Precise calculation of the decay rate of false vacuum with multi-field bounce

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ABSTRACT: We study the decay rate of a false vacuum in gauge theory at the one-loop level. We pay particular attention to the case where the bounce consists of an arbitrary number of scalar fields. With a multi-field bounce, which has a curved trajectory in the field space, the mixing among the gauge fields and the scalar fields evolves along the path of the bounce in the field space and the one-loop calculation of the vacuum decay rate becomes complicated. We consider the one-loop contribution to the decay rate with an arbitrary choice of the gauge parameter, and obtain a gauge invariant expression of the vacuum decay rate. We also give proper treatments of gauge zero modes and renormalization.

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Contents

1	Introduction	1
2	Setup and formulation	3
2.1	Lagrangian and bounce	3
2.2	Fluctuation operators	6
2.2.1	FP ghosts	8
2.2.2	Gauge bosons and scalars	8
2.3	Prefactor and functional determinant	10
3	Decomposition of solutions	11
3.1	$\ell > 0$	12
3.2	$\ell = 0$	13
4	Functional determinants	13
4.1	$\ell > 0$	13
4.2	$\ell = 0$	18
4.3	Background gauge	20
5	Zero modes	23
5.1	General issues	23
5.2	Gauge zero modes	27
5.3	Translational zero modes	30
6	Semi-analytic expression of the decay rate	32
6.1	Contributions of FP ghosts and transverse modes	32
6.2	Contributions of $(SL\varphi)$ modes	33
6.3	Background gauge	34
7	Renormalization	35
8	Conclusions and discussion	37
A	Evaluation of determinants	38
A.1	Alternative fluctuation operators	38
A.2	Evaluation of solutions for $\ell > 0$	40
A.2.1	Behavior at infinity	40
A.2.2	Translational zero modes	42
A.2.3	Functional determinant ($\ell > 1$)	43
A.2.4	Functional determinant ($\ell = 1$)	44
A.3	Evaluation of solutions for $\ell = 0$	45
A.3.1	Behavior at infinity	45

A.3.2	Gauge zero modes	47
A.3.3	Functional determinant	49
B	Use of alternative fluctuation operators	50
B.1	General discussion	50
B.1.1	Setup	50
B.1.2	Recursive formula	51
B.1.3	Error evaluation formula	51
B.2	Alternative fluctuation operators	54
B.2.1	Extended fluctuation operators	54
B.2.2	Linear approximation	56
B.3	$(SL\varphi)$ modes	58
B.4	$(c\bar{c})$ modes	59
B.5	$(\eta\lambda)$ modes	59
C	$\overline{\text{MS}}$ Counterterms	60

1 Introduction

The decay of a false vacuum has attracted theoretical and phenomenological interests in particle physics and cosmology. For example, in the standard model (SM) and models beyond the SM, there may exist a vacuum whose energy density is lower than that of the electroweak (EW) vacuum. If this is the case, the EW vacuum becomes a false vacuum and is not absolutely stable. Thus, the longevity of the EW vacuum often provides an important constraint on model parameters. In particular, assuming that the standard model is valid up to the Planck scale, the EW vacuum decays within a timescale shorter than the present cosmic age if the top-quark mass is too large or the Higgs mass is too small [1–12].¹ In addition, the decay of the false vacuum is also important for the studies of phase transitions in cosmological history, which may be related to inflation or the baryon asymmetry of the Universe. Thus, the precise calculation of the decay rate of the false vacuum is of great importance.

The calculation of a vacuum decay rate has been formulated in [27, 28], where the field configuration called the bounce plays a central role. The bounce is a saddle-point solution of the Euclidean equation of motion, which dominates the path integral for the decay process of the false vacuum. With the bounce configuration being given, the decay rate of a vacuum in unit volume is expressed as

$$\gamma = \mathcal{A}e^{-\mathcal{B}}, \quad (1.1)$$

where $\mathcal{B}$ is the action of the bounce and $\mathcal{A}$ contains the effects of the quantum corrections to the action. At the one-loop level, $\mathcal{A}$ is obtained by evaluating the functional determinants

¹For discussion of the absolute stability of the EW vacuum in the SM, see [13–26].

of the fluctuation operators around the false vacuum and those around the bounce. For the precise determination of a vacuum decay rate, the calculation of $\mathcal{A}$ is necessary not only because it fixes the overall factor but also because it cancels out the renormalization scale dependence of $\mathcal{B}$ at the one-loop level [29].²

If scalar fields responsible for the bounce couple to the gauge fields, the fluctuation operator generally depends on the gauge-fixing parameter (which we call ξ) and ξ appears everywhere in the calculation of the prefactor $\mathcal{A}$. On the other hand, the decay rate of the false vacuum should be independent of ξ because the effective action is gauge independent at its extrema [31, 32]. An explicit check of the gauge invariance at the one-loop level is quite formidable and the first calculation appeared only recently in [33, 34]. In these papers, a manifestly gauge-invariant expression of the decay rate has been obtained for the case where the bounce consists of a single field (single-field bounce). They also address another issue that arises when a gauge symmetry preserved in the false vacuum is broken by the bounce configuration. In such a case, there appears a flat direction of the action corresponding to the global part of the gauge symmetry; it can be seen as a gauge zero mode in the calculation of the functional determinant. Since the fluctuation toward such a flat direction cannot be treated with the saddle point method in the path integral, we need special treatment; a correct prescription for the gauge zero mode has been developed for the single-field bounce [34]. The prescriptions to calculate the decay rate of false vacuum given in [33, 34] are essential to perform a complete one-loop calculation of the decay rate of the EW vacuum in the SM [10–12], which update the previous result [1].³ In addition, they are also applied to models beyond the SM [12, 30].⁴

In this paper, we extend the results of [33, 34] to the case where the bounce consists of more than one field (multi-field bounce). We give a prescription to obtain a gauge-invariant expression of the vacuum decay rate, adopting two different gauge-fixing conditions; one is the Fermi gauge and the other is the background gauge. In the Fermi gauge, the treatment of the gauge zero mode can be understood easily, but the numerical calculation becomes difficult due to a severe cancellation. On the other hand, in the background gauge, the treatment of the gauge zero mode is complicated, but the numerical calculation becomes easier because of better behavior of fluctuation operators. Thus, we give a prescription to convert the result in the background gauge to that in the Fermi gauge which is guaranteed to be gauge invariant.

This paper is organized as follows. In section 2, we explain our basic setup for the calculation of the vacuum decay rate. We show that the vacuum decay rate (in particular, the prefactor $\mathcal{A}$) can be expressed by using solutions of a set of differential equations. In section 3, we provide a decomposition of the solutions. In section 4, we construct a set

²The numerical impact of the prefactor is demonstrated in [12, 29, 30].

³When the true vacuum appears as a result of the renormalization group improvement of the potential, a careful treatment is required to avoid the double-counting of quantum corrections. If the properties of the bounce are well described by the (renormalizable) Lagrangian with choosing relevant renormalization scale, however, the functional-determinant method can give the one-loop contribution to the prefactor without the double counting; examples include the standard model (see [10–12]).

⁴For other studies about the stability of the electroweak vacuum in models beyond the SM, see, for example, [35–53].

of solutions and calculate their asymptotic behavior, which is needed for the evaluation of $\mathcal{A}$. In section 5, we provide a general method to treat the zero modes and apply it to the zero modes in association with the gauge and the translational symmetries. In section 6, we summarize the analytic results. In section 7, issues related to the renormalization are discussed. Section 8 is devoted to conclusions and discussion.

2 Setup and formulation

2.1 Lagrangian and bounce

We consider a Euclidean four-dimensional gauge theory with a direct-product gauge group, $\mathfrak{G}$. We concentrate on the contributions from scalar fields and gauge fields; the effects of fermions, if they exist, can be taken into account separately. The Lagrangian is given by

$$\mathcal{L}_E = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2} (D_\mu \phi)_i (D_\mu \phi)_i + V(\phi) + \mathcal{L}^{(\text{GF})} + \mathcal{L}^{(\text{ghost})}, \quad (2.1)$$

where V is a scalar potential, while $\mathcal{L}^{(\text{GF})}$ and $\mathcal{L}^{(\text{ghost})}$ include the gauge fixing terms and the terms containing the ghosts, respectively, which will be defined later. In addition, $F_{\mu\nu}^a$ is the field strength of the gauge field A_μ^a (with a being the adjoint index of $\mathfrak{G}$), while ϕ_i (with $i = 1 - n_\phi$ being the index distinguishing scalars) are real scalar fields.⁵ Here and hereafter, the summation over the repeated scalar indices is implicit. The covariant derivative of ϕ is defined as

$$D_\mu \phi = (\partial_\mu + g_a A_\mu^a T^a) \phi. \quad (2.2)$$

In the real basis we are working with, the generators satisfy

$$(T^a)^T = -T^a, \quad (2.3)$$

$$[T^a, T^b] = -f^{abc} T^c, \quad (2.4)$$

where the superscript “ T ” denotes transpose and f^{abc} is the structure constant. The gauge coupling constants, denoted as g_a , can be different for different subgroups of $\mathfrak{G}$. We assume that the scalar potential, $V(\phi)$, has two minima, i.e., the false vacuum and the true vacuum (the false vacuum has higher potential energy than the true vacuum).

For the process of the false vacuum decay, the path integral is dominated by the field configuration called “bounce,” an $O(4)$ symmetric saddle-point solution of the Euclidean equations of motion [54, 55]. Due to the gauge symmetry, there exist an infinite number of solutions. We adopt one solution with $A_\mu = 0$. Then, the bounce configuration, which we denote as $\bar{\phi}(r)$, satisfies

$$\partial_r^2 \bar{\phi}_i + \frac{3}{r} \partial_r \bar{\phi}_i = \frac{\partial V}{\partial \phi_i} \Big|_{\phi \rightarrow \bar{\phi}}, \quad (2.5)$$

⁵In our convention, complex scalar fields are understood to be decomposed into pairs of real scalar fields.

with the following boundary conditions:

$$\partial_r \bar{\phi}_i(0) = 0, \quad (2.6)$$

$$\bar{\phi}_i(\infty) = v_i, \quad (2.7)$$

where v_i denotes the scalar amplitude at the false vacuum. Here, $r \equiv \sqrt{x_\mu x_\mu}$ is the radius from the center of the bounce, and ∂_r denotes the derivative with respect to r .

The bounce solution has the following properties. First, since the potential is symmetric under the infinitesimal gauge transformation $\phi \rightarrow \phi + \delta\theta^a T^a \phi$ with an arbitrary choice of transformation parameters $\delta\theta^a$,

$$\frac{\partial V}{\partial \phi_i} T_{ij}^a \phi_j = 0. \quad (2.8)$$

Differentiating it with respect to ϕ_k , we obtain

$$T_{ki}^a \frac{\partial V}{\partial \phi_i} = \frac{\partial^2 V}{\partial \phi_k \partial \phi_i} T_{ij}^a \phi_j. \quad (2.9)$$

Then, from eq. (2.5),

$$\partial_r^2 (T^a \bar{\phi}) + \frac{3}{r} \partial_r (T^a \bar{\phi}) = \Omega (T^a \bar{\phi}), \quad (2.10)$$

where

$$\Omega_{ij} = \left. \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \right|_{\phi \rightarrow \bar{\phi}}. \quad (2.11)$$

From eq. (2.10), we obtain

$$\frac{1}{r^3} \partial_r r^3 [(\partial_r \bar{\phi})^T T^a \bar{\phi}] = \frac{1}{2} \bar{\phi}^T [T^a, \Omega] \bar{\phi} = 0, \quad (2.12)$$

because Ω is gauge invariant. It implies that $(\partial_r \bar{\phi})^T T^a \bar{\phi}$ should be proportional to r^{-3} or zero. Since it should vanish at the origin,

$$(\partial_r \bar{\phi})^T T^a \bar{\phi} = 0. \quad (2.13)$$

For the later convenience, we define

$$M_{ia}(r) = -g_a T_{ik}^a \bar{\phi}_k(r), \quad (2.14)$$

which satisfies

$$\left(\partial_r^2 + \frac{3}{r} \partial_r - \Omega \right) M = 0. \quad (2.15)$$

Using eq. (2.13), the following relation holds:

$$M^T M' = (M^T)' M, \quad (2.16)$$

where $M' = \partial_r M$. Notice that the gauge boson mass matrix in the false vacuum is given by $M^T M(r \rightarrow \infty)$.

In our analysis, we concentrate on the case where the following conditions hold:

- The rank of the matrix $M^T M(r)$ is unchanged for $r < \infty$. (At the false vacuum, some of the broken gauge symmetries may be recovered so that the ranks of $M^T M(r < \infty)$ and $M^T M(\infty)$ may be different.)
- There are no zero modes except for the gauge zero modes and the translational zero modes.
- At a large r , $M^T M(r)$ approaches to $M^T M(\infty)$ exponentially.

The second condition is just for simplicity and our results can be extended to the cases with additional zero modes. The third condition is violated when the theory has the (approximate) scale invariance, which has already been discussed for the single-field case [10–12] and the multi-field case [30].

We define the subset of gauge fields that acquire masses from the bounce at $r < \infty$. Then, we define the following numbers:

- n_G : the number of the gauge bosons which acquire masses from the bounce at $r < \infty$.
- n_B : the number of the gauge bosons which remain massive at the false vacuum.
- n_U : the number of the gauge bosons which become massless at the false vacuum (but are massive at $r < \infty$).
- n_φ : the total number of scalar fields.
- n_H : the number of the scalar fields that do not mix with the gauge bosons at $r < \infty$.

Notice that $n_\varphi = n_G + n_H$ and $n_G = n_B + n_U$.

Now we introduce the gauge fixing terms. We consider the following two gauge fixing conditions:

1. Fermi gauge:

$$\mathcal{L}_{\text{Fermi}}^{(\text{GF})} = \frac{1}{2\xi} (\partial_\mu A_\mu^a)^2, \quad (2.17)$$

$$\mathcal{L}_{\text{Fermi}}^{(\text{ghost})} = -\bar{c} \partial_\mu D_\mu c. \quad (2.18)$$

2. Background gauge:

$$\mathcal{L}_{\text{BG}}^{(\text{GF})} = \frac{1}{2\xi} \left(\partial_\mu A_\mu^a + \xi g_a \phi^T T^a \bar{\phi} \right)^2, \quad (2.19)$$

$$\mathcal{L}_{\text{BG}}^{(\text{ghost})} = \bar{c}^a \left[-\partial_\mu D_\mu^{ab} + \xi (g_a T^a \bar{\phi})^T (g_b T^b \phi) \right] c^b. \quad (2.20)$$

These gauge fixing terms are consistent with the bounce of our choice since their contributions to the equation of motion vanish in the limit of $A_\mu \rightarrow 0$ and $\phi \rightarrow \bar{\phi}$. They are also consistent with the spacial translational invariance; it corresponds to the fluctuation of

$\delta\phi = \delta x_\mu \partial_\mu \bar{\phi}$. As for the global gauge transformation, $\delta\phi = \delta\theta^a T^a \bar{\phi}$, the gauge fixing term of the Fermi gauge is invariant [34, 56]. Then, in the Fermi gauge, a prescription to take care of the gauge zero modes is available. On the contrary, in the background gauge, the proper treatment of the gauge zero modes becomes more complicated, as we will discuss. The background gauge is, however, useful for numerical calculation especially with $\xi = 1$. Thus, we also discuss how the prefactors based on these two gauge fixing conditions are related in this paper.

2.2 Fluctuation operators

To evaluate the decay rate of the false vacuum, we integrate out the fluctuations around the bounce configuration. As is explicitly given in the next subsection, the prefactor is expressed with the functional determinant of the fluctuation operators, which are the second derivative of the action with respect to the fluctuations.

In discussing the contributions to the prefactor in our setup, we should consider the fluctuation operator for the bosons and that for the Faddeev-Popov (FP) ghosts. In the Fermi gauge, they are given by⁶

$$\mathcal{M}^{(A_\mu\varphi)} = \begin{pmatrix} (-\partial^2 + M^T M)\delta_{\mu\nu} + (1 - \frac{1}{\xi})\partial_\mu\partial_\nu & (\partial_\nu M)^T - M^T\partial_\nu \\ 2(\partial_\mu M) + M\partial_\mu & -\partial^2 + \Omega \end{pmatrix}, \quad (2.21)$$

and

$$\mathcal{M}^{(c\bar{c})} = -\partial^2, \quad (2.22)$$

respectively. Note that $\mathcal{M}^{(A_\mu\varphi)}$ is an $(n_G + n_\varphi) \times (n_G + n_\varphi)$ object and $\mathcal{M}^{(c\bar{c})}$ is $n_G \times n_G$. In addition, in the background gauge,

$$\mathcal{M}_{\text{BG}}^{(A_\mu\varphi)} = \begin{pmatrix} (-\partial^2 + M^T M)\delta_{\mu\nu} + (1 - \frac{1}{\xi})\partial_\mu\partial_\nu & 2(\partial_\nu M)^T \\ 2(\partial_\mu M) & -\partial^2 + \Omega + \xi M M^T \end{pmatrix}, \quad (2.23)$$

$$\mathcal{M}_{\text{BG}}^{(c\bar{c})} = -\partial^2 + \xi M^T M. \quad (2.24)$$

In order for the following discussion, let us define

$$\widehat{M} \equiv M(r \rightarrow \infty), \quad (2.25)$$

$$\widehat{\Omega} \equiv \Omega(r \rightarrow \infty). \quad (2.26)$$

Then, using eq. (2.15) as well as $M'(r \rightarrow \infty) = 0$, we obtain

$$\widehat{\Omega}\widehat{M} = 0. \quad (2.27)$$

⁶We do not put the subscript “Fermi” on the fluctuation operators in the Fermi gauge for notational simplicity while we put the subscript “BG” on those in the background gauge.

For later convenience, we choose the following basis of the gauge bosons and the scalars:

$$\widehat{M} = \left(\underbrace{\widehat{W}}_{n_B} \quad \underbrace{0}_{n_U} \right) \left\{ \begin{matrix} n_B \\ n_U + n_H \end{matrix} \right\}, \quad (2.28)$$

$$\widehat{\Omega} = \left(\underbrace{0}_{n_B} \quad \underbrace{0 \quad \widehat{m}^2}_{n_U + n_H} \right) \left\{ \begin{matrix} n_B \\ n_U + n_H \end{matrix} \right\}, \quad (2.29)$$

where $\widehat{W}$ and $\widehat{m}^2$ are full rank diagonal matrices.⁷ Such a choice always exists because of eq. (2.27). In this basis, we also define submatrices of M as

$$M(r) = \left(\underbrace{M_B(r)}_{n_B} \quad \underbrace{M_U(r)}_{n_U} \right). \quad (2.30)$$

In addition, we define

$$M_0 \equiv M(r = 0). \quad (2.31)$$

The fluctuation around the bounce can be expanded using the hyperspherical functions on S^3 , which are functions of $\hat{x}_\mu \equiv x_\mu/r$. The rotational Lie algebra of the four dimensional Euclidean space is equivalent to $SU(2)_L \times SU(2)_R$ and the hyperspherical functions, $\mathcal{Y}_P(\hat{x}_\mu)$, are labeled by indices $P = (\ell, m_A, m_B)$. Here, $\ell = 0, 1, 2, \dots$ is the azimuthal quantum number and m_A and m_B are the magnetic quantum numbers that take values of $-\ell/2, \ell/2 + 1, \dots, \ell/2$.⁸ Using mode functions that depend only on r and the hyperspherical functions, we expand fields around the bounce as

$$\phi(x) = \bar{\phi}(r) + \alpha_\varphi^P(r) \mathcal{Y}_P, \quad (2.32)$$

$$\begin{aligned} A_\mu^a(x) = & \alpha_S^{aP}(r) \frac{x_\mu}{r} \mathcal{Y}_P + \alpha_L^{aP}(r) \frac{r}{L} \partial_\mu \mathcal{Y}_P \\ & + \alpha_{T1}^{aP}(r) i \epsilon_{\mu\nu\rho\sigma} V_\nu^{(1)} L_{\rho\sigma} \mathcal{Y}_P + \alpha_{T2}^{aP}(r) i \epsilon_{\mu\nu\rho\sigma} V_\nu^{(2)} L_{\rho\sigma} \mathcal{Y}_P, \end{aligned} \quad (2.33)$$

$$c^a(x) = \alpha_c^{aP}(r) \mathcal{Y}_P, \quad (2.34)$$

$$\bar{c}^a(x) = \alpha_{\bar{c}}^{aP}(r) \mathcal{Y}_P, \quad (2.35)$$

where the summation over P is implicit. Here, $V_\nu^{(1)}$ and $V_\nu^{(2)}$ are arbitrary two independent vectors and

$$L_{\rho\sigma} = \frac{i}{\sqrt{2}} (x_\rho \partial_\sigma - x_\sigma \partial_\rho), \quad (2.36)$$

$$L = \sqrt{\ell(\ell + 2)}. \quad (2.37)$$

⁷We consider the case where there is no massless physical scalar in the false vacuum, and hence $\widehat{m}^2$ does not have zero eigenvalues.

⁸The variable J used in [33, 34] and ℓ are related as $\ell = 2J$.

Note that (L) and (T) modes do not have $\ell = 0$ mode. For the following discussion, it is convenient to define the derivative operator that corresponds to the Laplacian acting on the modes with the azimuthal quantum number ℓ :

$$\Delta_\ell = \partial_r^2 + \frac{3}{r}\partial_r - \frac{L^2}{r^2}. \quad (2.38)$$

Similarly, we define the fluctuation operators at the false vacuum, which can be obtained by replacement $M \rightarrow \widehat{M}$ and $\Omega \rightarrow \widehat{\Omega}$. We denote them as $\widehat{\mathcal{M}}^{(A_\mu\varphi)}$ and $\widehat{\mathcal{M}}^{(c\bar{c})}$.

In the following, we will show the hyperspherical expansion of the fluctuation operators around the bounce and around the false vacuum.

2.2.1 FP ghosts

The fluctuations of the FP ghosts can be expanded by using the hyperspherical functions. Correspondingly, the fluctuation operator for the FP ghosts can be block-diagonalized as

$$\mathcal{M}^{(c\bar{c})} = \bigoplus_{\ell=0}^{\infty} \left(\mathcal{M}_\ell^{(c\bar{c})} \right)^{2(\ell+1)^2}, \quad (2.39)$$

where the power comes from $(\ell+1)^2$ different choices of (m_A, m_B) and the complexity of the FP ghosts. In the Fermi gauge,

$$\mathcal{M}_\ell^{(c\bar{c})} = -\Delta_\ell, \quad (2.40)$$

while in the background gauge,

$$\mathcal{M}_{\ell, \text{BG}}^{(c\bar{c})} = -\Delta_\ell + \xi M^T M. \quad (2.41)$$

Notice that the fluctuation operators for the ghosts are $n_G \times n_G$ objects.

At the false vacuum, we have a similar block-diagonalization:

$$\widehat{\mathcal{M}}^{(c\bar{c})} = \bigoplus_{\ell=0}^{\infty} \left(\widehat{\mathcal{M}}_\ell^{(c\bar{c})} \right)^{2(\ell+1)^2}, \quad (2.42)$$

where, in the Fermi gauge,

$$\widehat{\mathcal{M}}_\ell^{(c\bar{c})} = -\Delta_\ell, \quad (2.43)$$

while in the background gauge,

$$\widehat{\mathcal{M}}_{\ell, \text{BG}}^{(c\bar{c})} = -\Delta_\ell + \xi \widehat{M}^T \widehat{M}. \quad (2.44)$$

2.2.2 Gauge bosons and scalars

Due to the mixing between the gauge bosons and the scalars, we cannot discuss their effects separately. Since only the mode functions with the same $P = (\ell, m_A, m_B)$ mix, the fluctuation operator $\mathcal{M}^{(A_\mu\varphi)}$ can be block-diagonalized as

$$\mathcal{M}^{(A_\mu\varphi)} = \mathcal{M}_0^{(S\varphi)} \oplus \left[\bigoplus_{\ell=1}^{\infty} \left(\mathcal{M}_\ell^{(SL\varphi)} \right)^{(\ell+1)^2} \right] \oplus \left[\bigoplus_{\ell=1}^{\infty} \left(\mathcal{M}_\ell^{(T)} \right)^{2(\ell+1)^2} \right]. \quad (2.45)$$

Here, independently of the choice of the gauge fixing, the fluctuation operator for the (T) modes is given by

$$\mathcal{M}_\ell^{(T)} = -\Delta_\ell + M^T M. \quad (2.46)$$

Meanwhile, the fluctuation operators of (S) , (L) , and (φ) modes depend on the gauge fixing. In the Fermi gauge,

$$\mathcal{M}_0^{(S\varphi)} = \begin{pmatrix} -\frac{1}{\xi}\Delta_1 + M^T M & (M')^T - M^T \partial_r \\ 2M' + M \frac{1}{r^3} \partial_r r^3 & -\Delta_0 + \Omega \end{pmatrix}, \quad (2.47)$$

$$\begin{aligned} \mathcal{M}_\ell^{(SL\varphi)} = & \begin{pmatrix} -\Delta_\ell + \frac{3}{r^2} + M^T M & -\frac{2L}{r^2} & (M')^T - M^T \partial_r \\ -\frac{2L}{r^2} & -\Delta_\ell - \frac{1}{r^2} + M^T M & -\frac{L}{r} M^T \\ 2M' + M \frac{1}{r^3} \partial_r r^3 & -\frac{L}{r} M & -\Delta_\ell + \Omega \end{pmatrix} \\ & + \left(1 - \frac{1}{\xi}\right) \begin{pmatrix} \partial_r^2 + \frac{3}{r} \partial_r - \frac{3}{r^2} - L \left(\frac{1}{r} \partial_r - \frac{1}{r^2}\right) & 0 \\ L \left(\frac{1}{r} \partial_r + \frac{3}{r^2}\right) & -\frac{L^2}{r^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.48)$$

Note that $\mathcal{M}_0^{(S\varphi)}$ is $(n_G + n_\varphi) \times (n_G + n_\varphi)$, while $\mathcal{M}_\ell^{(SL\varphi)}$ is $(2n_G + n_\varphi) \times (2n_G + n_\varphi)$.

