

On higher spin cubic interactions in $d = 3$

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ABSTRACT: In this paper we elaborate on higher spin cubic interactions for massless, massive and partially massless fields. We work in the gauge invariant frame-like multispinor formalism, combining Lagrangian and unfolded formulations.

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1 Introduction

Recently, a general classification of cubic interaction vertices for massive higher spin fields in flat three dimensional space has been developed in the light-cone formalism [1] (see also [2]). As is well known, massless higher spin fields in $d = 3$ do not have any local physical degrees of freedom and so they simply do not exist in the light-cone formalism. At the same time, it is still possible to construct non-trivial interactions for the massless fields the first such example being the famous Blencowe theory [3]. Similarly, interactions between massless and massive fields can lead to highly non-trivial theories the best known example being the Prokushkin-Vasiliev theory [4, 5], representing a set of self-consistent unfolded equations. Besides, the possibility to construct non-trivial interactions for partially massless fields was discussed recently in [6]. In this paper we elaborate on the higher spin cubic interactions for massless, massive and partially massless fields. We work in the gauge invariant frame-like

multispinor formalism [7–10] (see [11] for review), combining Lagrangian and unfolded formulations.

One of the characteristic features of any gauge invariant description for massive fields is the presence of the Stueckelberg fields. As it has been shown in the metric-like formalism for $d \geq 4$ [12] this fact leads to the following two important results. At first, there always exist enough field redefinitions as well as the field dependent redefinitions of the gauge parameters which allows one to transform any non-abelian interaction vertices (i.e. those that deform both gauge transformations and the algebra) into the abelian ones (i.e. those that deform gauge transformations, while the algebra remains abelian). At second, using further (higher derivative) field redefinition one can bring the vertex into the trivially gauge invariant form (i.e. expressed completely in terms of the gauge invariant objects of the free theory). Recently, on the two simple concrete examples (massive spin-3/2 and massive spin-2) we considered what happens if massless spin-2 field present in the system [13, 14]. In both cases we have found that there exist abelian vertices which are not equivalent to any trivially gauge invariant ones. Moreover, these vertices are necessary to reproduce the minimal gravitational interactions. One of the aims of the current work is to see what changes in $d = 3$ where massless fields do not have physical degrees of freedom and all their gauge invariant objects vanish on-shell.

This paper is organized as follows. We begin in section 2 with massless higher spin fields and show how the frame-like multispinor formalism allows easily reproduce the general classification of cubic interaction vertices for the bosonic fields developed in [15, 16] and also extend it including fermions. As a bonus, in section 3 we construct cubic interaction vertices for massless higher spin supermultiplets corresponding to the simplest $(1, 0)$ supersymmetry. These vertices correspond to the type II ones in the Metsaev’s classification of their four dimensional cousins [17, 18] (see also [19, 20] and references therein). Section 4 devoted to the massive fields. In the first two subsections we construct a very simple gauge invariant Lagrangians for bosons and fermions obtained from the initial ones by the (almost) maximal gauge fixing. In both cases each Lagrangian can be associated with the set of self-consistent unfolded equations. In subsection 4.3, both in Lagrangian and unfolded approach, we investigate cubic interactions for massive and massless fields. We consider all three possibilities: two massive and one massless bosons; two massive fermion and massless boson; massive boson and fermion and massless fermion. In all three cases we found that the gauge invariance implies the following relation on masses:

$$M_1 - M_2 = (s_1 - s_2)\lambda, \quad \lambda = \sqrt{-\Lambda}$$

which does not depend on the massless field spin. At last, in subsection 4.4 we discuss cubic vertices for the three massive fields. In section 5 we consider partially massless fields. For the case with maximal depth (the only one that has one physical degrees of freedom) we consider their interaction with massless fields and show that both Lagrangian and unfolded approach requires that spins of the partially massless fields must be equal. For the partially massless fields with non-maximal depth we show that our gauge invariant formalism can be rewritten in the simple and elegant form suggested previously in [6].

Notation and conventions. We mostly follow our review [11]. All objects are forms having a number of completely symmetric spinor indices which we denote $\alpha(n) = (\alpha_1 \alpha_2 \dots \alpha_n)$. A background AdS_3 space is described by the frame one-form $e^{\alpha(2)}$ and Lorentz covariant derivative D normalized so that

$$D \wedge D \zeta^\alpha = -\lambda^2 E^\alpha{}_\beta \zeta^\beta$$

Also we use two and three-forms defined as:

$$e^{\alpha(2)} \wedge e^{\beta(2)} = \varepsilon^{\alpha\beta} E^{\alpha\beta}, \quad E^{\alpha(2)} \wedge e^{\beta(2)} = \varepsilon^{\alpha\beta} \varepsilon^{\alpha\beta} E$$

In the main text wedge product sign \wedge will be omitted.

2 Massless fields

This section is devoted to the massless higher spin fields and their interactions. This material is rather well known, but in our gauge invariant formalism these fields serve as the building blocks for the massive and partially massless ones.

2.1 Free fields

Massless spin- s boson is described by a pair of one-forms $\Omega^{\alpha(2s-2)}$ and $f^{\alpha(2s-2)}$ with $2s-2$ completely symmetric local spinor indices which we collectively denote as $\alpha(2s-2)$. Free Lagrangian (three-form) for the massless field in AdS_3 background looks like:

$$\begin{aligned} (-1)^s \mathcal{L}_0 &= (s-1) \Omega_{\alpha(2s-3)\beta} e^\beta{}_\gamma \Omega^{\alpha(2s-3)\gamma} + \Omega_{\alpha(2s-2)} D f^{\alpha(2s-2)} \\ &+ \frac{(s-1)}{4} \lambda^2 f_{\alpha(2s-3)\beta} e^\beta{}_\gamma f^{\alpha(2s-3)\gamma}. \end{aligned} \quad (2.1)$$

This Lagrangian is invariant under the following local gauge transformations:

$$\begin{aligned} \delta_0 \Omega^{\alpha(2s-2)} &= D \eta^{\alpha(2s-2)} + \frac{\lambda^2}{4} e^\alpha{}_\beta \xi^{\alpha(2s-3)\beta}, \\ \delta_0 f^{\alpha(2s-2)} &= D \xi^{\alpha(2s-2)} + e^\alpha{}_\beta \eta^{\alpha(2s-3)\beta}, \end{aligned} \quad (2.2)$$

where parameters η, ξ are zero-forms. One can construct two gauge invariant two-forms (generalizing curvature and torsion in gravity):

$$\begin{aligned} \mathcal{R}^{\alpha(2s-2)} &= D \Omega^{\alpha(2s-2)} + \frac{\lambda^2}{4} e^\alpha{}_\beta f^{\alpha(2s-3)\beta}, \\ \mathcal{T}^{\alpha(2s-2)} &= D f^{\alpha(2s-2)} + e^\alpha{}_\beta \Omega^{\alpha(2s-3)\beta}, \end{aligned} \quad (2.3)$$

which we collectively call curvatures.

One of the specific properties of the three-dimensional case is the possibility to separate variables. Indeed, let us consider the following combinations of the fields and gauge parameters:

$$\begin{aligned} \Omega_\pm^{\alpha(2s-2)} &= \Omega^{\alpha(2s-2)} \pm \frac{\lambda}{2} f^{\alpha(2s-2)}, \\ \eta_\pm^{\alpha(2s-2)} &= \eta^{\alpha(2s-2)} \pm \frac{\lambda}{2} \xi^{\alpha(2s-2)}. \end{aligned} \quad (2.4)$$

In the new variables we obtain:

$$(-1)^s \mathcal{L}_0 = \frac{1}{2\lambda} [\mathcal{L}_+(\Omega_+) - \mathcal{L}_-(\Omega_-)], \quad (2.5)$$

where

$$\mathcal{L}_\pm(\Omega_\pm) = \Omega_{\pm, \alpha(2s-2)} D\Omega_\pm^{\alpha(2s-2)} \pm (s-1)\lambda \Omega_{\pm, \alpha(2s-3)\beta} e^\beta{}_\gamma \Omega_\pm^{\alpha(2s-3)\gamma} \quad (2.6)$$

Moreover, each Lagrangian \mathcal{L}_\pm is invariant under its own gauge transformations:

$$\delta\Omega_\pm^{\alpha(2s-2)} = D\eta_\pm^{\alpha(2s-2)} \pm \frac{\lambda}{2} e^\alpha{}_\beta \eta_\pm^{\alpha(2s-3)\beta}. \quad (2.7)$$

The same procedure works for the gauge invariant curvatures as well:

$$\mathcal{R}_\pm^{\alpha(2s-2)} = D\Omega_\pm^{\alpha(2s-2)} \pm \frac{\lambda}{2} e^\alpha{}_\beta \Omega_\pm^{\alpha(2s-3)\beta}. \quad (2.8)$$

In what follows we work with the + components only omitting the + sign.

Massless spin- $(s+1/2)$ fermion is described by one-form $\Phi^{\alpha(2s-1)}$. Free Lagrangian in AdS_3 looks like:

$$\frac{(-1)^s}{i} \mathcal{L}_0 = \frac{1}{2} \Phi_{\alpha(2s-1)} D\Phi^{\alpha(2s-1)} \pm \frac{(2s-1)}{4} \lambda \Phi_{\alpha(2s-2)\beta} e^\beta{}_\gamma \Phi^{\alpha(2s-2)\gamma}, \quad (2.9)$$

which is invariant under the following local gauge transformations:

$$\delta_0 \Phi^{\alpha(2s-1)} = D\zeta^{\alpha(2s-1)} \pm \frac{\lambda}{2} e^\alpha{}_\beta \zeta^{\alpha(2s-2)\beta}. \quad (2.10)$$

Here one can also construct a gauge invariant curvature:

$$\mathcal{F}^{\alpha(2s-1)} = D\Phi^{\alpha(2s-1)} \pm \frac{\lambda}{2} e^\alpha{}_\beta \Phi^{\alpha(2s-2)\beta} \quad (2.11)$$

For the fermions we also use the + sign case.

2.2 Cubic vertices

In the metric-like formalism a complete classification for the bosonic cubic vertices were elaborated in [15, 16]. In this subsection we show how these results appear in the frame-like formalism and extend them including fermionic fields.

Let us begin with the vertex for three bosons with spins $s_{1,2,3}$. The most general ansatz has the form:

$$\mathcal{L}_1 = g \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3)} \Omega_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)}. \quad (2.12)$$

Here the \hat{s}_i must satisfy

$$\hat{s}_2 + \hat{s}_3 = 2s_1 - 2, \quad \hat{s}_1 + \hat{s}_3 = 2s_2 - 2, \quad \hat{s}_1 + \hat{s}_2 = 2s_3 - 2 \quad (2.13)$$

and this gives

$$\hat{s}_3 = s_1 + s_2 - s_3 - 1, \quad \hat{s}_2 = s_1 + s_3 - s_2 - 1, \quad \hat{s}_1 = s_2 + s_3 - s_1 - 1. \quad (2.14)$$

From the requirement $\hat{s}_i \geq 0$ immediately follows that the three spins must satisfy a so-called strict triangular inequality $s_i < s_{i+1} + s_{i+2}$. Note also that all \hat{s}_i are simultaneously odd or even, so that

$$(-1)^{\hat{s}_1} = (-1)^{\hat{s}_2} = (-1)^{\hat{s}_3}. \quad (2.15)$$

Consider η_3 -transformations as an example:

$$\begin{aligned} \delta_0 \mathcal{L}_1 &= g \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) \left[D \eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1)} - \frac{\lambda}{2} e^{\beta \delta} \eta_{\beta(\hat{s}_2-1)\delta\gamma(\hat{s}_1)} - \frac{\lambda}{2} e^{\gamma \delta} \eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1-1)\delta} \right] \\ &= -g D \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) \eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1)} + g \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} D \Omega_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) \eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1)} \\ &\quad - \frac{g\lambda}{2} \hat{s}_2 \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2-1)\rho} \Omega_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) e^{\rho \delta} \eta_{\beta(\hat{s}_2-1)\delta\gamma(\hat{s}_1)} \\ &\quad - \frac{g\lambda}{2} \hat{s}_1 \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1-1)\rho} \alpha(\hat{s}_3) e^{\rho \delta} \eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1-1)\delta}. \end{aligned} \quad (2.16)$$

To compensate for these variations we introduce the following corrections to the gauge transformations:

$$\begin{aligned} \delta_1 \Omega_1^{\alpha(2s_1-2)} &= g_1 \Omega_2^{\alpha(\hat{s}_3)\gamma(\hat{s}_1)} \eta_3^{\alpha(\hat{s}_2)} \gamma(\hat{s}_1) \\ \delta_1 \Omega_2^{\alpha(2s_2-2)} &= g_2 \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \eta_3^{\alpha(\hat{s}_1)} \beta(\hat{s}_2). \end{aligned} \quad (2.17)$$

They produce:

$$\begin{aligned} \delta_1 \mathcal{L}_0 &= 2g_1 \frac{(2s_1-2)!}{\hat{s}_2! \hat{s}_3!} D \Omega^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) \eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1)} \\ &\quad + 2g_2 (-1)^{\hat{s}_1} \frac{(2s_2-2)!}{\hat{s}_1! \hat{s}_3!} \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} D \Omega_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) \eta_{\gamma(\hat{s}_1)\beta(\hat{s}_2)} \\ &\quad + g_1 \lambda \frac{(2s_1-2)!}{\hat{s}_3! (\hat{s}_2-1)!} \Omega_{1,\alpha(\hat{s}_3)\beta(\hat{s}_2-1)\delta} e^{\delta \rho} \Omega^{\alpha(\hat{s}_3)\gamma(\hat{s}_1)} \eta^{\beta(\hat{s}_2-1)\rho} \gamma(\hat{s}_1) \\ &\quad + g_2 \lambda \frac{(2s_2-2)!}{\hat{s}_3! (\hat{s}_1-1)!} \Omega_{2,\alpha(\hat{s}_3)\gamma(\hat{s}_1-1)\delta} e^{\delta \rho} \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \eta^{\gamma(\hat{s}_1-1)\rho} \beta(\hat{s}_2) \\ &\quad + g_1 \lambda \frac{(2s_1-2)!}{(\hat{s}_3-1)! \hat{s}_2!} \Omega_{1,\alpha(\hat{s}_3-1)\beta(\hat{s}_2)\delta} e^{\delta \rho} \Omega_2^{\alpha(\hat{s}_3-1)\rho\gamma(\hat{s}_1)} \eta^{\beta(\hat{s}_2)} \gamma(\hat{s}_1) \\ &\quad + g_2 \lambda \frac{(2s_2-2)!}{(\hat{s}_3-1)! \hat{s}_1!} \Omega_{2,\alpha(\hat{s}_3-1)\gamma(\hat{s}_1)\delta} e^{\delta \rho} \Omega_1^{\alpha(\hat{s}_3-1)\rho\beta(\hat{s}_2)} \eta^{\gamma(\hat{s}_1)} \beta(\hat{s}_2) \end{aligned} \quad (2.18)$$

