Fermi seas from Bose condensates in Chern-Simons matter theories and a bosonic exclusion principle

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ABSTRACT: We generalize previously obtained results for the (all orders in the 't Hooft coupling) thermal free energy of bosonic and fermionic large N Chern-Simons theories with fundamental matter, to values of the chemical potential larger than quasiparticle thermal masses. Building on an analysis by Geracie, Goykhman and Son, we present a simple explicit formula for the occupation number for a quasiparticle state of any given energy and charge as a function of the temperature and chemical potential. This formula is a generalization to finite 't Hooft coupling of the famous occupation number formula of Bose-Einstein statistics, and implies an exclusion principle for Chern-Simons coupled bosons: the total number of bosons occupying any particular state cannot exceed the Chern-Simons level. Specializing our results to zero temperature we construct the phase diagrams of these theories as a function of chemical potential and the UV parameters. At large enough chemical potential, all the bosonic theories we study transit into a compressible Bose condensed phase in which the runaway instability of free Bose condensates is stabilized by the bosonic exclusion principle. This novel Bose condensate is dual to — and reproduces the thermodynamics of — the fermionic Fermi sea.

KEYWORDS: $1/N$ Expansion, Anyons, Chern-Simons Theories, Duality in Gauge Field Theories

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1 Introduction

There is now considerable evidence to support the conjectured *Bose-Fermi* duality between families of $2+1$ dimensional Chern-Simons gauge theories coupled to scalars and (roughly level-rank) dual Chern-Simons gauge theories coupled to fermions [1–86]. In the simplest large N examples with the matter fields in the fundamental of the gauge group, the ‘elementary’ bosonic fields on the l.h.s. of the duality create stable bosonic particles that map, under duality, to the fermionic particles created by the ‘elementary’ fermionic fields on the r.h.s. of the duality.¹

That bosons and fermions can be dual to each other in this elementary fashion may seem surprising. In most familiar contexts, fermions have half-integer spin while bosons have integer spin.² While this distinction is completely sharp in $D \geq 4$ dimensions, it, however, blurs out in $D = 3$ where particle spins are allowed to take arbitrary value.³ Indeed, it turns out that the coupling to a Chern-Simons gauge field renormalizes the integer (resp. half-integer) spins of the bosons (resp. fermions) to general real values. For various specific conjectured Bose-Fermi dualities one can check that the renormalized boson spin does indeed match the renormalized fermion spin across the duality [86].

The matching of single particle spins does not address all paradoxes. Given the differences between Bose and Fermi statistics, we must still understand how the spectrum of multi-particle states can match across the duality. This question has already been investigated in the context of two particle scattering states in [20]. At least in large N limit, it turns out that the key to answer this lies in the colour quantum numbers of the bosonic and fermionic states.⁴

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¹In this sense the $2+1$ dimensional dualities at large N are qualitatively different from famous examples of bosonization in $1+1$ dimensions, in which particles created by the elementary fermionic field ψ map to particles created by a complicated composite (sometimes called solitonic) bosonic operator e^ϕ . This $1+1$ dimensional duality is, in some respects, more similar to the $N = 1$ case of the $2+1$ dimensional dualities [36–38] in which elementary fermions map under duality to vortices bound to a single boson. The role of the vortex is analogous to the role of the Jordan-Wigner ‘string’ in the $1+1$ dimensional context. We thank D. Radicevic for a discussion on this point.

²Bosonisation is less surprising in $1+1$ dimensions as there is insufficient room to rotate in a one dimensional space (more formally the massive little group, $SO(1)$, is trivial) and so massive excitations do not carry a spin quantum number.

³Massive particles in D dimensions transform in representations of the little group $SO(D-1)$ — or more precisely $Spin(D-1)$, the covering group of $SO(D-1)$. For $D \geq 3$, this group is non-abelian and its unitary representations are discrete. Fermions transform in spinorial representations (those in which a 2π rotation is implemented by the operator -1) while bosons transform in discretely distinct faithful representations (those in which a 2π rotation is implemented by the operator 1). In $D = 3$, this little group is $\mathbb{R}$ (the covering group of $SO(2)$). The unitary representations of $\mathbb{R}$ are labelled by one continuous parameter (the spin, an arbitrary real number) rather than a collection of discrete parameters as in higher dimensions.

⁴We have ignored the fact that the coupling to the Chern-Simons gauge field modifies the statistics of both bosonic and fermionic excitations. This is consistent in the large N limit as the effective anyonic phase in the scattering of two fundamental fermions or bosons is of order $1/N$ [20]. At finite values of N (in particular when $N = 1$) the modification of statistics plays a key role in reconciling the difference between bosonic and fermionic statistics and cannot be ignored.

Let us recall that the gauge groups are different on the bosonic and the fermionic sides of the duality.⁵ As the elementary bosons and fermions each transform in the fundamental representations of their respective gauge groups, it is clear that *all* single particle quantum numbers cannot match across the duality. While the ‘physical’ spacetime quantum numbers (mass and spin) match precisely, the ‘unphysical’ gauge quantum numbers lie in different spaces and so cannot be identified.

Of course, this ‘mismatch’ does not necessarily falsify duality. Suppose we work on three dimensional spacetime which is $S^2 \times \mathbb{R}$.⁶ Gauss’ law then forces the physical states to be colour singlets. Charged single-particle (or multi-particle) states are not by themselves physical, but are instead the building blocks for physical states. It was demonstrated in [20] that, under duality, two-particle states that are symmetric (antisymmetric) in colour indices map to two-particle states that are antisymmetric (symmetric) in colour. Thus, while dual pairs of bosonic and fermionic two-particle states have opposite statistics under the interchange of *all* quantum numbers, they have identical statistics under the interchange of all physical — i.e. non-gauge — quantum numbers.⁷ We see that, as promised, the ‘hidden’ colour indices are the key to reconciling duality with the difference between Bose and Fermi statistics.

In this paper we study four families of quantum field theories: the quasi-fermionic theories which are the regular fermion and the critical boson, and the quasi-bosonic theories which are the regular boson and the critical fermion.⁸ In our study of the quasi-fermionic theories, we adopt the conventions and regularization scheme of [65] (see section 1.1 of that paper including footnotes 5-8). Similarly, in our study of quasi-bosonic theories, we adopt the conventions and regularization scheme of [71] (see section 2.1 of that paper). The Euclidean actions for these theories are listed in section 2.1.

We will address three related but distinct questions. We first revisit the question of how Bose-Fermi duality can be reconciled with the difference between bosonic and fermionic statistics, but this time working with thermodynamic systems with macroscopic numbers of particles rather than few-particle scattering states. Specialising this discussion to zero

⁵ $SU(N_B)$ or $U(N_B)$ for the bosons, and $SU(N_F)$ or $U(N_F)$ for the fermions in the situation studied by [33, 40]. In general $N_F \neq N_B$.

⁶This is one way of regulating the spatial $\mathbb{R}^2$ in the IR. Another way would be to replace $\mathbb{R}^2$ with a disk of large radius. This is a more complicated regulator as it introduces boundary degrees of freedom — roughly the boundary WZW modes. We will not study such a regulator in this paper.

⁷So, for example, a two-particle fermion state that is antisymmetric in colour indices is forced, by Fermi statistics, to be symmetric under the interchange of all physical (non-gauge) quantum numbers like momenta. Its dual two-particle boson state is symmetric in colour indices and so is also forced by Bose statistics to be symmetric in all physical (non-gauge) quantum numbers.

⁸The names quasi-fermionic and quasi-bosonic [7, 8] indicate only the following: by definition the large N single-trace operator spectrum of quasi-fermionic theories is identical to the same spectrum of operators in a free, gauged, large N fermion theory. Similarly, the large N single-trace operator spectrum of quasi-bosonic theories is identical to the same spectrum of operators in a free, gauged, large N bosonic theory. The names carry no other significance. In particular, there is no sense in which all quasi-fermionic theories are ‘almost fermionic’. As an example, the ungauged large N bosonic Wilson-Fisher theory — certainly a bosonic theory from every other point of view — is quasi-fermionic with our definitions. We hope this terminology will not prove confusing. We thank A. Karch for a discussion on this point.

temperature, we then investigate how Bose condensates can replicate the behaviour of Fermi seas. Finally, we perform a detailed quantitative study of the zero temperature, finite chemical potential phase diagrams of the critical boson and regular boson theories (and hence, by duality, of the regular fermion and critical fermion respectively). This study is related to the previous question because Bose condensates (and their dual Fermi seas) play a key role in these phase diagrams. In the rest of this introduction we briefly describe our results on each of these counts in turn.

1.1 Reconciling statistics with duality

How do bosonic and fermionic thermodynamic ensembles⁹ manage to exhibit the same physics despite their apparent difference in statistics? A study of this question was initiated in the pioneering article of Geracie, Goykhman and Son [32]. The authors of [32] studied the analytically tractable theory of regular fermions coupled to $SU(N)$ Chern-Simons theories in the large N limit.¹⁰ The effective excitations in these grand canonical ensembles at finite temperature and finite chemical potential had previously been demonstrated to be parametrically long-lived fermionic excitations with a dynamically determined mass.¹¹ The authors of [32] (see section 4 below for a review and generalisations) demonstrated that the large N exact charge of the ensemble with temperature β^{-1} and chemical potential μ in the infinite volume limit is that of a non-interacting gas of these fermionic quasiparticles, with the average occupation number of each quasiparticle state with energy ϵ and charge q given by

$$\begin{aligned}\bar{n}_F(\epsilon, q) &= \frac{1}{2\pi|\lambda_F|} \int_{-\pi|\lambda_F|}^{\pi|\lambda_F|} d\alpha \frac{1}{e^{\beta(\epsilon - q\mu) - iq\alpha} + 1}, \\ &= \frac{1}{2} - \frac{1}{\pi|\lambda_F|} \tan^{-1} \left(\frac{e^{\beta(\epsilon - q\mu)} - 1}{e^{\beta(\epsilon - q\mu)} + 1} \tan \frac{\pi|\lambda_F|}{2} \right), \quad \text{with} \quad \tan^{-1}(\zeta) \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right].\end{aligned}\tag{1.1}$$

(λ_F is the 't Hooft coupling defined in (2.7); the first line of (1.1) was presented in [32]; the second line of (1.1) is derived in section 5.1 of this paper). The equation (1.1) is a

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⁹Our systems are defined on an S^2 with effective volume $\mathcal{V}_2$, where the volume is large ($\mathcal{V}_2 = VN$). While V is held fixed in the large N limit in general, we also study the separate limit $V \rightarrow \infty$ in some parts of our analysis.

¹⁰As we explain in appendix A, there are three simple Bose-Fermi dualities. The first of these asserts the equivalence between an $SU(N_F)$ gauged fermion theory and a $U(N_B)$ gauged bosonic theory. The second between a $U(N_F)$ gauged fermion theory and a $SU(N_B)$ gauged bosonic theory. The third between a $U(N_F)$ gauged fermion theory and a $U(N_B)$ gauged bosonic theory. On general grounds we expect $SU(N)$ and $U(N)$ theories — and so all three of these distinct dualities — to become indistinguishable in the large N limit. The reader may wonder how this expectation is consistent with the fact that $U(1)$ global symmetries of the $SU(N)$ theories are of the simple ‘phase all fields’ sort, while the $U(1)$ global symmetries of the $U(N)$ theories are the so called ‘topological’ symmetries (see appendix A.7). As explained in appendix A.7, the way this works is the following. If we integrate out the dynamical $U(1)$ factor from a large N $U(N)$ theory, the topological symmetry of the parent $U(N)$ theory turns into the usual symmetry of the resultant effective $SU(N)$ theory.

¹¹The mass of these quasiparticles is renormalized in the grand canonical ensemble; in other words it differs from the zero temperature and zero chemical potential pole masses at the same values of UV couplings.

finite- λ_F generalisation of the famous occupation number formula of Fermi-Dirac statistics

$$\bar{n}_F(\epsilon, q) = \frac{1}{e^{\beta(\epsilon - q\mu)} + 1}, \quad (1.2)$$

and (1.1) reduces to (1.2) in the limit $\lambda_F \rightarrow 0$.

What is the bosonic analogue of (1.1)? In addressing this question,¹² we face the following obstacle. The free energy of the bosonic theories (see (2.4) and (2.5) for the Lagrangians) and the fermionic theories (see (2.1) and (2.3) for the Lagrangians) have previously been computed only for a range of temperatures and chemical potentials: those values of these parameters for which the quasiparticle thermal masses turn out to be larger than the chemical potential.¹³ As the first step in the analysis of this paper we remedy this technical defect in (see section 3 below) by generalising previously obtained results to all values of the chemical potential and temperature.

The method we employ is to analytically continue previously computed results for the ‘off-shell free energy’ of matter Chern-Simons theories in the chemical potential. This procedure makes sense because off-shell free energy depends on the chemical potential only through a finite temperature determinant; as this determinant is effectively two dimensional we do not expect it to undergo a sharp phase transition as μ exceeds the thermal mass, see section 3.1.¹⁴ This analytic continuation is conveniently accomplished by modifying a contour of integration in the space of holonomy eigenvalues. The final results of this exercise are listed in detail in (3.35) and (3.36) for the bosonic theories in the infinite volume limit. For general finite volume phases, the formulas are in (3.46), (3.47) for the bosonic theories, and (3.44) and (3.45) for the fermionic theories.

Armed with exact results for the relevant thermodynamical formulae, it is not difficult to imitate the analysis of [32] for bosonic Chern-Simons-matter theories. Once again we find that the large N exact charge of the ensemble of temperature β^{-1} and chemical potential μ in the infinite volume limit¹⁵ is that of a non-interacting gas of these quasiparticles, with the average occupation number of each quasiparticle state of energy ϵ and charge q given by

$$\begin{aligned} \bar{n}_B(\epsilon, q) &= \frac{1}{2\pi|\lambda_B|} \int_{-\pi|\lambda_B|}^{\pi|\lambda_B|} d\alpha \frac{1}{e^{\beta(\epsilon - q\mu) - iq\alpha} - 1} + \frac{1}{|\lambda_B|} \Theta(q\mu - \epsilon), \\ &= \frac{1 - |\lambda_B|}{2|\lambda_B|} - \frac{1}{\pi|\lambda_B|} \tan^{-1} \left(\frac{e^{\beta(\epsilon - q\mu)} - 1}{e^{\beta(\epsilon - q\mu)} + 1} \cot \frac{\pi|\lambda_B|}{2} \right), \quad \text{with} \quad \tan^{-1}(\zeta) \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], \end{aligned} \quad (1.3)$$

where $\Theta(x)$ is the Heaviside step-function. As above, (5.21) may be regarded as the generalisation to finite 't Hooft coupling λ_B of the standard formula of Bose-Einstein statistics

$$\bar{n}_B(\epsilon, q) = \frac{1}{e^{\beta(\epsilon - q\mu)} - 1}. \quad (1.4)$$

¹²And also of the fermionic analogue of the same formula that holds in a general phase of the theory, i.e. away from the infinite volume limit.

¹³The same is true for fermionic theories in some phases that dominate away from the infinite volume limit.

¹⁴We thank O. Aharony and S. Wadia for discussions on this point.

¹⁵See section 3.4.3 for the results that apply in more general phases that dominate away from the infinite volume limit.

For quasiparticle energies $\epsilon > q\mu$, (1.3) reduces to (1.4) in the limit $\lambda_B \rightarrow 0$. Unlike (1.4), however, $\bar{n}_B(\epsilon, q)$ listed in (1.3) is positive at all values of ϵ (even when $\epsilon < q\mu$). In the limit $\lambda_B \rightarrow 0$, $\bar{n}_B(\epsilon, q)$ given in (1.3) diverges; this is the physically correct answer for a free Bose ensemble at $q\mu > \epsilon$ (see below for more discussion).

The modified occupation numbers (1.1) and (1.3) are one-parameter generalisations of the standard formulae (1.2) and (1.4) for fermions and bosons respectively, with the parameter being the 't Hooft coupling λ_F or λ_B respectively. As remarked in footnote 4, the anyonic phase obtained by exchanging two particles with each other or two antiparticles with each other is proportional to $1/N$ and hence is negligible in the large N limit.¹⁶ However the ensembles we study in this paper contain a thermodynamic number of particles since we are working in the large N and large volume limit of [15]. Consequently, the modified occupation number formulae (1.1) and (1.3) are presumably a consequence of these anyonic phases, whose net effect builds up to an $\mathcal{O}(1)$ number. It would be very interesting to understand this better, perhaps in the context of a Schrodinger type description of the non relativistic limit of this system, see [86].

Under the conjectured Bose-Fermi duality (see section 2.4 and in particular (2.6), (2.7) (2.8) and (2.10)), the single particle occupations numbers (1.3) and (1.1) of corresponding states are not equal but instead turn out to be related as

$$N_F \bar{n}_F(\epsilon, q) = N_B \bar{n}_B(\epsilon, q) . \quad (1.5)$$

The fact that $\bar{n}_F(\epsilon, q)$ and $\bar{n}_B(\epsilon, q)$ are not simply equal to each other may, at first, seem to contradict Bose-Fermi duality. In fact, the opposite is true. Equation (1.5) is *precisely* what is needed in order to ensure that the net occupation number of a particular fermionic state matches the net occupation number of bosonic state once we sum over the ‘invisible’ colour quantum numbers of these states. Effective occupation numbers as a function of physical quantum numbers agree exactly across the duality once we sum over the unphysical gauge quantum numbers. As in the case of two-particle scattering states reviewed earlier in this introduction, the ‘invisible’ colour quantum numbers play a key role in reconciling statistics with duality.

1.2 Fermi seas from Bose condensates

It is useful to focus attention on the ensemble in which the dichotomy between Bose and Fermi statistics intuitively appears most pronounced viz. finite chemical potential and zero temperature. In this limit we expect the fermions to arrange themselves into a Fermi sea while the bosons are expected to form a Bose condensate. We now explain how these two rather different phases manage to be dual to each other.

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¹⁶In contrast the anyonic phase obtained in taking a particle around an antiparticle in the so called singlet channel [20] is given $\pi|\lambda_F|$ or $\pi|\lambda_B|$ (for the fermions or bosons respectively), and so is of order unity in the large N limit.

In the zero temperature limit¹⁷ the formula for the Fermionic occupation number (1.1) simplifies to

$$\bar{n}_F(\epsilon, q) = \Theta(q\mu - \epsilon), \quad (1.6)$$

demonstrating that the fermionic phase is a vanilla Fermi sea at every value of λ_F (this was already noted in [32]).

On the other hand the formula for the bosonic occupation number (1.3) simplifies in the zero temperature limit to

$$\bar{n}_B(\epsilon, q) = \bar{n} \Theta(q\mu - \epsilon), \quad \text{with} \quad \bar{n} = \frac{1 - |\lambda_B|}{|\lambda_B|}. \quad (1.7)$$

In fact, the state described by (1.7) is a regularized Bose condensate. Recall that in a free bosonic theory with $\lambda_B = 0$, the partition function $\text{Tr} e^{-\beta(\epsilon - q\mu)}$ defines an ensemble which infinitely populates¹⁸ every single-particle state with energy $\epsilon < q\mu$.¹⁹ As a consequence, the ensemble with $q\mu$ strictly larger than the thermal mass c_B is simply ill-defined in the free theory. In agreement with this fact, $\bar{n}_B(\epsilon, q)$ diverges²⁰ when $\epsilon < \mu q$ in the limit $\lambda_B \rightarrow 0$. This divergence is regulated at nonzero λ_B . Quite remarkably, in the theories studied in this paper the resultant ‘non-perturbative’ phase is remarkably simple; it can be described by a collection of non-interacting quasi particles with occupation numbers given by (1.7).

We see that the regulated Bose condensate (1.7) is very similar to a Fermi sea with one key difference; every state with energy less than $q\mu$ in this phase is occupied not once (as in the case of a Fermi sea) but $\bar{n}$ times (see (1.7)). As in the previous subsection, the difference in occupation numbers between the Fermi sea and the Bose condensate is perfectly consistent with duality. It is easily verified (using the definition of the ’t Hooft couplings (2.7) and the duality map (2.8)) that

$$N_B \bar{n} = N_F, \quad (1.8)$$

and hence the occupation numbers agree once we sum on both sides over ‘invisible’ colour quantum numbers.

1.3 A bosonic exclusion principle

The discussion above suggests that the large N bosonic matter Chern-Simons theory appears to obey an effective exclusion principle. As the upper limit on the occupation number

¹⁷All statements here are accurate when we first take N to infinity and then take the temperature to zero. It is possible that the phase obtained by first taking T to zero and then taking N to infinity has additional complications. See section 8 for additional remarks.

¹⁸To forestall confusion we recall that the standard discussion of Bose condensates in free theories involves taking the limit $\mu q \rightarrow m_B$ (m_B is the boson mass) from below. This limit populates the boson zero mode in a macroscopic manner. $q\mu$ is never taken above m_B , precisely because the free ensemble with $q\mu > m_B$ infinitely populates some states — including the zero mode — and so yields an ill defined ensemble. We thank G. Mandal for a related discussion.

¹⁹Analytic continuation of this formula from values of $q\mu < \epsilon$ to values of $q\mu > \epsilon$ gets rid of the divergence, but evaluates to an unphysical negative number (see (1.4)).

²⁰The formula (1.7) applies both to the regular boson theory and the critical boson theory. It follows that the critical boson self-interactions are insufficient to cure the $\mu > c_B$ divergence of the free boson.

of single particle states,²¹

$$\bar{n} = \frac{1 - |\lambda_B|}{|\lambda_B|} = \frac{|k_B|}{N_B}, \quad (1.9)$$

(see around (2.6) for definitions) is not an integer, this exclusion principle does not operate at the level of single particle states with given colour quantum numbers. Instead, it asserts the total number of bosons one can add in any given single particle state after summing over all colour indices is the integer $N_B \bar{n} = |k_B|$. We pause to explore this result a little further.

Consider a collection of M bosons each of which occupies the same (not counting colour quantum numbers) single particle state. As the net wave function of these particles is symmetric under interchange, and as all non-colour quantum numbers are identical for each of the particles, the wave function of these M particles must be symmetric in colour. In other words, these M particles transform in the $SU(N_B)$ representation with a single row and M columns in the Young diagram as below

$$\underbrace{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \cdots \begin{array}{|c|} \hline \square \\ \hline \end{array}}_{M \text{ boxes}}$$

The exclusion principle above asserts that the length of the Young diagram row cannot exceed k_B .

The restriction on representations described in the previous paragraph is, of course, familiar. Recall that all integrable representations of $SU(N_B)_{k_B}$ WZW theory are described with Young diagram have with no more than k_B columns. It seems very likely to us that this well-known property of WZW theory will turn out to be the underlying explanation for the bosonic exclusion principle.

The discussion of the previous paragraph fits very neatly with the ideas of level-rank duality. Recall that M fermions in a given single particle state (ignoring colour quantum numbers) transform in an $SU(N_F)$ representation with a single column with M rows as below

$$M \text{ boxes} \left\{ \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \vdots \\ \hline \square \\ \hline \end{array} \right.$$

The fact that $M \leq N_F$ is just a statement of the Fermi exclusion principle. However, the representation with a single column and M rows maps, under level-rank duality, to a representation with a single row and M columns. The restriction $M \leq N_F$ is dual to $M \leq |k_B|$, i.e. the restriction described in the previous paragraph. We hope to return to this important and interesting point in future work.

1.4 Zero temperature phase diagrams at finite chemical potential

The exact (to all orders in 't Hooft coupling) large N results for the thermal free energy at finite chemical potential described above become completely explicit and fairly simple

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²¹We thank J. McGreevy for a very useful discussion on the topic of this subsection.

Figure 1. Phase diagram of the critical boson theory as a function of m_B^{cri} at $|\mu| = T = 0$. Here, s marks the origin of the m_B^{cri} axis at which point the theory undergoes a second order phase transition.

in the zero temperature limit.²² We now describe the detailed and quantitative phase diagrams for the bosonic and fermionic theories (see (2.4), (2.1), (2.5) and (2.3) for the Lagrangians) at zero temperature but arbitrary chemical potential (see sections 6 and 7 below) that follow from these results.

Before presenting our quantitative phase diagrams, it is useful to first address a preliminary qualitative question. What is the precise nature of the charged bosonic excitations that condense to form the Bose condensate? This question is more subtle than it might first seem, as we now explain.

At zero chemical potential and temperature, a $SU(N_B)_{k_B}$ Chern-Simons gauge theory coupled to fundamental bosons has two distinct phases as a function of the UV parameters. The critical boson theory (2.4) has one UV parameter: the mass deformation m_B^{cri} . At zero temperature and chemical potential, this theory lies in the unHiggsed phase $m_B^{\text{cri}} > 0$ but in the Higgsed phase for $m_B^{\text{cri}} < 0$ (m_B^{cri} is the mass parameter defined in (2.4)). There is a sharp second order phase transition between these two phases at $m_B^{\text{cri}} = 0$. The situation is qualitatively similar but quantitatively more complicated in the regular boson theory and will be addressed below.

The zero temperature, zero chemical potential phase diagram as a function of m_B^{cri} takes the form depicted in figure 1. In the ordinary or unHiggsed phase the excitations of the bosonic theory are spin zero scalars created by the elementary bosonic field ϕ . In the Higgsed phase the gauge symmetry is Higgsed from $SU(N_B)$ down to $SU(N_B - 1)$ due to $\bar{\phi}\phi$ acquiring a non-zero expectation value; the charged massive excitations in this phase are the spin one W -bosons. The transmutation of degrees of freedom from scalars to vectors is a consequence of the Higgs mechanism.

Given that the charged excitations are qualitatively distinct in the two phases depicted in figure 1, it might thus appear that there are two distinct Bose condensed phases; the condensate of scalar excitations about the unHiggsed vacuum and the condensate of spin one W -bosons around the Higgsed vacuum. At the level of calculations this is true at intermediate steps; the computations that determine the thermodynamics of these two phases appear completely different. Remarkably, it turns out that the final results for the thermodynamics in these two phases — at least at leading order in large N — turn out to be analytic continuations of each other. This result holds in both the critical boson theory (2.4) and the regular boson theory (2.5). It follows that these two apparently distinct Bose condensates are simply different but equivalent descriptions of the same

²²At $\mu = 0$ this fact was already observed in [65, 71, 73].

Figure 2. Phase diagram of critical boson theory as a function of m_B^{cri} at fixed μ . At the points $s_1 = |\mu|$, $s_2 = -|\mu|(\frac{2-|\lambda_B|}{|\lambda_B|})$ the theory undergoes a second order phase transition. The point inside the condensed phase corresponds to $m_B^{\text{cri}} = -|\mu|(\frac{1-|\lambda_B|}{|\lambda_B|})$ and denotes a change in description from that a phase of condensed scalars to one of phase of condensed W -bosons though the condensed phase is itself unique.

physical state.²³

The phase diagram of the critical boson theory as a function of its mass deformation parameter m_B^{cri} at zero temperature but fixed nonzero μ , is depicted in figure 2. As discussed in the previous subsection, the fermionic dual of the Bose condensed phase is a Fermi sea phase. The absence of phase transitions within the Fermi sea phase as the effective fermion mass passes through zero is hardly a surprise. Thus, Bose-Fermi duality can be thought of as a sort of ‘explanation’ for the lack of an invariant distinction between the Bose condensate of scalars and W bosons.

In the rest of this introduction we describe the more intricate phase diagram of the regular boson theory (2.5) (and so, of its dual, the critical fermion theory (2.3).) Let us first recall that in the large N limit, in addition to λ_B , the regular theory is labelled by the dimensionless UV parameter x_6^B , the (mass-)dimension one parameter b_4 and the dimension two parameter m_B^2 . These correspond to the sextic, quartic and mass couplings respectively. At zero temperature and zero chemical potential, and at fixed λ_B and x_6^B , the phase diagram of this theory is a function of b_4 and m_B^2 subject to the scaling equivalence

$$(b_4, m_B^2) \sim (\alpha b_4, \alpha^2 m_B^2), \quad \alpha > 0. \quad (1.10)$$

One way to incorporate the above equivalence while also keeping track of the signs of m_B^2 and b_4 is to consider the following section of the $(m_B^2, \lambda_B b_4)$ plane:

$$(m_B^2)^2 + (\lambda_B b_4)^4 = 1. \quad (1.11)$$

The phase diagram is then the above ellipse-like curve. We present this phase diagram (which was worked out in great detail in [71, 73]) in figure 3. When the chemical potential is non-zero, the phase diagram of the regular boson theory is two dimensional rather than one dimensional. It is efficiently parametrized by the two dimensionless variables

$$\left(\frac{m_B^2}{|\mu|^2}, \frac{\lambda_B b_4}{|\mu|} \right).$$

In the stable range of parameters ($\phi_u < x_6 < \phi_h$, see equation (7.14) for the definition of ϕ_u and ϕ_h) the phase diagram of our theory takes one of three qualitatively distinct forms depending on the precise range in which x_6^B lies. These three phase diagrams have been sketched in figure 4 below.

²³A similar phenomenon occurs at non-zero temperature, at every value of the chemical potential including zero, as noted in e.g. [65, 71].

Figure 3. Phase diagram for the regular boson theory at zero temperature and chemical potential for the stable range $\phi_h < x_6^B < \phi_u$ of the marginal parameter x_6^B (refer to (7.14) for the definitions of ϕ_h and ϕ_u). The letter u stands for the unHiggsed phase while the letter h stands for the Higgsed phase. The blue dots are second order transitions while the green dots are first order transitions.

Figure 4. Phase diagram for the regular boson theory at zero temperature for various subranges of the stable range $\phi_h < x_6^B < \phi_u$. Refer to (7.14) for the definitions of ϕ_u and ϕ_h . The blue lines are second order transitions while the green lines are first order transitions. Actual numerical plots of the above phase diagrams can be found in section 7.7 in figures 20, 21 and 22.

The phase diagrams in figure 4 allow one to determine the thermodynamical behaviour of any given regular boson theory (i.e. a theory at given fixed values of UV parameters) as a function of the chemical potential (see section 7.6 for details). When the chemical potential is small, the theory lies in the uncondensed phase²⁴ — unHiggsed or Higgsed — as depicted in figure 3 (b). As the (modulus of the) chemical potential is increased, for every value of UV parameters, the theory undergoes a single phase transition into the condensed phase and then stays in this phase at every higher value of the chemical potential. The phase transition is sometimes of second order and sometimes of first order depending on the precise value of UV parameters.

²⁴Note that the quantity plotted on the y axis scales like $\frac{1}{|\mu|}$ while the quantity plotted on the x axis scales like $\frac{1}{|\mu|^2}$. Going to small μ at fixed values of UV parameters therefore corresponds to moving along a parabola of the schematic form $y = \alpha\sqrt{x}$. Every such parabola moves into an uncondensed phase at infinity (corresponding to small $|\mu|$). See section 7.6 for details.

2 Theories and conjectured dualities

2.1 Theories

In this subsection, we present the Lagrangians for $SU(N)$ Chern-Simons theories coupled to bosonic and fermionic matter in the fundamental representation of $SU(N)$. Analogous expressions exist for Chern-Simons-matter theories for $U(N)$ gauge theories; see appendix A for a detailed discussion. In particular, the conjectured Bose-Fermi dualities involve theories with $U(N)$ gauge groups on at least one side of the duality. However, the results of this paper are accurate to leading order in $1/N$ in the 't Hooft large N limit in which case the distinction between $SU(N)$ and $U(N)$ appears only at subleading orders in the $1/N$ expansion and hence ignored.

2.1.1 The regular fermion

The Regular Fermion theory is defined by the Lagrangian (see e.g. [65])

$$S_{\text{RF}}[y, \psi] = \frac{i\tilde{\kappa}_F}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{Tr} \left(y_\mu \partial_\nu y_\rho - \frac{2i}{3} y_\mu y_\nu y_\rho \right) + \int d^3x \left(\bar{\psi} \gamma^\mu D_\mu \psi + m_F \bar{\psi} \psi \right), \quad (2.1)$$

where y is an $SU(N)$ gauge field and ψ is a Dirac spinor in the fundamental representation of $SU(N)$. The Chern-Simons level $\tilde{\kappa}_F$ is given by

$$\tilde{\kappa}_F = \kappa_F - \frac{1}{2} \text{sgn}(\kappa_F). \quad (2.2)$$

Note that the second term is subleading in the 't Hooft limit; hence, we shall ignore this $\mathcal{O}(1)$ shift in all subsequent calculations. The same comments apply for the critical fermion action below.

2.1.2 The critical fermion

The critical fermion theory is defined by the Lagrangian (see e.g. [71])

$$S_{\text{CF}}[y, \psi, \zeta_F] = \frac{i\tilde{\kappa}_F}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{Tr} \left(y_\mu \partial_\nu y_\rho - \frac{2i}{3} y_\mu y_\nu y_\rho \right) + \int d^3x \left(\bar{\psi} \gamma^\mu D_\mu \psi - \frac{4\pi}{\kappa_F} \zeta_F \left(\bar{\psi} \psi - \frac{\kappa_F y_2^2}{4\pi} \right) - \frac{4\pi y_4}{\kappa_F} \zeta_F^2 + \frac{(2\pi)^2}{\kappa_F^2} x_6^F \zeta_F^3 \right), \quad (2.3)$$

where y is an $SU(N)$ gauge field, ψ is a Dirac spinor in the fundamental representation of $SU(N)$ and ζ_F is an auxiliary field which is a singlet under $SU(N)$.

2.1.3 The critical boson

The critical boson theory is defined by the Lagrangian [65]

$$S_{\text{CB}}[y, \phi, \sigma] = \frac{i\kappa_B}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{Tr} \left(y_\mu \partial_\nu y_\rho - \frac{2i}{3} y_\mu y_\nu y_\rho \right) + \int d^3x \left(\overline{D_\mu \phi} D_\mu \phi + \sigma \left(\bar{\phi} \phi + \frac{N_B}{4\pi} m_B^{\text{cri}} \right) \right), \quad (2.4)$$

where y is an $SU(N)$ gauge field, ϕ is a scalar field in the fundamental representation of $SU(N)$ and σ is a gauge-singlet auxiliary field.

2.1.4 The regular boson

The regular boson theory is defined by the Lagrangian [71]

$$S_{\text{RB}}[y, \phi] = \frac{i\kappa_B}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{Tr} \left(y_\mu \partial_\nu y_\rho - \frac{2i}{3} y_\mu y_\nu y_\rho \right) + \int d^3x \left((D_\mu \bar{\phi})(D^\mu \phi) + m_B^2 \bar{\phi} \phi + \frac{4\pi b_4}{\kappa_B} (\bar{\phi} \phi)^2 + \frac{(2\pi)^2}{\kappa_B^2} (x_6^B + 1) (\bar{\phi} \phi)^3 \right), \quad (2.5)$$

where y is an $SU(N)$ gauge field and ϕ is a scalar field in the fundamental representation of $SU(N)$.

The dimensionless parameters x_6^F and x_6^B run under the renormalization group. However the beta functions of these parameters are of order $\frac{1}{N}$ (see [70] for a detailed study) and so vanish in the strict large N limit. In the strict large N limit it is thus consistent to treat x_6^F and x_6^B as freely tunable parameters — which one can later set equal to any desired values (e.g. values at the fixed points of beta functions). This is the attitude we adopt in this paper. See [70] and [71] for further discussion.

We have presented the actions (2.1), (2.3), (2.4) and (2.5) for the special case of $SU(N)$ gauge theories. The generalization of these actions to $U(N)$ theories is reviewed in detail in appendix A. At leading order in the large N limit, however, $SU(N)$ and $U(N)$ theories are identical, so the subtle distinction reviewed in appendix A will play no role in the discussions presented in the main text of this paper.

2.2 Symmetries and operators

Each of the theories described above enjoys invariance under a single $U(1)$ global symmetry. The precise nature of this symmetry depends on whether the gauge group of the theory in question is $SU(N)$ or $U(N)$. In the case that the gauge group is $SU(N)$, the symmetry is simply the ‘baryonic’ symmetry which corresponds to a uniform global phase rotation of all fundamental fields and a simultaneous uniform global rotation of all antifundamental fields by the inverse phase. In the case that the gauge group is $U(N)$, the global symmetry is the ‘topological $U(1)$ symmetry’ corresponding to the $U(1)$ part of the $U(N)$ gauge group. In this paper we will turn on a chemical potential corresponding to the unique global symmetry the theory enjoys (i.e. the baryonic symmetry if the theory is $SU(N)$ and the topological symmetry if the theory is $U(N)$). See appendix A.7 for more details about these symmetries.

At first sight the global symmetries of the $SU(N)$ and $U(N)$ theories appear to be rather different from each other. In appendix A.7 we argue, however, that this distinction goes away in the large N limit. For this reason, the thermodynamical formulae presented in this paper apply equally well to both the $SU(N)$ and the $U(N)$ theories.

At large N the gauge-invariant operators for each of the theories listed above falls into two classes.²⁵ The first class consists of products of single-trace (or, more accurately, single-

²⁵We thank T. Senthil and A. Vishwanath for discussions on this point.

sum) operators. A single-sum operator is given by the colour contraction of (derivatives acting on) a fundamental field and (derivatives acting on) an antifundamental field. Such operators generically have dimensions of order unity and are all uncharged under the $U(1)$ global symmetries that each of these theories enjoy.

In the case of the $SU(N)$ theories, the second class of operators is given by products of ‘baryons’ (operators with (derivatives of) N fundamental fields whose colour indices are contracted with an $SU(N)$ Levi-Civita tensor). In the case of $U(N)$ theories the second class of operators consists of monopole operators. In the simple case of a conformal theory these monopole operators are best understood via the state-operator correspondence. These operators correspond to states with one unit of flux, in, say, one of the diagonal entries of the $N \times N$ $U(N)$ gauge field matrix. The Chern-Simons equations of motion forces this configuration to be dressed by $|k|$ quanta of the fundamental field where k is the Chern-Simons level (see the next subsection). See appendix A.6 and [31, 33] for a detailed discussion of these operators.

The operators in this second class are ‘heavy’: their scaling dimensions are of order N in the large N limit. These operators — unlike those of the first class — are charged under the global $U(1)$ symmetries. Throughout this paper we work with global $U(1)$ symmetries normalised so that (roughly speaking) fundamental fields carry unit charge under this symmetry. This means, for instance, that the simplest baryonic operators carry charge N under this symmetry. Note that the $U(1)$ charges of these operators, like their scaling dimensions, are of order N in the large N limit.

2.3 A note on the Chern-Simons levels

We pause here to elaborate on the notation used in the actions (2.1), (2.3), (2.4) and (2.5) above.

When the fermions or bosons in the theories above are all massive, the low energy effective dynamics of our system is governed by a pure Chern-Simons theory. In the case of the bosonic (resp. fermionic) theories we use the symbols k_B (resp. k_F) to denote the level of the $SU(N)$ part of the WZW theory dual to the pure Chern-Simons theory obtained after integrating out the matter fields with a positive mass. The symbols κ_B and κ_F denote the ‘renormalized levels’ defined by

$$\kappa_B = \text{sgn}(k_B)(|k_B| + N_B), \quad \kappa_F = \text{sgn}(k_F)(|k_F| + N_F). \quad (2.6)$$

In this paper we will study the theories above in the large N limit. The true coupling constant of these theories in this limit are the ’t Hooft couplings λ_B (resp. λ_F) defined by

$$\lambda_F = \frac{N_F}{\kappa_F}, \quad \lambda_B = \frac{N_B}{\kappa_B}. \quad (2.7)$$

See [65] and [71] for further details of notations and conventions.

2.4 The conjectured Bose-Fermi duality map

The quasi-fermionic theories i.e. regular fermion (2.1) and critical boson (2.4) theories have been conjectured to be dual to each other. Similarly, the quasi-bosonic theories i.e. the

critical fermion (2.3) and regular boson (2.5) theories also have been conjectured to be dual to each other. For each of these conjectured dualities, the map between the renormalized levels and ranks of the CS gauge groups is given by

$$N_B = |\kappa_F| - N_F, \quad \kappa_B = -\kappa_F. \quad (2.8)$$

Equivalently, and perhaps more simply,

$$N_B = |k_F|, \quad k_B = -\text{sgn}(k_F)N_F. \quad (2.9)$$

The above relationship (2.8) is conjectured to be exact. In particular, they are precisely the rules for the well-established level-rank duality of WZW theories that are dual to the pure Chern-Simons theories that we obtain after integrating out the massive matter fields. From (2.7) it follows that the 't Hooft couplings of the dual pairs of theories are related by

$$\lambda_B = \lambda_F - \text{sgn}(\lambda_F). \quad (2.10)$$

In addition to the Chern-Simons levels and ranks, the quasi-fermionic theories are each characterised by an additional mass parameter, m_F and m_B^{cri} respectively for the regular fermion and critical boson theories. In the large N limit these mass parameters are conjectured to be related via the duality map

$$m_F = -\lambda_B m_B^{\text{cri}}. \quad (2.11)$$

The quasi-bosonic theories are similarly characterised by three UV parameters apart from the Chern-Simons ranks and levels. These correspond to the sextic, quartic and mass terms in the Lagrangian (see (2.5), (2.3)). In the large N limit these are conjectured to be related via the duality maps

$$x_6^F = x_6^B, \quad y_4 = b_4, \quad y_2^2 = m_B^2. \quad (2.12)$$

As mentioned above, while the relations (2.8) and (2.10) are expected to hold even at finite N , the relations (2.11) and (2.12) are conjectured to hold exactly at only infinite N and may receive finite- N corrections.

See e.g. [65, 71] for more details of the conjectured duality maps, including a detailed discussion of the levels of $U(1)$ factors, as well as the regulation scheme in which the duality map above is conjectured to hold.

3 Large N thermal partition functions

All four theories described in section 2.1 are effectively solvable in the large N limit. In particular, the free energy of each of these theories at finite temperature and finite chemical potential (in a range of chemical potentials, see below) has previously been computed at every value of the 't Hooft coupling in the large N limit. In this section we first review previously obtained results and then extend them to the range of chemical potentials of interest to this paper.

The partition function $\mathcal{Z}$ on $S^2 \times S^1$ of each of these theories is given by an expression of the sort

$$\mathcal{Z}_{S^2 \times S^1} = \int [dU]_{\text{CS}} e^{-\mathcal{V}_2 T^2 v[\rho]}, \quad (3.1)$$

where $[dU]_{\text{CS}}$ is the Chern-Simons modified Haar measure over $U(N)$ [14, 15]. At the physical level, the matrix U is the zero mode of the gauge field holonomy around the thermal circle S^1 . $\mathcal{V}_2$ is the volume of the two dimensional space S^2 and T is the temperature which is related to the radius β of the thermal circle S^1 as $T = \beta^{-1}$. The quantity ρ is the holonomy eigenvalue distribution function. As explained in e.g. section 1.2.1 in [65], the zero mode of the holonomy U of the gauge field around S^1 is completely specified (upto a permutation) by its eigenvalues, $e^{i\alpha_j}$ where, $j = 1, \dots, N$ and $\alpha_j \in (-\pi, \pi]$. In the large N limit, the eigenvalue locations on the circle are effectively specified by a continuous distribution function $\rho(\alpha)$ defined by

$$\rho(\alpha) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N \delta(\alpha - \alpha_j). \quad (3.2)$$

The quantity $v[\rho]$ that appears in (3.1) is simply the result of path integral over all non-holonomy modes including the gauge field and matter fields. Indeed,

$$e^{-\mathcal{V}_2 T^2 v[\rho]} = \int_{\mathbb{R}^2 \times S^1} [d\phi] e^{-S[\phi, \rho]}, \quad (3.3)$$

where $[d\phi]$ in (3.3) is the path integral measure over all matter and gauge degrees of freedom except the holonomy of the gauge field around the thermal circle. The subscript $\mathbb{R}^2 \times S^1$ indicates that the path integral is actually computed for the theory on $\mathbb{R}^2 \times S^1$. The information that we are in fact on $S^2 \times S^1$ goes into the Chern-Simons modified Haar measure $[dU]_{\text{CS}}$ in (3.1).