Meanwhile, in the background gauge,

$$\mathcal{M}_{0,\text{BG}}^{(S\varphi)} = \begin{pmatrix} -\frac{1}{\xi}\Delta_1 + M^T M & 2(M')^T \\ 2M' & -\Delta_0 + \Omega + \xi M M^T \end{pmatrix}, \quad (2.49)$$

$$\begin{aligned} \mathcal{M}_{\ell,\text{BG}}^{(SL\varphi)} = & \begin{pmatrix} -\Delta_\ell + \frac{3}{r^2} + M^T M & -\frac{2L}{r^2} & 2(M')^T \\ -\frac{2L}{r^2} & -\Delta_\ell - \frac{1}{r^2} + M^T M & 0 \\ 2M' & 0 & -\Delta_\ell + \Omega + \xi M M^T \end{pmatrix} \\ & + \left(1 - \frac{1}{\xi}\right) \begin{pmatrix} \partial_r^2 + \frac{3}{r} \partial_r - \frac{3}{r^2} - L \left(\frac{1}{r} \partial_r - \frac{1}{r^2}\right) & 0 \\ L \left(\frac{1}{r} \partial_r + \frac{3}{r^2}\right) & -\frac{L^2}{r^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.50)$$

At the false vacuum, similar block-diagonalizations hold. For the (T) mode,

$$\widehat{\mathcal{M}}_\ell^{(T)} = -\Delta_\ell + \widehat{M}^T \widehat{M}. \quad (2.51)$$

The fluctuation operator for $(SL\varphi)$ modes at the false vacuum can be further block-diagonalized thanks to the choice of the basis of eqs. (2.28) and (2.29). For $\ell > 0$, the fluctuation operator can be expressed as

$$\widehat{\mathcal{M}}_\ell^{(SL\varphi)} = \mathcal{M}_\ell^{(SL\varphi)} \Big|_{\bar{\phi} \rightarrow v} = \widehat{\mathcal{M}}_\ell^{(B)} \oplus \widehat{\mathcal{M}}_\ell^{(U)} \oplus \widehat{\mathcal{M}}_\ell^{(\sigma)}, \quad (2.52)$$

corresponding to the contributions from massive gauge bosons and Nambu-Goldstone (NG) bosons (due to broken gauge symmetry), massless gauge bosons (in association with unbroken gauge symmetry), and physical scalars. Firstly, contributions to $\widehat{\mathcal{M}}_\ell^{(B)}$ are from (S) and (L) modes of gauge bosons corresponding to broken symmetries ($a = 1, \dots, n_B$) and

the corresponding NG modes ($i = 1, \dots, n_B$). Then, $\widehat{\mathcal{M}}_\ell^{(B)}$ is a $3n_B \times 3n_B$ object and is given by

$$\begin{aligned} \widehat{\mathcal{M}}_\ell^{(B)} = & \begin{pmatrix} -\Delta_\ell + \frac{3}{r^2} + \widehat{W}^T \widehat{W} & -\frac{2L}{r^2} & -\widehat{W}^T \partial_r \\ -\frac{2L}{r^2} & -\Delta_\ell - \frac{1}{r^2} + \widehat{W}^T \widehat{W} & -\frac{L}{r} \widehat{W}^T \\ \widehat{W} \frac{1}{r^3} \partial_r r^3 & -\frac{L}{r} \widehat{W} & -\Delta_\ell \end{pmatrix} \\ & + \left(1 - \frac{1}{\xi}\right) \begin{pmatrix} \partial_r^2 + \frac{3}{r} \partial_r - \frac{3}{r^2} & -L \left(\frac{1}{r} \partial_r - \frac{1}{r^2}\right) & 0 \\ L \left(\frac{1}{r} \partial_r + \frac{3}{r^2}\right) & -\frac{L^2}{r^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.53)$$

Secondly, contributions to $\widehat{\mathcal{M}}_\ell^{(U)}$ are from massless gauge bosons in (S) and (L) modes. We obtain a $2n_U \times 2n_U$ fluctuation operator as

$$\widehat{\mathcal{M}}_\ell^{(U)} = \begin{pmatrix} -\Delta_\ell + \frac{3}{r^2} & -\frac{2L}{r^2} \\ -\frac{2L}{r^2} & -\Delta_\ell - \frac{1}{r^2} \end{pmatrix} + \left(1 - \frac{1}{\xi}\right) \begin{pmatrix} \partial_r^2 + \frac{3}{r} \partial_r - \frac{3}{r^2} & -L \left(\frac{1}{r} \partial_r - \frac{1}{r^2}\right) \\ L \left(\frac{1}{r} \partial_r + \frac{3}{r^2}\right) & -\frac{L^2}{r^2} \end{pmatrix}. \quad (2.54)$$

For the other massive scalars,

$$\widehat{\mathcal{M}}_\ell^{(\sigma)} = -\Delta_\ell + \widehat{m}^2, \quad (2.55)$$

which is an $(n_U + n_H) \times (n_U + n_H)$ object.

Similarly, for $\ell = 0$, we obtain

$$\widehat{\mathcal{M}}_0^{(S\varphi)} = \widehat{\mathcal{M}}_0^{(B)} \oplus \widehat{\mathcal{M}}_0^{(U)} \oplus \widehat{\mathcal{M}}_0^{(\sigma)}, \quad (2.56)$$

where

$$\widehat{\mathcal{M}}_0^{(B)} = \begin{pmatrix} -\frac{1}{\xi} \Delta_1 + \widehat{W}^T \widehat{W} & -\widehat{W}^T \partial_r \\ \widehat{W} \frac{1}{r^3} \partial_r r^3 & -\Delta_0 \end{pmatrix}, \quad (2.57)$$

$$\widehat{\mathcal{M}}_0^{(U)} = -\frac{1}{\xi} \Delta_1, \quad (2.58)$$

$$\widehat{\mathcal{M}}_0^{(\sigma)} = -\Delta_0 + \widehat{m}^2. \quad (2.59)$$

2.3 Prefactor and functional determinant

The prefactor $\mathcal{A}$ is expressed as

$$\mathcal{A} = \mathcal{A}^{(c\bar{c})} \mathcal{A}^{(A_\mu \varphi)}, \quad (2.60)$$

where $\mathcal{A}^{(c\bar{c})}$ denotes the contributions from the FP ghosts and $\mathcal{A}^{(A_\mu \varphi)}$ denotes those from the gauge bosons and the scalars. For the evaluation of the prefactor at the one-loop level,

the following quantities are necessary [28]:

$$\mathcal{A}^{(c\bar{c})} = \frac{\text{Det } \mathcal{M}^{(c\bar{c})}}{\text{Det } \widehat{\mathcal{M}}^{(c\bar{c})}} = \prod_{\ell=0}^{\infty} \left(\frac{\text{Det } \mathcal{M}_{\ell}^{(c\bar{c})}}{\text{Det } \widehat{\mathcal{M}}_{\ell}^{(c\bar{c})}} \right)^{(\ell+1)^2}, \quad (2.61)$$

$$\begin{aligned} \mathcal{A}'^{(A_{\mu}\varphi)} &= \left[\frac{\text{Det}' \mathcal{M}^{(A_{\mu}\varphi)}}{\text{Det } \widehat{\mathcal{M}}^{(A_{\mu}\varphi)}} \right]^{-1/2} \\ &= \left(\frac{\text{Det}' \mathcal{M}_0^{(S\varphi)}}{\text{Det } \widehat{\mathcal{M}}_0^{(S\varphi)}} \right)^{-1/2} \left(\frac{\text{Det}' \mathcal{M}_1^{(SL\varphi)}}{\text{Det } \widehat{\mathcal{M}}_1^{(SL\varphi)}} \right)^{-2} \\ &\quad \times \left[\prod_{\ell=2}^{\infty} \left(\frac{\text{Det } \mathcal{M}_{\ell}^{(SL\varphi)}}{\text{Det } \widehat{\mathcal{M}}_{\ell}^{(SL\varphi)}} \right)^{-(\ell+1)^2/2} \right] \left[\prod_{\ell=1}^{\infty} \left(\frac{\text{Det } \mathcal{M}_{\ell}^{(T)}}{\text{Det } \widehat{\mathcal{M}}_{\ell}^{(T)}} \right)^{-(\ell+1)^2} \right]. \end{aligned} \quad (2.62)$$

Here and hereafter, the “prime” is used for quantities after the proper subtraction of the zero modes if necessary. In particular, because of the translational invariance, the $\ell = 1$ contribution inevitably contains the effects of translational zero modes [28]. In addition, if $n_U > 0$, both $\mathcal{M}_0^{(S\varphi)}$ and $\mathcal{M}_{0,\text{BG}}^{(S\varphi)}$ have zero eigenvalues. More details about the zero mode subtraction and the relation between $\mathcal{A}^{(A_{\mu}\varphi)}$ and $\mathcal{A}'^{(A_{\mu}\varphi)}$ will be explained in section 5.

To evaluate the ratio of two functional determinants, we adopt the method developed in [57–61]. Let $\mathcal{M}_{\ell}^{(X)}$ and $\widehat{\mathcal{M}}_{\ell}^{(X)}$ be $n \times n$ fluctuation operators. We first prepare n linearly independent n -dimensional functions $\psi_{\ell}^{(X)(I)}(r)$ and $\widehat{\psi}_{\ell}^{(X)(I)}(r)$ (with $I = 1 - n$) that satisfy

$$\mathcal{M}_{\ell}^{(X)} \psi_{\ell}^{(X)(I)} = 0, \quad (2.63)$$

$$\widehat{\mathcal{M}}_{\ell}^{(X)} \widehat{\psi}_{\ell}^{(X)(I)} = 0, \quad (2.64)$$

and are regular at $r \rightarrow 0$. Then, we define $n \times n$ objects $\Psi_{\ell}^{(X)}(r)$ and $\widehat{\Psi}_{\ell}^{(X)}(r)$ as

$$\Psi_{\ell}^{(X)}(r) \equiv \left(\psi_{\ell}^{(X)(1)}(r) \cdots \psi_{\ell}^{(X)(n)}(r) \right), \quad (2.65)$$

$$\widehat{\Psi}_{\ell}^{(X)}(r) \equiv \left(\widehat{\psi}_{\ell}^{(X)(1)}(r) \cdots \widehat{\psi}_{\ell}^{(X)(n)}(r) \right), \quad (2.66)$$

with which the ratio of functional determinants can be evaluated as

$$\frac{\text{Det } \mathcal{M}_{\ell}^{(X)}}{\text{Det } \widehat{\mathcal{M}}_{\ell}^{(X)}} = \left(\frac{\det \Psi_{\ell}^{(X)}(r_0)}{\det \widehat{\Psi}_{\ell}^{(X)}(r_0)} \right)^{-1} \left(\frac{\det \Psi_{\ell}^{(X)}(r_{\infty})}{\det \widehat{\Psi}_{\ell}^{(X)}(r_{\infty})} \right), \quad (2.67)$$

where r_0 and r_{∞} are abbreviations of $r \rightarrow 0$ and $r \rightarrow \infty$, respectively.

Thus, for the evaluation of the prefactor $\mathcal{A}$, we need to understand the asymptotic behavior of $\Psi_{\ell}^{(X)}$ and $\widehat{\Psi}_{\ell}^{(X)}$ (and hence those of $\det \Psi_{\ell}^{(X)}(r)$ and $\det \widehat{\Psi}_{\ell}^{(X)}(r)$) at $r \rightarrow 0$ and $r \rightarrow \infty$. In particular, the behavior of $\det \Psi_{\ell}^{(SL\varphi)}$ and $\det \Psi_0^{(S\varphi)}$ is non-trivial because of the mixing among (S) , (L) , and (φ) modes. In the following sections, we discuss how we can evaluate those quantities.

3 Decomposition of solutions

In this section, we provide a decomposition of the set of solutions $\Psi_{\ell}^{(SL\varphi)}$ and $\Psi_0^{(S\varphi)}$, generalizing the results of [33, 34].

3.1 $\ell > 0$

Let us consider the solutions of the following equation in the Fermi gauge:

$$\mathcal{M}_\ell^{(SL\varphi)} \Psi_\ell^{(SL\varphi)} = 0. \quad (3.1)$$

Because $\mathcal{M}_\ell^{(SL\varphi)}$ is $(2n_G + n_\varphi) \times (2n_G + n_\varphi)$, there exist $(2n_G + n_\varphi)$ linearly independent solutions of eq. (3.1).

Solutions of eq. (3.1) can be decomposed by using four functions, χ , η , ζ , and λ :

$$\Psi_\ell^{(SL\varphi)} = \begin{pmatrix} \Psi_\ell^{(\text{top})} \\ \Psi_\ell^{(\text{mid})} \\ \Psi_\ell^{(\text{bot})} \end{pmatrix} = \begin{pmatrix} \partial_r \chi \\ \frac{L}{r} \chi \\ M \chi \end{pmatrix} + \begin{pmatrix} (M^T M)^{-1} \left[\frac{L}{r} \eta - 2(M')^T \lambda \right] \\ (M^T M)^{-1} \frac{1}{r^2} \partial_r r^2 \eta \\ \lambda \end{pmatrix} + \begin{pmatrix} [\partial_r (M^T M)^{-1}] \zeta \\ 0 \\ M (M^T M)^{-1} \zeta \end{pmatrix}. \quad (3.2)$$

Here, the functions χ , η , ζ , and λ are dependent on the azimuthal quantum number ℓ ; the subscript “ ℓ ” is omitted from these functions for notational simplicity. The shape of χ , η , and ζ is $n_G \times (2n_G + n_\varphi)$, while that of λ is $n_\varphi \times (2n_G + n_\varphi)$; here, the $(2n_G + n_\varphi)$ columns are linearly independent and are distinguished by the boundary conditions at $r = 0$.

The evolution of functions χ and ζ is governed by the following differential equations:

$$\begin{aligned} \mathcal{M}_\ell^{(c\bar{c})} \chi &= [\partial_r (M^T M)^{-1}] \frac{L}{r} \eta - \frac{2}{r^3} \partial_r r^3 (M^T M)^{-1} (M')^T \lambda \\ &\quad - \mathcal{M}_0^{(c\bar{c})} (M^T M)^{-1} \zeta - \frac{1}{r^3} \partial_r r^3 (M^T M)^{-1} \partial_r \zeta + \xi \zeta, \end{aligned} \quad (3.3)$$

$$\mathcal{M}_\ell^{(c\bar{c})} \zeta = 0, \quad (3.4)$$

while η and λ satisfy

$$\Delta_\ell \eta = M^T M \left[\eta - \{ \partial_r (M^T M)^{-1} \} \frac{1}{r^2} \partial_r r^2 \eta \right] - \frac{2L}{r} M'^T \lambda + \frac{L}{r} M^T M [\partial_r (M^T M)^{-1}] \zeta, \quad (3.5)$$

$$\begin{aligned} \Delta_\ell \lambda &= \Omega \lambda - 4M' (M^T M)^{-1} M'^T \lambda + \frac{2L}{r} M' (M^T M)^{-1} \eta - 2M' (M^T M)^{-1} \zeta' \\ &\quad + M \left[-\frac{2}{r^3} \partial_r r^3 (M^T M)^{-1} M'^T \lambda + \frac{L}{r} \{ \partial_r (M^T M)^{-1} \} \eta - \{ \partial_r (M^T M)^{-1} \} \zeta' \right]. \end{aligned} \quad (3.6)$$

In addition, λ satisfies the following constraint:

$$M^T \lambda = 0. \quad (3.7)$$

The above constraint is consistent with the evolution equations; one can derive $\Delta_\ell (M^T \lambda) = 0$, so that eq. (3.7) holds if it is satisfied at $r = 0$.

So far, we have discussed in the Fermi gauge. We note here that the decomposition given in eq. (3.2) is also applicable in the background gauge if we replace $\mathcal{M}_\ell^{(c\bar{c})}$ in eqs. (3.3) and (3.4) by $\mathcal{M}_{\ell, \text{BG}}^{(c\bar{c})}$ given in eq. (2.41). (In the background gauge, the relation $\mathcal{M}_{\ell, \text{BG}}^{(c\bar{c})} (M^T \lambda) = 0$ holds. Thus, the condition $M^T \lambda = 0$ is again consistent with the evolution equations.)

From eqs. (3.2)–(3.6), we can derive the following equations:

$$\mathcal{M}_\ell^{(c\bar{c})} \left(r \Psi_\ell^{(\text{mid})} \right) = -\frac{1}{r^3} \partial_r r^4 \eta + \xi L \zeta, \quad (3.8)$$

$$\Psi_\ell^{(\text{top})} = \frac{1}{L} \partial_r \left(r \Psi_\ell^{(\text{mid})} \right) - \frac{r}{L} \eta, \quad (3.9)$$

$$\Psi_\ell^{(\text{bot})} = \frac{r}{L} M \Psi_\ell^{(\text{mid})} - M (M^T M)^{-1} \frac{1}{Lr} \partial_r r^2 \eta + \lambda + M (M^T M)^{-1} \zeta. \quad (3.10)$$

These expressions are useful to derive the asymptotic behaviors of the solutions. Notice that, in the background gauge, eq. (3.8) is replaced by

$$\mathcal{M}_{\ell, \text{BG}}^{(c\bar{c})} \left(r \Psi_{\ell, \text{BG}}^{(\text{mid})} \right) = -\frac{1}{r^3} \partial_r r^4 \eta + \frac{\xi}{r} \partial_r r^2 \eta, \quad (3.11)$$

while eqs. (3.9) and (3.10) are unchanged.

3.2 $\ell = 0$

For $\ell = 0$ mode, we need solutions of

$$\mathcal{M}_0^{(S\varphi)} \Psi_0^{(S\varphi)} = 0. \quad (3.12)$$

The solutions can be decomposed as

$$\Psi_0^{(S\varphi)} = \begin{pmatrix} \Psi_0^{(\text{top})} \\ \Psi_0^{(\text{bot})} \end{pmatrix} = \begin{pmatrix} \partial_r \chi \\ M \chi \end{pmatrix} + \begin{pmatrix} -2(M^T M)^{-1} (M')^T \lambda \\ \lambda \end{pmatrix} + \begin{pmatrix} [\partial_r (M^T M)^{-1}] \zeta \\ M (M^T M)^{-1} \zeta \end{pmatrix}, \quad (3.13)$$

where χ , ζ , and λ for $\ell = 0$ satisfy eqs. (3.3), (3.4), and (3.6) with replacing $\ell \rightarrow 0$ and $\eta \rightarrow 0$. Notice that now χ and ζ are $n_G \times (n_G + n_\varphi)$ objects and λ is $n_\varphi \times (n_G + n_\varphi)$; the columns correspond to $(n_G + n_\varphi)$ independent choices of the boundary conditions at $r = 0$. Thus, $\Psi_0^{(S\varphi)}$ is an $(n_G + n_\varphi) \times (n_G + n_\varphi)$ object.

4 Functional determinants

In this section, we study the behavior of the functional determinants of the fluctuation operators in the Fermi gauge. Equivalence to the results in the background gauge is also discussed.

4.1 $\ell > 0$

Let us consider the case with $\ell > 0$. For the study of the functional determinants, we should first understand the behavior of the solutions of eq. (3.1). For this purpose, we define an r -dependent $n_\varphi \times n_H$ matrix V_H , which is given by

$$V_H = \begin{pmatrix} u_1 & \cdots & u_{n_H} \end{pmatrix}, \quad (4.1)$$

with u_p ($p = 1, \dots, n_H$) being zero eigenvectors of M^T , i.e.,

$$M^T u_p = 0, \quad u_p^T u_{p'} = \delta_{pp'}. \quad (4.2)$$

The columns of the solution of eq. (3.1) can be classified into the following three types:

- **Type 1:** $\eta^{(1)} = \lambda^{(1)} = \zeta^{(1)} = 0$.
- **Type 2:** $\zeta^{(2)} \neq 0$.
- **Type 3:** $\zeta^{(3)} = 0$, with non-vanishing $\eta^{(3)}$ and $\lambda^{(3)}$,

where the superscripts “(1),” “(2),” and “(3)” indicate solutions for the Type 1, 2, and 3, respectively. Note that the Type 1, 2, and 3 solutions include n_G , n_G , and n_φ independent solutions, respectively. The full matrix of solutions is constructed as

$$\eta = \left(\underbrace{\eta^{(1)}}_{n_G} \underbrace{\eta^{(2)}}_{n_G} \underbrace{\eta^{(3)}}_{n_\varphi} \right), \quad (4.3)$$

and similar for χ , λ , and ζ . For each type of solutions, the boundary conditions at $r \rightarrow 0$ are imposed as follows.

- **Type 1:** We take

$$\chi^{(1)}(r) = r^\ell I_G, \quad (4.4)$$

where I_G is the $n_G \times n_G$ unit matrix. Then, from eq. (3.2), we obtain

$$\Psi_\ell^{(SL\varphi)(1)}(r \rightarrow 0) \simeq \left(\underbrace{\begin{pmatrix} \ell r^{\ell-1} I_G \\ L r^{\ell-1} I_G \\ M r^\ell \end{pmatrix}}_{n_G} \right) \left. \begin{matrix} \} n_G \\ \} n_G \\ \} n_\varphi \end{matrix} \right. \quad (4.5)$$

- **Type 2:** We take

$$\zeta^{(2)}(r) = r^\ell I_G. \quad (4.6)$$

At $r \rightarrow 0$, $M' \propto r$ and $V_H V_H^T M' \propto r^2$. Thus, from eq. (3.5) and eq. (3.6), $\eta^{(2)}(r \rightarrow 0) = \mathcal{O}(r^{\ell+2})$ and $\lambda^{(2)}(r \rightarrow 0) = \mathcal{O}(r^{\ell+3})$. Consequently, using eqs. (3.8)–(3.10), we obtain

$$\Psi_\ell^{(SL\varphi)(2)}(r \rightarrow 0) \simeq \left(\underbrace{\begin{pmatrix} -\frac{\xi}{4} r^{\ell+1} I_G \\ -\frac{\xi \ell}{4L} r^{\ell+1} I_G \\ M(M^T M)^{-1} r^\ell \end{pmatrix}}_{n_G} \right) \left. \begin{matrix} \} n_G \\ \} n_G \\ \} n_\varphi \end{matrix} \right. \quad (4.7)$$

- **Type 3:** This type contains two classes of solutions. One is with $\eta^{(3\eta)}$ and $\lambda^{(3\eta)}$ which obey the following boundary conditions:

$$\eta^{(3\eta)}(r \rightarrow 0) \simeq r^\ell I_G, \quad \lambda^{(3\eta)}(r \rightarrow 0) \simeq 0. \quad (4.8)$$

The other is with $\eta^{(3\lambda)}$ and $\lambda^{(3\lambda)}$ whose boundary conditions are given by

$$\eta^{(3\lambda)}(r \rightarrow 0) \simeq 0, \quad \lambda^{(3\lambda)}(r \rightarrow 0) \simeq r^\ell V_H(r \rightarrow 0). \quad (4.9)$$

Combining two classes of solutions, we define $(2n_G + n_H) \times (n_G + n_H)$ object $\Psi_\ell^{(SL\varphi)(3)}$ whose behavior at $r \rightarrow 0$ is given by

$$\Psi_\ell^{(SL\varphi)(3)}(r \rightarrow 0) \simeq \left(\underbrace{\begin{pmatrix} \frac{\ell}{4L} r^{\ell+1} I_G & 0 \\ \frac{\ell+4}{4(\ell+2)} r^{\ell+1} I_G & 0 \\ -\frac{\ell+2}{L} M(M^T M)^{-1} r^\ell & r^\ell V_H \end{pmatrix}}_{\substack{n_G \\ n_H}} \right) \left. \begin{matrix} \} \\ \} \\ \} \end{matrix} \right\} \begin{matrix} n_G \\ n_G \\ n_\varphi \end{matrix}. \quad (4.10)$$

In our convention, because $M^T \bar{\phi}' = 0$, one of u_p is chosen to be proportional to $\bar{\phi}'$, which we denote as $u^{(\text{tr})}$:

$$u^{(\text{tr})} \equiv \frac{\bar{\phi}'}{\sqrt{\bar{\phi}'^T \bar{\phi}'}}. \quad (4.11)$$

For $\ell = 1$, it is related to the translational zero mode, and is important in eliminating the zero eigenvalues from the functional determinant. The column of $\Psi_1^{(SL\varphi)(3)}$ in association with $u^{(\text{tr})}$ is obtained by $\chi^{(\text{tr})} = \eta^{(\text{tr})} = 0$ and $\lambda^{(\text{tr})} \propto \bar{\phi}'$. We will come back to this issue in section 5.

