^{\bar{c}}(w, \bar{w}) \delta^{(2)}(z - w)$$

$$\approx \frac{R^{\bar{a}\bar{b}}_c}{k(2h_1 + 2h_2 - 3)}(1 - 2h_2)^2 W^{\bar{c}}_{h_1 + h_2 - 1}(w, \bar{w}) \delta^{(2)}(z - w) \,. \tag{3.17}$$

In the second line we have made use of the fact that $C_{h_2, h_1}{}^{h_1 + h_2 - 1} = 1$ (see equation (A.51)). This takes care of the first out of nine terms. Given this analysis, the rest of the calculation can be understood swiftly.

3.2.2 The second term

We consider the contribution from the second term in (3.2) involving the fermion current $\hat{j}$:

$$\frac{1}{k}(1 - h_1)\hat{j}(x_1; z)W^{\bar{a}}_1(x_1; z) \cdot c(w) \, e^{-\phi(w)} \, \psi(x_2; w) \, W^{\bar{b}}_2(x_2; w) \,. \tag{3.18}$$

The non-trivial operator product expansion involves $\hat{j}(x_1; z)$ with the fermion $\psi(x_2; w)$ and the operators $W^{\bar{a}}_1(x_1; z)$ and $W^{\bar{b}}_2(x_2; z)$. We neglect the $(x_1 - x_2)\partial_{x_2}\psi(x_2; w)$ term on the right hand side as it does not lead to a chiral ring element. The rest of the analysis parallels the one we already did in detail in the previous subsection. Once again, we have to pair with the right movers and we obtain the result:

$$T_2 \widetilde{T}_2 \approx \frac{R^{\bar{a}\bar{b}}{}_{\bar{c}}}{k(2h_1 + 2h_2 - 3)} \, (2 - 2h_1)^2 \, W^{\bar{c}}_{h_1 + h_2 - 1}(w, \bar{w}) \delta^{(2)}(z - w) \,. \tag{3.19}$$

3.2.3 The result

The third term T_3 in (3.2) involves the action of the supercharge $G^{S^1 + Y}_{-\frac{1}{2}}$ in the compact directions. Such a term cannot lead to NS$_2$ operators because of world sheet fermion number conservation — we shall discuss the action of this operator in subsection 3.8.

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[5]Note that we evaluate a pole that is associated to the $|z - w|$ dependence and it therefore differs from the poles in the structure constants, or the poles in the $|x_1 - x_2|$ dependence evaluated in [27].

Here, we combine the two contributions $(T_1 + T_2)$ from the left and right movers. Each factor of T_1 or $\widetilde{T}_1$ leads to a factor $(1 - 2h_2)$ while each factor of T_2 or $\widetilde{T}_2$ leads to the factor $(2 - 2h_1)$. Taking appropriate linear combinations leads us to:

$$(T_1 + T_2)\left(\widetilde{T}_1 + \widetilde{T}_2\right) \approx (2h_1 + 2h_2 - 3)\, R^{\bar{a}\bar{b}}{}_{\bar{c}}\, \frac{1}{k}\, W^{\bar{c}}_{h_1 + h_2 - 1}(w, \bar{w})\delta^{(2)}(z - w)\,. \qquad (3.20)$$

Integrating over z and recalling the other factor operators as well as the R-charge conservation equation $h_3 = h_1 + h_2 - 1$, we find the structure constant:

$$\mathcal{A}^{2\bar{a}\,2\bar{b}}{}_{2\bar{c}} = (2h_3 - 1)\, R^{\bar{a}\bar{b}}{}_{\bar{c}}\,. \qquad (3.21)$$

We iterate the fact that we have chosen the $\mathrm{NS}_2^{\bar{a}}$ vertex operators on both the left and right moving sector to derive this structure constant. This structure constant agrees with the structure constant computed in [9] for models with $\mathcal{N} = 4$ superconformal symmetry. The simple dependence of the structure constant on the space-time spin suggests both an interpretation in terms of a symmetric orbifold, as well as a possible non-renormalization theorem for this structure constant. Indeed, we found here that universal features known for the $\mathcal{N} = 4$ models extend to this structure constant in all $\mathcal{N} = 2$ superconformal backgrounds under consideration. In the following, we study to what extent these features persist for other structure constants.

3.3 The NS$_2$-NS$_1$ operator product expansion

Next, we consider the operator product expansion involving NS sector operators of type 1 and 2. We opt to have the NS$_2$ operator in the (0)-picture and wish to evaluate the on-shell space-time chiral primary contribution to:

$$\frac{1}{\sqrt{k}} \int d^2z \left(j(x_1; z) + (1 - h_1)\hat{j}(x_1; z) + \psi(x_1; z)G^{S^1 + Y}_{-1/2} \cdot \right) \Phi_{h_1}(x_1; z)e^{i\sqrt{\frac{2}{k}}(h_1 - 1)X(z)}V^{\bar{a}}_{a.c.p.}(z)$$

$$\times \frac{1}{\sqrt{k}\,(2h_2 - 1)}\, c\, e^{-\phi}\, \Phi_{h_2}(x_2; w)\, e^{i\sqrt{\frac{2}{k}}h_2 X(w)}\, V^b_{c.p.}(w)\,, \qquad (3.22)$$

when the integrated vertex operator approaches the fixed operator. The calculation is a simpler counterpart to the one in the previous subsection, so we shall be brief. We denote operator combinations as

$$W^{\bar{a}}_1(x_1; z) = \Phi_{h_1}(x_1; z)e^{i\sqrt{\frac{2}{k}}(h_1 - 1)X(z)}V^{\bar{a}}_{a.c.p.}(z) \qquad (3.23)$$

$$U^b_2(x_2; w) = \Phi_{h_2}(x_2; w)\, e^{i\sqrt{\frac{2}{k}}h_2 X(w)}\, V^b_{c.p.}(w)\,. \qquad (3.24)$$

The operator $W^{\bar{a}}_1$ has world sheet dimension 0 while the operator U^a_2 has dimension equal to $\frac{1}{2}$. The fermionic current as well as the $N = 1$ creation operator $G^{S^1 + Y}_{-1/2}$ appearing in the picture (0) operator do not lead to a spacetime chiral primary. We concentrate on the action of the bosonic current operator and consider the operator product expansion:

$$\frac{1}{k\,(2h_2 - 1)}\, \left(j\, W^{\bar{a}}_1 \right)(x_1; z) \cdot U^b_2(x_2; w)\,. \qquad (3.25)$$

We again use the formula (3.6) by first exchanging $z \leftrightarrow w$ and then computing the operator product expansion

$$\frac{1}{2\pi i \, k(2h_2 - 1)} \oint_w \frac{dy}{y - w} \left(\overbracket{U_2^b(x_2; z) \, j}(x_1; y) \ W_1^{\bar{a}}(x_1; w) + j(x_1; y) \ \overbracket{U_2^b(x_2; z) W}_1^{\bar{a}}(x_1; w) \right). \tag{3.26}$$

To carry out the indicated contractions we peruse the basic $\Phi\Phi$ and the $j\Phi$ operator product expansions in the AdS_3 sector. In addition we also need the operator product expansion in the circle direction and the chiral-anti-chiral operator product expansion in the Y-theory. In the first instance, we assume that the sum of the R-charges of the Y-vertex operators add up to a positive number. In such a scenario we have (see formula (B.26) in the appendix)

$$V_{c.p.}^b(z) V_{a.c.p.}^{\bar{a}}(w) \approx \frac{D^{b\bar{a}}{}_c}{(z - w)^{q_Y^b}} \ V_{c.p.}^c(w), \tag{3.27}$$

where the tensor $D^{b\bar{a}}{}_c = g^{b\bar{d}} \, R^{\bar{a}\bar{e}}{}_{\bar{d}} \, g_{c\bar{e}}$ is expressed in terms of the chiral ring coefficients R and the Zamolodchikov metric g of the theory Y — see appendix B. The conformal dimension of the world sheet chiral primary $V_{c.p.}^c$ is half of its R-charge, which in turn is the sum of the R-charges of the operators that are multiplied:

$$q_Y^c = q_Y^{\bar{a}} + q_Y^b = -2h_Y^{\bar{a}} + 2h_Y^b$$
$$= -1 + \frac{2}{k}(h_1 + h_2 - 1). \tag{3.28}$$

The basic operator product expansions that are needed to calculate the product are:

$$U_2^b(x_2; z) W_1^{\bar{a}}(x_1; w) \approx \int dh \, \frac{(x_1 - x_2)^{h - h_1 - h_2}}{(w - z)^{\frac{1}{2} + \frac{h(h-1)}{k} - \frac{(h_1 + h_2 - 1)^2}{k} - h_Y^c}} C_{h_1, h_2}{}^h \, D^{b\bar{a}}{}_c$$
$$\Phi_h(x_2; w) \, e^{i\sqrt{\frac{2}{k}}(h_1 + h_2 - 1)X} \, V_{c.p.}^c(w) \tag{3.29}$$

$$U_2^b(x_2; z) \, j(x_1; y) \approx -\frac{1}{z - y}((x_2 - x_1)^2 \partial_{x_1} - 2h_2(x_1 - x_2))U_2^b(x_2; z)$$
$$+ \left((x_2 - x_1)^2 \partial_{x_1} - 2h_2(x_1 - x_2)\right) \partial_z U_2^b(x_2; z). \tag{3.30}$$

From here onward the calculation closely parallels the calculation of the term t_1 in the NS$_2$-NS$_2$ operator product in subsection 3.2.1. We have expanded the U_2^b field near the point z. Once more, the world sheet derivative of that field will not lead to a spacetime chiral ring element and we omit this term in the subsequent analysis. For a non-trivial on-shell spacetime chiral ring element to be generated in this sector, we need the spin $h = h_3 = h_1 + h_2 - 1$. Precisely for this value of the spin h, we see from equation (3.28) and the left-moving momentum along the circle that we have a propagating on-shell state. Taking into account the contribution from the right-movers, we again evaluate the residue at the on-shell value for the integrand as a function of the spin h and obtain:

$$\frac{1}{k^2(2h_2 - 1)^2} \frac{k \, C_{h_2, h_1}{}^{h_1 + h_2 - 1} \, D^{b\bar{a}}{}_c}{2h_3 - 1} \, (2h_2 - 1)^2 \, U_{h_1 + h_2 - 1}^c(w, \bar{w}) \delta^{(2)}(z - w). \tag{3.31}$$