Partition functions at finite temperature and chemical potential are physical observables, and so should be the same for two dual theories. From the definition of the partition function in (3.1) it might at first seem that the requirement that thermal partition functions match across a duality requires that the quantities $v[\rho]$ match across the duality. However that is not quite the case; recall that unitary matrices are of different ranks on the bosonic and fermionic sides of the duality, and on each side one also has to take into account the contribution from the Chern-Simons modified Haar measure over the unitary matrices in (3.1), which depends on the 't Hooft couplings λ_B and λ_F in a non-trivial way. Nonetheless the bosonic and fermionic $v[\rho]$ functions must clearly be related in *some* way in order for the partition functions to match. It was demonstrated in [15] that the thermal partition functions given by (3.1) of the bosonic (regular or critical) and fermionic (critical or regular) theories *will* in fact agree with each other in the large N limit provided the functions $v[\rho]$ obey the more subtle relationship

$$v_{\text{RB}}[\rho_B] = v_{\text{CF}}[\rho_F], \quad v_{\text{CB}}[\rho_B] = v_{\text{RF}}[\rho_F], \quad (3.4)$$

where the holonomy distribution functions ρ_B and ρ_F are not equal to each other but are instead related by the equation

$$|\lambda_B| \rho_B(\pi - \alpha) + |\lambda_F| \rho_F(\alpha) = \frac{1}{2\pi}. \quad (3.5)$$

3.1 The off-shell free energy

In the large N 't Hooft limit, the path integral (3.3) is very effectively computed by the saddle point approximation when the volume $\mathcal{V}_2$ of the spatial S^2 on which we work is taken to be $\mathcal{V}_2 = NV$, and V and all other parameters (temperatures, chemical potentials, masses etc) are held fixed in the t' Hooft limit. This is the limit we work with all through this paper. In sections 5 onwards we also further specialize to limit $V \rightarrow \infty$. In this case we first take the limit $N \rightarrow \infty$ and then the limit $V \rightarrow \infty$.

The final results for $v[\rho]$, obtained this way in earlier work, are most conveniently presented in terms of a so-called ‘off-shell free energy’ [71]. The off-shell free energy is a quantity that depends on some auxiliary variables in addition to the UV parameters of the theory, the temperature, chemical potential and eigenvalue distribution ρ . It turns out that once we perform the path integral (3.3) via saddle point approximation and obtain an answer for $v[\rho]$ at each of the various large N saddle points, the whole procedure can be replaced by a simpler one: that of performing an ordinary integral over a few auxiliary variables, denoted collectively as φ_{aux} , with the integrand being a function of these variables rather than being a functional of fields:

$$e^{-\mathcal{V}_2 T^2 v[\rho]} = \int d\varphi_{\text{aux}} e^{-\mathcal{V}_2 \beta F(\varphi_{\text{aux}}; \rho)}, \quad (3.6)$$

where, the object $F(\varphi_{\text{aux}}; \rho)$ is the off-shell free energy per unit volume. The funny brackets $(\dots)$ indicate that F is still a functional of the holonomy distribution ρ while being an ordinary function of the auxiliary variables φ_{aux} . The off-shell free energy F has an explicit factor of N in its expression and hence, in the large N limit, the integral above may be evaluated by saddle point approximation. The quantity $v[\rho]$ is thus obtained from the off-shell free energy by extremizing the latter w.r.t. its auxiliary variables [71].

The existing results in the literature for the free energies $v[\rho]$ [5, 10, 12, 14–18, 32, 65, 70, 71, 73, 80] and $F(\varphi_{\text{aux}}; \rho)$ in the presence of chemical potential were carefully computed only for values of the chemical potential smaller than the thermal masses of the bosonic or fermionic excitations in the respective theories. We present these expressions for the free energy in the next subsection. It turns out that some interesting modifications are required in the expressions for the free energies in order to extend their validity for chemical potentials larger than the thermal masses. We describe these modifications in the subsection after next.

3.2 Previously known results

Note: throughout this paper, we use the convention that, given a quantity x of mass dimension a , the corresponding dimensionless hatted quantity $\hat{x}$ is defined to be the quantity x in the units of temperature, i.e., $\hat{x} = (\beta)^a x$. For example, the mass parameter m_B^{cri} of the critical boson theory has its dimensionless counterpart $\hat{m}_B^{\text{cri}} = \beta m_B^{\text{cri}}$.

3.2.1 The regular fermion

The off-shell free energy for the RF theory, presented as a function of the two auxiliary variables $\tilde{\mathcal{C}}$ and c_F , is [71, 73]

$$\begin{aligned}
 F_{\text{RF}}(c_F, \tilde{\mathcal{C}}) &= \frac{N_F}{6\pi\beta^3} \left[-8\lambda_F^2 \tilde{\mathcal{C}}^3 - 3\tilde{\mathcal{C}} \left(\hat{c}_F^2 - (2\lambda_F \tilde{\mathcal{C}} + \hat{m}_F)^2 \right) - 6\lambda_F \hat{m}_F \tilde{\mathcal{C}}^2 \right. \\
 &\quad \left. + \hat{c}_F^3 - 3 \int_{\hat{c}_F}^{\infty} d\hat{\epsilon} \hat{\epsilon} \int_{-\pi}^{\pi} d\alpha \rho_F(\alpha) \left(\log(1 + e^{-\hat{\epsilon} - \hat{\mu} - i\alpha}) + \log(1 + e^{-\hat{\epsilon} + \hat{\mu} + i\alpha}) \right) \right]. \quad (3.7)
 \end{aligned}$$

Extremization of (3.7) w.r.t. c_F and $\tilde{\mathcal{C}}$ gives, respectively,

$$\tilde{\mathcal{C}} = \mathcal{C}(c_F, \mu), \quad \text{and} \quad \hat{c}_F^2 = (2\lambda_F \tilde{\mathcal{C}} + \hat{m}_F)^2, \quad (3.8)$$

where $\mathcal{C}$ is a particular moment of the holonomy distribution given by

$$\mathcal{C}(\epsilon, \mu) \equiv \frac{1}{2} \int_{-\pi}^{\pi} d\alpha \rho_F(\alpha) \left(\log \left(2 \cosh \frac{\hat{\epsilon} + \hat{\mu} + i\alpha}{2} \right) + \log \left(2 \cosh \frac{\hat{\epsilon} - \hat{\mu} - i\alpha}{2} \right) \right). \quad (3.9)$$

The extremum value of the variable c_F gives the thermal mass of the fermionic excitations which is also denoted by the same symbol c_F . We do not know of a physical interpretation for the variable $\tilde{\mathcal{C}}$ except that it becomes the function $\mathcal{C}(c_F, \mu)$ on-shell.²⁶

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²⁶The logarithm in the free energy (3.7) and the moment function (3.9) — and indeed everywhere in this paper — is defined to have a branch cut for negative real arguments, so that the imaginary part of $\ln(x + iy)$ is $+\pi$ just above the negative x axis, but is $-\pi$ just below the negative x axis. In deriving the equation of motion (3.8) we have used that when $x > 0$

$$\begin{aligned}
 &\int_{-\pi}^{\pi} d\alpha \rho_F(\alpha) \log \left(2 \cosh \frac{x + i\alpha}{2} \right) \\
 &= \int_{-\pi}^{\pi} d\alpha \rho_F(\alpha) \left(\frac{x + i\alpha}{2} + \log(1 + e^{-x - i\alpha}) \right) \\
 &= \frac{x}{2} + \int_{-\pi}^{\pi} d\alpha \rho_F(\alpha) \left(\log(1 + e^{-x - i\alpha}) \right)
 \end{aligned} \quad (3.10)$$

We use this identity once with $x = \hat{c}_F + \hat{\mu}$ and once with $x = \hat{c}_F - \hat{\mu}$. The condition $x > 0$ is met in both cases because we have assumed $\hat{c}_F > |\hat{\mu}|$. In going from the first to the second line of (3.10) we use the fact that when the log-function is defined to be real on the positive real axis and with branch cut along the negative real axis

$$\log z_1 z_2 = \log z_1 + \log z_2, \quad \text{provided} \quad |\text{Arg}(z_1)| < \frac{\pi}{2}, \quad |\text{Arg}(z_2)| < \frac{\pi}{2}, \quad (3.11)$$

The conditions on the arguments of z_1 and z_2 in (3.11) are met in going from the first to the second line of (3.10), because we have assumed $x > 0$. In going from the second to the third line of (3.10) we have used the fact that $\rho_F(\alpha)$ is an even function that integrates to unity.

3.2.2 The critical fermion

The off-shell free energy for the CF theory was presented in [71] (see equation (A.3) of [71]) and is given by

$$\begin{aligned}
 F_{\text{CF}}(c_F, \zeta_F, \tilde{\mathcal{C}}) &= \frac{N_F}{6\pi\beta^3} \left[-8\lambda_F^2 \tilde{\mathcal{C}}^3 - 3\tilde{\mathcal{C}} \left(\hat{c}_F^2 - \left(2\lambda_F \tilde{\mathcal{C}} - \frac{4\pi\hat{\zeta}_F}{\kappa_F} \right)^2 \right) + 6\lambda_F \tilde{\mathcal{C}}^2 \left(\frac{4\pi\hat{\zeta}_F}{\kappa_F} \right) \right. \\
 &\quad + 3 \left(\frac{\hat{y}_2^2}{2\lambda_F} \frac{4\pi\hat{\zeta}_F}{\kappa_F} - \frac{\hat{y}_4}{2\lambda_F} \left(\frac{4\pi\hat{\zeta}_F}{\kappa_F} \right)^2 + \frac{x_6^F}{8\lambda_F} \left(\frac{4\pi\hat{\zeta}_F}{\kappa_F} \right)^3 \right) \\
 &\quad \left. + \hat{c}_F^3 - 3 \int_{\hat{c}_F}^{\infty} d\hat{\epsilon} \hat{\epsilon} \int_{-\pi}^{\pi} d\alpha \rho_F(\alpha) \left(\log(1 + e^{-\hat{\epsilon} - \hat{\mu} - i\alpha}) + \log(1 + e^{-\hat{\epsilon} + \hat{\mu} + i\alpha}) \right) \right]. \quad (3.12)
 \end{aligned}$$

In (3.12), c_F , ζ_F and $\tilde{\mathcal{C}}$ are auxiliary quantities (w.r.t. which we have to extremize F_{CF} in order to obtain $v_{\text{CF}}[\rho]$). The gap equations that follow from this extremization process are derived and described in detail in the section A.1 of [71]. We present the equations here for completeness. The extremization of (3.12) w.r.t. $\tilde{\mathcal{C}}$, c_F and ζ_F respectively gives the following equations

$$\begin{aligned}
 \hat{c}_F^2 &= \left(2\lambda_F \tilde{\mathcal{C}} - \frac{4\pi\hat{\zeta}_F}{\kappa_F} \right)^2, \\
 \tilde{\mathcal{C}} &= \mathcal{C}(c_F, \mu), \\
 \frac{3}{4} \left(\frac{4\pi\hat{\zeta}_F}{\kappa_F} \right)^2 x_6^F + \frac{8\pi\hat{\zeta}_F}{\kappa_F} (2\lambda_F \tilde{\mathcal{C}} - \hat{y}_4) - 4\lambda_F^2 \tilde{\mathcal{C}}^2 + \hat{y}_2^2 &= 0. \quad (3.13)
 \end{aligned}$$

As for the regular fermion, the extremum value of the variable c_F gives the thermal mass of the fermionic excitations and the extremum value of the variable $\tilde{\mathcal{C}}$ is $\mathcal{C}(c_F, \mu)$. The variable ζ_F in the critical fermion free energy (3.12) is precisely the (constant) expectation value of the auxiliary field $\zeta_F(x)$ that appears in the critical fermion Lagrangian (2.3). If we (classically) integrate out the variables c_F and $\tilde{\mathcal{C}}$ from the free energy (3.12) and perform the integral over the holonomy, the resultant function of ζ_F is the (large N exact) quantum effective potential of the theory as a function of ζ_F .

3.2.3 The critical boson

The off-shell free energy for the critical boson theory, presented in terms of the two auxiliary variables c_B and $\tilde{\mathcal{S}}$, is given by [71]

$$\begin{aligned}
 F_{\text{CB}}(c_B, \tilde{\mathcal{S}}) &= \frac{N_B}{6\pi\beta^3} \left[\frac{3}{2} \hat{c}_B^2 \hat{m}_B^{\text{cri}} - 4\lambda_B^2 \left(\tilde{\mathcal{S}} - \frac{1}{2} \hat{m}_B^{\text{cri}} \right)^3 + 6|\lambda_B| \hat{c}_B \left(\tilde{\mathcal{S}} - \frac{1}{2} \hat{m}_B^{\text{cri}} \right)^2 \right. \\
 &\quad \left. - \hat{c}_B^3 + 3 \int_{\hat{c}_B}^{\infty} d\hat{\epsilon} \hat{\epsilon} \int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) \left(\log(1 - e^{-\hat{\epsilon} + \hat{\mu} + i\alpha}) + \log(1 - e^{-\hat{\epsilon} - \hat{\mu} - i\alpha}) \right) \right]. \quad (3.14)
 \end{aligned}$$

Extremizing the free energy (3.14) w.r.t. $\tilde{\mathcal{S}}$ and c_B respectively gives

$$\begin{aligned} \left(\tilde{\mathcal{S}} - \frac{1}{2}\hat{m}_B^{\text{cri}}\right) \left(\frac{\hat{c}_B}{|\lambda_B|} - \left(\tilde{\mathcal{S}} - \frac{1}{2}\hat{m}_B^{\text{cri}}\right)\right) &= 0, \\ \left(\mathcal{S}(c_B, \mu) - \frac{1}{2}\hat{m}_B^{\text{cri}}\right) \hat{c}_B - |\lambda_B| \left(\tilde{\mathcal{S}} - \frac{1}{2}\hat{m}_B^{\text{cri}}\right)^2 &= 0, \end{aligned} \quad (3.15)$$

where the function $\mathcal{S}$ is defined by

$$\mathcal{S}(\epsilon, \mu) \equiv \frac{1}{2} \int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) \left(\log \left(2 \sinh \frac{\hat{\epsilon} + |\hat{\mu}| + i\alpha}{2} \right) + \log \left(2 \sinh \frac{\hat{\epsilon} - |\hat{\mu}| - i\alpha}{2} \right) \right). \quad (3.16)$$

(the logarithm in (3.16) is defined in exactly the same way as the logarithm in (3.9) in footnote 26).

The variable c_B has a direct physical interpretation: its extremum value is the thermal mass of the bosonic excitations. We do not have a similar physical interpretation for the variable $\tilde{\mathcal{S}}$ except that it becomes the function $\mathcal{S}(c_B, \mu)$ above.

3.2.4 The regular boson

As demonstrated in [71], the off-shell thermal free energy (per unit volume $\beta\mathcal{V}_2$) of the RB theory at a finite chemical potential μ is given by

$$\begin{aligned} F_{\text{RB}}(c_B, \sigma_B, \tilde{\mathcal{S}}) &= \frac{N_B}{6\pi\beta^3} \left[-3\hat{c}_B^2\hat{\sigma}_B + \lambda_B^2\hat{\sigma}_B^3 - 4\lambda_B^2(\tilde{\mathcal{S}} + \hat{\sigma}_B)^3 + 6|\lambda_B|\hat{c}_B(\tilde{\mathcal{S}} + \hat{\sigma}_B)^2 \right. \\ &\quad + 3(\hat{m}_B^2\hat{\sigma}_B + 2\lambda_B\hat{b}_4\hat{\sigma}_B^2 + (x_6 + 1)\lambda_B^2\hat{\sigma}_B^3) \\ &\quad \left. - \hat{c}_B^3 + 3 \int_{\hat{c}_B}^{\infty} \hat{\epsilon} d\hat{\epsilon} \int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) \left(\log(1 - e^{-\hat{\epsilon} + \hat{\mu} + i\alpha}) + \log(1 - e^{-\hat{\epsilon} - \hat{\mu} - i\alpha}) \right) \right]. \end{aligned} \quad (3.17)$$

In (3.17), c_B , σ_B and $\tilde{\mathcal{S}}$ are auxiliary quantities (w.r.t. which we have to extremize F_{RB} in order to obtain $v_{\text{RB}}[\rho]$). The gap equations that follow from this extremization process are derived in detail in equation (4.3) of [71], and for completeness, we list these here. Extremization of (3.17) w.r.t. $\tilde{\mathcal{S}}$, $\hat{c}_B$ and $\hat{\sigma}_B$ respectively gives the following equations

$$\begin{aligned} (\tilde{\mathcal{S}} + \hat{\sigma}_B)(\hat{c}_B - |\lambda_B|(\tilde{\mathcal{S}} + \hat{\sigma}_B)) &= 0, \\ (\mathcal{S}(c_B, \mu) + \hat{\sigma}_B)\hat{c}_B - |\lambda_B|(\tilde{\mathcal{S}} + \hat{\sigma}_B)^2 &= 0, \\ \hat{c}_B^2 - \hat{m}_B^2 - 4|\lambda_B|\hat{c}_B(\tilde{\mathcal{S}} + \hat{\sigma}_B) + \lambda_B(4\tilde{\mathcal{S}}^2\lambda_B - 4\hat{b}_4\hat{\sigma}_B + 8\lambda_B\hat{\sigma}_B\tilde{\mathcal{S}} - 3\lambda_Bx_6^B\hat{\sigma}_B^2) &= 0. \end{aligned} \quad (3.18)$$

The quantity σ_B in the regular boson free energy (3.17) also has a simple physical interpretation. It is related to the expectation value of the lightest gauge-invariant operator, $\bar{\phi}\phi$, of the regular theory as

$$\sigma_B = \frac{2\pi}{N_B} \langle \bar{\phi}\phi \rangle. \quad (3.19)$$

As for the critical boson, the variable c_B has the physical interpretation that, on shell, it is the thermal mass of the bosonic excitations. The variable σ_B also has an important physical

origin. In fact, if we (classically) integrate out the variables c_B and $\tilde{\mathcal{S}}$ in the regular boson free energy (3.17), and then substitute (3.19) for σ_B and finally perform the integral over the holonomy, the resultant function of $\langle\bar{\phi}\phi\rangle$ is simply the large- N -exact quantum effective potential of the theory as a function of its ‘lightest’ gauge invariant observable $\langle\bar{\phi}\phi\rangle$.

The objects $\tilde{\mathcal{S}}$ and $\mathcal{S}(c_B, \mu)$ are different off-shell: one is a variable while the other is a function of a different variable. However, on shell, we have

$$\tilde{\mathcal{S}} = \mathcal{S}(c_B, \mu), \quad (3.20)$$

which is easy to see by comparing the first two equations in (3.18).

3.2.5 Duality of off-shell variables and off-shell free energies

Under duality, the off-shell variables have to be appropriately mapped to each other in addition to the duality identifications between the coupling constants given in section 2.4. The duality map between the off-shell variables is as follows:

$$c_B = c_F, \quad \lambda_B \tilde{\mathcal{S}} = \lambda_F \tilde{\mathcal{C}} - \frac{1}{2} \text{sgn}(\lambda_F) c_F, \quad 2\lambda_B \sigma_B = -\frac{4\pi\zeta_F}{\kappa_F}. \quad (3.21)$$

It can be shown that the off-shell free energies of the critical fermion and regular boson map to each other under the duality map (3.21). Similarly, it can be shown that the off-shell free energies of the regular fermion and critical boson map to each other under the first two relations of the map (3.21) (the third relation in (3.21) is not required in this case). We relegate the demonstration of the matching of off-shell free energies to appendix B.

It is interesting that the off-shell free energies of the (conjecturally dual) bosonic and fermionic theories map precisely to each other at the full off-shell level rather than only on shell (i.e. only on the solutions to the equations of motion).²⁷ This stronger duality invariance suggests that the off-shell free energy is not just a computational device but is physically meaningful. It has previously been partially demonstrated that this is indeed the case: the authors of [71] demonstrated that the off-shell free energy of the regular boson theory as a function of σ_B (obtained after integrating out the variables c_B and $\tilde{\mathcal{S}}$) is closely related to the quantum effective action as a function of its lightest gauge invariant scalar operator $\bar{\phi}\phi$. It is possible that the full off-shell free energy is physical in a similar manner (i.e. it can be recast as the quantum effective action as a function of three suitable physical operators). The details of such a reformulation — if it exists — is an interesting problem for future research.

3.3 Conjecture for the off-shell free energy for $|\mu|$ greater than thermal mass

In section 3.2, we presented the expressions for the off-shell thermal free energies of the four classes of theories of interest to us in this paper. The results of section 3.2 apply whenever the thermal mass of the bosonic or fermionic excitations is larger than the modulus of the chemical potential. In this subsection, we will present a conjecture for the extension of

²⁷The latter weaker agreement would have been sufficient to ensure the equality of the free energies of the bosonic and fermionic theories under the duality map of section 2.4

these results to the situation in which the chemical potential is larger than the quasiparticle thermal masses.

In order to motivate our conjecture, consider the case of the bosonic theories. The effective squared mass of the boson ($c_B^2 - \mu^2$), vanishes when $\mu = c_B$ and becomes negative when $\mu^2 > c_B^2$, encouraging the boson to ‘condense’. As we will explain in detail in the next subsection, the off-shell free energies of these theories depend on the chemical potential only through a bosonic determinant. As this finite temperature determinant is two dimensional at long distances, however, we expect this ‘condensation’ to occur without a sharp phase transition (recall that spontaneous symmetry breaking of continuous symmetries is forbidden in two dimensions). Note this expectation continues to hold even in the large N limit, as the logarithm of a large N determinant depends on N only through an overall multiplicative factor of N which does not affect its analyticity properties.²⁸

The expectation on the fermionic side is similar: Fermi sea formation, while sharp at zero temperature, is a crossover at finite temperature.

At finite temperature, we then expect the off-shell free energies of both the bosonic and fermionic theories to be analytic functions of μ . *We thus conjecture that the correct formulae for the off-shell free energies for $|\mu| \geq c_B$ are simply the analytic continuation (in the variable μ) of the relevant formulae for $|\mu| < c_B$ (presented in sections 3.2.1–3.2.4).*

The conjecture presented in this section matches the expectations of Bose-Fermi duality more or less trivially. Since the fermionic and bosonic free energies have already been shown to match when the chemical potential is smaller than the quasiparticle masses, the analytic continuations of these two expressions necessarily agree at all values of the chemical potential.

3.4 Implementing the analytic continuation

3.4.1 Non-analyticity of the naive expressions

The off-shell free energies of the previous section were computed in the previous literature by evaluating the field theory partition functions (3.6) via the following (schematic) procedure. One first integrates out the gauge field (this is possible because it appears in the action in a quadratic manner). One then introduces various singlet Lagrange multiplier and Hubbard-Stratanovich fields chosen so as to ensure that the action is quadratic in the matter fields. The matter fields are then integrated out, yielding a determinant (which exponentiates to a log-determinant). The resultant action is a complicated non-local functional of the auxiliary singlet fields.

One then derives the large N saddle point equations for these singlet fields. Remarkably, it turns out to be possible to exactly solve the resultant non-linear integral equations. What is more remarkable is that, these complicated non-linear integral equations for the singlet fields can be recast into much simpler algebraic or transcendental equations for a few c -number variables rather than fields. These c -number variables are the ‘auxiliary’ variables that appear in the off-shell free energies (3.14), (3.17), (3.7) and (3.12), and the algebraic or transcendental equations involving these variables are the ones obtained by

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²⁸We thank S. Wadia for discussions on this point.

extremizing these simple off-shell free energies listed in these four equations w.r.t. their auxiliary variables as discussed in detail in section 3.2.

What is important for the current discussion is that the log-determinant described in the first paragraph above is the only part of the effective action that had its origin in a term in the Lagrangian involving derivatives acting on the matter fields. Therefore, it is the only part of this effective action that is sensitive to the chemical potential. This fact is also visible in the explicit expressions (3.14), (3.17), (3.7) and (3.12) (the log-determinant is the last line in each of those equations).

In summary, the general structure of the off-shell free energies (3.14), (3.17), (3.7) and (3.12) of the previous is

$$F = F_{\text{int}} + F_{\text{det}} , \quad (3.22)$$

where F_{det} is the log-determinant term and F_{int} (which is the rest of the free energy) is independent of μ . As reflected in the final answers (3.14), (3.17), (3.7) and (3.12), the determinants F_{det} take a simple universal form, correspond to the free energy of a single massive boson (of a mass c_B that remains to be determined) or a single massive fermion (of a mass c_F that remains to be determined) modified by its coupling to the holonomy distribution ρ .²⁹

Let us first focus on the bosonic determinant. At small enough values of μ , F_{det} is, of course, an analytic function of μ . At $|\mu| = c_B$ something special happens. At this value the effective mass of the bosonic field — which is proportional to $c_B^2 - \mu^2$ — vanishes. After Kaluza-Klein reduction on the thermal circle, this effectively massless scalar field reduces to an infinite collection of two dimensional fields of squared masses

$$\left(\frac{2\pi n - \alpha}{\beta} \right)^2 ,$$

where the Kaluza-Klein momentum n runs from $-\infty$ to ∞ , α is the holonomy and β is the inverse temperature. When $\alpha = 0$ the Kaluza-Klein mode with $n = 0$ is a massless two dimensional scalar field. The determinant over such a field has a logarithmic divergence. As a consequence we should expect our determinant to have a potential non-analyticity of the schematic form

$$\int d\alpha \rho(\alpha) \log(\beta(c_B - \mu - i\alpha)) ,$$

in the neighbourhood of $\alpha = 0$. Indeed the exact expression for the Bosonic determinant

$$\begin{aligned} & \beta \mathcal{V}_2 F_{B,\text{det}}(c_B, \mu) \\ &= \frac{N_B \mathcal{V}_2}{2\pi} \int_{c_B}^{\infty} d\epsilon \int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) \left(\log \left(2 \sinh \frac{\hat{\epsilon} + \hat{\mu} + i\alpha}{2} \right) + \log \left(2 \sinh \frac{\hat{\epsilon} - \hat{\mu} - i\alpha}{2} \right) \right) , \\ &= \frac{N_B \mathcal{V}_2}{2\pi} \left[-\frac{\beta c_B^3}{3} + \int_{c_B}^{\infty} d\epsilon \int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) \left(\log \left(1 - e^{-\hat{\epsilon} - \hat{\mu} - i\alpha} \right) + \log \left(1 - e^{-\hat{\epsilon} + \hat{\mu} + i\alpha} \right) \right) \right] , \end{aligned} \quad (3.23)$$

does have precisely such a singularity.

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²⁹Recall that this holonomy distribution function ρ_B or ρ_F which appears in the determinant is itself determined at the saddle-point by extremizing its effective action obtained by integrating out the matter fields.

Similar remarks apply to fermionic fields. Once again the field is effectively massless at $\mu = c_F$. Once again a Kaluza-Klein reduction at this value of μ gives an infinite number of two dimensional fields whose squared masses, this time, are

$$\left(\frac{2\pi(n + \frac{1}{2}) - \alpha}{\beta} \right)^2 .$$

(the shift of n by $\frac{1}{2}$ is a consequence of the antiperiodic boundary conditions of fermions around the thermal circle). This time the mode with $n = 0$ is effectively massless at $\alpha = \pi$ and so we should expect the fermionic determinant to have a potential non-analyticity of the form

$$\int d\alpha \rho(\alpha) \ln(\beta(c_F - \mu - i\alpha)) ,$$

in the neighbourhood of $\alpha = \pi$. Indeed, the exact expression for the fermionic determinant

$$\begin{aligned} & \beta \mathcal{V}_2 F_{F,\text{det}}(c_F, \mu) \\ &= -\frac{N_F \mathcal{V}_2}{2\pi} \int_{c_F}^{\infty} d\epsilon \int_{-\pi}^{\pi} d\alpha \rho_F(\alpha) \left(\log \left(2 \cosh \frac{\hat{\epsilon} + \hat{\mu} + i\alpha}{2} \right) + \log \left(2 \cosh \frac{\hat{\epsilon} - \hat{\mu} - i\alpha}{2} \right) \right) , \\ &= -\frac{N_F \mathcal{V}_2}{2\pi} \left[-\frac{\beta c_F^3}{3} + \int_{c_F}^{\infty} d\epsilon \int_{-\pi}^{\pi} d\alpha \rho_F(\alpha) \left(\log \left(1 + e^{-\hat{\epsilon} - \hat{\mu} - i\alpha} \right) + \log \left(1 + e^{-\hat{\epsilon} + \hat{\mu} + i\alpha} \right) \right) \right] . \end{aligned} \quad (3.24)$$

does have precisely such a singularity.³⁰

In summary, the bosonic determinant (3.23) is non-analytic at $|\mu| = c_B$ if $\rho_B(\alpha)$ is non-vanishing in a neighbourhood of $\alpha = 0$. In a similar way the fermionic determinant (3.24) is potentially non-analytic at $|\mu| = c_F$ if $\rho_F(\alpha)$ is non-vanishing in a neighbourhood of $\alpha = \pi$. In the rest of this subsection we will explore the nature of these non-analyticities and explain how we can analytically continue around them.

3.4.2 Analytic continuation in bosonic upper cap phases and fermionic lower gap phases

In this subsection, we consider phases of the fermionic theory [15] in which the eigenvalue distribution $\rho_F(\alpha)$ vanishes in an interval around $\alpha = \pi$. Correspondingly, the dual bosonic eigenvalue distribution $\rho_B(\alpha)$ saturates the upper cutoff $1/2\pi|\lambda_B|$ in an interval around $\alpha = 0$. Concretely, we focus on phases in which

$$\rho_F(\alpha) = 0 \quad \text{when} \quad |\alpha| > \pi - b, \quad \text{with} \quad b > 0 . \quad (3.25)$$

From the duality map relating ρ_B and ρ_F (3.5), it follows that the dual eigenvalue distribution obeys

$$\rho_B(\alpha) = \frac{1}{2\pi|\lambda_B|} \quad \text{when} \quad |\alpha| < b . \quad (3.26)$$

When the large N saddle-point eigenvalue distributions ρ_B and ρ_F satisfy (3.25) and (3.26), the fermionic theory is said to be in a lower gap phase while the bosonic theory is said to be in an upper cap phase [15]. We return to the study of more general phases in the next subsection.

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³⁰The second equality in both (3.23) and (3.24) holds in the dimensional regulation scheme.

Figure 5. The contour of integration in the z -plane for the integral (3.27). The angle α is related to z as $z = e^{i\alpha}$.

Fermions in the lower gap phase. Let us first examine the fermionic determinant (3.24). The term that depends on μ is the integral

$$\int_{c_F}^{\infty} d\epsilon \, \epsilon \int_{-\pi}^{\pi} d\alpha \, \rho_F(\alpha) \left(\log(1 + e^{-\hat{\epsilon} - \hat{\mu} - i\alpha}) + \log(1 + e^{-\hat{\epsilon} + \hat{\mu} + i\alpha}) \right). \quad (3.27)$$

We recast the integral over α into a contour integral over the unit circle in the complex z -plane with $z = e^{i\alpha}$, see figure 5. Let us study the branch cut structure of the logarithms (3.27). When $c_F > |\mu|$, the branch cut that appears in both logarithms does not intersect the unit circle in the z -plane (see figure 6(a)). As a consequence, the expression (3.27) is analytic in μ for $|\mu| < c_F$. On the other hand, when $|\mu| > c_F$, the unit circle intersects the branch cut of the second logarithm in (3.27) when ϵ lies in the range $c_F < \epsilon < |\mu|$. For a general distribution $\rho_F(\alpha)$, this intersection implies a breakdown of the analyticity of one of the logarithms in (3.27) as a function of μ at $\mu = c_F$ or at $\mu = -c_F$, depending on the sign of μ (recall that c_F is a positive quantity by definition).

However, when the fermion is in the lower gap phase (3.25) the integration contour is not the full unit circle but the open arc corresponding to

$$-(\pi - b) < \alpha < \pi - b. \quad (3.28)$$

The branch cut of the logarithms in (3.27) lies at $\alpha = \pi$ and so does not intersect the open contour specified by (3.28) (see figure 6(b)). It follows that when $\rho_F(\alpha)$ obeys (3.25), the expression (3.27) is analytic as a function of μ at all values of μ including $|\mu| \geq c_F$. Recall that (3.27) is the only term in the fermionic free energies that depends on μ . Now, since this term is analytic for all μ , we conclude that whenever the fermion is in a lower gap phase (3.25), the formulae for the fermionic off-shell free energies are fully analytic in μ and apply without modification for all values of μ including the range $|\mu| > c_F$.

Bosons in an upper cap phase. Let us now consider the bosonic determinant (3.23). The term that depends on μ is

$$\int_{c_B}^{\infty} d\epsilon \, \epsilon \int_{-\pi}^{\pi} d\alpha \, \rho_B(\alpha) \left(\log(1 - e^{-\hat{\epsilon} - \hat{\mu} - i\alpha}) + \log(1 - e^{-\hat{\epsilon} + \hat{\mu} + i\alpha}) \right). \quad (3.29)$$

When the boson is in an upper cap phase (3.26), the integral over α in (3.29) can be split up into the integral over the range $\alpha \in (-b, b)$ on the unit circle and the integral over the

Figure 6. Fermions in a lower gap phase: branch point is at $z = -e^{\hat{\epsilon}-|\hat{\mu}|}$ and is denoted by the black dot on the negative real axis. The green line is the branch cut for the logarithm appearing in the fermionic free energies. The dashed curve is the unit circle and the purple curve is the counterclockwise contour.

rest of the unit circle. The integral over the rest of the unit circle is manifestly analytic in μ for all values of μ . The part of (3.29) that is potentially non-analytic in μ is

$$\frac{1}{2\pi|\lambda_B|} \int_{c_B}^{|\mu|} \epsilon \, d\epsilon \int_{-b}^b d\alpha \log(1 - e^{-\hat{\epsilon}+|\hat{\mu}|+i\alpha}) . \quad (3.30)$$

The potentially non-analytic part is one of the logarithms of (3.29) depending on the sign of μ . In the above expression (3.30), we treat the two cases together by noticing that the non-analyticity in either case depends only on the absolute value $|\mu|$. As written, (3.30) is indeed non-analytic as a function of μ at $|\mu| = \epsilon$. However, it is easy to find a modification of (3.30) that agrees with (3.30) for $|\mu| < \epsilon$ but is analytic for all μ . We first note that (3.30) can be rewritten as an open contour integral

$$\frac{1}{2\pi|\lambda_B|} \int_{c_B}^{|\mu|} \epsilon \, d\epsilon \int_C \frac{dz}{iz} \log(1 - z e^{-\hat{\epsilon}+|\hat{\mu}|}) , \quad (3.31)$$

where the contour C starts at e^{-ib} and runs counterclockwise along the unit circle to e^{ib} (see figure 7). The modification of (3.31) that makes it analytic everywhere replaces the contour C with a contour C' that begins and ends at the same points, but no longer runs along the unit circle all the way. It lies on the unit circle except in the neighbourhood of the real axis where it makes a hairpin bend around the branch cut so that it cuts the positive real axis at a point $x_0 < e^{\hat{\epsilon}-|\hat{\mu}|}$ (see figure 7(b)). This condition on the contour C' is chosen to ensure that this contour never passes through the branch cut of the logarithm in (3.31). With this choice of contour, (3.31) defines a function that manifestly agrees with (3.30) when $|\mu| < c_B$ and is analytic everywhere.

Note that when z is slightly below (resp. above) the branch cut, the argument of the logarithm $1 - z e^{-\hat{\epsilon}+|\hat{\mu}|}$, has a small positive (resp. negative) imaginary part. It follows that the integral in (3.31) evaluates to

$$\frac{1}{2\pi|\lambda_B|} \int_{c_B}^{|\mu|} d\epsilon \int_{-b}^b d\alpha \log(1 - e^{-\hat{\epsilon}+|\hat{\mu}|+i\alpha}) - \frac{1}{|\lambda_B|} \Theta(|\mu| - c_B) \int_{c_B}^{|\mu|} d\epsilon \int_{e^{\hat{\epsilon}-|\hat{\mu}|}}^1 \frac{dx}{x} . \quad (3.32)$$

Figure 7. Bosons in the upper cap phase: branch point is at $z = +e^{\hat{\epsilon}-|\hat{\mu}|}$ and is denoted by the black dot on the positive real axis. Dashed curve is the unit circle. Orange curve is the contour (counterclockwise) over which the holonomy is integrated over in this phase. Red line is the branch cut for the logarithm appearing in the bosonic free energies.

The first term in (3.32) is the integral over the arc on the unit circle; the second term is the discontinuity across the branch cut of the logarithm. Performing the integral over x and then the integral over ϵ in the second term, we find that (3.32) reduces to

$$\frac{1}{2\pi|\lambda_B|} \int_{c_B}^{|\mu|} d\epsilon \int_{-b}^b d\alpha \log(1 - e^{-\hat{\epsilon} + \hat{\mu} + i\alpha}) - \beta\Theta(|\mu| - c_B) \frac{(|\mu| - c_B)^2(|\mu| + 2c_B)}{6|\lambda_B|}. \quad (3.33)$$

It follows from this analysis that whenever the boson is in the upper cap phase (i.e. when (3.26) holds), the analytic continuation of the log-determinant (3.23) is

$$\begin{aligned} & \beta\mathcal{V}_2 F_{B,\text{det}}(c_B, \mu) \\ &= \frac{N_B\mathcal{V}_2}{2\pi} \left[-\frac{\beta c_B^3}{3} + \int_{c_B}^{\infty} d\epsilon \int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) \left(\log(1 - e^{-\hat{\epsilon} - \hat{\mu} - i\alpha}) + \log(1 - e^{-\hat{\epsilon} + \hat{\mu} + i\alpha}) \right) \right. \\ & \quad \left. - \beta\Theta(|\mu| - c_B) \frac{(|\mu| - c_B)^2(|\mu| + 2c_B)}{6|\lambda_B|} \right]. \end{aligned} \quad (3.34)$$

The log-determinants in the last line of the critical boson free energy (3.14) and the regular boson free energy (3.17) have to be modified to include the additional term in (3.34) when $c_B < |\mu|$. We conjecture that the above procedure gives the correct expressions for the off-shell free energies of the critical boson and regular boson theories for all values of $\hat{\mu}$ in upper cap bosonic phases.

$$\begin{aligned} & F_{CB}(c_B, \tilde{\mathcal{S}}) \\ &= \frac{N_B}{6\pi\beta^3} \left[\frac{3}{2} \hat{c}_B^2 \hat{m}_B^{\text{cri}} - 4\lambda_B^2 \left(\tilde{\mathcal{S}} - \frac{1}{2} \hat{m}_B^{\text{cri}} \right)^3 + 6|\lambda_B| \hat{c}_B \left(\tilde{\mathcal{S}} - \frac{1}{2} \hat{m}_B^{\text{cri}} \right)^2 \right. \\ & \quad - \hat{c}_B^3 + 3 \int_{\hat{c}_B}^{\infty} d\hat{\epsilon} \hat{\epsilon} \int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) \left(\log(1 - e^{-\hat{\epsilon} + \hat{\mu} + i\alpha}) + \log(1 - e^{-\hat{\epsilon} - \hat{\mu} - i\alpha}) \right) \\ & \quad \left. - \Theta(|\mu| - c_B) \frac{(|\hat{\mu}| - \hat{c}_B)^2(|\hat{\mu}| + 2\hat{c}_B)}{2|\lambda_B|} \right]. \end{aligned} \quad (3.35)$$

$$\begin{aligned}
& F_{\text{RB}}(c_B, \sigma_B, \tilde{\mathcal{S}}) \\
&= \frac{N_B}{6\pi\beta^3} \left[-3\hat{c}_B^2 \hat{\sigma}_B + \lambda_B^2 \hat{\sigma}_B^3 - 4\lambda_B^2 (\tilde{\mathcal{S}} + \hat{\sigma}_B)^3 + 6|\lambda_B| \hat{c}_B (\tilde{\mathcal{S}} + \hat{\sigma}_B)^2 \right. \\
&\quad + 3(\hat{m}_B^2 \hat{\sigma}_B + 2\lambda_B \hat{b}_4 \hat{\sigma}_B^2 + (x_6 + 1)\lambda_B^2 \hat{\sigma}_B^3) \\
&\quad - \hat{c}_B^3 + 3 \int_{\hat{c}_B}^{\infty} d\hat{e} \, \hat{e} \int_{-\pi}^{\pi} d\alpha \, \rho_B(\alpha) \left(\log(1 - e^{-\hat{e} + \hat{\mu} + i\alpha}) + \log(1 - e^{-\hat{e} - \hat{\mu} - i\alpha}) \right) \\
&\quad \left. - \Theta(|\mu| - c_B) \frac{(|\hat{\mu}| - \hat{c}_B)^2 (|\hat{\mu}| + 2\hat{c}_B)}{2|\lambda_B|} \right]. \tag{3.36}
\end{aligned}$$

Note that the new term in (3.34) is proportional to $\frac{1}{\lambda_B}$ and hence is singular in the weak coupling limit $\lambda_B \rightarrow 0$. As we will see below, this term will play a key role in curing the runaway singularity of Bose condensation observed in the free bosonic theories.

3.4.3 Analytic continuation and off-shell free energies in general phases

Though the formulae of section 3.4.2 are all we will need later in this paper, in this subsection we take the opportunity to present the (conjecturally) correct expressions for the off-shell free energy for both the bosonic and the fermionic theories with $|\mu| > c_B$ and $|\mu| > c_F$ in more general phases than the bosonic upper cap phases and dual fermionic theory lower gap phases dealt with in the previous subsubsection.