The asymptotic behavior at $r \rightarrow \infty$ can be understood by carefully observing the differential equation. We leave the precise discussion to appendix A and, in this section, only show the results. Neglecting terms irrelevant for the calculation of the functional determinant of our interest, the columns of $\Psi_\ell^{(SL\varphi)}(r \rightarrow \infty)$ can be obtained by linear combinations of those of the following objects:

$$\tilde{\psi}^{(1)} = \left(\underbrace{\begin{pmatrix} \ell r^{\ell-1} I_G \\ L r^{\ell-1} I_G \\ M r^\ell \end{pmatrix}}_{n_G} \right) \left. \begin{matrix} \} \\ \} \\ \} \end{matrix} \right\} \begin{matrix} n_G \\ n_G \\ n_\varphi \end{matrix}, \quad (4.12)$$

$$\tilde{\psi}^{(2)} = \left(\underbrace{\begin{pmatrix} -\frac{\xi}{4} r^{\ell+1} I_B \\ 0 \\ -\frac{\xi \ell}{4L} r^{\ell+1} I_B \\ 0 \\ -\frac{\xi}{4(\ell+2)} r^{\ell+2} M_B \end{pmatrix}}_{n_B} \underbrace{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)} r^{\ell+1} I_U \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)} r^{\ell+1} I_U \\ 0 \end{pmatrix}}_{n_U} \right) \left. \begin{matrix} \} \\ \} \\ \} \\ \} \\ \} \end{matrix} \right\} \begin{matrix} n_B \\ n_U \\ n_B \\ n_U \\ n_\varphi \end{matrix}, \quad (4.13)$$

$$\tilde{\psi}^{(3B)} = \left(\underbrace{\begin{pmatrix} 0 \\ (M^T M)^{-1} \partial_r \tilde{\eta}^{(B)} \\ 0 \end{pmatrix}}_{n_B} \right) \left. \begin{matrix} \} \\ \} \\ \} \end{matrix} \right\} \begin{matrix} n_G \\ n_G \\ n_\varphi \end{matrix}, \quad (4.14)$$

$$\tilde{\psi}^{(3U)} = \left(\begin{array}{c} 0 \\ 0 \\ \underbrace{LM[\partial_r(M^T M)^{-1}]r^{-1}\tilde{\eta}^{(U)}}_{n_U} \end{array} \right) \left\{ \begin{array}{l} n_G \\ n_G \\ n_\varphi \end{array} \right\}, \quad (4.15)$$

$$\tilde{\psi}^{(3\lambda)} = \left(\begin{array}{c} 0 \\ 0 \\ \underbrace{\tilde{\lambda}}_{n_H} \end{array} \right). \quad (4.16)$$

Here, I_B and I_U are the $n_B \times n_B$ and $n_U \times n_U$ unit matrices, respectively, while $\tilde{\eta}^{(B)}$ and $\tilde{\eta}^{(U)}$ are $n_G \times n_B$ and $n_G \times n_U$ objects both of which satisfy

$$\left\{ -\Delta_\ell + (M^T M) - (M^T M)[\partial_r(M^T M)^{-1}]\frac{1}{r^2}\partial_r r^2 \right\} \tilde{\eta} = 0. \quad (4.17)$$

Among the solutions, $\tilde{\eta}^{(B)}$ corresponds to n_B solutions that exponentially grow at $r \rightarrow \infty$, while $\tilde{\eta}^{(U)}$ corresponds to n_U solutions that behave as r^{-2} at $r \rightarrow \infty$. In addition, $\tilde{\lambda}$ is $n_\varphi \times n_H$, satisfying

$$(-\Delta_\ell + \Omega)\tilde{\lambda} = 0, \quad (4.18)$$

and

$$M^T \tilde{\lambda} = 0. \quad (4.19)$$

Then, $\Psi_\ell^{(SL\varphi)}(r \rightarrow \infty)$ can be expressed as

$$\Psi_\ell^{(SL\varphi)}(r \rightarrow \infty) = \left(\tilde{\psi}^{(1)} \ \tilde{\psi}^{(2)} \ \tilde{\psi}^{(3B)} \ \tilde{\psi}^{(3U)} \ \tilde{\psi}^{(3\lambda)} \right) \begin{pmatrix} I_G & 0 & 0 \\ 0 & I_G & 0 \\ 0 & 0 & \tau^{(\eta\lambda)} \end{pmatrix} + (\text{irrelevant}), \quad (4.20)$$

with $\tau^{(\eta\lambda)}$ being an $n_\varphi \times n_\varphi$ orthogonal matrix. In the above expression, elements irrelevant for the following discussion are neglected.

In order to calculate the prefactor $\mathcal{A}$, we should also consider the fluctuation operator around the false vacuum. In particular, we need to derive solutions of $\widehat{\mathcal{M}}_\ell^{(SL\varphi)} \widehat{\Psi}_\ell^{(SL\varphi)} = 0$. For this purpose, we can use the fact that the fluctuation operator around the false vacuum can be block-diagonalized as eq. (2.52); the blocks are for the fluctuations of massive gauge bosons and NG bosons, for massless gauge bosons, and for physical scalars. Thus, we can discuss their contributions separately. Similarly to the discussion above, we define $\widehat{\Psi}_\ell^{(X)} (X = B, U, \sigma)$ obeying $\widehat{\mathcal{M}}_\ell^{(X)} \widehat{\Psi}_\ell^{(X)} = 0$, which describe independent solutions of the differential equation. Note that $\widehat{\Psi}_\ell^{(B)}$, $\widehat{\Psi}_\ell^{(U)}$, and $\widehat{\Psi}_\ell^{(\sigma)}$ are $3n_B \times 3n_B$, $2n_U \times 2n_U$, and $(n_U + n_H) \times (n_U + n_H)$, respectively.

Effects of the fluctuations of massive gauge boson and NG bosons around the false vacuum are embedded in $\widehat{\Psi}_\ell^{(B)}$, which behaves as

$$\widehat{\Psi}_\ell^{(B)}(r \rightarrow 0) = \left(\underbrace{\begin{pmatrix} \ell r^{\ell-1} I_B & -\frac{\xi}{4} r^{\ell+1} I_B & \frac{\ell}{4L} r^{\ell+1} I_B \\ L r^{\ell-1} I_B & -\frac{\xi}{4L} r^{\ell+1} I_B & \frac{\ell+4}{4(\ell+2)} r^{\ell+1} I_B \\ \widehat{W} r^\ell & \widehat{W}^{-1} r^\ell & -\frac{\ell+2}{L} \widehat{W}^{-1} r^\ell \end{pmatrix}}_{n_B} \right)_{n_B} \quad (4.21)$$

and

$$\widehat{\Psi}_\ell^{(B)}(r \rightarrow \infty) = \begin{pmatrix} \ell r^{\ell-1} I_B & -\frac{\xi}{4} r^{\ell+1} I_B & 0 \\ L r^{\ell-1} I_B & -\frac{\xi}{4L} r^{\ell+1} I_B & \widehat{W}^{-1} \widehat{\Psi}_\ell^{(\eta)} \\ \widehat{W} r^\ell & -\frac{\xi}{4(\ell+2)} r^{\ell+2} \widehat{W} & 0 \end{pmatrix}, \quad (4.22)$$

where $\widehat{\Psi}_\ell^{(\eta)}$ is an $n_B \times n_B$ function satisfying

$$\left(-\Delta_\ell + \widehat{W}^T \widehat{W} \right) \widehat{\Psi}_\ell^{(\eta)} = 0, \quad (4.23)$$

with

$$\widehat{\Psi}_\ell^{(\eta)}(r \rightarrow 0) = I_B r^\ell. \quad (4.24)$$

Next, we consider the fluctuations of massless gauge bosons. For those, we can obtain the solutions in the following form:

$$\widehat{\Psi}_\ell^{(U)} = \left(\underbrace{\begin{pmatrix} \ell r^{\ell-1} I_U & \frac{(\ell+2)\xi-\ell}{4L(\ell+2)} r^{\ell+1} I_U \\ L r^{\ell-1} I_U & \frac{(\ell+2)\xi-(\ell+4)}{4L(\ell+2)} r^{\ell+1} I_U \end{pmatrix}}_{n_U} \right)_{n_U}. \quad (4.25)$$

Furthermore, the solutions related to the physical scalars are given by an $(n_U + n_H) \times (n_U + n_H)$ object, $\widehat{\Psi}_\ell^{(\sigma)}$, whose evolution is governed by

$$\left(-\Delta_\ell + \widehat{m}^2 \right) \widehat{\Psi}_\ell^{(\sigma)} = 0, \quad (4.26)$$

with

$$\widehat{\Psi}_\ell^{(\sigma)}(r \rightarrow 0) = \begin{pmatrix} I_U & 0 \\ 0 & I_H \end{pmatrix} r^\ell, \quad (4.27)$$

where I_H is the $n_H \times n_H$ unit matrix. Then, $\det \widehat{\Psi}_\ell^{(SL\varphi)}(r)$ can be calculated as

$$\det \widehat{\Psi}_\ell^{(SL\varphi)}(r) = \det \left(\widehat{\Psi}_\ell^{(B)}(r) \right) \det \left(\widehat{\Psi}_\ell^{(U)}(r) \right) \det \left(\widehat{\Psi}_\ell^{(\sigma)}(r) \right). \quad (4.28)$$

By using $\det \Psi_\ell^{(SL\varphi)}$ and $\det \widehat{\Psi}_\ell^{(SL\varphi)}$, the functional determinant of our interest can be expressed as

$$\frac{\text{Det } \mathcal{M}_\ell^{(SL\varphi)}}{\text{Det } \widehat{\mathcal{M}}_\ell^{(SL\varphi)}} = \left(\frac{\det \Psi_\ell^{(SL\varphi)}(r_0)}{\det \widehat{\Psi}_\ell^{(SL\varphi)}(r_0)} \right)^{-1} \left(\frac{\det \Psi_\ell^{(SL\varphi)}(r_\infty)}{\det \widehat{\Psi}_\ell^{(SL\varphi)}(r_\infty)} \right). \quad (4.29)$$

In order to evaluate the above quantity, we define

$$\Psi_\ell^{(\eta\lambda)} = \begin{pmatrix} \eta^{(3\eta)} & \eta^{(3\eta)} \\ V_H^T \lambda^{(3\lambda)} & V_H^T \lambda^{(3\lambda)} \end{pmatrix}, \quad (4.30)$$

which can be obtained by solving eqs. (3.5) and (3.6) with taking the boundary conditions given in eq. (4.8) or (4.9). Then, by using the following relation:

$$\det \Psi_\ell^{(\eta\lambda)}(r_\infty) = \det \tilde{\eta}(r_\infty) \det \left[V_H^T(r_\infty) \tilde{\lambda}(r_\infty) \right], \quad (4.31)$$

we obtain

$$\begin{aligned} \frac{\text{Det } \mathcal{M}_\ell^{(SL\varphi)}}{\text{Det } \widehat{\mathcal{M}}_\ell^{(SL\varphi)}} &= \left(\frac{\ell}{r_\infty} \right)^{n_U} \left(\det M^T M \right)_\infty^{1/2} \left[\det \left(M_U^T M_U \right)' \right]_\infty^{-1} \\ &\quad \left(\det \Psi_\ell^{(\eta\lambda)}(r_\infty) \right) \left(\det \widehat{\Psi}_\ell^{(\eta)}(r_\infty) \right)^{-1} \left(\det \widehat{\Psi}_\ell^{(\sigma)}(r_\infty) \right)^{-1} \\ &\quad \left(\det M_0^T M_0 \right)^{1/2} \left(\det \widehat{W}^T \widehat{W} \right)^{-1}, \end{aligned} \quad (4.32)$$

where the subscript “ ∞ ” implies that the quantity should be evaluated at $r = r_\infty$; one can check that r_0 dependence and r_∞ dependence cancel out after taking $r_0 \rightarrow 0$ and $r_\infty \rightarrow \infty$. It is important to notice that eq. (4.32) does not contain the gauge parameter, ξ .

4.2 $\ell = 0$

Formally, the functional determinant of $\mathcal{M}_0^{(S\varphi)}$ can be obtained by a similar procedure as in the case of $\ell > 0$. The solutions of $\mathcal{M}_0^{(S\varphi)} \Psi_0^{(S\varphi)(I)} = 0$, corresponding to the Type 1, 2, and 3 solutions, are given in the following form:

- **Type 1:**

$$\Psi_0^{(S\varphi)(1)} = \underbrace{\begin{pmatrix} 0 \\ M \end{pmatrix}}_{n_G} \left\{ \begin{matrix} n_G \\ n_\varphi \end{matrix} \right\}. \quad (4.33)$$

- **Type 2:**

$$\Psi_0^{(S\varphi)(2)} = \underbrace{\begin{pmatrix} -\frac{\xi}{4} r I_G \\ 0 \end{pmatrix}}_{n_G} \left\{ \begin{matrix} n_G \\ n_\varphi \end{matrix} \right\} + (\text{irrelevant}), \quad (4.34)$$

where we neglect terms that are irrelevant for the calculation of the functional determinant of our interest.

- **Type 3:**

$$\Psi_0^{(S\varphi)(3)} = \underbrace{\begin{pmatrix} 0 \\ \tilde{\lambda} \end{pmatrix}}_{n_H} \left\{ \begin{matrix} n_G \\ n_\varphi \end{matrix} \right\} + (\text{irrelevant}), \quad (4.35)$$

where the function $\tilde{\lambda}$ satisfies eq. (3.6) with taking $\eta \rightarrow 0$ and $\zeta \rightarrow 0$, as well as $M^T \tilde{\lambda} = 0$.

Then, we define

$$\Psi_0^{(S\varphi)} \equiv \left(\Psi_0^{(S\varphi)(1)} \quad \Psi_0^{(S\varphi)(2)} \quad \Psi_0^{(S\varphi)(3)} \right), \quad (4.36)$$

which will be related to the functional determinant of the fluctuation operator around the bounce.

Recalling the block-diagonalization shown in eq. (2.56), we can solve $\widehat{\mathcal{M}}_0^{(X)} \widehat{\Psi}_0^{(X)} = 0$ ($X = B$ and U) to obtain

$$\widehat{\Psi}_0^{(B)}(r) = \left(\underbrace{\begin{pmatrix} 0 & -\frac{1}{4}\xi r I_B \\ \widehat{W} & 0 \end{pmatrix}}_{\substack{n_B \\ n_B}} \right) \left\{ \begin{matrix} n_B \\ n_B \end{matrix} \right\}, \quad (4.37)$$

and

$$\widehat{\Psi}_0^{(U)}(r) = \frac{\xi}{4} r I_U. \quad (4.38)$$

We also define $\widehat{\Psi}_0^{(\sigma)}(r)$ that satisfies

$$(\Delta_0 - \widehat{m}^2) \widehat{\Psi}_0^{(\sigma)} = 0, \quad (4.39)$$

with

$$\widehat{\Psi}_0^{(\sigma)}(r \rightarrow 0) = \begin{pmatrix} I_U & 0 \\ 0 & I_H \end{pmatrix}. \quad (4.40)$$

Using quantities defined above, the determinant is calculated as

$$\det \widehat{\Psi}_0^{(S\varphi)}(r) = \det \left(\widehat{\Psi}_0^{(B)}(r) \right) \det \left(\widehat{\Psi}_0^{(U)}(r) \right) \det \left(\widehat{\Psi}_0^{(\sigma)}(r) \right). \quad (4.41)$$

We obtain the following expression:

$$\frac{\text{Det } \mathcal{M}_0^{(S\varphi)}}{\text{Det } \widehat{\mathcal{M}}_0^{(S\varphi)}} = \left(\frac{\det \Psi_0^{(S\varphi)}(r_0)}{\det \widehat{\Psi}_0^{(S\varphi)}(r_0)} \right)^{-1} \left(\frac{\det \Psi_0^{(S\varphi)}(r_\infty)}{\det \widehat{\Psi}_0^{(S\varphi)}(r_\infty)} \right). \quad (4.42)$$

Importantly, however, extra treatment is needed when there exists an unbroken gauge symmetry at the false vacuum. If it exists, there shows up a gauge zero mode, which makes eq. (4.42) vanish. Indeed, it is easy to see the existence of zero modes explicitly. For $\ell = 0$, the Type 1 solution given in eq. (4.33) has n_G columns distinguished by the adjoint index of M . Because $M_{ia}(r \rightarrow \infty) = 0$ if the adjoint index a is for unbroken generators at the false vacuum, $\det \mathcal{M}_0^{(S\varphi)}$ vanishes if there exists an unbroken gauge symmetry at the false vacuum. In our setup, the number of gauge zero modes is n_U .

If there exist zero modes, a naive calculation of the prefactor $\mathcal{A}$ makes it divergent. Such a divergence is an artifact arising from the flat directions of the Euclidean action, implying the break down of the saddle point method in the path integral. The proper treatments of the zero modes will be discussed in section 5.

4.3 Background gauge

Although it is convenient to use the Fermi gauge to discuss the gauge invariance of the decay rate, the background gauge is useful in performing the numerical calculation of the decay rate. Here, we show that the calculations based on the Fermi and the background gauges give the same result for $\ell > 0$. (For $\ell = 0$, the zero mode subtraction is non-trivial in the background gauge. The treatments of the zero modes will be discussed in the next section.)

In order to discuss the functional determinants in the background gauge, we first define the function $\Psi_{\ell, \text{BG}}^{(c\bar{c})}$ which is $n_G \times n_G$. It obeys the following differential equation:

$$\left(-\Delta_\ell + \xi M^T M\right) \Psi_{\ell, \text{BG}}^{(c\bar{c})} = 0, \quad (4.43)$$

with the boundary condition

$$\Psi_{\ell, \text{BG}}^{(c\bar{c})}(r \rightarrow 0) = r^\ell I_G. \quad (4.44)$$

In our convention, because $M^T M(r \rightarrow \infty)$ reduces to the block-diagonal form (see eq. (2.28)), the relevant part of $\Psi_{\ell, \text{BG}}^{(c\bar{c})}$ for our analysis behaves as

$$\Psi_{\ell, \text{BG}}^{(c\bar{c})}(r \rightarrow \infty) \simeq \begin{pmatrix} f_{\ell, B}^{(c\bar{c})} & 0 \\ 0 & f_{\ell, U}^{(c\bar{c})} \end{pmatrix} \tau^{(c\bar{c})}, \quad (4.45)$$

where $f_{\ell, B}^{(c\bar{c})}$ and $f_{\ell, U}^{(c\bar{c})}$ are $n_B \times n_B$ and $n_U \times n_U$, respectively, while $\tau^{(c\bar{c})}$ is an $n_G \times n_G$ orthogonal matrix. Notice that $f_{\ell, B}^{(c\bar{c})}(r \rightarrow \infty)$ exponentially grows while $f_{\ell, U}^{(c\bar{c})}(r \rightarrow \infty)$ is approximately proportional to r^ℓ .

Similarly, we also define the function $\widehat{\Psi}_{\ell, \text{BG}}^{(c\bar{c})}$, obeying

$$\left(-\Delta_\ell + \xi \widehat{M}^T \widehat{M}\right) \widehat{\Psi}_{\ell, \text{BG}}^{(c\bar{c})} = 0, \quad (4.46)$$

and

$$\widehat{\Psi}_{\ell, \text{BG}}^{(c\bar{c})}(r \rightarrow 0) = r^\ell I_G. \quad (4.47)$$

Because of the block-diagonal nature of $\widehat{M}^T \widehat{M}$ in our convention, $\widehat{\Psi}_{\ell, \text{BG}}^{(c\bar{c})}$ can be expressed as

$$\widehat{\Psi}_{\ell, \text{BG}}^{(c\bar{c})} = \begin{pmatrix} \widehat{f}_{\ell, B}^{(c\bar{c})} & 0 \\ 0 & \widehat{f}_{\ell, U}^{(c\bar{c})} \end{pmatrix}, \quad (4.48)$$

where $\widehat{f}_{\ell, B}^{(c\bar{c})}$ is an $n_B \times n_B$ object, while $\widehat{f}_{\ell, U}^{(c\bar{c})} = r^\ell I_U$. The functional determinant of the fluctuation operator of the FP ghosts is then given by

$$\frac{\text{Det } \mathcal{M}_{\ell, \text{BG}}^{(c\bar{c})}}{\text{Det } \widehat{\mathcal{M}}_{\ell, \text{BG}}^{(c\bar{c})}} = \frac{\det \Psi_{\ell, \text{BG}}^{(c\bar{c})}(r_\infty)}{\det \widehat{\Psi}_{\ell, \text{BG}}^{(c\bar{c})}(r_\infty)}. \quad (4.49)$$

For the calculation of the functional determinant of $\mathcal{M}_{\ell, \text{BG}}^{(SL\varphi)}$, we should derive the solutions of the following differential equation:

$$\mathcal{M}_{\ell.\text{BG}}^{(SL\varphi)} \Psi_{\ell.\text{BG}}^{(SL\varphi)} = 0. \quad (4.50)$$

As in the case of the Fermi gauge, we can find a set of functions that are relevant for the determinant:

$$\tilde{\psi}_{\text{BG}}^{(1)} = \underbrace{\begin{pmatrix} \partial_r f_{\ell,B}^{(c\bar{c})} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{n_B} \underbrace{\begin{pmatrix} 0 \\ \ell f_{\ell,U}^{(c\bar{c})} \\ 0 \\ \frac{L}{r} f_{\ell,U}^{(c\bar{c})} \\ 0 \\ 0 \end{pmatrix}}_{n_U} \left. \begin{matrix} \vphantom{\begin{pmatrix} \partial_r f_{\ell,B}^{(c\bar{c})} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}} \right\} \begin{matrix} n_B \\ n_U \\ n_B \\ n_U \\ n_B \\ n_\varphi \end{matrix}, \quad (4.51)$$

$$\tilde{\psi}_{\text{BG}}^{(2)} = \left(\underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ M_B(M_B^T M_B)^{-1} f_{\ell,B}^{(c\bar{c})} \end{pmatrix}}_{n_B} \underbrace{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)} r f_{\ell,U}^{(c\bar{c})} \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)} r f_{\ell,U}^{(c\bar{c})} \\ 0 \end{pmatrix}}_{n_U} \right) \left. \begin{matrix} \} n_B \\ \} n_U \\ \} n_B, \\ \} n_U \\ \} n_\varphi \end{matrix} \right) \quad (4.52)$$

$$\tilde{\psi}_{\text{BG}}^{(3)} = \left(\tilde{\psi}^{(3B)} \quad \tilde{\psi}^{(3U)} \quad \tilde{\psi}^{(3\lambda)} \right), \quad (4.53)$$

where terms irrelevant for our discussion are neglected. With these functions, $\Psi_{\ell,\text{BG}}^{(SL\varphi)}(r \rightarrow \infty)$ can be expressed as

$$\Psi_{\ell, \text{BG}}^{(SL\varphi)}(r \rightarrow \infty) = \begin{pmatrix} \tilde{\psi}_{\text{BG}}^{(1)} & \tilde{\psi}_{\text{BG}}^{(2)} & \tilde{\psi}_{\text{BG}}^{(3)} \end{pmatrix} \begin{pmatrix} \tau_{\text{BG}}^{(c\bar{c})} & 0 & 0 \\ 0 & \tau_{\text{BG}}^{(c\bar{c})} & 0 \\ 0 & 0 & I_\varphi \end{pmatrix} \begin{pmatrix} I_G & 0 & 0 \\ 0 & I_G & 0 \\ 0 & 0 & \tau_{\text{BG}}^{(\eta\lambda)} \end{pmatrix} + (\text{irrelevant}), \quad (4.54)$$

with $\tau_{\text{BG}}^{(\eta\lambda)}$ being an $n_\varphi \times n_\varphi$ orthogonal matrix.