This gives:

$$g_1 = -\frac{(\hat{s}_2)! (\hat{s}_3)!}{2(\hat{s}_2 + \hat{s}_3)!} g, \quad g_2 = (-1)^{\hat{s}_1} \frac{(\hat{s}_1)! (\hat{s}_2)!}{2(\hat{s}_1 + \hat{s}_2)!} g \quad (2.19)$$

Similarly for two other gauge transformations. All corrections correspond to the following deformations of the gauge invariant curvatures:

$$\begin{aligned} \Delta \mathcal{R}_1^{\alpha(2s_1-2)} &= g_1 \Omega_2^{\alpha(\hat{s}_3)\gamma(\hat{s}_1)} \Omega_3^{\alpha(\hat{s}_2)} \gamma(\hat{s}_1) \\ \Delta \mathcal{R}_2^{\alpha(2s_2-2)} &= g_2 \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_3^{\alpha(\hat{s}_1)} \beta(\hat{s}_2) \\ \Delta \mathcal{R}_3^{\alpha(2s_3-2)} &= g_3 \Omega_1^{\alpha(\hat{s}_2)\beta(\hat{s}_3)} \Omega_2^{\alpha(\hat{s}_1)} \beta(\hat{s}_3) \end{aligned} \quad (2.20)$$

The vertex constructed does not have any definite parity. So to compare our results with the ones in the metric-like formalism, let us temporarily restore components with $-$ sign and consider

$$\mathcal{L}_1 = g_+ \Omega_{+1}^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_{+2}^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) \Omega_{+3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} + g_- \Omega_{-1}^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_{-2}^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) \Omega_{-3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \quad (2.21)$$

We obtain:

$$\begin{aligned} \mathcal{L}_1 = & (g_+ + g_-) \left[\Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) \Omega_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \right. \\ & + \frac{\lambda^2}{4} \left(\Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} f_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) f_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} + f_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} f_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) \Omega_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \right. \\ & \left. \left. + f_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) f_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \right) \right] \\ & + (g_+ - g_-) \frac{\lambda}{2} \left[\left(\Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) f_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} + \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} f_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) \Omega_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \right. \right. \\ & \left. \left. + f_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) \Omega_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \right) + \frac{\lambda^2}{4} f_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} f_2^{\gamma(\hat{s}_1)} \alpha(\hat{s}_3) f_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \right] \quad (2.22) \end{aligned}$$

Thus we obtain two independent vertices. The first one has three derivatives (and one derivative tail); it is parity even/odd when sum of the spins is odd/even. The second one has two derivatives (and zero derivative tail); it is parity even/odd when sum of the spins is even/odd. These results are in complete agreement with the classification given in [15, 16].

In the multispinor formalism it is easy to extend these results to include fermions. Let us consider the case for one boson with spin s_1 and two fermions with spins $s_2 + 1/2, s_3 + 1/2$. Then the vertex appears to be:

$$\mathcal{L}_1 = g \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Phi_2^{\gamma(\hat{s}_1+1)} \alpha(\hat{s}_3) \Phi_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1+1)} \quad (2.23)$$

where \hat{s}_i are the same as before.

In the frame-like formalism it is impossible to construct any vertices higher than cubic ones. Thus to construct consistent model one has to find such collection of massless fields and their cubic vertices that the gauge (super)algebra closes. As we have already mentioned in the Introduction the first such examples were constructed by Blencowe [3]. One more specific properties of three dimensional case is that there exist models with finite number of fields. The most simple and rather popular with the algebra $SL(n)$ [21] describes all integer spins $2, 3, \dots, n$. It is possible to truncate these models to even spins $2, 4, \dots, 2n$ only [22], the algebra being $Sp(2n)$. May be the most simple examples including fermions [23] correspond to superalgebras $OSp(1, 2n)$ and describe even spins $2, 4, \dots, 2n$ and one half-integer spin $n + 1/2$. The case $n = 1$ is just $(1, 0)$ supergravity, while $n = 2$ describes AdS_3 hypergravity with spin-2 $\omega^{\alpha(2)}$, spin-4 $\Sigma^{\alpha(6)}$ and spin-5/2 $\Psi^{\alpha(3)}$. Bosonic part of the vertex

$$\mathcal{L}_{1b} = \frac{g}{3} \omega_\alpha^\beta \omega_\beta^\gamma \omega_\gamma^\alpha + 3g \Sigma_{\alpha(5)\beta} \omega^\beta \gamma \Sigma^{\alpha(5)\gamma} + \frac{10\tilde{g}}{3} \Sigma_{\alpha(3)\beta(3)} \Sigma^{\beta(3)} \gamma(3) \Sigma^{\alpha(3)\gamma(3)} \quad (2.24)$$

where $\tilde{g} = \sqrt{10}g$, contains $(2-2-2)$, $(2-4-4)$ and $(4-4-4)$ subvertices, while fermionic part

$$\mathcal{L}_{1f} = \frac{3ig}{2} \Psi_{\alpha(2)\beta} \omega^{\beta\gamma} \Psi^{\alpha(2)\gamma} + \frac{3i\tilde{g}}{2} \Psi_{\alpha(3)} \Sigma^{\alpha(3)}_{\beta(3)} \Psi^{\beta(3)} \quad (2.25)$$

contains $(2-5/2-5/2)$ and $(4-5/2-5/2)$ ones. All of them follow the general pattern described above.

3 Massless supermultiplets

In four dimensions the complete classification of cubic vertices for the massless higher spin supermultiplets were developed in the light-cone formalism by Metsaev [17, 18]. Recall that the type I vertices in Metsaev’s classification have $N = s_1 + s_2 + s_3$ derivatives so in the covariant form they can be constructed in terms of the gauge invariant curvatures. Taking into account that in $d = 3$ all such curvatures for massless fields vanish on-shell, it is clear that such vertices do not exist in $d = 3$. The type II vertices, having $N = s_1 + s_2 - s_3$ derivatives, can be abelian or non-abelian, the later (where spins satisfy a so-called strict triangular inequality) in the covariant form were constructed in [19], using the results of [20]. In this section we show that in three dimensions one can construct cubic vertices similar to the ones obtained in [19].

3.1 Free supermultiplets

We work with the supermultiplets for the simplest $(1, 0)$ global superalgebra. Recall that in AdS_3 by global supertransformations we mean such that their spinor parameter satisfies

$$D\zeta^\alpha + \frac{\lambda}{2} e^\alpha_\beta \zeta^\beta = 0. \quad (3.1)$$

There exist two massless supermultiplets: with integer superspin $(s, s + 1/2)$ and with half-integer one $(s, s - 1/2)$. All we need is their explicit supertransformations such that the sum of the free bosonic and fermionic Lagrangians is invariant and the superalgebra closes on-shell (for more details see [11] and references therein).

Integer superspin $(s, s + 1/2)$.

$$\begin{aligned} \delta\Omega^{\alpha(2s-2)} &= i\sqrt{2s-1}\lambda\Phi^{\alpha(2s-2)\beta}\zeta_\beta \\ \delta\Phi^{\alpha(2s-1)} &= \frac{1}{\sqrt{2s-1}}\Omega^{\alpha(2s-2)}\zeta^\alpha \end{aligned} \quad (3.2)$$

Half-integer superspin $(s, s - 1/2)$.

$$\begin{aligned} \delta\Omega^{\alpha(2s-2)} &= \frac{i\lambda}{\sqrt{2s-2}}\Psi^{\alpha(2s-3)}\zeta^\alpha \\ \delta\Psi^{\alpha(2s-3)} &= \sqrt{2s-2}\Omega^{\alpha(2s-3)\beta}\zeta_\beta \end{aligned} \quad (3.3)$$

3.2 Cubic vertices

General procedure we use here is the same as we have already used in four dimensions [19]. Namely, having in our disposal three supermultiplets, i.e. three bosonic and three fermionic fields, we can construct four elementary vertices: one purely bosonic and three with fermions (schematically)

$$\mathcal{L}_1 \sim g_0 \Omega_1 \Omega_2 \Omega_3 + g_1 \Omega_1 \Phi_2 \Phi_3 + g_2 \Phi_1 \Omega_2 \Phi_3 + g_3 \Phi_1 \Phi_2 \Omega_3.$$

Thus we just have to adjust the four coupling constants so that the vertex be invariant under the global supertransformations. There are three type II vertices in Metsaev's classification: type IIa with three half-integer superspins and type IIb,c with two integer and one half-integer superspins. The difference for type IIb and type IIc comes from the fact that the number of derivatives in the four dimensional cubic vertices strongly depends on which fields has lowest spin. In three dimensions the spin ordering does not matter and this leaves us with just two possibilities, which we consider in turn.

Two integer and one half-integer superspins $(s_1, s_1 + 1/2)$, $(s_2, s_2 + 1/2)$, $(s_3, s_3 - 1/2)$. In this case a candidate for the supersymmetric vertex looks like:

$$\begin{aligned} \mathcal{L}_1 = & g_0 \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1)} \alpha_{(\hat{s}_3)} \Omega_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \\ & + i g_1 \lambda \Omega_1^{\alpha(\hat{s}_3+1)\beta(\hat{s}_2-1)} \Phi_2^{\gamma(\hat{s}_1)} \alpha_{(\hat{s}_3+1)} \Psi_{3,\beta(\hat{s}_2-1)\gamma(\hat{s}_1)} \\ & + i g_2 \lambda \Phi_1^{\alpha(\hat{s}_3+1)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1-1)} \alpha_{(\hat{s}_3+1)} \Psi_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1-1)} \\ & + i g_3 \lambda \Phi_1^{\alpha(\hat{s}_3+1)\beta(\hat{s}_2)} \Phi_2^{\gamma(\hat{s}_1)} \alpha_{(\hat{s}_3+1)} \Omega_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \end{aligned} \quad (3.4)$$

Note that in this case we must have $\hat{s}_{1,2} \geq 1$, $\hat{s}_3 \geq 0$. Let us calculate variations of the vertex containing two bosons and one fermion:

$$\begin{aligned} \frac{1}{i\lambda} \Delta_1 = & \left[\frac{g_0 \hat{s}_2}{\sqrt{2s_3 - 2}} - \frac{g_1(\hat{s}_3 + 1)}{\sqrt{2s_2 - 1}} \right] \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2-1)\delta} \Omega_2^{\gamma(\hat{s}_1)} \alpha_{(\hat{s}_3)} \Psi_{3,\beta(\hat{s}_2-1)\gamma(\hat{s}_1)} \zeta^\delta \\ & + \left[\frac{g_0 \hat{s}_1}{\sqrt{2s_3 - 2}} + \frac{g_2(\hat{s}_3 + 1)}{\sqrt{2s_1 - 1}} \right] \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1-1)\delta} \alpha_{(\hat{s}_3)} \Psi_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1-1)} \zeta^\delta \\ & - \left[\frac{g_1 \hat{s}_1}{\sqrt{2s_2 - 1}} + \frac{g_2 \hat{s}_2}{\sqrt{2s_1 - 1}} \right] \Omega_1^{\alpha(\hat{s}_3+1)\beta(\hat{s}_2-1)} \Omega_2^{\gamma(\hat{s}_1-1)} \alpha_{(\hat{s}_3+1)} \Psi_{3,\beta(\hat{s}_2-1)\gamma(\hat{s}_1-1)} \delta \zeta^\delta \\ & + \left[g_0 \sqrt{2s_1 - 1} + \frac{(\hat{s}_3 + 1)g_3}{\sqrt{2s_2 - 1}} \right] \Phi_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)\delta} \zeta^\delta \Omega_2^{\gamma(\hat{s}_1)} \alpha_{(\hat{s}_3)} \Omega_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \\ & + \left[-g_2 \sqrt{2s_3 - 2} + \frac{\hat{s}_1 g_3}{\sqrt{2s_2 - 1}} \right] \Phi_1^{\alpha(\hat{s}_3+1)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1-1)} \alpha_{(\hat{s}_3+1)} \Omega_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1-1)} \delta \zeta^\delta \end{aligned}$$

This gives:

$$\begin{aligned} g_1 &= \frac{\hat{s}_2}{(\hat{s}_3 + 1)} \sqrt{\frac{(2s_2 - 1)}{(2s_3 - 1)}} g_0 \\ g_2 &= -\frac{\hat{s}_1}{(\hat{s}_3 + 1)} \sqrt{\frac{(2s_1 - 1)}{(2s_3 - 1)}} g_0 \\ g_3 &= -\frac{\sqrt{(2s_1 - 2)(2s_2 - 2)}}{(\hat{s}_3 + 1)} g_0 \end{aligned} \quad (3.5)$$

Now, calculating the variations with three fermions we obtain:

$$\begin{aligned} \Delta_2 = & -\lambda^2 \left[g_1 \sqrt{2s_1 - 1} + \frac{g_3 \hat{s}_2}{\sqrt{2s_3 - 2}} \right] \Phi_1^{\alpha(\hat{s}_3+1)\beta(\hat{s}_2-1)\delta} \zeta_\delta \Phi_2^{\gamma(\hat{s}_1)} \Phi_{3,\beta(\hat{s}_2-1)\gamma(\hat{s}_1)} \\ & + \lambda^2 \left[-g_2 \sqrt{2s_2 - 1} + \frac{g_3 \hat{s}_1}{\sqrt{2s_3 - 2}} \right] \Phi_a^{\alpha(\hat{s}_3+1)\beta(\hat{s}_2)} \Phi_2^{\gamma(\hat{s}_1-1)\delta} \Phi_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1-1)} = 0 \end{aligned}$$

Three half-integer superspins ($s_i, s_i - 1/2$). In this case we consider

$$\begin{aligned} \mathcal{L}_1 = & g_0 \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Omega_2^{\gamma(\hat{s}_1)} \Omega_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \\ & + i g_1 \lambda \Omega_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \Psi_2^{\gamma(\hat{s}_1-1)} \Psi_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1-1)} \\ & + i g_2 \lambda \Psi_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2-1)} \Omega_2^{\gamma(\hat{s}_1)} \Psi_{3,\beta(\hat{s}_2-1)\gamma(\hat{s}_1)} \\ & + i g_3 \lambda \Psi_1^{\alpha(\hat{s}_3-1)\beta(\hat{s}_2)} \Psi_2^{\gamma(\hat{s}_1)} \Omega_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)}, \end{aligned} \quad (3.6)$$

where $\hat{s}_{1,2,3} \geq 1$. All calculations are quite similar to the previous case, so let us provide only the answer:

$$\begin{aligned} g_1 &= \frac{\hat{s}_1}{\sqrt{(2s_2 - 2)(2s_3 - 2)}} g_0 \\ g_2 &= \frac{\hat{s}_2}{\sqrt{(2s_1 - 2)(2s_3 - 2)}} g_0 \\ g_3 &= \frac{\hat{s}_3}{\sqrt{(2s_1 - 2)(2s_2 - 2)}} g_0 \end{aligned} \quad (3.7)$$

4 Massive fields

In this section we consider massive bosonic and fermionic higher spin fields. We work in the gauge invariant frame-like multispinor formalism and mostly follow [11].