When the boson is in an upper cap phase, the eigenvalue distribution ρ_B was constant in a finite interval around $\alpha = 0$ (see equation (3.26); its analytic continuation to the interior of the unit circle where the hairpin part of the contour was present (see figure 7(b)) was trivially the same constant. However, when the bosonic theory is not in an upper cap phase, this analytic continuation has to be done more carefully. For a general eigenvalue distribution ρ_B , it is still true [15] that there is a sufficiently small interval, say $|\alpha| < c$, in which $\rho_B(\alpha)$ is an analytic function. Similarly, the dual fermionic eigenvalue distribution ρ_F is also analytic in the interval $|\alpha| \geq \pi - c$. Within these intervals, these eigenvalue distributions may be expanded into Fourier series as

$$\begin{aligned}
\rho_B(\alpha) &= a_0 + \sum_{m=1}^{\infty} a_m \left(e^{im\alpha} + e^{-im\alpha} \right), \quad |\alpha| < c, \\
\rho_F(\alpha) &= b_0 + \sum_{m=1}^{\infty} b_m \left(e^{im\alpha} + e^{-im\alpha} \right), \quad |\alpha| > \pi - c. \tag{3.37}
\end{aligned}$$

We emphasize that the right hand sides of (3.37) — which are well-defined and analytic everywhere on the unit circle — are guaranteed to agree with the actual eigenvalue distributions only in the restricted intervals described above, and not necessarily on the whole unit circle.

We expect that the coefficients a_n and b_n in (3.37) decay with n at least as fast as ε^n for some number $\varepsilon < 1$ since the series are convergent on the unit circle $z = e^{i\alpha}$. It is then easy to see that the right hand sides can be analytically continued at least in the annulus

$$\varepsilon < |z| < \frac{1}{\varepsilon}, \tag{3.38}$$

Figure 8. Bosons in non-upper cap phases: the branch point is at $z = +e^{\hat{\epsilon}-|\hat{\mu}|}$ and is denoted by the black dot on the +ve real axis. The orange curve is the contour (counterclockwise) over which the holonomy is integrated over in this phase. The red line is the branch cut for the logarithm appearing in the bosonic free energies.

with the simple replacement $e^{i\alpha} \rightarrow z$. That is, consider the functions $\tilde{\rho}_B(z)$ and $\tilde{\rho}_F(z)$ defined by

$$\tilde{\rho}_B(z) = a_0 + \sum_m a_m \left(z^m + \frac{1}{z^m} \right), \quad \tilde{\rho}_F(z) = b_0 + \sum_m b_m \left(z^m + \frac{1}{z^m} \right). \quad (3.39)$$

These functions clearly agree with $\rho_B(\alpha)$ and $\rho_F(\alpha)$ in the restricted intervals shown in (3.37). Hence, $\tilde{\rho}_B(z)$ and $\tilde{\rho}_F(z)$ are the correct analytic continuations of the eigenvalue distributions ρ_B and ρ_F , at least in the strips

$$\text{Bosons : } \varepsilon < |z| < \frac{1}{\varepsilon}, \quad |\text{Arg}(z)| < c, \quad \text{Fermions : } \varepsilon < |z| < \frac{1}{\varepsilon}, \quad |\text{Arg}(z)| < \pi - c. \quad (3.40)$$

Of course, using the Bose-Fermi duality map of the holonomy distributions (3.5) we can relate the a_n and b_n coefficients in (3.37) which can then be plugged back into the analytic continuations $\tilde{\rho}_B$ and $\tilde{\rho}_F$ to get the duality relation

$$|\lambda_B| \tilde{\rho}_B(z) + |\lambda_F| \tilde{\rho}_F(-z) = \frac{1}{2\pi}. \quad (3.41)$$

With all these definitions in place, we can now imitate the analysis of the previous subsection to get expressions for the log-determinants that are analytic for all μ by deforming the contours of integration to go around the branch cuts as shown in figures 8 and 9.

$$\begin{aligned} & \beta \mathcal{V}_2 F_{B,\text{det}}(c_B, \mu) \\ &= \frac{N_B \mathcal{V}_2}{2\pi} \left[-\frac{\beta c_B^3}{3} + \int_{c_B}^{\infty} d\epsilon \int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) \left(\log \left(1 - e^{-\hat{\epsilon}-|\hat{\mu}|-i\alpha} \right) + \log \left(1 - e^{-\hat{\epsilon}+|\hat{\mu}+i\alpha} \right) \right) \right. \\ & \quad \left. - 2\pi \Theta(|\mu| - c_B) \int_{c_B}^{|\mu|} d\epsilon \int_{e^{\hat{\epsilon}-|\hat{\mu}|}}^1 dx \frac{\tilde{\rho}_B(x)}{x} \right], \end{aligned} \quad (3.42)$$

$$\begin{aligned} & \beta \mathcal{V}_2 F_{F,\text{det}}(c_F, \mu) \\ &= -\frac{N_F \mathcal{V}_2}{2\pi} \left[-\frac{\beta c_F^3}{3} + \int_{c_F}^{\infty} d\epsilon \int_{-\pi}^{\pi} d\alpha \rho_F(\alpha) \left(\log \left(1 + e^{-\hat{\epsilon}-|\hat{\mu}|-i\alpha} \right) + \log \left(1 + e^{-\hat{\epsilon}+|\hat{\mu}+i\alpha} \right) \right) \right. \\ & \quad \left. - 2\pi \Theta(|\mu| - c_F) \int_{c_F}^{|\mu|} d\epsilon \int_{e^{\hat{\epsilon}-|\hat{\mu}|}}^1 dx \frac{\tilde{\rho}_F(-x)}{x} \right]. \end{aligned} \quad (3.43)$$

Figure 9. Fermions in non-lower gap phases: the branch point is at $z = -e^{\hat{\epsilon}-|\hat{\mu}|}$ and is denoted by the black dot on the -ve real axis. The purple curve is the contour (counterclockwise) over which the holonomy is integrated over in this phase. The green line is the branch cut for the logarithm appearing in the fermionic free energies.

Finally, incorporating the modifications to the log-determinants into the off-shell free energies given in section 3.2, we get

$$\begin{aligned}
 & F_{\text{RF}}(c_F, \tilde{\mathcal{C}}) \\
 &= \frac{N_F}{6\pi\beta^3} \left[-8\lambda_F^2 \tilde{\mathcal{C}}^3 - 3\tilde{\mathcal{C}} \left(\hat{c}_F^2 - (2\lambda_F \tilde{\mathcal{C}} + \hat{m}_F)^2 \right) - 6\lambda_F \hat{m}_F \tilde{\mathcal{C}}^2 \right. \\
 &\quad + \hat{c}_F^3 - 3 \int_{\hat{c}_F}^{\infty} d\hat{\epsilon} \hat{\epsilon} \int_{-\pi}^{\pi} d\alpha \rho_F(\alpha) \left(\log(1 + e^{-\hat{\epsilon}-\hat{\mu}-i\alpha}) + \log(1 + e^{-\hat{\epsilon}+\hat{\mu}+i\alpha}) \right) \\
 &\quad \left. + 6\pi\Theta(|\mu| - c_F) \int_{\hat{c}_F}^{|\hat{\mu}|} d\hat{\epsilon} \hat{\epsilon} \int_{e^{\hat{\epsilon}-\hat{\mu}}}^1 dx \frac{\tilde{\rho}_F(-x)}{x} \right]. \quad (3.44)
 \end{aligned}$$

$$\begin{aligned}
 & F_{\text{CF}}(c_F, \zeta_F, \tilde{\mathcal{C}}) \\
 &= \frac{N_F}{6\pi\beta^3} \left[-8\lambda_F^2 \tilde{\mathcal{C}}^3 - 3\tilde{\mathcal{C}} \left(\hat{c}_F^2 - \left(2\lambda_F \tilde{\mathcal{C}} - \frac{4\pi\hat{\zeta}_F}{\kappa_F} \right)^2 \right) + 6\lambda_F \tilde{\mathcal{C}}^2 \left(\frac{4\pi\hat{\zeta}_F}{\kappa_F} \right) \right. \\
 &\quad + 3 \left(\frac{\hat{y}_2^2}{2\lambda_F} \frac{4\pi\hat{\zeta}_F}{\kappa_F} - \frac{\hat{y}_4}{2\lambda_F} \left(\frac{4\pi\hat{\zeta}_F}{\kappa_F} \right)^2 + \frac{x_6^F}{8\lambda_F} \left(\frac{4\pi\hat{\zeta}_F}{\kappa_F} \right)^3 \right) \\
 &\quad + \hat{c}_F^3 - 3 \int_{\hat{c}_F}^{\infty} d\hat{\epsilon} \hat{\epsilon} \int_{-\pi}^{\pi} d\alpha \rho_F(\alpha) \left(\log(1 + e^{-\hat{\epsilon}-\hat{\mu}-i\alpha}) + \log(1 + e^{-\hat{\epsilon}+\hat{\mu}+i\alpha}) \right) \\
 &\quad \left. + 6\pi\Theta(|\mu| - c_F) \int_{\hat{c}_F}^{|\hat{\mu}|} d\hat{\epsilon} \hat{\epsilon} \int_{e^{\hat{\epsilon}-\hat{\mu}}}^1 dx \frac{\tilde{\rho}_F(-x)}{x} \right]. \quad (3.45)
 \end{aligned}$$

$$\begin{aligned}
 & F_{\text{CB}}(c_B, \tilde{\mathcal{S}}) \\
 &= \frac{N_B}{6\pi\beta^3} \left[\frac{3}{2} \hat{c}_B^2 \hat{m}_B^{\text{cri}} - 4\lambda_B^2 \left(\tilde{\mathcal{S}} - \frac{1}{2} \hat{m}_B^{\text{cri}} \right)^3 + 6|\lambda_B| \hat{c}_B \left(\tilde{\mathcal{S}} - \frac{1}{2} \hat{m}_B^{\text{cri}} \right)^2 \right. \\
 &\quad - \hat{c}_B^3 + 3 \int_{\hat{c}_B}^{\infty} d\hat{\epsilon} \hat{\epsilon} \int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) \left(\log(1 - e^{-\hat{\epsilon}+\hat{\mu}+i\alpha}) + \log(1 - e^{-\hat{\epsilon}-\hat{\mu}-i\alpha}) \right) \\
 &\quad \left. - 6\pi\Theta(|\mu| - c_B) \int_{\hat{c}_B}^{|\hat{\mu}|} d\hat{\epsilon} \hat{\epsilon} \int_{e^{\hat{\epsilon}-\hat{\mu}}}^1 dx \frac{\tilde{\rho}_B(x)}{x} \right]. \quad (3.46)
 \end{aligned}$$

$$\begin{aligned}
 & F_{\text{RB}}(c_B, \sigma_B, \tilde{\mathcal{S}}) \\
 &= \frac{N_B}{6\pi\beta^3} \left[-3\hat{c}_B^2 \hat{\sigma}_B + \lambda_B^2 \hat{\sigma}_B^3 - 4\lambda_B^2 (\tilde{\mathcal{S}} + \hat{\sigma}_B)^3 + 6|\lambda_B| \hat{c}_B (\tilde{\mathcal{S}} + \hat{\sigma}_B)^2 \right. \\
 &\quad + 3(\hat{m}_B^2 \hat{\sigma}_B + 2\lambda_B \hat{b}_4 \hat{\sigma}_B^2 + (x_6 + 1)\lambda_B^2 \hat{\sigma}_B^3) \\
 &\quad - \hat{c}_B^3 + 3 \int_{\hat{c}_B}^{\infty} d\hat{e} \hat{e} \int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) (\log(1 - e^{-\hat{e} + \hat{\mu} + i\alpha}) + \log(1 - e^{-\hat{e} - \hat{\mu} - i\alpha})) \\
 &\quad \left. - 6\pi\Theta(|\mu| - c_B) \int_{\hat{c}_B}^{|\hat{\mu}|} d\hat{e} \hat{e} \int_{e^{\hat{e} - \hat{\mu}}}^1 dx \frac{\tilde{\rho}_B(x)}{x} \right]. \quad (3.47)
 \end{aligned}$$

The expressions (3.35) and (3.36) for the bosonic free energies in an upper cap phase can be obtained by setting $\tilde{\rho}_B(x) = 1/2\pi|\lambda_B|$ in the above bosonic formulae (3.46) and (3.47). Similarly, upon setting $\tilde{\rho}_F(-x) = 0$, the fermionic free energies (3.44) and (3.45) respectively reduce to the original expressions (3.7) and (3.12) which are the correct free energies for a lower gap phase. We conjecture that the above expressions (3.44)–(3.47) are the correct expressions for the off-shell free energies of the bosonic and fermionic theories for all values of μ .

3.4.4 Gap equations

For chemical potentials less than the gap c_B or c_F , the equations that followed from varying the off-shell free energy were presented for the regular fermions in (3.8), for the critical fermions in (3.13), for the critical bosons in (3.15) and for the regular bosons in (3.18). In general, these equations of motion are modified when the chemical potential is greater than c_B or c_F since the off-shell free energies are themselves modified in this case.

The modified equations of motion can be obtained by varying the modified off-shell free energies presented in the previous subsection. Alternatively, note that the only explicit occurrences of μ in the equations are in the quantities $\mathcal{S}(c_B, \mu)$ and $\mathcal{C}(c_F, \mu)$ which are the same factors which appear in the log-determinants (3.23) and (3.24) under the ϵ -integral. Thus, the analytic continuation procedure for the equations is identical to the one used for the off-shell free energies. The final result is that the equations continue to hold for $|\mu| > c_B$ and $|\mu| > c_F$ provided we make the replacements

$$\begin{aligned}
 \mathcal{C}(c_F, \mu) &\rightarrow \mathcal{C}(c_F, \mu) - \pi\Theta(|\mu| - c_F) \int_{e^{\hat{c}_F - |\hat{\mu}|}}^1 dx \frac{\tilde{\rho}_F(-x)}{x}, \\
 \mathcal{S}(c_B, \mu) &\rightarrow \mathcal{S}(c_B, \mu) - \pi\Theta(|\mu| - c_B) \int_{e^{\hat{c}_B - |\hat{\mu}|}}^1 dx \frac{\tilde{\rho}_B(x)}{x}, \quad (3.48)
 \end{aligned}$$

where, recall that $\tilde{\rho}_F(x)$ and $\tilde{\rho}_B(x)$ are the analytic continuation of the holonomy distributions $\rho_F(\alpha)$ and $\rho_B(\alpha)$ to a small neighbourhood of the unit circle. In an upper cap phase for the bosons (3.26), we can use the second equation in (3.48) after setting $\tilde{\rho}_B(x) = 1/2\pi|\lambda_B|$, i.e. make the replacement

$$\mathcal{S}(c_B, \mu) \rightarrow \mathcal{S}(c_B, \mu) - \Theta(|\mu| - c_B) \frac{|\hat{\mu}| - \hat{c}_B}{2|\lambda_B|}. \quad (3.49)$$

The equation (3.49) was already guessed in [65] (see equation A.5 of [65]).

4 Quasiparticle occupation numbers

The usual thermodynamical formulae inform us that the charge of an ensemble is obtained by varying the free energy $\mathcal{F}$ w.r.t. the chemical potential μ while keeping β and the spatial volume $\mathcal{V}_2$ fixed. In equations,

$$Q = - \left(\frac{\partial \mathcal{F}}{\partial \mu} \right)_\beta . \quad (4.1)$$

The quantity $\mathcal{F}$ that appears in (4.1) is the thermodynamic free energy related to the partition function $\mathcal{Z}$ as

$$\mathcal{Z} = \exp(-\beta \mathcal{F}) . \quad (4.2)$$

The free energy $\mathcal{F}$ is typically a rather complicated function. As we have discussed in the previous section, in the large N limit, it is obtained by extremizing the effective action functional $s[\rho]$ w.r.t. the holonomy distribution ρ . This action functional consists of two pieces. The first is the contribution $v[\rho]$ that we discuss in this paper due to integrating out all modes except the gauge field holonomy ρ and the second is the contribution due to the Chern-Simons modified path integral measure (discussed extensively in [15]). Note, of course, that this measure is independent of μ .

We have seen in section 3.1, the final (usually complicated) answer for $v[\rho]$ can be obtained by extremizing a rather simple ‘off-shell’ free energy function $F(\varphi_{\text{aux}}; \rho]$ with respect to the auxiliary variables denoted collectively as φ_{aux} . Thus the full free energy $\mathcal{F}$ is obtained by extremizing the effective action functional $\mathcal{S}(\varphi_{\text{aux}}; \rho]$ which consists of the ‘off-shell’ free energy $F(\varphi_{\text{aux}}; \rho]$ and the contribution due to the measure which depends only on ρ but not the other auxiliary variables (and is also independent of μ).

Let the full set of variables (auxiliary variables as well as holonomies) be denoted by $\{\varphi_i\}$. The on-shell values φ_i^* of the variables φ_i are determined by extremizing the effective action $\mathcal{S}(\varphi_{\text{aux}}; \rho]$ w.r.t. each of the φ_i :

$$\left. \frac{\partial \mathcal{S}}{\partial \varphi_i} \right|_{\varphi_i = \varphi_i^*} = 0 . \quad (4.3)$$

The final free energy is obtained by evaluating $\mathcal{S}(\varphi_{\text{aux}}; \rho]$ at these extremum values.

In general, the extremum values of φ_i are functions of μ . Hence, the final free energy $\mathcal{F}$ has two sources of dependence on μ : (a) the explicit dependence on μ coming from the off-shell free energy F (b) the implicit dependence coming from the fact that the extremum values φ_i^* are functions of μ . In other words,

$$\mathcal{F}(\mu) \equiv \mathcal{S}(\varphi_i^*(\mu), \mu) . \quad (4.4)$$

where $\mathcal{S}(\varphi_i^*(\mu), \mu)$ is the effective action evaluated at one of its extrema. It follows that

$$-Q = \left(\frac{\partial \mathcal{F}(\mu)}{\partial \mu} \right)_\beta = \left. \frac{\partial \mathcal{S}}{\partial \mu} \right|_{\varphi_i = \varphi_i^*} + \sum_i \left. \frac{\partial \mathcal{S}}{\partial \varphi_i} \right|_{\varphi_i = \varphi_i^*} \frac{\partial \varphi_i^*}{\partial \mu} . \quad (4.5)$$

Now, we use the equations of motion (4.3) to set the second term above to zero. Note that the only explicit dependence on μ in the effective action $\mathcal{S}$ is through the off-shell free

energy F . Thus, the formula for the charge (4.5) simplifies to

$$Q = - \frac{\partial F}{\partial \mu} \Big|_{\varphi_i = \varphi_i^*} . \quad (4.6)$$

Moreover the quantity F in (4.6) can further be replaced by (the corrected form of) F_{det} as this is the only part of the off-shell free energy that depends on μ .

4.1 Chemical potential smaller than quasiparticle thermal mass

When the chemical potential is smaller than the thermal masses of the excitations, the off-shell free energy formulae are given in section 3.2. Recall that the only explicit dependence on μ of these formulae is in the log-determinants in the last line of these expressions. Evaluating the derivative w.r.t. μ (see (4.6)) we find that for both the regular (3.7) and critical fermionic (3.12) theories³¹

$$Q_F(c_F, \mu) = \frac{\mathcal{V}_2 N_F}{2\pi} \int_{c_F}^{\infty} d\epsilon \, \epsilon \int_{-\pi}^{\pi} d\alpha \, \rho_F(\alpha) \left(\frac{1}{e^{\beta(\epsilon-\mu)-i\alpha} + 1} - \frac{1}{e^{\beta(\epsilon+\mu)+i\alpha} + 1} \right) . \quad (4.7)$$

Now recall that the number dn of single-particle states of any fixed colour is given by

$$dn = \frac{\mathcal{V}_2 d^2 k}{(2\pi)^2} = \frac{\mathcal{V}_2 k dk}{2\pi} = \frac{\mathcal{V}_2 \epsilon d\epsilon}{2\pi} . \quad (4.8)$$

(where we have used the usual relativistic dispersion relation $\epsilon^2 = c_F^2 + \vec{k}^2$). It follows from (4.8) that (4.7) can be rewritten as

$$Q_F(c_F, \mu) = N_F \int dn \int_{-\pi}^{\pi} d\alpha \, \rho_F(\alpha) \left(\frac{1}{e^{\beta(\epsilon-\mu)-i\alpha} + 1} - \frac{1}{e^{\beta(\epsilon+\mu)+i\alpha} + 1} \right) . \quad (4.9)$$

The expression (4.9) has an obvious interpretation. At least as far as the charge goes, the system under study consists of a collection of effectively non-interacting particles where the occupation number of a single-particle state at energy ϵ and charge q is given by $\bar{n}_F(\epsilon, q)$ where

$$\bar{n}_F(\epsilon, q) \equiv \int_{-\pi}^{\pi} d\alpha \, \rho_F(\alpha) \frac{1}{e^{\beta(\epsilon-q\mu)-iq\alpha} + 1} . \quad (4.10)$$

where $q = 1$ for fundamental excitations and $q = -1$ for antifundamental excitations. The two terms under the integral in (4.9) then represent the occupation numbers of the fundamental quasiparticle (charge +1) and antifundamental (charge -1) states respectively.

Note: the formula (4.10) and the interpretation proposed here, first appeared in the paper [32] in the form

$$\begin{aligned} n_F(\vec{p}) &= \left(\bar{n}_F(\epsilon_{\vec{p}}, q = +1) - \bar{n}_F(\epsilon_{\vec{p}}, q = -1) \right) , \\ &= \frac{1}{2} \int_{-\pi}^{\pi} d\alpha \rho_F(\alpha) \left[\tanh \frac{\beta(\epsilon_{\vec{p}} + \mu) + i\alpha}{2} - \tanh \frac{\beta(\epsilon_{\vec{p}} - \mu) - i\alpha}{2} \right] , \end{aligned} \quad (4.11)$$

---

³¹We restore the two-dimensional volume factor $\mathcal{V}_2$ and explicit factors of β in the free energy expressions in this subsection.

where the notation $n_F(\vec{p})$ stands for the sum of the charges of a fundamental quasiparticle state and its antifundamental counterpart with momentum $\vec{p}$ and energy $\epsilon_{\vec{p}}$. It is easy to rewrite the formula (4.7) for the charge above to match the one in [32] using the trigonometric identity

$$\frac{1}{e^{x-y} + 1} - \frac{1}{e^{x+y} + 1} = \frac{1}{2} \left(\tanh \frac{x+y}{2} - \tanh \frac{x-y}{2} \right). \quad (4.12)$$

The analysis of the bosonic theories proceeds along completely identical lines. Differentiating the free energies for the critical boson (3.14) and the regular boson (3.17) w.r.t μ , we find a formula for the net charge that admits an interpretation in terms of effectively non-interacting particle states of energy ϵ and charge q with average occupation number given by the formula

$$\bar{n}_B(\epsilon, q) \equiv \int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) \frac{1}{e^{\beta(\epsilon - q\mu) - iq\alpha} - 1}. \quad (4.13)$$

The charge of the bosonic ensemble is then given by

$$Q_B(c_B, \mu) = N_B \int dn \int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) \left(\frac{1}{e^{\beta(\epsilon - \mu) - i\alpha} - 1} - \frac{1}{e^{\beta(\epsilon + \mu) + i\alpha} - 1} \right), \quad (4.14)$$

or, in terms of the occupation number $\bar{n}(\epsilon, q)$,

$$Q_B(c_B, \mu) = N_B \int dn \left(\bar{n}_B(\epsilon, q = +1) - \bar{n}_B(\epsilon, q = -1) \right). \quad (4.15)$$

4.2 Chemical potential larger than quasiparticle mass

The generalisation of the fermionic (4.10) and bosonic (4.13) occupation numbers to $|\mu| \geq c_F$ and $|\mu| \geq c_B$ may be obtained in one of two ways. First, we could recompute the charge of the ensemble by differentiating the corrected off-shell free energy formulae (3.44), (3.45) for the fermions and (3.46), (3.47) for the bosons w.r.t. μ and then reinterpret the result in terms of a modified occupation number for each single-particle state. Alternatively (and more simply as well as more informatively), we could simply analytically continue (4.10) and (4.13) to values of μ larger than c_F and c_B . We briefly outline this second method (which proceeds along the lines of section 3.4.3).

The formula (4.10) for the fermionic occupation number is analytic at all values of μ such that $\text{sgn}(q\mu) = -1$. On the other hand this expression has a potential non-analyticity³² at $q\mu = \epsilon$ (recall $q = \pm 1$). In order to see this non-analyticity, we rewrite the integral over holonomies in (4.10) as a contour integral over $z = e^{i\alpha}$:

$$\bar{n}_F(\epsilon, q) \equiv \oint_C \frac{dz}{iz} \tilde{\rho}_F(z) \frac{1}{e^{\beta(\epsilon - q\mu)} z^{-1} + 1}. \quad (4.16)$$

The contour C in (4.16) is initially the unit circle, but with this definition (4.16) is not analytic in μ at $q\mu = \epsilon$. We obtain the analytic continuation of (4.16) by deforming the contour C to the contour C' (see figure 10(a)). C' is chosen so that it continues to circle the origin, but cuts the negative x axis at a point closer to the origin than $-e^{\beta(\epsilon - q\mu)}$. The formula (4.16), with C replaced by C' , is thus the correct analytic continuation of (4.16) to

Figure 10. Fermions: pole is at $z = -e^{\beta(\epsilon-q\mu)}$ and is denoted by the black dot on the -ve real axis. The big circle is the unit circle. The purple curve is the contour C' (counterclockwise) over which the holonomy is integrated over.

values of $\hat{\mu}$ s.t. $q\mu > \epsilon$. The contour C' can be deformed to the sum of two contours without changing the value of the contour integral: the first contour is the original contour C while the second is a second smaller loop that runs clockwise around the pole at $z = -e^{\beta(\epsilon-q\mu)}$ (see figure 10(b)). The contribution of the small loop encircling the pole is

$$-2\pi\Theta(q\mu - \epsilon) \tilde{\rho}_F(-e^{\beta(\epsilon-q\mu)}), \quad (4.17)$$

where the Heaviside Θ -function is to indicate that this additional contribution is present only when $q\mu > \epsilon$. It follows that the analytic continuation of (4.10) — and hence the true fermionic occupation number of a quasiparticle state of energy ϵ — is obtained by subtracting the contribution due to the pole:

$$\bar{n}_F(\epsilon, q) = \int_{-\pi}^{\pi} d\alpha \left(\rho_F(\alpha) \frac{1}{e^{\beta(\epsilon-q\mu)-iq\alpha} + 1} \right) - 2\pi\Theta(q\mu - \epsilon) \tilde{\rho}_F(-e^{\beta(\epsilon-q\mu)}). \quad (4.18)$$

In the special case of a lower gap fermionic phase, $\tilde{\rho}_F$ vanishes, and so, (4.18) reduces to (4.10).

The analysis of the bosonic case proceeds along similar lines. The occupation number (4.13) can be rewritten as a contour integral as

$$\bar{n}_B(\epsilon, q) = \int_C \frac{dz}{iz} \tilde{\rho}_B(z) \frac{1}{e^{\beta(\epsilon-q\mu)} z^{-1} - 1}. \quad (4.19)$$

Again, the analytic continuation of (4.13) (and hence the correct formula for the occupation number of a bosonic quasiparticle state at energy ϵ) is obtained by subtracting the contribution due to the pole at $z = +e^{\beta(\epsilon-q\mu)}$. The calculation can be followed pictorially in figure 11. The modified occupation number is then

$$\bar{n}_B(\epsilon, q) = \int_{-\pi}^{\pi} d\alpha \left(\rho_B(\alpha) \frac{1}{e^{\hat{\epsilon}-q\hat{\mu}-i\alpha q} - 1} \right) + 2\pi\Theta(q\mu - \epsilon) \tilde{\rho}_B(e^{-|\hat{\mu}|+\hat{\epsilon}}). \quad (4.20)$$

³²However, the non-analyticity in the integrand of the holonomy integral in the expression of occupation number is a pole, whereas the analogous non-analyticity in the case of the free energy is a branch cut.

Figure 11. Bosons: the pole is at $z = +e^{\beta(\epsilon - q\mu)}$ and is denoted by the black dot on the +ve real axis. Radius of the big circle is unity. The orange curve is the contour C' (counterclockwise) over which the holonomy is integrated over.

In the particular case of a bosonic upper cap phase, (4.20) simplifies to

$$\bar{n}_B(\epsilon, q) = \int_{-\pi}^{\pi} d\alpha \left(\rho_B(\alpha) \frac{1}{e^{\hat{\epsilon} - q\hat{\mu} - iq\alpha} - 1} \right) + \frac{1}{|\lambda_B|} \Theta(q\mu - \epsilon). \quad (4.21)$$

5 The thermodynamic limit

In the rest of the paper we specialise to the study of the quasi-fermionic theories and quasi-bosonic theories in the thermodynamic limit defined by taking the limit $V \rightarrow \infty$ (recall that all through this paper we choose the volume of our S^2 to be given by $\mathcal{V}_2 = NV$ and N is taken to ∞ first before any other limit). In this limit, the dominant phase of the holonomy eigenvalues is one in which the bosonic and fermionic eigenvalue distributions take the ‘tabletop’ form which we reproduce below:

$$\rho_F(\alpha) = \frac{1}{2\pi|\lambda_F|} \Theta(\pi|\lambda_F| - |\alpha|), \quad \rho_B(\alpha) = \frac{1}{2\pi|\lambda_B|} \Theta(\pi|\lambda_B| - |\alpha|). \quad (5.1)$$

We will further be interested in the so-called ‘thermodynamic, zero-temperature limit’ obtained by first taking the limit $V \rightarrow \infty$ and only then taking the limit $T \rightarrow 0$. The importance of the above order of limits is the following. When we take the limit $V \rightarrow \infty$ the eigenvalue distribution is frozen to (5.1). The zero-temperature limit is then taken with the eigenvalue distribution frozen at (5.1).

In this section, we first give expressions for the occupation numbers in terms of explicit well-known functions in the thermodynamic limit. We then consider the ‘thermodynamic, zero-temperature limit’ of those expressions. We also provide the expressions for the off-shell free energies in the limit since they will be used extensively in the subsequent sections.

5.1 Occupation numbers in the infinite volume limit

In this subsection we specialise the formulae for occupation numbers presented in section 4 to the $V \rightarrow \infty$ limit where the eigenvalue distributions take the universal ‘tabletop’ profile (5.1).

It turns out to be convenient to phrase the discussion in terms of the variable x defined as

$$x = e^{\beta(\epsilon - q\mu)} . \quad (5.2)$$

We shall encounter the multivalued function $\tan^{-1} \zeta$ in the formulas of this section. This is defined unambiguously by choosing the following branch:

$$-\frac{\pi}{2} < \tan^{-1} \zeta < \frac{\pi}{2} . \quad (5.3)$$

Note that with this definition of $\tan^{-1}(\zeta)$, we have $\tan^{-1}(-\zeta) = -\tan^{-1}(\zeta)$.

5.1.1 The fermionic occupation number

The fermionic occupation number is given by equation (4.10) where $\rho_F(\alpha)$ takes the universal tabletop form

$$\rho_F(\alpha) = \frac{1}{2\pi|\lambda_F|} \Theta(\pi|\lambda_F| - |\alpha|) . \quad (5.4)$$

In terms of the variable x defined in (5.2), the formula for the occupation number of a quasiparticle state at energy ϵ and of charge $q = \pm 1$ simplifies to

$$\begin{aligned} \bar{n}_F(\epsilon, q) &= \frac{1}{2\pi|\lambda_F|} \int_{-\pi|\lambda_F|}^{\pi|\lambda_F|} d\alpha \frac{1}{xe^{-i\alpha} + 1} , \\ &= \frac{1}{2\pi|\lambda_F|} \int_{-\pi|\lambda_F|}^{\pi|\lambda_F|} d\alpha \frac{(x \cos \alpha + 1)}{1 + x^2 + 2x \cos \alpha} , \end{aligned} \quad (5.5)$$

(an equivalent formula first appeared in [32]). In going from the first to the second line of (5.5) we have used the fact that the integration limit is symmetric about $\alpha = 0$. The integral on the r.h.s. of the second line of (5.5) can be performed explicitly and we find the relatively simple analytic expression

$$\bar{n}_F(\epsilon, q) = \frac{1}{2} - \frac{1}{\pi|\lambda_F|} \tan^{-1} \left(\frac{x-1}{x+1} \tan \frac{\pi|\lambda_F|}{2} \right) , \quad (5.6)$$

where $x = e^{\beta(\epsilon - q\mu)}$ is defined in (5.2). As indicated in (5.3), the multivalued function $\tan^{-1}(\zeta)$ that appears in (5.6) is unambiguously defined by choosing the branch $-\frac{\pi}{2} < \tan^{-1}(\zeta) < \frac{\pi}{2}$. Substituting (5.2) in (5.6) it follows that

$$\bar{n}_F(\epsilon, q) = \frac{1}{2} - \frac{1}{\pi|\lambda_F|} \tan^{-1} \left(\frac{e^{\beta(\epsilon - q\mu)} - 1}{e^{\beta(\epsilon - q\mu)} + 1} \tan \frac{\pi|\lambda_F|}{2} \right) . \quad (5.7)$$

(5.6) and (5.7) are the main results of this subsection. In the rest of this subsection we will study various limits of (5.7).

Zero-temperature limit. In the zero temperature limit (i.e., $\beta \rightarrow \infty$), we have

$$x = e^{\beta(\epsilon - q\mu)} = \begin{cases} 0, & \epsilon < q\mu , \\ 1, & \epsilon = q\mu , \\ \infty, & \epsilon > q\mu . \end{cases} \quad (5.8)$$

It follows immediately from (5.7), that in this limit,

$$\bar{n}_F(\epsilon, q) = \begin{cases} 1, & \epsilon < q\mu, \\ \frac{1}{2}, & \epsilon = q\mu, \\ 0, & \epsilon > q\mu. \end{cases} \quad (5.9)$$

Note in particular that all states with energy smaller than $q\mu$ are occupied with unit occupancy. In the zero temperature limit, in other words, the fermions form a free Fermi sea at every value of λ_F (this fact was already noted in [32]).

Small- λ_F expansion. Working at arbitrary temperature and μ , (5.7) is easily expanded in a power series expansion in λ_F about zero; we find

$$\bar{n}_F(\epsilon, q) = \frac{1}{e^{\beta(\epsilon - q\mu)} + 1} - \frac{\pi^2 \lambda_F^2}{6} \frac{e^{\beta(\epsilon - q\mu)} (e^{\beta(\epsilon - q\mu)} - 1)}{(e^{\beta(\epsilon - q\mu)} + 1)^3} + \mathcal{O}(\lambda_F^4), \quad (5.10)$$

(as the expression (5.7) is an even function of λ_F , the power series on the r.h.s. of (5.10) has only even powers in λ_F). The expansion (5.10) applies at all values of the energy ϵ .

Note that the first term on the r.h.s. of (5.10) is, of course, the familiar formula of Fermi Statistics for the occupation number of a single particle state. Subsequent terms in this expansion represents universal ‘anyonic’ corrections to this familiar formula.

Small- x expansion. When the energy of our quasiparticle state lies well below $q\mu$ in units of the temperature, then $x = e^{\beta(\epsilon - q\mu)} \ll 1$. Equations (5.6) or (5.7) takes the following form

$$\begin{aligned} \bar{n}_F(\epsilon, q) &= 1 - \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n\pi\lambda_F)}{n\pi\lambda_F} x^n, \\ &= 1 - \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n\pi\lambda_F)}{n\pi\lambda_F} e^{n\beta(\epsilon - q\mu)}. \end{aligned} \quad (5.11)$$

The first term on the r.h.s. of (5.11) is unity indicating that at leading order in this limit, all states are occupied with unit occupancy. The first correction to unity on the r.h.s. of (5.11), $-\left(\frac{\sin \pi \lambda_F}{\pi \lambda_F}\right)x$, gives the exponentially suppressed probability of this state becoming unfilled (equivalently, of a ‘quasihole’ being created). The intriguing factor of $\frac{\sin \pi \lambda_F}{\pi \lambda_F}$ (which is unity in the free limit $\lambda_F \rightarrow 0$) indicates that the formation at a hole is less likely at nonzero λ_F than in the free limit. It would be very interesting to understand why this is the case directly from the Schrodinger equation in an appropriate non-relativistic limit.

Large- x expansion. On the other hand when the energy of our quasiparticle state lies well above $q\mu$ in units of the temperature, then $x = e^{\beta(\epsilon - q\mu)} \gg 1$. (5.6) or (5.7) takes the form (5.7) takes the following form

$$\begin{aligned} \bar{n}_F(\epsilon, q) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n\pi\lambda_F)}{n\pi\lambda_F} x^{-n}, \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n\pi\lambda_F)}{n\pi\lambda_F} e^{-n\beta(\epsilon - q\mu)}. \end{aligned} \quad (5.12)$$

(5.12) tells us that the occupation number of a state with such energies is exponentially suppressed. Once again note the interesting suppression of this probability by the factor of $\frac{\sin \pi \lambda_F}{\pi \lambda_F}$, which once again calls out for a non-relativistic explanation.

5.1.2 The bosonic occupation number

Recall the formula for the bosonic occupation number in an upper cap phase (4.21):

$$\bar{n}_B(\epsilon, q) = \int_{-\pi}^{\pi} \rho_B(\alpha) d\alpha \frac{1}{e^{\beta(\epsilon - q\mu) - i\alpha} - 1} + \frac{1}{|\lambda_B|} \Theta(q\mu - \epsilon). \quad (5.13)$$

Using the tabletop form of the bosonic eigenvalue distribution (5.1), we find the following formula for the bosonic occupation number in the thermodynamic limit:

$$\begin{aligned} \bar{n}_B(\epsilon, q) &= \frac{1}{2\pi|\lambda_B|} \int_{-\pi|\lambda_B|}^{\pi|\lambda_B|} d\alpha \frac{1}{xe^{-i\alpha} - 1} + \frac{1}{|\lambda_B|} \Theta(q\mu - \epsilon), \\ &= \frac{1}{2\pi|\lambda_B|} \int_{-\pi|\lambda_B|}^{\pi|\lambda_B|} d\alpha \frac{(x \cos \alpha - 1)}{1 + x^2 - 2x \cos \alpha} + \frac{1}{|\lambda_B|} \Theta(q\mu - \epsilon). \end{aligned} \quad (5.14)$$

As in the case of fermions, the integral on the r.h.s. of the second line of (5.14) can be exactly evaluated and we find³³

$$\bar{n}_B(\epsilon, q) = -\frac{1}{2} + \frac{1}{\pi|\lambda_B|} \tan^{-1} \left(\frac{x+1}{x-1} \tan \frac{\pi|\lambda_B|}{2} \right) + \frac{1}{|\lambda_B|} \Theta(q\mu - \epsilon). \quad (5.15)$$

Using the fact that

$$\begin{aligned} x &> 1 && \text{for } \epsilon > q\mu, \\ x &= 1 && \text{for } \epsilon = q\mu, \\ x &< 1 && \text{for } \epsilon < q\mu, \end{aligned} \quad (5.16)$$

the Θ -function appearing in (5.15) can also be rewritten as $\Theta(1 - x)$. So, (5.15) can be rewritten as

$$\bar{n}_B(\epsilon, q) = -\frac{1}{2} + \frac{1}{\pi|\lambda_B|} \tan^{-1} \left(\frac{x+1}{x-1} \tan \frac{\pi|\lambda_B|}{2} \right) + \frac{1}{|\lambda_B|} \Theta(1 - x). \quad (5.17)$$

The $\tan^{-1}$ and the Θ terms on the r.h.s. above discontinuous at $x = 1$. However these discontinuities cancel each other and the final result is perfectly smooth at $x = 1$. In order to see this we proceed as follows. Using

$$\Theta(1 - x) = \frac{1}{2} - \frac{1}{2} \text{sgn}(x - 1), \quad (5.18)$$

and the identity

$$\tan^{-1}(a) + \tan^{-1} \frac{1}{a} = \text{sgn}(a) \frac{\pi}{2}, \quad (5.19)$$

we see that (5.17) can be rewritten as

$$\bar{n}_B(\epsilon, q) = \frac{1 - |\lambda_B|}{2|\lambda_B|} - \frac{1}{\pi|\lambda_B|} \tan^{-1} \left(\frac{x-1}{x+1} \cot \left(\frac{\pi|\lambda_B|}{2} \right) \right), \quad (5.20)$$

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³³As indicated in (5.3), the function $\tan^{-1}(\zeta)$ used in that equation is defined so that $-\frac{\pi}{2} < \tan^{-1}(\zeta) < \frac{\pi}{2}$.

keeping in mind the choice of branch (5.3) of the $\tan^{-1}$ function. Inserting the definition of x (5.2), we obtain

$$\bar{n}_B(\epsilon, q) = \frac{1 - |\lambda_B|}{2|\lambda_B|} - \frac{1}{\pi|\lambda_B|} \tan^{-1} \left(\frac{e^{\beta(\epsilon - q\mu)} - 1}{e^{\beta(\epsilon - q\mu)} + 1} \cot \frac{\pi|\lambda_B|}{2} \right). \quad (5.21)$$

(5.20) and (5.21) are the main results of this subsection. In the rest of this subsection we study the various limits of these formulae.

Zero-temperature limit. Using (5.8), it follows that in the limit $\beta \rightarrow \infty$.

$$\bar{n}_B(\epsilon, q) = \begin{cases} \frac{1 - |\lambda_B|}{|\lambda_B|}, & \epsilon < q\mu, \\ \frac{1 - |\lambda_B|}{2|\lambda_B|}, & \epsilon = q\mu, \\ 0, & \epsilon > q\mu. \end{cases} \quad (5.22)$$

It follows from (5.22), in particular, that all states with energies less than $q\mu$ (states that would have been infinitely occupied by a Bose condensate in the free limit) have an occupation number given by

$$\frac{1 - |\lambda_B|}{|\lambda_B|}.$$

Note that the additional piece in the occupation number of the boson (the second term in the formula for $\bar{n}_B$ in (5.13)) — plays a key role in the formula (5.22). Without this term we would have found $\bar{n}_B(\epsilon, q) = -1$; this is the nonsensical ‘divergent’ occupation number of a theory of free bosons. The second term in (5.13) ‘regulates’ this free result into a physically sensible answer, albeit one that does diverge in the limit $\lambda_B \rightarrow 0$.

Small- λ_B expansion. Working at arbitrary values of x , (5.20) is easily expanded in a power series in λ_B . We find

$$\bar{n}_B(\epsilon, q) = \frac{1}{|\lambda_B|} \Theta(q\mu - \epsilon) + \frac{1}{e^{\beta(\epsilon - q\mu)} - 1} - \frac{\pi^2 \lambda_B^2}{6} \frac{e^{\beta(\epsilon - q\mu)} (e^{\beta(\epsilon - q\mu)} + 1)}{(e^{\beta(\epsilon - q\mu)} - 1)^3} + \mathcal{O}(\lambda_B^4). \quad (5.23)$$

The above equation can be rewritten more explicitly as

$$\bar{n}_B(\epsilon, q) = \begin{cases} \frac{1}{e^{\beta(\epsilon - q\mu)} - 1} - \frac{\pi^2 \lambda_B^2}{6} \frac{e^{\beta(\epsilon - q\mu)} (e^{\beta(\epsilon - q\mu)} + 1)}{(e^{\beta(\epsilon - q\mu)} - 1)^3} + \mathcal{O}(\lambda_B^4), & \epsilon > q\mu, \\ \frac{1}{|\lambda_B|} + \frac{1}{e^{\beta(\epsilon - q\mu)} - 1} - \frac{\pi^2 \lambda_B^2}{6} \frac{e^{\beta(\epsilon - q\mu)} (e^{\beta(\epsilon - q\mu)} + 1)}{(e^{\beta(\epsilon - q\mu)} - 1)^3} + \mathcal{O}(\lambda_B^4), & \epsilon < q\mu. \end{cases} \quad (5.24)$$

The first term on the first line of (5.24) is, of course, the familiar expression for the occupation number of a single particle state with $\epsilon > q\mu$ using the formulae of Bose

statistics; the remaining terms on this line are the universal³⁴ ‘anyonic’ corrections to this formula.