We once more use the value of the structure constant $C_{h_2 h_1}{}^{h_1 + h_2 - 1} = 1$ and absorb a factor of $\left(\sqrt{k}\,(2h_3 - 1)\right)^{-2}$ to obtain the correctly normalized closed string NS_1 operator. We thereby obtain the structure constant:

$$\mathcal{A}^{2\bar{a}\,1b}{}_{1c} = (2h_3 - 1)\, g^{b\bar{d}}\, R^{\bar{a}\bar{e}}{}_{\bar{d}}\, g_{c\bar{e}}\,. \tag{3.32}$$

Let us also show that the product of an NS_2 operator with an NS_1 operator can not give rise to an NS_2 operator. We have already analyzed the bosonic and fermionic currents appearing in the zero picture operator in (3.22). It remains to consider the $G^{S^1 + Y}$ term: the S^1 term in the $N = 1$ supercurrent will not contribute since the fermion associated to the tangent space to the circle is absent from the list of spacetime chiral primaries. The G^Y term will also not contribute to the operator product expansion. This follows from the identity

$$\langle (G^+_{-1/2} V^{\bar{a}}_{a.c.p.})(z_1) V^b_{c.p.}(z_2) V^c_{c.p.}(z_3) \rangle = 0\,. \tag{3.33}$$

This is a consequence of pulling off the G^+ supercurrent across the sphere.[6] The identity implies that one cannot produce an anti-chiral primary in the operator product expansion of $(G^+_{-1/2} V_{a.c.p.})(z_1)$ and $V_{c.p.}(z_2)$. Thus, we have proven the vanishing

$$\mathcal{A}^{1\bar{a}\,2b}{}_{2\bar{c}} = 0\,. \tag{3.34}$$

3.4 The NS_1-NS_1 operator product expansion

We turn to the remaining operator product expansion in the NS sector involving two operators of type 1. Up to normalization factors we have the operator product:

$$\int d^2 z\, \left(\frac{2}{k} \psi_A D^A_{x_1} + G^{S^1 + Y}_{-1/2} \cdot \right) \Phi_{h_1}(x_1; z)\, e^{i\sqrt{\frac{2}{k}} h_1 X}\, V^a_{c.p.}(z) \tag{3.35}$$

$$\times\, c\, e^{-\phi}\, \Phi_{h_2}(x_2; w)\, e^{i\sqrt{\frac{2}{k}} h_2 X(w)}\, V^b_{c.p.}(w)\,. \tag{3.36}$$

The first term can only potentially give rise to an NS_2 vertex operator in the minus one picture as it has an AdS_3 fermion. However, note that the resulting vertex operator in the (-1) picture will necessarily have a chiral primary in the Y sector and such an operator is not in the list of chiral primaries. Therefore, we find that

$$\mathcal{A}^{1a\,1b}{}_{2\bar{c}} = 0\,. \tag{3.37}$$

Let us similarly consider the second term in (3.35); this can potentially give rise to a NS_1 vertex operator in the minus one picture. As before, the term G^{S^1} does not contribute. We therefore focus on the G^Y term. In the Y sector, we have an operator product that is determined by the three-point function:

$$\langle (G^-_{-1/2} V^a_{c.p.})(z_1) V^b_{c.p.}(z_2) V^{\bar{c}}_{a.c.p.}(z_3) \rangle = 0\,. \tag{3.38}$$

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[6] An alternative in our context is to argue that due to the bound on the world sheet R-charge of the anti-chiral primaries that can appear in our space-time chiral primaries, a non-zero correlator (3.33) would violate R-charge conservation.

By utilizing the superconformal Ward identity one can indeed show that this correlation function vanishes.[7] Therefore we have the vanishing $\mathcal{N} = 2$ chiral ring structure constant:

$$\mathcal{A}^{1a\,1b}{}_{1c} = 0 \,. \tag{3.39}$$

The tallying reader realizes that in the NS sector, there is one remaining structure constant to compute in which NS_2 operators produce a NS_1 operator. We will discuss this possibility in subsection 3.8.

3.5 The NS$_2$-R operator product expansion

We turn to operator product expansions involving one NS sector and one Ramond sector vertex operator. We use the integrated Ramond vertex operator in the $(-\frac{1}{2})$ picture and the unintegrated NS_2 operator in the (-1) picture. The result is an unintegrated Ramond sector operator in the $(-\frac{3}{2})$ picture. We study the operator product:

$$\frac{1}{\sqrt{k}} \int d^2 z \; e^{-\frac{\phi}{2}} \Phi_{h_1}(x_1; z) \; S(x_1; z) \; e^{i\sqrt{\frac{2}{k}}\left(h_1 - \frac{1}{2}\right)X(z)} \; \Sigma^{\bar{a}}$$
$$\times \frac{1}{\sqrt{k}} \; c \; e^{-\phi} \; \Phi_{h_2}(x_2; w) \; \psi(x_2; w) \; e^{i\sqrt{\frac{2}{k}}(h_2 - 1)X(w)} \; V^{\bar{b}}_{a.c.p.}(w) \,. \tag{3.40}$$

In addition to the operator product expansions between the AdS_3 operators Φ_{h_i} and the operator product expansions in the circle sector, we use the expansions:

$$e^{-\frac{\phi(z)}{2}} \cdot e^{-\phi(w)} \approx \frac{e^{-\frac{3\phi(w)}{2}}}{(z-w)^{\frac{1}{2}}} \tag{3.41}$$

$$\frac{1}{\sqrt{k}} \psi(x_1; z) S(x_2; w) \approx 2(x_1 - x_2) \frac{\widetilde{S}(x_2; w)}{(z-w)^{\frac{1}{2}}} \tag{3.42}$$

$$V^{\bar{b}}_{a.c.p.}(z) \Sigma^{\bar{a}}(w) \approx \frac{R^{\bar{b}\bar{a}}{}_{\bar{c}} \Sigma^{\bar{c}}(w)}{(z-w)^{h^{\bar{b}}_Y}} \,. \tag{3.43}$$

In the last operator product we use the fact that $h^{\bar{b}}_Y = \frac{h_2 - 1}{k}$. We once again add the right moving contribution and proceed as before. In the limit $x_1 \to x_2$, we find that the leading non-zero contribution arises from the spin $h = h_3 = h_1 + h_2 - 1$. This precisely leads to a spacetime chiral primary generated by a world sheet Ramond sector operator in the $(-\frac{3}{2})$ picture. The contribution by this on-shell state is obtained from the residue in the h-integral and leads to a δ-function that localizes the z-integral. We obtain the closed string vertex operator:

$$\frac{4 C_{h_1,h_2}{}^{h_1+h_2-1} R^{\bar{b}\bar{a}}{}_c}{(2h_3 - 1)} \; c \, \bar{c} \, e^{-\frac{3\phi}{2} - \frac{3\tilde{\phi}}{2}} \; \Phi_{h_1+h_2-1}(x_2; w, \bar{w}) e^{i\left(h_1+h_2-\frac{3}{2}\right)X(w,\bar{w})} \; \widetilde{S}(x_2; w, \bar{w}) \; \Sigma^a(w, \bar{w}) \,. \tag{3.44}$$

By keeping track of the normalization of the $(-\frac{3}{2}, -\frac{3}{2})$ picture operator (see (2.64)) and using $C^{h_1+h_2-1}_{h_1 h_2} = 1$, we obtain the structure constant

$$\mathcal{B}^{2\bar{a}\,\bar{b}}{}_{\bar{c}} = (2h_3 - 1) R^{\bar{b}\bar{a}}{}_{\bar{c}} \,. \tag{3.45}$$

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[7]See e.g. [28] for the detailed derivation.

If we consider the operator product expansion of the Ramond operator with the NS_1 vertex operator we do not find a space-time chiral ring element in the $(-\frac{3}{2})$-picture as the spin field retains its chirality. We conclude that this structure constant is zero:

$$\mathcal{B}^{1\,a\,b}{}_c = 0 \, . \tag{3.46}$$

3.6 The R-R operator product expansion

Finally, we evaluate the operator product expansion involving one integrated Ramond vertex operator with picture number $(-\frac{1}{2})$ and an unintegrated Ramond vertex operator with picture number $(-\frac{1}{2})$. The left moving contribution is given by

$$\frac{1}{\sqrt{k}} \int d^2 z \; e^{-\frac{\phi}{2}} \Phi_{h_1}(x_1; z) \; S(x_1; z) \; e^{i\sqrt{\frac{2}{k}}\left(h_1 - \frac{1}{2}\right)X(z)} \; \Sigma^{\bar{a}}(z)$$

$$\times \frac{1}{\sqrt{k}} \; c \; e^{-\frac{\phi}{2}} \; \Phi_{h_2}(x_2; w) \; S(x_2; w) \; e^{i\sqrt{\frac{2}{k}}\left(h_2 - \frac{1}{2}\right)X(w)} \; \Sigma^{\bar{b}}(w) \, . \tag{3.47}$$

The bosonic ghost is transcribed trivially while the operator product expansions in the bosonic $AdS_3 \times S^1$ sectors are the same as before. The new operator product expansions involved in this calculation are

$$e^{-\frac{\phi(z)}{2}} e^{-\frac{\phi(w)}{2}} \approx \frac{e^{-\phi(w)}}{(z-w)^{\frac{1}{4}}} \tag{3.48}$$

$$S(x_1; z) S(x_2; w) \approx -\frac{x_1 - x_2}{(z-w)^{\frac{1}{2}}} \tag{3.49}$$

$$\Sigma^{\bar{a}}(z) \Sigma^{\bar{b}}(w) \approx \frac{R^{\bar{a}\bar{b}}{}_{\bar{c}} M^{\bar{c}}{}_c V^c_{c.p.}}{(z-w)^{h^c_Y - \frac{c_Y}{12}}} \, . \tag{3.50}$$

Here $M^{\bar{c}}{}_c$ is the real structure associated to the $N = 2$ superconformal theory on Y [24]. In a second step, we include once more the contribution of the right-movers and in the $x_1 \to x_2$ limit the leading non-zero term picks out the spin $j = h_1 + h_2 - 1$ in the Φ-Φ operator product expansion. Regarding the operator product expansion in the Y-sector we see that, provided that the R-charges of the Ramond ground states add up to a positive real number, we obtain a world sheet chiral primary whose conformal dimension is half of its R-charge (which in turn is determined by the sum of the R-charges of the Ramond ground states). Proceeding as before, we obtain the operator product expansion coefficient

$$\mathcal{C}^{\bar{a}\bar{b}}{}_{1c} = (2h_3 - 1) \; R^{\bar{a}\bar{b}}{}_{\bar{c}} M^{\bar{c}}{}_c \, . \tag{3.51}$$

It is clear from the derivation that it is impossible to obtain an AdS_3 fermion on the right hand side. Thus the structure constant associated to the R-R to NS_2 operator product is zero, $\mathcal{C}^{\bar{a}\bar{b}}{}_{2\bar{c}} = 0$.