Meanwhile, the behavior of $\Psi_{\ell}^{(SL\varphi)}(\text{BG})$ around $r = 0$ is given by

$$\Psi_{\ell, \text{BG}}^{(SL\varphi)}(r \rightarrow 0) \simeq \left(\underbrace{\begin{pmatrix} \ell r^{\ell-1} I_G & 0 \\ L r^{\ell-1} I_G & 0 \end{pmatrix}}_{n_G} \underbrace{\begin{pmatrix} M_0(M_0^T M_0)^{-1} r^\ell & -\frac{\ell+2}{L} M_0(M_0^T M_0)^{-1} r^\ell \end{pmatrix}}_{n_G} \underbrace{\begin{pmatrix} \frac{\ell-\xi(\ell+2)}{4L} r^{\ell+1} I_G \\ \frac{(\ell+4)-\xi(\ell+2)}{4(\ell+2)} r^{\ell+1} I_G \end{pmatrix}}_{n_G} \underbrace{\begin{pmatrix} 0 \\ 0 \\ V_H r^\ell \end{pmatrix}}_{n_H} \right) \left. \begin{matrix} \vphantom{\Psi_{\ell, \text{BG}}^{(SL\varphi)}} \\ \vphantom{\Psi_{\ell, \text{BG}}^{(SL\varphi)}} \\ \vphantom{\Psi_{\ell, \text{BG}}^{(SL\varphi)}} \end{matrix} \right\} \begin{matrix} n_G \\ n_G \\ n_\varphi \end{matrix}. \quad (4.55)$$

Asymptotic behavior of the function $\widehat{\Psi}_{\ell,BG}^{(SL\varphi)}$, which obeys

$$\widehat{\mathcal{M}}_{\ell,BG}^{(SL\varphi)} \widehat{\Psi}_{\ell,BG}^{(SL\varphi)} = 0, \quad (4.56)$$

can be again obtained using the block-diagonalization given in eq. (2.52). We define $\widehat{\Psi}_{\ell,BG}^{(X)}(X = B, U, \sigma)$ through $\widehat{\mathcal{M}}_{\ell,BG}^{(X)} \widehat{\Psi}_{\ell,BG}^{(X)} = 0$, which describe independent solutions of the differential equation. For $\widehat{\Psi}_{\ell,BG}^{(B)}$, we obtain

$$\widehat{\Psi}_{\ell,BG}^{(B)}(r) = \left(\underbrace{\begin{pmatrix} \partial_r \widehat{f}_{\ell,B}^{(c\bar{c})} & 0 & 0 \\ 0 & 0 & \widehat{W}^{-2} \partial_r \widehat{\Psi}_{\ell}^{(\eta)} \\ 0 & \widehat{W}^{-1} \widehat{f}_{\ell,B}^{(c\bar{c})} & 0 \end{pmatrix}}_{\substack{n_B \\ n_B \\ n_B}} \right) \left. \begin{matrix} \\ \\ \end{matrix} \right\} \begin{matrix} n_B \\ n_B \\ n_B \end{matrix}. \quad (4.57)$$

For the others, we can find $\widehat{\Psi}_{\ell,BG}^{(U)} = \widehat{\Psi}_{\ell}^{(U)}$ and $\widehat{\Psi}_{\ell,BG}^{(\sigma)} = \widehat{\Psi}_{\ell}^{(\sigma)}$. Combining all of them, $\det \widehat{\Psi}_{\ell,BG}^{(SL\varphi)}(r)$ is given by

$$\det \widehat{\Psi}_{\ell,BG}^{(SL\varphi)}(r) = \det \left(\widehat{\Psi}_{\ell,BG}^{(B)}(r) \right) \det \left(\widehat{\Psi}_{\ell,BG}^{(U)}(r) \right) \det \left(\widehat{\Psi}_{\ell,BG}^{(\sigma)}(r) \right). \quad (4.58)$$

Finally, we can find

$$\frac{\det \Psi_{\ell,BG}^{(SL\varphi)}(r_{\infty})}{\det \widehat{\Psi}_{\ell,BG}^{(SL\varphi)}(r_{\infty})} = \left(\frac{\det \Psi_{\ell,BG}^{(c\bar{c})}(r_{\infty})}{\det \widehat{\Psi}_{\ell,BG}^{(c\bar{c})}(r_{\infty})} \right)^2 \frac{\det \Psi_{\ell}^{(SL\varphi)}(r_{\infty})}{\det \widehat{\Psi}_{\ell}^{(SL\varphi)}(r_{\infty})}, \quad (4.59)$$

and consequently,

$$\frac{\text{Det } \mathcal{M}_{\ell,BG}^{(SL\varphi)}}{\text{Det } \widehat{\mathcal{M}}_{\ell,BG}^{(SL\varphi)}} = \left(\frac{\det \Psi_{\ell,BG}^{(c\bar{c})}(r_{\infty})}{\det \widehat{\Psi}_{\ell,BG}^{(c\bar{c})}(r_{\infty})} \right)^2 \frac{\text{Det } \mathcal{M}_{\ell}^{(SL\varphi)}}{\text{Det } \widehat{\mathcal{M}}_{\ell}^{(SL\varphi)}}. \quad (4.60)$$

In the Fermi and the background gauges, the functional determinants of the fluctuation operators of the $(SL\varphi)$ modes differ from each other. In the calculation of the prefactor $\mathcal{A}$, the difference is compensated by the contributions from the FP ghosts. Indeed, we can find

$$\left(\frac{\text{Det } \mathcal{M}_{\ell,BG}^{(SL\varphi)}}{\text{Det } \widehat{\mathcal{M}}_{\ell,BG}^{(SL\varphi)}} \right)^{-1/2} \frac{\text{Det } \mathcal{M}_{\ell,BG}^{(c\bar{c})}}{\text{Det } \widehat{\mathcal{M}}_{\ell,BG}^{(c\bar{c})}} = \left(\frac{\text{Det } \mathcal{M}_{\ell}^{(SL\varphi)}}{\text{Det } \widehat{\mathcal{M}}_{\ell}^{(SL\varphi)}} \right)^{-1/2} \frac{\text{Det } \mathcal{M}_{\ell}^{(c\bar{c})}}{\text{Det } \widehat{\mathcal{M}}_{\ell}^{(c\bar{c})}}. \quad (4.61)$$

The above relation guarantees the equivalence of the total functional determinants (on which the prefactor $\mathcal{A}$ depends) in the Fermi and the background gauges for $\ell \geq 1$.

In performing the calculation in the background gauge, we should also study the $\ell = 0$ modes. The boundary condition for $\ell = 0$ (for the study of $\mathcal{M}_{0,\text{BG}}^{(S\varphi)}$) are also obtained as

$$\Psi_{0,\text{BG}}^{(S\varphi)(1)}(r \rightarrow 0) \simeq \underbrace{\begin{pmatrix} \frac{\xi}{4} r (M_0^T M_0) \\ M_0 \end{pmatrix}}_{n_G} \left\{ \begin{matrix} n_G \\ n_\varphi \end{matrix} \right\}, \quad (4.62)$$

$$\Psi_{0,\text{BG}}^{(S\varphi)(2)}(r \rightarrow 0) \simeq \underbrace{\begin{pmatrix} 0 \\ M_0 (M_0^T M_0)^{-1} \end{pmatrix}}_{n_G} \left\{ \begin{matrix} n_G \\ n_\varphi \end{matrix} \right\}, \quad (4.63)$$

$$\Psi_{0,\text{BG}}^{(S\varphi)(3)}(r \rightarrow 0) \simeq \underbrace{\begin{pmatrix} 0 \\ V_H \end{pmatrix}}_{n_G} \left\{ \begin{matrix} n_G \\ n_\varphi \end{matrix} \right\}. \quad (4.64)$$

For $\ell = 0$, special care is necessary when there exist gauge zero modes. In such a case, both $\text{Det } \mathcal{M}_0^{(S\varphi)}$ and $\text{Det } \mathcal{M}_{0,\text{BG}}^{(S\varphi)}$ vanish and the comparison between the calculations in the Fermi and the background gauges is non-trivial (see the discussion in the next section).

5 Zero modes

When the action has flat directions around the bounce, mode functions corresponding to those directions become zero modes of fluctuation operators. Here, we consider the zero modes in association with continuous symmetries of the theory. In particular, in the case of our interest, the translational and the gauge symmetries result in zero modes and their proper treatments are essential to calculate the prefactor, $\mathcal{A}$. In this section, we discuss how we can deal with these zero modes.

5.1 General issues

Let us first discuss the treatment of zero modes in general. As we have mentioned in the previous section, the ratio of the functional determinants of $n \times n$ fluctuation operators, $\mathcal{M}$ and $\widehat{\mathcal{M}}$,⁹ is evaluated as

$$\frac{\text{Det } \mathcal{M}}{\text{Det } \widehat{\mathcal{M}}} = \left(\frac{\det \Psi(r_0)}{\det \widehat{\Psi}(r_0)} \right)^{-1} \left(\frac{\det \Psi(r_\infty)}{\det \widehat{\Psi}(r_\infty)} \right), \quad (5.1)$$

where $\Psi(r)$ and $\widehat{\Psi}(r)$ satisfy $\mathcal{M}\Psi = 0$ and $\widehat{\mathcal{M}}\widehat{\Psi} = 0$, respectively. Notice that Ψ and $\widehat{\Psi}$ are $n \times n$ matrices, whose columns are linearly independent solutions, i.e.,

$$\Psi(r) = \left(\psi^{(1)}(r) \cdots \psi^{(n)}(r) \right), \quad (5.2)$$

$$\widehat{\Psi}(r) = \left(\widehat{\psi}^{(1)}(r) \cdots \widehat{\psi}^{(n)}(r) \right), \quad (5.3)$$

⁹Here, we omit the superscripts and the subscript for notational simplicity.

with $\mathcal{M}\psi^{(I)} = 0$ and $\widehat{\mathcal{M}}\widehat{\psi}^{(I)} = 0$. When $\mathcal{M}$ has zero eigenvalues, $\det \Psi(r_\infty) = 0$ and some of $\psi^{(I)}(r_\infty)$'s are not independent. Here, the number of the dependent columns of $\Psi(r_\infty)$ matches that of the zero modes, which we denote as n_{zero} . In this subsection, we take a basis in which

$$\psi^{(I \leq n_{\text{zero}})}(r_\infty) = 0, \quad \psi^{(I > n_{\text{zero}})}(r_\infty) \neq 0. \quad (5.4)$$

(In the following subsections, we may use a different convention.)

The functional determinant is due to the use of the saddle point method in the path integral. To evaluate the path integral, we first expand the fields around the bounce configuration. Let us denote the fields as Φ ; in the case of our interest, Φ contains the gauge and scalar fields, i.e., $\Phi \ni (A_\mu, \phi)$. Then, we expand Φ around the bounce configuration (which is denoted as $\bar{\Phi}$) as

$$\Phi = \bar{\Phi} + \sum_a c_a \mathcal{G}_a(x), \quad (5.5)$$

where c_a 's are expansion coefficients and $\mathcal{G}_a(x)$'s denote the eigenfunctions of the fluctuation operator:

$$\mathcal{M}\mathcal{G}_a(x) = \omega_a \mathcal{G}_a(x), \quad (5.6)$$

with ω_a 's being the corresponding eigenvalues. The eigenfunctions should satisfy $\mathcal{G}_a(r_\infty) = 0$, and are normalized as¹⁰

$$\langle \mathcal{G}_a | \mathcal{G}_b \rangle = \delta_{ab}. \quad (5.7)$$

Then, the path integral is evaluated as

$$\int \mathcal{D}\Phi e^{-S[\Phi]} \simeq e^{-S[\bar{\Phi}]} \int \left(\prod_a \frac{dc_a}{\sqrt{2\pi}} e^{-\frac{1}{2}\omega_a c_a^2} \right). \quad (5.8)$$

By performing the Gaussian integrals, we obtain

$$\int \mathcal{D}\Phi e^{-S[\Phi]} \simeq e^{-S[\bar{\Phi}]} \left(\prod_a \omega_a \right)^{-1/2} \equiv e^{-S[\bar{\Phi}]} (\text{Det } \mathcal{M})^{-1/2}. \quad (5.9)$$

If there exist zero modes, some of ω_a 's vanish and the above result diverges. In such a case, we cannot use the naive saddle point method to evaluate the path integral.

When the zero mode is related to the symmetry of the theory, we may properly eliminate the zero modes and avoid the divergence of the transition amplitude mentioned above, as discussed in [28] for the case of translational zero modes. Let us denote a generic symmetry transformation of the bounce configuration as

$$\bar{\Phi} \rightarrow \bar{\Phi} + \sum_A z_A \mathcal{F}_A(x) + \mathcal{O}(z^2), \quad (5.10)$$

¹⁰The inner product is defined as

$$\langle \mathcal{G}_a | \mathcal{G}_b \rangle = \int d^4x \mathcal{G}_a \cdot \mathcal{G}_b.$$

where z_A denotes the transformation parameter. With the symmetry transformation, the bounce action is invariant so that $\mathcal{F}_A$ satisfies

$$\mathcal{M}\mathcal{F}_A = 0, \quad (5.11)$$

with

$$\lim_{r \rightarrow \infty} \mathcal{F}_A(x) = 0. \quad (5.12)$$

Thus, the following relation holds

$$\sum_{a \in \mathcal{I}_0} c_a \mathcal{G}_a(x) = \sum_A z_A \mathcal{F}_A(x), \quad (5.13)$$

where $\mathcal{I}_0$ denotes the set of indices for the zero modes, resulting in

$$c_a = \sum_A \langle \mathcal{G}_a | \mathcal{F}_A \rangle z_A \quad : \quad a \in \mathcal{I}_0. \quad (5.14)$$

Using

$$\langle \mathcal{F}_A | \mathcal{F}_B \rangle = \sum_{a \in \mathcal{I}_0} \langle \mathcal{F}_A | \mathcal{G}_a \rangle \langle \mathcal{G}_a | \mathcal{F}_B \rangle, \quad (5.15)$$

the Jacobian to convert the variable c_a to z_A is found to be

$$\mathcal{J} = \sqrt{\det_{AB} \langle \mathcal{F}_A | \mathcal{F}_B \rangle}, \quad (5.16)$$

where $\det_{AB}$ denotes the determinant of the matrix with the indices A and B .

Based on the above argument, we reinterpret the path integral containing zero modes related to the symmetry as

$$\int \mathcal{D}\Phi e^{-S[\Phi]} \rightarrow e^{-S[\bar{\Phi}]} \int \left(\prod_A \frac{dz_A}{\sqrt{2\pi}} \right) \mathcal{J} (\text{Det}' \mathcal{M})^{-1/2}, \quad (5.17)$$

where $\text{Det}' \mathcal{M}$ denotes the minor determinant of $\mathcal{M}$ with its zero eigenvalues eliminated:

$$\text{Det}' \mathcal{M} \equiv \prod_{a \notin \mathcal{I}_0} \omega_a. \quad (5.18)$$

With properly interpreting the integrals over z_A 's, the divergence originating from the zero modes may be avoided.

The minor determinant, $\text{Det}' \mathcal{M}$, can be calculated by regulating the fluctuation operator as

$$\mathcal{M}_{\text{reg}} \equiv \mathcal{M} + \nu, \quad (5.19)$$

where ν is a constant. With ν being small enough, non-zero eigenvalues of $\mathcal{M}$ are (almost) unchanged while the zero eigenvalues are lifted by ν . Consequently,

$$\text{Det}' \mathcal{M} = \lim_{\nu \rightarrow 0} \frac{1}{\nu^{n_{\text{zero}}}} \text{Det} \mathcal{M}_{\text{reg}}. \quad (5.20)$$

We can calculate $\text{Det } \mathcal{M}_{\text{reg}}$ by using the procedure mentioned above. We can obtain a solution of $\mathcal{M}_{\text{reg}} \Psi_{\text{reg}} = 0$, where Ψ_{reg} is $n \times n$ and has n linearly independent columns:

$$\Psi_{\text{reg}}(r) = \left(\psi_{\text{reg}}^{(1)}(r) \cdots \psi_{\text{reg}}^{(n)}(r) \right). \quad (5.21)$$

Then, $\text{Det } \mathcal{M}_{\text{reg}}$ can be obtained by using Ψ_{reg} .

Because our purpose is to evaluate $\det \Psi_{\text{reg}}(r_\infty)$ up to $O(\nu^{n_{\text{zero}}})$, we only need to calculate the columns in association with the zero eigenvalues up to $O(\nu)$; for the other columns, we can take $\psi_{\text{reg}}^{(I)} \simeq \psi^{(I)}$ ($I > n_{\text{zero}}$). We can calculate $\psi_{\text{reg}}^{(I)}$ related to the zero modes by treating ν as a perturbation. We introduce the function $\check{\psi}^{(I)}$ as

$$\psi_{\text{reg}}^{(I \leq n_{\text{zero}})} = \psi^{(I \leq n_{\text{zero}})} + \nu \check{\psi}^{(I \leq n_{\text{zero}})} + O(\nu^2), \quad (5.22)$$

where the superscript “ $(I \leq n_{\text{zero}})$ ” indicates the columns in association with the zero modes. Then, $\check{\psi}^{(I \leq n_{\text{zero}})}$ should satisfy

$$\mathcal{M} \check{\psi}^{(I \leq n_{\text{zero}})} = -\psi^{(I \leq n_{\text{zero}})}, \quad (5.23)$$

with $\psi^{(I \leq n_{\text{zero}})}(r_0) = 0$. With $\check{\psi}^{(I \leq n_{\text{zero}})}$ being obtained by solving the above equation, we can take care of the zero modes as

$$\left[\frac{\text{Det } \mathcal{M}}{\text{Det } \widehat{\mathcal{M}}} \right]^{-1/2} \rightarrow \int \left(\prod_A \frac{dz_A}{\sqrt{2\pi}} \right) \mathcal{J} \left[\frac{\text{Det}' \mathcal{M}}{\text{Det } \widehat{\mathcal{M}}} \right]^{-1/2}, \quad (5.24)$$

where

$$\frac{\text{Det}' \mathcal{M}}{\text{Det } \widehat{\mathcal{M}}} = \left[\frac{\det \Psi(r_0)}{\det \widehat{\Psi}(r_0)} \right]^{-1} \lim_{\nu \rightarrow 0} \frac{1}{\nu^{n_{\text{zero}}}} \left[\frac{\det \{ \Psi(r_\infty) + \nu \check{\Psi}(r_\infty) \}}{\det \widehat{\Psi}(r_\infty)} \right], \quad (5.25)$$

with $\check{\Psi}$ being $n \times n$ function containing $\check{\psi}^{(I \leq n_{\text{zero}})}$:

$$\check{\Psi} \equiv \begin{pmatrix} \check{\psi}^{(1)} \cdots \check{\psi}^{(n_{\text{zero}})} & 0 \cdots 0 \end{pmatrix}. \quad (5.26)$$

Then, using the fact that

$$\Psi(r_\infty) + \nu \check{\Psi}(r_\infty) = \left(\nu \check{\psi}^{(1)}(r_\infty) \cdots \nu \check{\psi}^{(n_{\text{zero}})}(r_\infty) \quad \psi^{(n_{\text{zero}}+1)}(r_\infty) \cdots \psi^{(n)}(r_\infty) \right), \quad (5.27)$$

we obtain

$$\frac{\text{Det}' \mathcal{M}}{\text{Det } \widehat{\mathcal{M}}} = \left[\frac{\det \Psi(r_0)}{\det \widehat{\Psi}(r_0)} \right]^{-1} \left[\frac{\det \{ \Psi(r_\infty) + \check{\Psi}(r_\infty) \}}{\det \widehat{\Psi}(r_\infty)} \right]. \quad (5.28)$$

The actual calculations of $\text{Det}' \mathcal{M}$ for the gauge and translational zero modes will be discussed in the following subsections.

5.2 Gauge zero modes

We first consider the gauge zero modes. As we have mentioned in the previous section, if a gauge symmetry, which is broken by the bounce, is restored at the false vacuum, there show up zero modes in $\ell = 0$ fluctuation operators. In this subsection, we apply the discussion in the previous subsection to the gauge zero modes.

The gauge zero modes can be given in the following form (see eq. (4.33)):

$$\psi^{(A)} = \begin{pmatrix} 0 \\ M \end{pmatrix} \mathcal{U}^{(A)}, \quad (5.29)$$

where $\mathcal{U}^{(A)}$'s are defined as

$$\left(\mathcal{U}^{(1)} \dots \mathcal{U}^{(n_U)} \right) \equiv \begin{pmatrix} 0 \\ I_U \end{pmatrix} \left\{ \begin{matrix} n_B \\ n_U \end{matrix} \right\}. \quad (5.30)$$

Notice that $\psi^{(A)}$ is a column of $\Psi_0^{(S\varphi)(1)}$ and hence satisfy $\mathcal{M}_0^{(S\varphi)} \psi^{(A)} = 0$. Since the last n_U columns of M vanish at $r \rightarrow r_\infty$, $\psi^{(A)}(r_\infty) = 0$ and hence $\psi^{(A)}$ are eigenfunctions of $\mathcal{M}_0^{(S\varphi)}$ with zero eigenvalues.

In order to follow the prescription given in the previous subsection, we first calculate the function $\check{\psi}^{(A)}$ that satisfies

$$\mathcal{M}_0^{(S\varphi)} \check{\psi}^{(A)} = -\psi^{(A)}, \quad (5.31)$$

with the boundary condition of

$$\check{\psi}^{(A)}(r_0) = 0. \quad (5.32)$$

The function $\check{\psi}^{(A)}$ can be decomposed as

$$\check{\psi}^{(A)} = \begin{pmatrix} \partial_r \check{\chi}^{(A)} \\ M \check{\chi}^{(A)} \end{pmatrix} + \begin{pmatrix} -2(M^T M)^{-1} (M')^T \check{\lambda}^{(A)} \\ \check{\lambda}^{(A)} \end{pmatrix} + \begin{pmatrix} [\partial_r (M^T M)^{-1}] \check{\zeta}^{(A)} \\ M (M^T M)^{-1} \check{\zeta}^{(A)} \end{pmatrix}, \quad (5.33)$$

where $\check{\chi}^{(A)}$, $\check{\zeta}^{(A)}$, and $\check{\lambda}^{(A)}$ evolve as

$$\Delta_0 \check{\chi}^{(A)} = \Delta_0 \chi|_{\chi \rightarrow \check{\chi}^{(A)}, \zeta \rightarrow \check{\zeta}^{(A)}, \lambda \rightarrow \check{\lambda}^{(A)}}, \quad (5.34)$$

$$\Delta_0 \check{\zeta}^{(A)} = M^T M \mathcal{U}^{(A)}, \quad (5.35)$$

$$\Delta_0 \check{\lambda}^{(A)} = \Delta_0 \lambda|_{\chi \rightarrow \check{\chi}^{(A)}, \zeta \rightarrow \check{\zeta}^{(A)}, \lambda \rightarrow \check{\lambda}^{(A)}}, \quad (5.36)$$

and $\check{\lambda}$ satisfies

$$M^T \check{\lambda} = 0. \quad (5.37)$$

One can solve eq. (5.35) to obtain

$$\check{\zeta}^{(A)}(r) = \int_0^r dr_1 r_1^{-3} \int_0^{r_1} dr_2 r_2^3 M^T M(r_2) \mathcal{U}^{(A)} + (\text{irrelevant}). \quad (5.38)$$

As r becomes large, $\check{\zeta}^{(A)}(r)$ becomes constant because $M^T M \mathcal{U}^{(A)}$ approaches to zero exponentially. Then, the relevant part of $\check{\psi}^{(A)}$ for the calculation of the prefactor $\mathcal{A}$ is given by

$$\check{\psi}^{(A)}(r) = \begin{pmatrix} 0 \\ M_U \mathcal{I}_U \mathcal{X}_U \mathcal{U}^{(A)} \end{pmatrix}, \quad (5.39)$$

where $\mathcal{I}_U$ and $\mathcal{X}_U^{(A,B)}$ are $n_U \times n_U$ objects whose (A, B) elements are given by

$$\mathcal{I}_U^{(A,B)}(r) \equiv \int^r dr r^{-3} \mathcal{U}^{(A)T} (M^T M)^{-1} \mathcal{U}^{(B)}, \quad (5.40)$$

and

$$\mathcal{X}_U^{(A,B)} \equiv \frac{1}{2\pi^2} \int d^4x \mathcal{U}^{(A)T} M^T M \mathcal{U}^{(B)}. \quad (5.41)$$

Thus, because

$$\text{Det}' \mathcal{M}_0^{(S\varphi)} \propto \det \begin{pmatrix} 0 & 0 & -\frac{1}{4} \xi r I_G & 0 \\ M_B & M_U \mathcal{I}_U \mathcal{X}_U & 0 & \tilde{\lambda} \end{pmatrix} \Big|_{r \rightarrow \infty}, \quad (5.42)$$

with $\tilde{\lambda}$ being the function introduced in eq. (4.35), we can find

$$\frac{\text{Det}' \mathcal{M}_0^{(S\varphi)}}{\text{Det} \widehat{\mathcal{M}}_0^{(S\varphi)}} = \det_{AB} \mathcal{X}_U^{(A,B)} \det_{AB} \mathcal{I}_U^{(A,B)} \left[\frac{\det \widehat{\Psi}_0^{(\sigma)}(r_\infty)}{\det \widehat{\Psi}_0^{(\sigma)}(r_0)} \right]^{-1} \left[\frac{\det \Psi_0^{(\lambda)}(r_\infty)}{\det \Psi_0^{(\lambda)}(r_0)} \right] \left[\frac{\det(M^T M)_\infty}{\det(M_0^T M_0)} \right]^{1/2}, \quad (5.43)$$

with

$$\Psi_0^{(\lambda)} \equiv V_H^T \tilde{\lambda}. \quad (5.44)$$

The gauge zero modes are associated with the gauge symmetries that are restored at the false vacuum; the NG bosons in association with the global gauge transformations are not eaten by the gauge bosons and appear as the zero modes. Thus, the path integration over the gauge zero modes can be replaced by the integration over the gauge volume of the unbroken gauge symmetry at the false vacuum.