The second line of (5.24) deals with the more interesting case of a state whose energy is less than $q\mu$. Any such state would be infinitely occupied in a free Bose ensemble. The first terms on the r.h.s. of (5.24) tell us that finite λ_B effects regulate this infinite occupation to a finite occupation number given by $\frac{1}{|\lambda_B|}$. The remaining subleading terms in this expansion give us the (temperature dependent) corrections to this result.

The corrections to the leading order results in (5.24) are of order $\frac{\lambda_B}{\beta(\epsilon - q\mu)}$; at generic values of ϵ , these corrections are small if λ_B is small. No matter how small λ_B is, however, these corrections are non-negligible when

$$\beta(\epsilon - q\mu) \sim \lambda_B . \quad (5.25)$$

At small λ_B , it follows that $\bar{n}_B(\epsilon, q)$ takes the value $\frac{1}{|\lambda_B|}$ (for $\epsilon < q\mu$) and $\frac{1}{e^{\beta(\epsilon - q\mu)} - 1}$ for $\epsilon > q\mu$. The transition between these two values takes place in the range of energies (5.25). In fact at leading order in small λ_B , $\bar{n}_B(\epsilon, q)$ is everywhere (i.e. at all energies) well approximated by

$$\bar{n}_B(\epsilon, q) \approx \frac{1}{2|\lambda_B|} - \frac{1}{\pi|\lambda_B|} \tan^{-1} \left(\frac{e^{\beta(\epsilon - q\mu)} - 1}{\pi|\lambda_B|} \right) \quad (5.26)$$

Note that the r.h.s. of tends to zero when $\beta(\epsilon - q\mu) \gg |\lambda_B|$ but tends to $\frac{1}{|\lambda_B|}$ for $\beta(\epsilon - q\mu) \ll -|\lambda_B|$.

Small- x expansion. As in the case of fermions, x is small for states whose energies lie much below $q\mu$ in units of the temperature. The occupation number for such states is well approximated by the small- x expansion of (5.20), given by

$$\begin{aligned} \bar{n}_B(\epsilon, q) &= \frac{1 - |\lambda_B|}{|\lambda_B|} - \sum_{n=1}^{\infty} \frac{\sin(n\pi\lambda_B)}{n\pi\lambda_B} x^n , \\ &= \frac{1 - |\lambda_B|}{|\lambda_B|} - \sum_{n=1}^{\infty} \frac{\sin(n\pi\lambda_B)}{n\pi\lambda_B} e^{n\beta(\epsilon - q\mu)} . \end{aligned} \quad (5.27)$$

We see that at leading order in this expansion, all states are occupied with occupation number $\frac{1 - |\lambda_B|}{|\lambda_B|}$. The probability of a quasihole appearing in this condensate is, at leading order, proportional to $\frac{\sin \pi \lambda_B}{\pi \lambda_B} x$. As in the case of fermions it would be interesting to find a physical interpretation of the prefactor $\frac{\sin \pi \lambda_B}{\pi \lambda_B}$.

Large- x expansion. x is large for states whose energies lie much above $q\mu$ in units of the temperature. The occupation number of such states is well approximated expanding (5.20)

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³⁴These corrections are universal in the following sense. They apply both to the critical and the regular boson theory, and all values of the UV parameters of these theories. Note, however, that these formulae only apply in the infinite volume limit where the holonomy eigenvalue distribution takes the tabletop form.

in a power series in $1/x$. This expansion is given by

$$\begin{aligned}\bar{n}_B(\epsilon, q) &= \sum_{n=1}^{\infty} \frac{\sin(n\pi\lambda_B)}{n\pi\lambda_B} \frac{1}{x^n}, \\ &= \sum_{n=1}^{\infty} \frac{\sin(n\pi\lambda_B)}{n\pi\lambda_B} e^{-n\beta(\epsilon-q\mu)}.\end{aligned}\quad (5.28)$$

We see that the probability of occupation for any of these states is given by $\frac{\sin \pi \lambda_B}{\pi \lambda_B} x^{-1}$ at leading order. The factor of x^{-1} is the familiar Boltzmann suppression of a high energy state. The universal prefactor is more mysterious and requires an explanation.

5.1.3 Duality of explicit forms of occupation number

In this brief subsection we directly verify that the explicit expressions (5.7) and (5.21) are dual to each other. Using (5.7) and (5.21) we see that

$$\begin{aligned}|\lambda_B|\bar{n}_B - |\lambda_F|\bar{n}_F &= \frac{1 - |\lambda_B| - |\lambda_F|}{2} \\ &- \frac{1}{\pi} \left(\tan^{-1} \left(\frac{e^{\beta(\epsilon-q\mu)} - 1}{e^{\beta(\epsilon-q\mu)} + 1} \cot \left(\frac{\pi|\lambda_B|}{2} \right) \right) - \tan^{-1} \left(\frac{e^{\beta(\epsilon-q\mu)} - 1}{e^{\beta(\epsilon-q\mu)} + 1} \tan \left(\frac{\pi|\lambda_F|}{2} \right) \right) \right).\end{aligned}\quad (5.29)$$

The duality map of the coupling constants³⁵ $|\lambda_F| = 1 - |\lambda_B|$ implies that $\tan \left(\frac{\pi|\lambda_F|}{2} \right) = \cot \left(\frac{\pi|\lambda_B|}{2} \right)$. It then follows that the right hand side of (5.29) vanishes. We conclude that

$$|\lambda_B|\bar{n}_B - |\lambda_F|\bar{n}_F = 0. \quad (5.30)$$

as required by duality.

5.2 Free energies at zero temperature

The expressions for the free energies in section 3.4.2 simplify in the limit $\beta \rightarrow \infty$ in that the holonomy integrals become almost trivial. Consider the holonomy integral in the bosonic case:

$$3 \int_{\hat{c}_B}^{\infty} d\hat{\epsilon} \hat{\epsilon} \int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) \left(\log \left(1 - e^{-\hat{\epsilon} + |\hat{\mu}| + i\alpha} \right) + \log \left(1 - e^{-\hat{\epsilon} + |\hat{\mu}| + i\alpha} \right) \right), \quad (5.31)$$

In the limit $\beta \rightarrow \infty$ with ϵ and μ fixed, the quantities $\hat{\epsilon}$ and $\hat{\mu}$ go to ∞ . It follows that the second logarithm above in (5.31) simply vanishes. The first logarithm also vanishes when ϵ is larger than $|\mu|$. On the other hand, when $\epsilon < \mu$, we have

$$\log(1 - e^{-\hat{\epsilon} + \hat{\mu} + i\alpha}) = \hat{\mu} - \hat{\epsilon} + i\theta_B(\alpha), \quad (5.32)$$

where

$$\theta_B(\alpha) = \alpha - \pi \quad \text{when} \quad \alpha > 0 \quad \text{and} \quad \theta_B(\alpha) = \alpha + \pi \quad \text{when} \quad \alpha < 0. \quad (5.33)$$

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³⁵Throughout this section and in other sections, (unless mentioned explicitly) we have implicitly used the fact that $c_F = c_B$, so that the single particle energy spectra, $\epsilon = \sqrt{p^2 + c_F^2} = \sqrt{p^2 + c_B^2}$, are identical for both fermions and bosons.

Using the fact that $\rho_B(\alpha)$ is an even function of α , it follows that

$$\int_{-\pi}^{\pi} d\alpha \rho_B(\alpha) \theta_B(\alpha) = 0 .$$

The holonomy integral (5.31) thus simplifies to

$$3 \Theta(|\mu| - c_B) \int_{\hat{c}_B}^{|\hat{\mu}|} d\hat{\epsilon} \hat{\epsilon} (\hat{\mu} - \hat{\epsilon}) = \Theta(|\mu| - c_B) \frac{(|\hat{\mu}| - \hat{c}_B)^2 (|\hat{\mu}| + 2\hat{c}_B)}{2} . \quad (5.34)$$

The same calculation works for the fermionic case except for one simplification: there is no additional choice of branch for the logarithm in (5.33) since there is no additional minus sign in the logarithm $\log(1 + e^{-\hat{\epsilon} + |\hat{\mu}| + i\alpha})$ for the fermions.

Thus, in the zero-temperature limit, the off-shell free energies for the bosonic and fermionic theories become

$$F_{\text{RF}}(c_F, \tilde{C}) = \frac{N_F}{6\pi\beta^3} \left[-8\lambda_F^2 \tilde{C}^3 - 3\tilde{C} \left(\hat{c}_F^2 - (2\lambda_F \tilde{C} + \hat{m}_F)^2 \right) - 6\lambda_F \hat{m}_F \tilde{C}^2 + \hat{c}_F^3 - \frac{1}{2} \Theta(|\mu| - c_F) (|\hat{\mu}| - \hat{c}_F)^2 (|\hat{\mu}| + 2\hat{c}_F) \right] . \quad (5.35)$$

$$F_{\text{CF}}(c_F, \zeta_F, \tilde{C}) = \frac{N_F}{6\pi\beta^3} \left[-8\lambda_F^2 \tilde{C}^3 - 3\tilde{C} \left(\hat{c}_F^2 - \left(2\lambda_F \tilde{C} - \frac{4\pi\hat{\zeta}_F}{\kappa_F} \right)^2 \right) + 6\lambda_F \tilde{C}^2 \left(\frac{4\pi\hat{\zeta}_F}{\kappa_F} \right) + 3 \left(\frac{\hat{y}_2^2}{2\lambda_F} \frac{4\pi\hat{\zeta}_F}{\kappa_F} - \frac{\hat{y}_4}{2\lambda_F} \left(\frac{4\pi\hat{\zeta}_F}{\kappa_F} \right)^2 + \frac{x_6^F}{8\lambda_F} \left(\frac{4\pi\hat{\zeta}_F}{\kappa_F} \right)^3 \right) + \hat{c}_F^3 - \frac{1}{2} \Theta(|\mu| - c_F) (|\hat{\mu}| - \hat{c}_F)^2 (|\hat{\mu}| + 2\hat{c}_F) \right] . \quad (5.36)$$

$$F_{\text{CB}}(c_B, \tilde{S}) = \frac{N_B}{6\pi\beta^3} \left[\frac{3}{2} \hat{c}_B^2 \hat{m}_B^{\text{cri}} - 4\lambda_B^2 \left(\tilde{S} - \frac{1}{2} \hat{m}_B^{\text{cri}} \right)^3 + 6|\lambda_B| \hat{c}_B \left(\tilde{S} - \frac{1}{2} \hat{m}_B^{\text{cri}} \right)^2 - \hat{c}_B^3 + \frac{1}{2} \left(1 - \frac{1}{|\lambda_B|} \right) \Theta(|\mu| - c_B) (|\hat{\mu}| - \hat{c}_B)^2 (|\hat{\mu}| + 2\hat{c}_B) \right] . \quad (5.37)$$

$$F_{\text{RB}}(c_B, \sigma_B, \tilde{S}) = \frac{N_B}{6\pi\beta^3} \left[-3\hat{c}_B^2 \hat{\sigma}_B + \lambda_B^2 \hat{\sigma}_B^3 - 4\lambda_B^2 (\tilde{S} + \hat{\sigma}_B)^3 + 6|\lambda_B| \hat{c}_B (\tilde{S} + \hat{\sigma}_B)^2 + 3(\hat{m}_B^2 \hat{\sigma}_B + 2\lambda_B \hat{b}_4 \hat{\sigma}_B^2 + (x_6 + 1) \lambda_B^2 \hat{\sigma}_B^3) - \hat{c}_B^3 + \frac{1}{2} \left(1 - \frac{1}{|\lambda_B|} \right) \Theta(|\mu| - c_B) (|\hat{\mu}| - \hat{c}_B)^2 (|\hat{\mu}| + 2\hat{c}_B) \right] . \quad (5.38)$$

6 Quasi-fermionic theories

In this section, we analyse in detail the thermodynamical zero-temperature phase structure of the quasi-fermionic theories i.e. the regular fermion and the critical boson. We mainly work in the bosonic theory and appeal to Bose-Fermi duality in order obtain the corresponding results for the fermions.

The off-shell free energy of the critical boson theory in the zero-temperature limit was obtained in (5.37) and is reproduced here for convenience:

$$F_{\text{CB}}(c_B, \tilde{\mathcal{S}}) = \frac{N_B}{6\pi\beta^3} \left[\frac{3}{2} \hat{c}_B^2 \hat{m}_B^{\text{cri}} - 4\lambda_B^2 \left(\tilde{\mathcal{S}} - \frac{1}{2} \hat{m}_B^{\text{cri}} \right)^3 + 6|\lambda_B| \hat{c}_B \left(\tilde{\mathcal{S}} - \frac{1}{2} \hat{m}_B^{\text{cri}} \right)^2 - \hat{c}_B^3 + \frac{1}{2} \left(1 - \frac{1}{|\lambda_B|} \right) \Theta(|\mu| - c_B) (|\hat{\mu}| - \hat{c}_B)^2 (|\hat{\mu}| + 2\hat{c}_B) \right]. \quad (6.1)$$

The equation of motion that follows by varying (6.1) w.r.t. $\tilde{\mathcal{S}}$ is

$$\frac{c_B}{|\lambda_B|} \left(\beta^{-1} \tilde{\mathcal{S}} - \frac{1}{2} m_B^{\text{cri}} \right) - \left(\beta^{-1} \tilde{\mathcal{S}} - \frac{1}{2} m_B^{\text{cri}} \right)^2 = 0, \quad (6.2)$$

and the equation of motion from varying c_B is

$$\begin{cases} \frac{c_B}{|\lambda_B|} \left(\frac{c_B}{2} - \frac{1}{2} m_B^{\text{cri}} \right) - \left(\beta^{-1} \tilde{\mathcal{S}} - \frac{1}{2} m_B^{\text{cri}} \right)^2 = 0 & c_B > |\mu|, \\ \frac{c_B}{|\lambda_B|} \left(\frac{|\mu|}{2} - \frac{1}{2|\lambda_B|} (|\mu| - c_B) - \frac{1}{2} m_B^{\text{cri}} \right) - \left(\beta^{-1} \tilde{\mathcal{S}} - \frac{1}{2} m_B^{\text{cri}} \right)^2 = 0 & c_B < |\mu|. \end{cases} \quad (6.3)$$

Comparing (6.2) and (6.3), we see that $\tilde{\mathcal{S}}$ is given on-shell by

$$\beta^{-1} \tilde{\mathcal{S}} = \begin{cases} \frac{c_B}{2} & c_B > |\mu|, \\ \frac{|\mu|}{2} - \frac{1}{2|\lambda_B|} (|\mu| - c_B) & c_B < |\mu|. \end{cases} \quad (6.4)$$

Plugging this back into the off-shell free energy (6.1), we get, for $c_B < |\mu|$,

$$F_{\text{condensed}} = \frac{N_B}{12\pi} \left[\lambda_B^2 (m_B^{\text{cri}})^3 + 3|\lambda_B| (1 - |\lambda_B|) (m_B^{\text{cri}})^2 |\mu| + 3(1 - |\lambda_B|)^2 m_B^{\text{cri}} |\mu|^2 - (2 - |\lambda_B|) (1 - |\lambda_B|) |\mu|^3 \right], \quad (6.5)$$

and for $c_B > |\mu|$,

$$F_{\text{uncondensed}} = \frac{N_B}{12\pi} \left[\lambda_B^2 (m_B^{\text{cri}})^3 + 3|\lambda_B| (1 - |\lambda_B|) (m_B^{\text{cri}})^2 c_B + 3(1 - |\lambda_B|)^2 m_B^{\text{cri}} c_B^2 - (2 - |\lambda_B|) (1 - |\lambda_B|) c_B^3 \right]. \quad (6.6)$$

Plugging the on-shell value of $\tilde{\mathcal{S}}$ into the equation of motion for c_B (6.3), we get

$$\begin{cases} \left(\frac{2 - |\lambda_B|}{|\lambda_B|} c_B + m_B^{\text{cri}} \right) (c_B - m_B^{\text{cri}}) = 0 & c_B > |\mu|, \\ c_B^2 = |\lambda_B|^2 \left(\left(1 - \frac{1}{|\lambda_B|} \right) |\mu| - m_B^{\text{cri}} \right)^2 & c_B < |\mu|. \end{cases} \quad (6.7)$$

In the uncondensed case (corresponding to $c_B > |\mu|$), there are two different solutions to the equation of motion for c_B . These two saddle points correspond to the uncondensed

Higgsed and uncondensed unHiggsed phases of the free energy:

$$\begin{aligned} \text{uncondensed, unHiggsed : } c_B &= m_B^{\text{cri}}, \\ \text{uncondensed, Higgsed : } c_B &= -\frac{|\lambda_B|}{2 - |\lambda_B|} m_B^{\text{cri}}. \end{aligned} \quad (6.8)$$

These phases occur for the following ranges of m_B^{cri} that we obtain from the restriction $c_B > |\mu|$:

$$\begin{aligned} \text{uncondensed, unHiggsed : } m_B^{\text{cri}} &> |\mu|, \\ \text{uncondensed, Higgsed : } m_B^{\text{cri}} &< -\frac{2 - |\lambda_B|}{|\lambda_B|} |\mu|. \end{aligned} \quad (6.9)$$

On the contrary, in the condensed phase (corresponding to $c_B < |\mu|$), there seems to be a unique solution:

$$\text{condensed : } c_B = |\lambda_B| \left| \left(1 - \frac{1}{|\lambda_B|} \right) |\mu| - m_B^{\text{cri}} \right|. \quad (6.10)$$

This phase exists for the following range of m_B^{cri} obtained from the restriction $c_B < |\mu|$:

$$-\frac{2 - |\lambda_B|}{|\lambda_B|} |\mu| < m_B^{\text{cri}} < |\mu|. \quad (6.11)$$

There are two possibilities of reaching the condensed phase from an uncondensed phase by raising the chemical potential above c_B : either from the uncondensed unHiggsed side or from the uncondensed Higgsed side. Recall that the excitations in the unHiggsed phase are scalars while those in the Higgsed phase are W -bosons. Depending on the side we approach from, the description of the condensed phase is either in terms of condensed W bosons or condensed scalars. This can be seen in the formula for c_B (6.10) which is non-analytic in the coupling constant m_B^{cri} due to the absolute value on the right hand side. The sign of the quantity inside the modulus on the right hand side distinguishes between these two cases:

$$\begin{aligned} \text{condensed scalars : } m_B^{\text{cri}} &> -\frac{1 - |\lambda_B|}{|\lambda_B|} |\mu|, \\ \text{condensed } W \text{ bosons : } m_B^{\text{cri}} &< -\frac{1 - |\lambda_B|}{|\lambda_B|} |\mu|. \end{aligned} \quad (6.12)$$

Though the description of the condensed branch in terms of condensed scalars or condensed W bosons seems manifestly different, it turns out that there is no real distinction between these two descriptions. This can be seen from the free energy in the condensed phase (6.5) which does not depend on the variable c_B at all! Thus, the non-analyticity in the equation of motion for c_B does not affect the free energy in the condensed phase.

Plugging the solutions for each of the uncondensed phases into (6.6), we find the free energy in the unHiggsed and Higgsed phases to be

$$\begin{aligned} F_{\text{unHiggsed}} &= \frac{N_B}{12\pi} (m_B^{\text{cri}})^3, \\ F_{\text{Higgsed}} &= \frac{N_B}{12\pi} \frac{|\lambda_B|^2}{(2 - |\lambda_B|)^2} (m_B^{\text{cri}})^3. \end{aligned} \quad (6.13)$$

Figure 12. Phase diagram of critical boson theory as a function of m_B^{cri} at fixed μ . At $s_1 = |\mu|$ and $s_2 = -|\mu|(\frac{2-|\lambda_B|}{|\lambda_B|})$ the system undergoes a sharp second order phase transition from a condensed to an uncondensed phase. The black dot inside the condensed phase marks the point where the description of the physically unique condensed phase changes from one of a phase of condensed scalars to a phase of condensed W bosons; this occurs at $m_B^{\text{cri}} = -|\mu|(\frac{1-|\lambda_B|}{|\lambda_B|})$.

6.1 Phase diagram

Note that the conditions (6.9) and (6.11) on m_B^{cri} are mutually non-overlapping. Together they give a single cover of the range of masses

$$-\infty < m_B^{\text{cri}} < \infty.$$

At any given value of $|\mu|$ and m_B^{cri} , there exists exactly one solution of the gap equations. At every value of parameters, therefore, the system has a unique phase. It follows that the phase diagram of the system is as depicted in figure 12. As depicted in figure 12, our system undergoes a sharp second order phase transition at the edges of the range (6.11). Within the range (6.11), our system is in a compressible phase, i.e. the charge is a non-trivial function of the chemical potential. Outside the range (6.11), the system is in an incompressible phase: its charge is independent of μ (and vanishes in particular).

Within the compressible phase, the charge (per unit volume) as a function of μ is given by the formula

$$Q = \frac{N_B}{4\pi} \text{sgn}(\mu) \left(\frac{1-|\lambda_B|}{|\lambda_B|} \right) (\mu^2 - c_B^2). \quad (6.14)$$

Note that, the charge vanishes at both the edges of the condensed range for m_B^{cri} (6.11), but that its first derivative is non-vanishing at these boundary points. This discontinuity in the second derivative of the free energy demonstrates that our system undergoes a second order phase transition at the edges of the range (6.11). The number of single particle states (per unit volume) in the condensed phase, upto the energy $|\mu|$ is given by

$$\int \frac{d^2k}{(2\pi)^2} = \int_{c_B}^{|\mu|} \frac{\epsilon d\epsilon}{2\pi} = \frac{1}{4\pi} (\mu^2 - c_B^2). \quad (6.15)$$

Thus, we see that (6.14) can be thought of as the product of the number of single particle states (6.15) and the occupation number $\bar{n}$ listed in (1.7).

Plugging in the expression for c_B as a function of μ into this expression ((6.10)) we find an explicit expression for the charge as a function of UV parameters and the chemical potential: μ -dependence, the charge can be written as

$$Q = \frac{N_B}{4\pi} \text{sgn}(\mu) \Theta(|\mu| - c_B) (1 - |\lambda_B|) \left((2 - |\lambda_B|) \mu^2 - 2(1 - |\lambda_B|) m_B^{\text{cri}} |\mu| - |\lambda_B| (m_B^{\text{cri}})^2 \right). \quad (6.16)$$

It is clear from (6.16) that the charge Q is a quadratic function of μ . The compressibility of this phase is given by

$$\frac{\partial Q}{\partial \mu} = \frac{N_B}{2\pi} (1 - |\lambda_B|) \Theta(|\mu| - c_B) \left((2 - |\lambda_B|) |\mu| - (1 - |\lambda_B|) m_B^{\text{cri}} \right). \quad (6.17)$$

Note that $\frac{\partial Q}{\partial \mu}$ is always positive in the condensed phase and vanishes at the edges.

7 Quasi-bosonic theories

In this section, we consider the detailed phase structure of the quasi-bosonic theories viz. the critical fermion and the regular boson. We explicitly work in the bosonic setting and use the Bose-Fermi duality map discussed previously to obtain the analogous results for the critical fermion. The off-shell free energy for the regular boson at zero temperature is given by (5.38) and is reproduced below:

$$\begin{aligned} F_{\text{RB}}(c_B, \sigma_B, \tilde{\mathcal{S}}) = \frac{N_B}{6\pi} & \left[-3c_B^2 \sigma_B + 4\lambda_B^2 \sigma_B^3 + 3 \left(m_B^2 \sigma_B + 2\lambda_B b_4 \sigma_B^2 + x_6^B \lambda_B^2 \sigma_B^3 \right) \right. \\ & - 4\lambda_B^2 (\beta^{-1} \tilde{\mathcal{S}} + \sigma_B)^3 + 6|\lambda_B| c_B (\beta^{-1} \tilde{\mathcal{S}} + \sigma_B)^2 \\ & \left. - c_B^3 + \frac{1}{2} \Theta(|\mu| - c_B) \left(1 - \frac{1}{|\lambda_B|} \right) (|\mu| - c_B)^2 (|\mu| + 2c_B) \right]. \quad (7.1) \end{aligned}$$

The variation w.r.t. c_B and $\tilde{\mathcal{S}}$ give the equations

$$\begin{aligned} \beta \frac{\partial}{\partial \tilde{\mathcal{S}}} : \quad & c_B (\beta^{-1} \tilde{\mathcal{S}} + \sigma_B) - |\lambda_B| (\beta^{-1} \tilde{\mathcal{S}} + \sigma_B)^2 = 0, \\ \frac{\partial}{\partial c_B} : \quad & \begin{cases} c_B \left(\frac{c_B}{2} + \sigma_B \right) - |\lambda_B| (\beta^{-1} \tilde{\mathcal{S}} + \sigma_B)^2 = 0 & c_B > |\mu|, \\ c_B \left(\frac{c_B}{2} + \frac{1}{2} \left(1 - \frac{1}{|\lambda_B|} \right) (|\mu| - c_B) + \sigma_B \right) - |\lambda_B| (\beta^{-1} \tilde{\mathcal{S}} + \sigma_B)^2 = 0 & c_B < |\mu|. \end{cases} \end{aligned} \quad (7.2)$$

Consistency of the two equations of motion gives the following on-shell condition for $\tilde{\mathcal{S}}$ in the zero temperature limit:

$$\beta^{-1} \tilde{\mathcal{S}} = \begin{cases} \frac{c_B}{2} & c_B > |\mu|, \\ \frac{|\mu|}{2} - \frac{1}{2|\lambda_B|} (|\mu| - c_B) & c_B < |\mu|. \end{cases} \quad (7.3)$$

We substitute the above expression for $\tilde{\mathcal{S}}$ back into the off-shell free energy (7.1) and the c_B equation of motion in the second line of (7.2) respectively. The off-shell free energy

becomes

$$F_{\text{RB}}(c_B, \sigma_B) = \begin{cases} \frac{N_B}{2\pi} \left[x_6^B \lambda_B^2 \sigma_B^3 + 2(\lambda_B b_4 + |\lambda_B|(1 - |\lambda_B|)c_B) \sigma_B^2 \right. \\ \quad \left. + (m_B^2 - (1 - |\lambda_B|)^2 c_B^2) \sigma_B - \frac{1}{6}(1 - |\lambda_B|)(2 - |\lambda_B|)c_B^3 \right] & c_B > |\mu|, \\ \frac{N_B}{2\pi} \left[x_6^B \lambda_B^2 \sigma_B^3 + 2(\lambda_B b_4 + |\lambda_B|(1 - |\lambda_B|)|\mu|) \sigma_B^2 \right. \\ \quad \left. + (m_B^2 - (1 - |\lambda_B|)^2 |\mu|^2) \sigma_B - \frac{1}{6}(1 - |\lambda_B|)(2 - |\lambda_B|)|\mu|^3 \right] & c_B < |\mu|, \end{cases} \quad (7.4)$$

and the equation of motion for c_B becomes

$$\frac{\partial}{\partial c_B} : \begin{cases} \left(\frac{2 - |\lambda_B|}{2|\lambda_B|} c_B - \sigma_B \right) \left(\frac{c_B}{2} + \sigma_B \right) = 0 & c_B > |\mu|, \\ \left(\frac{c_B}{2|\lambda_B|} \right)^2 - \left(\sigma_B + \frac{|\mu|}{2} \left(1 - \frac{1}{|\lambda_B|} \right) \right)^2 = 0 & c_B < |\mu|. \end{cases} \quad (7.5)$$

Each of the equations in (7.5) is quadratic in c_B . It may thus seem that the function $c_B(\sigma)$ obtained from the solution of (7.5) is multivalued with upto four branches. In actuality this is not the case as we now explain. Solutions to the equation in the second of (7.5) are valid only when

$$0 \leq c_B \leq |\mu|. \quad (7.6)$$

It follows that the second equation in (7.5) has at most one valid solution - the so called condensed phase solution - given by

$$c_B = 2|\lambda_B| \left| \sigma_B + \frac{|\mu|}{2} \left(1 - \frac{1}{|\lambda_B|} \right) \right|. \quad (7.7)$$

This solution is valid if and only if σ_B satisfies

$$\text{condensed : } -\frac{|\mu|}{2} < \sigma_B < \frac{2 - |\lambda_B|}{2|\lambda_B|} |\mu|. \quad (7.8)$$

The expression for c_B in (7.7) as a function of σ_B is not analytic at $\sigma_B = -\frac{|\mu|}{2} \left(1 - \frac{1}{|\lambda_B|} \right)$. Intuitively, this non-analyticity is a consequence of the fact that the solution migrates from the condensed unHiggsed to the condensed Higgsed phase at this value of σ_B . Remarkably enough, however, the free energy in the second line of (7.4) is independent of c_B , and so the non-analyticity of c_B as a function of σ_B does not translate into a non-analyticity of the free energy as a function of σ_B .

Notice that the above non-analytic point of c_B is at $\sigma_B = 0$ when the chemical potential is zero. This was the point at which the free energy was non-analytic and there was a phase transition between the unHiggsed and Higgsed phases. The point $\sigma_B = -\frac{|\mu|}{2} \left(1 - \frac{1}{|\lambda_B|} \right)$ is the analog of such a transition when we turn on a chemical potential. The discussion in

the previous paragraph indicates that, at least in the large N limit, there is no distinction between the condensed unHiggsed and the condensed Higgsed phases; we just have one single condensed phase. The contribution of this phase to the free energy as a function of σ_B is simply the second line of (7.4). This expression for the free energy is valid when (7.8) is obeyed.

There are clearly two different branches of solutions to the first line of (7.5); these are the solutions

$$\text{unHiggsed} : c_B = -2\sigma_B, \quad \text{Higgsed} : c_B = \frac{|\lambda_B|}{2 - |\lambda_B|} 2\sigma_B. \quad (7.9)$$

These solutions are only valid when $c_B > |\mu|$ which translates to the following conditions on σ_B :

$$\text{unHiggsed} : \sigma_B < -\frac{|\mu|}{2}, \quad \text{Higgsed} : \sigma_B > \frac{2 - |\lambda_B|}{2|\lambda_B|} |\mu|. \quad (7.10)$$

Plugging the solutions (7.9) into the first line of (7.4) yields the free energy as a function of σ_B in the ranges (7.10).

Notice that the ranges of σ_B in (7.10) are non-overlapping, and are also exactly complementary to the range (7.8). It follows that the three ranges listed in (7.8) and (7.10) together cover the σ_B line in a non-overlapping manner. The transitions between the three ranges take place at the points

$$\begin{aligned} \text{condensed} \leftrightarrow \text{unHiggsed} : \quad \sigma_B &= -\frac{|\mu|}{2}, \\ \text{condensed} \leftrightarrow \text{Higgsed} : \quad \sigma_B &= \frac{2 - |\lambda_B|}{2|\lambda_B|} |\mu|. \end{aligned} \quad (7.11)$$

We may set $|\mu| = 1$ in the above expressions since it just corresponds to a choice of scale and we will do so in the intermediate steps of the analysis below. However, it will be useful in the end to restore the $|\mu|$ dependence; for this we just rescale all dimensionful quantities with appropriate powers of $|\mu|$:

$$m_B^2 \rightarrow \frac{m_B^2}{|\mu|^2}, \quad b_4 \rightarrow \frac{b_4}{|\mu|}, \quad \sigma_B \rightarrow \frac{\sigma_B}{|\mu|}, \quad F_{\text{RB}} \rightarrow \frac{F_{\text{RB}}}{|\mu|^3}. \quad (7.12)$$

In summary, the off-shell free energy as a function of the single variable σ_B is given by piecewise analytic expressions in the three ranges (7.8) and (7.10):

$$U_{\text{eff}}(\sigma_B) = \frac{N_B}{2\pi} \times \begin{cases} (x_6^B - \phi_u) \lambda_B^2 \sigma_B^3 + 2\lambda_B b_4 \sigma_B^2 + m_B^2 \sigma_B & \sigma_B < -\frac{1}{2}, \\ x_6^B \lambda_B^2 \sigma_B^3 + 2\lambda_B \left(b_4 + (\text{sgn}(\lambda_B) - \lambda_B) \right) \sigma_B^2 \\ + \left(m_B^2 - (1 - |\lambda_B|)^2 \right) \sigma_B - \frac{1}{6} (1 - |\lambda_B|) (2 - |\lambda_B|) & -\frac{1}{2} < \sigma_B < \frac{2 - |\lambda_B|}{2|\lambda_B|}, \\ (x_6^B - \phi_h) \lambda_B^2 \sigma_B^3 + 2\lambda_B b_4 \sigma_B^2 + m_B^2 \sigma_B & \sigma_B > \frac{2 - |\lambda_B|}{2|\lambda_B|}. \end{cases} \quad (7.13)$$

The quantities ϕ_h and ϕ_u are functions of $|\lambda_B|$ given by

$$\phi_h = \frac{4}{3} \left(\frac{1}{(2 - |\lambda_B|)^2} - 1 \right), \quad \phi_u = \frac{4}{3} \left(\frac{1}{\lambda_B^2} - 1 \right), \quad (7.14)$$

and satisfy, for all $|\lambda_B| \leq 1$,

$$\phi_h < 0 < \phi_u. \quad (7.15)$$

Recall that the quantity $U_{\text{eff}}(\sigma_B)$ has a clear physical interpretation. The variable change in equation (3.19) given by $\sigma_B = \frac{2\pi}{N_B} \langle \bar{\phi}\phi \rangle$ turns the free energy into the quantum effective potential for the lightest gauge invariant operator $\bar{\phi}\phi$ of our theory (see [71]). We frequently refer to $U_{\text{eff}}(\sigma_B)$ as the effective potential while keeping in mind the above fact.

Notice that $U_{\text{eff}}(\sigma_B)$ presented in (7.13) is a piecewise cubic function of σ_B . Independently in each of the three intervals in (7.13), the equation $U'_{\text{eff}}(\sigma_B) = 0$ is quadratic in σ_B . Real solutions to the quadratic equations exist only when their discriminants are positive. They first start existing on the curves along which the discriminants vanish i.e. the parabolas

$$\begin{aligned} D_u &\equiv 4\lambda_B^2 b_4^2 - 3m_B^2 \lambda_B^2 (x_6^B - \phi_u) = 0, \\ D_c &\equiv 4 \left(\lambda_B b_4 + |\lambda_B| (1 - |\lambda_B|) \right)^2 - 3\lambda_B^2 x_6^B \left(m_B^2 - (1 - |\lambda_B|)^2 \right) = 0, \\ D_h &\equiv 4\lambda_B^2 b_4^2 - 3m_B^2 \lambda_B^2 (x_6^B - \phi_h) = 0. \end{aligned} \quad (7.16)$$

The real solutions that are created are double zeroes of $U'_{\text{eff}}(\sigma_B)$ ³⁶ and are located on the σ_B line respectively at

$$\begin{aligned} D_u : \quad \sigma_B &= -\frac{2\lambda_B b_4}{3(x_6^B - \phi_u)\lambda_B^2}, \\ D_c : \quad \sigma_B &= -\frac{2\lambda_B (b_4 + (\text{sgn}(\lambda_B) - \lambda_B))}{3x_6^B \lambda_B^2}, \\ D_h : \quad \sigma_B &= -\frac{2\lambda_B b_4}{3(x_6^B - \phi_h)\lambda_B^2}. \end{aligned} \quad (7.17)$$

Crossing any of the curves listed in (7.16) signals the (dis)appearance of new extrema in the corresponding branch of the effective potential $U_{\text{eff}}(\sigma_B)$. Of course the discriminant curves (7.16) are physically relevant only when the solutions (7.17) lie within their respective domains of validity (7.8), (7.10). In other words, only certain parts of the discriminant curves (7.16) are relevant for the phase diagram.

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³⁶For a generic potential $U(\sigma) = A\sigma^3 + B\sigma^2 + C\sigma + D$, the extrema of this function are given by $U'(\sigma) = 3A\sigma^2 + 2B\sigma + C = 0$. The solutions of this quadratic equation are

$$\sigma_{\pm} = \frac{-B \pm \sqrt{B^2 - 3AC}}{3A}.$$

When the discriminant $B^2 - 3AC$ vanishes, this simplifies to $\sigma_{\pm} = -\frac{B}{3A} = -\frac{C}{B}$.

7.1 Strategy to obtain the phase diagram

In the rest of this paper, we work out the phase diagram of the theory that follows from (7.13). We split the analysis into four cases depending on the range of the marginal parameter x_6^B :

$$\begin{aligned} \text{Case A : } x_6^B < \phi_h, & \quad \text{Case C : } 0 < x_6^B < \phi_u, \\ \text{Case B : } \phi_h < x_6^B < 0, & \quad \text{Case D : } \phi_u < x_6^B. \end{aligned} \quad (7.18)$$

Looking at the potential (7.13), it is clear that the potential is unbounded below for large and positive σ_B when $x_6^B < \phi_h$ (Case A) and is unbounded below for large and negative σ_B when $x_6^B > \phi_u$ (Case D). When x_6^B is in the range $\phi_h < x_6^B < \phi_u$ corresponding to Cases B and C, the potential is bounded below for both positive and negative σ_B . Thus, the potential is stable for cases B and C while it is unstable for cases A and D. We treat each of the cases in (7.18) separately below. The stable cases are addressed in section 7.4 while the analysis for the unstable cases is pushed to the appendix and is presented in appendix D.

The phase diagram of the theory is two-dimensional and can be conveniently described by the two dimensionless ratios

$$\frac{m_B^2}{|\mu|^2} \quad \text{and} \quad \frac{\lambda_B b_4}{|\mu|}. \quad (7.19)$$

We have one phase diagram for every range of x_6^B in (7.18). In drawing these diagrams, we place the variable $\frac{\lambda_B b_4}{|\mu|}$ along the Y-axis and $\frac{m_B^2}{|\mu|^2}$ along the X-axis. The phase of the theory at any particular point on this diagram is decided by the dominant minimum of the potential $U_{\text{eff}}(\sigma_B)$ with the values of the parameters m_B^2 and $\lambda_B b_4$ fixed by the coordinates of the point on the phase diagram.

We obtain the detailed structure of the potential for each range of x_6^B at each point of the phase diagram. Our strategy is as follows. First, recall that there are two values of σ_B in the neighbourhood of which the potential (7.13) is non-analytic. These points, which we designate ℓ and r , are

$$\ell : \quad \sigma_B = -\frac{1}{2}, \quad r : \quad \sigma_B = \frac{2 - |\lambda_B|}{2|\lambda_B|}. \quad (7.20)$$

The point ℓ is the junction between the unHiggsed and the condensed branches of the potential, while the point r is the junction between the condensed and the Higgsed branches of the potential.

In order to work out the full phase diagram we study the behaviour of the potential near each of the non-analytic points above. To study the potential near a non-analytic point, say $\sigma_B = \ell$, we shift our origin in σ_B to that point. Then, we study the behaviour of the effective potential (7.13) as a function of m_B^2 and $\lambda_B b_4$ by pretending that the effective potential has only the unHiggsed and the condensed branches which meet at $\sigma_B = \ell$ i.e. ignoring the existence of the Higgsed branch that, in reality, cuts off the condensed phase at $\sigma_B = r$. This description is not expected to hold for all values of m_B^2 and $\lambda_B b_4$

but only for a small region in the phase diagram where the $\sigma_B = r$ cutoff is not important. Similarly, we study the behaviour of the potential near $\sigma_B = r$ by ignoring the existence of the unHiggsed branch of the potential which cuts off the condensed branch at $\sigma_B = \ell$.

The two exercises above furnish two ‘local patches’ of the behaviour of the effective potential on the phase diagram. We then appeal to the simplicity of the form of the effective potential to interpolate between these two patches and obtain a global picture of the behaviour of the effective potential. It is then a simple exercise to determine the dominant minimum of the potential and thus the phase structure of the theory. We now provide the expressions for the potential in the neighbourhood of the two non-analytic points in σ_B described above in (7.20).

7.2 Effective potential in the local patches around non-analytic points

7.2.1 Near ℓ : $\sigma_B = -\frac{1}{2}$

First, let $\sigma'_B = \sigma_B + \frac{1}{2}$. Note that $\sigma'_B = 0$ when $\sigma_B = \ell$. In the local patch around $\sigma'_B = 0$, we have

$$U_{\text{eff}}(\sigma'_B) = \frac{N_B}{2\pi} \times \begin{cases} (x_6^B - \phi_u)\lambda_B^2\sigma_B'^3 + 2\left(\lambda_B b_4 - \frac{3}{4}\lambda_B^2(x_6^B - \phi_u)\right)\sigma_B'^2 & \sigma'_B < 0, \\ \quad + \left(m_B^2 - 2\lambda_B b_4 + \frac{3}{4}\lambda_B^2(x_6^B - \phi_u)\right)\sigma'_B & \\ x_6^B\lambda_B^2\sigma_B'^3 + 2\left(\lambda_B b_4 - \frac{3}{4}\lambda_B^2(x_6^B - \phi_u) - (1 - |\lambda_B|)\right)\sigma_B'^2 & \sigma'_B > 0. \\ \quad + \left(m_B^2 - 2\lambda_B b_4 + \frac{3}{4}\lambda_B^2(x_6^B - \phi_u)\right)\sigma'_B & \end{cases} \quad (7.21)$$

There is an additional common constant to both the expressions above which is unimportant for understanding the behaviour near $\sigma'_B = 0$:

$$c' = \frac{1}{8} \left(-4m_B^2 + 4\lambda_B b_4 - \lambda_B^2(x_6^B - \phi_u) \right). \quad (7.22)$$

The potential (7.21) matches the general form of the potential C.1 considered in appendix C which we reproduce here for convenience:

$$U_{\text{eff}}(\sigma) = \begin{cases} A\sigma^3 + B\sigma^2 + C\sigma & \sigma < 0, \\ (A+a)\sigma^3 + (B+b)\sigma^2 + C\sigma & \sigma > 0. \end{cases} \quad (7.23)$$

The quantities A , a , B , b and C in the present case are given by

$$\begin{aligned} A &= x_6^B - \phi_u, & a &= \phi_u, & B &= 2(\lambda_B b_4 - \frac{3}{4}\lambda_B^2(x_6^B - \phi_u)), \\ b &= -(1 - |\lambda_B|), & C &= m_B^2 - 2\lambda_B b_4 + \frac{3}{4}\lambda_B^2(x_6^B - \phi_u). \end{aligned} \quad (7.24)$$

As we have mentioned above, we use $\lambda_B b_4 = 0$ and $m_B^2 = 0$ as the X -axis and Y -axis for the full phase diagram. However, the local coordinate axes for the potential (7.21) are

different are obtained by setting the quantities B and C in (7.24) to zero:

$$X'\text{-axis: } \lambda_B b_4 = \frac{3}{4} \lambda_B^2 (x_6 - \phi_u), \quad Y'\text{-axis: } m_B^2 - 2\lambda_B b_4 + \frac{3}{4} \lambda_B^2 (x_6 - \phi_u) = 0. \quad (7.25)$$

The origin of the local coordinate axes above lies at $B = C = 0$, given by

$$O_\ell: \quad m_B^2 = \lambda_B b_4 = \frac{3}{4} \lambda_B^2 (x_6 - \phi_u). \quad (7.26)$$

There is another point on this phase diagram in which the effective potential (7.21) becomes particularly simple: the point L in the phase diagram at which $B = -b$ and $C = 0$ in (7.24):

$$L: \begin{cases} m_B^2 = \frac{3}{4} \lambda_B^2 (x_6^B - \phi_u) + 2(1 - |\lambda_B|), \\ \lambda_B b_4 = \frac{3}{4} \lambda_B^2 (x_6^B - \phi_u) + (1 - |\lambda_B|). \end{cases} \quad (7.27)$$

Borrowing results from the general analysis of appendix C, we see that this ‘local patch’ of the phase diagram has the following special curves.