3.7 The structure constants from topological data

Let us collect the results we have computed and simplify them by redefining the class of NS_1 operators using the Zamolodchikov metric:

$$O^{(-1)}_{\bar{a}, NS_1} = g_{\bar{a}b} \; O^{b\,(-1)}_{NS_1} \, . \tag{3.52}$$

In a more condensed notation, the non-vanishing structure constants of the spacetime chiral ring computed so far are:

$$\text{NS}_2{}^{\bar{a}} \times \text{NS}_2{}^{\bar{b}} \longrightarrow (2h_3 - 1) \, R^{\bar{a}\bar{b}}{}_{\bar{c}} \, \text{NS}_2{}^{\bar{c}} \tag{3.53}$$

$$\text{NS}_2{}^{\bar{a}} \times \text{NS}_{1_{\bar{b}}} \longrightarrow (2h_3 - 1) \, R^{\bar{a}\bar{c}}{}_{\bar{b}} \, \text{NS}_{1_{\bar{c}}} \tag{3.54}$$

$$\text{NS}_2{}^{\bar{a}} \times \text{Ra}^{\bar{b}} \longrightarrow (2h_3 - 1) \, R^{\bar{a}\bar{b}}{}_{\bar{c}} \, \text{Ra}^{\bar{c}} \tag{3.55}$$

$$\text{Ra}^{\bar{a}} \times \text{Ra}^{\bar{b}} \longrightarrow (2h_3 - 1) \, R^{\bar{a}\bar{b}}{}_{\bar{c}} \, \eta^{\bar{c}\bar{d}} \, \text{NS}_{1_{\bar{d}}} \,, \tag{3.56}$$

where $h_3 = h_1 + h_2 - 1$ and $\text{Ra}^{\bar{a}}$ denotes the Ramond sector operators. We have defined the topological metric $\eta^{\bar{c}\bar{d}} = M^{\bar{c}}{}_c \, g^{c\bar{d}}$ which is the two point function of world sheet chiral primaries in the twisted topological theory Y [24]. We conclude that these four classes of structure constants are entirely captured by the world sheet anti-chiral ring of theory Y.

Finally, we recall that our notation is compact. For instance, in equation (3.53), when we consider a space-time (c, c) ring element, we can combine left- and right moving operators, and would find the index structure:

$$\text{NS}_2{}^{\bar{a}_L, \bar{a}_R} \times \text{NS}_2{}^{\bar{b}_L, \bar{b}_R} \longrightarrow (2h_3 - 1) \, R^{\bar{a}_L \bar{b}_L}{}_{\bar{c}_L} \, R^{\bar{a}_R \bar{b}_R}{}_{\bar{c}_R} \, \text{NS}_2{}^{\bar{c}_L \bar{c}_R} \,, \tag{3.57}$$

where the ring structure constants correspond to the (a, a) world sheet ring and the indices need to be chosen such that the left-right world sheet vertex operator occurs in the chosen modular invariant string spectrum. This is but one example of how to unpack our compact notation in equations (3.53) to (3.56).

3.8 The NS$_2$-NS$_2$ to NS$_1$ structure constants

We are ready to study the odd duckling left over from the NS$_2$-NS$_2$ operator product expansion. The NS$_2$-NS$_2$ into NS$_1$ structure constant is the only case in which we have to compute the action of the supercurrent on the operators in the theory Y in order to compute the space-time chiral ring relation. Therefore, in this case, the space-time structure constant depends not only on the quantum numbers and world sheet chiral ring operators, but also on the three-point function of a world sheet superconformal descendant. To compute the structure constant, we consider explicit models. A first model is the counterpart to the $\mathcal{N} = 4$ superconformal model $AdS_3 \times S^3$ — we consider the T-dual background $AdS_3 \times S^1 \times MM \times M^4$ where the minimal model MM has central charge $c_{MM} = 3 - 6/k$ and M^4 represents a central charge six world sheet conformal field theory.[8] We then generalize our analysis to models $AdS_3 \times S^1 \times \prod_{i=1}^{3} MM_i$ where the three minimal model levels satisfy $\sum_{i=1}^{3} k_i^{-1} = k^{-1}$ for criticality.

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[8]For simplicity we ignore orbifold identifications.

3.8.1 The $\mathcal{N} = 4$ model revisited

The calculation of the structure constant proceeds as in previous subsections. We begin with the operator product

$$\frac{1}{\sqrt{k}} \int d^2z \, \Phi_{h_1}(x_1; z) e^{i\sqrt{\frac{2}{k}}(h_1-1)X(z)} \, \psi(x_1; z) \, \left(G^Y_{-1/2} \cdot V^{\bar{a}}_{a.c.p.} \right)(z)$$

$$\times \frac{1}{\sqrt{k}} \, c \, e^{-\phi} \, \psi(x_2; w) \, \Phi_{h_2}(x_2; w) \, e^{i\sqrt{\frac{2}{k}}(h_2-1)X(w)} \, V^{\bar{b}}_{a.c.p.}(w) \,. \tag{3.58}$$

The ghost and superghost factors, the bosonic factors and the AdS_3 fermions all have operator products that we have already encountered. We focus on the new operator product that arises in the Y-sector. We consider a theory $Y = MM \times M^4$ where M^4 is geometrical and therefore $M^4 = K3$ or $M^4 = T^4$. These theories have no (anti)-chiral primary of dimension smaller than one half, and therefore only the minimal model theory provides non-trivial operators in the vertex operators describing space-time chiral primaries.

Our conventions for the minimal model are those of [29].[9] The super parafermion fields are $\psi_{j,n,s}$ where j is the $su(2)$ spin, n is a spin component and s denotes the fermion number. The world sheet conformal dimension and $U(1)_R$-charge of the parafermions are given by

$$h(j, n, s) = \frac{j(j+1)}{k} - \frac{n^2}{4k} + \frac{s^2}{8} \,, \qquad q(j, n, s) = \frac{s}{2} - \frac{n}{k} \,. \tag{3.59}$$

The anti-chiral primaries that appear in (3.58) correspond to the fields $\psi_{j,2j,0}$. The $su(2)$ spin of the colliding operators are related to the AdS_3 spin h_i by the relation $j_i = h_i - 1$. From the operator product it is clear that in the Y sector we need the coupling between the descendant of an anti-chiral field and an anti-chiral primary field. For an NS_1 operator to appear as a result, the fusion should give rise to a chiral primary field. The relevant operator product is given by

$$(G^+_{-1/2}\psi_{j_1,2j_1,0})(z)\psi_{j_2,2j_2,0}(w) \approx \frac{C^{su(2)}{}_{j_1 j_2}{}^{\frac{k}{2}-\tilde{j}_3-1}}{(z-w)^{\frac{\tilde{j}_3-j_1-j_2}{k}+\frac{1}{2}}} \, \psi_{\frac{k}{2}-\tilde{j}_3-1,-2\tilde{j}_3+k,2}(w) + \dots \,. \tag{3.60}$$

In the parafermion variables a chiral primary corresponds to the field $\psi_{j,-2j,0}$. However there is an equivalence relation $\psi_{j,n,s} \equiv \psi_{k/2-j-1,n+k,s+2}$ which allows us to identify the state on the right hand side of (3.60) as a chiral primary $\psi_{\tilde{j}_3,-2\tilde{j}_3,0}$. The action of the operator $G^+_{-1/2}$ on the anti-chiral primary is to augment the s quantum number by two. Moreover, by R-charge conservation we obtain

$$1 - \frac{2j_1 + 2j_2}{k} = \frac{2\tilde{j}_3}{k} \,. \tag{3.61}$$

We define the spin $j_3 = k/2 - \tilde{j}_3 - 1$ in terms of which we find the simpler relation $j_3 = j_1 + j_2 - 1$. Furthermore, the conformal dimension of the chiral primary field in terms of the new variable is given by

$$h_{c.p.} = \frac{1}{2} - \frac{j_3 + 1}{k} = \frac{1}{2} - \frac{j_1 + j_2}{k} \,. \tag{3.62}$$

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[9]However, we work with a supersymmetric level $k = k_{bos} + 2$.

To summarize: from all possible fusions, we pick the one labelled by the spin $\tilde{j}_3$ as it leads to a chiral primary in the minimal model, which is crucial to obtain an NS_1 operator on the right hand side. The parafermion structure constant reduces to an $su(2)$ Wess-Zumino-Witten model structure constant [31]. The relevant $su(2)$ structure constant is recorded in equation (A.53) in appendix A. Given the fusion in the compact sector, we proceed as before and take the limit $x_1 \to x_2$. The leading non-zero contribution arises when the intermediate AdS_3 spin equals $h_3 = h_1 + h_2 - 2 = j_1 + j_2$, where we have used the relation between the spins in the AdS_3 and the compact sectors. Comparing with equation (3.62) and using the momentum along the S^1 direction we see that we have obtained a propagating on-shell state that corresponds to a spacetime chiral primary of the type NS_1 (2.24).