4.1 Free boson

For the gauge invariant description of massive boson we need a number of one-forms ($\Omega^{\alpha(2k)}, f^{\alpha(2k)}$), $1 \leq k \leq s - 1$, one-form A and zero-forms ($B^{\alpha(2)}, \pi^{\alpha(2)}, \varphi$). Free Lagrangian in AdS₃ background has the form:

$$\begin{aligned} \mathcal{L}_0 = & \sum_{k=1} (-1)^{k+1} \left[k \Omega_{\alpha(2k-1)\beta} e^\beta{}_\gamma \Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)} D f^{\alpha(2k)} \right] \\ & + E B_{\alpha(2)} B^{\alpha(2)} - B_{\alpha(2)} e^{\alpha(2)} D A - E \pi_{\alpha(2)} \pi^{\alpha(2)} + \pi_{\alpha(2)} E^{\alpha(2)} D \varphi \\ & + \sum_{k=1} (-1)^{k+1} a_k \left[-\frac{(k+2)}{k} \Omega_{\alpha(2k)\beta(2)} e^{\beta(2)} f^{\alpha(2k)} + \Omega_{\alpha(2k)} e_{\beta(2)} f^{\alpha(2k)\beta(2)} \right] \\ & + 2a_0 \Omega_{\alpha(2)} e^{\alpha(2)} A - a_0 f_{\alpha\beta} E^\beta{}_\gamma B^{\alpha\gamma} + 2M s \pi_{\alpha(2)} E^{\alpha(2)} A \\ & + \sum_{k=1} (-1)^{k+1} b_k f_{\alpha(2k-1)\beta} e^\beta{}_\gamma f^{\alpha(2k-1)\gamma} + \frac{M s a_0}{2} f_{\alpha(2)} E^{\alpha(2)} \varphi + \frac{3}{2} a_0^2 E \varphi^2 \end{aligned} \quad (4.1)$$

Here the coefficients (a, b) are determined by the requirement that the Lagrangian must be gauge invariant and appear to be:

$$\begin{aligned}
 a_k^2 &= \frac{k(s+k+1)(s-k-1)}{2(k+1)(k+2)(2k+3)} [M^2 - (k+1)^2 \lambda^2], \\
 a_0^2 &= \frac{(s+1)(s-1)}{3} [M^2 - \lambda^2], \\
 b_k &= \frac{M^2 s^2}{4k(k+1)^2}, \quad M^2 = m^2 + (s-1)^2 \lambda^2.
 \end{aligned} \tag{4.2}$$

Gauge transformations leaving this Lagrangian invariant look like:

$$\begin{aligned}
 \delta \Omega^{\alpha(2k)} &= D\eta^{\alpha(2k)} + \frac{(k+2)}{k} a_k e_{\beta(2)} \eta^{\alpha(2k)\beta(2)} + \frac{a_{k-1}}{k(2k-1)} e^{\alpha(2)} \eta^{\alpha(2k-2)} + \frac{b_k}{k} e^\alpha{}_\beta \xi^{\alpha(2k-1)\beta}, \\
 \delta f^{\alpha(2k)} &= D\xi^{\alpha(2k)} + a_k e_{\beta(2)} \xi^{\alpha(2k)\beta(2)} + \frac{(k+1)a_{k-1}}{k(k-1)(2k-1)} e^{\alpha(2)} \xi^{\alpha(2k-2)} + e^\alpha{}_\beta \eta^{\alpha(2k-1)\beta}, \\
 \delta \Omega^{\alpha(2)} &= D\eta^{\alpha(2)} + 3_1 e_{\beta(2)} \eta^{\alpha(2)\beta(2)} + b_1 e^\alpha{}_\beta \xi^{\alpha\beta}, \\
 \delta f^{\alpha(2)} &= D\xi^{\alpha(2)} + e^\alpha{}_\beta \eta^{\alpha\beta} + a_1 e_{\beta(2)} \xi^{\alpha(2)\beta(2)} + 2a_0 e^{\alpha(2)} \xi, \\
 \delta B^{\alpha(2)} &= 2a_0 \eta^{\alpha(2)}, \quad \delta A = D\xi + \frac{a_0}{4} e_{\alpha(2)} \xi^{\alpha(2)}, \\
 \delta \pi^{\alpha(2)} &= \frac{M s a_0}{2} \xi^{\alpha(2)}, \quad \delta \varphi = -2M s \xi.
 \end{aligned} \tag{4.3}$$

Unfortunately, in this general case it is impossible to make a separation of variables similarly to the massless case. But this becomes possible after the partial gauge fixing. Indeed, let us set $\varphi = 0$ and solve its equation:

$$\varphi = 0 \quad \Rightarrow \quad A = \frac{1}{2Ms} e_{\alpha(2)} \pi^{\alpha(2)} \tag{4.4}$$

Resulting Lagrangian (after the rescaling $\pi \Rightarrow \frac{Ms}{2} \pi$) acquires the form:

$$\begin{aligned}
 \mathcal{L}_0 &= \sum_{k=1} (-1)^{k+1} \left[k \Omega_{\alpha(2k-1)\beta} e^\beta{}_\gamma \Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)} D f^{\alpha(2k)} \right. \\
 &\quad \left. + a_k \left(-\frac{(k+2)}{k} \Omega_{\alpha(2k)\beta(2)} e^{\beta(2)} f^{\alpha(2k)} + \Omega_{\alpha(2k)} e_{\beta(2)} f^{\alpha(2k)\beta(2)} \right) \right] \\
 &\quad + E B_{\alpha(2)} B^{\alpha(2)} - B_{\alpha\beta} E^\beta{}_\gamma D \pi^{\alpha\gamma} - 2a_0 \Omega_{\alpha\beta} E^\beta{}_\gamma \pi^{\alpha\gamma} - a_0 f_{\alpha\beta} E^\beta{}_\gamma B^{\alpha\gamma} \\
 &\quad + \sum_{k=1} (-1)^{k+1} \frac{M^2 s^2}{4k(k+1)^2} f_{\alpha(2k-1)\beta} e^\beta{}_\gamma f^{\alpha(2k-1)\gamma} + \frac{M^2 s^2}{4} E \pi_{\alpha(2)} \pi^{\alpha(2)}.
 \end{aligned} \tag{4.5}$$

Now let us introduce:

$$\begin{aligned}
 \Omega_\pm^{\alpha(2k)} &= \Omega^{\alpha(2k)} \pm \frac{Ms}{2k(k+1)} f^{\alpha(2k)}, \\
 B_\pm^{\alpha(2)} &= B^{\alpha(2)} \pm \frac{Ms}{2} \pi^{\alpha(2)}
 \end{aligned} \tag{4.6}$$

and similarly for the gauge parameters. Then we obtain:

$$\mathcal{L}_0 = \frac{1}{2M_s} [\mathcal{L}_+(\Omega_+, B_+) - \mathcal{L}_-(\Omega_-, B_-)], \quad (4.7)$$

$$\begin{aligned} \mathcal{L}_\pm = \sum_{k=1} (-1)^{k+1} & \left[k(k+1) \Omega_{\pm, \alpha(2k)} D\Omega_{\pm}^{\alpha(2k)} \pm M_s k \Omega_{\pm, \alpha(2k-1)\beta} e^\beta{}_\gamma \Omega_{\pm}^{\alpha(2k-1)\gamma} \right. \\ & \left. - 2(k+1)(k+2) a_k \Omega_{\pm, \alpha(2k)\beta(2)} e^{\beta(2)} \Omega_{\pm}^{\alpha(2k)} \right] \\ & - B_{\pm, \alpha\beta} E^\beta{}_\gamma D B_{\pm}^{\alpha\gamma} \pm M_s E B_{\pm, \alpha(2)} B_{\pm}^{\alpha(2)} - 4a_0 \Omega_{\pm, \alpha\beta} E^\beta{}_\gamma B_{\pm}^{\alpha\gamma}. \end{aligned} \quad (4.8)$$

Each Lagrangian \mathcal{L}_\pm is invariant under its own set of gauge transformations:

$$\begin{aligned} \delta\Omega_{\pm}^{\alpha(2k)} &= D\eta_{\pm}^{\alpha(2k)} + \frac{(k+2)}{k} a_k e_{\beta(2)} \eta_{\pm}^{\alpha(2k)\beta(2)} + \frac{a_{k-1}}{k(2k-1)} e^{\alpha(2)} \eta_{\pm}^{\alpha(2k-2)} \\ &\quad \pm \frac{M_s}{2k(k+1)} e^\alpha{}_\beta \eta_{\pm}^{\alpha(2k-1)\beta}, \\ \delta\Omega_{\pm}^{\alpha(2)} &= D\eta_{\pm}^{\alpha(2)} + 3a_1 e_{\beta(2)} \eta_{\pm}^{\alpha(2)\beta(2)} \pm \frac{M_s}{4} e^\alpha{}_\beta \eta_{\pm}^{\alpha\beta}, \\ \delta B_{\pm}^{\alpha(2)} &= 2a_0 \eta_{\pm}^{\alpha(2)}. \end{aligned} \quad (4.9)$$

As in the massless case, from now on we work only with + components, omitting the + sign.

Now we try to construct a complete set of gauge invariant objects (we still call all them curvatures though now they are two and one-forms). For the one-forms the structure of the gauge invariant two-forms can be easily read from the structure of gauge transformations:

$$\begin{aligned} \mathcal{R}^{\alpha(2k)} &= D\Omega^{\alpha(2k)} + \frac{(k+2)}{k} a_k e_{\beta(2)} \Omega^{\alpha(2k)\beta(2)} + \frac{a_{k-1}}{k(2k-1)} e^{\alpha(2)} \Omega^{\alpha(2k-2)} \\ &\quad + \frac{M_s}{2k(k+1)} e^\alpha{}_\beta \Omega^{\alpha(2k-1)\beta}, \\ \mathcal{R}^{\alpha(2)} &= D\Omega^{\alpha(2)} + 3a_1 e_{\beta(2)} \Omega^{\alpha(2)\beta(2)} + \frac{M_s}{4} e^\alpha{}_\beta \Omega^{\alpha\beta} + c_0 E^\alpha{}_\beta B^{\alpha\beta}. \end{aligned} \quad (4.10)$$

But to construct a gauge invariant one-form for $B^{\alpha(2)}$ we have to introduce an extra zero-form $B^{\alpha(4)}$:

$$\mathcal{B}^{\alpha(2)} = D B^{\alpha(2)} - 2a_0 \Omega^{\alpha(2)} + 3a_1 e^\alpha{}_\beta B^{\alpha\beta} + \frac{M_s}{4} e_{\beta(2)} B^{\alpha(2)\beta(2)}, \quad (4.11)$$

where we postulate

$$\delta B^{\alpha(4)} = 2a_0 \eta^{\alpha(4)} \quad (4.12)$$

Then to construct a curvature for $B^{\alpha(4)}$ we need $B^{\alpha(6)}$ and so on. The procedure stops with the set of extra zero-forms with $2 \leq k \leq s-1$

$$\delta B^{\alpha(2k)} = 2a_0 \eta^{\alpha(2k)}, \quad (4.13)$$

with their curvatures (one-forms) being:

$$\begin{aligned} \mathcal{B}^{\alpha(2k)} &= D B^{\alpha(2k)} - 2a_0 \Omega^{\alpha(2k)} + \frac{M_s}{2k(k+1)} e^\alpha{}_\beta B^{\alpha(2k-1)\beta} \\ &\quad + \frac{(k+2)}{k} a_k e_{\beta(2)} B^{\alpha(2k)\beta(2)} + \frac{a_{k-1}}{k(2k-1)} e^{\alpha(2)} B^{\alpha(2k-2)} \end{aligned} \quad (4.14)$$

With the help of all these gauge invariant curvatures, one can rewrite the Lagrangian in the explicitly gauge invariant form:

$$\mathcal{L}_0 = \sum_{k=1}^{s-1} (-1)^k \frac{k(k+1)}{2a_0} \mathcal{R}_{\alpha(2k)} \mathcal{B}^{\alpha(2k)}, \quad (4.15)$$

where coefficients are determined by the so-called extra field decoupling condition:

$$\frac{\delta \mathcal{L}_0}{\delta B^{\alpha(2k)}} = 0, \quad k > 1.$$

A large number of fields involved in the description of free field make the investigation of their interactions very cumbersome. To simplify investigations, we use the procedure of (almost) maximal gauge fixing. In more details, let us use $\eta^{\alpha(2)}$ transformations to set $B^{\alpha(2)} = 0$ and set to zero its gauge invariant one-form:

$$B^{\alpha(2)} = 0 \quad \Rightarrow \quad \Omega^{\alpha(2)} = \frac{3a_1}{2a_0} e_{\beta(2)} B^{\alpha(2)\beta(2)}, \quad (4.16)$$

then it easy to check that

$$\begin{aligned} \mathcal{R}^{\alpha(2)} &= -\frac{3a_1}{2a_0} \left[e_{\beta(2)} D B^{\alpha(2)\beta(2)} - 2a_0 e_{\beta(2)} \Omega^{\alpha(2)\beta(2)} + M s E_{\beta(2)} B^{\alpha(2)\beta(2)} \right] \\ &= -\frac{3a_1}{2a_0} e_{\beta(2)} \mathcal{B}^{\alpha(2)\beta(2)}, \end{aligned} \quad (4.17)$$

so that $\mathcal{R}^{\alpha(2)}$ is not an independent object and can be omitted. Proceeding in this way, at the last step we set

$$B^{\alpha(2s-4)} = 0 \quad \Rightarrow \quad \Omega^{\alpha(2s-4)} = \frac{s a_{s-2}}{2(s-2)a_0} e_{\beta(2)} B^{\alpha(2s-4)\beta(2)}. \quad (4.18)$$

Here also we find that

$$\mathcal{R}^{\alpha(2s-4)} = \frac{s a_{s-2}}{2(s-2)a_0} e_{\beta(2)} \mathcal{B}^{\alpha(2s-4)\beta(2)} \quad (4.19)$$

is not independent any more.