We parameterize such a global gauge transformation of the bounce configuration as

$$\bar{\phi} \rightarrow \bar{\phi} + \sum_A \theta^A \tilde{T}^A \bar{\phi} + \mathcal{O}(\theta^2), \quad (5.45)$$

where $\tilde{T}^A$'s are generators of unbroken gauge symmetry at the false vacuum with $\tilde{T}^A \bar{\phi}$ being required to be orthogonal, i.e.,

$$\int d^4x (\tilde{T}^A \bar{\phi})^T \tilde{T}^B \bar{\phi} = 0 \quad (\text{for } A \neq B). \quad (5.46)$$

Here, the generators $\tilde{T}^A$ are introduced to set a diagonal basis for the integration over the gauge volume, and are given by linear combinations of the generators of our original choice, i.e.,

$$\tilde{T}^A = \kappa_{AB} T^B. \quad (5.47)$$

Note that the ambiguity in κ is absorbed into the normalization of the gauge volume $\mathcal{V}_U$ (see the following discussion).

Since the fluctuation operator in the Fermi gauge is invariant under the transformation given in eq. (5.45), the path integral over the gauge zero modes can be interpreted as

$$\int \prod_a \frac{dc_a^{(\text{gauge})}}{\sqrt{2\pi}} \rightarrow \mathcal{J}^{(\text{gauge})} \int \prod_A \frac{d\theta^A}{\sqrt{2\pi}} \equiv \frac{\mathcal{J}^{(\text{gauge})}}{(2\pi)^{n_U/2}} \mathcal{V}_U, \quad (5.48)$$

where $c_a^{(\text{gauge})}$'s are the expansion coefficients in association with the gauge zero modes, $\mathcal{V}_U$ is the volume of the moduli space arising from the spontaneous symmetry breaking, and

$$\mathcal{J}^{(\text{gauge})} \equiv \det \kappa \left[\det_{AB} \int d^4x (T^A \bar{\phi})^T T^B \bar{\phi} \right]^{1/2} = \left(\prod_A \frac{g_A^2}{2\pi^2} \right)^{-1/2} \det \kappa \left(\det_{AB} \mathcal{X}_U^{(A,B)} \right)^{1/2}. \quad (5.49)$$

Then, using $\det \hat{\Psi}_0^{(\sigma)}(r_0) = \det \Psi_0^{(\lambda)}(r_0) = 1$, the right-hand side of eq. (5.43) becomes

$$\frac{\text{Det}' \mathcal{M}_0^{(S\varphi)}}{\text{Det} \widehat{\mathcal{M}}_0^{(S\varphi)}} = \left(\frac{1}{2r_\infty^3} \right)^{n_U} \left(\prod_A \frac{g_A^2}{2\pi^2} \right) \left(\frac{\mathcal{J}^{(\text{gauge})}}{\det \kappa} \right)^2 (\det W'_U)^{-1}_\infty \frac{\det \Psi_0^{(\lambda)}(r_\infty)}{\det \hat{\Psi}_0^{(\sigma)}(r_\infty)} \left[\frac{\det \widehat{W}^T \widehat{W}}{\det(M_0^T M_0)} \right]^{1/2}, \quad (5.50)$$

and hence the $\ell = 0$ contribution, containing the path integral over the gauge zero modes, can be written as

$$\begin{aligned} \int \prod_a \frac{dc_a^{(\text{gauge})}}{\sqrt{2\pi}} \left(\frac{\text{Det}' \mathcal{M}_0^{(S\varphi)}}{\text{Det} \widehat{\mathcal{M}}_0^{(S\varphi)}} \right)^{-1/2} &\rightarrow \left(\frac{r_\infty^3}{\pi} \right)^{n_U/2} \left(\prod_A \frac{g_A^2}{2\pi^2} \right)^{-1/2} \mathcal{V}_U \det \kappa \\ &\quad (\det W'_U)^{1/2} \left[\frac{\det \Psi_0^{(\lambda)}(r_\infty)}{\det \hat{\Psi}_0^{(\sigma)}(r_\infty)} \right]^{-1/2} \left[\frac{\det \widehat{W}^T \widehat{W}}{\det(M_0^T M_0)} \right]^{-1/4}. \end{aligned} \quad (5.51)$$

Next, we consider the background gauge. The fluctuation operator in the background gauge, i.e., $\mathcal{M}_{0,\text{BG}}^{(S\varphi)}$ given in eq. (2.49), also has zero eigenvalues in association with the unbroken gauge symmetry in the false vacuum. The existence of the zero eigenvalues of $\mathcal{M}_{0,\text{BG}}^{(S\varphi)}$ can be understood from the fact that the following relation holds:

$$\mathcal{M}_{0,\text{BG}}^{(S\varphi)} \begin{pmatrix} \partial_r \Psi_0^{(c\bar{c})} \\ M \Psi_0^{(c\bar{c})} \end{pmatrix} = 0, \quad (5.52)$$

and that some of the columns of $\Psi_0^{(c\bar{c})}$ become constant at $r \rightarrow \infty$ if there exists an unbroken gauge symmetry at the false vacuum (see eq. (4.43)). The proper treatment of such zero modes is, however, complicated because the relation between the path integral over such zero modes and the integration over the gauge parameter is non-trivial in the background gauge. Here, we consider a way to reconstruct the result in the Fermi gauge from that in the background gauge. Even though the study in the Fermi gauge is enough for

the gauge invariant formulation of the decay rate, the background gauge is advantageous for numerically calculating the decay rate. This is because the numerical calculation in the background gauge shows a better convergence at $r \rightarrow \infty$. Thus, it is desirable to understand the procedure to transform the results in the background gauge to those in the Fermi gauge. We will explicitly show such a transformation in the following.

In discussing the treatment of the gauge zero modes in the background gauge, we first define

$$\mathcal{M}_{0,\text{BG},\text{reg}}^{(S\varphi)} = \mathcal{M}_{0,\text{BG}}^{(S\varphi)} + \nu \begin{pmatrix} \xi^{-1} I_G & 0 \\ 0 & I_\varphi \end{pmatrix}, \quad (5.53)$$

$$\widehat{\mathcal{M}}_{0,\text{BG},\text{reg}}^{(S\varphi)} = \widehat{\mathcal{M}}_{0,\text{BG}}^{(S\varphi)} + \nu \begin{pmatrix} \xi^{-1} I_G & 0 \\ 0 & I_\varphi \end{pmatrix}. \quad (5.54)$$

Here, the regulator is chosen so that the final result becomes simple. Then, we can find (see appendix A)

$$\frac{\text{Det}' \mathcal{M}_0^{(S\varphi)}}{\text{Det} \widehat{\mathcal{M}}_0^{(S\varphi)}} = \frac{1}{\det_{AB} \mathcal{K}_{AB}} \left(\prod_A \frac{g_A^2}{2\pi^2} \right) \left(\frac{\mathcal{J}^{(\text{gauge})}}{\det \kappa} \right)^2 \left(\frac{\text{Det} \mathcal{M}_{0,\text{BG}}^{(c\bar{c})}}{\text{Det} \widehat{\mathcal{M}}_{0,\text{BG}}^{(c\bar{c})}} \right)^{-2} \lim_{\nu \rightarrow 0} \frac{1}{\nu^{n_U}} \frac{\text{Det} \mathcal{M}_{0,\text{BG},\text{reg}}^{(S\varphi)}}{\text{Det} \widehat{\mathcal{M}}_{0,\text{BG},\text{reg}}^{(S\varphi)}}, \quad (5.55)$$

where

$$\mathcal{K}_{AB} = \lim_{r \rightarrow \infty} \frac{r^3}{\xi} \mathcal{U}^{(A)T} \left(\partial_r \Psi_0^{(c\bar{c})} \right) \left(\Psi_0^{(c\bar{c})} \right)^{-1} \mathcal{U}^{(B)}. \quad (5.56)$$

In numerical calculations, eq. (5.55) can be used to obtain the Fermi gauge result from the calculation with the fluctuation operators in the background gauge. Here, one should note that, even though our regulator for the background gauge becomes equal to that of eq. (5.19) when $\xi = 1$,

$$\left(\frac{\text{Det}' \mathcal{M}_{0,\text{BG}}^{(S\varphi)}}{\text{Det} \widehat{\mathcal{M}}_{0,\text{BG}}^{(S\varphi)}} \right)^{-1/2} \frac{\text{Det} \mathcal{M}_{0,\text{BG}}^{(c\bar{c})}}{\text{Det} \widehat{\mathcal{M}}_{0,\text{BG}}^{(c\bar{c})}} \Big|_{\xi=1} \neq \left(\frac{\text{Det}' \mathcal{M}_0^{(S\varphi)}}{\text{Det} \widehat{\mathcal{M}}_0^{(S\varphi)}} \right)^{-1/2} \frac{\text{Det} \mathcal{M}_0^{(c\bar{c})}}{\text{Det} \widehat{\mathcal{M}}_0^{(c\bar{c})}}. \quad (5.57)$$

The above inequality is not surprising because the gauge transformations relating different bounce configurations differ in two choices of the gauge fixing. In particular, in the background gauge, different bounces are related by a local, not global, gauge transformation [34]. Thus, if we worked only in the background gauge, it would become very non-trivial to relate the path integral over the zero modes to the integration over the gauge volume.

5.3 Translational zero modes

Next, let us consider the zero modes due to the translational invariance. These zero modes are related to

$$\psi^{(\text{tr})} \equiv \frac{r_0}{\sqrt{\bar{\phi}'^T(r_0) \bar{\phi}'(r_0)}} \begin{pmatrix} 0 \\ 0 \\ \bar{\phi}' \end{pmatrix}. \quad (5.58)$$

One can see that $\psi^{(\text{tr})}$ satisfies $\mathcal{M}_1^{(SL\varphi)}\psi^{(\text{tr})} = 0$ and $\psi^{(\text{tr})}(r \rightarrow \infty) = 0$. We note here that $\psi^{(\text{tr})}$ belongs to the Type 3 solutions (see eq. (4.9)). It corresponds to the solution obtained with $\chi^{(\text{tr})} = \eta^{(\text{tr})} = 0$ and $\lambda^{(\text{tr})} \propto \bar{\phi}'$; the relation $(\Delta_1 - \Omega)\bar{\phi}' = 0$ holds and that $\lambda^{(\text{tr})} \propto \bar{\phi}'$ satisfies the constraint imposed on the function λ , i.e., $M^T \lambda^{(\text{tr})} = 0$ (see eq. (2.13)). The normalization of $\psi^{(\text{tr})}$ is determined so that $\psi^{(\text{tr})}$ can be embedded into $\Psi_\ell^{(SL\varphi)(3)}$ (see eq. (4.10)).

We need to eliminate the zero eigenvalues from $\text{Det } \mathcal{M}_1^{(SL\varphi)}$. For the calculation of $\text{Det}' \mathcal{M}_1^{(SL\varphi)}$, we should obtain the function satisfying

$$\mathcal{M}_1^{(SL\varphi)} \check{\psi}^{(\text{tr})} = -\psi^{(\text{tr})}. \quad (5.59)$$

The solution of the above equation can be expressed as

$$\check{\psi}^{(\text{tr})} = \begin{pmatrix} \partial_r \check{\chi}^{(\text{tr})} \\ \frac{L}{r} \check{\chi}^{(\text{tr})} \\ M \check{\chi}^{(\text{tr})} \end{pmatrix} + \begin{pmatrix} (M^T M)^{-1} \left[\frac{L}{r} \check{\eta}^{(\text{tr})} - 2(M')^T \check{\lambda}^{(\text{tr})} \right] \\ (M^T M)^{-1} \frac{1}{r^2} \partial_r r^2 \check{\eta}^{(\text{tr})} \\ \check{\lambda}^{(\text{tr})} \end{pmatrix}, \quad (5.60)$$

where $\check{\chi}^{(\text{tr})}$, $\check{\eta}^{(\text{tr})}$, and $\check{\lambda}^{(\text{tr})}$ are functions satisfying

$$\Delta_1 \check{\chi}^{(\text{tr})} = \Delta_1 \chi|_{\chi \rightarrow \check{\chi}^{(\text{tr})}, \eta \rightarrow \check{\eta}^{(\text{tr})}, \lambda \rightarrow \check{\lambda}^{(\text{tr})}, \zeta \rightarrow 0}, \quad (5.61)$$

$$\Delta_1 \check{\eta}^{(\text{tr})} = \Delta_1 \eta|_{\chi \rightarrow \check{\chi}^{(\text{tr})}, \eta \rightarrow \check{\eta}^{(\text{tr})}, \lambda \rightarrow \check{\lambda}^{(\text{tr})}, \zeta \rightarrow 0}, \quad (5.62)$$

$$\Delta_1 \check{\lambda}^{(\text{tr})} = \Delta_1 \lambda|_{\chi \rightarrow \check{\chi}^{(\text{tr})}, \eta \rightarrow \check{\eta}^{(\text{tr})}, \lambda \rightarrow \check{\lambda}^{(\text{tr})}, \zeta \rightarrow 0} - \frac{r_0}{\sqrt{\bar{\phi}'^T(r_0) \bar{\phi}'(r_0)}} \bar{\phi}'. \quad (5.63)$$

By solving the above equations, $\check{\chi}^{(\text{tr})}$, $\check{\eta}^{(\text{tr})}$, and $\check{\lambda}^{(\text{tr})}$ can be obtained. Then, we can evaluate $\check{\psi}^{(\text{tr})}(r \rightarrow \infty)$, with which $\text{Det}' \mathcal{M}_1^{(SL\varphi)}$ can be calculated.

The bounce is localized in space-time, and the shift of the position of the bounce does not change the action. Thus, in four space-time dimensions, there are four translational zero modes which can be parameterized as

$$\begin{pmatrix} \bar{A}_\mu \\ \bar{\phi}(x) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{A}_\mu \\ \bar{\phi}(x) \end{pmatrix} + dx_\mu \begin{pmatrix} 0 \\ \partial_\mu \bar{\phi} \end{pmatrix} + \mathcal{O}(dx^2), \quad (5.64)$$

or as

$$\begin{pmatrix} \bar{A}_\mu \\ \bar{\phi}(x) \end{pmatrix} \rightarrow \begin{pmatrix} \bar{A}_\mu \\ \bar{\phi}(x) \end{pmatrix} + c_{(1,m_A,m_B)}^{(\text{tr})} \begin{pmatrix} 0 \\ \mathcal{N}^{(\text{tr})} \bar{\phi}' \mathcal{Y}_{(1,m_A,m_B)} \end{pmatrix} + \mathcal{O}(c^2), \quad (5.65)$$

with $\mathcal{N}^{(\text{tr})}$ being the normalization constant:

$$\mathcal{N}^{(\text{tr})} = \frac{1}{\sqrt{\int dr r^3 \bar{\phi}'^2}}. \quad (5.66)$$

The path integral over the translational zero modes can be understood as the integration over the position of the center of the bounce as [28]:

$$\prod_{m_A, m_B} \frac{dc_{(1,m_A,m_B)}^{(\text{tr})}}{\sqrt{2\pi}} \rightarrow \frac{\mathcal{B}^2}{4\pi^2} \prod_\mu dx_\mu. \quad (5.67)$$

Consequently, the functional determinant from the path integral is interpreted as

$$\left[\frac{\text{Det} \mathcal{M}_1^{(SL\varphi)}}{\text{Det} \widehat{\mathcal{M}}_1^{(SL\varphi)}} \right]^{-1/2} \rightarrow \frac{\mathcal{B}^2}{4\pi^2} \left[\frac{\text{Det}' \mathcal{M}_1^{(SL\varphi)}}{\text{Det} \widehat{\mathcal{M}}_1^{(SL\varphi)}} \right]^{-1/2} \prod_{\mu} dx_{\mu}. \quad (5.68)$$

Integration over the space-time volume will disappear from the expression of the decay rate per unit volume (i.e., the transition probability per unit time and unit volume). We note that, in the background gauge, we can obtain eq. (5.68) with the fluctuation operators in the Fermi gauge being replaced by those in the background gauge.

6 Semi-analytic expression of the decay rate

Now, we summarize the semi-analytic expression of the decay rate of the false vacuum. We are particularly interested in the expression of the prefactor $\mathcal{A}$ and its gauge invariance. The prefactor $\mathcal{A}$ can be given as a product of the contributions of various fluctuations as

$$\mathcal{A} = \mathcal{A}^{(c\bar{c})} \mathcal{A}^{(T)} \mathcal{A}^{(SL\varphi)}, \quad (6.1)$$

where $\mathcal{A}^{(c\bar{c})}$, $\mathcal{A}^{(T)}$, and $\mathcal{A}^{(SL\varphi)}$ denote the contributions of the FP ghosts, transverse mode of the gauge fields, and other bosonic fluctuations, respectively. (If there exist fermions coupled to the bounce, their contributions should be also taken into account.) Furthermore, because of the spherical nature of the bounce configuration, each contribution can be decomposed into those with different (ℓ, m_A, m_B) :

$$\mathcal{A}^{(X)} = \prod_{\ell} \left[\mathcal{A}_{\ell}^{(X)} \right]^{(\ell+1)^2}, \quad (6.2)$$

with $(X) = (c\bar{c})$, (T) , or $(SL\varphi)$. (Here, $\mathcal{A}_0^{(SL\varphi)}$ should be understood as $\mathcal{A}_0^{(S\varphi)}$.) For $\mathcal{A}^{(c\bar{c})}$ and $\mathcal{A}^{(SL\varphi)}$, $0 \leq \ell < \infty$ and for $\mathcal{A}^{(T)}$, $1 \leq \ell < \infty$. The final result $\mathcal{A}$ does not depend on the choice of the gauge fixing (as far as the gauge zero modes are irrelevant), although $\mathcal{A}^{(c\bar{c})}$ and $\mathcal{A}^{(SL\varphi)}$ in the Fermi and the background gauges differ from each other. In the following, we first summarize the results in the Fermi gauge and then discuss those in the background gauge.

6.1 Contributions of FP ghosts and transverse modes

First, we consider the contributions of the FP ghosts. In the Fermi gauge, the FP ghosts do not couple to the bounce, and hence the fluctuation operator of FP ghosts does not contain $\bar{\phi}$. Consequently,

$$\mathcal{A}^{(c\bar{c})} = 1. \quad (6.3)$$

For the contributions of the transverse modes of the gauge fields, we should solve

$$\mathcal{M}_{\ell}^{(T)} \Psi_{\ell}^{(T)} = 0, \quad (6.4)$$

$$\widehat{\mathcal{M}}_{\ell}^{(T)} \widehat{\Psi}_{\ell}^{(T)} = 0, \quad (6.5)$$

where $\mathcal{M}_\ell^{(T)}$ and $\widehat{\mathcal{M}}_\ell^{(T)}$ are the fluctuation operators given in eqs. (2.46) and (2.51), respectively. In addition, $\Psi_\ell^{(T)}$ and $\widehat{\Psi}_\ell^{(T)}$ are $n_G \times n_G$ objects, and their boundary conditions are given by

$$\Psi_\ell^{(T)}(r \rightarrow 0) \simeq r^\ell I_G, \quad (6.6)$$

$$\widehat{\Psi}_\ell^{(T)}(r \rightarrow 0) \simeq r^\ell I_G. \quad (6.7)$$

With these quantities,

$$\frac{\det \mathcal{M}_\ell^{(T)}}{\det \widehat{\mathcal{M}}_\ell^{(T)}} = \frac{\det \Psi_\ell^{(T)}(r_\infty)}{\det \widehat{\Psi}_\ell^{(T)}(r_\infty)}. \quad (6.8)$$

Then,

$$\mathcal{A}^{(T)} = \prod_{\ell=1}^{\infty} \left[\frac{\det \Psi_\ell^{(T)}(r_\infty)}{\det \widehat{\Psi}_\ell^{(T)}(r_\infty)} \right]^{-(\ell+1)^2/2}. \quad (6.9)$$

6.2 Contributions of $(SL\varphi)$ modes

Contributions of the $(SL\varphi)$ modes are complicated especially because of the zero modes. The prefactor originating from the $(SL\varphi)$ modes can be written as

$$\mathcal{A}^{(SL\varphi)} = \mathcal{A}_0^{(S\varphi)} \prod_{\ell=1}^{\infty} [\mathcal{A}_\ell^{(SL\varphi)}]^{(\ell+1)^2}. \quad (6.10)$$

For $\ell > 1$, we do not expect zero modes. Based on eq. (4.32), we obtain

$$\begin{aligned} \mathcal{A}_{\ell>1}^{(SL\varphi)} = & \left(\frac{r_\infty}{\ell} \right)^{n_U/2} \left[\det(M^T M)_\infty \det(M_0^T M_0) \right]^{-1/4} \left[\det(1/W_U^T W_U)'_\infty \right]^{-1/2} \left[\det \widehat{W}^T \widehat{W} \right]^{1/4} \\ & \left[\det \Psi_\ell^{(\eta\lambda)}(r_\infty) \right]^{-1/2} \left[\det \widehat{\Psi}_\ell^{(\eta)}(r_\infty) \right]^{1/2} \left[\det \widehat{\Psi}_\ell^{(\sigma)}(r_\infty) \right]^{1/2}, \end{aligned} \quad (6.11)$$

where $\Psi_\ell^{(\eta\lambda)}$ is give by eq. (4.30), while $\widehat{\Psi}_\ell^{(\eta)}$ and $\widehat{\Psi}_\ell^{(\sigma)}$ are defined by eqs. (4.23) and (4.24) and eqs. (4.26) and (4.27), respectively.

For $\ell = 1$, the translational zero modes exist. The zero eigenvalues from the translational zero modes are removed by replacing $\eta^{(\text{tr})}$ and $\lambda^{(\text{tr})}$ by $\check{\eta}^{(\text{tr})}$ and $\check{\lambda}^{(\text{tr})}$, respectively, in $\det \Psi_1^{(SL\varphi)}(r \rightarrow \infty)$, with taking account of the Jacobian (see eq. (5.68)). As a result, $\mathcal{A}_1^{(SL\varphi)}$ is obtained as

$$\begin{aligned} \mathcal{A}_1^{(SL\varphi)} = & \frac{\mathcal{B}^2}{4\pi^2} r_\infty^{n_U/2} \left[\det(M^T M)_\infty \det(M_0^T M_0) \right]^{-1/4} \left[\det(1/W_U^T W_U)'_\infty \right]^{-1/2} \left[\det \widehat{W}^T \widehat{W} \right]^{1/4} \\ & \left[\det(\Psi_1^{(\eta\lambda)}(r_\infty) + \check{\Psi}_1^{(\eta\lambda)}(r_\infty)) \right]^{-1/2} \left[\det \widehat{\Psi}_1^{(\eta)}(r_\infty) \right]^{1/2} \left[\det \widehat{\Psi}_1^{(\sigma)}(r_\infty) \right]^{1/2}, \end{aligned} \quad (6.12)$$

where

$$\Psi_1^{(\eta\lambda)} + \check{\Psi}_1^{(\eta\lambda)} = \Psi_1^{(\eta\lambda)} \Big|_{\psi^{(\text{tr})} \rightarrow \check{\psi}^{(\text{tr})}}. \quad (6.13)$$

Here, the definitions of $\psi^{(\text{tr})}$ and $\check{\psi}^{(\text{tr})}$ are given in eqs. (5.58) and (5.59), respectively. Notice that the space-time integral in eq. (5.68) is eliminated in calculating the decay rate per unit volume.