1. The Y' -axis (7.25) is divided into three segments by the local origin O_ℓ (7.26) and the point L (7.27). These segments correspond to $B > -b$, $0 < B < -b$ and $B < 0$ in (7.24). These translate to the following conditions on $\lambda_B b_4$:

$$\begin{aligned} S': \quad & \frac{3}{4} \lambda_B^2 (x_6^B - \phi_u) + (1 - |\lambda_B|) \leq \lambda_B b_4, \\ N': \quad & \frac{3}{4} \lambda_B^2 (x_6^B - \phi_u) \leq \lambda_B b_4 \leq \frac{3}{4} \lambda_B^2 (x_6^B - \phi_u) + (1 - |\lambda_B|), \\ M': \quad & \lambda_B b_4 \leq \frac{3}{4} \lambda_B^2 (x_6^B - \phi_u). \end{aligned} \quad (7.28)$$

Please refer to section C.1 for a detailed description of these segments and the role they play in understanding the structure of the potential.

2. We also have the discriminant parabolas (7.16) for the unHiggsed and condensed branches of the potential:

$$\begin{aligned} D_u: \quad & 4\lambda_B^2 b_4^2 - 3m_B^2 \lambda_B^2 (x_6^B - \phi_u) = 0, \\ D_c: \quad & 4(\lambda_B b_4 + |\lambda_B|(1 - |\lambda_B|))^2 - 3\lambda_B^2 x_6^B (m_B^2 - (1 - |\lambda_B|)^2) = 0. \end{aligned} \quad (7.29)$$

Note that the curve D_c passes through the point L at the junction of S' and N' ; in fact, this parabola is tangent to the Y' -axis (7.25) at L . In a similar manner the curve D_u is tangent to the Y' -axis at the local origin O_ℓ (7.26) which is at the junction of N' and M' .

3. In addition to the curves we have drawn above, there may be first order transition lines which have to be determined numerically.

7.2.2 Near r : $\sigma_B = \frac{2-|\lambda_B|}{2|\lambda_B|}$

We look at the effective potential (7.13) in the neighbourhood of $\sigma_B = \frac{2-|\lambda_B|}{2|\lambda_B|}$ by setting $\sigma_B'' = \sigma_B - \frac{2-|\lambda_B|}{2|\lambda_B|}$ and expanding the relevant expressions for U_{eff} . Again, note that $\sigma_B'' = 0$ when $\sigma_B = r$. In a neighbourhood around $\sigma_B'' = 0$, we have

$$U_{\text{eff}}(\sigma_B'') = \frac{N_B}{2\pi} \times \begin{cases} x_6^B \lambda_B^2 \sigma_B''^3 + 2 \left(\lambda_B b_4 + \frac{3}{4} |\lambda_B| (2 - |\lambda_B|) (x_6^B - \phi_h) - \frac{|\lambda_B| (1 - |\lambda_B|)}{2 - |\lambda_B|} \right) \sigma_B''^2 \\ \quad + \left(m_B^2 + 2 \frac{2 - |\lambda_B|}{|\lambda_B|} \lambda_B b_4 + \frac{3}{4} (2 - |\lambda_B|)^2 (x_6^B - \phi_h) \right) \sigma_B'' & \sigma_B'' < 0, \\ (x_6^B - \phi_h) \lambda_B^2 \sigma_B''^3 + 2 \left(\lambda_B b_4 + \frac{3}{4} |\lambda_B| (2 - |\lambda_B|) (x_6^B - \phi_h) \right) \sigma_B''^2 \\ \quad + \left(m_B^2 + 2 \frac{2 - |\lambda_B|}{|\lambda_B|} \lambda_B b_4 + \frac{3}{4} (2 - |\lambda_B|)^2 (x_6^B - \phi_h) \right) \sigma_B'' & \sigma_B'' > 0. \end{cases} \quad (7.30)$$

Again, there is an additional constant in both expressions which is unimportant while looking at the behaviour near $\sigma_B'' = 0$:

$$c'' = \frac{2 - |\lambda_B|}{2|\lambda_B|} \left(m_B^2 + 2 \lambda_B b_4 \frac{2 - |\lambda_B|}{2|\lambda_B|} + \left(\frac{2 - |\lambda_B|}{2|\lambda_B|} \right)^2 \lambda_B^2 (x_6^B - \phi_h) \right). \quad (7.31)$$

Again, the potential in (7.30) is of the form of the general potential (C.1) in appendix C (also reproduced in (7.23) above) with the following identification of the various parameters:

$$\begin{aligned} A &= x_6^B, & a &= -\phi_h, & B &= \lambda_B b_4 + \frac{3}{4} |\lambda_B| (2 - |\lambda_B|) (x_6^B - \phi_h) - \frac{|\lambda_B| (1 - |\lambda_B|)}{2 - |\lambda_B|}, \\ b &= \frac{|\lambda_B| (1 - |\lambda_B|)}{2 - |\lambda_B|}, & C &= m_B^2 + 2 \frac{2 - |\lambda_B|}{|\lambda_B|} \lambda_B b_4 + \frac{3}{4} (2 - |\lambda_B|)^2 (x_6^B - \phi_h). \end{aligned} \quad (7.32)$$

The local coordinate axes for the above potential in the $(m_B^2, \lambda_B b_4)$ plane are given by setting the quantities B and C to zero:

$$\begin{aligned} X''\text{-axis} : \lambda_B b_4 &= -\frac{3}{4} |\lambda_B| (2 - |\lambda_B|) (x_6^B - \phi_h) - \frac{|\lambda_B| (1 - |\lambda_B|)}{2 - |\lambda_B|}, \\ Y''\text{-axis} : m_B^2 + 2 \frac{2 - |\lambda_B|}{|\lambda_B|} \lambda_B b_4 + \frac{3}{4} (2 - |\lambda_B|)^2 (x_6^B - \phi_h) &= 0. \end{aligned} \quad (7.33)$$

The origin of the above local coordinate axes is at the point R specified by $B = C = 0$ in (7.32):

$$R : \begin{cases} m_B^2 = \frac{3}{4} (2 - |\lambda_B|)^2 (x_6^B - \phi_h) - 2(1 - |\lambda_B|), \\ \lambda_B b_4 = -\frac{3}{4} |\lambda_B| (2 - |\lambda_B|) (x_6^B - \phi_h) + \frac{|\lambda_B| (1 - |\lambda_B|)}{2 - |\lambda_B|}. \end{cases} \quad (7.34)$$

There is another point on this phase diagram in which the effective potential (7.30) becomes particularly simple. This is the point P_r at which $B = -b$ and $C = 0$ in (7.32):

$$P_r : \begin{cases} m_B^2 = \frac{3}{4} (2 - |\lambda_B|)^2 (x_6^B - \phi_h), \\ \lambda_B b_4 = -\frac{3}{4} |\lambda_B| (2 - |\lambda_B|) (x_6^B - \phi_h). \end{cases} \quad (7.35)$$

Appealing to the results of appendix C, we see that this ‘local patch’ of the phase diagram has the following special curves.

1. The Y'' -axis (7.33) is divided into three segments S'' , N'' and M'' by the local origin R (7.34) and the point P_r (7.35). The importance of these segments are elucidated in section C.1 in the appendix. The segments are again specified by the following inequalities on $\lambda_B b_4$:

$$\begin{aligned}
 S'' : \quad & -\frac{3}{4}|\lambda_B|(2-|\lambda_B|)(x_6^B - \phi_h) + \frac{|\lambda_B|(1-|\lambda_B|)}{2-|\lambda_B|} \leq \lambda_B b_4, \\
 N'' : \quad & 0 \leq \lambda_B b_4 + \frac{3}{4}|\lambda_B|(2-|\lambda_B|)(x_6^B - \phi_h) \leq \frac{|\lambda_B|(1-|\lambda_B|)}{2-|\lambda_B|}, \\
 M'' : \quad & \lambda_B b_4 < -\frac{3}{4}|\lambda_B|(2-|\lambda_B|)(x_6^B - \phi_h).
 \end{aligned} \tag{7.36}$$

2. The parabolas D_c and D_h were discussed around equation (7.16) and correspond to the discriminants in the condensed and the Higgsed branches.

$$\begin{aligned}
 D_c : \quad & 4(\lambda_B b_4 + |\lambda_B|(1-|\lambda_B|))^2 - 3\lambda_B^2 x_6^B (m_B^2 - (1-|\lambda_B|)^2) = 0, \\
 D_h : \quad & 4\lambda_B^2 b_4^2 - 3m_B^2 \lambda_B^2 (x_6^B - \phi_h) = 0.
 \end{aligned} \tag{7.37}$$

Note that the curve D_c passes through the local origin R at the junction of S'' and N'' ; in fact, this parabola is tangent to the Y'' -axis (7.33) at the point R . In a similar manner the curve D_h is tangent to the Y'' -axis at the point P_r which is at junction of N'' and M'' .

3. In addition to the curves we have described above, there may be first order transition lines which have to be determined numerically.

7.3 Some broad features of the phase diagram

As we have mentioned above, we will present the phase diagram of our theory on a plane whose Y -axis is $m_B^2 = 0$ and whose X -axis is $\lambda_B b_4 = 0$. With these conventions we note the following general properties of the curves we have defined above.

7.3.1 Relative locations of the local patches

The local patches of the phase diagram corresponding to the behaviour of the potential near $\sigma_B = \ell$ and $\sigma_B = r$ are described separately in the previous subsection. Now, we try to understand their relative locations in the full phase diagram.

1. The Y' -axis in (7.25) has slope $\frac{1}{2}$. Substituting the coordinates of the point R (7.34) into the expression for Y' in (7.25), we get

$$Y'(m_B^2, \lambda_B b_4) \Big|_{(m_B^2, \lambda_B b_4)=R} = 3x_6^B. \tag{7.38}$$

Thus, the point R lies to the right of the Y' -axis when x_6^B is positive, and to its left when x_6^B is negative.

2. The slope of the Y'' -axis in (7.33) is $-\frac{|\lambda_B|}{2(2-|\lambda_B|)}$. Similar to the previous case, substituting the coordinates of the point L (7.27) into the expression for Y'' , we get

$$Y''(m_B^2, \lambda_B b_4) \Big|_{(m_B^2, \lambda_B b_4)=L} = 3x_6^B. \quad (7.39)$$

Thus, the point L lies to the right of Y'' when $x_6^B > 0$ and to its left when $x_6^B < 0$.

3. The points L and R are coincident when $x_6^B = 0$ and are located at

$$m_B^2 = (1 - |\lambda_B|)^2, \quad \lambda_B b_4 = -|\lambda_B|(1 - |\lambda_B|). \quad (7.40)$$

7.3.2 The phase near the origin of the phase diagram

Let us focus on the origin of the phase diagram $(m_B^2, \lambda_B b_4) = (0, 0)$. The effective potential (7.13) becomes

$$U_{\text{eff}}(\sigma_B) = \frac{N_B}{2\pi} \times \begin{cases} (x_6^B - \phi_u) \lambda_B^2 \sigma_B^3 & \sigma_B < -\frac{1}{2}, \\ x_6^B \lambda_B^2 \sigma_B^3 + 2|\lambda_B|(1 - |\lambda_B|) \sigma_B^2 & -\frac{1}{2} < \sigma_B < \frac{2 - |\lambda_B|}{2|\lambda_B|}, \\ -(1 - |\lambda_B|)^2 \sigma_B - \frac{1}{6}(1 - |\lambda_B|)(2 - |\lambda_B|) & \\ (x_6^B - \phi_h) \lambda_B^2 \sigma_B^3 & \sigma_B > \frac{2 - |\lambda_B|}{2|\lambda_B|}. \end{cases} \quad (7.41)$$

The potential is monotonic in the unHiggsed and Higgsed branches since it contains only the cubic term. The only extrema possible are then in the condensed branch. Depending on the choice of x_6^B , there may be extrema or not in the range of validity of σ_B of the condensed branch. Thus, the origin of the phase diagram is either in the condensed phase when an extremum exists or in a ‘no phase’ region if the potential is monotonic in the condensed branch as well. The same logic applies if m_B^2 and $\lambda_B b_4$ are non-zero but are small when compared to the chemical potential. Thus, a neighbourhood of the origin lies in the condensed phase or in a ‘no phase’ region.

7.3.3 The behaviour for $|m_B^2| \gg |\mu|^2, |\lambda_B b_4| \gg |\mu|$

When the parameters m_B^2 and $\lambda_B b_4$ are much larger in modulus than $|\mu|^2$ and $|\mu|$ respectively, the effective potential (7.13) takes the simple form

$$U_{\text{eff}}(\sigma_B) \approx \frac{N_B}{2\pi} \times \begin{cases} (x_6^B - \phi_u) \lambda_B^2 \sigma_B^3 + 2\lambda_B b_4 \sigma_B^2 + m_B^2 \sigma_B & \sigma_B < -\frac{|\mu|}{2}, \\ x_6^B \lambda_B^2 \sigma_B^3 + 2\lambda_B b_4 \sigma_B^2 + m_B^2 \sigma_B & -\frac{|\mu|}{2} < \sigma_B < \frac{2 - |\lambda_B|}{2|\lambda_B|} |\mu|, \\ (x_6^B - \phi_h) \lambda_B^2 \sigma_B^3 + 2\lambda_B b_4 \sigma_B^2 + m_B^2 \sigma_B & \sigma_B > \frac{2 - |\lambda_B|}{2|\lambda_B|} |\mu|. \end{cases} \quad (7.42)$$

Since the extrema of the potential scale like $\lambda_B b_4$ or $\sqrt{m_B^2}$, they will not lie in the condensed branch of the potential whose range of validity is $\mathcal{O}(|\mu|)$ around $\sigma_B = 0$. Thus, when m_B^2 and $\lambda_B b_4$ are both large, we can ignore the presence of the condensed branch. In this case the analysis reduces to that of the regular boson without chemical potential [71] whose effective potential has only two branches viz. the unHiggsed and the Higgsed.

Suppose we go to large values of m_B^2 and $\lambda_B b_4$ along the parabolas

$$m_B^2 = \alpha(\lambda_B b_4)^2. \quad (7.43)$$

Then the above analysis tells us that, after a certain point on *any* parabola of the type given above, the system is in one of the two uncondensed phases i.e. either the unHiggsed or the Higgsed phase (or in a no-phase region if x_6^B is in one of the unstable ranges).

For the regular boson without chemical potential with x_6^B in the stable range $\phi_h < x_6^B < \phi_u$, there is a first order curve for $\lambda_B b_4 < 0$ along the following parabola:

$$m_B^2 = \nu(x_6^B, |\lambda_B|) (\lambda_B b_4)^2, \quad (7.44)$$

where $\nu(x_6^B, |\lambda_B|)$ is a numerically determined function. The relevant information about ν that is needed is that it is monotonically decreasing as a function of x_6^B from ϕ_h to ϕ_u and is zero when $x_6^B = \frac{1}{2}(\phi_u + \phi_h)$. It can be easily seen from the expressions for ϕ_h and ϕ_u in (7.14) that $\phi_h + \phi_u$ is always positive for any value of $|\lambda_B| \leq 1$. Thus, the above parabola is in the fourth quadrant when $\phi_h < x_6^B < \frac{1}{2}(\phi_u + \phi_h)$ and is in the third quadrant when $\frac{1}{2}(\phi_u + \phi_h) < x_6^B < \phi_u$. From the discussion above, it is clear that we should see the above first order curve even in phase diagrams of the current paper for sufficiently large $\lambda_B b_4$ and m_B^2 .

7.4 The phase diagram

In this subsection, we focus on the ranges of x_6^B in (7.18) that correspond to the potential being stable i.e. it is bounded below for large and positive σ_B as well as for large and negative σ_B . This corresponds to the cases B and C in (7.18). The unstable ranges of x_6^B are addressed in appendix D.

Figure 13. Structure of the potential for $\phi_h < x_6^B < 0$. The green curve is a first order transition while the blue curves are second order transitions. We have restored the $|\mu|$ dependence of the phase diagram by rescaling m_B^2 and $\lambda_B b_4$ by appropriate powers of $|\mu|$.

Figure 14. Phase diagram for $\phi_h < x_6^B < 0$. The blue lines are second order transitions and the green line is a first order transition. We have restored the $|\mu|$ dependence of the phase diagram by rescaling m_B^2 and $\lambda_B b_4$ by appropriate powers of $|\mu|$.

7.4.1 Case B: $\phi_h < x_6^B < 0$

Case III of appendix C is pertinent near the non-analytic point ℓ since the quantity A in (7.24) satisfies $A < 0$ and $A + a < 0$, while Case II of section C is pertinent near the point r since the quantity A in (7.32) satisfies $A < 0$ and $A + a > 0$. This implies that the global potential is bounded for both positive and negative σ_B . The unboundedness of the local potential near $\sigma'_B = 0$ for large and positive σ'_B is not very serious since it is cutoff at $\sigma'_B = 1/|\lambda_B|$.

The following conditions specify the segments of D_u , D_c and D_h that appear in this phase diagram:

$$D_u : \lambda_B b_4 \leq \frac{3}{4} \lambda_B^2 (x_6^B - \phi_u), \quad D_h : \lambda_B b_4 \leq -\frac{3}{4} (2 - |\lambda_B|) (x_6^B - \phi_h), \quad (7.45)$$

$$D_c : \frac{3}{4} \lambda_B^2 (x_6^B - \phi_u) - (1 - |\lambda_B|) \leq \lambda_B b_4 \leq -\frac{3}{4} |\lambda_B| (2 - |\lambda_B|) (x_6^B - \phi_h) + \frac{|\lambda_B| (1 - |\lambda_B|)}{2 - |\lambda_B|}. \quad (7.46)$$

We display the detailed structure of the potential near each local patch in figure 13(a) and put together these two local behaviours of the potential in figure 13(b). The final phase structure is displayed in figure 14. The first order transition lines shown in solid green in figures 13 and 14 are determined numerically. The part of the first order line that separates the unHiggsed and Higgsed phases is determined by comparing the value of the potential at the respective minima in each branch. As was discussed in section 7.3.3 (also see the discussion in [71] and [70]), this line goes to infinity in the fourth quadrant when $x_6^B < \frac{1}{2}(\phi_u + \phi_h)$ and in the third quadrant when $x_6^B > \frac{1}{2}(\phi_u + \phi_h)$. Since $\phi_h + \phi_u$ is always positive, it follows that the first order line always curves towards infinity in the fourth quadrant for all values of x_6^B such that $\phi_h < x_6^B < 0$.

7.4.2 Case C: $0 < x_6^B < \phi_u$

For this range of x_6^B , Case II of section C applies for the potential near $\sigma_B = \ell$ and Case I applies for the potential near $\sigma_B = r$. The conditions specifying which segments of the discriminant parabolas appear in this phase diagram are

$$D_u : \lambda_B b_4 \leq \frac{3}{4} \lambda_B^2 (x_6^B - \phi_u), \quad D_h : \lambda_B b_4 \leq -\frac{3}{4} (2 - |\lambda_B|) (x_6^B - \phi_h), \quad (7.47)$$

$$D_c : -\frac{3}{4} |\lambda_B| (2 - |\lambda_B|) (x_6^B - \phi_h) + \frac{|\lambda_B| (1 - |\lambda_B|)}{2 - |\lambda_B|} \leq \lambda_B b_4 \leq \frac{3}{4} \lambda_B^2 (x_6^B - \phi_u) - (1 - |\lambda_B|). \quad (7.48)$$

The detailed structure of the potential is displayed in figure 15. In figure 15(a), we provide a small inset plot of the local potential near $\sigma_B = \ell$ and r in each region demarcated by the special lines of the potentials (7.21) and (7.30). In figure 15(b) we show the global potential in each of the above regions. Finally, we display the phase structure of the theory in figure 16.

The part of the first order line that separates the unHiggsed and Higgsed phases goes to infinity in the fourth quadrant when $x_6^B < \frac{1}{2}(\phi_u + \phi_h)$ as explained in section 7.3.3. As indicated there, $\phi_u + \phi_h$ is always positive for all values of $|\lambda_B|$. It follows that the first order line curves towards infinity in the fourth quadrant for x_6^B such that $0 < x_6^B < \frac{1}{2}(\phi_u + \phi_h)$ and goes to infinity in the third quadrant when $\frac{1}{2}(\phi_u + \phi_h) < x_6^B < \phi_u$. These two cases are displayed in figure 16.

Figure 15. Structure of the potential for $0 < x_6^B < \phi_u$. We have restored the $|\mu|$ dependence of the phase diagram by rescaling m_B^2 and $\lambda_B b_4$ by appropriate powers of $|\mu|$. The green curve is a first order transition while the blue curves are second order transitions. The above plot is for the subrange $0 < x_6^B < \frac{1}{2}(\phi_h + \phi_u)$ where the first order curve goes to infinity in the fourth quadrant. The analysis proceeds similarly when the first order curve goes to infinity in the third quadrant for $\frac{1}{2}(\phi_h + \phi_u) < x_6^B < \phi_u$.

Figure 16. Phase diagram for $0 < x_6^B < \phi_u$. We have restored the $|\mu|$ dependence of the phase diagram by rescaling m_B^2 and $\lambda_B b_4$ by appropriate powers of $|\mu|$. The blue lines are second order transitions and the green line is a first order transition. The figure on the left is for the range $0 < x_6^B < \frac{1}{2}(\phi_h + \phi_u)$ where the first order line goes to infinity in the fourth quadrant. The figure on the right corresponds to $\frac{1}{2}(\phi_h + \phi_u) < x_6^B < \phi_u$ where the first order line goes to infinity in the third quadrant. At the special value $x_6^B = \frac{1}{2}(\phi_h + \phi_u)$ the green first order phase transition line from T downwards runs down the $\lambda_B b_4$ axis.

Figure 17. Different possibilities for the first order curve F' , labelled as (1), (2) and (3), in relation to the first order curve F . The two subfigures differ in the relative placements of the second order line (blue) and the first order line.

7.5 A closer view at the triple point

In the stable cases, Case B: $\phi_h < x_6^B < 0$ and Case C: $0 < x_6^B < \phi_u$, there is an interesting point labelled T on the corresponding phase diagrams figures 14 and 16. At this point, all three phases — the condensed, the (uncondensed) unHiggsed and the (uncondensed) Higgsed — of the system coexist. Equivalently, this ‘triple point’ is the intersection of two first order lines and one second order line. In this subsection, we study the relative orientations of the two first order lines that meet at the ‘triple point’.

The first order curve F separating the unHiggsed and the Higgsed phases takes the shape of a parabola as discussed in section 7.3.3 which starts at the triple point T and goes off to infinity. The first order curve F' which separates the condensed phase and one of the uncondensed phases (Higgsed for Case B while unHiggsed for Case C) has to be determined numerically and is generally more complicated than a parabola. However, there are some features which can be understood analytically.

Suppose we continue the first order curve F that separates the unHiggsed and Higgsed phases slightly further past the triple point and closer to the origin. Then, the question is as follows: does the other first order curve F' go (1) to the left of, (2) along, or (3) to the right of the extended curve F ? We have illustrated the possibilities in figure 17. It is in fact quite simple to figure out which of the above possibilities actually occurs in our situation given the forms of the effective potential (7.21) and (7.30) near the two non-analytic points $\sigma_B = \ell$ and $\sigma_B = r$ respectively.

We study Case B for concreteness — the results for Case C can be obtained by swapping the unHiggsed and Higgsed phases. Recall that F' is the curve that separates the condensed phase and the Higgsed phase. On the extended line F , the Higgsed and unHiggsed phases coexist with equal free energy:

$$U_{\text{Higgsed}}(\sigma_1) = U_{\text{unHiggsed}}(\sigma_2), \quad (7.49)$$

where σ_1 is the minimum in the Higgsed branch of the potential while σ_2 is the minimum in the unHiggsed branch. On this line and very close to the triple point, we compare the free energy at the condensed phase minimum and at the Higgsed phase minimum (which

is the same as the free energy unHiggsed phase minimum). If these free energies are the same, then F' coincides with the extended F . On the other hand, if the condensed phase minimum has lower free energy, the extended curve F is in fact in the condensed phase, implying that the first order curve F' is to the left of the extended F . Analogously, if the condensed phase minimum has the higher free energy, then the first order curve F' is to the right of the extended F .

The triple point T is on the second order line separating the condensed and the unHiggsed phases. From the general analysis of section C.1 in the appendix and the expression for the potential (7.21) in section 7.2.1, we know that the minimum in the condensed phase and the minimum in the unHiggsed phase are both situated at the junction between the branches $\sigma'_B = 0$. When we vary the parameters to go slightly into the condensed phase along the extended F , both the condensed and unHiggsed minima go to the right of $\sigma'_B = 0$,³⁷ and if we are really close to the triple point, then we can take these minima to be at the same value of $\sigma'_B = \sigma_* \approx 0$. In this case, the free energies of the unHiggsed and condensed minima are given by

$$\begin{aligned} U_{\text{unHiggsed}}(\sigma_*) &= A(\sigma_*)^3 + B(\sigma_*)^2 + C\sigma_*, \\ U_{\text{condensed}}(\sigma_*) &= (A + a)(\sigma_*)^3 + (B + b)(\sigma_*)^2 + C\sigma_*, \end{aligned} \quad (7.50)$$

where the parameters A, a, B, b and C are given in equation (7.24). The difference between the condensed and unHiggsed phase minimal free energies is then given by

$$\Delta U = U_{\text{condensed}}(\sigma^*) - U_{\text{unHiggsed}}(\sigma^*) = b(\sigma_*)^2, \quad (7.51)$$

where we have discarded the $(\sigma_*)^3$ term since σ_* is very close to 0. Since the parameter $b = -(1 - |\lambda_B|)$ is negative for the potential (7.21), it follows that the condensed phase has lower free energy than the unHiggsed phase and hence, also lower free energy than the Higgsed phase since we are on the extended F line. Thus, possibility (1) in figure 17 is realised for Case B.

In the same way, for Case C, one must compare the free energies at the condensed and Higgsed phase minima

$$\begin{aligned} U_{\text{condensed}}(\sigma'_*) &= A(\sigma'_*)^3 + B(\sigma'_*)^2 + C\sigma'_*, \\ U_{\text{Higgsed}}(\sigma'_*) &= (A + a)(\sigma'_*)^3 + (B + b)(\sigma'_*)^2 + C\sigma'_*, \end{aligned} \quad (7.52)$$

where the parameters A, a, B, b and C are given in (7.32) corresponding to the potential (7.30). Again, the sign of the parameter b decides the phase with lower free energy. It is easy to see from (7.32) that the parameter b is positive, and hence the condensed phase has lower free energy than the Higgsed phase, and consequently, than the unHiggsed phase since we are on the extended F line. Thus, possibility (3) in figure 17 is realised for Case C.

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³⁷Note that the unHiggsed minimum is not inside the range of validity $\sigma'_B < 0$ of the unHiggsed branch of the potential and is not useful for determining the phase. However, it is indeed useful for the ongoing analysis.

7.6 Varying the chemical potentials for fixed UV parameters

In the analysis above we have studied the phase diagram of our system as a function of the three dimensionful parameters, $\lambda_B b_4$, m_B^2 and $|\mu|$. While the first two of these parameters appear in the Lagrangian of the theory (and so define its behaviour in the deep ultraviolet), the third parameter (the chemical potential) specifies the ensemble of the theory, and so the ‘vacuum’ about which we work.

A question of clear physical interest is the following: what is the phase structure of any particular theory as a function of the modulus of the chemical potential? What phases, in other words, do we encounter if we start at $|\mu| = 0$ and then steadily raise the chemical potential to ∞ , all the while keeping the UV parameters of the theory fixed?

The phase diagrams in figures 14 and 16 allow us to answer the questions posed in the previous paragraph rather easily. Recall that we have plotted $\frac{m_B^2}{|\mu|^2}$ and $\frac{\lambda_B b_4}{|\mu|}$ on the X -axis and Y -axis respectively of the phase diagrams referred to above. Now consider the following two sets of parabolas

$$y = \alpha\sqrt{x} \quad \text{with } x > 0, \quad y = \gamma\sqrt{-x} \quad \text{with } x < 0. \quad (7.53)$$

Note that the curve corresponding to the first equation in (7.53) lies in the first quadrant of the x - y plane when α is positive and in the fourth quadrant when α is negative. Similarly the second curve in (7.53) lies in the second quadrant when γ is positive but in the third quadrant when γ is negative.

$$\begin{aligned} \alpha > 0 : & \quad \text{First quadrant}, & \alpha < 0 : & \quad \text{Fourth quadrant}, \\ \gamma > 0 : & \quad \text{Second quadrant}, & \gamma < 0 : & \quad \text{Third quadrant}. \end{aligned} \quad (7.54)$$

These four sets of (half-)parabolas foliate the x - y plane and we shall refer to them as *foliating parabolas*.

By substituting the relations $y = \frac{\lambda_B b_4}{|\mu|}$ and $x = \frac{m_B^2}{|\mu|^2}$ we see that

$$\alpha = \frac{\sqrt{m_B^2}}{\lambda_B b_4} \quad \text{with } m_B^2 > 0, \quad \gamma = \frac{\sqrt{-m_B^2}}{\lambda_B b_4} \quad \text{with } m_B^2 < 0. \quad (7.55)$$

The important point here is that α and γ depend only on (and are fixed by) the UV parameters of the theory but are independent of $|\mu|$. In other words, as we move along the foliating parabolas (7.53), we are working in the same UV theory but at different values of $|\mu|$. Each of the parabolas run from the origin of the phase diagram to infinity. When $|\mu| = 0$ we start out at infinity. As $|\mu|$ is increased we move in towards the origin, reaching the origin only at $|\mu| = \infty$. Tracing this path in the phase diagrams in figures 34, 14, 16 and 36 gives the phase diagram as a function of μ in any particular theory given by the fixed UV parameters. In the rest of this subsection we explain in more detail how this works for values of x_6^B that lie in the ‘stable range’.

7.6.1 $\phi_h \leq x_6 \leq 0$

In this case the phase diagram of our system is depicted in figure 14 and reproduced in figure 18. There are two interesting parabolas of the form (7.53) associated with this phase

Figure 18. Foliating parabolas in gray for the phase diagram with x_6^B in the range $\phi_h < x_6^B < 0$. Increasing the chemical potential at fixed values of UV parameters corresponds to moving along one of these parabolas (the values of the UV parameters determine which one). The parabolas from T to F (green line) and the parabola that pass through R are distinguished. They separate parabolas across which the phase transition to a compressible phase is of second order from those along which the same transition is of first order.

diagram. The first of these passes through the triple point T in the fourth quadrant and goes along the green first order phase transition line F that starts at T in figure 18. This curve is precisely a parabola of the form of the first of (7.53) with a particular value of $\alpha = \tilde{\alpha}$, as explained in section 7.3.3. Note that $\tilde{\alpha}$ is a negative number.³⁸

A second interesting parabola associated with the same phase diagram is a parabola that passes through the point R in the second quadrant in figure 18. This is of the form of the second curve in (7.53) with a specific positive value for γ given by $\gamma = \gamma^*$. The value of γ^* can be determined by plugging in the coordinates of the point R (7.34) in the formula for γ in (7.55):

$$\gamma^* = \frac{\left(2(1 - |\lambda_B|) - \frac{3}{4}(2 - |\lambda_B|)^2(x_6^B - \phi_h)\right)^{1/2}}{-\frac{3}{4}|\lambda_B|(2 - |\lambda_B|)(x_6^B - \phi_h) + \frac{|\lambda_B|(1 - |\lambda_B|)}{2 - |\lambda_B|}}. \quad (7.56)$$

A glance at figure 18 will convince the reader that if our UV parameters are such that we are on the foliating parabolas with $\alpha^* < \alpha < \infty$, our theory starts out in the unHiggsed phase at small $|\mu|$. As $|\mu|$ is increased, the theory undergoes a single order second order phase transition into a condensed phase and then stays in this phase for all larger values of $|\mu|$.

If the UV parameters are such that we are on the foliating parabolas with $\gamma^* < \gamma < \infty$, then our theory starts out in the Higgsed phase at small $|\mu|$. As $|\mu|$ is increased the theory

³⁸As explained in section 7.3.3, this curve is a parabola because it represents a phase transition between the uncondensed Higgsed and the uncondensed unHiggsed phase, and so is completely insensitive to μ . The value of $\tilde{\alpha}$ is also insensitive to μ and was evaluated in [70] (see around figures 25 and 26).

Figure 19. Foliating parabolas for the range $0 < x_6^B < \phi_u$. The significance of these parabolas was recounted in the caption to the previous figure. The distinguished parabolas (see caption to the previous figure) are the curves TF (green line) and the parabola that passes through L .

undergoes a single order second order phase transition into a condensed phase and then stays in this phase for all larger values of $|\mu|$.

Finally, if the UV parameters are such that we are on the parabolas with $-\infty < \alpha < \alpha^*$ or quadrants with $-\infty < \gamma < \gamma^*$, then our theory starts out in the Higgsed phase at small $|\mu|$, and then undergoes a first order phase transition to the condensed phase at a particular value of $|\mu|$. The theory then stays in this phase for all larger values of $|\mu|$.

7.6.2 $0 \leq x_6^B \leq \phi_u$

In this case the phase diagram of our system is depicted in figure 16 and reproduced in figure 19. Once again, there are two interesting foliating parabolas (7.53) associated with this phase diagram. The first of these passes through the triple point T and continues along the first order curve F that separates the unHiggsed and Higgsed phases. As above, this curve is precisely a parabola. For $0 < x_6^B < \frac{1}{2}(\phi_h + \phi_u)$, this curve is an α -parabola with a particular value of $\alpha = \tilde{\alpha}$. For the complementary range $\frac{1}{2}(\phi_h + \phi_u) < x_6^B < \phi_u$, the curve is a γ parabola with a particular value of $\gamma = \tilde{\gamma}$. Note that both $\tilde{\alpha}$ and $\tilde{\gamma}$ are negative are determined numerically in general.

A second interesting parabola associated with the phase diagram is an α -parabola which passes through the point L in the first quadrant of figure 19. This corresponds to a positive value of α given by $\alpha = \alpha^*$. We have an analytic expression for α^* that we obtain by plugging in the coordinates of the point L (7.27) in the formula for α in (7.55):

$$\alpha^* = \frac{\left(\frac{3}{4}\lambda_B^2(x_6^B - \phi_u) + 2(1 - |\lambda_B|)\right)^{1/2}}{\frac{3}{4}\lambda_B^2(x_6^B - \phi_u) + (1 - |\lambda_B|)} . \quad (7.57)$$

For the range $0 < x_6^B < \frac{1}{2}(\phi_h + \phi_u)$, suppose our UV parameters are such that we are on a γ -parabola with any value of γ , or an α -parabola with $-\infty < \alpha < \tilde{\alpha}$. Then, our theory

Figure 20. The phase diagram of the regular boson theory for $|\lambda_B| = \frac{1}{4}$ and $x_6^B = -0.8$ such that it is in the stable range $\phi_h < x_6^B < 0$.

starts out in the Higgsed phase at small $|\mu|$. As $|\mu|$ is increased the theory undergoes a single second order phase transition into the condensed phase and then stays in this phase for all larger values of $|\mu|$.

On the other hand if the UV parameters are such that we are on α -parabolas with $\alpha^* < \alpha < \infty$ then our theory starts out in the unHiggsed phase at small $|\mu|$ and as $|\mu|$ is increased it undergoes a single order second order phase transition into a condensed phase and then stays in this phase for all larger values of $|\mu|$. When $\tilde{\alpha} < \alpha < \alpha^*$, the theory starts out in the unHiggsed phase at small $|\mu|$, undergoes a first order phase transition at an intermediate value of $|\mu|$ into the condensed phase and stays in that phase for all higher values of $|\mu|$.

A similar analysis can be performed in a straightforward way for the case $\frac{1}{2}(\phi_h + \phi_u) < x_6^B < \phi_u$ depicted on the right in figure 19.

7.7 Numerical plots

In this section we have so far presented only schematic phase diagrams for the quasi bosonic theories. However the analysis that led up to these phase diagrams was completely quantitative. Consequently, it is not difficult, at any given value of $x_6 \lambda_B$ to plot fully quantitative phase diagrams (the precise shapes of the first order phase transitions in these diagrams can be determined only numerically).

To illustrate this fact, in this subsection we present three sample numerical plots, in figures 20, 21 and 22 for the three subranges in the stable range of x_6^B for the choice of 't Hooft coupling $|\lambda_B| = 1/4$. The schematic plots corresponding to these are in figures 14, 16(a) and 16(b).

Figure 21. The phase diagram of the regular boson theory for $|\lambda_B| = \frac{1}{4}$ and $x_6^B = 8$ such that it is in the stable range $0 < x_6^B < \frac{1}{2}(\phi_h + \phi_u)$.

Figure 22. The phase diagram of the regular boson theory for $|\lambda_B| = \frac{1}{4}$ and $x_6^B = 11$ such that it is in the stable range $\frac{1}{2}(\phi_h + \phi_u) < x_6^B < \phi_u$.

8 Discussion and conclusions

Following the analysis of [32], in this paper we have presented completely explicit formulae

$$\begin{aligned}\bar{n}_F(\epsilon, q) &= \frac{1}{2} - \frac{1}{\pi|\lambda_F|} \tan^{-1} \left(\frac{e^{\beta(\epsilon-q\mu)} - 1}{e^{\beta(\epsilon-q\mu)} + 1} \tan \frac{\pi|\lambda_F|}{2} \right), \\ \bar{n}_B(\epsilon, q) &= \frac{1 - |\lambda_B|}{2|\lambda_B|} - \frac{1}{\pi|\lambda_B|} \tan^{-1} \left(\frac{e^{\beta(\epsilon-q\mu)} - 1}{e^{\beta(\epsilon-q\mu)} + 1} \cot \frac{\pi|\lambda_B|}{2} \right).\end{aligned}\tag{8.1}$$

for the occupation numbers of fermionic and bosonic quasiparticle states in large N matter Chern-Simons theories at infinite volume as a function of temperature and chemical potentials. These formulae reduce to the famous formulae (1.2) and (1.4) in the weak coupling limit, and may be regarded as ‘anyonic’ generalisations of these (undergraduate) textbook formulae. In our opinion the formulae (8.1) are very interesting, and deserve to be understood and rederived from many points of view.

Specialising (8.1) to zero temperature we have demonstrated that the runaway instability in the Bose condensate phase of free (or large N Wilson-Fisher) bosons at values of the chemical potential larger than the thermal mass is cured by the coupling to Chern-Simons gauge theory in large N matter Chern-Simons theories. The final stabilised Bose condensate is extremely simple: it can be viewed as a theory of free bosonic quasiparticles, with each quasiparticle state of energy less than $|\mu|$ occupied with an occupation number given by (1.7). This regulated Bose condensate is the dual of an effectively free Fermi sea of fermions coupled to an analogous Chern-Simons gauge theory.

We find it fascinating that bosonic quasiparticle states appear to obey a sort of modified exclusion principle, as a consequence of which the Bose condensed phase is conceptually very similar to a Fermi sea and should display many of its characteristic features. As in a Fermi sea, in this phase every single particle state with energy less than $|\mu|$ is fully occupied (i.e. has the maximum allowed occupation number $\bar{n}$). As a consequence, excitations about this Bose condensate are qualitatively similar to those about a Fermi sea. In particular this Bose condensate hosts charged hole type excitations. It also hosts uncharged particle hole pair excitations. As in the case of the Fermi sea, when the particle carries momentum $\vec{k}_p$ with $|\vec{k}_p|$ just larger than k_F (k_F is the modulus of the momentum of the last occupied state) and the hole carries momentum $\vec{k}_h$ with $|\vec{k}_h|$ just smaller than k_F then this particle-hole state carries nonzero momentum but arbitrarily small energy. As in the case of a Fermi sea, the modulus of the momentum of these particle hole excitations is always smaller than $2k_F$. As a consequence, we expect to see $2k_F$ singularities — similar to those in Fermi liquid theory — in correlation functions of operators like the stress tensor that are built out of bilinears of bosonic operators. The bosonic theory should also host collective excitations of its condensate analogous to the zero sound of a Fermi surface.³⁹

It would be very interesting to understand whether and how the modified ‘bosonic’ exclusion principle manifests itself in dynamical processes. Let us recall that for usual bosons

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\tag{8.2}$$

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³⁹We thank D. Tong for a discussion on this point.

(here $|n\rangle$ and $|n+1\rangle$ are the normalised states corresponding to n particles occupying a given single particle energy level). The factor of $\sqrt{n+1}$ in this formula tells us that bosons are gregarious; they preferentially transit to states that are highly occupied. It would be fascinating to investigate what the analogue of (8.2) (if it exists) is for Chern-Simons coupled bosons. At small λ_B one would intuitively expect (8.2) to continue to apply for $n \ll \bar{n}$ (see (1.7) for a definition: recall $\bar{n}$ is large when λ_B is small). However (8.2) will certainly be significantly modified for values of n of order $\bar{n}$. The computation of scattering processes at finite chemical potential and temperature (which may be technically feasible at large N along the lines of [20]) could shed light on this question, and could also make connection with Haldane's characterisation of statistics in terms of occupation numbers [87].⁴⁰ The discussion of section 1.3 suggests that these S-matrices could also make contact with $SU(N_B)_{k_B}$ representation theory.

The results of this paper have been obtained starting from expression for the leading order in large N (but exact to all orders in the 't Hooft coupling) partition function of the system at finite temperature and chemical potential. Explicit all-orders expressions for this partition function had previously been derived at values of the (modulus of the) chemical potential smaller than the quasiparticle masses. In the current paper we have extended these known formulae to values of the chemical potential larger than quasiparticle masses (see section 3.3). We obtained the new free energy formulae of this paper by analytically continuing previously obtained formulae in μ . It would be interesting to confirm these results from a direct analysis of the path integral at values of the modulus of the chemical potential larger than the thermal mass.