As a result, we obtain really minimal Lagrangian (rescaling $B^{\alpha(2s-2)}$ for convenience):

$$\begin{aligned} \frac{(-1)^s}{s(s-1)} \mathcal{L} &= \Omega_{\alpha(2s-2)} D \Omega^{\alpha(2s-2)} + M \Omega_{\alpha(2s-3)\beta} e^\beta{}_\gamma \Omega^{\alpha(2s-3)\gamma} \\ &\quad - B_{\alpha(2s-3)\beta} E^\beta{}_\gamma D B^{\alpha(2s-3)\gamma} + \frac{M s}{(s-1)} E B_{\alpha(2s-2)} B^{\alpha(2s-2)} \\ &\quad - 2\tilde{m} \Omega_{\alpha(2s-3)\beta} E^\beta{}_\gamma B^{\alpha(2s-3)\gamma}, \end{aligned} \quad (4.20)$$

where

$$\tilde{m} = \sqrt{\frac{2}{(s-1)}} m.$$

This Lagrangian follows the general pattern for the gauge invariant description for the massive fields. Indeed, it is invariant under the only remaining gauge transformations:

$$\begin{aligned}\delta\Omega^{\alpha(2s-2)} &= D\eta^{\alpha(2s-2)} + \frac{M}{2(s-1)}e^\alpha{}_\beta\eta^{\alpha(2s-3)\beta}, \\ \delta B^{\alpha(2s-2)} &= \tilde{m}\eta^{\alpha(2s-2)}.\end{aligned}\tag{4.21}$$

Moreover, we still have a couple of gauge invariant curvatures:

$$\begin{aligned}\mathcal{R}^{\alpha(2s-2)} &= D\Omega^{\alpha(2s-2)} + \frac{M}{2(s-1)}e^\alpha{}_\beta\Omega^{\alpha(2s-3)\beta} - \frac{\tilde{m}}{2(s-1)}E^\alpha{}_\beta B^{\alpha(2s-3)\beta}, \\ \mathcal{B}^{\alpha(2s-2)} &= DB^{\alpha(2s-2)} - \tilde{m}\Omega^{\alpha(2s-2)} + \frac{M}{2(s-1)}e^\alpha{}_\beta B^{\alpha(2s-3)\beta}.\end{aligned}\tag{4.22}$$

They satisfy the following differential identities:

$$\begin{aligned}D\mathcal{R}^{\alpha(2s-2)} &= -\frac{M}{2(s-1)}e^\alpha{}_\beta\mathcal{R}^{\alpha(2s-3)\beta} - \frac{\tilde{m}}{2(s-1)}E^\alpha{}_\beta\mathcal{B}^{\alpha(2s-3)\beta}, \\ D\mathcal{B}^{\alpha(2s-2)} &= -\tilde{m}\mathcal{R}^{\alpha(2s-2)} - \frac{M}{2(s-1)}e^\alpha{}_\beta\mathcal{B}^{\alpha(2s-3)\beta}.\end{aligned}\tag{4.23}$$

Naturally, these curvatures appear in the variation of the Lagrangian under the arbitrary variations of Ω and B fields:

$$\delta\mathcal{L} \sim \mathcal{R}_{\alpha(2s-2)}\delta\Omega^{\alpha(2s-2)} + \mathcal{B}_{\alpha(2s-3)\beta}E^\beta{}_\gamma\delta B^{\alpha(2s-3)\gamma}.\tag{4.24}$$

At last, but not least, the Lagrangian can be nicely rewritten in the explicitly gauge invariant form:

$$\mathcal{L} \sim \frac{1}{2\tilde{m}}\mathcal{R}_{\alpha(2s-2)}\mathcal{B}^{\alpha(2s-2)}.\tag{4.25}$$

With this Lagrangian formulation we can associate a self-consistent set of unfolded equations (compare [24–26]):

$$\begin{aligned}0 &= D\Omega^{\alpha(2s-2)} + \frac{M}{2(s-1)}e^\alpha{}_\beta\Omega^{\alpha(2s-3)\beta} - \frac{\tilde{m}}{2(s-1)}E^\alpha{}_\beta B^{\alpha(2s-3)\beta}, \\ 0 &= DB^{\alpha(2s-2)} - \tilde{m}\Omega^{\alpha(2s-2)} + \frac{M}{2(s-1)}e^\alpha{}_\beta B^{\alpha(2s-3)\beta} + e_{\beta(2)}W^{\alpha(2s-2)\beta(2)}, \\ 0 &= DW^{\alpha(2k)} + e_{\beta(2)}W^{\alpha(2k)\beta(2)} + \alpha_k e^\alpha{}_\beta W^{\alpha(2k-1)\beta} + \beta_k e^{\alpha(2)}W^{\alpha(2k-2)},\end{aligned}\tag{4.26}$$

where $W^{\alpha(2k)}$, $k \geq s$ is an infinite set of gauge invariant zero-forms and

$$\alpha_k = \frac{Ms}{2k(k+1)}, \quad \beta_k = -\frac{(k^2 - s^2)}{2(4k^2 - 1)} \left[\frac{M^2}{k^2} - \lambda^2 \right].\tag{4.27}$$

Note, that the equations for $W^{\alpha(2k)}$ have exactly the same form as in the general case [25, 26] and this serves as an additional conformation that after all these gauge fixing we still have one physical degree of freedom.

4.2 Free fermion

For the gauge invariant description of massive spin- $(s + 1/2)$ fermion we need a set of one-forms $\Phi^{\alpha(2k+1)}$, $(0 \leq k \leq s - 1)$ and zero-form ϕ^α . Free Lagrangian looks like:

$$\begin{aligned} \frac{1}{i}\mathcal{L}_0 &= \sum_{k=0}^{s-1} (-1)^{k+1} \left[\frac{1}{2} \Phi_{\alpha(2k+1)} D \Phi^{\alpha(2k+1)} \right] + \frac{1}{2} \phi_\alpha E^\alpha{}_\beta D \phi^\beta \\ &+ \sum_{k=1}^{s-1} (-1)^{k+1} c_k \Phi_{\alpha(2k-1)\beta(2)} e^{\beta(2)} \Phi^{\alpha(2k-1)} + c_0 \Phi_\alpha E^\alpha{}_\beta \phi^\beta \\ &+ \sum_{k=0}^{s-1} (-1)^{k+1} \frac{d_k}{2} \Phi_{\alpha(2k)\beta} e^\beta{}_\gamma \Phi^{\alpha(2k)\gamma} - \frac{3d_0}{2} E \phi_\alpha \phi^\alpha, \end{aligned} \quad (4.28)$$

where

$$\begin{aligned} d_k &= \frac{(2s+1)}{(2k+3)} M, \quad M^2 = m^2 + \left(s - \frac{1}{2}\right)^2 \lambda^2, \\ c_k^2 &= \frac{(s+k+1)(s-k)}{2(k+1)(2k+1)} \left[M^2 - (2k+1)^2 \frac{\lambda^2}{4} \right], \\ c_0^2 &= 2s(s+1) \left[M^2 - \frac{\lambda^2}{4} \right]. \end{aligned} \quad (4.29)$$

This Lagrangian is invariant under the following gauge transformations:

$$\begin{aligned} \delta_0 \Phi^{\alpha(2k+1)} &= D \zeta^{\alpha(2k+1)} + \frac{d_k}{(2k+1)} e^\alpha{}_\beta \zeta^{\alpha(2k)\beta} + \frac{c_k}{k(2k+1)} e^{\alpha(2)} \zeta^{\alpha(2k-1)} \\ &+ c_{k+1} e_{\beta(2)} \zeta^{\alpha(2k+1)\beta(2)}, \\ \delta_0 \phi^\alpha &= c_0 \zeta^\alpha. \end{aligned} \quad (4.30)$$

As in the bosonic case, to construct a complete set of the gauge invariant curvatures:

$$\begin{aligned} \mathcal{F}^{\alpha(2k+1)} &= D \Phi^{\alpha(2k+1)} + \frac{d_k}{(2k+1)} e^\alpha{}_\beta \Phi^{\alpha(2k)\beta} \\ &+ \frac{c_k}{k(2k+1)} e^{\alpha(2)} \Phi^{\alpha(2k-1)} + c_{k+1} e_{\beta(2)} \Phi^{\alpha(2k+1)\beta(2)}, \\ \mathcal{F}^\alpha &= D \Phi^\alpha + d_0 e^\alpha{}_\beta \Phi^\beta + c_1 e_{\beta(2)} \Phi^{\alpha\beta(2)} - c_0 E^\alpha{}_\beta \phi^\beta, \\ \mathcal{C}^\alpha &= D \phi^\alpha - c_0 \Phi^\alpha + d_0 e^\alpha{}_\beta \phi^\beta + c_1 e_{\beta(2)} \phi^{\alpha\beta(2)}, \\ \mathcal{C}^{\alpha(2k+1)} &= D \phi^{\alpha(2k+1)} - c_0 \Phi^{\alpha(2k+1)} + \frac{d_k}{(2k+1)} e^\alpha{}_\beta \phi^{\alpha(2k)\beta}, \\ &+ \frac{c_k}{k(2k+1)} e^{\alpha(2)} \phi^{\alpha(2k-1)} + c_{k+1} e_{\beta(2)} \phi^{\alpha(2k+1)\beta(2)}, \end{aligned} \quad (4.31)$$

we have to introduce a number of extra zero-forms $\phi^{\alpha(2k+1)}$, $1 \leq k \leq s - 1$, where

$$\delta \phi^{\alpha(2k+1)} = c_0 \zeta^{\alpha(2k+1)}. \quad (4.32)$$

Such Lagrangian can also be rewritten in terms of the curvatures:

$$\mathcal{L}_0 = \sum_{k=0}^{s-1} (-1)^k \frac{i}{2c_0} \mathcal{F}_{\alpha(2k+1)} \mathcal{C}^{\alpha(2k+1)}, \quad (4.33)$$

where coefficients are again determined by the extra field decoupling conditions:

$$\frac{\delta \mathcal{L}_0}{\delta \phi^{\alpha(2k+1)}} = 0, \quad k > 0. \quad (4.34)$$

As in the bosonic case, we now apply the (almost) maximal gauge fixing. We begin by setting

$$\phi^\alpha = 0 \quad \Rightarrow \quad \Phi^\alpha = \frac{c_1}{c_0} e_{\beta(2)} \phi^{\alpha\beta(2)} \quad (4.35)$$

and checking that

$$\mathcal{F}^\alpha = -\frac{c_1}{c_0} e_{\beta(2)} \mathcal{C}^{\alpha\beta(2)}. \quad (4.36)$$

Proceeding in this way, at the last step we set:

$$\phi^{\alpha(2s-3)} = 0 \quad \Rightarrow \quad \Phi^{\alpha(2s-3)} = \frac{c_{s-1}}{c_0} e_{\beta(2)} \phi^{\alpha(2s-3)\beta(2)} \quad (4.37)$$

and check that

$$\mathcal{F}^{\alpha(2s-3)} = -\frac{c_{s-1}}{c_0} e_{\beta(2)} \mathcal{C}^{\alpha(2s-3)\beta(2)}. \quad (4.38)$$

Thus we obtain the minimal Lagrangian (rescaling $\phi^{\alpha(2s-1)}$ for simplicity):

$$\begin{aligned} \frac{(-1)^s}{i} \mathcal{L} = & \frac{1}{2} \Phi_{\alpha(2s-1)} D \Phi^{\alpha(2s-1)} - 2 \phi_{\alpha(2s-2)\beta} E^\beta{}_\gamma D \phi^{\alpha(2s-2)\gamma} \\ & - 4 \tilde{m} \Phi_{\alpha(2s-2)\beta} E^\beta{}_\gamma \phi^{\alpha(2s-2)\gamma} \\ & + \frac{M}{2} \Phi_{\alpha(2s-2)\beta} e^\beta{}_\gamma \Phi^{\alpha(2s-2)\gamma} - \frac{2(2s+1)}{(2s-1)} M E \phi_{\alpha(2s-1)} \phi^{\alpha(2s-1)}, \end{aligned} \quad (4.39)$$

where

$$\tilde{m} = \sqrt{\frac{m}{(2s-1)}}. \quad (4.40)$$

This Lagrangian is still invariant under the only remaining gauge transformations:

$$\begin{aligned} \delta_0 \Phi^{\alpha(2s-1)} &= D \zeta^{\alpha(2s-1)} + \frac{M}{(2s-1)} e^\alpha{}_\beta \zeta^{\alpha(2s-2)\beta}, \\ \delta_0 \phi^{\alpha(2s-1)} &= \tilde{m} \zeta^{\alpha(2s-1)} \end{aligned} \quad (4.41)$$

Also, we still have a couple of gauge invariant curvatures:

$$\begin{aligned} \mathcal{F}^{\alpha(2s-1)} &= D \Phi^{\alpha(2s-1)} + \frac{M}{(2s-1)} e^\alpha{}_\beta \Phi^{\alpha(2s-2)\beta} - \frac{4c_{s-1}}{(2s-1)} E^\alpha{}_\beta \phi^{\alpha(2s-2)\beta}, \\ \mathcal{C}^{\alpha(2s-1)} &= D \phi^{\alpha(2s-1)} - c_{s-1} \Phi^{\alpha(2s-1)} + \frac{M}{(2s-1)} e^\alpha{}_\beta \phi^{\alpha(2s-2)\beta} \end{aligned} \quad (4.42)$$

satisfying the following differential identities:

$$\begin{aligned} D \mathcal{F}^{\alpha(2s-1)} &= -\frac{M}{(2s-1)} e^\alpha{}_\beta \mathcal{F}^{\alpha(2s-2)\beta} - \frac{4c_{s-1}}{(2s-1)} E^\alpha{}_\beta \mathcal{C}^{\alpha(2s-2)\beta}, \\ D \mathcal{C}^{\alpha(2s-1)} &= -c_{s-1} \mathcal{F}^{\alpha(2s-1)} - \frac{M}{(2s-1)} e^\alpha{}_\beta \mathcal{C}^{\alpha(2s-2)\beta}. \end{aligned} \quad (4.43)$$

Variation of the Lagrangian under the arbitrary variations of Φ and ϕ has the form:

$$\delta\mathcal{L} \sim \mathcal{F}_{\alpha(2s-1)}\delta\Phi^{\alpha(2s-1)} + 4\mathcal{C}_{\alpha(2s-2)\beta}E^\beta_\gamma\delta\phi^{\alpha(2s-2)\gamma}, \quad (4.44)$$

while the Lagrangian can be rewritten simply as

$$\mathcal{L} \sim \frac{1}{2\tilde{m}}\mathcal{F}_{\alpha(2s-1)}\mathcal{C}^{\alpha(2s-1)}. \quad (4.45)$$

Here there also exists a set of self-consistent unfolded equations (compare [26]):

$$\begin{aligned} 0 &= D\Phi^{\alpha(2s-1)} + \frac{M}{(2s-1)}e^\alpha_\beta\Phi^{\alpha(2s-2)\beta} - \frac{4c_{s-1}}{(2s-1)}E^\alpha_\beta\phi^{\alpha(2s-2)\beta}, \\ 0 &= D\phi^{\alpha(2s-1)} - c_{s-1}\Phi^{\alpha(2s-1)} + \frac{M}{(2s-1)}e^\alpha_\beta\phi^{\alpha(2s-2)\beta} + e_{\beta(2)}V^{\alpha(2s-1)\beta(2)}, \\ 0 &= DV^{\alpha(2k+1)} + e_{\beta(2)}V^{\alpha(2k+1)\beta(2)} + \alpha_k e^\alpha_\beta V^{\alpha(2k)\beta} + \beta_k e^{\alpha(2)}V^{\alpha(2k-1)}, \end{aligned} \quad (4.46)$$

where $V^{\alpha(2k+1)}$, $k \geq s$ is an infinite set of gauge invariant fermionic zero-forms, while

$$\alpha_k = \frac{(2s+1)M}{(2k+1)(2k+3)}, \quad \beta_k = -\frac{(k-s)(k+s+1)}{2k(k+1)}\left[\frac{M^2}{(2k+1)^2} - \frac{\lambda^2}{4}\right]. \quad (4.47)$$

Let us stress once again, that the equations for V have the same form as in the general formalism without any gauge fixing.