For $\ell = 0$, there may exist gauge zero modes. Based on eq. (5.51), we obtain

$$\mathcal{A}_0^{(S\varphi)} = \left(\frac{r_\infty^3}{\pi}\right)^{n_U/2} \left(\prod_A \frac{g_A^2}{2\pi^2}\right)^{-1/2} \mathcal{V}_U \det \kappa (\det W'_U)^{1/2} \left[\frac{\det \Psi_0^{(\lambda)}(r_\infty)}{\det \widehat{\Psi}_0^{(\sigma)}(r_\infty)}\right]^{-1/2} \left[\frac{\det \widehat{W}^T \widehat{W}}{\det(M_0^T M_0)}\right]^{-1/4}, \quad (6.14)$$

where $\mathcal{V}_U$ is the gauge volume and κ is the matrix orthogonalizing $T^A \bar{\phi}$ (see eq. (5.47)). In addition, $\Psi_0^{(\lambda)}$ is given in eq. (5.44).

6.3 Background gauge

Here, we summarize the relation between the results in the Fermi gauge and in the background gauge. In the background gauge, the FP contribution becomes

$$\mathcal{A}_{\text{BG}}^{(c\bar{c})} = \prod_{\ell=0}^{\infty} \left[\mathcal{A}_{\ell,\text{BG}}^{(c\bar{c})}\right]^{(\ell+1)^2} = \prod_{\ell=0}^{\infty} \left[\frac{\det \Psi_{\ell,\text{BG}}^{(c\bar{c})}}{\det \widehat{\Psi}_{\ell,\text{BG}}^{(c\bar{c})}}\right]^{(\ell+1)^2}. \quad (6.15)$$

The functions $\Psi_{\ell,\text{BG}}^{(c\bar{c})}$ and $\widehat{\Psi}_{\ell,\text{BG}}^{(c\bar{c})}$ are defined in eqs. (4.43) and (4.44) and eqs. (4.46) and (4.47), respectively.

Since the fluctuation operators for the (T) modes are identical to those in the Fermi gauge,

$$\mathcal{A}_{\ell,\text{BG}}^{(T)} = \mathcal{A}_\ell^{(T)}. \quad (6.16)$$

As we have seen in eq. (4.61), the following relation holds for $\ell > 0$:

$$\mathcal{A}_{\ell,\text{BG}}^{(SL\varphi)} \mathcal{A}_{\ell,\text{BG}}^{(c\bar{c})} = \mathcal{A}_\ell^{(SL\varphi)} \mathcal{A}_\ell^{(c\bar{c})}. \quad (6.17)$$

Thus, we can explicitly see that the Fermi and the background gauges give the same decay rate as far as the modes with $\ell > 0$ are concerned.

For $\ell = 0$, the proper treatment of the gauge zero modes in the background gauge is non-trivial, as we have explained in section 5. Thus, in the background gauge, we just introduce a convenient way of regulating the fluctuation operator with gauge zero modes and relate its functional determinant with that in the Fermi gauge. With the regulated fluctuation operators given in eqs. (5.53) and (5.54), we can obtain

$$\mathcal{A}_0^{(S\varphi)} = \mathcal{V}_U \left(\det \mathcal{K}_{AB}\right)^{1/2} \det \kappa \left(\prod_A \frac{g_A^2}{\pi}\right)^{-1/2} \left(\frac{\text{Det} \mathcal{M}_{0,\text{BG}}^{(c\bar{c})}}{\text{Det} \widehat{\mathcal{M}}_{0,\text{BG}}^{(c\bar{c})}}\right) \left(\lim_{\nu \rightarrow 0} \frac{1}{\nu^{n_U}} \frac{\text{Det} \mathcal{M}_{0,\text{BG,reg}}^{(S\varphi)}}{\text{Det} \widehat{\mathcal{M}}_{0,\text{BG,reg}}^{(S\varphi)}}\right)^{-1/2}, \quad (6.18)$$

with $\mathcal{K}_{AB}$ being given in eq. (5.56). The above equality is useful for numerical calculations: the right-hand side of eq. (6.18) is written only with the quantities in the background gauge and hence we can calculate them numerically.

7 Renormalization

The functional determinants diverge once the contributions of all ℓ are taken into account, and the renormalization is necessary. Here, we discuss how the divergences are removed, adopting the $\overline{\text{MS}}$ scheme.

In calculating the one-loop contribution to the decay rate of the false vacuum, we are interested in the ratio of the functional determinants in the following form:

$$\mathcal{A}^{(X)} = \left(\frac{\text{Det } \mathcal{M}^{(X)}}{\text{Det } \widehat{\mathcal{M}}^{(X)}} \right)^{-1/2} = \prod_{\ell=0}^{\infty} \left(\frac{\text{Det } \mathcal{M}_{\ell}^{(X)}}{\text{Det } \widehat{\mathcal{M}}_{\ell}^{(X)}} \right)^{-(\ell+1)^2/2}, \quad (7.1)$$

where $\mathcal{M}^{(X)}$ and $\widehat{\mathcal{M}}^{(X)}$ are fluctuation operators around the bounce and that around the false vacuum, respectively. Our procedure of the renormalization is as follows [1]:

- (i) We first identify terms which contain the divergence in $\ln \mathcal{A}^{(X)}$ (which we call $\delta \ln \mathcal{A}_{\text{div}}^{(X)}$).
- (ii) Next, we perform the $\overline{\text{MS}}$ subtraction of $\delta \ln \mathcal{A}_{\text{div}}^{(X)}$. The result is denoted as $\delta \ln \mathcal{A}_{\text{div}, \overline{\text{MS}}}^{(X)}$.

Then, the divergence is removed as [1]

$$\ln \mathcal{A}^{(X)} \rightarrow \ln \mathcal{A}_{\overline{\text{MS}}}^{(X)} \equiv \ln \mathcal{A}^{(X)} - \delta \ln \mathcal{A}_{\text{div}}^{(X)} + \delta \ln \mathcal{A}_{\text{div}, \overline{\text{MS}}}^{(X)}. \quad (7.2)$$

Notice that, in the above expression, $\ln \mathcal{A}^{(X)}$ and $\delta \ln \mathcal{A}_{\text{div}}^{(X)}$ are both divergent, while $\ln \mathcal{A}_{\text{div}}^{(X)} - \delta \ln \mathcal{A}_{\text{div}, \overline{\text{MS}}}^{(X)}$ and $\delta \ln \mathcal{A}_{\text{div}, \overline{\text{MS}}}^{(X)}$ are finite.

The divergent part of the functional determinant can be obtained by a diagrammatic consideration. Let us define

$$\delta \mathcal{M} \equiv \mathcal{M}^{(X)} - \widehat{\mathcal{M}}^{(X)}, \quad (7.3)$$

and $\delta \mathcal{M}_{\ell} \equiv \mathcal{M}_{\ell}^{(X)} - \widehat{\mathcal{M}}_{\ell}^{(X)}$. Then, treating $\delta \mathcal{M}$ as a perturbation, $\ln \mathcal{A}^{(X)}$ can be expressed as

$$\ln \mathcal{A}^{(X)} = - \sum_{I=1}^{\infty} s_I^{(X)}, \quad (7.4)$$

where $s_I^{(X)}$ indicates the contribution of diagrams with I being the number of insertions of $\delta \mathcal{M}$.

Before proceeding further, we comment on the choice of the gauge fixing which affects the calculation of $\text{Det } \mathcal{M}_{\ell}^{(SL\varphi)}$. The calculation can be performed in any gauge; as we have discussed in section 4, the Fermi and the background gauge calculations give the same result of $\text{Det } \mathcal{M}_{\ell}^{(SL\varphi)}$ for $\ell \geq 1$ and hence the calculation of the divergent part is possible in either gauge. In practice, however, it is convenient to work in the background gauge because the number of diagrams necessary to calculate the divergent part is reduced. This is because, in the background gauge, $\delta \mathcal{M}$ does not contain derivative operators; thus, only $s_1^{(SL\varphi)}$ and $s_2^{(SL\varphi)}$ are divergent. (In the Fermi gauge, in general, $s_{I \leq 4}^{(SL\varphi)}$ are divergent.) Thus, we adopt the background gauge to calculate the divergent part. In the following, we consider the case where only $s_1^{(X)}$ and $s_2^{(X)}$ are divergent while $s_{I \geq 3}^{(X)}$'s are finite.

We have calculated the functional determinants of fluctuation operators by using the functions satisfying $\mathcal{M}_\ell^{(X)} \Psi_\ell^{(X)} = 0$ and $\widehat{\mathcal{M}}_\ell^{(X)} \widehat{\Psi}_\ell^{(X)} = 0$. We can express $\Psi_\ell^{(X)}$ in the following form:

$$\Psi_\ell^{(X)} = \widehat{\Psi}_\ell^{(X)} + \sum_{I=1}^{\infty} \delta \Psi_{\ell,I}^{(X)}, \quad (7.5)$$

where the following equations are satisfied:

$$\widehat{\mathcal{M}}_\ell^{(X)} \delta \Psi_{\ell,1}^{(X)} = -\delta \mathcal{M}_\ell \widehat{\Psi}_\ell^{(X)}, \quad (7.6)$$

$$\widehat{\mathcal{M}}_\ell^{(X)} \delta \Psi_{\ell,I+1}^{(X)} = -\delta \mathcal{M}_\ell \delta \Psi_{\ell,I}^{(X)} \quad (I \geq 1), \quad (7.7)$$

with $\delta \Psi_{\ell,I}^{(X)}(r \rightarrow 0) \simeq 0$. Notice that $\delta \Psi_{\ell,I}^{(X)}$ can be understood as the term which is of the order of $(\delta \mathcal{M})^I$. Then, $\ln \mathcal{A}^{(X)}$ becomes

$$\ln \mathcal{A}^{(X)} = -\lim_{r \rightarrow \infty} \sum_{\ell=0}^{\infty} \frac{(\ell+1)^2}{2} \left[\text{Tr} \ln \left(\widehat{\Psi}_\ell^{(X)} + \sum_{I=1}^{\infty} \delta \Psi_{\ell,I}^{(X)} \right) - \text{Tr} \ln \widehat{\Psi}_\ell^{(X)} \right], \quad (7.8)$$

from which we obtain

$$s_1^{(X)} = \lim_{r \rightarrow \infty} \sum_{\ell=0}^{\infty} \frac{(\ell+1)^2}{2} \left[\text{Tr} [\widehat{\Psi}_\ell^{(X)}]^{-1} \delta \Psi_{\ell,1}^{(X)} \right], \quad (7.9)$$

$$s_2^{(X)} = \lim_{r \rightarrow \infty} \sum_{\ell=0}^{\infty} \frac{(\ell+1)^2}{2} \left[\text{Tr} [\widehat{\Psi}_\ell^{(X)}]^{-1} \delta \Psi_{\ell,2}^{(X)} - \frac{1}{2} \text{Tr} [\widehat{\Psi}_\ell^{(X)}]^{-1} \delta \Psi_{\ell,1}^{(X)} [\widehat{\Psi}_\ell^{(X)}]^{-1} \delta \Psi_{\ell,1}^{(X)} \right]. \quad (7.10)$$

In order to perform the $\overline{\text{MS}}$ subtraction from the divergent part, we can use

$$\ln \mathcal{A}^{(X)} = -\frac{1}{2} \text{Tr} [\widehat{\mathcal{M}}^{(X)}]^{-1} \delta \mathcal{M} + \frac{1}{4} \text{Tr} [\widehat{\mathcal{M}}^{(X)}]^{-1} \delta \mathcal{M} [\widehat{\mathcal{M}}^{(X)}]^{-1} \delta \mathcal{M} + \mathcal{O}(\delta \mathcal{M}^3), \quad (7.11)$$

to obtain

$$s_1^{(X)} = \frac{1}{2} \text{Tr} [\widehat{\mathcal{M}}^{(X)}]^{-1} \delta \mathcal{M}, \quad (7.12)$$

$$s_2^{(X)} = -\frac{1}{4} \text{Tr} [\widehat{\mathcal{M}}^{(X)}]^{-1} \delta \mathcal{M} [\widehat{\mathcal{M}}^{(X)}]^{-1} \delta \mathcal{M}. \quad (7.13)$$

These quantities can be calculated by using the ordinary diagrammatic method. We calculate them adopting dimensional regularization, and perform $\overline{\text{MS}}$ subtraction to obtain the renormalized quantities, $s_{1,\overline{\text{MS}}}^{(X)}$ and $s_{2,\overline{\text{MS}}}^{(X)}$; their explicit forms are given in appendix C.¹¹

The prefactor after the $\overline{\text{MS}}$ subtraction can be obtained as

$$\ln \mathcal{A}_{\overline{\text{MS}}}^{(X)} = \left[\ln \mathcal{A} + s_1^{(X)} + s_2^{(X)} \right] - s_{1,\overline{\text{MS}}}^{(X)} - s_{2,\overline{\text{MS}}}^{(X)}, \quad (7.14)$$

where $s_1^{(X)}$ and $s_2^{(X)}$ in the square bracket should be understood as those given in eqs. (7.9) and (7.10), respectively. Notice that the quantity in the square bracket can be evaluated order by order in ℓ . At a large ℓ , the contribution behaves as ℓ^{-2} and the sum over ℓ converges. One may truncate higher ℓ , or fit the terms with large enough ℓ and sum them up for better precision.

¹¹There was an error in the calculation of the counterterms in [34], which is corrected in the expression in appendix C.

8 Conclusions and discussion

In this paper, we have studied the decay rate of the false vacuum in gauge theory for the case with a multi-field bounce. If more than one scalar fields contribute to the bounce, the mixing among the gauge fields and the scalars becomes r -dependent, making it complicated to calculate the one-loop contributions to the decay rate. We have extended the results of [33, 34] to perform a gauge invariant calculation of the decay rate at the one-loop level. The one-loop contributions to the decay rate, which are denoted as $\mathcal{A}$ in this article, are given by the ratio of the functional determinants of the fluctuation operators around the bounce and those around the false vacuum (see eqs. (2.61) and (2.62)). Using the fact that the functional determinant of the fluctuation operator, $\mathcal{M}^{(X)}$, can be related to the asymptotic behavior of the function, $\psi^{(X)}$, obeying $\mathcal{M}^{(X)}\psi^{(X)} = 0$, we have derived an expression of the decay rate at the one-loop level, which is manifestly independent of the gauge parameter, ξ . In particular, we have discussed how the functional determinant of $\mathcal{M}^{(A_\mu\varphi)}$, i.e. the fluctuation operator for the gauge and scalar fields, can be evaluated in a gauge invariant way. Our main results are summarized in section 6.

In our study, we have worked in both the Fermi gauge and the background gauge. We have shown that the Fermi gauge has an advantage to show the gauge invariance of the result especially when there exist gauge zero modes. The path integral over the gauge zero modes should be replaced by the integration over the gauge volume; the rule of the replacement can be straightforwardly obtained in the Fermi gauge. The background gauge has, however, an advantage in numerically calculating the decay rate because of the better convergence. In order to utilize the calculation in the background gauge, we derived a prescription to translate the result in the background gauge to that in the Fermi gauge.

We have also discussed how we can remove ultraviolet divergences from the decay rate. We have shown a procedure to subtract the divergences from the one-loop results and to perform the renormalization in the $\overline{\text{MS}}$ scheme (see section 7).

Our results apply to various types of models with many scalar fields having gauge charges; in such models, it is often the case that there shows up a true vacuum whose energy density is lower than that of the electroweak vacuum. Our results can be used to calculate the decay rate of the electroweak vacuum in such models. Phenomenological application of our results may be discussed elsewhere.

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A Evaluation of determinants

A.1 Alternative fluctuation operators

One of the important steps in our calculation is to rewrite $\det \Psi_\ell^{(SL\varphi)}$ using η and λ at $r \sim \infty$. Since the fluctuation matrices approach those at the false vacuum as $r \rightarrow \infty$, one may expect that the behavior of $\Psi_\ell^{(SL\varphi)}$ at $r \sim \infty$ is the same as that of the solution at the false vacuum, $\widehat{\Psi}_\ell^{(SL\varphi)}$. However, it is not always the case because they are related as

$$\Psi_\ell^{(SL\varphi)}(r \rightarrow \infty) = \widehat{\Psi}_\ell^{(SL\varphi)}(r)G(r), \quad (\text{A.1})$$

where G is a square matrix. Since the elements of $\widehat{\Psi}_\ell^{(SL\varphi)}$ generally become hierarchical as $r \rightarrow \infty$, larger elements easily contaminate smaller elements due to the mixing induced by G . The crucial problem here is that the contamination cannot be avoided just by taking a linear combination of the solutions since G is r -dependent. Such contamination leads to a severe cancellation when we calculate the determinant of $\Psi_\ell^{(SL\varphi)}$. In the following, we discuss a way to avoid this problem.

We first define the projection operators as

$$P_H = V_H V_H^T, \quad (\text{A.2})$$

$$P_G = M(M^T M)^{-1} M^T, \quad (\text{A.3})$$

which satisfy $P_H + P_G = I_\varphi$ and $P_H P_G = P_G P_H = 0$. Then, we define r -dependent orthogonal matrices $O_\varphi(r)$ and $O_G(r)$ having the following properties:

$$O_\varphi^T(r) M(r) O_G(r) = \underbrace{\begin{pmatrix} W(r) \\ 0 \end{pmatrix}}_{n_G} \Bigg\}_{n_H} = \begin{pmatrix} W_B(r) & 0 \\ 0 & W_U(r) \\ 0 & 0 \end{pmatrix} \Bigg\}_{n_B}^{n_U} \Bigg\}_{n_H}, \quad (\text{A.4})$$

$$O_\varphi^T(r) P_H(r) \Omega(r) P_H(r) O_\varphi(r) = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & m^2(r) \end{pmatrix}}_{n_G} \Bigg\}_{n_H}, \quad (\text{A.5})$$

where m^2 , W , W_B and W_U are diagonal matrices. Since we work in the field basis given in eqs. (2.28) and (2.29), we can take

$$\lim_{r \rightarrow \infty} O_\varphi(r) = I_\varphi, \quad \lim_{r \rightarrow \infty} O_G(r) = I_G. \quad (\text{A.6})$$

The basic idea is to use the following matrices instead of M and Ω .

$$M_{\text{alt}} = \begin{cases} M, & r \ll r_*, \\ O_\varphi^T M O_G, & r \gg r_*, \end{cases} \quad (\text{A.7})$$

$$\Omega_{\text{alt}} = \begin{cases} \Omega, & r \ll r_*, \\ O_\varphi^T P_H \Omega P_H O_\varphi + O_\varphi^T P_G \Omega P_G O_\varphi, & r \gg r_*. \end{cases} \quad (\text{A.8})$$

Here, r_* is taken to be much larger than the typical scale of the bounce. Then, the behavior of the solution at $r \rightarrow \infty$ can be understood easily since the fluctuation operators become block-diagonalized. The behavior at $r \sim r_*$ will be defined in appendix B.

The decomposed solution in section 3 cannot be used as it is since M_{alt} does not satisfy eqs. (2.15) and (2.16) for $r \gtrsim r_*$. However, we find that the solution becomes exact if we add extra scalar mass terms to the fluctuation operators as

$$\mathcal{M}_{\ell,\text{alt}}^{(SL\varphi)} = \mathcal{M}_{\ell}^{(SL\varphi)} \Big|_{M \rightarrow M_{\text{alt}}, \Omega \rightarrow \Omega_{\text{alt}} + \delta\Omega}, \quad (\text{A.9})$$

$$\mathcal{M}_{0,\text{alt}}^{(S\varphi)} = \mathcal{M}_0^{(S\varphi)} \Big|_{M \rightarrow M_{\text{alt}}, \Omega \rightarrow \Omega_{\text{alt}} + \delta\Omega}, \quad (\text{A.10})$$

where

$$\delta\Omega = -M_{\text{alt}}(M_{\text{alt}}^T M_{\text{alt}})^{-1} \varepsilon^T, \quad (\text{A.11})$$

with

$$\varepsilon = -\Delta_0 M_{\text{alt}} + \Omega_{\text{alt}} M_{\text{alt}}. \quad (\text{A.12})$$

Then, the solutions of

$$\mathcal{M}_{\ell,\text{alt}}^{(SL\varphi)} \Psi_{\ell,\text{alt}}^{(SL\varphi)} = 0, \quad (\text{A.13})$$

and

$$\mathcal{M}_{0,\text{alt}}^{(S\varphi)} \Psi_{0,\text{alt}}^{(S\varphi)} = 0, \quad (\text{A.14})$$

are given by eqs. (3.2)–(3.6) and (3.13) with replacing $M \rightarrow M_{\text{alt}}$ and $\Omega \rightarrow \Omega_{\text{alt}}$. Since the fluctuation operators are not modified for $r \lesssim r_*$, we take

$$\Psi_{\ell,\text{alt}}^{(SL\varphi)}(r \lesssim r_*) = \Psi_{\ell}^{(SL\varphi)}(r \lesssim r_*), \quad (\text{A.15})$$

$$\Psi_{0,\text{alt}}^{(S\varphi)}(r \lesssim r_*) = \Psi_0^{(S\varphi)}(r \lesssim r_*). \quad (\text{A.16})$$

Similarly, we define the solutions, $\Psi_{\ell,\text{alt}}^{(c\bar{c})}$ and $\Psi_{\ell,\text{alt}}^{(\eta\lambda)}$, which satisfy the same differential equations as $\Psi_{\ell}^{(c\bar{c})}$ and $\Psi_{\ell}^{(\eta\lambda)}$ but with $M \rightarrow M_{\text{alt}}$ and $\Omega \rightarrow \Omega_{\text{alt}}$.

For later convenience, we also define

$$P_{G,\text{alt}} = M_{\text{alt}}(M_{\text{alt}}^T M_{\text{alt}})^{-1} M_{\text{alt}}^T, \quad (\text{A.17})$$

$$P_{H,\text{alt}} = I_{\varphi} - P_{G,\text{alt}}. \quad (\text{A.18})$$

The deformation at $r \sim r_*$ should be smooth enough and the decomposition of the solutions should be well-defined for the entire region of r . In addition, we need to ensure the use of these alternative fluctuation operators does not affect the final results. We will discuss these issues in appendix B.

A.2 Evaluation of solutions for $\ell > 0$

A.2.1 Behavior at infinity

Here, we evaluate $\Psi_{\ell,\text{alt}}^{(SL\varphi)}$ at $r \rightarrow \infty$. For this purpose, we can use the fact that $\Psi_{\ell,\text{alt}}^{(c\bar{c})}$ and $\Psi_{\ell,\text{alt}}^{(\eta\lambda)}$ behave as follows:

$$\Psi_{\ell,\text{alt}}^{(c\bar{c})}(r \rightarrow \infty) \simeq \begin{pmatrix} \frac{e\sqrt{\xi}W_B r}{r^{3/2}} I_B & 0 \\ 0 & r^\ell I_U \end{pmatrix} \mathcal{T}_\ell^{(c\bar{c})}, \quad (\text{A.19})$$

and

$$\Psi_{\ell,\text{alt}}^{(\eta\lambda)}(r \rightarrow \infty) \simeq \begin{pmatrix} \frac{eW_B r}{r^{3/2}} I_B & 0 & 0 \\ 0 & \frac{1}{r^2} I_U & 0 \\ 0 & 0 & \frac{e^{mr}}{r^{3/2}} I_H \end{pmatrix} \mathcal{T}_\ell^{(\eta\lambda)}, \quad (\text{A.20})$$

where $\mathcal{T}_\ell^{(c\bar{c})}$ and $\mathcal{T}_\ell^{(\eta\lambda)}$ are constant square matrices.

We examine the behavior of each type of the solutions at $r \rightarrow \infty$ below.