We perform a similar analysis for the right-movers and execute the h-integral as before. We obtain the structure constant [9]:

$$\mathcal{A}^{1\bar{a}2\bar{a}}{}_{3c} = (2h_3 - 1) \, C_{h_1,h_2}{}^{h_1+h_2-2} C^{su(2)}{}_{j_1 j_2}{}^{j_1+j_2-1} . \tag{3.63}$$

We note that since $h_3 = h_1 + h_2 - 2$, unlike the previous cases, we encounter a non-trivial structure constant (A.52) in the AdS_3 sector. At the same time we also have a non-trivial structure constant arising from the compact sector. As observed in [9] in the $\mathcal{N} = 4$ supersymmetric models, a key point is that the AdS_3 and $su(2)$ structure constants cancel up to a factor of ν^{-1}, and we are left with the structure constant proportional to the spins, namely $2h_3 - 1$. Of course, we obtain a non-zero structure constant only when the fusion is allowed by the $su(2)$ fusion rules [9].

3.8.2 A generalization

It would certainly be interesting to compute the NS_2-NS_2-NS_1 structure constant for an arbitrary choice of theory Y. To obtain the structure constant, we must evaluate the correlator

$$\langle G^{+,Y}_{-1/2} \cdot V^a_{a.c.p.}(z_1) V^b_{a.c.p.}(z_2) V^c_{a.c.p.}(z_3) \rangle \tag{3.64}$$

in theory Y. When we have a factorized theory Y, the $G^{+,Y}$ current is a sum of super currents in the individual factors. Then we must have that the operators in all factors but one are the identity operator. Otherwise world sheet R-charge conservation in the factor not containing the $G^+_{-1/2}$ term will set the structure constant to zero.

We shall settle for remarking how to compute the structure constant in an infinite but restricted class of $\mathcal{N} = 2$ theories. We choose the theory Y to be a product of three minimal models at levels $k_{i=1,2,3}$. This represents a large class of models since we can allow any levels k_i for the three minimal models as long as we choose $k^{-1} = \sum_{i=1}^{3} k_i^{-1}$. The operator $G^{+,Y}_{-1/2}$ in the theory Y becomes a sum of operators, each non-trivial in a given factor:

$$G^{+,Y}_{-1/2} = G^{+,MM_1}_{-1/2} + G^{+,MM_2}_{-1/2} + G^{+,MM_3}_{-1/2} . \tag{3.65}$$

As we argued, a set of chiral primary operators with non-zero structure constants must correspond to the identity operators in all but one minimal model. We label the single

non-trivial minimal model factor that enters a particular calculation by an index l. The on-shell constraints for the operators in the structure constant calculation read:

$$\text{NS}_1: \quad \frac{\tilde{j}_3}{k_l} = \frac{1}{2} - \frac{h_3}{k} = \frac{1}{2} - \frac{j_3 + 1}{k_l}$$

$$\text{NS}_2: \quad \frac{j_{1,2}}{k_l} = \frac{h_{1,2} - 1}{k} . \tag{3.66}$$

Space-time R-charge conservation still implies that $h_3 = h_1 + h_2 - 2$ while world sheet R-charge conservation enforces $j_3 = j_1 + j_2 - 1$. The structure constant is computed as in the previous subsection, and reads:

$$\mathcal{A}^{1\bar{a}2\bar{a}}{}_{3c}(l) = (2h_3 - 1)C_{h_1,h_2}{}^{h_1+h_2-2}C^{su(2)_{k_l}}{}_{j_1 j_2}{}^{j_1+j_2-1} . \tag{3.67}$$

To evaluate the structure constant, we make an important observation. The structure constant for the AdS_3 model simplifies drastically and generically when the space-time R-charge constraint is taken into account:[10]

$$C_{h_1,h_2}{}^{h_1+h_2-2}C^{su(2)_{k_l}}{}_{j_1 j_2}{}^{j_1+j_2-1} = \nu^{-1} \frac{\gamma\left(\frac{2h_1-2}{k}\right)\gamma\left(\frac{2h_2-2}{k}\right)}{\gamma\left(\frac{2h_1+2h_2-4}{k}\right)} \times \frac{\gamma\left(\frac{2j_1+2j_2}{k_l}\right)}{\gamma\left(\frac{2j_1}{k_l}\right)\gamma\left(\frac{2j_2}{k_l}\right)} = \nu^{-1} . \tag{3.68}$$

Thus, for this large class of $\mathcal{N} = 2$ models, we again find the same structure constant after a still more intricate cancellation. It would be interesting to calculate these structure constants for more general models Y or to prove that the cancellation we observed for our class of models persists in all cases.

3.9 Additional observations on the structure constants

We saw that with a convenient normalization of the vertex operators, four of the structure constants are proportional to $2h_3 - 1$ where h_i are the world sheet spins of the operators and $h_3 = h_1 + h_2 - 1$. The dependence on the Y theory was through the structure constants of its (anti-) chiral ring. This generalizes the structure constants in the literature [9] for the $\mathcal{N} = 4$ superconformal models to a large class of $\mathcal{N} = 2$ superconformal models. In particular, the previously computed chiral ring structure constants, e.g. in [9, 10, 23] were computed in backgrounds which contain at least one S^3 factor which geometrically represents the R-symmetry of an $N = 4$ superconformal algebra. On the one hand, our approach is based only on operator product expansions and thus simplifies the calculations; on the other hand they are valid for all backgrounds containing an $AdS_3 \times S^1$ factor and are therefore valid in a much larger class of models. Indeed, this is the reason why we had to derive new results in the $N = 2$ superconformal world sheet theory in appendix B. Without these new results in the worldsheet theory it would not have been possible to obtain the spacetime operator product expansions in our larger class of models. We stress the fact that we never needed to invoke a cancellation of poles in three-point function correlators,

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[10]This is independent of whether $2h_i - 1$ is an integer.

nor did we make use of the physical two-point functions. With respect to the treasure trove of useful results for correlators of string theory in AdS_3 [4, 5, 33–50], we make the following remarks. Firstly, our results are efficient in concentrating on only those ingredients that are necessary to compute space-time chiral ring structure constants. Secondly, we obtain the explicit final result for the structure constants of the $\mathcal{N} = 2$ superconformal field theories in space-time for the first time. Previously, these results have been marshalled to compute the chiral ring structure constants only for theories with a larger space-time symmetry group.

Introducing the number $n_i = 2h_i - 1 + kw_i$, we see that the universal part of the structure constant equals $n_1 + n_2 - 1$, which we recognize as a combinatorial structure constant arising in symmetric orbifold conformal field theory, where the numbers n_i have the interpretation as the lengths of permutation cycles. This is described in [30] (and combinatorially in e.g. [7]).

In the case of $\mathcal{N} = 4$ space-time superconformal symmetry, a non-renormalization theorem was proven that shows that the structure constants of the chiral ring are covariantly constant on the moduli space [28]. This allows for a natural matching of structure constants between a bulk NSNS string point and a symmetric orbifold conformal field theory [9, 10]. The non-renormalization theorem shows that in $\mathcal{N} = 4$ theories, all marginal deformations can be constructed from anti-chiral primaries, and in these directions, structure constants of the chiral ring are covariantly constant [28]. The proof of the theorem makes use of the full $\mathcal{N} = 4$ algebra and therefore does not necessarily hold in backgrounds with a $\mathcal{N} = 2$ superconformal symmetry.

Our results for $\mathcal{N} = 2$ theories, for which in principle structure constants can depend on marginal deformations built on space-time chiral primaries, beg the question of why they seem similarly universal, after explicit calculation. Of course, there is a factorized dependence on the world sheet chiral ring structure constants of the theory Y which can vary under space-time marginal deformations, and relevant deformation in Y. The combinatorial prefactor $n_1 + n_2 - 1$ however, appears to be universal.

For the other structure constant we computed, we confirmed that in $\mathcal{N} = 4$ theories, it is proportional to $n_1 + n_2 - 3$, associated to different and interesting combinatorics in the symmetric orbifold [30].[11] Moreover, for a class of $\mathcal{N} = 2$ superconformal models that arise as a tensor product of minimal models, we showed that the same combinatorial prefactor appears in the structure constant. This raises the question as to whether these properties will continue to be true for a generic $\mathcal{N} = 2$ model.

4 Conclusions

In this paper, we studied the bulk string theory background $AdS_3 \times S^1 \times Y$ where Y corresponds to a conformal field theory with $N = 2$ superconformal symmetry on the world sheet. This is a class of string theory backgrounds dual to a space-time conformal field theory with $\mathcal{N} = 2$ superconformal symmetry. The background with Neveu-Schwarz-Neveu-Schwarz flux is exactly solvable in the inverse string tension expansion and allows a calculation of the spectrum and correlation functions exactly in the parameter α' divided

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[11]See also [7] for the relation to a co-product in the topological conformal field theory in the $\mathcal{N} = 4$ context.

by the radius of curvature squared. This is a large class of backgrounds with extended supersymmetry and they therefore have a chiral ring. These backgrounds and observables form a wonderful testing ground to attempt to prove a topological subsector of the anti-de Sitter/conformal field theory correspondence.

Firstly, we fully determined the spectrum of space-time chiral primaries using the exact world sheet description of the string propagating in $AdS_3 \times S^1 \times Y$. Our results classify the space-time chiral primaries in terms of the chiral primaries, anti-chiral primaries, and Ramond ground states of the $N = 2$ superconformal world sheet theory Y. An upper bound on the world sheet R-charge of the operators that enter the calculation emerges. It lays bare a close connection between the space-time chiral primaries and supersymmetry preserving relevant deformations of theory Y. Secondly, we computed a subset of the structure constants of the chiral primary ring. Our method of calculation consisted in determining the leading order in the chiral operator product expansion by exploiting the operator product of the corresponding world sheet vertex operators. The method concentrates on purely topological data in the space-time theory and gives rise to results that are simple and universal. We applied the general method to all structure constants involving operators with zero spectral flow.

We demonstrated that the space-time chiral ring structure constants involved the world sheet chiral ring structure constants, the Zamolodchikov metric in the chiral sector as well as the world sheet real structure that relates two canonical bases of Ramond ground states. When expressed in terms of suitably chosen space-time chiral primaries, we could show that a subset of space-time structure constants only depend on the topological data of the theory Y. We also noted a dependence of a structure constant on a global superconformal descendant. In the $\mathcal{N} = 4$ supersymmetric setting, it was demonstrated that this structure constant is tied to a co-product defined in terms of the central charge six factor of theory Y. It is a definite challenge to extend this algebraic structure to the $\mathcal{N} = 2$ setting.