4.3 Two massive and one massless fields

In this subsection we consider possible cubic vertices for two massive and one massless fields. Let us begin with the purely bosonic case: two massive bosons with spins $s_{1,2}$ and one massless boson with spin s_3 . Any consistent non-abelian corrections to the gauge transformations must be related with consistent deformations for all gauge invariant curvatures [10] (schematically):

$$\begin{aligned} \Delta\mathcal{R}_1 &\sim \Omega_2\Omega_3 + eB_2\Omega_3, & \Delta\mathcal{B}_1 &\sim B_2\Omega_3, \\ \Delta\mathcal{R}_2 &\sim \Omega_1\Omega_3 + eB_1\Omega_3, & \Delta\mathcal{B}_2 &\sim B_2\Omega_3. \end{aligned}$$

The main consistency requirement here is that the deformed curvatures $\hat{\mathcal{R}}_{1,2} = \mathcal{R}_{1,2} + \Delta\mathcal{R}_{1,2}$, $\hat{\mathcal{B}}_{1,2} = \mathcal{B}_{1,2} + \Delta\mathcal{B}_{1,2}$ transform covariantly:

$$\begin{aligned} \delta\hat{\mathcal{R}}_1 &\sim \mathcal{R}_2\eta_3 + e\mathcal{B}_2\eta_3, & \delta\hat{\mathcal{B}}_1 &\sim \mathcal{B}_2\eta_3, \\ \delta\hat{\mathcal{R}}_2 &\sim \mathcal{R}_1\eta_3 + e\mathcal{B}_1\eta_3, & \delta\hat{\mathcal{B}}_2 &\sim \mathcal{B}_1\eta_3. \end{aligned}$$

Due to the presence of zero-form fields, there exists a pair of possible field redefinitions:

$$\Omega_1 \Rightarrow \Omega_1 + \kappa_1 B_2 \Omega_3, \quad \Omega_2 \Rightarrow \Omega_2 + \kappa_2 B_1 \Omega_3.$$

It appears that using these field redefinitions one can completely remove all possible non-abelian corrections. Taking into account that in three dimensions it is not possible to

construct any trivially gauge invariant vertex (simply because it must be four-form as a minimum), the only remaining possibility is an abelian vertex:

$$\mathcal{L}_1 = g\mathcal{B}_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)}\mathcal{B}_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3)}\Omega_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)}. \quad (4.48)$$

Here \hat{s}_i are the same as in the massless case and it means that three spins $s_{1,2,3}$ also must satisfy a strict triangular inequality. Consider variation of the vertex under the gauge transformations of the massless field:

$$\begin{aligned} \frac{1}{g}\delta\mathcal{L}_1 &= \mathcal{B}_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)}\mathcal{B}_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3)}\left[D\eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1)} - \frac{\lambda}{2}\left(e_{\beta}{}^{\delta}\eta_{\beta(\hat{s}_2-1)\delta\gamma(\hat{s}_1)} + e_{\gamma}{}^{\delta}\eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1-1)\delta}\right)\right] \\ &= \tilde{m}_1\mathcal{R}_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)}\mathcal{B}_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3)}\eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1)} - \tilde{m}_2\mathcal{B}_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)}\mathcal{R}_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3)}\eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1)} \\ &\quad + \frac{M_1}{(2s_1-2)}\left(e_{\delta}{}^{\alpha}\mathcal{B}_1^{\alpha(\hat{s}_3-1)\delta\beta(\hat{s}_2)} + e^{\beta}{}_{\delta}\mathcal{B}_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2-1)\delta}\right)\mathcal{B}_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3)}\eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1)} \\ &\quad - \frac{M_2}{(2s_2-2)}\mathcal{B}_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)}\left(e_{\delta}{}^{\gamma}\mathcal{B}_2^{\gamma(\hat{s}_1-1)\delta}{}_{\alpha(\hat{s}_3)} - e_{\alpha}{}^{\delta}\mathcal{B}_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3-1)\delta}\right)\eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1)} \\ &\quad - \frac{\lambda}{2}\mathcal{B}_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)}\mathcal{B}_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3)}\left(e_{\beta}{}^{\delta}\eta_{\beta(\hat{s}_2-1)\delta\gamma(\hat{s}_1)} + e_{\gamma}{}^{\delta}\eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1-1)\delta}\right). \end{aligned}$$

Terms with the $\mathcal{R}_{1,2}$ vanish on-shell or, if one likes, can be compensated by the abelian corrections to the gauge transformations:

$$\begin{aligned} \delta\Omega_1^{\alpha(2s_1-2)} &\sim \mathcal{B}_2^{\alpha(\hat{s}_3)\gamma(\hat{s}_1)}\eta^{\alpha(\hat{s}_2)}{}_{\gamma(\hat{s}_1)}, \\ \delta\Omega_2^{\alpha(2s_1-2)} &\sim \mathcal{B}_1^{\alpha(\hat{s}_3)\gamma(\hat{s}_2)}\eta^{\alpha(\hat{s}_1)}{}_{\gamma(\hat{s}_2)}, \end{aligned} \quad (4.49)$$

which correspond to abelian deformations for curvatures:

$$\begin{aligned} \Delta\mathcal{R}_1^{\alpha(2s_1-2)} &\sim \mathcal{B}_2^{\alpha(\hat{s}_3)\gamma(\hat{s}_1)}\Omega_3^{\alpha(\hat{s}_2)}{}_{\gamma(\hat{s}_1)}, \\ \Delta\mathcal{R}_2^{\alpha(2s_1-2)} &\sim \mathcal{B}_1^{\alpha(\hat{s}_3)\gamma(\hat{s}_2)}\Omega_3^{\alpha(\hat{s}_1)}{}_{\gamma(\hat{s}_2)}. \end{aligned} \quad (4.50)$$

As for the terms quadratic in \mathcal{B} , we use the fact that on-shell

$$\mathcal{B}^{\alpha(2s-2)} \approx e_{\beta(2)}W^{\alpha(2s-2)\beta(2)}.$$

Then these terms are reduced to

$$\begin{aligned} \Delta &\sim \left[\frac{\hat{s}_2 + \hat{s}_3}{(2s_1-2)}M_1 - \frac{\hat{s}_1 + \hat{s}_3}{(2s_2-2)}M_2 + (\hat{s}_1 - \hat{s}_2)\frac{\lambda}{2}\right]W_1^{\alpha(\hat{s}_3+2)\beta(\hat{s}_2)}W_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3+2)}\eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1)} \\ &= [M_1 - M_2 - (s_1 - s_2)\lambda]W_1^{\alpha(\hat{s}_3+2)\beta(\hat{s}_2)}W_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3+2)}\eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1)}. \end{aligned} \quad (4.51)$$

Thus the gauge invariance imposes the relation on the masses:

$$M_1 - M_2 = (s_1 - s_2)\lambda. \quad (4.52)$$

In the same way one can construct cubic vertices for two other cases: two massive fermions and one massless boson

$$\mathcal{L}_1 = g\mathcal{C}_1^{\alpha(\hat{s}_3+1)\beta(\hat{s}_2)}\mathcal{C}_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3+1)}\Omega_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \quad (4.53)$$

or massive boson and massive and massless fermions

$$\mathcal{L}_1 = g\mathcal{B}_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)}\mathcal{C}_2^{\gamma(\hat{s}_1+1)}{}_{\alpha(\hat{s}_3)}\Phi_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1+1)} \quad (4.54)$$

with the same relations on the masses.

Note, that for the first time a simple example of such mass relation appeared in our construction of unfolded formulation for massive higher spin supermultiplets [26]. There we consider deformations of the unfolded equations for massive boson (M_1, s_1) and massive fermion (M_2, s_2) in the presence of a background massless spin-3/2 field:

$$0 = D\Phi^\alpha + \frac{\lambda}{2}e^\alpha{}_\beta\Phi^\beta. \quad (4.55)$$

For the gauge invariant part of the unfolded equations such ansatz has the form:

$$\begin{aligned} 0 = & DW^{\alpha(2k)} + e_{\beta(2)}W^{\alpha(2k)\beta(2)} + \alpha_k e^\alpha{}_\beta W^{\alpha(2k-1)\beta} + \beta_k e^{\alpha(2)}W^{\alpha(2k-2)} \\ & + f_{k,0}\Phi_\beta V^{\alpha(2k)\beta} + f_{k,1}\Phi^\alpha V^{\alpha(2k-1)}, \end{aligned} \quad (4.56)$$

$$\begin{aligned} 0 = & DV^{\alpha(2k+1)} + e_{\beta(2)}V^{\alpha(2k+1)\beta(2)} + \gamma_k e^\alpha{}_\beta V^{\alpha(2k)\beta} + \delta_k e^{\alpha(2)}V^{\alpha(2k-1)} \\ & + g_{k,0}\Phi_\beta W^{\alpha(2k+1)\beta} + g_{k,1}\Phi^\alpha W^{\alpha(2k)}. \end{aligned} \quad (4.57)$$

Self-consistency of these deformations requires

$$s_2 = s_1 + \Delta, \quad M_2 = M_1 + \Delta\lambda, \quad \Delta = \pm\frac{1}{2}, \quad (4.58)$$

the solution being:

$$f_{k,0} = f_0, \quad f_{k,1} = \mp\frac{(k \mp s_1)}{2k(2k+1)}[M_1 \mp k\lambda]f_0, \quad (4.59)$$

$$g_{k,0} = g_0, \quad g_{k,1} = \pm\frac{(k \pm s_1 + 1)}{2(k+1)(2k+1)}[M_1 \pm (k+1)\lambda]g_0. \quad (4.60)$$

This construction can be easily generalized to the massless fermion with arbitrary spin s (see appendix A). We have found a complete solution for the case $s = \frac{5}{2}$ and obtain the same relations as in (4.58) but with $\Delta = \pm\frac{1}{2}, \pm\frac{3}{2}$.

It is instructive to compare our results with what happens in the Prokushkin-Vasiliev theory [4, 5], describing interaction of higher spin massless fields with massive spin 0 and spin 1/2. The unfolded equations for these massive fields [27] in our current notation look like:

$$0 = DW^{\alpha(2k)} + e_{\beta(2)}W^{\alpha(2k)\beta(2)} + \beta_k e^{\alpha(2)}W^{\alpha(2k-2)} \quad (4.61)$$

$$\begin{aligned} \beta_k = & -\frac{1}{2(4k^2-1)}[m_0^2 - (k^2-1)\lambda^2] \\ 0 = & DV^{\alpha(2k+1)} + e_{\beta(2)}V^{\alpha(2k+1)\beta(2)} + \alpha_k e^\alpha{}_\beta V^{\alpha(2k)\beta} + \beta_k e^{\alpha(2)}V^{\alpha(2k-1)} \end{aligned} \quad (4.62)$$

$$\alpha_k = \frac{m}{(2k+1)(2k+3)}, \quad \beta_k = -\frac{1}{2(2k+1)^2}\left[m^2 - (2k+1)^2\frac{\lambda^2}{4}\right]$$

Trying to solve consistency relations given in appendix A iteratively by l already at $l = 2$ we have found that the solution exists only when

$$m_0^2 = m^2 - m\lambda - \frac{3}{4}\lambda^2 \quad (4.63)$$

It is crucial that this relation does not depend on the spin of the massless fermion so that the whole infinite set of massless fields with different spins can simultaneously interact with one and the same massive supermultiplet $(\frac{1}{2}, 0)$. Recall that the same relation we have found for this massive supermultiplet in [26].

Similarly, one can consider interaction for massless boson with two massive bosons (see appendix B) or two massive fermions. Let us provide here a couple of explicit examples.

For the massless spin 2 and two massive bosons with arbitrary masses and spins one can consider

$$0 = DW_1^{\alpha(2k)} + e_{\beta(2)}W_1^{\alpha(2k)\beta(2)} + \alpha_{1,k}e^\alpha{}_\beta W_1^{\alpha(2k-1)\beta} + \beta_{1,k}e^{\alpha(2)}W_1^{\alpha(2k-2)} \\ + f_{k,0}\Omega_{\beta(2)}W_2^{\alpha(2k)\beta(2)} + f_{k,1}\Omega^\alpha{}_\beta W_2^{\alpha(2k-1)\beta} + f_{k,2}\Omega^{\alpha(2)}W_2^{\alpha(2k-2)} \quad (4.64)$$

and similarly for the second boson. We again obtain the same relations with $\Delta = 0, \pm 1$, where

$\Delta = 0$:

$$f_{k,0} = f_0, \quad f_{k,1} = \alpha_k + \frac{\lambda}{2}, \quad f_{k,2} = \beta_k; \quad (4.65)$$

$\Delta = \pm 1$:

$$f_{k,0} = f_0, \quad f_{k,1} = \mp \frac{(k \mp s_1)}{2k(k+1)}[M_1 \mp k\lambda]f_0, \\ f_{k,2} = \frac{(k \mp s_1 - 1)(k \mp s_1)}{2k^2(4k^2 - 1)}[M_1 \mp k\lambda][M_1 \mp (k-1)\lambda]f_0. \quad (4.66)$$

We have found a complete solution for the massless spin 3 and indeed found $\Delta = 0, \pm 1, \pm 2$.