- Type 1

In the Fermi gauge, $\chi^{(1)} = r^\ell I_G$. Then, from eq. (3.2),

$$\Psi_{\ell,\text{alt}}^{(SL\varphi)(1)}(r \rightarrow \infty) \simeq \underbrace{\begin{pmatrix} \frac{\ell}{r} I_G \\ \frac{L}{r} I_G \\ W \\ 0 \end{pmatrix}}_{n_G} \left\{ \begin{array}{l} n_G \\ n_G \\ n_G \\ n_H \end{array} \right\} r^\ell. \quad (\text{A.21})$$

In the background gauge, $\chi^{(1)} = \Psi_{\ell,\text{alt}}^{(c\bar{c})}$. Using eq. (3.2),

$$\Psi_{\ell,\text{BG,alt}}^{(SL\varphi)(1)}(r \rightarrow \infty) \simeq \underbrace{\begin{pmatrix} \sqrt{\xi}W_B & 0 \\ 0 & \frac{\ell}{r} I_U \\ \frac{L}{r} I_B & 0 \\ 0 & \frac{L}{r} I_U \\ W_B & 0 \\ 0 & W_U \\ 0 & 0 \end{pmatrix}}_{\substack{n_B \\ n_U}} \underbrace{\begin{pmatrix} 0 \\ \frac{\ell}{r} I_U \\ 0 \\ \frac{L}{r} I_U \\ 0 \\ W_U \\ 0 \end{pmatrix}}_{n_U} \left\{ \begin{array}{l} n_B \\ n_U \\ n_B \\ n_U \\ n_B \\ n_U \\ n_H \end{array} \right\} \Psi_{\ell,\text{alt}}^{(c\bar{c})}. \quad (\text{A.22})$$

- Type 2

Since $P_{H,\text{alt}} M'_{\text{alt}} = 0$ for $r \gtrsim r_*$, $\lambda^{(2)} = 0$ from eq. (3.6). In the Fermi gauge, $\zeta^{(2)} = r^\ell$, and

$$\eta^{(2)}(r \rightarrow \infty) = \underbrace{\begin{pmatrix} \mathcal{O}(r^{-1}W'_B) \\ 0 \end{pmatrix}}_{n_B} \underbrace{\begin{pmatrix} 0 \\ \frac{L}{\ell+2} + \mathcal{O}(W_U^2) \end{pmatrix}}_{n_U} \left\{ \begin{array}{l} n_B \\ n_U \end{array} \right\} r^\ell. \quad (\text{A.23})$$

Then, from eqs. (3.8)–(3.10),

$$\Psi_{\ell,\text{alt}}^{(SL\varphi)(2)}(r \rightarrow \infty) = \left(\underbrace{\begin{pmatrix} -\frac{\xi}{4}rI_B \\ 0 \\ -\frac{\xi\ell}{4L}rI_B \\ 0 \\ -\frac{\xi}{4(\ell+2)}r^2W_B \\ 0 \\ 0 \end{pmatrix}}_{n_B} \underbrace{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)}rI_U \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)}rI_U \\ 0 \\ \mathcal{O}(r^2W_U) \\ 0 \end{pmatrix}}_{n_U} \right) \begin{matrix} \left. \vphantom{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)}rI_U \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)}rI_U \\ 0 \\ \mathcal{O}(r^2W_U) \\ 0 \end{pmatrix}} \right\} n_B \\ \left. \vphantom{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)}rI_U \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)}rI_U \\ 0 \\ \mathcal{O}(r^2W_U) \\ 0 \end{pmatrix}} \right\} n_U \\ \left. \vphantom{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)}rI_U \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)}rI_U \\ 0 \\ \mathcal{O}(r^2W_U) \\ 0 \end{pmatrix}} \right\} n_B \\ \left. \vphantom{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)}rI_U \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)}rI_U \\ 0 \\ \mathcal{O}(r^2W_U) \\ 0 \end{pmatrix}} \right\} n_U \\ \left. \vphantom{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)}rI_U \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)}rI_U \\ 0 \\ \mathcal{O}(r^2W_U) \\ 0 \end{pmatrix}} \right\} n_H \end{matrix} r^\ell. \quad (\text{A.24})$$

In the background gauge, $\zeta^{(2)} = \Psi_{\ell,\text{alt}}^{(c\bar{c})}$. Then,

$$\eta^{(2)}(r \rightarrow \infty) = \left(\underbrace{\begin{pmatrix} \mathcal{O}(r^{-1}W'_B) \\ 0 \end{pmatrix}}_{n_B} \underbrace{\begin{pmatrix} 0 \\ \frac{L}{\ell+2} + \mathcal{O}(W_U^2) \end{pmatrix}}_{n_U} \right) \begin{matrix} \left. \vphantom{\begin{pmatrix} 0 \\ \frac{L}{\ell+2} + \mathcal{O}(W_U^2) \end{pmatrix}} \right\} n_B \\ \left. \vphantom{\begin{pmatrix} 0 \\ \frac{L}{\ell+2} + \mathcal{O}(W_U^2) \end{pmatrix}} \right\} n_U \end{matrix} \Psi_{\ell,\text{alt}}^{(c\bar{c})}, \quad (\text{A.25})$$

and

$$\Psi_{\ell,\text{BG,alt}}^{(SL\varphi)(2)}(r \rightarrow \infty) = \left(\underbrace{\begin{pmatrix} \mathcal{O}(W'_B) \\ 0 \\ \mathcal{O}(W'_B) \\ 0 \\ W_B^{-1} \\ 0 \\ 0 \end{pmatrix}}_{n_B} \underbrace{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)}rI_U \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)}rI_U \\ 0 \\ \mathcal{O}(r^2W_U) \\ 0 \end{pmatrix}}_{n_U} \right) \begin{matrix} \left. \vphantom{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)}rI_U \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)}rI_U \\ 0 \\ \mathcal{O}(r^2W_U) \\ 0 \end{pmatrix}} \right\} n_B \\ \left. \vphantom{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)}rI_U \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)}rI_U \\ 0 \\ \mathcal{O}(r^2W_U) \\ 0 \end{pmatrix}} \right\} n_U \\ \left. \vphantom{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)}rI_U \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)}rI_U \\ 0 \\ \mathcal{O}(r^2W_U) \\ 0 \end{pmatrix}} \right\} n_B \\ \left. \vphantom{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)}rI_U \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)}rI_U \\ 0 \\ \mathcal{O}(r^2W_U) \\ 0 \end{pmatrix}} \right\} n_U \\ \left. \vphantom{\begin{pmatrix} 0 \\ \frac{\ell-(\ell+2)\xi}{4(\ell+2)}rI_U \\ 0 \\ \frac{\ell[(\ell+4)-(\ell+2)\xi]}{4L(\ell+2)}rI_U \\ 0 \\ \mathcal{O}(r^2W_U) \\ 0 \end{pmatrix}} \right\} n_H \end{matrix} \Psi_{\ell,\text{alt}}^{(c\bar{c})}. \quad (\text{A.26})$$

- Type 3

When gauge bosons become massive at the false vacuum, it is convenient to use the following relations:

$$\Delta_\ell \left(r\Psi_\ell^{(\text{top})} \right) = L\eta - \frac{\xi}{r}\partial_r r^2 Y, \quad (\text{A.27})$$

$$\Psi_\ell^{(\text{mid})} = \frac{1}{Lr^2}\partial_r r^2 \left(r\Psi_\ell^{(\text{top})} \right) + \frac{r\xi}{L}Y, \quad (\text{A.28})$$

$$\Psi_\ell^{(\text{bot})} = M\chi + \lambda + M(M^T M)^{-1}\zeta, \quad (\text{A.29})$$

where

$$Y = \begin{cases} \zeta & : \text{Fermi} \\ -M^T M\chi & : \text{BG} \end{cases}. \quad (\text{A.30})$$

Using eqs. (3.3) and (3.8)–(3.10) as well as the above relations,

$$\Psi_{\ell,\text{alt}}^{(SL\varphi)(3)}(r \rightarrow \infty) = \left(\underbrace{\begin{pmatrix} \frac{L}{r} W_B^{-2} \\ 0 \\ W_B^{-1} \\ 0 \\ \mathcal{O}(r^{-1} W'_B) \\ 0 \\ 0 \end{pmatrix}}_{n_B} \underbrace{\begin{pmatrix} 0 \\ -\frac{r}{L} I_U \\ 0 \\ -\frac{2r}{L^2} I_U \\ 0 \\ \frac{L}{2r} (W'_U)^{-1} \\ 0 \end{pmatrix}}_{n_U} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I_H \end{pmatrix}}_{n_H} \right) \Psi_{\ell,\text{alt}}^{(\eta\lambda)}. \quad (\text{A.31})$$

A.2.2 Translational zero modes

Here, we discuss the translational zero modes. The following discussion applies to both the Fermi gauge and the background gauge. With the alternative fluctuation operator, $\mathcal{M}_{1,\text{alt}}^{(SL\varphi)}$, the existence of the translational zero modes is not guaranteed; $\det \Psi_{1,\text{alt}}^{(SL\varphi)}(r \rightarrow \infty)$ can be non-vanishing. However, it does not cause a problem since it approaches zero as $r_* \rightarrow \infty$ and we always take $\nu \rightarrow 0$ after $r_* \rightarrow \infty$.

Let us start with the original fluctuation matrix (the one without “alt”). To construct the solution explicitly, we regularize the translational zero modes using

$$\mathcal{M}_{1,\text{reg}}^{(SL\varphi)} = \mathcal{M}_1^{(SL\varphi)} + \nu \left(\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & P_H \end{pmatrix}}_{\substack{2n_G \\ n_\varphi}} \right) \left\{ \begin{matrix} 2n_G \\ n_\varphi \end{matrix} \right\}, \quad (\text{A.32})$$

instead of $\mathcal{M}_1^{(SL\varphi)} + \nu$. Since

$$\mathcal{M}_{1,\text{reg}}^{(SL\varphi)} \psi^{(\text{tr})} = \nu \psi^{(\text{tr})}, \quad (\text{A.33})$$

we can use the same discussion for the zero modes as in section 5. Then, we define its alternative as

$$\mathcal{M}_{1,\text{reg,alt}}^{(SL\varphi)} = \mathcal{M}_{1,\text{reg}}^{(SL\varphi)} \Big|_{M \rightarrow M_{\text{alt}}, \Omega \rightarrow \Omega_{\text{alt}} + \delta\Omega, P_H \rightarrow P_{H,\text{alt}}}. \quad (\text{A.34})$$

Since $\psi^{(\text{tr})}$ is included in $\Psi_1^{(SL\varphi)(3)}$, it is enough to calculate

$$\mathcal{M}_{1,\text{reg,alt}}^{(SL\varphi)} \check{\Psi}_{1,\text{alt}}^{(SL\varphi)(3)} = -\Psi_{1,\text{alt}}^{(SL\varphi)(3)}. \quad (\text{A.35})$$

The solution is given by

$$\check{\Psi}_{1,\text{alt}}^{(SL\varphi)(3)} = \Psi_{1,\text{alt}}^{(SL\varphi)(3)} \Big|_{\Psi_{1,\text{alt}}^{(\eta\lambda)} \rightarrow \check{\Psi}_{1,\text{alt}}^{(\eta\lambda)}}, \quad (\text{A.36})$$

where

$$\check{\Psi}_{1,\text{alt}}^{(\eta\lambda)} = \begin{pmatrix} \check{\eta} \\ V_{H,\text{alt}}^T \check{\lambda} \end{pmatrix}. \quad (\text{A.37})$$

Here, $\check{\eta}$ and $\check{\lambda}$ satisfy

$$\Delta_1 \check{\eta} = \Delta_1 \eta|_{\eta \rightarrow \check{\eta}, \zeta \rightarrow 0, \lambda \rightarrow \check{\lambda}, M \rightarrow M_{\text{alt}}, \Omega \rightarrow \Omega_{\text{alt}}}, \quad (\text{A.38})$$

$$\Delta_1 \check{\lambda} = \Delta_1 \lambda|_{\eta \rightarrow \check{\eta}, \zeta \rightarrow 0, \lambda \rightarrow \check{\lambda}, M \rightarrow M_{\text{alt}}, \Omega \rightarrow \Omega_{\text{alt}}} + \begin{pmatrix} 0 & 0 & V_{H, \text{alt}} \end{pmatrix} \Psi_{1, \text{alt}}^{(SL\varphi)}. \quad (\text{A.39})$$

A.2.3 Functional determinant ($\ell > 1$)

Here, we calculate the determinant of the solutions obtained above.

In the Fermi gauge,

$$\begin{aligned} \det \Psi_{\ell, \text{alt}}^{(SL\varphi)}(r \rightarrow \infty) &= \det \left(\Psi_{\ell, \text{alt}}^{(SL\varphi)(1)} \quad \Psi_{\ell, \text{alt}}^{(SL\varphi)(2)} \quad \Psi_{\ell, \text{alt}}^{(SL\varphi)(3)} \right) \\ &\simeq \left(-\frac{\xi r}{2(\ell+2)} \right)^{n_B} \left(\frac{\ell[\ell + \xi(\ell+2)]}{4(\ell+2)r} \right)^{n_U} r^{2\ell n_G} \frac{\det \Psi_{\ell, \text{alt}}^{(\eta\lambda)}}{\det W'_U}, \end{aligned} \quad (\text{A.40})$$

and

$$\begin{aligned} \det \hat{\Psi}_{\ell}^{(SL\varphi)}(r \rightarrow \infty) &= \det \hat{\Psi}_{\ell}^{(B)} \det \hat{\Psi}_{\ell}^{(U)} \det \hat{\Psi}_{\ell}^{(\lambda)} \\ &\simeq \left(-\frac{\xi r}{2(\ell+2)} \right)^{n_B} r^{2\ell n_B} \det \hat{\Psi}_{\ell}^{(\eta)} \left(-\frac{\ell + \xi(\ell+2)}{2(\ell+2)} r^{2\ell} \right)^{n_U} \det \hat{\Psi}_{\ell}^{(\lambda)}. \end{aligned} \quad (\text{A.41})$$

Thus,

$$\frac{\det \Psi_{\ell, \text{alt}}^{(SL\varphi)}(r \rightarrow \infty)}{\det \hat{\Psi}_{\ell}^{(SL\varphi)}(r \rightarrow \infty)} = \left(-\frac{\ell}{2r_{\infty}} \right)^{n_U} \frac{\det \Psi_{\ell, \text{alt}}^{(\eta\lambda)}}{\det \hat{\Psi}_{\ell}^{(\eta)} \det \hat{\Psi}_{\ell}^{(\lambda)} \det W'_U}. \quad (\text{A.42})$$

Keeping only the leading terms in r , we can interchange $r \rightarrow \infty$ and $r_* \rightarrow \infty$ and

$$\frac{\det \Psi_{\ell}^{(SL\varphi)}(r \rightarrow \infty)}{\det \hat{\Psi}_{\ell}^{(SL\varphi)}(r \rightarrow \infty)} = \left(-\frac{\ell}{2r_{\infty}} \right)^{n_U} \frac{\det \Psi_{\ell}^{(\eta\lambda)}}{\det \hat{\Psi}_{\ell}^{(\eta)} \det \hat{\Psi}_{\ell}^{(\lambda)} \det W'_U}. \quad (\text{A.43})$$

In the background gauge,

$$\begin{aligned} \det \Psi_{\ell, \text{BG}, \text{alt}}^{(SL\varphi)}(r \rightarrow \infty) &= \det \left(\Psi_{\ell, \text{BG}, \text{alt}}^{(SL\varphi)(1)} \quad \Psi_{\ell, \text{BG}, \text{alt}}^{(SL\varphi)(2)} \quad \Psi_{\ell, \text{BG}, \text{alt}}^{(SL\varphi)(3)} \right) \\ &\simeq \left(-\sqrt{\xi} \right)^{n_B} \frac{1}{\det \widehat{W}} \left(\frac{\ell[\ell + \xi(\ell+2)]}{4(\ell+2)r} \right)^{n_U} \frac{[\det \Psi_{\ell, \text{alt}}^{(c\bar{c})}]^2 \det \Psi_{\ell, \text{alt}}^{(\eta\lambda)}}{\det W'_U}, \end{aligned} \quad (\text{A.44})$$

and

$$\begin{aligned} \det \hat{\Psi}_{\ell, \text{BG}}^{(SL\varphi)}(r \rightarrow \infty) &= \det \hat{\Psi}_{\ell}^{(B)} \det \hat{\Psi}_{\ell}^{(U)} \det \hat{\Psi}_{\ell}^{(\lambda)} \\ &\simeq \left(-\sqrt{\xi} \right)^{n_B} \frac{1}{\det \widehat{W}} \left[r^{-\ell n_U} \det \hat{\Psi}_{\ell}^{(c\bar{c})} \right]^2 \det \hat{\Psi}_{\ell}^{(\eta)} \left(-\frac{\ell + \xi(\ell+2)}{2(\ell+2)} r^{2\ell} \right)^{n_U} \det \hat{\Psi}_{\ell}^{(\lambda)}. \end{aligned} \quad (\text{A.45})$$

Here, we have used $W_B(r \rightarrow \infty) = \widehat{W}$. Thus,

$$\frac{\det \Psi_{\ell, \text{BG}, \text{alt}}^{(SL\varphi)}(r \rightarrow \infty)}{\det \hat{\Psi}_{\ell, \text{BG}}^{(SL\varphi)}(r \rightarrow \infty)} = \lim_{r \rightarrow \infty} \left(-\frac{\ell}{2r_{\infty}} \right)^{n_U} \frac{\det \Psi_{\ell, \text{alt}}^{(\eta\lambda)}}{\det \hat{\Psi}_{\ell}^{(\eta)} \det \hat{\Psi}_{\ell}^{(\lambda)} \det W'_U} \left(\frac{\det \Psi_{\ell, \text{alt}}^{(c\bar{c})}}{\det \hat{\Psi}_{\ell}^{(c\bar{c})}} \right)^2, \quad (\text{A.46})$$

and hence

$$\frac{\det \Psi_{\ell, \text{BG}}^{(SL\varphi)}(r \rightarrow \infty)}{\det \widehat{\Psi}_{\ell, \text{BG}}^{(SL\varphi)}(r \rightarrow \infty)} = \left(-\frac{\ell}{2r_\infty}\right)^{n_U} \frac{\det \Psi_{\ell}^{(\eta\lambda)}}{\det \widehat{\Psi}_{\ell}^{(\eta)} \det \widehat{\Psi}_{\ell}^{(\lambda)} \det W'_U} \left(\frac{\det \Psi_{\ell}^{(c\bar{c})}}{\det \widehat{\Psi}_{\ell}^{(c\bar{c})}}\right)^2. \quad (\text{A.47})$$

From eqs. (4.5), (4.7) and (4.10),

$$\lim_{r \rightarrow 0} \frac{\det \Psi_{\ell}^{(SL\varphi)}(r)}{\det \widehat{\Psi}_{\ell}^{(SL\varphi)}(r)} = \frac{\det \widehat{W}}{\sqrt{\det M_0^T M_0}}, \quad (\text{A.48})$$

independently of the choice of the gauge fixing. Taking the ratio between the determinants with $r \rightarrow 0$ and $r \rightarrow \infty$, we get eq. (4.32).

A.2.4 Functional determinant ($\ell = 1$)

In the Fermi gauge,

$$\begin{aligned} \det \Psi_{1, \text{reg, alt}}^{(SL\varphi)}(\nu; r \rightarrow \infty) &= \det \left(\Psi_{1, \text{alt}}^{(SL\varphi)(1)} \quad \Psi_{1, \text{alt}}^{(SL\varphi)(2)} \quad \Psi_{1, \text{alt}}^{(SL\varphi)(3)} + \nu \check{\Psi}_{1, \text{alt}}^{(SL\varphi)(3)} \right) \\ &\simeq \left(-\frac{\xi r}{6}\right)^{n_B} \left(\frac{1+3\xi}{12r}\right)^{n_U} r^{2n_G} \frac{\det [\Psi_{1, \text{alt}}^{(\eta\lambda)} + \nu \check{\Psi}_{1, \text{alt}}^{(\eta\lambda)}]}{\det W'_U} \\ &\quad + \mathcal{O}(\nu^2), \end{aligned} \quad (\text{A.49})$$

and

$$\det \widehat{\Psi}_{1, \text{reg}}^{(SL\varphi)}(\nu; r \rightarrow \infty) \simeq \left(-\frac{\xi r}{6}\right)^{n_B} r^{2n_B} \det \widehat{\Psi}_1^{(\eta)} \left(-\frac{1+3\xi}{6} r^2\right)^{n_U} \det \widehat{\Psi}_1^{(\lambda)} + \mathcal{O}(\nu). \quad (\text{A.50})$$

Thus,

$$\frac{\det \Psi_{1, \text{reg, alt}}^{(SL\varphi)}(\nu; r \rightarrow \infty)}{\det \widehat{\Psi}_{1, \text{reg}}^{(SL\varphi)}(\nu; r \rightarrow \infty)} = \left(-\frac{1}{2r_\infty}\right)^{n_U} \frac{\det [\Psi_{1, \text{alt}}^{(\eta\lambda)} + \nu \check{\Psi}_{1, \text{alt}}^{(\eta\lambda)}]}{\det \widehat{\Psi}_1^{(\eta)} \det \widehat{\Psi}_1^{(\lambda)} \det W'_U} + \mathcal{O}(\nu^2, \mathcal{E}\nu), \quad (\text{A.51})$$

where

$$\mathcal{E} = \lim_{\nu \rightarrow 0} \frac{\det \Psi_{1, \text{reg, alt}}^{(SL\varphi)}(\nu; r \rightarrow \infty)}{\det \widehat{\Psi}_{1, \text{reg}}^{(SL\varphi)}(\nu; r \rightarrow \infty)}. \quad (\text{A.52})$$

Notice that $\mathcal{E}$ can be non-vanishing when the translational zero modes disappear in the alternative fluctuation operator. As $r_* \rightarrow \infty$, $\mathcal{E}$ should approach zero. Keeping only the leading terms in r , we can interchange $r \rightarrow \infty$ and $r_* \rightarrow \infty$. Then,

$$\lim_{\nu \rightarrow 0} \frac{1}{\nu} \frac{\det \Psi_1^{(SL\varphi)}(\nu; r \rightarrow \infty)}{\det \widehat{\Psi}_1^{(SL\varphi)}(\nu; r \rightarrow \infty)} = \lim_{\nu \rightarrow 0} \frac{1}{\nu} \left(-\frac{1}{2r_\infty}\right)^{n_U} \frac{\det [\Psi_1^{(\eta\lambda)} + \nu \check{\Psi}_1^{(\eta\lambda)}]}{\det \widehat{\Psi}_1^{(\eta)} \det \widehat{\Psi}_1^{(\lambda)} \det W'_U}. \quad (\text{A.53})$$

In the background gauge,

$$\begin{aligned} \det \Psi_{1, \text{reg, BG, alt}}^{(SL\varphi)}(\nu; r \rightarrow \infty) &= \det \left(\Psi_{1, \text{BG, alt}}^{(SL\varphi)(1)} \quad \Psi_{1, \text{BG, alt}}^{(SL\varphi)(2)} \quad \Psi_{1, \text{BG, alt}}^{(SL\varphi)(3)} + \nu \check{\Psi}_{1, \text{BG, alt}}^{(SL\varphi)(3)} \right) \\ &\simeq \left(-\sqrt{\xi}\right)^{n_B} \frac{1}{\det \widehat{W}} \left(\frac{1+3\xi}{12r}\right)^{n_U} \frac{[\det \Psi_{1, \text{alt}}^{(c\bar{c})}]^2 \det [\Psi_{1, \text{alt}}^{(\eta\lambda)} + \nu \check{\Psi}_{1, \text{alt}}^{(\eta\lambda)}]}{\det W'_U} + \mathcal{O}(\nu^2), \end{aligned} \quad (\text{A.54})$$

and

$$\begin{aligned} \det \widehat{\Psi}_{1,\text{reg,BG}}^{(SL\varphi)}(\nu; r \rightarrow \infty) &= \det \widehat{\Psi}_1^{(B)} \det \widehat{\Psi}_1^{(U)} \det \widehat{\Psi}_1^{(\lambda)} \\ &\simeq \left(-\sqrt{\xi}\right)^{n_B} \frac{1}{\det \widehat{W}} \left[r^{-1n_U} \det \widehat{\Psi}_1^{(c\bar{c})}\right]^2 \det \widehat{\Psi}_1^{(\eta)} \left(-\frac{1+3\xi}{6}r^2\right)^{n_U} \det \widehat{\Psi}_1^{(\lambda)} + \mathcal{O}(\nu). \end{aligned} \quad (\text{A.55})$$

Here, we have used $W_B(r \rightarrow \infty) = \widehat{W}$. Thus,

$$\begin{aligned} \lim_{\nu \rightarrow 0} \frac{1}{\nu} \frac{\det \Psi_{1,\text{reg,BG,alt}}^{(SL\varphi)}(\nu; r \rightarrow \infty)}{\det \widehat{\Psi}_{1,\text{reg,BG}}^{(SL\varphi)}(\nu; r \rightarrow \infty)} \\ = \lim_{\nu \rightarrow 0} \lim_{r \rightarrow \infty} \frac{1}{\nu} \left(-\frac{1}{2r_\infty}\right)^{n_U} \frac{\det \left[\Psi_{1,\text{alt}}^{(\eta\lambda)} + \nu \check{\Psi}_{1,\text{alt}}^{(\eta\lambda)}\right]}{\det \widehat{\Psi}_1^{(\eta)} \det \widehat{\Psi}_1^{(\lambda)} \det W'_U} \left(\frac{\det \Psi_{1,\text{alt}}^{(c\bar{c})}}{\det \widehat{\Psi}_1^{(c\bar{c})}}\right)^2, \end{aligned} \quad (\text{A.56})$$

and hence

$$\lim_{\nu \rightarrow 0} \frac{1}{\nu} \frac{\det \Psi_{1,\text{reg,BG}}^{(SL\varphi)}(\nu; r \rightarrow \infty)}{\det \widehat{\Psi}_{1,\text{reg,BG}}^{(SL\varphi)}(\nu; r \rightarrow \infty)} = \lim_{\nu \rightarrow 0} \frac{1}{\nu} \left(-\frac{1}{2r_\infty}\right)^{n_U} \frac{\det \left[\Psi_1^{(\eta\lambda)} + \nu \check{\Psi}_1^{(\eta\lambda)}\right]}{\det \widehat{\Psi}_1^{(\eta)} \det \widehat{\Psi}_1^{(\lambda)} \det W'_U} \left(\frac{\det \Psi_1^{(c\bar{c})}}{\det \widehat{\Psi}_1^{(c\bar{c})}}\right)^2. \quad (\text{A.57})$$

Taking the ratio between the determinants with $r \rightarrow 0$ and $r \rightarrow \infty$, we obtain eq. (5.68).