Our extended free energy formulae involved the addition of extra terms to the free energies. These new terms qualitatively modify the dynamics of the Bose condensate, stabilising it. Equivalent new terms in the free energy expression for fermions are absent in the infinite volume limit that we have focused on in this paper; as a consequence the dynamics of the fermionic compressible phase in this limit is essentially identical to that of a free Fermi sea. Away from the infinite volume limit, however, the modification to the free energy does not vanish in other phases [15] of the fermionic theory (see subsection 3.4.2). This observation strongly suggests that the thermodynamics of fermions in such phases differs from that of a non-interacting Fermi sea; plausibly each single particle fermion state has an occupation number less than unity in such phases. This phenomenon deserves analysis, both from a direct study of the relevant free energy formulae, as well as from a study of the Schrodinger equation in the appropriate non-relativistic limits.

The fermionic and bosonic compressible phases described in this paper can both be thought of as collections of quasiparticles which occupy all states with energy smaller than the chemical potential with a given occupation number (unity for the fermions and $\bar{n}$ (1.7) for the bosons at zero temperature). The fact that each of these quasiparticles each carry a definite spin⁴¹ suggests that each of these phases have extensive angular momenta (i.e. have definite angular momenta per unit area). In appendix E, we have verified that this is indeed

⁴⁰We thank D. Radicevic for bringing this paper, and its relevance to our current work, to our attention.

⁴¹Note that this spin is a number and not a multiplet in $D = 3$.

true of the free Fermi sea. It would be interesting to compute the angular momentum per unit volume of all the compressible phases studied in this paper as a function of the chemical potential and the 't Hooft coupling.

Since the results of this paper apply at every value of the chemical potential μ and the quasiparticle mass c_B , they apply, in particular, in the limit $\frac{|\mu|-c_B}{c_B} \ll 1$. In this limit, the particles that make up the Bose condensate are all non-relativistic and the Bose condensate described in this paper should admit a description as a solution to the multi-particle Schrodinger equation [86]. It would be very interesting to reproduce the ‘bosonic exclusion principle’ formula of section 1.3 (also the more general formula (8.1)) from this point of view. Such an analysis would likely furnish qualitatively new understanding.

The finite chemical potential phase diagram for the regular boson theory, presented e.g. in figures 14 and 16, is rather intricate and has many interesting features. At fixed values of the UV parameters, upon increasing the chemical potential we always encounter phase transition indicating the formation of a Bose condensate. For a range of values of the chemical potential, this transition turns out to be of first order rather than second order. It would be interesting to better understand the physics of this switch of phase transitions between first and second order.

Finally, we have studied compressible phases by first taking $N \rightarrow \infty$ and then taking $T \rightarrow 0$. In other words, the results of this paper apply at temperatures that are much smaller than the chemical potential but much larger than (for instance) the chemical potential divided by any given positive power of N . It is possible that, at still lower temperatures (temperatures of order of the chemical potential divided by a positive power of N), new infrared divergences invalidate the naive $1/N$ expansion (so that diagrams at apparently different orders in $1/N$ all end up actually contributing at the same order, see e.g. [88]) leading to very interesting low temperature dynamics. It would be very interesting to investigate this further.

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A Bose-Fermi dualities at finite rank and level

In the present paper, we work in the 't Hooft large N limit in which the rank N and level κ are both taken to infinity with the ratio N/κ fixed. In this limit, the distinction between $U(N)$ and $SU(N)$ gauge groups is lost. At finite N and κ , it is important to distinguish

between $U(N)$ and $SU(N)$ groups and Bose-Fermi duality works correctly only when the groups as well as the corresponding levels are correctly chosen.

In this appendix we will carefully describe three distinct classes of Chern-Simons theories: theories based on the $SU(N)$ gauge group and two classes of $U(N)$ theories — the so-called Type II and Type I theories — all of which become identical in the large N limit. In keeping with the focus of this paper, we will pay special attention to the global symmetries of these theories. We will explain that though the global symmetries of the $SU(N)$ and $U(N)$ theories are structurally very different from each other, they become functionally identical in the large N limit. We will also describe Bose-Fermi duality conjectures at finite rank and level, keeping careful track of all $U(1)$ factors [33]. The material presented in this appendix is largely a review of well-known results (see e.g. [40]), and is included here in order clearly define all conventions and also in an attempt to make the current paper self-contained.

A.1 Pure Chern-Simons theories

A.1.1 Conventions

Throughout this paper we define the level k of an $SU(N)$ pure Chern-Simons theory to equal (upto sign) the level of its dual ‘boundary’ $SU(N)_k$ WZW theory.

We now turn to pure $U(N)$ Chern-Simons theories. Recall the definition of the $U(N)$ group

$$U(N) = \frac{SU(N) \times U(1)}{\mathbb{Z}_N} . \quad (\text{A.1})$$

In words, $U(N)$ has a simple $SU(N)$ factor and an abelian $U(1)$ factor, with the centre $\mathbb{Z}_N$ of $SU(N)$ identified with a corresponding $\mathbb{Z}_N$ subgroup of the $U(1)$.⁴² It follows that pure $U(N)$ Chern-Simons theories are characterised by two levels, one each for the $SU(N)$ and the $U(1)$ factors. We use the notation $U(N)_{k,k'}$ to denote these levels where k and k' are defined as follows.

$$U(N)_{k,k'} \equiv SU(N)_k \times U(1)_{Nk'} . \quad (\text{A.2})$$

k is the level of the $SU(N)$ part of the gauge group as defined at the beginning of this subsection. Nk' is the level of the $U(1)$ part of the gauge group (normalised in the convention in which all charged Wilson lines have integer charges). In fact, Nk' is (upto sign) the level of the ‘boundary’ dual $U(1)$ WZW theory, normalised so that all primary operators have integer charges. As we explain later (see section A.1.3 below), the allowed values for k' are

$$k' = k + mN , \quad (\text{A.3})$$

where m is an arbitrary integer.

In this appendix, we will be particularly interested in two $U(N)$ Chern-Simons theories, the Type II theories ($m = 0$ in (A.3)) and Type I ($m = \text{sgn}(k)$ in (A.3))

$$\text{Type II : } U(N)_{k,k} , \quad \text{Type I : } U(N)_{k,\kappa} . \quad (\text{A.4})$$

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⁴²In the rest of this paper we often suppress writing the quotient by the diagonal $\mathbb{Z}_N$ subgroup.

where κ is the so called ‘renormalized level’, defined as

$$\kappa = \text{sgn}(k)(|k| + N) . \quad (\text{A.5})$$

The authors of [40] have also studied another interesting class of theories — we could call these Type I’ theories - with $m = -\text{sgn}(k)$. The properties of Type I’ theories are analogous to those of the Type I theory (and Type I’ theories also obey Bose-Fermi dualities closely analogous to those of Type I theories). In this appendix we do not explicitly discuss Type I’ theories. However the interested reader will find it easy to modify the discussion of Type I theories presented in this appendix to Type I’ theories. Further, the authors of [41] generalise the dualities to the entire family of theories with $U(1)$ levels in (A.3), of which the Type II, Type I and Type I’ theories are special cases.

We end this brief subsection with a notational remark. Following the conventions of [40], dynamical gauge fields shall be denoted by lowercase letters and background gauge fields by uppercase letters. The $U(N)$ gauge field is generically denoted by x_μ , and it is often useful to split it into its traceless $SU(N)$ part and a pure-trace $U(1)$ part:

$$x_\mu = y_\mu + a_\mu \mathbb{1}_N , \quad \text{Tr } y_\mu = 0 , \quad (\text{A.6})$$

where $\mathbb{1}_N$ is the $N \times N$ identity matrix. We will adopt the notation (A.6) in the rest of this appendix. In other words, we will always reserve the symbol x_μ for a general $N \times N$ matrix valued gauge field, the symbol y_μ for the traceless $N \times N$ matrix valued gauge field, and a_μ for the $U(1)$ gauge field normalised so that $U(N)$ Wilson lines in the fundamental / antifundamental of $U(N)$ carry charge ± 1 under a_μ . The authors of [40] work with the $U(1)$ gauge field $\text{Tr } x_\mu = N a_\mu$ under which Wilson lines in the fundamental / antifundamental representation of $U(N)$ carry a fractional charge of $\pm 1/N$.

A.1.2 Lagrangians

In this subsection we will explain how the levels for pure Chern-Simons theories, defined in the previous subsection, enter the Lagrangians for these theories. The discussion is very easy for the $U(1)$ part of the Lagrangian. For a $U(N)_{k,k'}$ the Lagrangian for the $U(1)$ factor is simply

$$\mathcal{S}_{U(1)}[a] = \frac{ik'N}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho . \quad (\text{A.7})$$

This Lagrangian captures the dynamics of the $U(1)$ part of a $U(N)_{k,k'}$ theory in every reasonable regulation scheme.

The discussion is a bit more complicated for the $SU(N)$ part of the Lagrangian. In this case the value of the Chern-Simons path integral depends on the regularization scheme used. The $SU(N)_k$ theory is defined by the action

$$\mathcal{S}_{SU(N)}[y] = \frac{ic}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{Tr} \left(y_\mu \partial_\nu y_\rho - \frac{2i}{3} y_\mu y_\nu y_\rho \right) , \quad (\text{given reg. scheme}) , \quad (\text{A.8})$$

where the value of c depends on the regularization scheme employed. It turns out $c = k$ when the path integral is regulated by the addition of a Yang-Mills term in the Lagrangian

with an infinitesimal coefficient (a very large Yang-Mills coupling) — we refer to this as the YM scheme. On the other hand $c = \kappa$ when the path integral is regularized in the dimensional regularization scheme (we call this the dim-reg scheme).

The Lagrangians presented in the main text of this paper are always used in conjunction with the dimensional regularization scheme (see [65, 71] for more details of the regularization scheme).⁴³

We establish a bit of notation before we proceed. The actions may be written very succinctly by employing differential forms. The $U(1)$ and $SU(N)$ actions in (A.7) and (A.8) respectively are written as

$$\mathcal{S}_{U(1)}[a] = \frac{ik'N}{4\pi} \int ada, \quad \mathcal{S}_{SU(N)}[y] = \frac{ic}{4\pi} \int \text{Tr} \left(ydy - \frac{2i}{3} y^3 \right), \quad (\text{A.9})$$

where $ada = d^3x \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho$ and $y^3 = d^3x \epsilon^{\mu\nu\rho} y_\mu y_\nu y_\rho$.

We summarize the explicit Lagrangians in the different regularization schemes below. The $SU(N)$ theory is defined as follows:

$$\begin{aligned} \mathcal{S}_{SU(N)}[y] &= \frac{i\kappa}{4\pi} \int \text{Tr} \left(ydy - \frac{2i}{3} y^3 \right), \quad (\text{dim-reg}), \\ &= \frac{ik}{4\pi} \int \text{Tr} \left(ydy - \frac{2i}{3} y^3 \right), \quad (\text{YM-reg}). \end{aligned} \quad (\text{A.10})$$

The Type I $U(N)_{k,\kappa}$ Chern-Simons theory is defined by the Lagrangians

$$\begin{aligned} \mathcal{S}_{\text{Type I}}[y, a] &= \frac{i\kappa}{4\pi} \int \text{Tr} \left(ydy - \frac{2i}{3} y^3 \right) + \frac{i\kappa N}{4\pi} \int ada, \quad (\text{dim-reg}), \\ &= \frac{ik}{4\pi} \int \text{Tr} \left(ydy - \frac{2i}{3} y^3 \right) + \frac{i\kappa N}{4\pi} \int ada, \quad (\text{YM-reg}). \end{aligned} \quad (\text{A.11})$$

Similarly, the Type II $U(N)_{k,k}$ theory is defined by the Lagrangians

$$\begin{aligned} \mathcal{S}_{\text{Type II}}[y, a] &= \frac{i\kappa}{4\pi} \int \text{Tr} \left(ydy - \frac{2i}{3} y^3 \right) + \frac{ikN}{4\pi} \int ada, \quad (\text{dim-reg}), \\ &= \frac{ik}{4\pi} \int \text{Tr} \left(ydy - \frac{2i}{3} y^3 \right) + \frac{ikN}{4\pi} \int ada, \quad (\text{YM-reg}). \end{aligned} \quad (\text{A.12})$$

A.1.3 Quantization of the $U(1)$ Chern-Simons levels

The level of the gauge action for the $U(1)$ factor of a $U(N)$ Chern-Simons theory is generally of the form Nk' with $k' = k + nN$ where m is an arbitrary integer. In this subsection, we justify this constraint on the $U(1)$ level.

⁴³The general rule is that we get the same physical $SU(N)$ theory in the dim-reg and YM schemes if the coefficients c in the actions used with these schemes differ by

$$c_{\text{dim-reg}} - c_{\text{YM-reg}} = \text{sgn}(c_{\text{YM-reg}})N.$$

[89] As far as $U(1)$ factors are concerned, we get the same theories in the two regulation schemes if the two actions are simply equal. The value of c in the YM scheme equals the level of the corresponding WZW theory.

The Lagrangian of a $U(N)_{k,k'}$ Chern-Simons theory is given by

$$\mathcal{S}[y, a] = \frac{ik}{4\pi} \int \text{Tr} \left(y dy - \frac{2i}{3} y^3 \right) + \frac{iNk'}{4\pi} \int a da . \quad (\text{A.13})$$

Rewriting the above Lagrangian in terms of the $U(N)$ gauge field $x = y + a \mathbb{1}_N$ in (A.6), we have

$$\mathcal{S}[x, a] = \frac{ik}{4\pi} \int \text{Tr} \left(x dx - \frac{2i}{3} x^3 \right) + \frac{iN(k' - k)}{4\pi} \int a da . \quad (\text{A.14})$$

This action should be well-defined (and have all the physically expected properties) for the $U(N)$ gauge field x .

Consider a $U(N)$ gauge configuration where only the gauge field corresponding to one of the $U(1)$ subgroups of $U(N)$ is turned on, say the first entry on the diagonal of the $U(N)$ gauge field. Suppose this $U(1)$ gauge field (call it b_μ) has one unit of spatial flux and a constant time component b_0 with a holonomy of 2π in the time direction. This configuration is gauge equivalent (via a large gauge transformation) to flux plus no holonomy; thus, it must yield the same contribution to the path integral as this second configuration. However, the action of the second configuration is zero. Consequently the action of our original configuration must equal an integral multiple of $2\pi i$. Using the fact that a $U(1)$ field b_μ the above sort (namely with one unit of spatial flux and one unit of timelike holonomy) has action⁴⁴

$$\frac{1}{4\pi} \int b db = 2\pi ,$$

it follows that the first term in (A.14) evaluates to $2\pi i k$ which is $2\pi i$ times an integer as we required. We now turn to the second term in (A.14). Recall that the $U(1)$ gauge field is related to the $U(N)$ gauge field as

$$a = \frac{1}{N} \text{Tr } x .$$

Then, in the configuration above with only b_μ turned on, the field a_μ is given by $\frac{1}{N} b_\mu$ and thus has $1/N$ units of flux and a holonomy equal to $2\pi/N$. It follows that the second term in (A.14) evaluates to $2\pi i/N^2$. The total action for the flux plus holonomy configuration is then

$$2\pi i k + N(k' - k) \frac{2\pi i}{N^2} . \quad (\text{A.15})$$

Thus, for this configuration to be $2\pi i \times \text{integer}$ in order to reproduce the same path integral as the configuration with flux plus no holonomy, the quantity $\frac{k' - k}{N}$ must be an integer. Thus, we have the condition

$$k' = k + mN , \quad (\text{A.16})$$

where m is an arbitrary integer.

---

⁴⁴See footnote 15 of [90] and the discussion above equation (5.15) of the lecture notes [91] for an explanation of a tricky factor of 2 here.

Type I-Type I	$U(N)_{k,\kappa} \longleftrightarrow U(k)_{\epsilon N, \epsilon \kappa}$
Type II- $SU(N)$	$U(N)_{k,k} \longleftrightarrow SU(k)_{\epsilon N}$

Table 1. Level rank dualities between pure Chern-Simons theories. The quantity ϵ is given by $\epsilon = -\text{sgn}(k)$. Theories on the same row are dual to each other. Here $\kappa = k + \text{sgn}(k)N$.

A.1.4 Level-rank dualities

Pure Chern-Simons theories are well known to enjoy level-rank dualities (see e.g. the relatively recent paper [40] and references therein). Level-rank duality asserts the equivalence of two Type I $U(N)$ theories or of a Type II $U(N)$ theory and an $SU(N)$ theory, when their ranks and levels are related as in table 1 below. Type I' theories also enjoy analogous level-rank dualities [40] that we do not explicitly enumerate here.

A.2 Chern-Simons-matter theories

A.2.1 Conventions for levels and ranks

In this paper we study Chern-Simons theories coupled to either scalars or fermions in the fundamental representation of the gauge group. Like their pure Chern-Simons counterparts, our Chern-Simons-matter theories are labelled by the levels and ranks of the Chern-Simons gauge theory.

Throughout this appendix we will assume that the number of fundamental scalars is smaller than N , and the number of fundamental fermions is smaller than $|k|$. Under these conditions we adopt the following convention for the definitions of the rank and level of a Chern-Simons-matter theory. A Chern-Simons-matter theory with an SU gauge group is said to have rank⁴⁵ N and level k if the following is satisfied: the low energy theory obtained by massing up all scalars with a positive squared mass (or a positive m_B^{cri} in the case of a critical boson theory) and by massing up all fermion fields with a mass whose sign is the same as that of the level k , is pure $SU(N)_k$ Chern-Simons theory. A similar convention applies to the Type II $U(N)_{k,k}$ and the Type I $U(N)_{k,\kappa}$ theories.

Now consider a Chern-Simons-matter theory coupled to n_{fer} fundamental fermions and n_{bos} fundamental boson fields. We assume that all our matter fields are massive. However not all fields need have masses of the same sign. Let $n_{\text{fer}}^- \leq n_{\text{fer}}$ fundamental fermions have (generic) masses whose sign is opposite to that of k . Also let $n_{\text{bos}}^- \leq n_{\text{bos}}$ fundamental bosons have (generic) negative squared masses (or negative m_B^{cri} in the case of the critical boson theory). Under these circumstances, the low energy dynamics of the $SU(N)_k$, Type II $U(N)_{k,k}$ or Type I $U(N)_{k,\kappa}$ theories is governed by pure $SU(N_{\text{eff}})_{k_{\text{eff}}}$, Type II $U(N_{\text{eff}})_{k_{\text{eff}},k_{\text{eff}}}$ or Type I $U(N_{\text{eff}})_{k_{\text{eff}},\kappa_{\text{eff}}}$ Chern-Simons theories with rank N_{eff} and level k_{eff} where

$$N_{\text{eff}} = N - n_{\text{bos}}^-, \quad k_{\text{eff}} = k - n_{\text{fer}}^-, \quad \kappa_{\text{eff}} = k_{\text{eff}} + \text{sgn}(k_{\text{eff}})N_{\text{eff}}. \quad (\text{A.17})$$

The assumption spelt out at the beginning of the previous paragraph ensures that N_{eff} is always positive and that k_{eff} has the same sign as k .

⁴⁵The rank of the $SU(N)$ group is $N - 1$ and *not* N . However, we persist with this mild imprecision for the sake of brevity.

The first of (A.17) follows because the negative mass fundamental scalars induce a Higgs mechanism which breaks the $SU(N)$ gauge group down to $SU(N_{\text{eff}})$. The second of (A.17) follows because integrating out fermions shifts the low energy Chern-Simons level in a manner that depends the sign of the masses of the integrated-out fermions (more below).

A.2.2 Lagrangians

Consider a Chern-Simons-matter theory as defined in the previous subsection. Let us define⁴⁶

$$\tilde{k} = k - \text{sgn}(k) \frac{n_{\text{fer}}}{2}, \quad \tilde{\kappa} = \kappa - \text{sgn}(\kappa) \frac{n_{\text{fer}}}{2}. \quad (\text{A.18})$$

The gauge field Lagrangians for the $SU(N)$, Type II and Type I Chern-Simons-matter theories are given by the Lagrangians in section A.1.2 once we make the replacements

$$k \rightarrow \tilde{k}, \quad \kappa \rightarrow \tilde{\kappa}. \quad (\text{A.19})$$

Note that the ‘levels’ that appear in the Lagrangians depend only on the total number of fermionic fields and not on their masses. This is what we would expect of a UV Lagrangian.⁴⁷

Within either the Yang-Mills or the dimensional regularization schemes, integrating out a massive boson leaves all Chern-Simons levels unchanged. On the other hand integrating out a massive fermion of mass m_F leaves the rank of Chern-Simons theory ($SU(N)$, Type II or Type I) unchanged but shifts the coefficient c (see (A.8)) of the Chern-Simons part of the Lagrangian⁴⁸ according to the formula

$$c \longrightarrow c + \frac{1}{2} \text{sgn}(m_F). \quad (\text{A.20})$$

After this renormalization, the effective IR level of the Chern-Simons theory is given by

$$k_{\text{eff}} = k - n_{\text{fer}}^-, \quad (\text{A.21})$$

giving an explanation of (A.17) (see around (A.17) for the definition of n_{fer}^-).

A.2.3 Level-rank duality

An $SU(N)_k$, Type II $U(N)_{k,k}$, Type I $U(N)_{k,\kappa}$ Chern-Simons-matter theory at rank N and level k with n_{ferm} fundamental fermions and n_{bos} fundamental bosons is conjectured to be dual to a Type II $U(N')_{k',k'}$, $SU(N')$, Type I $U(N')_{k',\kappa'}$ Chern-Simons-matter theory respectively with n'_{ferm} fundamental fermions and n'_{bos} bosons where

$$k' = -\text{sgn}(k)N, \quad N' = |k|, \quad n'_{\text{bos}} = n_{\text{fer}}, \quad n'_{\text{fer}} = n_{\text{bos}}. \quad (\text{A.22})$$

These conjectures are summarised in table 2 below. Upon integrating out massive matter,

⁴⁶Note that $\tilde{k}$ is the average of the level of the low energy Chern-Simons theory (A.17) obtained by massing up all fermions with a mass that has the same sign as k , and the level of the low energy Chern-Simons theory (A.17) obtained by massing up all fermions with a mass that has the opposite sign as k .

⁴⁷One should be able to deform a UV CFT by mass terms of either sign, so the UV definition of CFT itself should be insensitive to the sign of the masses.

⁴⁸The $U(1)$ levels are also shifted in the manner needed so as to keep the Type I or Type II theory respectively Type I or Type II.

Type I - Type I	$U(N)_{k,\kappa}$	$U(N')_{k',\kappa'}$
Type II - $SU(N)$	$U(N)_{k,k}$	$SU(N')_{k'}$
$SU(N)$ - Type II	$SU(N)_k$	$U(N')_{k',k'}$

Table 2. This table lists conjecturally dual pairs of theories. Theories in the same row of the table are conjecturally dual provided that the levels, ranks and the number of fermionic and bosonic fields are related as described in (A.22) this subsection. See earlier in this appendix for a detailed discussion of the conventions for the levels in Chern-Simons-matter theories.

the theories listed on the two sides of the dualities listed in 2 each reduce to pure Chern-Simons theories of the same type (i.e. $SU(N)$, Type II or Type I) as their parent matter theories, but with level and rank $(N_{\text{eff}}, k_{\text{eff}})$ and $(N'_{\text{eff}}, k'_{\text{eff}})$ respectively. For every choice of matter masses it is easily verified that

$$k'_{\text{eff}} = -\text{sgn}(k_{\text{eff}})N_{\text{eff}}, \quad N'_{\text{eff}} = |k_{\text{eff}}|. \quad (\text{A.23})$$

As a consequence, these low energy topological theories can be independently verified to be dual to each other using the known level-rank dualities listed in table 1. This consistency check of the Chern-Simons-matter dualities was pointed out in [11].

A.3 Global Symmetries

Each of the Chern-Simons-matter theories described above enjoys invariance under a $U(1)$ global symmetry. In this subsection we will describe the action of each of these symmetries by describing how each of the Lagrangians above is modified when we turn on a background field C_μ that couples to the global $U(1)$ symmetry.⁴⁹

In all the theories studied in this paper, the Lagrangian for the $SU(N)$ part y of the dynamical gauge field is unaffected by the coupling to C : there are no direct couplings between C and any term in the Lagrangian involving y . For this reason, in all the Lagrangians presented in this section we omit the Lagrangian for the $SU(N)$ gauge fields; the relevant terms continue to be given by (A.11), (A.12) with the replacements (A.19). However, as the $U(1)$ part of the dynamical gauge field in the $U(N)$ theories does couple to C , we keep all terms in the Lagrangian involving dynamical $U(1)$ gauge fields in the formulae below. As these terms are the same in the dimensional regularization and the Yang-Mills regularization schemes (see (A.11), (A.12)) the discussions of this section apply in both schemes. In places where the $SU(N)$ Chern-Simons action is relevant, we work with the Yang-Mills regularization scheme in this subsection.

A.3.1 $SU(N)$ theories

The $SU(N)$ Chern-Simons-matter theories in this paper are coupled to the background gauge field C as follows. The matter fields that the gauge fields in (A.25) interact with are each assigned a charge q under the gauge field C . The full Lagrangian of our theory —

⁴⁹The chemical potential we have used in the main text of this paper is the imaginary holonomy of the gauge field C .

including couplings of matter fields to C_μ — is obtained as follows. All derivatives acting on matter fields are modified to

$$D_\mu = \partial_\mu - iqC_\mu .$$

In particular, for the Chern-Simons matter theories in this paper we assign the charge $q = 1$ to every fundamental field and $q = -1$ to every antifundamental field. In other words, we replace the $SU(N)$ gauge fields y_μ that appear in the Chern-Simons-matter Lagrangians (2.3), (2.3), (2.4), (2.5) by

$$z_\mu = y_\mu + C_\mu \mathbb{K}_N , \quad (\text{A.24})$$

(where y_μ is an $SU(N)$ matrix and $\mathbb{K}_N$ is the $N \times N$ identity matrix). Note that the gauge field z_μ is a gauge field for the $U(N)$ gauge group with its trace being the non-dynamical, background gauge field C .

In addition to the terms coupling C with matter above, our Lagrangians are required to have a pure C ‘contact’ Lagrangian:

$$\mathcal{S}[\dots; C] = \frac{i\tilde{L}}{4\pi} \int C dC , \quad (\text{A.25})$$

where the ellipsis $\dots$ depicts the dependence on the matter fields and the $SU(N)$ gauge field which we have suppressed above. The level $\tilde{L}$ must have certain integrality properties but also has some arbitrariness which, when fixed, gives the precise definition of the (contact terms in) the symmetry current correlators of the theory. We pause now to explain in more detail the extent to which $\tilde{L}$ is fixed and the degree of its ambiguity.

From requirements of consistency, $\tilde{L}$ is constrained to take the form

$$\tilde{L} = \tilde{k}N + LN^2 . \quad (\text{A.26})$$

To see why this is the case, let us integrate out all fermions which we assume to be massive (for simplicity we assume we are in a phase without any Higgsing due to the bosonic matter). Since the fermions are minimally coupled to the dynamical $SU(N)$ gauge field and the background field C , this integrating-out procedure replaces $\tilde{k}$ with k_{eff} both in (A.26) and in the $SU(N)$ part of the Lagrangian. Restoring the $SU(N)$ part of the lagrangian (and working in the Yang-Mills regularization scheme), plugging in (A.26) results in the effective low energy Lagrangian

$$\mathcal{S}[\dots, y; C] = \frac{ik_{\text{eff}}}{4\pi} \int \text{Tr} \left(y dy - \frac{2i}{3} y^3 \right) + \frac{iN(k_{\text{eff}} + LN)}{4\pi} \int C dC , \quad (\text{A.27})$$

or, in terms of the $U(N)$ gauge field $z = y + C\mathbb{K}_N$ in (A.24),

$$\mathcal{S}[\dots, z; C] = \frac{ik_{\text{eff}}}{4\pi} \int \text{Tr} \left(z dz - \frac{2i}{3} z^3 \right) + \frac{iLN^2}{4\pi} \int C dC , \quad \text{with} \quad C_\mu = \frac{1}{N} \text{Tr} z_\mu . \quad (\text{A.28})$$

The above action is formally the same as the action for the $U(N)$ theory in (A.14) though the $U(1)$ field C_μ here is non-dynamical as opposed to the dynamical $U(1)$ gauge field a_μ

in (A.14). The same arguments that led to the quantization rule (A.16) tell us that the level L in (A.27) should be an integer.

Recall that the gauge field configuration that was considered in section (A.1.3) consisted of one unit of spatial magnetic flux and one unit of holonomy around the time circle for the trace of the $U(N)$ gauge field i.e. $\text{Tr } z_\mu = NC_\mu$. Thus, as in section A.1.3, the gauge field C_μ has a flux of $1/N$ and a holonomy of $2\pi/N$. The fact that the background field C has fractional flux on admissible gauge configurations tells us that it is not normalised correctly to be viewed as a $U(1)$ gauge field. It follows from the previous discussion that the correctly normalised $U(1)$ gauge field is⁵⁰

$$\tilde{C}_\mu = NC_\mu . \quad (\text{A.29})$$

In terms of this field the low energy action (A.27) takes the form

$$\mathcal{S}[\dots, z; \tilde{C}] = \frac{ik_{\text{eff}}}{4\pi} \int \text{Tr} \left(z dz - \frac{2i}{3} z^3 \right) + \frac{iL}{4\pi} \int \tilde{C} d\tilde{C}, \quad \text{with} \quad \tilde{C}_\mu = \text{Tr } z_\mu . \quad (\text{A.30})$$

Wording the action in terms of the $SU(N)$ gauge field y_μ and the background field C_μ and suppressing the dependence on the $SU(N)$ gauge field as mentioned earlier, we find that the action is

$$\mathcal{S}[\dots; \tilde{C}] = \frac{i}{4\pi} \left(\frac{k_{\text{eff}}}{N} + L \right) \int \tilde{C} d\tilde{C} . \quad (\text{A.31})$$

The above equation (A.31), is, of course, just (A.25) re-expressed in terms of the field $\tilde{C}$. Apart from the fractional shift of $\frac{k_{\text{eff}}}{N}$ that has its origins in coupling with the $SU(N)$ part, L is simply the level of the $U(1)$ Chern-Simons action for the background field $\tilde{C}$.

There is another way to understand why $\tilde{C}$, rather than C , is the correctly normalised $U(1)$ field. Under the field C each fundamental / antifundamental field carries charges ± 1 .⁵¹ However, fundamental / antifundamental states are not $SU(N)$ gauge-invariant. Every charged operator that is $SU(N)$ gauge-invariant (and every charged state that is $SU(N)$ gauge-invariant on S^2) is built out of $SU(N)$ Levi-Civita tensors and so carries charge that are quantized in units of N under the gauge field C_μ (see below for further discussion of these *baryonic* operators). On the other hand, these states carry charges quantized in units of 1 under the gauge field $\tilde{C}$. This explains why $\tilde{C}$, rather than C , is the correctly normalised $U(1)$ gauge field.

A.3.2 Type II theories

The $U(1)$ global symmetries of the Type II and Type I theories are of the ‘topological’ sort. As a consequence, the background field C interacts only with the dynamical gauge

⁵⁰Concretely, $\tilde{C}$ transforms under background global gauge transformations as $\tilde{C}_\mu \rightarrow \tilde{C}_\mu + \partial_\mu \alpha$ where α is $U(1)$ valued, i.e. obeys $\alpha \sim \alpha + 2\pi$.

⁵¹Note that the above statement is accurate for fermions and only for scalar excitations in the unHiggsed phase of the bosonic theories. The W boson excitations in the Higgsed phase carry charge $\frac{N_B}{N_B-1}$ under the gauge field C ; see section 2.9.7 of [65]. It follows that $SU(N_B-1)$ baryons constructed out of N_B-1 W bosons continue to carry charge N_B under the gauge field C , or charge unity under the rescaled gauge field $\tilde{C}$.

fields of theory and not directly with matter fields. Recall that the $U(1)$ part of the gauge Lagrangian for Type II theory is

$$\mathcal{S}[a] = \frac{i\tilde{k}N}{4\pi} \int a da, \quad (\text{A.32})$$

where a is the dynamical $U(1)$ gauge field and $\tilde{k}$ was defined in (A.18). After coupling with the background field C , this ‘UV’ action takes the form

$$\mathcal{S}[a; C] = \frac{i\tilde{k}N}{4\pi} \int a da - \frac{i\tilde{k}N}{2\pi} \int C da + \frac{iL_{\text{II}}k^2}{4\pi} \int C dC. \quad (\text{A.33})$$

In the above, the gauge field C couples to the topological current

$$J^\mu = \frac{kN}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu a_\rho = \frac{k}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu \text{Tr } x_\rho, \quad (\text{A.34})$$

where x_μ is the dynamical $U(N)$ gauge field. The normalisation of the current is such that for a monopole of flux +1 w.r.t. one of the $U(1)$ subgroups of $U(N)$, say the one with generator $\text{diag}(1, 0, \dots, 0)$, the electric charge corresponding to the current (A.34) is k . More generally, the charges of all operators in this theory are quantized in units of k . As in the discussion around (A.29), it follows that C_μ in (A.33) is not quite correctly normalised as a $U(1)$ gauge field, but that the rescaled field

$$\tilde{C}_\mu = kC_\mu, \quad (\text{A.35})$$

is indeed correctly normalised (note that the charge of monopole operators is quantized in units of unity under the field $\tilde{C}_\mu$). The action (A.33) can thus be rewritten as

$$\mathcal{S}_{\text{Type II}}[a; C] = \frac{i\tilde{k}N}{4\pi} \int a da - \frac{iN}{2\pi} \int \tilde{C} da + \frac{iL_{\text{II}}}{4\pi} \int \tilde{C} d\tilde{C}, \quad (\text{A.36})$$

As $\tilde{C}_\mu$ is a properly normalised $U(1)$ gauge field, it follows that L_{II} must be an integer.

Integrating out matter fields renormalizes $\tilde{k}$ to k_{eff} and N to N_{eff} according to (A.17) and we obtain a low-energy effective action of the form

$$\mathcal{S}[a'; C] = \frac{ik_{\text{eff}}N_{\text{eff}}}{4\pi} \int a' da' - \frac{ik_{\text{eff}}N_{\text{eff}}}{2\pi} \int C da' + \frac{iL_{\text{II}}k^2}{4\pi} \int C dC, \quad (\text{A.37})$$

where a' is the $U(1)$ gauge field given by $1/N_{\text{eff}}$ times the trace of the $U(N_{\text{eff}})$ gauge field. Note that this related to the $U(1)$ gauge field part of the $U(N)$ gauge field a by $a' = \frac{N_{\text{eff}}}{N}a$. Once again, we may re-express the above action in terms of the correctly normalised $\tilde{C}$ (A.35)

$$\mathcal{S}[a'; C] = \frac{ik_{\text{eff}}N_{\text{eff}}}{4\pi} \int a' da' - \frac{iN_{\text{eff}}}{2\pi} \int \tilde{C} da' + \frac{iL_{\text{II}}}{4\pi} \int \tilde{C} d\tilde{C}. \quad (\text{A.38})$$

The a' equation of motion that follows from the above action (A.37) ensures that a single unit of $U(N_{\text{eff}})$ flux traps k_{eff} fundamental fields while the $U(1)$ charge of the single unit of flux is k . It follows that each fundamental field effectively carries the charge k/k_{eff}

under the field C and the charge $1/k_{\text{eff}}$ under the correctly normalised background field $\tilde{C}$. Sometimes it will also be useful to define another incorrectly normalised background field $k_{\text{eff}}C' = \tilde{C} = kC$ under which the fundamental field charge 1. In terms of this field C' , we have

$$\mathcal{S}[a'; C'] = \frac{i k_{\text{eff}} N_{\text{eff}}}{4\pi} \int a' da' - \frac{i k_{\text{eff}} N_{\text{eff}}}{2\pi} \int C' da' + \frac{i L_{\text{II}} k_{\text{eff}}^2}{4\pi} \int C' dC' . \quad (\text{A.39})$$

Specialising to the case of the regular fermion and critical fermion theories with one flavour of fermion with mass m_F , it follows that the fundamental fermion effectively carries unit charge under C when $m_F k_F > 0$ but carries charge $\frac{|k_F|}{|k_F|-1}$ under this field when $m_F k_F < 0$, matching the charge of scalar excitations and W boson excitations under $SU(N)$ – Type II duality. In the case of the critical boson or regular boson theories with one flavour of boson, on the other hand, $k_{\text{eff}} = k_B$ in both phases, so the global charge of scalar excitations in the unHiggsed phase and the W bosons in the Higgsed phase are both unity. This charge assignment is also consistent with $SU(N_F)$ – Type II duality (recall that $SU(N_F)$ fermions carry unit global charge for both signs of the fermion mass).

A.3.3 Type I theories

The $U(1)$ part of the gauge Lagrangian for the Type I theory is

$$\mathcal{S}[a] = \frac{i\tilde{\kappa}N}{4\pi} \int a da . \quad (\text{A.40})$$

($\tilde{\kappa}$ was defined in (A.18)). Upon coupling to the background gauge field C_μ , this action is modified in the UV to

$$\mathcal{S}_{\text{Type I}}[a; C] = \frac{i\tilde{\kappa}N}{4\pi} \int a da - \frac{i\kappa N}{2\pi} \int C da + \frac{i\kappa^2 L_{\text{I}}}{4\pi} \int C dC , \quad (\text{A.41})$$

The $U(1)$ global charge of a monopole of flux $+1$ w.r.t. one of the $U(1)$ subgroups of $U(N)$ carries charge κ , and so all gauge-invariant operators in this theory carry global $U(1)$ charges that are quantized in units of κ . It follows that C in (A.34) is not correctly normalised, and the correctly normalised $U(1)$ field is the rescaled field

$$\tilde{C}_\mu = \kappa C_\mu . \quad (\text{A.42})$$

The coupling (A.41) can be rewritten in terms of the field $\tilde{C}$ as

$$\mathcal{S}[a; \tilde{C}] = \frac{i\tilde{\kappa}N}{4\pi} \int a da - \frac{iN}{2\pi} \int \tilde{C} da + \frac{iL_{\text{I}}}{4\pi} \int \tilde{C} d\tilde{C} . \quad (\text{A.43})$$

Since $\tilde{C}$ is a correctly normalised field, it follows that the quantity L_{I} can be an arbitrary integer.

Accounting for the shifts in level and rank (A.17) coming from integrating out massive fermions and bosons, the action becomes

$$\mathcal{S}[a'; C] = \frac{i\tilde{\kappa}_{\text{eff}} N_{\text{eff}}}{4\pi} \int a' da' - \frac{i\kappa N_{\text{eff}}}{2\pi} \int C da' + \frac{i\kappa^2 L_{\text{I}}}{4\pi} \int C dC , \quad (\text{A.44})$$

where, once again, a' is the $U(1)$ gauge field associated to the $U(N_{\text{eff}})$ gauge field and is related to the $U(1)$ part a of the $U(N)$ gauge field as $a' = \frac{N}{N_{\text{eff}}}a$. The above action (A.44) can also be rewritten in terms of $\tilde{C}$ yielding

$$\mathcal{S}[a'; \tilde{C}] = \frac{i\tilde{\kappa}_{\text{eff}}N_{\text{eff}}}{4\pi} \int a' da' - \frac{iN_{\text{eff}}}{2\pi} \int \tilde{C} da' + \frac{iL_I}{4\pi} \int \tilde{C} d\tilde{C}, \quad (\text{A.45})$$

Proceeding along the lines of the previous subsection we see that a fundamental excitation carries an effective $U(1)$ charge of magnitude $\frac{\kappa}{\kappa_{\text{eff}}}$ under the gauge field C . As κ and κ_{eff} both switch sign under duality, charges of dual fundamental excitations match under duality.⁵² It is helpful to define another incorrectly normalised background field C' given by $\kappa_{\text{eff}}C' = \tilde{C} = \kappa C$ such that the fundamental excitations have charge 1 under C' . In terms of this field, the action (A.45) becomes

$$\mathcal{S}[a'; C'] = \frac{i\tilde{\kappa}_{\text{eff}}N_{\text{eff}}}{4\pi} \int a' da' - \frac{i\kappa_{\text{eff}}N_{\text{eff}}}{2\pi} \int C' da' + \frac{iL_I\kappa_{\text{eff}}^2}{4\pi} \int C' dC'. \quad (\text{A.46})$$

A.4 A precise statement of duality including background gauge fields

We present the actions including Lagrangians for background gauge fields for the dual pairs $SU(N)_k \longleftrightarrow U(N')_{k',k'}$ and $U(N)_{k,\kappa} \longleftrightarrow U(N')_{k',\kappa'}$. Recall that the duality of ranks and levels are given by (A.22) which we reproduce here for convenience:

$$k' = -\text{sgn}(k)N, \quad N' = |k|. \quad (\text{A.47})$$

A.4.1 $SU(N)$ and Type II

The precise statement of the $SU(N)$ – Type II duality is summarised by the following two equivalent equations

$$\frac{i\tilde{k}N}{4\pi} \int C dC \longleftrightarrow \frac{i\tilde{k}'N'}{4\pi} \int a da - \frac{iN}{2\pi} \int \tilde{C} da, \quad \text{with } \tilde{C} = k'C, \quad (\text{A.48})$$

where we have again suppressed the gauge action for the $SU(N)$ and $SU(N')$ gauge fields on the l.h.s. and r.h.s. respectively and the dependence on the matter fields. Substituting $\tilde{C} = k'C$ on the right hand side of (A.48), we get

$$\frac{i\tilde{k}N}{4\pi} \int C dC \longleftrightarrow \frac{i\tilde{k}'N'}{4\pi} \int a da - \frac{iN'k'}{2\pi} \int C da. \quad (\text{A.49})$$

The above equations have the following meaning. They assert that the path integral of a bosonic (resp. fermionic) theory coupled to the $SU(N)_k$ Chern-Simons theory on the l.h.s. is the same as the path integral of the fermionic (resp. bosonic theory) coupled to the Type II $U(N')_{k',k'}$ theory on the r.h.s. .

Let us summarise (A.49) and (A.48) in words. They assert that the bosonic (resp. fermionic) $SU(N)_k$ theory with background level $L = 0$ (see (A.26)) is dual to the fermionic

⁵²More concretely, fundamental excitations carry unit charge in the Type I bosonic theories with unbroken gauge group and the Type I fermionic theories in which m_F has the same sign as k_F . On the other hand fundamental excitations carry charge $\frac{|\kappa_F|}{|\kappa_F|-1}$ in Type I bosonic theories with broken gauge group or Type I fermionic theories in which the sign of m_F is opposite to that of k_F .