4.4 Three massive fields

In this subsection we consider cubic vertices for three massive fields. Again we begin with the purely bosonic case. First of all note, that here we have a number of possible field redefinitions

$$\Omega_1 \Rightarrow \Omega_1 + \kappa_1\Omega_2B_3 + \kappa_2B_2\Omega_3$$

and similarly for two other one-forms. As in the previous case, using these redefinitions one can completely remove all possible non-abelian corrections to the gauge transformations. Further, any abelian vertex of the form $\mathcal{B}\mathcal{B}\Omega$ by using the substitution

$$\Omega^{\alpha(2s-2)} \Rightarrow -\frac{1}{\tilde{m}} \left[\mathcal{B}^{\alpha(2s-2)} - DB^{\alpha(2s-2)} - \frac{M}{2(s-1)}e^\alpha{}_\beta B^{\alpha(2s-3)\beta} \right]$$

and differential identities for the curvatures can be shown to be equivalent to the combination of the trivially gauge invariant ones and abelian vertices of the form $\mathcal{B}\mathcal{B}\mathcal{B}$, which can not be made gauge invariant. This leaves us with the only possibility — trivially gauge invariant vertex:

$$\mathcal{L}_1 = g\mathcal{B}_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)}\mathcal{B}_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3)}\mathcal{B}_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \quad (4.67)$$

Similarly, for two massive fermions and one massive boson we can construct:

$$\mathcal{L}_1 = g\mathcal{C}_1^{\alpha(\hat{s}_3+1)\beta(\hat{s}_2)}\mathcal{C}_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3+1)}\mathcal{B}_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)} \quad (4.68)$$

In both cases a strict triangular inequality must hold. Each term in these vertices are separately gauge invariant so one can freely consider both + and – components combining them into parity even/odd combinations similarly to the massless case.

An important remaining open question is the relation of such vertices with the classification obtained recently in [1] (see also [2]).

5 Partially massless fields

To discuss the partially massless fields let us return back to the initial Lagrangian (4.1) for massive boson. It is unitary not only in AdS₃ and flat Minkowski spaces, but in dS₃ space as well, provided

$$M^2 = m^2 - (s - 1)^2\Lambda \geq 0. \quad (5.1)$$

At the boundary of the unitarity region lives the only partially massless field which has one physical degree of freedom. Inside the region we find a number of partially massless fields described by a set of one-forms $\Omega^{\alpha(2k)}$, $f^{\alpha(2k)}$, $s - 1 - t \leq k \leq s - 1$, where t is a depth of partially masslessness (we define depth so that $t = 0$ corresponds to the massless case). These fields do not have any physical degrees of freedom and must be considered separately.

5.1 Maximal depth

Here $M = 0$ and zero-forms $\pi^{\alpha(2)}$ and φ decouple. The Lagrangian becomes:

$$\begin{aligned} \mathcal{L}_0 = & \sum_{k=1} (-1)^{k+1} \left[k\Omega_{\alpha(2k-1)\beta} e^\beta{}_\gamma \Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)} Df^{\alpha(2k)} \right] \\ & + EB_{\alpha(2)} B^{\alpha(2)} - B_{\alpha(2)} e^{\alpha(2)} DA \\ & + \sum_{k=1} (-1)^{k+1} a_k \left[-\frac{(k+2)}{k} \Omega_{\alpha(2k)\beta(2)} e^{\beta(2)} f^{\alpha(2k)} + \Omega_{\alpha(2k)} e_{\beta(2)} f^{\alpha(2k)\beta(2)} \right] \\ & + 2a_0 \Omega_{\alpha(2)} e^{\alpha(2)} A - a_0 f_{\alpha\beta} E^\beta{}_\gamma B^{\alpha\gamma}, \end{aligned} \quad (5.2)$$

where

$$\begin{aligned} a_k^2 &= \frac{k(k+1)(s+k+1)(s-k-1)}{2(k+2)(2k+3)} \Lambda, \\ a_0^2 &= \frac{(s+1)(s-1)}{3} \Lambda. \end{aligned} \quad (5.3)$$

The gauge transformations are the same as before (but without π and φ). The gauge invariant two-forms now:

$$\begin{aligned}
 \mathcal{R}^{\alpha(2k)} &= D\Omega^{\alpha(2k)} + \frac{(k+2)}{k} a_k e_{\beta(2)} \Omega^{\alpha(2k)\beta(2)} + \frac{a_{k-1}}{k(2k-1)} e^{\alpha(2)} \Omega^{\alpha(2k-2)}, \\
 \mathcal{T}^{\alpha(2k)} &= Df^{\alpha(2k)} + a_k e_{\beta(2)} f^{\alpha(2k)\beta(2)} + \frac{(k+1)a_{k-1}}{k(k-1)(2k-1)} e^{\alpha(2)} f^{\alpha(2k-2)} + e^\alpha{}_\beta \Omega^{\alpha(2k-1)\beta}, \\
 \mathcal{R}^{\alpha(2)} &= D\Omega^{\alpha(2)} + 3a_1 e_{\beta(2)} \Omega^{\alpha(2)\beta(2)}, \\
 \mathcal{T}^{\alpha(2)} &= Df^{\alpha(2)} + e^\alpha{}_\beta \Omega^{\alpha\beta} + a_1 e_{\beta(2)} f^{\alpha(2)\beta(2)} + 2a_0 e^{\alpha(2)} A, \\
 \mathcal{R} &= DA + \frac{a_0}{4} e_{\alpha(2)} f^{\alpha(2)} - E_{\alpha(2)} B^{\alpha(2)}.
 \end{aligned} \tag{5.4}$$

As for the one-forms, we obtain:

$$\begin{aligned}
 \mathcal{B}^{\alpha(2)} &= DB^{\alpha(2)} - 2a_0 \Omega^{\alpha(2)} + 3a_1 e_{\beta(2)} B^{\alpha(2)\beta(2)}, \\
 \mathcal{B}^{\alpha(2k)} &= DB^{\alpha(2k)} - 2a_0 \Omega^{\alpha(2k)} + \frac{a_{k-1}}{k(2k-1)} e^{\alpha(2)} B^{\alpha(2k-2)} + \frac{(k+2)}{k} a_k e_{\beta(2)} B^{\alpha(2k)\beta(2)}, \\
 \mathcal{B}^{\alpha(2s-2)} &= DB^{\alpha(2s-2)} - 2a_0 \Omega^{\alpha(2s-2)} + \frac{a_{s-2}}{(s-1)(2s-3)} e^{\alpha(2)} B^{\alpha(2s-4)}.
 \end{aligned} \tag{5.5}$$

There also exists a set of self-consistent unfolded equations [25], namely, all curvatures except $\mathcal{B}^{\alpha(2s-2)}$ are zero, while

$$\begin{aligned}
 0 &= DB^{\alpha(2s-2)} - 2a_0 \Omega^{\alpha(2s-2)} + \frac{a_{s-2}}{(s-1)(2s-3)} e^{\alpha(2)} B^{\alpha(2s-4)} + e_{\beta(2)} W^{\alpha(2s-2)\beta(2)}, \\
 0 &= DW^{\alpha(2k)} + e_{\beta(2)} W^{\alpha(2k)\beta(2)} + \beta_k e^{\alpha(2)} W^{\alpha(2k-2)},
 \end{aligned} \tag{5.6}$$

where

$$\beta_k = -\frac{(k^2 - s^2)}{2(4k^2 - 1)} \Lambda. \tag{5.7}$$

Two partially massless and one massless fields. First of all let us note that in this case we have twice as many one-forms as Stueckelberg zero-forms so our trick with maximal gauge fixing can not be straightforwardly applied here. Leaving the general analysis to the future work, we assume that in this case non-abelian corrections can also be transformed into abelian ones by appropriate field redefinitions and consider the following ansatz:

$$\mathcal{L}_1 = g \mathcal{B}_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \mathcal{B}_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3)} f_{3,\beta(\hat{s}_2)\gamma(\hat{s}_1)}, \tag{5.8}$$

where \hat{s}_i are the same as before and the spins s_i again must satisfy a strict triangular inequality. Taking into account that

$$DB^{\alpha(2s-2)} = -2a_0 \mathcal{R}^{\alpha(2s-2)} - \frac{a_{s-2}}{(s-1)(2s-3)} e^{\alpha(2)} \mathcal{B}^{\alpha(2s-4)} \approx 0$$

the only non-zero on-shell contribution appears to be:

$$\delta \mathcal{L}_1 \sim \mathcal{B}_1^{\alpha(\hat{s}_3)\beta(\hat{s}_2)} \mathcal{B}_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3)} \left(e_\beta{}^\delta \eta_{\beta(\hat{s}_2-1)\delta\gamma(\hat{s}_1)} + e_\gamma{}^\delta \eta_{\beta(\hat{s}_2)\gamma(\hat{s}_1-1)\delta} \right) \tag{5.9}$$

From the unfolded equations it follows that $\mathcal{B}^{\alpha(2s-2)} \approx e_{\beta(2)} W^{\alpha(2s-2)\beta(2)}$ so we obtain:

$$\delta\mathcal{L}_1 \sim (s_1 - s_2) W_1^{\alpha(\hat{s}_3+2)\beta(\hat{s}_2)} W_2^{\gamma(\hat{s}_1)}{}_{\alpha(\hat{s}_3+2)\eta\beta(\hat{s}_2)\gamma(\hat{s}_1)}. \quad (5.10)$$

Thus gauge invariance requires that the spins of the two partially massless fields must be equal $s_1 = s_2$. Indeed, this is just the particular case of the general relation (4.52) when $M_1 = M_2 = 0$. Anyway, the result is rather strong, so as an independent check we again consider the deformation of the unfolded equations for the partially massless fields due to the massless spin- s field satisfying ($n = 2s - 2$)

$$\begin{aligned} 0 &= D\Omega^{\alpha(n)} - \frac{\Lambda}{4} e^\alpha{}_\beta f^{\alpha(n-1)\beta} \\ 0 &= Df^{\alpha(n)} + e^\alpha{}_\beta \Omega^{\alpha(n-1)\beta} \end{aligned} \quad (5.11)$$

The most general ansatz for the parity even deformation has the form:

$$\begin{aligned} 0 &= DW_1^{\alpha(2k)} + e_{\beta(2)} W_1^{\alpha(2k)\beta(2)} + \beta_{1,k} e^{\alpha(2)} W_1^{\alpha(2k-2)} \\ &+ \sum_{l=0}^{n/2} f_{k,l} f^{\alpha(2l)}{}_{\beta(n-2l)} W_2^{\alpha(2k-2l)\beta(n-2l)} \\ &+ \sum_{l=0}^{n/2-1} g_{k,l} \Omega^{\alpha(2l+1)}{}_{\beta(n-2l-1)} W_2^{\alpha(2k-2l-1)\beta(n-2l-1)}. \end{aligned} \quad (5.12)$$

Self-consistency of such deformations provide a large number of equations on the parameters $f_{k,l}$ and $g_{k,l}$ which can be solved iteratively on l (see appendix C). For $l = 0$ they give:

$$f_{k+1,0} = f_{k,0}, \quad g_{k+1,0} = g_{k,0}, \quad g_{k+1,0} = \frac{n}{2} f_{k,0}$$

so we set

$$f_{k,0} = f_0 = 1, \quad g_{k,0} = \frac{n}{2}$$

Equations with $l = 1$ for the field f look like:

$$\begin{aligned} 0 &= 2f_{k+1,1} - (2k-1)\beta_{1,k} + (2k-1+n)\beta_{2,k+n/2} + \frac{n^2}{8}\Lambda \\ 0 &= f_{k+1,1} + k(2k-1)\beta_{1,k} - \frac{(2k+n)(2k+n-1)}{2}\beta_{2,k+n/2} - \frac{kn}{4}\Lambda \\ 0 &= f_{k+1,1} - f_{k,1} + \beta_{1,k} - \beta_{2,k+n/2} \end{aligned}$$

For $f_{k,1}$ from the first equation the second one satisfies identically, while the third one requires

$$s_1 = s_2$$

Similarly, equations for Ω with $l = 1$ look like

$$\begin{aligned} 0 &= g_{k+1,1} - n(k-1)\beta_{1,k} + n(2k-3+n)\beta_{2,k-1+n/2}/2 - \frac{n}{2}f_{k,1} \\ 0 &= g_{k+1,1} - g_{k,1} + \frac{3n}{2}(\beta_{1,k} - \beta_{2,k-1+n/2}) \\ 0 &= g_{k+1,1} + \frac{n(k-1)(2k-1)}{2}\beta_{1,k} - \frac{(2k+n-2)(2k+n-3)}{4}\beta_{2,k-1+n/2} + (2k-1)f_{k,1} \end{aligned}$$

and again solution exists only when $s_1 = s_2$. Let us provide here complete solutions for a couple of simple cases.

Massless spin 2. In this case equations are:

$$\begin{aligned}
 0 = & DW_1^{\alpha(2k)} + e_{\beta(2)}W_1^{\alpha(2k)\beta(2)} + \beta_k e^{\alpha(2)}W_1^{\alpha(2k-2)} \\
 & + f_{k,0}f_{\beta(2)}W_2^{\alpha(2k)\beta(2)} + g_{k,0}\Omega^\alpha{}_\beta W_2^{\alpha(2k-1)\beta} + f_{k,1}\Phi^{\alpha(2)}W_2^{\alpha(2k-2)}
 \end{aligned} \tag{5.13}$$

and the solution:

$$f_{k,0} = g_{k,0} = 1, \quad f_{k,1} = \beta_k$$

Massless spin 3. Now we have

$$\begin{aligned}
 0 = & DW_1^{\alpha(2k)} + e_{\beta(2)}W_1^{\alpha(2k)\beta(2)} + \beta_k e^{\alpha(2)}W_1^{\alpha(2k-2)} \\
 & + f_{k,0}f_{\beta(4)}W_2^{\alpha(2k)\beta(4)} + f_{k,1}f^{\alpha(2)}{}_{\beta(2)}W_2^{\alpha(2k-2)\beta(2)} + f_{k,2}f^{\alpha(4)}W_3^{\alpha(2k-4)} \\
 & + g_{k,0}\Omega^\alpha{}_{\beta(3)}W_2^{\alpha(2k-1)\beta(3)} + g_{k,1}\Omega^{\alpha(3)}{}_\beta W_2^{\alpha(2k-3)\beta}
 \end{aligned} \tag{5.14}$$

and obtain:

$$\begin{aligned}
 f_{k,0} = f_0 = 1, \quad f_{k,1} = 2[(k+2)\beta_{k+1} - (k-1)\beta_k], \quad f_{k,2} = 6\beta_k\beta_{k-1} \\
 g_{k,0} = g_0 = 2f_0, \quad g_{k,1} = 6\beta_k
 \end{aligned}$$