There is a clear open problem at hand, which is to compute all possible structure constants among the full set of chiral primaries, including all operators with non-zero winding. It is tempting to speculate that the number $n_i = 2h_i - 1$ appearing in the structure constants will be generalized to $n_i = 2h_i - 1 + kw_i$ for operators including winding, but it is important to demonstrate this explicitly.

If non-renormalization theorems exist for these backgrounds, they need to be proven. They are obviously crucial in understanding the matching of structure constants at different points in the moduli space, accessible in the bulk or boundary theory.

There is an intriguing global question to be answered. Clearly, the algebraic $N = 2$ structure of the theory Y, including its relevant deformations, structure constants and Zamolodchikov metric enter the topological theory in spacetime. Can we express the space-time chiral ring in terms of these world sheet structures? Our work gives a partial answer to this question, but more work is needed to characterize the precise relation. The full answer may invoke the topological-anti-topological structure of the theory Y, and may involve a symmetric group Frobenius algebra. Finally, we can ask once more whether there is a reformulation of the space-time theory that renders the calculation of these structure constants even simpler by projecting onto space-time chiral primaries at every step.

A The world sheet theory on $AdS_3 \times S^1$

In this appendix, we gather data on the world sheet conformal field theories that are necessary to perform the calculations in the bulk of the paper. The $AdS_3 \times S^1 \times Y$ target space gives rise to largely factorized world sheet conformal field theories. We discuss these in turn.

A.1 The supersymmetric AdS_3 factor

We define the affine currents J^A of a supersymmetric $sl(2,\mathbb{R})$ WZW model at level k, with operator product expansions:[12]

$$J^A(z)J^B(w) \approx \frac{k}{2}\frac{\eta^{AB}}{(z-w)^2} + \frac{i\epsilon^{ABC}\eta_{CD}J^D(w)}{z-w} \tag{A.1}$$

$$J^A(z)\psi^B(w) \approx \frac{i\epsilon^{ABC}\eta_{CD}\psi^D(w)}{z-w} \tag{A.2}$$

$$\psi^A(z)\psi^B(w) \approx \frac{k}{2}\frac{\eta^{AB}}{z-w}, \tag{A.3}$$

with $\eta^{AB} = \mathrm{diag}(1,1,-1)$ and $\epsilon^{123} = 1$. One can define a combination of currents j^A such that they form a bosonic $sl(2,\mathbb{R})$ affine algebra at level $k+2$:

$$j^A = J^A + \frac{i}{k}\epsilon^{ABC}\eta_{BD}\eta_{CE}\psi^D\psi^E. \tag{A.4}$$

The bosonic currents j^A have regular operator product expansion with the fermions. The three fermions ψ^A generate an $sl(2,\mathbb{R})$ algebra at level -2. The algebra is generated by the currents

$$\hat{j}^A = -\frac{i}{k}\epsilon^{ABC}\eta_{BD}\eta_{CE}\psi^D\psi^E. \tag{A.5}$$

In component form the currents read:

$$\hat{j}^1 = \frac{2i}{k}\psi^2\psi^3, \qquad \hat{j}^2 = \frac{2i}{k}\psi^3\psi^1, \qquad \hat{j}^3 = -\frac{2i}{k}\psi^1\psi^2. \tag{A.6}$$

The total current J^A is then the sum of the bosonic and fermionic ones:

$$J^A = j^A + \hat{j}^A. \tag{A.7}$$

We transform the fermions and the currents into a light cone $(+,-,3)$ basis by defining the combinations

$$\psi^{\pm} = \psi^1 \pm i\psi^2, \qquad \hat{j}^{\pm} = \hat{j}^1 \pm i\hat{j}^2. \tag{A.8}$$

Then the components are

$$\hat{j}^+ = -\frac{2}{k}\psi^3\psi^+, \qquad \hat{j}^3 = \frac{1}{k}\psi^+\psi^-, \qquad \hat{j}^- = \frac{2}{k}\psi^3\psi^-. \tag{A.9}$$

In the new basis, the non-vanishing operator product expansions of the fermions read:

$$\psi^+(z)\psi^-(0) \approx \frac{k}{z} \tag{A.10}$$

$$\psi^3(z)\psi^3(0) \approx -\frac{k}{2z}. \tag{A.11}$$

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[12]We follow the notations and conventions of [8].

The operator product expansions between the currents and the fermions take the form:

$$\hat{j}^+(z)\psi^+(0) \approx O(z)\,, \qquad \hat{j}^+(z)\psi^3(0) \approx -\frac{\psi^+(0)}{z}\,, \qquad \hat{j}^+(z)\psi^-(0) \approx -\frac{2\psi^3(0)}{z}$$

$$\hat{j}^3(z)\psi^+(0) \approx \frac{\psi^+(0)}{z}\,, \qquad \hat{j}^3(z)\psi^3(0) \approx O(z)\,, \qquad \hat{j}^3(z)\psi^-(0) \approx -\frac{\psi^-(0)}{z}$$

$$\hat{j}^-(z)\psi^+(0) \approx \frac{2\psi^3(0)}{z}\,, \qquad \hat{j}^-(z)\psi^3(0) \approx \frac{\psi^-(0)}{z}\,, \qquad \hat{j}^-(z)\psi^-(0) \approx O(z)\,. \qquad \text{(A.12)}$$

The metric on the tangent space in the $(+,-,3)$ basis is off-diagonal:

$$g_{AB} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\,. \qquad \text{(A.13)}$$

Thus we have $\psi_+ = \frac{1}{2}\psi^-$ and $\psi_- = \frac{1}{2}\psi^+$, $\psi_3 = -\psi^3$ and similarly for the currents. The stress tensor and its $N = 1$ superpartner for the supersymmetric AdS_3 factor are

$$T = \frac{1}{k}j^A j_A - \frac{1}{k}\psi^A \psi_A\,, \qquad G = \frac{2}{k}\left(\psi^A j_A + \frac{2i}{k}\psi^1\psi^2\psi^3\right)\,. \qquad \text{(A.14)}$$

These are standard structures in $N = 1$ supersymmetric Wess-Zumino-Witten models.

A.2 The circle sector

In addition to the three world sheet fermions associated to the space tangent to the three-dimensional anti-de Sitter space, we introduce a fermion ψ^0 that is tangent to the circle S^1. Its bosonic partner J^0 is the world sheet current that arises from translation invariances along the circle associated to the space-time $\mathrm{U}(1)_R$ charge. These world sheet fields have the operator product expansions

$$J^0(z)J^0(w) \approx \frac{1}{(z-w)^2} \qquad \text{(A.15)}$$

$$\psi^0(z)\psi^0(w) \approx \frac{1}{z-w} \qquad \text{(A.16)}$$

$$J^0(z)\psi^0(w) \approx 0\,. \qquad \text{(A.17)}$$

This factor world sheet conformal field theory is free.

A.2.1 Spacetime R-charge from the circle direction

We recall that the space-time $\mathcal{N} = 2$ superalgebra includes a $\mathrm{U}(1)_R$ current. The spacetime R-charge operator which we denote by $\mathcal{Q}_0$, corresponds to the zero mode along the S^1 direction [8]

$$\mathcal{Q}_0 = \sqrt{2k}\oint J^0 = \oint e^{-\psi}\psi^0\,. \qquad \text{(A.18)}$$

When we bosonize the current $J^0(z)$:

$$J^0 = i\partial X\,, \qquad \text{(A.19)}$$

any vertex operator on the world sheet with spacetime R-charge Q_{st} can be represented in the form

$$V_{Q_{\text{st}}} = e^{i\frac{Q_{\text{st}}}{\sqrt{2k}}X} \,. \tag{A.20}$$

We make good use of this fact in the bulk of the paper.

A.2.2 The bosonization of four fermions

The four fermions can be paired up and it is useful to bosonize them in order to write down vertex operators in the Ramond sector. We define bosons $H_{0,1}$ through the formulas

$$\partial H_1 = -\frac{2}{k}\psi^1\psi^2 \,, \qquad \partial H_0 = -i\sqrt{\frac{2}{k}}\psi^0\psi^3 \,. \tag{A.21}$$

The bosons H_I satisfy the operator product expansions

$$H_I(z)H_J(w) \approx -\delta_{IJ}\ln(z-w) \quad \text{for} \quad I, J = 0, 1 \,. \tag{A.22}$$

The fermions are exponentials in the bosons:

$$\frac{1}{\sqrt{k}}\psi^\pm = \frac{1}{\sqrt{k}}(\psi^1 \pm i\psi^2) = e^{\pm iH_1} \tag{A.23}$$

$$\frac{1}{\sqrt{2}}\psi^0 \mp \frac{1}{\sqrt{k}}\psi^3 = e^{\pm i\pi N_1}e^{\pm iH_0} \,, \tag{A.24}$$

where we have included the cocycle factor that depends on the number operator $N_1 = i\oint \partial H_1$. This factor ensures that the fermion vertex operators anti-commute appropriately. Their presence invites us to define hatted scalars

$$\hat{H}_1 = H_1 \qquad \hat{H}_0 = H_0 + \pi\,N_1 \,. \tag{A.25}$$

In terms of the hatted scalars, the bosonization formulae simplify. The fermionic currents $\hat{j}^A$ can be written as:

$$\hat{j}^3 = i\,\partial\hat{H}_1 \,, \qquad \hat{j}^\pm = \pm e^{\pm i\hat{H}_1}(e^{-i\hat{H}_0} - e^{+i\hat{H}_0}) \,. \tag{A.26}$$

Thus, we have conveniently expressed essential ingredients in the $AdS_3 \times S^1$ conformal field theories in terms of bosons only.