(resp. bosonic) Type II $U(N')_{k',k'}$ theory with background level $L_{\text{II}} = 0$ (see (A.36)). More generally, the bosonic (resp. fermionic) $SU(N)$ theory with contact integer L and the fermionic (resp. bosonic) Type II theory with contact integer $L_{\text{II}} = 0$ are dual provided $L = L_{\text{II}}$.

A.4.2 Type I and Type I

The precise statement of Type I – Type I duality is

$$\frac{i\tilde{\kappa}N}{4\pi} \int ada - \frac{iN\kappa}{2\pi} \int adH \longleftrightarrow \frac{i\tilde{k}'N'}{4\pi} \int a'da' - \frac{iN'\kappa'}{2\pi} \int a'dH + \frac{i \operatorname{sgn}(\kappa')(\kappa')^2}{4\pi} \int H dH . \quad (\text{A.50})$$

Equivalently, in terms of the properly normalised $U(1)$ field⁵³ $\tilde{H} = \kappa H = -\kappa' H$, we have

$$\frac{i\tilde{\kappa}N}{4\pi} \int ada - \frac{iN}{2\pi} \int ad\tilde{H} \longleftrightarrow \frac{i\tilde{k}'N'}{4\pi} \int a'da' + \frac{iN'}{2\pi} \int a'd\tilde{H} + \frac{i \operatorname{sgn}(\kappa')}{4\pi} \int \tilde{H} d\tilde{H} , \quad (\text{A.51})$$

In words a bosonic (resp. fermionic) Type I theory with rank and level (N, k) and with background level $L_{\text{I}} = 0$ is dual to the fermionic (resp. bosonic) Type I theory with rank and level (N', k') and with background level $L'_{\text{I}} = \operatorname{sgn}(k')$. More generally, the two Type I theories are dual to each other provided the background levels on the l.h.s. and r.h.s. satisfy

$$L'_{\text{I}} - L_{\text{I}} = \operatorname{sgn}(k') = -\operatorname{sgn}(k) .$$

A.5 Consistency check for conjectured dualities

In this subsection we will present two consistency checks of the dualities described in the previous subsection. The consistency checks presented below are essentially identical to those presented in [40], and are included here for completeness, and also to have a clear presentation of the dualities in the notation used in this paper.

The consistency checks essentially involve the following operations on both sides of a given duality. We promote the background field C to a dynamical gauge field c , add the coupling $\int H dc$ to a new background gauge field and finally integrate over c in the path integral. If the old duality is consistent, we expect to recover another duality with the correct actions including the correct background levels on either side of the new duality.

A.5.1 Self-consistency of $SU(N)$ – Type II duality

We start with an $SU(N)_k \longleftrightarrow U(N')_{k',k'}$ duality. It follows from (A.48) that

$$\begin{aligned} \frac{i\tilde{k}N}{4\pi} \int cdc - \frac{iN}{2\pi} \int cd\tilde{H} &\longleftrightarrow \frac{i\tilde{k}'N'}{4\pi} \int ada - \frac{iN'k'}{2\pi} \int cda - \frac{iN}{2\pi} \int cd\tilde{H} , \\ &\longleftrightarrow \frac{i\tilde{k}'N'}{4\pi} \int ada - \frac{iN'}{2\pi} \int \tilde{c}da - \frac{i \operatorname{sgn}(k')}{2\pi} \int \tilde{c}d\tilde{H} , \\ &\longleftrightarrow \frac{i\tilde{k}'N'}{4\pi} \int ada , \quad N'a = -\operatorname{sgn}(k')\tilde{H} , \\ &\longleftrightarrow \frac{i\tilde{k}'N'}{4\pi} \int H dH , \quad N'H = \tilde{H} = -\operatorname{sgn}(k')N'a = ka . \end{aligned} \quad (\text{A.52})$$

⁵³Note that fundamental fields carry unit charge under the field H_μ on both sides of the duality. Under the properly normalised gauge field $\tilde{H}$, on the other hand, fundamental fields carry charge $\frac{1}{\kappa}$ on the l.h.s. of the duality, while fundamental fields on the r.h.s. of the duality carry the opposite charge $\frac{1}{\kappa'} = -\frac{1}{\kappa}$.

We have started with (A.48), renamed the gauge field C_μ as c_μ , added the same contact term⁵⁴ $-\frac{iN}{2\pi} \int cd\tilde{H}$ to both sides and then we have path integrated over c (i.e. made it a dynamical gauge field). In going from the first to the second line of (A.52) we have made the substitution $k'c = \tilde{c}$ and have used the fact that $k' = \text{sgn}(k')N$. In going from the second to the third line we have integrated out the Lagrange multiplier field $\tilde{c}_\mu$ (note that the equation $N'a' = -\text{sgn}(k')\tilde{H}$ is sensible because both sides of the equation involved a properly normalised $U(1)$ gauge field). In going from the third to the fourth line we have substituted the incorrectly normalised background field H in place of the correctly normalised field $\tilde{H}$.

The final result of the manipulations in (A.52) is that

$$\frac{i\tilde{k}N}{4\pi} \int cdc - \frac{iN}{2\pi} \int cd\tilde{H} \longleftrightarrow \frac{i\tilde{k}'N'}{4\pi} \int HdH. \quad (\text{A.53})$$

This is simply (A.48) with the replacement

$$(k, N) \leftrightarrow (k', N').$$

Let us summarize. The starting point of the manipulations (A.52) was a duality in which the matter on the l.h.s. is $SU(N)$ coupled with $L = 0$ and the matter on the r.h.s. is $U(N')_{k',k'}$ coupled with $L_{\text{II}} = 0$. The end point is the duality in which the matter on the r.h.s. is $SU(N')$ gauged with $L = 0$ and the matter on the l.h.s. is coupled to a $U(N)_{k,k}$ theory with $L_{\text{II}} = 0$.

It follows that the discussion of this section can be regarded as a consistency check of the proposed $SU(N)$ -Type II duality.

A.5.2 Type I – Type I from $SU(N)$ – Type II

As the starting point of the analysis of this section we use the $SU(N)$ -Type II duality (A.49) with contact term

$$\frac{i \text{sgn}(k)N^2}{4\pi} \int CdC = -\frac{i \text{sgn}(k')(k')^2}{4\pi} \int CdC,$$

added to both sides,⁵⁵ i.e. with

$$\frac{i\tilde{k}N}{4\pi} \int CdC \longleftrightarrow \frac{i\tilde{k}'N'}{4\pi} \int ada - \frac{iN'k'}{2\pi} \int Cda - \frac{i \text{sgn}(k')(k')^2}{4\pi} \int CdC. \quad (\text{A.54})$$

Renaming C as c , adding $-\frac{iN}{2\pi} \int cd\tilde{H}$ to both sides of (A.54) and then promoting c to a dynamical field (i.e. path integrating over it) we find

$$\begin{aligned} \frac{i\tilde{k}N}{4\pi} \int cdc - \frac{iN}{2\pi} \int cd\tilde{H} \longleftrightarrow \\ \frac{i\tilde{k}'N'}{4\pi} \int ada - \frac{iN'k'}{2\pi} \int cda - \frac{i \text{sgn}(k')(k')^2}{4\pi} \int cdc - \frac{i \text{sgn}(k')k'}{2\pi} \int cd\tilde{H}. \end{aligned} \quad (\text{A.55})$$

⁵⁴ $\tilde{H}$ is a properly normalised $U(1)$ gauge field; note that the integrality properties of the coefficients of all terms in the first line of (A.52) are consistent with this.

⁵⁵In other words the $SU(N)$ Type II duality with $L = L_{\text{II}} = \text{sgn}(k) = -\text{sgn}(k')$.

The right hand side of (A.55) can be simplified as follows:

$$\begin{aligned}
\text{r.h.s.} &= \frac{i\tilde{k}'N'}{4\pi} \int ada - \frac{iN'}{2\pi} \int \tilde{c}da - \frac{i \operatorname{sgn}(k')}{4\pi} \int \tilde{c}d\tilde{c} - \frac{i \operatorname{sgn}(k')}{2\pi} \int \tilde{c}d\tilde{H}, \\
&= \frac{i\tilde{k}'N'}{4\pi} \int ada + \frac{i \operatorname{sgn}(k')}{4\pi} \int \left(\tilde{H} + \operatorname{sgn}(k')N'a \right) d \left(\tilde{H} + \operatorname{sgn}(k')N'a \right), \\
&= \frac{i\tilde{k}'N'}{4\pi} \int ada + \frac{iN'}{2\pi} \int a'd\tilde{H} + \frac{i \operatorname{sgn}(k')}{4\pi} \int \tilde{H}d\tilde{H}.
\end{aligned} \tag{A.56}$$

In going from (A.55) to the first line of (A.56) we have moved to the new integration variable $\tilde{c} = k'c$. In going from the first to the second line of (A.56) we have completed the square in $\tilde{c}$ and have also use the fact that the path integral

$$-\frac{i \operatorname{sgn}(k')}{4\pi} \int fdf, \quad f = \tilde{c} - \tilde{H} + \operatorname{sgn}(k')N'a',$$

is trivial and can be dropped⁵⁶ [40]. Moving to the background gauge field H_μ defined by $\tilde{H} = \kappa H = -\kappa' H$, we get

$$\frac{i\tilde{\kappa}N}{4\pi} \int cdc - \frac{iN\kappa}{2\pi} \int cdH \sim \frac{i\tilde{k}'N'}{4\pi} \int ada - \frac{iN'\kappa'}{2\pi} \int adH + \frac{i \operatorname{sgn}(\kappa')(\kappa')^2}{4\pi} \int HdH, \tag{A.57}$$

where now, the dynamical gauge field c on the left hand side is the $U(1)$ part of the $U(N)$ gauge field of a Type I theory. Similarly, the gauge field a on the right hand side, which was originally the $U(1)$ gauge field in a Type II theory, now becomes a $U(1)$ gauge field of a Type I theory due to the change in its level. Thus, (A.57) is in perfect agreement with the statement of the Type I–Type I duality (A.50), including the correct relative values of the background levels L_I and L'_I . Starting from the $SU(N)$ –Type II duality we have thus been able to derive the Type I–Type I duality.

A.6 Charged gauge-invariant operators

As we have mentioned in the main text in section 2.2, the spectrum of gauge invariant operators in the theories under study include ‘single-sum’ operators (operators that are constructed from the gauge contractions of derivatives acting on a fundamental and other derivatives acting on an antifundamental) whose dimension remains finite in the large N limit and more complicated operators whose dimension goes to infinity in the large N limit. In each case the single sum operators are uncharged under the global $U(1)$ symmetry. The only charged operators in the theory are the more complicated operators referred to above, whose nature we now describe.

A.6.1 $SU(N)$ theories

The complicated operators for $SU(N)$ theories are baryonic in nature. Baryon operators are given by the contraction of N fundamental fields with the colour Levi-Civita tensor

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⁵⁶The fact that the path integral for a level ± 1 $U(1)$ theory is trivial is essentially the statement that this theory is empty. This is familiar from many points of view. For instance, the Hilbert space of this theory on the torus has only one state. The vacuum is the only primary operator of the WZW theory dual to this Chern-Simons theory.

(and in general, with different numbers of derivatives acting on each of the N fields). There are many distinct baryon operators, labelled by the N distinct ‘letters’ (the structure of the derivative operators acting on the fundamental fields) used to construct the operator.

In an $SU(N)$ theory coupled to a single fundamental boson, the baryon operators are all bosonic in nature i.e. they have integral spins (all arising from the derivatives in the operator in the case of scalars and also additionally from the spin of the W bosons in the Higgsed phase). On the other hand, in an $SU(N)$ theory coupled to a single fundamental fermion, the baryon operators are bosonic when N is even and fermionic when N is odd.

Baryon operators built out of fundamental fields carry charge N under the gauge field C_μ in (A.25), but carry unit charge under the gauge field $\tilde{C}_\mu$ defined in (A.29).

A.6.2 Type II $U(N)$ theories

In the case of Type II $U(N)_{k,k}$ theories, the more complicated operators are monopole operators. A basic monopole operator consists of unit magnetic flux for trace of the $U(N)$ gauge field. The Chern-Simons coupling endows the operator with a electric gauge charge which is in the ‘totally symmetric’ $SU(N)$ colour representation with $|k|$ boxes and a charge of k for the $U(1)$ factor of the gauge group (i.e. this representation carries charge k under the gauge field a_μ in (A.32)). In order to make this gauge invariant, we have to attach $|k|$ copies of the fundamental (or antifundamental) field in the colour representation

$$\underbrace{\square \square \cdots \square}_{|k| \text{ boxes}}, \quad (\text{A.58})$$

(and in general, different numbers of derivatives acting on each of the $|k|$ fields). As in the case of the baryon operators, there are many distinct monopole operators labelled by the $|k|$ distinct ‘letters’ (derivatives acting on fundamental fields).

The spin of a monopole operator is given by the sum of the intrinsic spin s_{int} and the statistical spin s_{stat}

$$s = s_{\text{int}} + s_{\text{stat}}. \quad (\text{A.59})$$

The intrinsic spin is the sum of the spin of all the letters that go into the construction of the operator. The ‘statistical spin’ is given by the formula

$$s_{\text{stat}} = -\frac{C_2(R)}{\kappa} - \frac{Q^2}{2Nk}, \quad (\text{A.60})$$

where $C_2(R)$ is the quadratic Casimir of the $SU(N)$ representation R with Young diagram (A.58) and Q is the $U(1)$ charge of this representation. It is not difficult to verify that [86]

$$C_2(R) = \frac{1}{2} \left(k^2 - |k| + N|k| - \frac{k^2}{N} \right), \quad Q = |k|, \quad \text{and hence} \quad s_{\text{stat}} = -\frac{k}{2}. \quad (\text{A.61})$$

In the case of the theory with only bosonic matter the intrinsic spin s_{int} of this operator is always integral (and comes only from the derivatives for scalar excitations and also additionally from the W bosons in the Higgsed phase). On the other hand, the statistical

spin s_{stat} given in (A.61) is integral or half-integral depending on whether k is even or odd. It follows that the monopole operator has integer spin (and hence bosonic) or half-integer spin (and hence fermionic) depending on whether k is even or odd respectively.

In the case of the theory with only fermionic matter, the intrinsic spin of the monopole operator is integral or half-integral depending on whether k is even or odd respectively. The statistical spin of the operator continues to be integral or half-integral depending on whether k is even or odd respectively. Thus, monopole operators in Type II theories coupled to fermions are always bosonic.

Note that this maps perfectly to the statistics of the baryon operators of the dual $SU(N')$ theories with $N' = |k|$. When the Type II $U(N)$ theory has only bosonic (resp. fermionic) matter, the $SU(N')$ theory has only fermionic (resp. bosonic) matter. In the case that the Type II theory was coupled to fermionic matter, the monopole operator is bosonic when k is even and fermionic when k odd. The same is true of the baryon operators for the $SU(N')$ theory with bosonic matter since $N' = |k|$. Similarly, the Type II theory with fermionic matter has only bosonic monopole operators, which is in follows that, in this case, the in perfect agreement that the baryon operators of the $SU(N')$ theory with bosonic matter are always bosonic.

The monopole operator discussed in this subsection carries charge k under the gauge field C_μ , and charge unity under the rescaled gauge field $\tilde{C}_\mu$.

We re-emphasize that the monopole operator defined above has half-integer spin in the theory with bosonic matter when k is an odd integer. This tells us that the spectrum on S^2 of Chern-Simons theories coupled to bosonic fundamental matter includes fermions when k is odd. It follows that Chern-Simons theories coupled to bosons in the fundamental representation are secretly fermionic theories, and are well-defined only on manifolds with a specified spin structure.

The observations of the previous paragraph are a reflection of the following facts about pure Chern-Simons theory [40]. Despite the fact that the monopole operator described in this subsection appears to transform in the representation (A.58), it is actually gauge invariant (the gauge charges of the matter fields (A.58) are screened by the monopole flux). This screening is a reflection of the fact that the only representation that appears in the completely symmetric fusion of k fundamental primaries of WZW is the identity representation. The authors of [40] noted that Wilson lines in the representation (A.58) are ‘invisible’ in the sense of monodromy, but are not quite nothing because they carry half integral spin. These otherwise invisible fermionic lines have to be added to the spectrum, and make pure Chern Simons theory at odd k a theory that is only well defined on a spin manifold with a specified spin structure.

A.6.3 Type I $U(N)$ theories

The complicated operators in Type I $U(N)_{k,\kappa}$ are monopole operators. The electric gauge charge of these operators under the $U(1)$ factor of the $U(N)$ gauge group is now κ rather than k , while the $SU(N)$ representation is still the totally symmetric representation with $|k|$ boxes. The $SU(N)$ representation can be neutralised by attaching $|k|$ fundamental (or antifundamental) fields in the representation (A.58), but this is not sufficient to neutralise

the $U(1)$ charge. The solution is to take N additional fundamental fields but now transforming in a gauge singlet (in other words, a baryon-type operator). In other words, we have the following representations of the matter field attachments:

$$\underbrace{\square \square \cdots \square}_{|k| \text{ boxes}} \oplus N \text{ boxes} \left\{ \begin{array}{c} \square \\ \square \\ \vdots \\ \square \end{array} \right. \quad (\text{A.62})$$

The intrinsic spins of these operators get a contribution from the totally symmetric $|k|$ number of fundamental fields and an additional contribution from the totally antisymmetric baryon-type attachment with N fundamental fields. The statistical spin is now given by the formula

$$s_{\text{stat}} = -\frac{C_2(R)}{\kappa} - \frac{Q^2}{2(N\kappa)}, \quad (\text{A.63})$$

with $C_2(R)$ being the quadratic Casimir for the representation (A.58) whose value is displayed in (A.61), and Q is the $U(1)$ electric charge of the monopole operator which is given by κ . Plugging these into (A.63), we get [86]

$$s_{\text{stat}} = -\text{sgn}(k) \frac{|k| + 1}{2}. \quad (\text{A.64})$$

In the theory with bosonic matter, the baryon-type attachment as well as the fundamental fields in the totally symmetric representation (A.58) are bosonic and hence have integral intrinsic spin. The statistical spin is half-integral (resp. integral) when k is even (resp. odd). Thus, the monopole operator is a fermion when k is even and a boson when k is odd.

In the theory with fermionic matter, the baryon-type attachment has half-integral (resp. integral) intrinsic spin when N is odd (resp. even) and the $|k|$ fundamental fields in the totally symmetric representation have half-integral (resp. integral) spin when k is odd (resp. even). The statistical spin is integral (resp. half-integral) when k is odd (resp. even). Thus the contribution to the total spin from the $|k|$ fundamental fields in the totally symmetric representation and the statistical spin is always half-integral for any value of k . Thus, the monopole operator in fermionic theories are bosons when N is odd and fermions when N is even.

Since $|k|$ is mapped to N' and bosonic matter is mapped to fermionic matter under the Bose-Fermi duality, it is straightforward to check that the above statements are consistent with duality. The monopole operator discussed in this subsection carries charge κ under the gauge field C_μ , and charge unity under the rescaled gauge field $\tilde{C}_\mu$.

As in the previous subsection, the formula (A.64) tells us that even Type I theories coupled to bosonic matter have fermions in their spectrum, and so are well defined only on spin manifolds (and need a spin structure for their definition) when k is even. As in the previous subsection this fact is closely related to the fact that pure Type I Chern Simons theories have otherwise invisible fermionic line operators in their spectrum when k is even [40].

A.7 $SU(N)$ and $U(N)$ theories at finite chemical potential

In this brief subsection we will explain that even though the global symmetries for Type II and Type I theories are structurally very different from that of the $SU(N)$ theory, it is nonetheless the case that a finite chemical potential ensemble for Type II and Type I theories are identical to their $SU(N)$ counterparts in the large N limit.

Consider the Type II theory (A.37) in the presence of a nontrivial background gauge field C_μ . In this section, we work with the incorrectly normalised field $C' = \frac{k}{k_{\text{eff}}}C$ as defined above equation (A.39) such that the fundamental excitations of a particular phase of the theory have unit charge. We reproduce the action here for convenience:

$$\mathcal{S}[a'; C'] = \frac{ik_{\text{eff}}N_{\text{eff}}}{4\pi} \int a' da' - \frac{ik_{\text{eff}}N_{\text{eff}}}{2\pi} \int C' da' + \frac{iL_{\text{II}}k_{\text{eff}}^2}{4\pi} \int C' dC' . \quad (\text{A.65})$$

The equation of motion for the dynamical $U(1)$ gauge field a'_μ is

$$\frac{iN_{\text{eff}}k_{\text{eff}}}{2\pi} \epsilon^{\mu\nu\rho} (\partial_\nu a'_\rho - \partial_\nu C'_\rho) = J^\mu , \quad (\text{A.66})$$

where J^μ is the matter $U(1)$ current that couples to the dynamical gauge field a_μ . The solution to (A.66) clearly takes the form

$$a'_\mu = C'_\mu + \tilde{a}_\mu , \quad (\text{A.67})$$

where $\tilde{a}_\mu$ satisfies⁵⁷

$$\frac{iN_{\text{eff}}k_{\text{eff}}}{4\pi} \epsilon^{\mu\nu\rho} \tilde{f}_{\nu\rho} \equiv \frac{iN_{\text{eff}}k_{\text{eff}}}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu \tilde{a}_\rho = J^\mu . \quad (\text{A.68})$$

Turning on a chemical potential μ amounts to setting $C'_0 = i\mu$. It follows that

$$a_\nu = i\mu \delta_{\nu 0} + \tilde{a}_\nu . \quad (\text{A.69})$$

Had we been dealing with an $SU(N)$ rather than a Type II theory, a_μ would have been the background field and $a_\nu = i\mu \delta_{\nu 0}$ would have corresponded to turning on a chemical potential μ . The term $\tilde{a}_\nu$ on the r.h.s. of (A.69) is therefore captures the difference between turning on a chemical potential of magnitude μ in the Type II theory and the $SU(N)$ theory.

In order to understand the nature and magnitude of this difference, we note that turning on a chemical potential of order unity gives rise to a charge density of order N_{eff} . Two things immediately follow by consulting (A.68). First, $\tilde{a}_\nu$ corresponds to the gauge field for a uniform magnetic field, at least approximately since the charge density on the r.h.s. of (A.68) is approximately uniform. In other words, the effect of turning on a chemical potential for the Type II theory is similar to turning on both a chemical potential and a background magnetic field for the $SU(N)$ theory. Second, the magnitude of this magnetic field scales like $1/k_{\text{eff}}$. When k_{eff} is of order unity, this magnetic field plays a key role in

⁵⁷In order to get (A.68), we have used the fact that, in Euclidean space, $\epsilon_{\alpha\beta\rho}\epsilon^{\mu\nu\rho} = \delta_\alpha^\mu\delta_\beta^\nu - \delta_\beta^\mu\delta_\alpha^\nu$ with the choice $\epsilon_{123} = \epsilon^{123} = +1$.

the dynamics of the Type II theory at finite chemical potential. On the other hand, in the 't Hooft large N limit, this magnetic field is parametrically small and can be ignored.

Of course the discussion of a chemical potential in the Type I theory is very similar; in this case $\tilde{a}_\nu$ is the field strength corresponding to a magnetic field whose magnitude is of order $\frac{1}{\kappa_{\text{eff}}}$. It follows that there is effectively no difference between $SU(N)$, Type II and Type I theories at finite chemical potential at leading order in the 't Hooft large N limit.

B Duality of off-shell free energies and occupation numbers

We show that the off-shell free energies of the appropriate fermionic and bosonic theories map to each other under the duality map. This requires the mapping of levels and ranks (2.8)

$$N_B = |\kappa_F| - N_F, \quad \kappa_B = -\kappa_F, \quad (\text{B.1})$$

the identification of the UV parameters (2.11) for the quasi-fermionic theories

$$\lambda_B m_B^{\text{cri}} = -m_F, \quad (\text{B.2})$$

and (2.12) for the quasi-bosonic theories

$$x_6^B = x_6^F, \quad b_4 = y_4, \quad m_B^2 = y_2^2, \quad (\text{B.3})$$

the map between the holonomy distributions (3.5)

$$|\lambda_B| \rho_B(\pi - \alpha) + |\lambda_F| \rho_F(\alpha) = \frac{1}{2\pi}, \quad (\text{B.4})$$

and the following identification of the off-shell variables (3.21):

$$c_B = c_F, \quad \lambda_B \tilde{\mathcal{S}} = \lambda_F \tilde{\mathcal{C}} - \frac{1}{2} \text{sgn}(\lambda_F) c_F, \quad 2\lambda_B \sigma_B = -\frac{4\pi \zeta_F}{\kappa_F}. \quad (\text{B.5})$$

We divide the checking of duality into two parts. Note that the holonomy distributions appear only in one term in each of the free energies of the two bosonic and two fermionic theories. We first check that the terms independent of the holonomy distribution in the free energies map to each other under duality. We then separately check that the terms involving the holonomy distributions map to each other as well.

It is easy to check that the terms independent of the holonomy in the free energies of the critical boson and regular fermion map to each other provided we make the duality identifications (B.1), (B.2) and (B.5) (see e.g. around equations A.6 in [65]). Similarly, it is easy to check (see e.g. [71]) that the regular boson and critical fermion free energies (3.17) and (3.12)) map to each other once we make the duality identifications (B.1), (B.3) and (B.5). Next, we check the duality of the holonomy-dependent term in the free energies.

B.1 Duality of the off-shell free energy when $c_B > |\mu|$

It is useful to introduce the following notation for the holonomy integral in the last line of the each of the bosonic (see equations (3.14) and (3.17)) and fermionic (see equations (3.7) and (3.12)) free energies.

$$\begin{aligned}\mathcal{C}_1(\epsilon, |\mu|) &\equiv \int_{-\pi}^{\pi} d\alpha \, \rho_F(\alpha) \left(\log(1 + e^{-\hat{\epsilon} - |\hat{\mu}| - i\alpha}) + \log(1 + e^{-\hat{\epsilon} + |\hat{\mu}| + i\alpha}) \right), \\ \mathcal{S}_1(\epsilon, |\mu|) &\equiv \int_{-\pi}^{\pi} d\alpha \, \rho_B(\alpha) \left(\log(1 - e^{-\hat{\epsilon} - |\hat{\mu}| - i\alpha}) + \log(1 - e^{-\hat{\epsilon} + |\hat{\mu}| + i\alpha}) \right),\end{aligned}\quad (\text{B.6})$$

where, $\mathcal{C}_1$ is the holonomy integral in the last line of the free energies for both the regular (in (3.12)) and critical fermion (in (3.7)) while $\mathcal{S}_1$ is the holonomy integral for both the regular (in (3.17)) and critical bosonic (in (3.14)) theories. It is then easy to verify using the map (B.4) between the holonomy distributions that

$$|\lambda_B| \mathcal{S}_1(\epsilon, |\mu|) + |\lambda_F| \mathcal{C}_1(\epsilon, |\mu|) = 0 \quad \text{when} \quad \epsilon > |\mu|. \quad (\text{B.7})$$

To verify (B.7), we start with the expression for $\mathcal{S}_1(\epsilon, |\mu|)$ presented in the second line of (B.6), substitute ρ_B in terms of ρ_F using (B.4) and make the change of variables

$$\alpha = \pi - \tilde{\alpha}, \quad \text{when } 0 < \alpha < \pi, \quad \text{and} \quad \alpha = -\pi + \tilde{\alpha}, \quad \text{when } -\pi < \alpha < 0. \quad (\text{B.8})$$

We find

$$\mathcal{S}_1(\epsilon, |\mu|) = \frac{\lambda_F}{\lambda_B} \mathcal{C}_1(\epsilon, |\mu|) + \frac{1}{|\lambda_B|} \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \left(\log(1 - e^{-\hat{\epsilon} - |\hat{\mu}| - i\alpha}) + \log(1 - e^{-\hat{\epsilon} + |\hat{\mu}| + i\alpha}) \right). \quad (\text{B.9})$$

When $\epsilon > |\mu|$, the integral in the second term on the r.h.s. vanishes. One can see this by power-series-expanding each of the logarithms in $e^{-\hat{\epsilon} \pm (|\hat{\mu}| + i\alpha)}$ (which is less than 1 since $\hat{\epsilon} > |\hat{\mu}|$) and integrating term by term. Alternatively, we may convert the integral over α into a contour integral which nowhere intersects the branch cut of the log because $\epsilon > \mu$, and use Cauchy's theorem. We have thus established (B.7).

There is a further integral of $\mathcal{C}_1$ or $\mathcal{S}_1$ over the energy ϵ in each of the bosonic and fermionic free energies (3.14), (3.17), (3.12) and (3.7). The range of the ϵ integral is from $\hat{c}_B$ to infinity for the bosons and from $\hat{c}_F$ to infinity for the fermions. Since we work with values of μ such that $c_B > |\mu|$ and $c_F > |\mu|$, it follows that all the values of ϵ that appear in these integrals obey the condition $\epsilon > |\mu|$ and hence (B.7) applies. Thus, the terms with the holonomy integrals also map to each other under Bose-Fermi duality.

In summary, it follows that the off-shell free energies of the regular fermion and critical boson theories map to each other under duality and similarly for the critical fermion and regular boson theories. It follows that the expressions for the full thermal free energies map to each other under duality, at least when $|\mu| \leq c_B$ and $|\mu| \leq c_F$. In the next subsection, we check the duality of the modified off-shell free energies in the situation when $|\mu| > c_B$ or $|\mu| > c_F$.

B.2 Duality of the modified off-shell free energies for $c_B < |\mu|$

The holonomy dependent parts of the modified bosonic free energies (3.47) and (3.46)

$$\begin{aligned} \mathcal{S}'_1(\epsilon, |\mu|) = \int_{-\pi}^{\pi} d\alpha \, \rho_B(\alpha) \left(\log(1 - e^{-\hat{\epsilon} + |\hat{\mu}| + i\alpha}) + \log(1 - e^{-\hat{\epsilon} - |\hat{\mu}| - i\alpha}) \right) \\ - 2\pi\Theta(|\mu| - \epsilon) \int_{e^{\hat{\epsilon} - |\hat{\mu}|}}^1 dx \, \frac{\tilde{\rho}_B(x)}{x}, \end{aligned} \quad (\text{B.10})$$

and those of the modified fermionic free energies (3.44) and (3.45) are

$$\begin{aligned} \mathcal{C}'_1(\epsilon, |\mu|) = \int_{-\pi}^{\pi} d\alpha \, \rho_F(\alpha) \left(\log(1 + e^{-\hat{\epsilon} + |\hat{\mu}| + i\alpha}) + \log(1 + e^{-\hat{\epsilon} - |\hat{\mu}| - i\alpha}) \right) \\ - 2\pi\Theta(|\mu| - \epsilon) \int_{e^{\hat{\epsilon} - |\hat{\mu}|}}^1 dx \, \frac{\tilde{\rho}_F(-x)}{x}. \end{aligned} \quad (\text{B.11})$$

Similar to the discussion in the previous subsection, it is clearly sufficient to show that

$$|\lambda_B| \mathcal{S}'_1(\epsilon, |\mu|) + |\lambda_F| \mathcal{C}'_1(\epsilon, |\mu|) = 0, \quad (\text{B.12})$$

in order to establish that (B.10) and (B.11) map to each other under duality. We will now proceed to verify (B.12). We need the duality relations between the holonomies ρ_B , ρ_F (B.4) and the analytically continued holonomies $\tilde{\rho}_B$, $\tilde{\rho}_F$ (3.41) which we reproduce below:

$$|\lambda_B| \rho_B(\alpha) + |\lambda_F| \rho_F(\pi - \alpha) = \frac{1}{2\pi}, \quad |\lambda_B| \tilde{\rho}_B(z) + |\lambda_F| \tilde{\rho}_F(-z) = \frac{1}{2\pi}. \quad (\text{B.13})$$

It is then not difficult to verify that the l.h.s. of (B.12) simplifies to

$$\int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \left(\log(1 - e^{-\hat{\epsilon} + |\hat{\mu}| + i\alpha}) + \log(1 - e^{-\hat{\epsilon} - |\hat{\mu}| - i\alpha}) \right) - \Theta(|\mu| - \epsilon) \int_{e^{\hat{\epsilon} - |\hat{\mu}|}}^1 \frac{dx}{x}. \quad (\text{B.14})$$

The above equation can be rewritten using the variable change $z = e^{i\alpha}$ as

$$\oint_C \frac{dz}{2\pi i z} \left(\log(1 - z e^{-\hat{\epsilon} + |\hat{\mu}|}) + \log(1 - z e^{-\hat{\epsilon} - |\hat{\mu}|}) \right) - \Theta(|\mu| - \epsilon) \int_{e^{\hat{\epsilon} - |\hat{\mu}|}}^1 \frac{dx}{x}, \quad (\text{B.15})$$

where the contour C runs over the unit circle. The second logarithm in the first term of (B.15) is analytic inside the unit circle (its branch cut starts outside the unit circle) and so, by Cauchy's theorem, the contour integral involving this logarithm vanishes. It follows that (B.15) simplifies to

$$\oint_C \frac{dz}{2\pi i z} \log(1 - z e^{-\hat{\epsilon} + |\hat{\mu}|}) - \Theta(|\mu| - \epsilon) \int_{e^{\hat{\epsilon} - |\hat{\mu}|}}^1 \frac{dx}{x}. \quad (\text{B.16})$$

Now the contour integral involving the logarithm in (B.16) does not vanish, but receives contributions only from the discontinuity across the cut of this logarithm. Evaluating this discontinuity we find that (B.16) becomes

$$\Theta(|\mu| - \epsilon) \int_{e^{\hat{\epsilon} - |\hat{\mu}|}}^1 \frac{dx}{x} - \Theta(|\hat{\mu}| - \hat{\epsilon}) \int_{e^{\hat{\epsilon} - |\hat{\mu}|}}^1 \frac{dx}{x} = 0, \quad (\text{B.17})$$

establishing (B.12).

B.3 Duality of the occupation numbers

We will now examine how the occupation numbers $\bar{n}_B(\epsilon, q)$ (as computed in (4.20)) and $\bar{n}_F(\epsilon, q)$ (as computed in (4.18)) map to each other under duality. We have

$$\begin{aligned}
 & |\lambda_B| \bar{n}_B(\epsilon, q) - |\lambda_F| \bar{n}_F(\epsilon, q) \\
 &= \int_{-\pi}^{\pi} d\alpha |\lambda_B| \rho_B(\alpha) \frac{1}{e^{\hat{\epsilon}-q\hat{\mu}-i\alpha} - 1} - \int_{-\pi}^{\pi} d\alpha |\lambda_F| \rho_F(\alpha) \frac{1}{e^{\hat{\epsilon}-q\hat{\mu}-i\alpha} + 1} \\
 &\quad + 2\pi\Theta(q\mu - \epsilon) \left(|\lambda_B| \tilde{\rho}_B(e^{-|\hat{\mu}|+\hat{\epsilon}}) + |\lambda_F| \tilde{\rho}_F(-e^{-|\hat{\mu}|+\hat{\epsilon}}) \right), \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\alpha \frac{1}{e^{\hat{\epsilon}-q\hat{\mu}-i\alpha} - 1} + \Theta(q\mu - \epsilon), \\
 &= 0.
 \end{aligned} \tag{B.18}$$

We have used

$$|\lambda_B| \tilde{\rho}_B(z) + |\lambda_F| \tilde{\rho}_F(-z) = \frac{1}{2\pi}, \tag{B.19}$$

to argue that the third line of (B.18) evaluates to the second term on the fourth line of the same equation. We have also broken up the integral in the second term in the second line (the term involving ρ_F) into an integral from $-\pi$ to 0 and 0 to π , and then made the change of variables (B.8) and used $e^{\pm i\pi q} = -1$ and (B.13) to demonstrate that the second line of (B.18) reduces to the first term on the fourth line of the same equation. In going from fourth line to the last line of (B.18), we have used

$$\int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \frac{1}{e^{\hat{\epsilon}-q\hat{\mu}-i\alpha q} - 1} = -\Theta(q\mu - \epsilon). \tag{B.20}$$

It thus follows from (B.18) that

$$N_B \bar{n}_B(\epsilon, q) = N_F \bar{n}_F(\epsilon, q). \tag{B.21}$$

C Curve sketching

In this appendix we study a toy model which is governed the following quantum effective potential

$$U_{\text{eff}}(\sigma) = \begin{cases} A\sigma^3 + B\sigma^2 + C\sigma & \sigma < 0, \\ (A+a)\sigma^3 + (B+b)\sigma^2 + C\sigma & \sigma > 0, \end{cases} \tag{C.1}$$

Note that $U_{\text{eff}}(\sigma)$ is analytic (and polynomial) everywhere away from $\sigma = 0$. At $\sigma = 0$, $U_{\text{eff}}(\sigma)$ is continuous and once-differentiable, but its second and third derivatives are discontinuous.

We designate the $\sigma < 0$ branch in (C.1) as the ‘ $-$ ’ branch and the $\sigma > 0$ branch as the ‘ $+$ ’ branch. Whenever a local minimum exists in the $-$ or $+$ branch and we choose our theory to be in that vacuum, the theory is said to be in the $-$ or $+$ phase.

The potential $U_{\text{eff}}(\sigma)$ has 5 parameters, A , B , C , a and b . We will also impose the condition $a > 0$ ⁵⁸ but allow all the other four parameters to have either sign.⁵⁹ In this section we will study the phase diagram of the model (C.1) at fixed values of A , a , b , but as functions of the two parameters B and C . Thus, the phase diagram will be two dimensional, with C parametrizing the x -axis and B parametrizing the y -axis.

It turns out that the phase diagrams in question are qualitatively different depending on the signs of A , $A + a$ and b . As a consequence we will present $2^3 - 2$ different phase diagrams, one for each of the *allowed* signs for these three ‘parameters’ (since we have $a > 0$, the case $A > 0$, $A + a < 0$ with either sign of b is excluded from the analysis).

The sign of A (resp. $A + a$) decides the boundedness of the potential for large negative (resp. positive) σ . We thus consider the three cases separately:⁶⁰

$$\text{Case I: } A > 0, \ A + a > 0, \quad \text{Case II: } A < 0, \ A + a > 0, \quad \text{Case III: } A < 0, \ A + a < 0. \quad (\text{C.2})$$

Cases I and III are unbounded below (for negative σ and positive σ respectively) while Case II is stable. Within each case, we consider two subcases corresponding to $b > 0$ and $b < 0$.

C.1 Some analytic features of the phase diagram

The extremization equation of the effective potential, $U'_{\text{eff}}(\sigma) = 0$, is separately quadratic both when $\sigma < 0$ and $\sigma > 0$. The discriminants of these two quadratic equations are given by

$$D_- = B^2 - 3AC, \quad D_+ = (B + b)^2 - 3(A + a)C. \quad (\text{C.3})$$

and will play a key role in our analysis below. When these discriminants are negative, there are no extrema in the corresponding branch of the potential. The B -axis i.e. the $C = 0$ line is tangent to the discriminant curves D_- and D_+ in (C.3).

The $C = 0$ line in the phase diagram is very interesting analytically. On the $C = 0$ line, there are extrema in both the $-$ and $+$ branches since the discriminants D_- and D_+ in (C.3) are both positive. The extrema for the $-$ and $+$ branches are at

$$\begin{aligned} - \text{ branch : } \quad \sigma &= 0, \quad \sigma = -\frac{2B}{3A}, \\ + \text{ branch : } \quad \sigma &= 0, \quad \sigma = -\frac{2(B + b)}{3(A + a)}. \end{aligned} \quad (\text{C.4})$$

The extremum at $\sigma = 0$ is always a legitimate one for both branches. However, the other extremum must be discarded if it occurs outside the range of validity $\sigma \lesseqgtr 0$ for the $-$ or $+$

⁵⁸Simply because this condition always turns out to be true in the case of physical interest to us, see section 7.2.

⁵⁹Under the change of variables $\sigma \rightarrow \alpha\sigma$ (with $\alpha > 0$), these 5 variables scale as

$$A \rightarrow \alpha^3 A, \quad a \rightarrow \alpha^3 a, \quad B \rightarrow \alpha^2 B, \quad b \rightarrow \alpha^2 b, \quad C \rightarrow \alpha C$$

so in actuality our toy model (C.1) has 4 continuous parameters.

⁶⁰When one of the $\sigma < 0$ or $\sigma > 0$ branches in (C.1) corresponds to the condensed branch of the regular boson effective potential (7.13) which exists only for a finite range of σ , the question of boundedness is not very relevant since σ is not allowed to grow too large.

Figure 23. Plots of the potential $U_{\text{eff}}(\sigma)$ at $C = 0$ and at different values of the parameter B (the x -axis of the figure). The blue solid line indicates that the potential has a minimum at $\sigma = 0$, the red dotted line indicates that it has a point of inflection and the black dashed-dotted line indicates that it has a maximum.

branches. The second derivative of the potential in either branch at the extremum $\sigma = 0$ is given by

$$- \text{ branch : } U''_{\text{eff}}(0) = 2B, \quad + \text{ branch : } U''_{\text{eff}}(0) = 2(B + b). \quad (\text{C.5})$$

Based on the above facts, we deduce three different kinds of transitions in the behaviour of the potential (C.1) depending on the range of B . We give the details separately for $b > 0$ and $b < 0$. The potential is sketched in figure 23 for each of the different cases.

1. $b > 0$

- (a) $B > 0$: the full potential has a minimum at $\sigma = 0$ i.e. both branches of the potential have a minimum at $\sigma = 0$.
- (b) $-b < B < 0$: a maximum occurs in the $-$ branch at $\sigma = 0$ while a minimum occurs in the $+$ branch. Thus, the potential has a point of inflection at $\sigma = 0$.
- (c) $B < -b$: the full potential has a maximum at $\sigma = 0$ since both branches of the potential have a maximum at $\sigma = 0$.

2. $b < 0$

- (a) $B > -b$: both branches have a minimum at $\sigma = 0$ and hence the full potential has a minimum at this point.
- (b) $0 < B < -b$: a minimum occurs in the $-$ branch at $\sigma = 0$ while a maximum occurs in the $+$ branch. Thus, the potential has a point of inflection at $\sigma = 0$.
- (c) $B < 0$: both branches of the potential have a maximum at $\sigma = 0$, thus giving rise to a maximum for the full potential at $\sigma = 0$.

So far, we have looked at the behaviour on the potential by setting $C = 0$ and varying B . Suppose we now broaden our focus to a small neighbourhood of the $C = 0$ line. To begin with, note that the slope of the potential $U_{\text{eff}}(\sigma)$ at $\sigma = 0$ is given by C in both branches of the potential. Let us study the potential as we take C through zero from positive to negative values by choosing B to be in one of the three ranges discussed previously. When B is such that we are on the blue line in figure 23, a local minimum crosses from the $-$ branch to the $+$ branch. When we choose B to be on the black line in figure 23, a local maximum crosses from the $+$ branch to the $-$ branch.