5.2 Non-maximal depth

Recently, a very simple and elegant representation for partially massless fields of non-maximal depth in AdS₃ were suggested [6]. The aim of this subsection is to show that this representation can be reproduced starting with our gauge invariant formalism. First of all we have to move to AdS. These partially massless fields are non-unitary there, so if we formally set $\Lambda = -\lambda^2$ we obtain non-hermitian Lagrangian. To avoid this, we use the same trick as we have already used in our construction for the partially massless supermultiplets in AdS₄ [28]. Namely, we switch off alternating signs for the kinetic terms. Then we obtain the Lagrangian ($n = s - t - 1$):

$$\begin{aligned}
 \mathcal{L}_0 = & \sum_{k=n}^{s-1} \left[k\Omega_{\alpha(2k-1)\beta} e^\beta{}_\gamma \Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)} Df^{\alpha(2k)} + b_k f_{\alpha(2k-1)\beta} e^\beta{}_\gamma f^{\alpha(2k-1)\gamma} \right. \\
 & \left. - \frac{(k+2)}{k} a_k \Omega_{\alpha(2k)\beta(2)} e^{\beta(2)} f^{\alpha(2k)} + a_k \Omega_{\alpha(2k)} e_{\beta(2)} f^{\alpha(2k)\beta(2)} \right],
 \end{aligned} \tag{5.15}$$

where all coefficients are real and such that

$$a_k^2 = \frac{k(s^2 - (k+1)^2)((k+1)^2 - n^2)}{2(k+1)(k+2)(2k+3)} \lambda^2, \quad b_k = \frac{s^2 n^2}{4k(k+1)^2} \lambda^2.$$

This Lagrangian is invariant under the following gauge transformations:

$$\begin{aligned}
 \delta\Omega^{\alpha(2k)} = & D\eta^{\alpha(2k)} - \frac{(k+2)}{k} a_k e_{\beta(2)} \eta^{\alpha(2k)\beta(2)} + \frac{a_{k-1}}{k(2k-1)} e^{\alpha(2)} \eta^{\alpha(2k-2)} + \frac{b_k}{k} e^\alpha{}_\beta \xi^{\alpha(2k-1)\beta}, \\
 \delta f^{\alpha(2k)} = & D\xi^{\alpha(2k)} - a_k e_{\beta(2)} \xi^{\alpha(2k)\beta(2)} + \frac{(k+1)a_{k-1}}{k(k-1)(2k-1)} e^{\alpha(2)} \xi^{\alpha(2k-2)} + e^\alpha{}_\beta \eta^{\alpha(2k-1)\beta},
 \end{aligned}$$

while the gauge invariant two-forms look like:

$$\begin{aligned}\mathcal{R}^{\alpha(2k)} &= D\Omega^{\alpha(2k)} - \frac{(k+2)}{k} a_k e_{\beta(2)} \Omega^{\alpha(2k)\beta(2)} + \frac{a_{k-1}}{k(2k-1)} e^{\alpha(2)} \Omega^{\alpha(2k-2)} + \frac{b_k}{k} e^\alpha{}_\beta f^{\alpha(2k-1)\beta}, \\ \mathcal{T}^{\alpha(2k)} &= Df^{\alpha(2k)} + e^\alpha{}_\beta \Omega^{\alpha(2k-1)\beta} - a_k e_{\beta(2)} f^{\alpha(2k)\beta(2)} + \frac{(k+1)a_{k-1}}{k(k-1)(2k-1)} e^{\alpha(2)} f^{\alpha(2k-2)}.\end{aligned}$$

Now to separate the variables we introduce:

$$\Omega_\pm^{\alpha(2k)} = \Omega^{\alpha(2k)} \pm \frac{sn\lambda}{2k(k+1)} f^{\alpha(2k)} \quad (5.16)$$

Working with the + components and omitting the + sign we obtain:

$$\begin{aligned}\mathcal{R}^{\alpha(2k)} &= D\Omega^{\alpha(2k)} - \frac{(k+2)}{k} a_k e_{\beta(2)} \Omega^{\alpha(2k)\beta(2)} \\ &+ \frac{a_{k-1}}{k(2k-1)} e^{\alpha(2)} \Omega^{\alpha(2k-2)} + \frac{sn\lambda}{2k(k+1)} e^\alpha{}_\beta \Omega^{\alpha(2k-1)\beta}\end{aligned} \quad (5.17)$$

As an illustration let us consider the simplest case $t = 1$. In this case we have only two fields $\Omega^{\alpha(2s-2)}$ and $\Omega^{\alpha(2s-4)}$ with curvatures

$$\begin{aligned}\mathcal{R}^{\alpha(2s-2)} &= D\Omega^{\alpha(2s-2)} + \frac{2a_{s-2}}{(s-1)(2s-3)} \frac{\lambda}{2} e^{\alpha(2)} \Omega^{\alpha(2s-4)} + \frac{(s-2)\lambda}{(s-1)2} e^\alpha{}_\beta \Omega^{\alpha(2s-3)\beta}, \\ \mathcal{R}^{\alpha(2s-4)} &= D\Omega^{\alpha(2s-4)} - \frac{2sa_{s-2}}{(s-2)2} \frac{\lambda}{2} e_{\beta(2)} \Omega^{\alpha(2s-4)\beta(2)} + \frac{s}{(s-1)2} \frac{\lambda}{2} e^\alpha{}_\beta \Omega^{\alpha(2s-5)\beta},\end{aligned} \quad (5.18)$$

where we explicitly show multipliers $\frac{\lambda}{2}$ so that

$$a_{s-2}^2 = \frac{(s-2)(2s-3)}{2s(s-1)}. \quad (5.19)$$

Now we show that both these curvatures can be combined into

$$\hat{\mathcal{R}}^{\alpha(2s-3),\beta} = D\hat{\Omega}^{\alpha(2s-3)\beta} + \frac{\lambda}{2} e^\alpha{}_\gamma \hat{\Omega}^{\alpha(2s-4)\gamma,\beta} - \frac{\lambda}{2} e^\beta{}_\gamma \hat{\Omega}^{\alpha(2s-3),\gamma}. \quad (5.20)$$

For this we introduce an ansatz:

$$\hat{\Omega}^{\alpha(2s-3),\beta} = \Omega^{\alpha(2s-3)\beta} + \kappa_1 \Omega^{\alpha(2s-4)} \varepsilon^{\alpha\beta} \quad (5.21)$$

Let us begin with the $\Omega^{\alpha(2s-2)}$ contribution (here and further on all calculations are up to $\frac{\lambda}{2}$):

$$\Delta_0 = e^\alpha{}_\gamma \Omega^{\alpha(2s-4)\beta\gamma} - e^\beta{}_\gamma \Omega^{\alpha(2s-3)\gamma} \quad (5.22)$$

Using an identity:

$$e^\alpha{}_\gamma \Omega^{\alpha(2s-4)\beta\gamma} - (2s-3) e^\beta{}_\gamma \Omega^{\alpha(2s-3)\gamma} = \varepsilon^{\alpha\beta} e_{\gamma(2)} \Omega^{\alpha(2s-4)\gamma(2)},$$

we can rewrite this contribution as

$$\Delta_0 = \frac{(s-2)}{(s-1)} \left[e^\alpha{}_\gamma \Omega^{\alpha(2s-4)\beta\gamma} + e^\beta{}_\gamma \Omega^{\alpha(2s-3)\gamma} \right] + \frac{1}{(s-1)} \varepsilon^{\alpha\beta} e_{\gamma(2)} \Omega^{\alpha(2s-4)\gamma(2)}. \quad (5.23)$$

The coefficient before the first term is exactly what we need, while for the second contribution be correct we must have

$$\frac{1}{(s-1)} = -\kappa_1 \frac{2s}{(s-2)} a_{s-2} \implies \kappa_1 = -\frac{a_{s-2}}{(2s-3)}. \quad (5.24)$$

Now we consider $\Omega^{\alpha(2s-4)}$ contribution:

$$\Delta_1 = e^\alpha_\gamma \Omega^{\alpha(2s-5)\gamma} \varepsilon^{\alpha\beta} - 2e^{\alpha\beta} \Omega^{\alpha(2s-4)} \quad (5.25)$$

Using one more identity:

$$e^\alpha_\gamma \Omega^{\alpha(2s-5)\gamma} \varepsilon^{\alpha\beta} = 2e^{\alpha(2)} \Omega^{\alpha(2s-5)\beta} - 2(s-2)e^{\alpha\beta} \Omega^{\alpha(2s-4)},$$

we obtain

$$\Delta_1 = \frac{s}{(s-1)} e^\alpha_\gamma \Omega^{\alpha(2s-5)\gamma} \varepsilon^{\alpha\beta} - \frac{2}{(s-1)} \left(e^{\alpha(2)} \Omega^{\alpha(2s-5)\beta} + e^{\alpha\beta} \Omega^{\alpha(2s-4)} \right). \quad (5.26)$$

Taking into account that

$$-\kappa_1 \frac{2}{(s-1)} = \frac{2a_{s-2}}{(s-1)(2s-3)}$$

we find that both coefficients are correct.

Now we turn to the general case and show that the whole set of the gauge invariant curvatures can be packed into

$$\hat{\mathcal{R}}^{\alpha(2s-2-t),\beta(t)} = D\hat{\Omega}^{\alpha(2s-2-t),\beta(t)} + \frac{\lambda}{2} e^\alpha_\gamma \hat{\Omega}^{\alpha(2s-3-t)\gamma,\beta(t)} - \frac{\lambda}{2} e^\beta_\gamma \hat{\Omega}^{\alpha(2s-2-t),\beta(t-1)\gamma}. \quad (5.27)$$

For this purpose we introduce the following ansatz:

$$\hat{\Omega}^{\alpha(2s-2-t),\beta(t)} = \Omega^{\alpha(2s-2-t)\beta(t)} + \sum_{l=1}^t \kappa_l \Omega^{\alpha(2s-2-t-l)\beta(t-l)} (\varepsilon^{\alpha\beta})^l, \quad (5.28)$$

where $(\varepsilon^{\alpha\beta})^l$ denotes a product of l copies of $\varepsilon^{\alpha\beta}$. It is enough to consider a contribution for one particular field $\Omega^{\alpha(2s-2-2l)}$. For convenience, let us explicitly show the terms we have to reproduce:

$$\mathcal{R}^{\alpha(2s-2l)} = \dots + \frac{2a_{s-l-1}}{(s-l)(2s-2l-1)} e^{\alpha(2)} \Omega^{\alpha(2s-2-2l)} + \dots \quad (5.29)$$

$$\mathcal{R}^{\alpha(2s-2-2l)} = \dots + \frac{s(s-t-1)}{(s-l)(s-l-1)} e^\alpha_\beta \Omega^{\alpha(2s-3-2l)\beta} \quad (5.30)$$

$$\mathcal{R}^{\alpha(2s-4-2l)} = \dots - \frac{2(s-l)}{(s-l-2)} a_{s-l-2} e_{\beta(2)} \Omega^{\alpha(2s-4-2l)\beta(2)} + \dots \quad (5.31)$$

This particular contribution appears to be:

$$\begin{aligned} \Delta_l = & \left[e^\alpha_\gamma \Omega^{\alpha(2s-3-t-l)\gamma\beta(t-l)} - e^\beta_\gamma \Omega^{\alpha(2s-2-t-l)\beta(t-l-1)\gamma} \right] (\varepsilon^{\alpha\beta})^l \\ & - 2l e^{\alpha\beta} \Omega^{\alpha(2s-2-t-l)\beta(t-l)} (\varepsilon^{\alpha\beta})^{l-1} \end{aligned} \quad (5.32)$$

Further on we omit common multiplier $(\varepsilon^{\alpha\beta})^{l-1}$. Using a pair of identities

$$\begin{aligned} e^\alpha_\gamma \Omega^{\alpha(2s-3-t-l)\gamma\beta(t-l)} \varepsilon^{\alpha\beta} &= 2(t-l+1)e^{\alpha(2)} \Omega^{\alpha(2s-3-t-l)\beta(t-l+1)} \\ &\quad - (2s-2-t-l)e^{\alpha\beta} \Omega^{\alpha(2s-2-t-l)\beta(t-l)}, \\ e^\beta_\gamma \Omega^{\alpha(2s-2-t-l)\beta(t-l-1)\gamma} \varepsilon^{\alpha\beta} &= -2(2s-1-t-l)e^{\beta(2)} \Omega^{\alpha(2s-1-t-l)\beta(t-l-1)} \\ &\quad + (t-l)e^{\alpha\beta} \Omega^{\alpha(2s-2-t-l)\beta(t-l)}, \end{aligned}$$

one can straightforwardly show that

$$\begin{aligned} \Delta_{l,1} &= \left[\rho_1 e^\alpha_\gamma \Omega^{\alpha(2s-3-t-l)\beta(t-l)\gamma} + \rho_2 e^\beta_\gamma \Omega^{\alpha(2s-2-t-l)\beta(t-l-1)\gamma} \right] \varepsilon^{\alpha\beta} \\ &\quad - 2l e^{\alpha\beta} \Omega^{\alpha(2s-2-t-l)\beta(t-l)}, \end{aligned} \quad (5.33)$$

where

$$\rho_1 = -\frac{l(2s-t-l-1)}{(s-l)(2s-2l-1)}, \quad \rho_2 = \frac{l(t-l+1)}{(s-l)(2s-2l-1)},$$

can be rewritten as:

$$\begin{aligned} \Delta_{l,1} &= -\frac{l(2s-t-l-1)}{(s-l)(2s-2l-1)} \left[e^{\alpha(2)} \Omega^{\alpha(2s-3-t-l)\beta(t-l+1)} \right. \\ &\quad \left. + e^{\alpha\beta} \Omega^{\alpha(2s-2-t-l)\beta(t-l)} + e^{\beta(2)} \Omega^{\alpha(2s-1-t-l)\beta(t-l-1)} \right]. \end{aligned} \quad (5.34)$$

To correctly reproduce the term in (5.29) we must have

$$-\kappa_l \frac{l(2s-t-l-1)}{(s-l)(2s-2l-1)} = \kappa_{l-1} \frac{2a_{s-l-1}}{(s-l)(2s-2l-1)}.$$

This gives us a recurrent relation on κ_l :

$$\kappa_l = -\kappa_{l-1} \frac{a_{s-l-1}}{l(t-l+1)(2s-t-l-1)}. \quad (5.35)$$

Taking into account that $\kappa_0 = 1$ we found a solution

$$\kappa_l = (-1)^l \frac{(t-l)!(2s-2-t-l)!}{l!t!(2s-2-t)!} \prod_{m=1}^l a_{s-1-m}. \quad (5.36)$$

This leaves us with (omitting the last $\varepsilon^{\alpha\beta}$)

$$\Delta_{l,2} = (1-\rho_1)e^\alpha_\gamma \Omega^{\alpha(2s-3-t-l)\gamma\beta(t-l)} - (1+\rho_2)e^\beta_\gamma \Omega^{\alpha(2s-2-t-l)\beta(t-l-1)\gamma}.$$

Using the last identity

$$\begin{aligned} (t-l)e^\alpha_\gamma \Omega^{\alpha(2s-3-t-l)\gamma\beta(t-l)} &= (2s-2-t-l)e^\beta_\gamma \Omega^{\alpha(2s-2-t-l)\beta(t-l-1)\gamma} \\ &\quad + \varepsilon^{\alpha\beta} e_{\gamma(2)} \Omega^{\alpha(2s-3-t-l)\beta(t-l-1)\gamma(2)}, \end{aligned}$$

we obtain:

$$\begin{aligned} \Delta_{l,2} &= \frac{s(s-t-1)}{(s-l)(s-l-1)} \left[e^\alpha_\gamma \Omega^{\alpha(2s-3-t-l)\gamma\beta(t-l)} + e^\beta_\gamma \Omega^{\alpha(2s-2-t-l)\beta(t-l-1)\gamma} \right] \\ &\quad + \frac{(2s-l-1)}{(s-l-1)(2s-2l-1)} \varepsilon^{\alpha\beta} e_{\gamma(2)} \Omega^{\alpha(2s-3-t-l)\beta(t-l-1)\gamma(2)}. \end{aligned}$$

Here the first coefficients is exactly what we need for (5.30), while to reproduce (5.31) we must have

$$\kappa_l \frac{(2s-l-1)}{(s-l-1)(2s-2l-1)} = -\kappa_{l+1} \frac{2(s-l)}{(s-l-2)} a_{s-l-2}.$$

Using explicit expressions for κ_l and a_{s-2} it is straightforward to check that this equation is fulfilled.