A.3 Operator products in the spin-field basis

In this section we list the operator product expansions between the spin fields and fermions used in the main text. We define a column matrix S^α which spans the vector space of the four-dimensional spin fields:

$$S^\alpha \leftrightarrow \begin{pmatrix} e^{\frac{i}{2}\hat{H}_1 + \frac{i}{2}\hat{H}_0} \\ e^{-\frac{i}{2}\hat{H}_1 + \frac{i}{2}\hat{H}_0} \\ e^{+\frac{i}{2}\hat{H}_1 - \frac{i}{2}\hat{H}_0} \\ e^{-\frac{i}{2}\hat{H}_1 - \frac{i}{2}\hat{H}_0} \end{pmatrix} \,. \tag{A.27}$$

The spin fields represent the Clifford algebra of the fermion zero modes:

$$\frac{1}{\sqrt{k}}\psi^m(z)S^\alpha(w) \approx \frac{(\gamma^m)^\alpha{}_\beta S^\beta}{(z-w)^{\frac{1}{2}}} \,. \tag{A.28}$$

The spin fields multiply to form the charge conjugation matrix C:

$$S^\alpha(z)S^\beta(w) \approx \frac{C^{\alpha\beta}}{(z-w)^{\frac{1}{2}}} \,. \tag{A.29}$$

The $so(3,1)$ Clifford algebra

$$\{\gamma^m,\gamma^n\} = 2\eta^{mn} \,, \tag{A.30}$$

with metric $\eta = \mathrm{diag}(+,+,-,+)$ is explicitly represented by

$$\gamma_1 = \sigma_1 \otimes \mathbb{I}_2 \,, \qquad \gamma_2 = \sigma_2 \otimes \mathbb{I}_2 \,, \qquad \gamma_3 = -i\sigma_3 \otimes \sigma_2 \,, \qquad \gamma_4 = \sigma_3 \otimes \sigma_1 \,, \tag{A.31}$$

where σ_a are the usual Pauli matrices and $\mathbb{I}_2$ is the two by two identity matrix. The charge conjugation matrix C satisfies

$$C\,\gamma_m\,C^{-1} = -(\gamma_m)^T \tag{A.32}$$

where T denotes the transpose. The charge conjugation matrix in our representation is:

$$C = \sigma_2 \otimes \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \,. \tag{A.33}$$

To obtain these results one needs to take care in choosing appropriate branch cuts for the spin fields and their operator product expansions. For example, the leading term in a spin field operator product $e^{i\frac{\hat{H}}{2}}(z)e^{-i\frac{\hat{H}}{2}}(w) \approx (z-w)^{-\frac{1}{4}}e^{\frac{i}{2}(\hat{H}(z)-\hat{H}(w))}$ picks up a fourth root of -1 when we exchange the positions of the operators. These roots must be chosen consistently in order to reproduce the claimed operator expansions for the spin fields.

A.4 The operators and operator products in the $sl(2,\mathbb{R})$ model

An affine primary field $\Phi_h(x;z)$ of spin h with respect to the $sl(2,\mathbb{R})$ current algebra satisfies the expansion:

$$j^A(z)\Phi_h(x;z) \approx -\frac{D_x^A\Phi_h(x;z)}{z-w} \,, \tag{A.34}$$

where the differential operators D_x^A are defined as

$$D_x^- = \partial_x \,, \qquad D_x^3 = x\frac{\partial}{\partial x} + h \,, \qquad D_x^+ = x^2\partial_x + 2h\,x \,. \tag{A.35}$$

The differential operators D_x^A represent the global $sl(2,\mathbb{R})$ subalgebra. Using the variable x is a handy way to parameterize states in $sl(2,\mathbb{R})$ representations. For instance, it is possible to form x dependent linear combinations of the currents, fermions and spin fields

that render manifest their transformation properties under the $sl(2, \mathbb{R})$ symmetry. One defines [4]

$$j(x; z) = -j^+(z) + 2xj^3(z) - x^2 j^+(z). \tag{A.36}$$

Using equation (A.34) one verifies that

$$j(x_1; z)\Phi_h(x_2; w) \approx \frac{1}{z - w}\left((x_1 - x_2)^2 \partial_{x_2} - 2h(x_1 - x_2)\right)\Phi_h(x_2; w). \tag{A.37}$$

A similar combination of currents can also be used to define $\hat{j}(x; z)$ and $J(x; z)$ for the fermionic and supersymmetric currents respectively — indeed, they transform in the same three-dimensional representation. The operator product expansion between the currents can be rewritten as

$$J(x_1, z)J(x_2, w) \approx k\frac{(x_1 - x_2)^2}{(z - w)^2} + \frac{1}{z - w}\left((x_1 - x_2)^2 \partial_{x_2} + 2(x_1 - x_2)\right)J(x_2, w). \tag{A.38}$$

The three fermions can similarly be combined:

$$\psi(x; z) = -\psi^+(z) + 2x\psi^3(z) - x^2\psi^-(z). \tag{A.39}$$

Then, first of all one can verify the operator product expansion

$$\hat{j}^A(z)\psi(x; z) \approx -\frac{D_x^A \psi(x; z)}{z - w}, \tag{A.40}$$

where the D_x^A are as in (A.35) with $h = -1$. This shows that $\psi(x)$ is an $sl(2, \mathbb{R})$ primary field with spin equal to -1. One can then rewrite the fermion operator product expansions:

$$\psi(x_1; z_1)\psi(x_2; z_2) \approx k\frac{(x_{12})^2}{z_{12}} \tag{A.41}$$

$$\hat{j}(x_1; z)\psi(x_2, w) \approx \frac{1}{z - w}\left((x_1 - x_2)^2 \partial_{x_2} + 2(x_1 - x_2)\right)\psi(x_2, w). \tag{A.42}$$

For the spin fields, we define the linear combinations:

$$S(x; z) = x\, e^{-\frac{i}{2}\hat{H}_1 + \frac{i}{2}\hat{H}_0} + e^{\frac{i}{2}\hat{H}_1 - \frac{i}{2}\hat{H}_0} \tag{A.43}$$

$$\widetilde{S}(x; z) = -x\, e^{-\frac{i}{2}\hat{H}_1 - \frac{i}{2}\hat{H}_0} + e^{\frac{i}{2}\hat{H}_1 + \frac{i}{2}\hat{H}_0}. \tag{A.44}$$

One can check that these transforms as spin $h = -\frac{1}{2}$ fields. The fields S and $\tilde{S}$ have opposite four-dimensional chirality. Using the operator product expansions (A.28) and (A.29), one derives the operator product expansions for the fields in the x-basis:

$$S(x_1; z)S(x_2; w) \approx -\frac{x_1 - x_2}{(z - w)^{\frac{1}{2}}} \tag{A.45}$$

$$\frac{1}{\sqrt{k}}\psi(x_1; z)S(x_2; w) \approx 2(x_1 - x_2)\frac{\widetilde{S}(x_2; w)}{(z - w)^{\frac{1}{2}}}. \tag{A.46}$$

A.4.1 The operator products of bosonic primary fields

The primary fields Φ_h satisfy the mutual operator product expansion — to avoid clutter we are only presenting the holomorphic dependence on $x_1 - x_2$ and $z - w$:

$$\Phi_{h_1}(x_1; z)\Phi_{h_2}(x_2; w) \approx \int dh \, \frac{(x_1 - x_2)^{h-h_1-h_2}}{(z-w)^{-\frac{h_1(h_1-1)}{k} - \frac{h_2(h_2-1)}{k} + \frac{h(h-1)}{k}}} C_{h_1 h_2}{}^h \, \Phi_h(x_2; w) + \dots \tag{A.47}$$

The AdS_3 structure constant $C_{h_1 h_2}{}^{h_3}$ is given by

$$C_{h_1 h_2}{}^{h_3} = \frac{\nu^{1-h_1-h_2-h_3}\Upsilon(b)}{\Upsilon(b(h_1+h_2+h_3)-b)} \frac{\Upsilon(2bh_1-b)\Upsilon(2bh_2-b)\Upsilon(2bh_3-b)}{\Upsilon(bh_1+bh_2-bh_3)\Upsilon(bh_2+bh_3-bh_1)\Upsilon(bh_3+bh_1-bh_2)} \tag{A.48}$$

where $b^2 = \frac{1}{k}$, ν is a constant, and $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$. The special function Υ introduced in [32] satisfies special shift properties

$$\Upsilon(x+b) = b^{1-2bx} \gamma(bx) \, \Upsilon(x) \,, \qquad \Upsilon\left(x + \frac{1}{b}\right) = b^{-1+\frac{2x}{b}}\gamma\left(\frac{x}{b}\right) \Upsilon(x) \,, \tag{A.49}$$

whenever x lies outside the range $0 < \text{Re}(x) < b + \frac{1}{b}$. The structure constant is obtained from the three point functions $C_H(h_1, h_2, h_3)$ of the theory on Euclidean AdS_3 derived in [33] by the relation:

$$C_{h_1 h_2}{}^{h_3} = \frac{C_H(h_1, h_2, h_3)}{C_H(1, h_3, h_3)} \,. \tag{A.50}$$

We will be interested in the form of the operator product expansion coefficient for two particular cases:

Case 1. For $h_3 = h_1 + h_2 - 1$ we find the structure constant

$$C_{h_1 h_2}{}^{h_1+h_2-1} = 1 \,. \tag{A.51}$$

It is crucial that this is a universal constant independent of the spins of the primaries.

Case 2. For $h_3 = h_1 + h_2 - 2$ we find the structure constant:

$$C_{h_1 h_2}{}^{h_1+h_2-2} = \frac{1}{\nu\gamma\left(\frac{1}{k}\right)} \frac{\gamma\left(\frac{(2h_1-2)}{k}\right)\gamma\left(\frac{(2h_2-2)}{k}\right)}{\gamma\left(\frac{2h_3}{k}\right)} \,. \tag{A.52}$$

This structure constant depends on the spins.

A.5 the operator product coefficients of $su(2)$

The operator product coefficients of $su(2)$ at level k can be gleaned from [9] after a convenient renormalization of the operators:

$$C^{su(2)}{}_{j_1 j_2}{}^{j_3} = N_{j_1 j_2}{}^{j_3} P(j+1) \prod_{i=1}^{3} \frac{P(j-2j_i)}{P(2j_i)} \frac{P(2j_3)}{P(2j_3+1)} \,. \tag{A.53}$$

Here the function $P(s)$ is defined as a finite product:

$$P(s) = \prod_{n=1}^{s} \gamma\left(\frac{n}{k}\right), \tag{A.54}$$

with $P(0) = 1$, and $j = j_1 + j_2 + j_3$. As noted in [9], whenever $2h_i$ and $h_1 + h_2 + h_3$ are integer valued, the structure constants (A.48) and (A.53) multiply into a constant when $j_i = h_i - 1$. In the bulk of the paper, we encounter a further generalization of this property. We also exploit the fact that the structure constants of the super parafermions are essentially given by those of the $su(2)$ Wess-Zumino-Witten model [31].