Finally, when we choose B such that we are on the red line segment, two kinds of behaviour are possible depending on the sign of b . First, consider the case $b > 0$. The potential is monotonic near the origin when $C > 0$, has a point of inflection at $C = 0$ and develops a local maximum in the $-$ branch and a local minimum in the $+$ branch for $C < 0$. Second, when $b < 0$, the potential starts with a local minimum in the $-$ branch and a local maximum in the $+$ branch for $C > 0$, has a point of inflection at $C = 0$ and is monotonic near $\sigma = 0$ for $C < 0$.

We stress that the above description applies to only the local behaviour of the potential near $\sigma = 0$, the junction of the two branches. In particular, there may be other local extrema of the potential as one goes deeper into either of its branches. A full analysis of the potential is required in order to understand its global structure, which is what we turn to next.

C.2 Case I: $A > 0$, $A + a > 0$

The potential is bounded below for $\sigma > 0$ and unbounded below for $\sigma < 0$. We have the following cases:

1. $C > 0$: displayed in figures 24(a)-(d). The potential in the $-$ branch is monotonic when $B < 0$ or ($B > 0$ with $D_- < 0$) and has extrema when ($B > 0$ with $D_- > 0$). Similarly, the potential is monotonic in the $+$ branch when $B + b > 0$ or ($B + b < 0$ with $D_+ < 0$) and has extrema when ($B + b < 0$ with $D_+ > 0$). Then, we have the following possibilities.
  - (a) Either $B < 0$ or ($B > 0$ with $D_- < 0$); either $B + b > 0$ or ($B + b < 0$ with $D_+ < 0$): monotonic in both $+$ and $-$ branches.
  - (b) ($B > 0$ with $D_- > 0$); either $B + b > 0$ or ($B + b < 0$ with $D_+ < 0$): two extrema present in the $-$ branch, no extremum in the $+$ branch.
  - (c) Either $B < 0$ or ($B > 0$ with $D_- < 0$); ($B + b < 0$ with $D_+ > 0$): no extremum in the $-$ branch, two extrema present in the $+$ branch.
  - (d) ($B > 0$ with $D_- > 0$); ($B + b < 0$ with $D_+ > 0$): maximum-minimum pairs in both Higgsed and unHiggsed phases. Depending on the relative magnitudes of the parameters B and C , the minimum in one of the $+$ or $-$ branches will be deeper and hence more dominant. A first order transition is possible when the dominant minimum switches from one branch to the other. We depict the two situations in figure 24(d).

Figure 24. Effective potential for Case I: $A > 0$, $A + a > 0$.

Figure 25. Detailed structure of the potential $U_{\text{eff}}(\sigma)$ (C.1) for Case I: $A > 0$, $A + a > 0$. The solid blue (resp. dashed-dotted black) lines signify a local minimum (resp. local maximum) crossing from one branch to the other through $\sigma = 0$. The red dashed lines D_+ and D_- signify the (dis)appearance of new extrema in the potential (cf. (C.3)). The dotted red line segment between $B = 0$ and $B = -b$ signifies a local maximum in one branch and a local minimum in the other meeting at $\sigma = 0$. The solid green line is a first order phase transition line which signifies the presence of two equal local minima of the potential, one in each branch.

Figure 26. Phase diagram for Case I: $A > 0$, $A + a > 0$. The solid blue line is a second order phase transition between the $-$ and $+$ phases. The solid red lines demarcate regions of monotonic potential. The solid green line is a first order phase transition line between the $-$ and $+$ phases.

2. $C < 0$: in figure 24(e). There is always a minimum in the $-$ branch and a maximum in the $+$ branch, irrespective of the values of the parameter B .

C.3 Case II: $A < 0$, $A + a > 0$

The potential is bounded below for both $\sigma < 0$ and $\sigma > 0$.

Figure 27. Effective potential for Case II: $A < 0$, $A + a > 0$.

Figure 28. Detailed structure of the potential for Case II: $A < 0$, $A + a > 0$. Refer to the caption of figure 24 for details on the various marked lines.

1. $C > 0$, B either sign
  - (a) either $B + b > 0$ or ($B + b < 0$ with $D_+ < 0$): minimum in $-$ branch, monotonic in $+$ branch. Shown in figure 27(a).
  - (b) ($B + b < 0$ with $D_+ > 0$): minimum in $-$ branch, a maximum-minimum pair in the $+$ branch. Shown in figure 27(b). The minimum in the $+$ branch could also be lower than the minimum in the $-$ branch though it is not depicted in figure 27(b).
2. $C < 0$, $B + b$ either sign
  - (a) $B > 0$ or ($B < 0$ with $D_- < 0$): monotonic in the $-$ branch, minimum in the $+$ branch. Shown in figure 27(c).
  - (b) ($B < 0$ with $D_- > 0$): minimum in the $+$ branch, a maximum-minimum pair in the $-$ branch. Shown in figure 27(d). The minimum in the $-$ branch could also be lower than the minimum in the $+$ branch though it is not depicted in figure 27(d).

C.4 Case III: $A < 0$, $A + a < 0$

In this case, the effective potential is bounded below for $\sigma < 0$ and unbounded below for $\sigma > 0$.

Figure 29. Phase diagram for Case II: $A < 0$, $A + a > 0$. The solid blue line is a second order phase transition between the $-$ and $+$ phases. The solid green line is a first order phase transition line between the $-$ and $+$ phases.

Figure 30. Effective potential for Case III: $A < 0$, $A + a < 0$.

1. $C < 0$. The potential in the $-$ branch is monotonic when $B > 0$ or ($B < 0$ with $D_- < 0$) and has two extrema when ($B < 0$ with $D_- > 0$). Similarly, the potential in the $+$ branch is monotonic when $B + b < 0$ or ($B + b > 0$ with $D_+ < 0$) and has two extrema when ($B + b > 0$ with $D_+ > 0$). We now consider the following possibilities.
  - (a) Either $B + b < 0$ or ($B + b > 0$ with $D_+ < 0$); either $B > 0$ or ($B < 0$ with $D_- < 0$): potential is monotonic.
  - (b) Either $B + b < 0$ or ($B + b > 0$ with $D_+ < 0$); ($B < 0$ with $D_- > 0$): two extrema in the $-$ branch, monotonic in the $+$ branch.
  - (c) ($B + b > 0$ with $D_+ > 0$); either $B > 0$ or ($B < 0$ with $D_- < 0$): two extrema in the $+$ branch, monotonic in the $-$ branch.
  - (d) ($B + b > 0$ with $D_+ > 0$); ($B < 0$ with $D_- > 0$): maximum-minimum pairs in both branches. The minimum in one of the branches is typically lower than the other. There will be a first order phase transition when the dominant minimum switches from one branch to the other. We have shown the two possibilities in figure 30(d).
2. $C > 0$: there is always a minimum in $-$ branch and a maximum in the $+$ branch, irrespective of the value of the parameter B .

Figure 31. Detailed structure of the potential for Case III: $A < 0$, $A + a < 0$. Refer to the caption of figure 24 for details on the various marked lines.

Figure 32. Phase diagram for Case III: $A < 0$, $A + a < 0$. Refer to the caption of figure 26 for details on the various marked lines.

D The phase diagram of the regular boson theory with an unstable potential

In this section, we present the analysis of the effective potential (7.13) for the unstable ranges of x_6^B viz. Case A and Case D in (7.18):

$$\text{Case A : } x_6^B < \phi_h, \quad \text{Case D : } \phi_u < x_6^B. \quad (\text{D.1})$$

As we have remarked earlier, since the potential is unbounded below either for large and positive σ_B (Case A) or for large and negative σ_B (Case D), the phases that may exist are metastable at best. However, we present the calculations simply because it is informative and for the sake of completeness. The analysis of the current section may also be useful in the study of the $\mathcal{N} = 2$ supersymmetric Chern-Simons theory with one fundamental chiral multiplet in the presence of chemical potential.⁶¹

⁶¹The phase diagram of this theory with zero chemical potential was computed in [73]. The main strategy in that paper was to use the results for the effective potential for the regular boson to understand the local behaviour of the effective potential for the $\mathcal{N} = 2$ theory near its non-analytic points (analogous to what we do in this paper). The local behaviour of the potential was sometimes decided by the unstable cases of the regular boson, though the final global potential was stable. The global stability of the potential essentially

Recall that we work only at leading order in the large N limit in which case the parameter x_6^B is exactly marginal. However, this is no longer the case when the subleading corrections in $1/N$ for the β -function of x_6^B are taken into account [70]. The current analysis may be useful in this situation since the fixed points of x_6^B may well end up being in one of the above unstable ranges (see [70] for more details).

D.1 Case A: $x_6^B < \phi_h$

For this range of x_6^B , both local expressions of the potential (7.21) and (7.30) are unbounded below for large and positive σ'_B and σ''_B respectively since the quantity A satisfies $A < 0$ and $A + a < 0$ for both (7.21) and (7.30). Thus, Case III of appendix C applies to the potential near both non-analytic points. We must remark that the unboundedness of (7.21) is not very serious since it occurs in the condensed branch which is eventually cut off at $\sigma'_B = 1/|\lambda_B|$.

We plot the special lines corresponding to the potential near both non-analytic points in figure 33. The segments of D_u , D_c and D_h which are present in this phase diagram are specified by the following conditions:

$$D_u: \quad \lambda_B b_4 \leq \frac{3}{4} \lambda_B^2 (x_6^B - \phi_u), \quad D_h: \quad -\frac{3}{4} (2 - |\lambda_B|) (x_6^B - \phi_h) \leq \lambda_B b_4, \quad (\text{D.2})$$

$$D_c: \quad \frac{3}{4} \lambda_B^2 (x_6^B - \phi_u) - (1 - |\lambda_B|) \leq \lambda_B b_4 \leq -\frac{3}{4} |\lambda_B| (2 - |\lambda_B|) (x_6^B - \phi_h) + \frac{|\lambda_B| (1 - |\lambda_B|)}{2 - |\lambda_B|}. \quad (\text{D.3})$$

For each region demarcated by the special lines in figure 33(a), we display the shape of the potential in the neighbourhood of the points $\sigma_B = \ell$ and $\sigma_B = r$ as a small inset figure (with the plot on the left (resp. right) corresponding to ℓ (resp. r)). In figure 33(b), we combine the information from the plots near ℓ and r to give a global shape of the potential in each region demarcated by the special lines. Note that one is able to obtain this information just by comparing the common region of the two plots and without any detailed quantitative analysis of the potential.

Since the potential is unbounded from below for $\sigma_B \gg 0$, all the phases for this range of x_6^B are metastable at best. We show this “phase” diagram in figure 34.

D.2 Case D: $\phi_u < x_6^B$

Case I of appendix C applies to the behaviour of the potential near both non-analytic points. The conditions on $\lambda_B b_4$ which specify the segments of the discriminant parabolas that appear in the phase diagram are

$$D_u: \quad \lambda_B b_4 \geq \frac{3}{4} \lambda_B^2 (x_6^B - \phi_u), \quad D_h: \quad \lambda_B b_4 \leq -\frac{3}{4} (2 - |\lambda_B|) (x_6^B - \phi_h), \quad (\text{D.4})$$

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meant that the unboundedness of the local potential was replaced by a very deep minimum in the global potential which was situated outside the range of applicability of the local potential.

Figure 33. Structure of the potential for $x_6^B < \phi_h$. Refer to sections 7.2.1 and 7.2.2 for the labels of the various points and lines. We have restored the $|\mu|$ dependence of the phase diagram by rescaling m_B^2 and $\lambda_B b_4$ by appropriate powers of $|\mu|$.

Figure 34. Phase diagram for $x_6^B < \phi_h$. The blue lines are second order transitions, the green line is a first order transition while the red lines demarcate regions of runaway potential from the regions where metastable phases exist. Recall that the potential is unstable and all the phases displayed above are metastable at best. We have restored the $|\mu|$ dependence of the phase diagram by rescaling m_B^2 and $\lambda_B b_4$ by appropriate powers of $|\mu|$.

Figure 35. Structure of the potential for $\phi_u < x_6^B$. Recall that the effective potential is unbounded from below in this case so that the ‘phases’ displayed here are dominant metastable phases when they exist. To the right of the red curve the effective potential is monotonic so no metastable phases exist. The green curve is a first order transition (between dominant metastable phases) while the blue curves are second order transitions between such phases. We have restored the $|\mu|$ dependence of the phase diagram by rescaling m_B^2 and $\lambda_B b_4$ by appropriate powers of $|\mu|$.

$$D_c : \quad -\frac{3}{4}|\lambda_B|(2 - |\lambda_B|)(x_6^B - \phi_h) + \frac{|\lambda_B|(1 - |\lambda_B|)}{2 - |\lambda_B|} \leq \lambda_B b_4 \leq \frac{3}{4}\lambda_B^2(x_6^B - \phi_u) - (1 - |\lambda_B|). \quad (\text{D.5})$$

The detailed structure of the potential is given as small inset plots in figure 35(a) in each region demarcated by the special lines of the two local potentials (7.21) and (7.30). The global structure of the potential is displayed in each of these regions in figure 35(b). Since the potential is unbounded below in this range of x_6^B , the phases in this case are metastable and are displayed in figure 36.

E Preliminary discussions on angular momentum

The 2 + 1 dimensional bosons and fermions that we study in the main text of this paper do not appear in spin multiplets but instead each carry the same spin in the rest frame. For this reason we expect a macroscopic collection of these particles to carry angular momentum that is extensive (i.e. scales like the volume).

In this appendix, we will demonstrate that — at least with one reasonable definition of the angular momentum — this expectation is indeed correct for the Fermi sea of a collection of free fermions of unit charge in 2 + 1 dimensions. We leave the definition and computation of the angular momentum of thermodynamical ensembles of Chern-Simons-matter theories to future work.⁶²

⁶²One additional complication that the Chern-Simons gauge field would introduce is the presence of WZW boundary degrees of freedom, which, by themselves, could carry extensive angular momentum.

Figure 36. Phase diagram for $\phi_u < x_6^B$. Recall that the effective potential is unbounded from below in this case so that the ‘phases’ displayed here are dominant metastable phases when they exist. To the right of the red curve the effective potential is monotonic so no meta-table phases exist. The blue lines are second order transitions and the green curve is a first order transition between metastable phases. To the ‘right’ of the red curve no metastable phases exist. We have restored the $|\mu|$ dependence of the phase diagram by rescaling m_B^2 and $\lambda_B b_4$ by appropriate powers of $|\mu|$.

E.1 The free Dirac equation and its symmetries

The free Dirac equation

$$(\gamma^\mu \partial_\mu + m_F) \Psi(x^\alpha) = 0, \quad (\text{E.1})$$

follows from the action

$$\mathcal{S} = i \int d^3x \bar{\Psi} (\gamma^\mu \partial_\mu + m_F) \Psi. \quad (\text{E.2})$$

We make the following choice of gamma matrices

$$\gamma^0 \equiv \gamma^t = i\sigma^3, \quad \gamma^1 \equiv \gamma^x = \sigma^1, \quad \gamma^2 \equiv \gamma^y = \sigma^2. \quad (\text{E.3})$$

Note that

$$\gamma^0 \gamma^i = -\epsilon^{ij} \gamma_j, \quad (\text{E.4})$$

where $\epsilon^{12} = -\epsilon^{21} = 1$.

E.1.1 Symmetries of the Lagrangian

Parity: it is easy to check that the field redefinition

$$\chi(x, y, t) = \gamma^y \Psi(x, -y, t) \equiv (\mathcal{P}^y \cdot \Psi)(x, y, t), \quad (\text{E.5})$$

turns the Dirac action (E.2) into

$$\mathcal{S} = i \int d^3x \bar{\chi} (\gamma^\mu \partial_\mu - m_F) \chi. \quad (\text{E.6})$$

In other words the parity operation (E.5) (corresponding to $y \rightarrow -y$) flips the sign of the mass term. Equivalently, the parity operation combined with a flip in the sign of the fermion mass is a symmetry of the system.

Charge conjugation: in a similar vein, it is easy to check that the field redefinition

$$\chi(x, y, t) = \gamma^x \Psi^*(x, y, t) \equiv (\mathcal{C} \cdot \Psi)(x, y, t), \quad (\text{E.7})$$

turns the Dirac action (E.2) into

$$\mathcal{S} = i \int d^3x \bar{\chi} (\gamma^\mu \partial_\mu + m_F) \chi. \quad (\text{E.8})$$

In other words, complex conjugation (E.7) preserves the Dirac action and is a symmetry of the system.

Time reversal: finally, note that the field redefinition

$$\chi(x, y, t) = \gamma^0 \Psi(x, y, -t) \equiv (\mathcal{T}' \cdot \Psi)(x, y, t), \quad (\text{E.9})$$

turns the Dirac Lagrangian (E.2) into

$$\mathcal{S} = -i \int d^3x \bar{\chi} (\gamma^\mu \partial_\mu - m_F) \chi \quad (\text{E.10})$$

The ‘time reversal transformation’ $\mathcal{T}'$ (E.9)⁶³ and a flip in the sign of the fermion mass together flips the overall sign of the Lagrangian and so preserves the Dirac equation.

E.1.2 The Hamiltonian, angular momentum and symmetries

The (single-particle) Dirac Hamiltonian

$$H(m_F) = i\gamma^0 \gamma^i \partial_i + i m_F \gamma^0. \quad (\text{E.11})$$

The single-particle angular momentum operator is given by

$$J = i\epsilon^{ij} x_i \partial_j + i \frac{\gamma^0}{2} = i(x \partial_y - y \partial_x) + i \frac{\gamma^0}{2} = i \partial_\phi - \frac{\sigma^3}{2}. \quad (\text{E.12})$$

Note that

$$[H, J] = -\gamma^0 \gamma_i \epsilon^{ik} \partial_k - \gamma^0 \gamma^i \gamma^0 \partial_i = \epsilon_{ij} \epsilon^{ik} \gamma^j \partial_k - \gamma^i \partial_i = 0, \quad (\text{E.13})$$

and

$$[J, x^k] = i\epsilon_{ik} x^i, \quad [J, \gamma^k] = i\epsilon_{mk} \gamma^m. \quad (\text{E.14})$$

As expected, x^i and γ^i rotate the same way under rotations (both are vectors in the spatial plane). It follows that the operators $x^i \gamma_i$ and γ^0 both commute with J .

We now study the interplay between solutions of the Dirac equation and the discrete symmetries discussed in the previous subsection.

First, let us look at parity. The analysis around (E.6) tells us that if $\Psi(x, y, t)$ is a solution of the Dirac equation then $\gamma^y \Psi(x, -y, t)$ is also a solution of the same equation with flipped mass. Applying this result to solutions of the form $\psi(x, y) e^{-iEt}$, we conclude that if $\psi(x, y)$ is an eigenstate of the Dirac Hamiltonian $H(m_F)$ with energy E , then

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⁶³More precisely (E.9) is a $\mathcal{CT}$ transformation.

$\gamma^y \psi(x, -y)$ is an eigenstate of the Dirac Hamiltonian $H(-m_F)$ with the same energy. This is easy to check directly. Let $\psi(x, y)$ be an eigenstate of $H(m_F)$ with energy E i.e.

$$\left(i\gamma^0 \gamma^i \partial_i + im_F \gamma^0 \right) \psi(x, y) = E \psi(x, y) . \quad (\text{E.15})$$

Then it is easily verified that

$$\left(i\gamma^0 \gamma^i \partial_i - im_F \gamma^0 \right) \gamma^y \psi(x, -y) = E \gamma^y \psi(x, -y) . \quad (\text{E.16})$$

Then, the above discussion can be phrased in terms of the parity operator $\mathcal{P}^y$ as

$$\mathcal{P}^y H(m_F) = H(-m_F) \mathcal{P}^y . \quad (\text{E.17})$$

Again, it is easily verified that⁶⁴

$$J(\gamma^y \psi(x, -y)) = -\gamma^y (J\psi(x, y))|_{\alpha=x, \beta=-y} . \quad (\text{E.20})$$

The above equation can be recast in terms of the parity operator P^y as

$$(JP^y \psi)(x, y) = -(\mathcal{P}^y J\psi)(x, y) , \quad \text{i.e.} \quad JP^y = -\mathcal{P}^y J . \quad (\text{E.21})$$

It follows that parity operator flips angular momentum; it maps an angular momentum eigenstate with eigenvalue j to an eigenstate of eigenvalue $-j$. Consider a simultaneous eigenstate of the Dirac Hamiltonian $H(m_F)$ and the angular momentum operator J with eigenvalues E and j respectively. Then the parity operator $\mathcal{P}^y$ maps this eigenstate to an eigenstate of $H(-m_F)$ and J with eigenvalues E and $-j$ respectively.

We can repeat this discussion for the charge conjugation operation. It follows from the discussion around (E.8) that if $\psi(x, y)e^{-iEt}$ is a solution to the Dirac equation with mass m_F , then $\gamma^x \psi^*(x, y)e^{iEt}$ is also a solution to the Dirac equation with mass m_F . In other words, if $\psi(x, y)$ is an eigenstate of $H(m_F)$ with energy E then $\gamma^x \psi^*(x, y)$ is another eigenstate of $H(m_F)$ but with energy $-E$. Again this is easy to check explicitly. Given that (E.15) holds, it follows that

$$\left(i\gamma^0 \gamma^i \partial_i + im_F \gamma^0 \right) \gamma^x \psi^*(x, y) = -E \gamma^x \psi^*(x, y) , \quad \text{i.e.} \quad \mathcal{C}H(m_F) = -H(m_F)\mathcal{C} . \quad (\text{E.22})$$

Similarly, it is easy to verify that⁶⁵

$$J(\gamma^x \psi^*(x, y)) = -\gamma^x (J\psi(x, y))^* , \quad \text{i.e.} \quad J\mathcal{C} = -\mathcal{C}J . \quad (\text{E.24})$$

---

⁶⁴If $\Psi(x, y, t)$ is an eigenstate of the angular momentum operator (E.12) with eigenvalue j i.e.

$$J\Psi(x, y, t) = j\Psi(x, y, t) , \quad (\text{E.18})$$

then,

$$J(\gamma^y \Psi(x, -y, t)) = -j(\gamma^y \Psi(x, -y, t)) \quad (\text{E.19})$$

⁶⁵More explicitly, if $\Psi(x, y, t)$ is an eigenstate of the angular momentum operator (E.12) with eigenvalue j , then, it is easy to convince oneself that

$$J(\gamma^x \Psi^*(x, y, t)) = -j(\gamma^x \Psi^*(x, y, t)) \quad (\text{E.23})$$

In summary, if the state ψ has energy E and angular momentum J , then the state $\gamma^x \psi^*$ has energy $-E$ and angular momentum $-J$.

Finally, the discussion around (E.10) tells us that if $\psi(x, y)e^{-iEt}$ is a solution to the Dirac equation with mass m_F , then $\gamma^0 \psi(x, y)e^{iEt}$ is also a solution to the Dirac equation with mass $-m_F$. It follows that if $\psi(x, y)$ is an eigenstate of the Dirac Hamiltonian $H(m_F)$ with energy E , then $\gamma^0 \psi(x, y)$ is another eigenstate of $H(-m_F)$ with energy $-E$. Again this is easy to check explicitly:

$$\left(i\gamma^0 \gamma^i \partial_i + im_F \gamma^0 \right) \psi(x, y, t) = i\partial_t \psi(x, y, t) . \quad (\text{E.25})$$

$$i\gamma^0 \left(\gamma^i \partial_i - m_F \right) \gamma^0 \psi(x, y) = -E \gamma^0 \psi(x, y) , \quad \text{i.e.} \quad \mathcal{T}' H(m_F) = -H(-m_F) \mathcal{T}' . \quad (\text{E.26})$$

Also it is easy to verify that⁶⁶

$$J \left(\gamma^0 \psi(x, y) \right) = \gamma^0 (J \psi(x, y)) , \quad \text{i.e.} \quad J \mathcal{T}' = \mathcal{T}' J . \quad (\text{E.28})$$

In summary, if a simultaneous eigenstate ψ of $H(m_F)$ and J has energy E and angular momentum j respectively, then the state $\gamma^0 \psi$ is a simultaneous eigenstate of $H(-m_F)$ and J with energy $-E$ and angular momentum j respectively.

E.2 Boundary conditions at infinity

Since the angular momentum of a Fermi sea on a truly infinite space will turn out to be infinite, we need an IR regulator to get a sensible answer. We adopt the following strategy. Let us define the radial spatial variable r and the unit position vector by

$$r^2 = x^2 + y^2 , \quad \hat{x}^i = \frac{x^i}{r} . \quad (\text{E.29})$$

We study the Dirac on the cut off plane $r \leq R$, subject to one of the two the boundary conditions

$$\gamma^i \hat{x}^i \psi = \pm \psi \quad \text{at} \quad r = R . \quad (\text{E.30})$$

The boundary conditions with the $\pm$ signs in (E.30) define two distinct Hilbert Spaces $\mathcal{H}^+$ and $\mathcal{H}^-$ respectively.

We will now discuss the interplay between the boundary conditions (E.30) and the symmetry operations (E.5), (E.7) and (E.9). Consider solutions $\psi_{\pm} \in \mathcal{H}^{\pm}$ that obeys the boundary conditions (E.30). It is easy to check that

$$\begin{aligned} \gamma^i \hat{x}^i (\gamma^y \psi_{\pm}(x, -y)) &= \mp \gamma^y \psi_{\pm}(x, -y) \\ \gamma^i \hat{x}^i (\gamma^x \psi_{\pm}^*(x, y)) &= \pm \gamma^x \psi_{\pm}^*(x, y) \\ \gamma^i \hat{x}^i (\gamma^0 \psi_{\pm}(x, y)) &= \mp \gamma^0 \psi_{\pm}(x, y) \end{aligned} \quad (\text{E.31})$$

Thus, parity and time reversal map $\mathcal{H}^+$ to $\mathcal{H}^-$ and vice versa (recall that these two operations also flip the sign of the mass parameter m_F). On the other hand the charge conjugation operation acts separately within $\mathcal{H}^+$ and $\mathcal{H}^-$.

⁶⁶In other words, if $\Psi(x, y, t)$ is an eigenstate of the angular momentum operator (E.12) with eigenvalue j , then, it follows that

$$J(\gamma^0 \Psi(x, y, -t)) = j(\gamma^0 \Psi(x, y, -t)) \quad (\text{E.27})$$

E.3 A convenient definition of the angular momentum of the Fermi sea

Let $J^\pm(\mu, m_F)$ be the angular momentum of the Fermi sea with chemical potential μ of fermions with mass m_F in the Hilbert Space $\mathcal{H}^\pm$. The fact that the charge conjugation operation maps every state of energy E and angular momentum j in $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) to another state of energy $-E$ and angular momentum $-j$ in $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) (and the fact that a antiparticle is the removal of a particle in a state of negative energy⁶⁷ and therefore has minus the charges of the negative energy state in question) tells us that the spectrum of particle and antiparticle states are exactly isomorphic. For every particle state of charge 1, energy E and angular momentum j , there is a corresponding antiparticle state of charge -1 , energy E and angular momentum j . In other words

$$J^\pm(\mu, m_F) = J^\pm(-\mu, m_F) . \quad (\text{E.32})$$

The parity operator tells us that for every eigenstate of $H(m_F)$ of energy E and angular momentum j in $\mathcal{H}^+$ there is a corresponding eigenstate of $H(-m_F)$ with energy E and angular momentum $-j$ in $\mathcal{H}^-$. It follows that

$$J^\pm(\mu, m_F) = -J^\mp(\mu, -m_F) . \quad (\text{E.33})$$

Finally, the time reversal operation tells us that for every eigenstate of $H(m_F)$ with energy E and angular momentum j in $\mathcal{H}^+$ there is a corresponding eigenstate of $H(-m_F)$ with energy $-E$ and angular momentum j in $\mathcal{H}^-$. It follows that

$$J^\pm(\mu, m_F) = -J^\mp(-\mu, -m_F) . \quad (\text{E.34})$$

Clearly (E.34) carries no new information, but follows from (E.32) and (E.33).

A symmetric definition of the angular momentum — one that we will adopt in the rest of this appendix — is

$$J(\mu, m_F) = \frac{J^+(\mu, m_F) + J^-(\mu, m_F)}{2} . \quad (\text{E.35})$$

It follows immediately from (E.32), (E.33) and (E.34) that

$$J(\mu, m_F) = J(-\mu, m_F) = -J(\mu, -m_F) = -J(-\mu, -m_F) . \quad (\text{E.36})$$

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⁶⁷In the first quantized formalism in which we work in this appendix, the fermion has both positive and negative energy states. The vacuum of the theory at $\mu = 0$ is the Dirac Sea in which all negative energy states are filled and all positive energy states are unfilled. The positive μ Fermi sea is the state in which all negative energy states continue to be filled, but in addition, positive energy states with energies less than μ are also filled. At negative μ , on the other hand, no positive energy states are filled and, in addition, some negative energy states — those with $\epsilon > \mu$ — are also now empty (the other negative energy states continue to be filled). In contrast to this appendix, in the main text we have adopted the second quantized viewpoint in which all fermions states have positive energy, but there are two kinds of fermion states; those corresponding to positive charge ‘particles’ (these map to the positive energy states of the Dirac Sea picture) and those corresponding to negative charge ‘antiparticles’ (these map to the removal of negative energy states in the Dirac Sea picture).

E.4 Solutions of the Dirac equation

With the choice of γ matrices (E.3), it is not difficult to verify that a basis of regular solutions to the Dirac equation in polar coordinates is given — upto normalisation — by

$$\Psi(r, \phi, t) = \begin{pmatrix} e^{in\phi} J_{|n|}(pr) \\ -\text{sgn}(n)\eta\sqrt{\frac{|E|+\eta m_F}{|E|-\eta m_F}} e^{i(n+1)\phi} J_{|n+1|}(pr) \end{pmatrix} e^{-i\eta|E|t} \quad (\text{E.37})$$

where n is an integer, positive or negative and the energy is given by $E = \eta|E| = \eta\sqrt{p^2 + m_F^2}$. $\eta = \pm 1$ for positive/negative energy solutions, respectively.⁶⁸ Explicitly, the positive energy solutions are given by

$$\Psi(r, \phi, t) = \begin{pmatrix} e^{in\phi} J_{|n|}(pr) \\ -\text{sgn}(n)\sqrt{\frac{|E|+m_F}{|E|-m_F}} e^{i(n+1)\phi} J_{|n+1|}(pr) \end{pmatrix} e^{-i|E|t} \quad (\text{E.40})$$

And the negative energy solutions are given by

$$\Psi(r, \phi, t) = \begin{pmatrix} e^{in\phi} J_{|n|}(pr) \\ \text{sgn}(n)\sqrt{\frac{|E|-m_F}{|E|+m_F}} e^{i(n+1)\phi} J_{|n+1|}(pr) \end{pmatrix} e^{i|E|t} \quad (\text{E.41})$$

Together with the integer n , the solution (E.39) is parametrized by the continuous radial momentum p . The momentum p determines the energy of the solution via

$$E^2 = p^2 + m_F^2. \quad (\text{E.42})$$

The integer n determines the angular momentum of the solution via the equation

$$j = -\left(n + \frac{1}{2}\right). \quad (\text{E.43})$$

Let ψ_{E,j,m_F} denote the eigenfunction of the Dirac Hamiltonian $H(m_F)$, energy E and angular momentum j (E and j both range over positive and negative values in our notation). The explicit form of ψ_{E,j,m_F} is easily read off from (E.37), (E.42) and (E.43). As a check on our algebra we have verified that, these explicit solutions obey

$$\begin{aligned} \gamma^y \psi_{E,j,m_F}(r, -\phi) &\propto \psi_{E,-j,-m_F}(r, \phi), \\ \gamma^x \psi_{E,j,m_F}^*(r, \phi) &\propto \psi_{-E,-j,m_F}(r, \phi), \\ \gamma^0 \psi_{E,j,m_F}(r, \phi) &\propto \psi_{-E,j,-m_F}(r, \phi). \end{aligned} \quad (\text{E.44})$$

as predicted on general grounds in section E.1.2.

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⁶⁸More explicitly the solutions are given in terms of positive integers n by

$$\Psi(r, \phi, t) = \begin{pmatrix} e^{i|n|\phi} J_{|n|}(pr) \\ -\eta\sqrt{\frac{|E|+\eta m_F}{|E|-\eta m_F}} e^{i(|n|+1)\phi} J_{|n|+1}(pr) \end{pmatrix} e^{-i\eta|E|t} \quad (\text{E.38})$$

and

$$\Psi(r, \phi, t) = \begin{pmatrix} e^{-i|n|\phi} J_{|n|}(pr) \\ \eta\sqrt{\frac{|E|+\eta m_F}{|E|-\eta m_F}} e^{i(-|n|+1)\phi} J_{|n|-1}(pr) \end{pmatrix} e^{-i\eta|E|t} \quad (\text{E.39})$$

E.5 Boundary conditions on the solutions

In writing (E.37) we have not yet imposed any boundary conditions at $r = R$. Turning to the boundary conditions (E.30) now, it is easy to check that the solution (E.37) belongs to $\mathcal{H}^\pm$ provided

$$\chi_{n,\eta}^\pm(p) \equiv J_{|n|+\text{sgn}(n)}(pr) \pm \text{sgn}(n)\eta \sqrt{\frac{|E| - \eta m_F}{|E| + \eta m_F}} J_{|n|}(pr) = 0. \quad (\text{E.45})$$

Explicitly, the boundary condition equations for positive energy solutions are given by

$$\chi_{n,+}^\pm(p) \equiv J_{|n|+\text{sgn}(n)}(pr) \pm \text{sgn}(n) \sqrt{\frac{|E| - m_F}{|E| + m_F}} J_{|n|}(pr) = 0, \quad (\text{E.46})$$

and for negative energy solutions

$$\chi_{n,-}^\pm(p) \equiv J_{|n|+\text{sgn}(n)}(pr) \mp \text{sgn}(n) \sqrt{\frac{|E| + m_F}{|E| - m_F}} J_{|n|}(pr) = 0. \quad (\text{E.47})$$

E.6 Computation of the angular momentum

We wish to enumerate the states in both $\mathcal{H}^+$ and $\mathcal{H}^-$ graded by their angular momenta that contribute to the Fermi sea of chemical potential μ . In the computations presented in the rest of this appendix we will assume that $\mu > 0$ so that we need to deal with only particles with positive energy states rather than ‘antiparticles’ with positive energy (i.e. states in which particles of negative energy are removed or rather, absent). As we know that $J^\pm(\mu, m_F)$ is an even function of μ (see (E.32)), this restriction results in no loss of generality.

We proceed with our computation as follows. Let $N_n^\pm(\mu, m_F)$ denote the number of eigenstates of the Dirac Hamiltonian $H(m_F)$ in the Hilbert Space $\mathcal{H}^\pm$, with discrete label n and with energy less than μ . These are states such that

$$p^2 + m_F^2 < \mu^2,$$

i.e with $p < p_F$ where p_F is defined by the equation

$$p_F^2 + m_F^2 = \mu^2. \quad (\text{E.48})$$

It follows that

$$J^\pm(\mu, m_F) = - \sum_{n=-\infty}^{\infty} \left(n + \frac{1}{2}\right) N_n^\pm(\mu, m_F), \quad (\text{E.49})$$

and so (using (E.35))

$$J(\mu, m_F) = -\frac{1}{2} \left(\sum_{n=-\infty}^{\infty} \left(n + \frac{1}{2}\right) N_n^+(\mu, m_F) + \sum_{n=-\infty}^{\infty} \left(n + \frac{1}{2}\right) N_n^-(\mu, m_F) \right). \quad (\text{E.50})$$

E.6.1 Rough estimate of angular momentum

The angular momentum $J(\mu, m_F)$ is a function of R . We are interested in computing it at large R while keeping only those terms that are at least extensive i.e. that scale like R^2 or faster. The following rough estimate gives a sense of the scales involved. It is not too difficult to verify that, at large n , the Bessel function $J_n(x)$ is everywhere well approximated by a WKB approximation (e.g., see appendix E.2 in [92]) as follows. Let

$$n_* = \sqrt{4n^2 - 1}, \quad f(n, x) = \frac{n_*}{2} \sin^{-1} \frac{n_*}{2x} + \sqrt{x^2 - \frac{n_*^2}{4}}. \quad (\text{E.51})$$

Then, we have

$$J_n(x) = \begin{cases} \frac{1}{\sqrt{2\pi x}} \frac{1}{\left(\frac{n_*^2}{4x^2} - 1\right)^{1/4}} \left(\frac{n_*}{2x} - \sqrt{\frac{n_*^2}{4x^2} - 1}\right)^{n_*/2} e^{\sqrt{\frac{n_*^2}{4} - x^2}}, & 0 < x \ll \frac{n_*}{2} - n_*^{1/3}, \\ \sqrt{\frac{2}{x}} \left(\frac{n_*}{4}\right)^{1/6} \text{Ai}\left(-\left(4/n_*\right)^{1/3}\left(x - \frac{n_*}{2}\right)\right), & \frac{n_*}{2} - \frac{n_*}{6} \ll x \ll \frac{n_*}{2} + \frac{n_*}{6}, \\ \sqrt{\frac{2}{\pi x}} \frac{1}{\left(1 - \frac{n_*^2}{4x^2}\right)^{1/4}} \cos\left(\frac{\pi}{4} + \frac{n_*\pi}{4} - f(n, x)\right), & x \gg \frac{n_*}{2} + n_*^{1/3}. \end{cases} \quad (\text{E.52})$$

The first line in (E.52) is the WKB approximation to this function in the classically disallowed region. The second line in (E.52) is the Airy form⁶⁹ that occurs in the WKB approximation in the transition from the allowed to the disallowed region. The last line is the WKB approximation in the classically allowed region. Equation (E.52) leads to the following rough approximation (valid at leading order in R assuming n has a generic value, i.e. $n \sim R$)

$$N_{|n|}^{\pm}(\mu, m_F) = \frac{1}{\pi} \Theta(p_F R - |n|) \left(|n| \sin^{-1} \left(\frac{|n|}{p_F R} \right) + \sqrt{(p_F R)^2 - |n|^2} - \frac{|n|\pi}{2} \right). \quad (\text{E.53})$$

We see that $N_{|n|}^{\pm} \sim R$. Also, the number of n values that contribute (before the Θ function in (E.53) kills the contribution) is also of order R . Given that generic $n \sim R$, it follows that the contribution of positive n to the sums (E.49) and (E.54) is of order R^3 , and so is super-extensive.

This super-extensive term is killed by a cancellation between terms of positive and negative angular momenta, as we now explain. To see this, note that the sum in (E.49)

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⁶⁹Here, $\text{Ai}(x)$ is the standard Airy function of the first kind.

can be reorganised in three useful ways, each of which we now list:

$$\begin{aligned}
 J^\pm(\mu, m_F) &= - \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) \left(N_n^\pm(\mu, m_F) - N_{-n-1}^\pm(\mu, m_F) \right), \\
 &= - \sum_{n=0}^{\infty} n \left(N_n^\pm(\mu, m_F) - N_{-n-2}^\pm(\mu, m_F) \right) - \frac{1}{2} \sum_{n=0}^{\infty} \left(N_n^\pm(\mu, m_F) - 3N_{-n-2}^\pm(\mu, m_F) \right), \\
 &= \frac{1}{2} N_0^\pm(\mu, m_F) - \sum_{n=0}^{\infty} n \left(N_n^\pm(\mu, m_F) - N_{-n}^\pm(\mu, m_F) \right) - \frac{1}{2} \sum_{n=0}^{\infty} \left(N_n^\pm(\mu, m_F) + N_{-n}^\pm(\mu, m_F) \right).
 \end{aligned} \tag{E.54}$$

In particular note that the difference between N values in the first line in (E.54) is of order unity rather than order R (it is easy to check that this quantity vanishes when $m_F = 0$ using the boundary condition equations (E.45)). This establishes that the angular momentum is actually of order R^2 rather than R^3 .

E.7 Angular momentum in the non-relativistic limit

The utility of the second line of (E.54) is that it allows us to see that the angular momentum of the Fermi sea is simply $\frac{1}{2}$ times the number of occupied single particle states in the non-relativistic limit at positive mass $m_F > 0$. To see this, we note that, for large enough n , it follows from (E.53) that $N_n^\pm = N_{-n-2}^\pm$. And so, we get from the second line of (E.54)

$$J^\pm(\mu, m_F) = \sum_{n=0}^{\infty} N_n^\pm(\mu, m_F). \tag{E.55}$$

It follows that the total angular momentum (E.50) is given by

$$J(\mu, m_F) = \frac{1}{2} \sum_{n=0}^{\infty} \left(N_n^+(\mu, m_F) + N_n^-(\mu, m_F) \right) \tag{E.56}$$

and that, in this limit, the angular momentum is exactly half of the charge of the Fermi sea.

To see this explicitly in formulas, we note that at large n , we can replace this summation by an integral and use (E.53) to get

$$J(\mu, m_F) = \frac{p_F R}{\pi} \int_0^1 \left(\alpha \sin^{-1} \alpha + \sqrt{1 - \alpha^2} - \frac{\pi \alpha}{2} \right) d\alpha \tag{E.57}$$

where, we have defined $\alpha = n/(p_F R)$ for brevity. Performing the integral, the final result is

$$J(\mu, m_F) = \frac{1}{8} (p_F R)^2 \tag{E.58}$$

This result is exactly the $1/2$ times number of occupied single-particle states⁷⁰ in the non-relativistic limit at positive mass $m_F > 0$.

⁷⁰This can be easily obtained from the phase-space counting, $\int_{p \leq p_F} \frac{\mathcal{V}_2 d^2 p}{(2\pi)^2} = \frac{(p_F R)^2}{4}$, where, $\mathcal{V}_2 = \pi R^2$ is the volume of the two dimensional space with a radial cutoff at R .

In the same way, the third line allows us to see that the angular momentum of the Fermi sea is simply $-\frac{1}{2}$ times the number of occupied single-particle states in the non-relativistic limit at negative mass $m_F < 0$. The argument is similar to the positive mass case. The final answer in this case is given by

$$J(\mu, m_F) = -\frac{1}{8}(p_F R)^2. \quad (\text{E.59})$$

Combining (E.58) and (E.59), we see that in the non-relativistic limit

$$J(\mu, m_F) = \frac{\text{sgn}(m_F)}{2} \frac{(p_F R)^2}{4}. \quad (\text{E.60})$$

Note, in particular, that this angular momentum is extensive.

E.8 General form of angular momentum

Away from the non-relativistic limit the angular momentum of the Fermi sea, defined in this appendix, takes the form

$$J(\mu, m_F) = \frac{1}{2} \frac{(p_F R)^2}{4} h\left(\frac{m_F}{|\mu|}\right), \quad (\text{E.61})$$

where $h(x)$ is a yet to be carefully computed function which has the properties

$$h(-1) = -1, \quad h(0) = 0, \quad h(1) = 1. \quad (\text{E.62})$$

A very crude estimation — one that makes approximations that we have not attempted to systematically justify — suggests

$$h(x) = \frac{2x}{1 + |x|}. \quad (\text{E.63})$$

We leave the verification or improvement of (E.63) (a relatively easy exercise) and its generalisation to nonzero 't Hooft coupling (a more interesting exercise) to future work.

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