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A Massless fermion and massive boson and fermion

Let us consider massive boson (M_1, s_1) and massive fermion (M_2, s_2) interacting with massless spin- s fermion satisfying

$$0 = D\Phi^{\alpha(n)} + \frac{\lambda}{2} e^{\alpha}_{\beta} \Phi^{\alpha(n-1)\beta}, \quad n = 2s - 2$$

Deformation for the bosonic equations has the form:

$$\begin{aligned} 0 = & DW^{\alpha(2k)} + e_{\beta(2)} W^{\alpha(2k)\beta(2)} + \alpha_{1,k} e^{\alpha}_{\beta} W^{\alpha(2k-1)\beta} + \beta_{1,k} e^{\alpha(2)} W^{\alpha(2k-2)} \\ & + \sum_{l=0}^n f_{k,l} \Phi^{\alpha(l)}_{\beta(n-l)} V^{\alpha(2k-l)\beta(n-l)} \end{aligned} \quad (\text{A.1})$$

The consistency requirement leads to the number of equations which can (in-principle) be solved iteratively by l :

$l = 0$

$$\begin{aligned} 0 &= f_{k+1,0} - f_{k,0} \quad \Rightarrow \quad f_{k,0} = f_0 \\ 0 &= 2f_{k+1,1} + (2k+n)\alpha_{2,k+(n-1)/2} - 2k\alpha_{1,k} - \frac{n\lambda}{2} \\ 0 &= f_{k+1,1} - f_{k,1} + \alpha_{1,k} - \alpha_{2,k+(n-1)/2} \end{aligned}$$

$1 \leq l \leq n-1$:

$$\begin{aligned} 0 &= 2f_{k+1,l+1} + \left[(2k-2l+n)\alpha_{2,k-l+(n-1)/2} - (2k-2l)\alpha_{1,k} - n\frac{\lambda}{2} \right] f_{k,l} \\ &\quad - l(2k-l)\beta_{1,k} f_{k-1,l-1} + l(2k-2l+n+1)\beta_{2,k-l+(n+1)/2} f_{k,l-1} \\ 0 &= f_{k+1,l+1} - f_{k,l+1} + (l+1)(\alpha_{1,k} - \alpha_{2,k-l+(n-1)/2}) f_{k,l} \\ &\quad + \frac{l(l+1)}{2} \left[\beta_{1,k} f_{k-1,l-1} - \beta_{2,k-l+(n+1)/2} f_{k,l-1} \right] \\ 0 &= 2f_{k+1,l+1} - (2k-l+1)(2\alpha_{1,k} - \lambda) f_{k,l} + (2k-l)(2k-l+1)\beta_{1,k} f_{k-1,l-1} \\ &\quad - (2k-2l+n+2)(2k-2l+n+1)\beta_{2,k-l+(n+1)/2} f_{k,l-1} \end{aligned} \quad (\text{A.2})$$

$l = n$:

$$\begin{aligned}
 0 &= (\alpha_{1,k} - \alpha_{2,k-(n+1)/2})f_{k,n} + \frac{n}{2} \left[\beta_{1,k}f_{k-1,n-1} - \beta_{2,k-(n-1)/2}f_{k,n-1} \right] \\
 0 &= (2\alpha_{1,k} - \lambda)f_{k,n} + (2k - n)\beta_{1,k}f_{k-1,n-1} - (2k - n + 2)\beta_{2,k-(n-1)/2}f_{k,n-1} \\
 0 &= \beta_{1,k}f_{k-1,n} - \beta_{2,k-(n-1)/2}f_{k,n}
 \end{aligned}$$

Similarly, deformation for the fermionic equations can be written as:

$$\begin{aligned}
 0 &= DV^{\alpha(2k+1)} + e_{\beta(2)}V^{\alpha(2k+1)\beta(2)} + \alpha_{2,k}e^{\alpha}_{\beta}V^{\alpha(2k)\beta} + \beta_{2,k}e^{\alpha(2)}V^{\alpha(2k-1)} \\
 &\quad + \sum_{l=0}^n \Phi^{\alpha(l)}_{\beta(n-l)}W^{\alpha(2k+1-l)\beta(n-l)}
 \end{aligned} \tag{A.3}$$

and their consistency leads to:

$l = 0$

$$\begin{aligned}
 0 &= g_{k+1,0} - g_{k,0}, \quad \Rightarrow \quad g_{k,0} = g_0 \\
 0 &= g_{k+1,1} - g_{k,1} + \alpha_{2,k} - \alpha_{1,k+(n+1)/2} \\
 0 &= 2g_{k+1,1} + (2k + 1 + n)\alpha_{1,k+(n+1)/2} - (2k + 1)\alpha_{2,k} - \frac{n\lambda}{2}
 \end{aligned}$$

$1 \leq l \leq n - 1$

$$\begin{aligned}
 0 &= 2g_{k+1,l+1} + \left[(2k + 1 - 2l + n)\alpha_{1,k-l+(n+1)/2} - (2k + 1 - 2l)\alpha_{2,k} - n\frac{\lambda}{2} \right] g_{k,l} \\
 &\quad + l(2k + 2 - 2l + n)\beta_{1,k-l+(n+3)/2}g_{k,l-1} - l(2k + 1 - l)\beta_{2,k}g_{k-1,l-1} \\
 0 &= g_{k+1,l+1} - g_{k,l+1} + (l + 1)(\alpha_{2,k} - \alpha_{1,k-l+(n+1)/2})g_{k,l} \\
 &\quad + \frac{l(l+1)}{2} \left[\beta_{2,k}g_{k-1,l-1} - \beta_{1,k-l+(n+3)/2}g_{k,l-1} \right] \\
 0 &= 2g_{k+1,l+1} - (2k + 2 - l)(2\alpha_{2,k} - \lambda)g_{k,l} + (2k + 2 - l) \left[(2k + 1 - l)\beta_{2,k}g_{k-1,l-1} \right. \\
 &\quad \left. - (2k - 2l + n + 3)(2k - 2l + n + 2)\beta_{1,k-l+(n+3)/2}g_{k,l-1} \right]
 \end{aligned} \tag{A.4}$$

$l = n$

$$\begin{aligned}
 0 &= (\alpha_{2,k} - \alpha_{1,k-(n-1)/2})g_{k,n} + \frac{n}{2}(\beta_{2,k}g_{k-1,n-1} - \beta_{1,k-(n-3)/2}g_{k,n-1}) \\
 0 &= (2\alpha_{2,k} - \lambda)g_{k,n} - (2k + 1 - n)\beta_{2,k}g_{k-1,n-1} + (2k + 3 - n)\beta_{1,k-(n-3)/2}g_{k,n-1} \\
 0 &= \beta_{2,k}g_{k-1,n} - \beta_{1,k-(n-1)/2}g_{k,n}
 \end{aligned}$$

B One massless and two massive bosons

Deformation for the first boson:

$$\begin{aligned}
 0 &= DW_1^{\alpha(2k)} + e_{\beta(2)}W_1^{\alpha(2k)\beta(2)} + \alpha_{1,k}e^{\alpha}_{\beta}W_1^{\alpha(2k-1)\beta} + \beta_{1,k}e^{\alpha(2)}W_1^{\alpha(2k-2)} \\
 &\quad + \sum_{l=0}^n f_{k,l}\Omega^{\alpha(l)}_{\beta(n-l)}W_2^{\alpha(2k-l)\beta(n-l)}
 \end{aligned} \tag{B.1}$$

Consistency requirement leads to:

$$\begin{aligned}
 0 &= f_{k+1,0} - f_{k,0} \quad \Rightarrow \quad f_{k,0} = f_0 = 1 \\
 0 &= 2f_{k+1,1} + (2k+n)\alpha_{2,k+n/2} - 2k\alpha_{1,k} - n\frac{\lambda}{2} \\
 0 &= f_{k+1,1} - f_{k,1} + \alpha_{1,k} - \alpha_{2,k+n/2}
 \end{aligned}$$

$$1 \leq l \leq n-1$$

$$\begin{aligned}
 0 &= 2f_{k+1,l+1} - (2k-l+1)(2\alpha_{1,k} - \lambda)f_{k,l} + (2k-l)(2k-l+1)\beta_{1,k}f_{k-1,l-1} \\
 &\quad - (2k-2l+n+2)(2k-2l+n+1)\beta_{2,k-l+1+n/2}f_{k,l-1} \\
 0 &= f_{k+1,l+1} - f_{k,l+1} + (l+1) \left[\alpha_{1,k}f_{k,l} - \alpha_{2,k-l+n/2}f_{k,l} \right] \\
 &\quad + \frac{l(l+1)}{2} \left[\beta_{1,k}f_{k-1,l-1} - \beta_{2,k-l+1+n/2}f_{k,l-1} \right] \\
 0 &= 2f_{k+1,l+1} + \left[(2k-2l+n)\alpha_{2,k-l+n/2} - (2k-2l)\alpha_{1,k} - n\frac{\lambda}{2} \right] f_{k,l} \\
 &\quad - l(2k-l)\beta_{1,k}f_{k-1,l-1} + l(2k-2l+n+1)\beta_{2,k-l+1+n/2}f_{k,l-1}
 \end{aligned} \tag{B.2}$$

$$l = n$$

$$\begin{aligned}
 0 &= (2\alpha_{1,k} - \lambda)f_{k,n} - (2k-n)\beta_{1,k}f_{k-1,n-1} + (2k-n+2)\beta_{2,k+1-n/2}f_{k,n-1} \\
 0 &= 2(\alpha_{1,k} - \alpha_{2,k-l+n/2})f_{k,n} + n \left[\beta_{1,k}f_{k-1,n-1} - \beta_{2,k+1-n/2}f_{k,n-1} \right] \\
 0 &= \beta_{1,k}f_{k-1,n} - \beta_{2,k-n/2}f_{k,n}
 \end{aligned}$$

C One massless and two partially massless bosons

Deformation for the first partially massless one:

$$\begin{aligned}
 0 &= DW_1^{\alpha(2k)} + e_{\beta(2)}W_1^{\alpha(2k)\beta(2)} + \beta_{1,k}e^{\alpha(2)}W_1^{\alpha(2k-2)} \\
 &\quad + \sum_{l=0}^{n/2} f_{k,l}f^{\alpha(2l)}_{\beta(n-2l)}W_2^{\alpha(2k-2l)\beta(n-2l)} \\
 &\quad + \sum_{l=1}^{n/2-1} g_{k,l}\Omega^{\alpha(2l+1)}_{\beta(n-2l-1)}W_2^{\alpha(2k-2l-1)\beta(n-2l-1)}
 \end{aligned} \tag{C.1}$$

Terms with the $f^{\alpha(n)}$ field lead to:

$$f_{k+1,0} = f_{k,0} \quad \Rightarrow \quad f_{k,0} = 0$$

$$1 \leq l \leq n/2$$

$$\begin{aligned} 0 &= 2f_{k,l} - (2l-1) \left[(2k-2l-1)\beta_{1,k-1}f_{k-2,l-1} \right. \\ &\quad \left. - (2k-4l+1+n)\beta_{2,k-2l+1+n/2}f_{k-2,l-1} \right] + \frac{n\Lambda}{4}g_{k-1,l-1} \\ 0 &= f_{k+1,l} - f_{k,l} + l(2l-1) \left[\beta_{1,k}f_{k-1,l-1} - \beta_{2,k-2l+2+n/2}f_{k,l-1} \right] \\ 0 &= f_{k,l} - \frac{(2k-4l+n+2)(2k-4l+n+1)}{2}\beta_{2,k-2l+1+n/2}f_{k-1,l-1} \\ &\quad + (k-l)(2k-2l-1)\beta_{1,k-1}f_{k-2,l-1} - (k-l)\frac{\Lambda}{2}g_{k-1,l-1} \\ 0 &= \beta_{1,k}f_{k-1,n/2} - \beta_{2,k-n/2}f_{k,n/2} \end{aligned}$$

Terms with the $\Omega^{\alpha(n)}$ field lead to:

$$g_{k+1,0} - g_{k,0} = 0, \quad g_{k+1,0} - \frac{n}{2}f_{k,0} = 0$$

$$1 \leq l \leq n/2 - 1$$

$$\begin{aligned} 0 &= g_{k,l} - l \left[2(k-l-1)\beta_{1,k-1}g_{k-2,l-1} - (2k-4l-1+n)\beta_{2,k-2l+n/2}g_{k-1,l-1} \right] - \frac{n}{2}f_{k-1,l} \\ 0 &= g_{k+1,l} - g_{k,l} + l(2l+1) \left[\beta_{1,k}g_{k-1,l-1} - \beta_{2,k-2l+1+n/2}g_{k,l-1} \right] \\ 0 &= g_{k,l} - \frac{(2k-4l+n)(2k-4l+n-1)}{2}\beta_{2,k-2l+n/2}g_{k-1,l-1} \\ &\quad + (2k-2l-1)(k-l-1)\beta_{1,k-1}g_{k-2,l-1} + (2k-2l-1)f_{k-1,l} \\ \beta_{1,k}g_{k-1,n/2-1} - \beta_{2,k-n/2+1}g_{k,n/2-1} &= 0, \quad \beta_{2,k-n/2+1}g_{k,n/2-1} - f_{k,n/2} = 0 \end{aligned}$$

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