B Extended superconformal field theory

The conformal field theory Y is an $N = (2,2)$ superconformal field theory with central charge $c_Y = 9 - 6/k$. The left moving superconformal symmetry is generated by the stress tensor T, the supercurrents $G^{\pm}$ and a $U(1)_R$ current J_R. In this appendix we omit the explicit Y indices that would denote the fact that symbols refer to the internal conformal field theory Y alone. To set the stage, we begin with a brief review of (anti-)chiral rings and Ramond ground states.

B.1 Chiral primaries, spectral flow and a real structure

Chiral and anti-chiral primary states are defined by the conditions [15]:

$$G^{+}_{-\frac{1}{2}} V_{c.p.}^{a} = 0, \qquad G^{\pm}_{n+\frac{1}{2}} V_{c.p.}^{a} = 0, \qquad \text{for} \quad n \geq 0 \tag{B.1}$$

$$G^{-}_{-\frac{1}{2}} V_{a.c.p.}^{\bar{a}} = 0, \qquad G^{\pm}_{n+\frac{1}{2}} V_{a.c.p.}^{\bar{a}} = 0, \qquad \text{for} \quad n \geq 0. \tag{B.2}$$

From the supersymmetry algebra and unitarity one can check that they satisfy the extremal conditions $h^a = \frac{q^a}{2}$ and $h^{\bar{a}} = -\frac{q^{\bar{a}}}{2}$ respectively. These operators have regular operator product expansions and one can define a chiral and anti-chiral ring [15]:

$$V_{c.p.}^{a}(z) V_{c.p.}^{b}(w) \approx R^{ab}{}_{c} V_{c.p.}^{c}(w), \tag{B.3}$$

$$V_{a.c.p.}^{\bar{a}}(z) V_{a.c.p.}^{\bar{b}}(w) \approx R^{\bar{a}\bar{b}}{}_{\bar{c}} V_{a.c.p.}^{\bar{c}}(w), \tag{B.4}$$

where the ring structure constants $R^{ab}{}_{c}$ and $R^{\bar{a}\bar{b}}{}_{\bar{c}}$ are related by complex conjugation (namely the assumed $\mathbb{Z}_2$ symmetry that charge conjugates chiral into anti-chiral primaries).

In this section we write down the basic operator product expansions between chiral primaries, anti-chiral primaries and Ramond ground states in the Y-theory. There are a few important ingredients that enter the derivations. The first is the existence of spectral flow in $N = 2$ theories. Spectral flow by an amount η acts on operators and shifts their R-charges and conformal dimensions according to the formulae:

$$J_{R,0} \to J_{R,0} + \frac{c}{3}\eta, \qquad L_m \to L_m + \eta J_{R,m} + \frac{c}{6}\eta^2 \delta_{m,0}. \tag{B.5}$$

The $J_{R,m}$ and L_m are the modes of the R-current and stress tensor respectively. The NS and Ramond sectors are related by spectral flow by an amount $\eta = \pm\frac{1}{2}$ [15]. In particular,

Ramond ground states are related by spectral flow by $\eta = \pm \frac{1}{2}$ to (anti-)chiral primary operators in the NS sector.

The second ingredient is the bosonization of the R-current in terms of a canonically normalized scalar Z:

$$J_R(z) = \sqrt{\frac{c}{3}}\, i\, \partial Z\,. \tag{B.6}$$

Thus, an operator with worldsheet R-charge q can be represented as the product

$$O_q = e^{i\sqrt{\frac{3}{c}}q\,Z}\, \widetilde{O}\,, \tag{B.7}$$

where the operator $\widetilde{O}$ has zero world sheet R-charge. We shall use this fact extensively. An (anti-) chiral primary of charge q^a ($q^{\bar{a}}$) can therefore be written in the product form:

$$V_{c.p.}^a = e^{i\sqrt{\frac{3}{c}}q^a Z}\Pi^a\,, \qquad V_{a.c.p.}^{\bar{a}} = e^{i\sqrt{\frac{3}{c}}q^{\bar{a}} Z}\Pi^{\bar{a}}\,. \tag{B.8}$$

We assume that the operators $\Pi^a, \Pi^{\bar{a}}$ decouple from the $\mathrm{U}(1)_R$ sector. Note that the unique generalized volume operator of charge $\frac{c}{3}$ is purely in the $\mathrm{U}(1)_R$ sector.

The third ingredient is a useful way to think of the connection between NS and Ramond sectors of $(2,2)$ superconformal theories that is advanced in [24]. If we start with the lowest R-charge Ramond ground state, we can generate all other ground states by acting on it with the chiral ring:

$$\begin{aligned} \Sigma^a &= V_{c.p.}^a \cdot \Sigma_0 \\ &= e^{i\sqrt{\frac{3}{c}}\left(q^a - \frac{c}{6}\right)Z}\,\Pi^a\,, \end{aligned} \tag{B.9}$$

where we have used the fact that the Ramond ground state $\Sigma_0 = e^{-i\sqrt{\frac{c}{12}}Y}$ is purely written in terms of the $\mathrm{U}(1)_R$ sector. In the final expression in (B.9) we also see how the Ramond ground state is related to the chiral primary in the NS sector by spectral flow by half a unit. Alternatively if we start with the highest R-charge Ramond sector ground state, we generate all other ground states by acting with the anti-chiral ring:

$$\begin{aligned} \Sigma^{\bar{a}} &= V_{a.c.p.}^{\bar{a}} \cdot \bar{\Sigma}_0 \\ &= e^{i\sqrt{\frac{3}{c}}\left(q^{\bar{a}} + \frac{c}{6}\right)Z}\,\Pi^{\bar{a}}\,, \end{aligned} \tag{B.10}$$

where $\bar{\Sigma}_0 = e^{i\sqrt{\frac{c}{12}}Z}$. This means that Ramond ground states can be canonically associated to chiral ring elements (a) or to anti-chiral ring elements $(\bar{a})$.

The last ingredient is a relation between these two choices of basis of Ramond ground states. They are related by a real structure, $M^a{}_{\bar{a}}$ [24]:

$$\Sigma^a = M^a{}_{\bar{a}}\,\Sigma^{\bar{a}}\,. \tag{B.11}$$

The basis change $M^a{}_{\bar{a}}$ carries R-charge equal to $2q^a - \frac{c}{3}$. One can similarly define its complex conjugate $M^{\bar{a}}{}_a$ which has R-charge $-2q_a + \frac{c}{3}$. The complex conjugate matrix equals the inverse matrix:

$$M^a{}_{\bar{a}}M^{\bar{a}}{}_b = \delta^a{}_b\,. \tag{B.12}$$

Given the real structure M and the fact that the chiral and anti-chiral primaries are related to the Ramond ground states by half a unit of spectral flow, we also obtain another basis of (anti-)chiral primaries labelled by $\bar{a}$ (a):

$$V_{a.c.p.}^a = \text{Flow by minus one unit}(V_{c.p.}^a) = M^a{}_{\bar{a}}V_{a.c.p.}^{\bar{a}}$$
$$V_{c.p.}^{\bar{a}} = \text{Flow by one unit}(V_{a.c.p.}^{\bar{a}}) = M^{\bar{a}}{}_a V_{c.p.}^a \,. \tag{B.13}$$

These operators have charges $\mp(\frac{c}{3} - q_a)$ respectively. Given these ingredients, we are ready to derive the necessary operator product expansions starting from the chiral ring relations.

B.2 Operator products and a metric

Our derivation of various terms in operator product expansions is based on the chiral ring relation

$$V_{c.p.}^a(z)V_{c.p.}^b(w) \approx R^{ab}{}_c V_{c.p.}^c(w) + O(z-w)\,. \tag{B.14}$$

Firstly, we derive an analogue of the chiral ring relation in which we factor out the $\mathrm{U}(1)_R$ charge of the chiral primary operators. We use the form (B.8) of the chiral ring elements and write the product of operators as:

$$e^{i\sqrt{\frac{3}{c}}q^a Z}\Pi^a(z)e^{i\sqrt{\frac{3}{c}}q^b Z}\Pi^b(w) \approx (z-w)^{\frac{3}{c}q^a q^b}e^{i\sqrt{\frac{3}{c}}(q^a Z(z)+q^b Z(w))}\Pi^a(z)\Pi^b(w)\,. \tag{B.15}$$

In order to be consistent with the ring relation (B.14) we identify

$$V_{c.p.}^c(w) = e^{i\sqrt{\frac{3}{c}}(q^a+q^b)Z(w))}\Pi^c(w)\,. \tag{B.16}$$

and reconstruct the leading term in the operator product expansion that the fields Π^a must satisfy:

$$\Pi^a(z)\Pi^b(w) \approx (z-w)^{-\frac{3}{c}q^a q^b}\left(R^{ab}{}_c\Pi^c(w) + O(z-w)\right)\,. \tag{B.17}$$

We can think of the fields Π^a as R-parafermions. Now that we have the Π^a operator product expansion, we more easily obtain the leading term in the operator product expansion of the Ramond ground states:

$$\Sigma^a(z)\Sigma^b(w) \approx e^{i\sqrt{\frac{3}{c}}(q^a-\frac{c}{6})Z}\Pi^a(z)e^{i\sqrt{\frac{3}{c}}(q^b-\frac{c}{6})Z}\Pi^b(z)$$
$$\approx (z-w)^{-\frac{1}{2}(q^a+q^b)+\frac{c}{12}}\left(e^{i\sqrt{\frac{3}{c}}\left(q^a+q^b-\frac{c}{3}\right)Z}R^{ab}{}_c\Pi^c(w) + O(z-w)\right)\,. \tag{B.18}$$

We have assumed that the charges q^a corresponds to the spectrum of R-charges of the chiral primaries — they are therefore positive. We once more define the chiral primary operator $V_{c.p.}^c$ as in (B.16) and compute:

$$\Sigma^a(z)\Sigma^b(w) \approx (z-w)^{-\frac{1}{2}(q^a+q^b)+\frac{c}{12}}\left(e^{-i\sqrt{\frac{c}{3}}Z} \ R^{ab}{}_c \ V_{c.p.}^c(w) + O(z-w)\right)$$
$$\approx (z-w)^{-\frac{1}{2}(q^a+q^b)+\frac{c}{12}}\left(R^{ab}{}_c\left[M^c{}_{\bar{c}}V_{a.c.p.}^{\bar{c}}\right] + O(z-w)\right)\,. \tag{B.19}$$

In going from the first to the second line we observe that we have spectrally flowed a chiral primary by minus one unit and this leads to the anti-chiral primary (B.13). We note

that the dimension of the anti-chiral primary with the non-canonical labelling $V^c_{a.c.p.}$ is given by $\frac{c}{6} - \frac{q^c}{2}$.

A second equation can similarly be found by starting out with Ramond ground states labelled by anti-chiral primaries and performing the operator products:

$$\Sigma^{\bar{a}}(z)\Sigma^{\bar{b}}(w) \approx (z-w)^{\frac{1}{2}\left(q^{\bar{a}}+q^{\bar{b}}\right)+\frac{c}{12}} \left(R^{\bar{a}\bar{b}}{}_{\bar{c}} \left[M^{\bar{c}}{}_c V^c_{c.p.} \right] + O(z-w) \right). \qquad (B.20)$$

We find a chiral primary obtained by unit spectral flow from an anti-chiral primary, as in equation (B.13), and it has world sheet dimension $\frac{c}{6} + \frac{q^{\bar{c}}}{2}$.

We summarize that in the operator product of Ramond ground states one can get either a chiral primary or an anti-chiral primary as the leading term depending on the R-charges of the Ramond ground states involved. The relevant formulas are (B.19) and (B.20).

Since the Ramond ground state can be written as a (anti-)chiral ring element times the states with lowest (highest) R-charge, it is straightforward to compute the leading term in the operator product expansion between a (anti-)chiral primary and a Ramond ground state:

$$\begin{aligned}
V^a_{c.p.}(z)\Sigma^b(w) &\approx e^{i\sqrt{\frac{3}{c}}q^a Z}\Pi^a(z) e^{i\sqrt{\frac{3}{c}}(q^b - \frac{c}{6})Z}\Pi^b(z) \\
&\approx (z-w)^{-\frac{q^a}{2}} e^{i\sqrt{\frac{3}{c}}(q^a + q^b - \frac{c}{6})Z} R^{ab}{}_c \Pi^c(w) + \dots \\
&\approx \frac{R^{ab}{}_c \Sigma^c(w)}{(z-w)^{\frac{q^a}{2}}} + \dots
\end{aligned} \qquad (B.21)$$

Again, this holds whenever the structure constant is non-zero. Similarly we find that

$$V^{\bar{a}}_{a.c.p.}(z)\Sigma^{\bar{b}}(w) \approx \frac{R^{\bar{a}\bar{b}}{}_{\bar{c}}\Sigma^{\bar{c}}(w)}{(z-w)^{-\frac{q^{\bar{a}}}{2}}} + \dots \qquad (B.22)$$

Lastly we derive the operator products between the chiral and anti-chiral ring elements. The two point function of a chiral and anti-chiral primary operator is defined to be

$$\langle V^a_{c.p.}(z) V^{\bar{b}}_{a.c.p.}(w) \rangle = \frac{g^{a\bar{b}}}{(z-w)^{2h_a}}, \qquad (B.23)$$

where $g^{a\bar{b}}$ is the Zamolodchikov metric and we have chosen a basis in which it is diagonal in the sense that $q^{\bar{b}} = -q^a$. In order to obtain the operator product coefficient of interest we shall compute a three-point function in two different ways [28]. The three point function of interest is given by

$$\langle V^a_{c.p.} V^{\bar{b}}_{a.c.p.} V^{\bar{c}}_{a.c.p.} \rangle = \frac{X^{a\bar{b}\bar{c}}}{(z_a - z_b)^{q^b}(z_c - z_a)^{q^c}}. \qquad (B.24)$$

Here we have used the usual definition of the three point function and the charge conservation equation $q^a = q^b + q^c$. In the first manner of calculation, we use the chiral ring relation and obtain

$$\begin{aligned}
\langle V^a_{c.p.} V^{\bar{b}}_{a.c.p.} V^{\bar{c}}_{a.c.p.} \rangle &= R^{\bar{b}\bar{c}}{}_{\bar{d}} \langle V^a_{c.p.} V^{\bar{d}}_{a.c.p.} \rangle \\
&= \frac{R^{\bar{b}\bar{c}}{}_{\bar{d}} \, g^{a\bar{d}}}{(z_a - z_c)^{2h^a}}.
\end{aligned} \qquad (B.25)$$

Thus we have fixed the structure constant to be $X^{ab\bar{c}} = R^{\bar{b}\bar{c}}{}_{\bar{d}} \, g^{a\bar{d}}$. Equivalently we can use the operator product expansion

$$V^a_{c.p.}(z_a)V^{\bar{b}}_{a.c.p.}(z_b) \approx \frac{D^{a\bar{b}}{}_c}{(z_a - z_b)^{q^b}} \, V^c_{c.p.}(z_a) \tag{B.26}$$

and obtain

$$\langle V^a_{c.p.} V^{\bar{b}}_{a.c.p.} V^{\bar{c}}_{a.c.p.} \rangle = \frac{D^{a\bar{b}}{}_c}{(z_a - z_b)^{q^b}} \frac{g^{c\bar{c}}}{(z_a - z_c)^{q^c}} \, . \tag{B.27}$$

We therefore find — see also appendix E of [28] –:

$$D^{a\bar{b}}{}_c = g^{a\bar{d}} \, R^{\bar{b}\bar{e}}{}_{\bar{d}} \, g_{c\bar{e}} \, . \tag{B.28}$$

We can write this equality as $D = gRg^{-1}$ where we indicate with g the Zamolodchikov metric with upper indices.

C The spacetime global superconformal algebra

The global $\mathcal{N} = 2$ superconformal algebra is described by the commutation relations:

$$\begin{aligned}
\{\mathcal{G}_r^+, \mathcal{G}_s^-\} &= 2\mathcal{L}_{r+s} + (r-s)\mathcal{Q}_0 \\
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n} \\
[\mathcal{L}_m, \mathcal{G}_r^\pm] &= (\frac{m}{2} - r)\mathcal{G}_{m+r}^\pm \\
[\mathcal{Q}_0, \mathcal{G}_r^\pm] &= \pm\mathcal{G}_r^\pm \, ,
\end{aligned} \tag{C.1}$$

for $r, s = \pm\frac{1}{2}$ and $m, n = 0, \pm 1$. Using the currents of the worldsheet theory it is possible to write down integrated operators on the worldsheet that correspond to these global generators [8]. The zero modes of the bosonic currents generate the isometries of the target space. If we define

$$\mathcal{L}_0 = -\oint J^3 = -\oint e^{-\phi}\psi^3 \, , \tag{C.2}$$

$$\mathcal{L}_\pm = -\oint (J_1 \pm iJ_2) = \oint e^{-\phi}(\psi_1 \pm i\psi_2) \, , \tag{C.3}$$

then these generators satisfy the global Virasoro algebra. The $U(1)_R$ charge in spacetime was defined in equation (A.18)

$$\mathcal{Q}_0 = \sqrt{2k} \oint J^0 = \oint e^{-\psi}\psi^0 \, . \tag{C.4}$$

It therefore remains to write down the spacetime supercharges. In the $(-\frac{1}{2})$ picture they read:

$$\mathcal{G}_r^\pm = \oint e^{-\frac{1}{2}\phi} \, S_r^\pm \, , \quad \text{with} \quad r = \pm\frac{1}{2} \, . \tag{C.5}$$

The operator ϕ is the bosonized superghost and the exponential has worldsheet conformal dimension $\frac{3}{8}$. Thus the vertex operator $S_r^\pm$ must have dimension equal to $\frac{5}{8}$. The key to

writing down the vertex operators for the spacetime supercharges is to keep track of the charges under the bosonic spacetime symmetries. In particular from the spacetime algebra it is clear that the spacetime supercharges have ± 1 charge under the spacetime R-current $\mathcal{Q}_0$ and must therefore include the factor $e^{\pm\frac{i}{\sqrt{2k}}X}$ in their vertex operators. Similarly, their Lorentz transformation properties fix their form to be

$$S_r^{\pm} = e^{-ir(H_1 \mp H_0)} e^{\pm \frac{i}{\sqrt{2k}}X} \Sigma^{\pm}\,, \quad \text{with} \quad r = \pm\frac{1}{2}\,. \tag{C.6}$$

The operator factors $\Sigma^{\pm}$ are purely in the Y-theory. By using the constraint that the worldsheet dimension of $S_r^{\pm}$ should be $\frac{5}{8}$ we see that

$$\Delta(\Sigma^{\pm}) = \frac{3}{8} - \frac{1}{4k} = \frac{c_Y}{24}\,. \tag{C.7}$$

Thus we conclude that these must be Ramond ground states of the theory Y. A priori there are many Ramond ground states in Y but, as shown in [8], the Ramond ground states that appear in the spacetime supercharge are those with the highest or lowest worldsheet R-charge $\pm\frac{c_Y}{6}$. These can be expressed purely in terms of the boson Z that captures the world sheet $\mathrm{U}(1)_R$ current direction:

$$S_r^{\pm} = e^{-ir(H_1 \mp H_0)} e^{\pm \frac{i}{\sqrt{2k}}X} e^{\pm i\sqrt{\frac{c_Y}{12}}Z}\,. \tag{C.8}$$

Using the free field operator products, one checks that the integrated vertex operators $\mathcal{L}_n, \mathcal{G}_r^{\pm}$ and $\mathcal{Q}_0$ indeed satisfy the spacetime $\mathcal{N} = 2$ global supersymmetry algebra (C.1).

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