A.3 Evaluation of solutions for $\ell = 0$

A.3.1 Behavior at infinity

The behavior of $\Psi_{0,\text{alt}}^{(c\bar{c})}$ and $\Psi_{0,\text{alt}}^{(\lambda)}$ is given by

$$\Psi_{0,\text{alt}}^{(c\bar{c})}(r \rightarrow \infty) \simeq \begin{pmatrix} \frac{e\sqrt{\xi}W_B r}{r^{3/2}} I_B & 0 \\ 0 & I_U + \mathcal{O}(r^{-2}) \end{pmatrix} \mathcal{T}_0^{(c\bar{c})}, \quad (\text{A.58})$$

and

$$\Psi_{0,\text{alt}}^{(\lambda)}(r \rightarrow \infty) \simeq \frac{e^{mr}}{r^{3/2}} I_H \mathcal{T}_0^{(\lambda)}, \quad (\text{A.59})$$

where $\mathcal{T}_0^{(c\bar{c})}$ and $\mathcal{T}_0^{(\lambda)}$ are constant square matrices.

Next, we consider the behavior of $\Psi_{0,\text{alt}}^{(SL\varphi)}$ at $r \rightarrow \infty$.

- Type 1

In the Fermi gauge, $\chi^{(1)} = I_G$. Then, from eq. (3.13),

$$\Psi_{0,\text{alt}}^{(S\varphi)(1)}(r \rightarrow \infty) \simeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ W_B & 0 \\ 0 & W_U \\ 0 & 0 \end{pmatrix}}_{\substack{n_B \\ n_U}} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\substack{n_B \\ n_U \\ n_U \\ n_H}}. \quad (\text{A.60})$$

In the background gauge, $\chi^{(1)} = \Psi_{0,\text{alt}}^{(c\bar{c})}$, and

$$\Psi_{0,\text{BG,alt}}^{(S\varphi)(1)}(r \rightarrow \infty) \simeq \underbrace{\begin{pmatrix} \sqrt{\xi}W_B & 0 \\ 0 & I_U \partial_r \\ W_B & 0 \\ 0 & W_U \\ 0 & 0 \end{pmatrix}}_{\substack{n_B \\ n_U}} \left\{ \begin{matrix} n_B \\ n_U \\ n_B \\ n_U \\ n_H \end{matrix} \right\} \Psi_{0,\text{alt}}^{(c\bar{c})}. \quad (\text{A.61})$$

- Type 2

There are some useful relations:

$$\frac{1}{r^3} \partial_r r^3 \Psi_0^{(\text{top})} = -\xi Y, \quad (\text{A.62})$$

$$\Psi_0^{(\text{bot})} = \lambda + M\Gamma, \quad (\text{A.63})$$

where Y is given in eq. (A.30) and Γ is a function obeying

$$\partial_r \Gamma = \Psi_0^{(\text{top})} + (M^T M)^{-1} [\partial_r \zeta + 2(M')^T \lambda]. \quad (\text{A.64})$$

In the Fermi gauge, $\zeta^{(2)} = I_G$. From eqs. (A.62) and (A.63), we get

$$\Psi_{0,\text{alt}}^{(S\varphi)(2)}(r \rightarrow \infty) \simeq \underbrace{\begin{pmatrix} -\frac{\xi}{4} r I_G \\ -\frac{\xi}{8} r^2 W \\ 0 \end{pmatrix}}_{n_G} \left\{ \begin{matrix} n_G \\ n_G \\ n_H \end{matrix} \right\}. \quad (\text{A.65})$$

In the background gauge, $\zeta^{(2)} = \Psi_{0,\text{alt}}^{(c\bar{c})}$, and

$$\chi^{(2)}(r \rightarrow \infty) \simeq \underbrace{\begin{pmatrix} \mathcal{O}(W'_B) & 0 \\ 0 & -W_U^{-2} \end{pmatrix}}_{\substack{n_B \\ n_U}} \left\{ \begin{matrix} n_B \\ n_U \end{matrix} \right\} \Psi_{0,\text{alt}}^{(c\bar{c})}. \quad (\text{A.66})$$

The behavior of W_U is given by

$$W_U(r \rightarrow \infty) \propto \frac{e^{W'_U(W_U)^{-1}r}}{r^{3/2}}. \quad (\text{A.67})$$

where $Z_B^{(A)}$ and $Z_U^{(A)}$ become constant at $r \rightarrow \infty$. Then, from eqs. (A.62) and (A.63), we obtain

$$\check{\Psi}_{0,\text{alt}}^{(S\varphi)(1U)}(r \rightarrow \infty) \simeq \underbrace{\begin{pmatrix} \frac{\xi}{4r} Z_B^{(A)} \\ \frac{\xi}{4r} Z_U^{(A)} \\ \frac{\xi}{4} W_B Z_B^{(A)} \ln r \\ -\frac{1}{2r^3} (W_U')^{-1} Z_U^{(A)} \\ 0 \end{pmatrix}}_{n_U} \left\{ \begin{array}{l} n_B \\ n_U \\ n_B \\ n_U \\ n_H \end{array} \right\}, \quad (\text{A.74})$$

with which we define

$$\check{\Psi}_{0,\text{alt}}^{(S\varphi)(1U)} = \left(\check{\psi}_{\text{alt}}^{(1)} \cdots \check{\psi}_{\text{alt}}^{(n_U)} \right). \quad (\text{A.75})$$

Let us move on to the background gauge. The gauge zero modes are given by

$$\psi_{\text{BG,alt}}^{(A)} = \begin{pmatrix} I_G \partial_r \\ M_{\text{alt}} \end{pmatrix} \Psi_{0,\text{alt}}^{(c\bar{c})} \mathcal{U}_{\text{BG}}^{(A)}, \quad (\text{A.76})$$

where

$$\mathcal{U}_{\text{BG}}^{(A)} = \left[\mathcal{T}_0^{(c\bar{c})} \right]^{-1} \mathcal{U}^{(A)}. \quad (\text{A.77})$$

We need to solve

$$\mathcal{M}_{0,\text{BG,alt}}^{(S\varphi)} \check{\psi}_{0,\text{BG,alt}}^{(A)} = -\psi_{0,\text{BG,alt}}^{(A)}. \quad (\text{A.78})$$

Notably, $\check{\psi}_{0,\text{BG,alt}}^{(A)}$ can be expressed as eq. (5.33) with $\check{\zeta} = \check{\lambda} = 0$, while $\check{\chi}^{(A)}$ satisfies

$$\Delta_0 \check{\chi}^{(A)} = \Delta_0 \chi|_{\chi \rightarrow \check{\chi}^{(A)}, \zeta \rightarrow 0, \lambda \rightarrow 0, M \rightarrow M_{\text{alt}}, \Omega \rightarrow \Omega_{\text{alt}}} + \Psi_{0,\text{alt}}^{(c\bar{c})} \mathcal{U}_{\text{BG}}^{(A)}. \quad (\text{A.79})$$

The following relation also holds.

$$\frac{1}{r^3} \partial_r r^3 \check{\Psi}_0^{(\text{top})(A)} = \xi M^T M \check{\chi}^{(A)} + \Psi_0^{(c\bar{c})} \mathcal{U}_{\text{BG}}^{(A)}. \quad (\text{A.80})$$

From eq. (A.79),

$$\check{\chi}^{(A)}(r \rightarrow \infty) = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & \frac{r^2}{8} I_U \end{pmatrix}}_{\substack{n_B \\ n_U}} \left\{ \begin{array}{l} n_B \\ n_U \end{array} \right\} \Psi_{0,\text{alt}}^{(c\bar{c})} \mathcal{U}_{\text{BG}}^{(A)}. \quad (\text{A.81})$$

From eqs. (A.80) and (A.63), we obtain

$$\check{\psi}_{0,\text{BG,alt}}^{(A)}(r \rightarrow \infty) \simeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & \frac{r}{4} I_U \\ 0 & 0 \\ 0 & \frac{r^2}{8} W_U \\ 0 & 0 \end{pmatrix}}_{\substack{n_B \\ n_U}} \left\{ \begin{array}{l} n_B \\ n_U \\ n_B \\ n_U \\ n_H \end{array} \right\} \Psi_{0,\text{alt}}^{(c\bar{c})} \mathcal{U}_{\text{BG}}^{(A)}, \quad (\text{A.82})$$

with which we define

$$\check{\Psi}_{0,\text{BG,alt}}^{(S\varphi)(1U)} = \sum_B \check{\psi}_{0,\text{BG,alt}}^{(B)} \mathcal{U}^{(B)T} \mathcal{T}_0^{(c\bar{c})}. \quad (\text{A.83})$$

A.3.3 Functional determinant

Here, we calculate the determinant of the solutions obtained above.

In the Fermi gauge,

$$\begin{aligned} \det \Psi_{0,\text{reg,alt}}^{(S\varphi)}(\nu; r \rightarrow \infty) &= \det \left(\Psi_{0,\text{alt}}^{(S\varphi)(1B)} \Psi_{0,\text{alt}}^{(S\varphi)(1U)} + \nu \check{\Psi}_{0,\text{alt}}^{(S\varphi)(1U)} \Psi_{0,\text{alt}}^{(S\varphi)(2)} \Psi_{0,\text{alt}}^{(S\varphi)(3)} \right) \\ &\quad + \mathcal{O}(\nu^{n_U+1}) \\ &\simeq \nu^{n_U} \left(\frac{\xi}{4} r \right)^{n_B} \det W_B \left(-\frac{\xi}{8r^2} \right)^{n_U} \frac{\det Z_U}{\det W'_U} \det \Psi_{0,\text{alt}}^{(\lambda)} + \mathcal{O}(\nu^{n_U+1}), \end{aligned} \quad (\text{A.84})$$

and

$$\begin{aligned} \det \widehat{\Psi}_{0,\text{reg}}^{(S\varphi)}(\nu; r \rightarrow \infty) &= \det \Psi_0^{(B)} \det \Psi_0^{(U)} \det \Psi_0^{(\lambda)} \\ &\simeq \left(\frac{\xi}{4} r \right)^{n_B} \det \widehat{W} \left(\frac{\xi}{4} r \right)^{n_U} \det \widehat{\Psi}_0^{(\lambda)} + \mathcal{O}(\nu). \end{aligned} \quad (\text{A.85})$$

Thus,

$$\frac{1}{\nu^{n_U}} \frac{\det \Psi_{0,\text{reg,alt}}^{(S\varphi)}(\nu; r \rightarrow \infty)}{\det \widehat{\Psi}_{0,\text{reg}}^{(S\varphi)}(\nu; r \rightarrow \infty)} = \left(-\frac{1}{2r_\infty^3} \right)^{n_U} \frac{\det Z_U}{\det W'_U} \frac{\det \Psi_{0,\text{alt}}^{(\lambda)}}{\det \widehat{\Psi}_0^{(\lambda)}} + \mathcal{O}(\nu), \quad (\text{A.86})$$

and

$$\lim_{\nu \rightarrow 0} \frac{1}{\nu^{n_U}} \frac{\det \Psi_{0,\text{reg}}^{(S\varphi)}(\nu; r \rightarrow \infty)}{\det \widehat{\Psi}_{0,\text{reg}}^{(S\varphi)}(\nu; r \rightarrow \infty)} = \left(-\frac{1}{2r_\infty^3} \right)^{n_U} \frac{\det \mathcal{X}_U}{\det W'_U} \frac{\det \Psi_0^{(\lambda)}}{\det \widehat{\Psi}_0^{(\lambda)}}. \quad (\text{A.87})$$

In the background gauge,

$$\begin{aligned} \det \Psi_{0,\text{reg,BG,alt}}^{(SL\varphi)}(\nu; r \rightarrow \infty) &= \det \left(\Psi_{0,\text{BG,alt}}^{(S\varphi)(1)} + \nu \check{\Psi}_{0,\text{BG,alt}}^{(S\varphi)(1)} \Psi_{0,\text{BG,alt}}^{(S\varphi)(2)} \Psi_{0,\text{BG,alt}}^{(S\varphi)(3)} \right) + \mathcal{O}(\nu^{n_U+1}) \\ &\simeq \nu^{n_U} \left(\sqrt{\xi} \right)^{n_B} \left(-\frac{\xi}{8r^2} \right)^{n_U} \frac{\det \mathcal{K}_{\text{alt}}}{\det W'_U} \left[\det \Psi_{0,\text{alt}}^{(c\bar{c})} \right]^2 \det \Psi_{\text{alt}}^{(\lambda)} + \mathcal{O}(\nu^{n_U+1}), \end{aligned} \quad (\text{A.88})$$

and

$$\begin{aligned} \det \widehat{\Psi}_{0,\text{reg,BG}}^{(SL\varphi)}(\nu; r \rightarrow \infty) &= \det \Psi_0^{(B)} \det \Psi_0^{(U)} \det \Psi_0^{(\lambda)} \\ &\simeq \left(\sqrt{\xi} \right)^{n_B} \left[\det \widehat{\Psi}_0^{(c\bar{c})} \right]^2 \left(\frac{\xi}{4} r \right)^{n_U} \det \widehat{\Psi}_0^{(\lambda)} + \mathcal{O}(\nu). \end{aligned} \quad (\text{A.89})$$

Thus,

$$\frac{1}{\nu^{n_U}} \frac{\det \Psi_{0,\text{reg,BG,alt}}^{(S\varphi)}(\nu; r \rightarrow \infty)}{\det \widehat{\Psi}_{0,\text{reg,BG}}^{(S\varphi)}(\nu; r \rightarrow \infty)} = \left(-\frac{1}{2r_\infty^3} \right)^{n_U} \frac{\det \mathcal{K}_{\text{alt}}}{\det W'_U} \frac{\det \Psi_{0,\text{alt}}^{(\lambda)}}{\det \widehat{\Psi}_0^{(\lambda)}} \left(\frac{\det \Psi_{0,\text{alt}}^{(c\bar{c})}}{\det \widehat{\Psi}_0^{(c\bar{c})}} \right)^2 + \mathcal{O}(\nu), \quad (\text{A.90})$$

and

$$\lim_{\nu \rightarrow 0} \frac{1}{\nu^{n_U}} \frac{\det \Psi_{0,\text{reg,BG}}^{(S\varphi)}(\nu; r \rightarrow \infty)}{\det \widehat{\Psi}_{0,\text{reg,BG}}^{(S\varphi)}(\nu; r \rightarrow \infty)} = \left(-\frac{1}{2r_\infty^3} \right)^{n_U} \frac{\det \mathcal{K}}{\det W'_U} \frac{\det \Psi_0^{(\lambda)}}{\det \widehat{\Psi}_0^{(\lambda)}} \left(\frac{\det \Psi_0^{(c\bar{c})}}{\det \widehat{\Psi}_0^{(c\bar{c})}} \right)^2. \quad (\text{A.91})$$

From eqs. (4.33), (4.34) and (4.35),

$$\lim_{r \rightarrow 0} \frac{\det \Psi_0^{(S\varphi)}(r)}{\det \widehat{\Psi}_0^{(S\varphi)}(r)} = \frac{\sqrt{\det M_0^T M_0}}{\det \widehat{W}}, \quad (\text{A.92})$$

independently of the choice of the gauge fixing. Based on the above results, eq. (5.51) is obtained.

B Use of alternative fluctuation operators

In this appendix, we justify the use of the alternative fluctuation operators in the evaluation of the determinants at $r \rightarrow \infty$.

B.1 General discussion

B.1.1 Setup

We compare the determinants of the two functions, $\mathfrak{F}$ and $\mathfrak{F}_{\text{alt}}$, which satisfy

$$\left[-\partial_r^2 - \Lambda(r)\partial_r - \delta\Lambda(r)\partial_r + \Xi(r) + \delta\Xi(r) \right] \mathfrak{F}(r) = 0, \quad (\text{B.1})$$

$$\left[-\partial_r^2 - \Lambda(r)\partial_r + \Xi(r) \right] \mathfrak{F}_{\text{alt}}(r) = 0, \quad (\text{B.2})$$

where $\Lambda, \delta\Lambda, \Xi, \delta\Xi, \mathfrak{F}$ and $\mathfrak{F}_{\text{alt}}$ are $n \times n$ matrices. The support of $\delta\Lambda(r)$ and $\delta\Xi(r)$ is $r > r_*$ and we take

$$\mathfrak{F}(r) = \mathfrak{F}_{\text{alt}}(r) \quad : \quad r < r_*. \quad (\text{B.3})$$

We assume that there exist constants, $C_{\mathfrak{F}}, C_{\delta\Lambda}$ and $C_{\delta\Xi}$, such that

$$\|\mathfrak{F}'_{\text{alt}}(r)\mathfrak{F}_{\text{alt}}^{-1}(r)\| < C_{\mathfrak{F}}, \quad \|\delta\Lambda(r)\| < \frac{C_{\delta\Lambda}}{r^3}, \quad \|\delta\Xi(r)\| < \frac{C_{\delta\Xi}}{r^3}, \quad (\text{B.4})$$

where $\|A\|$ is the induced (spectral) norm of the real matrix A , which is defined as

$$\|A\| = \max_{x \in \mathbb{R}^n, |x| \neq 0} \frac{|Ax|}{|x|}, \quad (\text{B.5})$$

with $|x| = \sqrt{\sum_i x_i^2}$. In addition, we assume

$$\mu\left(-\mathfrak{F}'_{\text{alt}}(r)\mathfrak{F}_{\text{alt}}^{-1}(r)\right) \leq 0, \quad \mu(-\Lambda(r)) < -\frac{k}{r}, \quad (\text{B.6})$$

where $k > 2$ is a non-integer constant and $\mu(A)$ is the logarithmic norm of matrix A , which is defined as

$$\mu(A) = \lim_{h \rightarrow 0^+} \frac{\|I + hA\| - 1}{h}. \quad (\text{B.7})$$

Notice that $\mu(A)$ gives the largest eigenvalue of $\frac{1}{2}(A + A^T)$.

The goal of this subsection is to show that the following quantity has an upper bound:

$$\lim_{r \rightarrow \infty} \left| \ln \frac{\det \mathfrak{F}(r)}{\det \mathfrak{F}_{\text{alt}}(r)} \right|.$$

B.1.2 Recursive formula

Let us construct a formal solution of eq. (B.1) using $\mathfrak{F}_{\text{alt}}$. We express $\mathfrak{F}$ as

$$\mathfrak{F}(r) = \mathfrak{F}_{\text{alt}}(r) \mathcal{P} \exp \left[\int_{r_*}^r ds \mathfrak{F}_{\text{alt}}^{-1}(s) \Theta(s) \mathfrak{F}_{\text{alt}}(s) \right], \quad (\text{B.8})$$

where Θ is a function satisfying

$$\Theta' + \left(\mathfrak{F}_{\text{alt}}' \mathfrak{F}_{\text{alt}}^{-1} + \Lambda \right) \Theta + \Theta \mathfrak{F}_{\text{alt}}' \mathfrak{F}_{\text{alt}}^{-1} = \delta \Xi - \delta \Lambda \mathfrak{F}_{\text{alt}}' \mathfrak{F}_{\text{alt}}^{-1} - \Theta^2 - \delta \Lambda \Theta. \quad (\text{B.9})$$

Here, the path-ordered exponential of matrix $A(s)$ is defined as¹²

$$\mathcal{P} \exp \left[\int_{r_i}^{r_f} ds A(s) \right] = 1 + \sum_{p=1}^{\infty} \int_{r_i}^{r_f} ds_p \int_{r_i}^{s_p} ds_{p-1} \cdots \int_{r_i}^{s_2} ds_1 A(s_p) A(s_{p-1}) \cdots A(s_1). \quad (\text{B.10})$$

Treating $\delta \Lambda$ and $\delta \Xi$ as perturbations, we expand Θ as

$$\Theta = \Theta^{(1)} + \Theta^{(2)} + \cdots, \quad (\text{B.11})$$

where $\Theta^{(p)}$ is the p -th order term with respect to fluctuations. Formally, we obtain

$$\Theta^{(p)}(r) = \int_{r_*}^r ds \mathcal{P} \exp \left[\int_s^r dt \left(-\mathfrak{F}_{\text{alt}}'(t) \mathfrak{F}_{\text{alt}}^{-1}(t) - \Lambda(t) \right) \right] \Delta^{(p)}(s) \mathcal{P} \exp \left[\int_r^s dt \mathfrak{F}_{\text{alt}}'(t) \mathfrak{F}_{\text{alt}}^{-1}(t) \right], \quad (\text{B.12})$$

where

$$\Delta^{(p)} = \begin{cases} \delta \Xi - \delta \Lambda \mathfrak{F}_{\text{alt}}' \mathfrak{F}_{\text{alt}}^{-1} & : p = 1 \\ -\delta \Lambda \Theta^{(p-1)} - \sum_{i=1}^{p-1} \Theta^{(i)} \Theta^{(p-i)} & : p > 1 \end{cases}. \quad (\text{B.13})$$

This recursive formula can be solved order by order and we can obtain Θ if the sum over p converges.

B.1.3 Error evaluation formula

Let us first evaluate the right path-ordered exponential in eq. (B.12). Using eq. (B.5) as well as

$$\mathcal{P} \exp \left[\int_r^s dt \mathfrak{F}_{\text{alt}}'(t) \mathfrak{F}_{\text{alt}}^{-1}(t) \right] y_R(r) = y_R(s), \quad (\text{B.14})$$

the following relation holds:

$$\left\| \mathcal{P} \exp \left[\int_r^s dt \mathfrak{F}_{\text{alt}}'(t) \mathfrak{F}_{\text{alt}}^{-1}(t) \right] \right\| = \max_{|y_R(r)| \neq 0} \frac{|y_R(s)|}{|y_R(r)|}, \quad (\text{B.15})$$

¹²One may also write the path-ordered exponential as

$$\mathcal{P} \exp \left[\int_{r_i}^{r_f} ds A(s) \right] = \lim_{N \rightarrow \infty} e^{A(r_f) \delta s} e^{A(r_f - \delta s) \delta s} e^{A(r_f - 2\delta s) \delta s} \cdots e^{A(r_f - N\delta s) \delta s},$$

with

$$\delta s = \frac{r_f - r_i}{N}.$$

where $y_R(t)$ satisfies

$$y'_R(t) = \mathfrak{F}'_{\text{alt}}(t) \mathfrak{F}_{\text{alt}}^{-1}(t) y_R(t). \quad (\text{B.16})$$

Then,

$$\begin{aligned} D^- |y_R(t)| &= \lim_{h \rightarrow 0^+} \frac{|y_R(t)| - |y_R(t-h)|}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{|y_R(t)| - |y_R(t) - h \mathfrak{F}'_{\text{alt}}(t) \mathfrak{F}_{\text{alt}}^{-1}(t) y_R(t)|}{h} \\ &\geq -\mu \left(-\mathfrak{F}'_{\text{alt}}(t) \mathfrak{F}_{\text{alt}}^{-1}(t) \right) |y_R(t)|, \end{aligned} \quad (\text{B.17})$$

where D^- is the left-hand derivative. After integration, we obtain

$$\ln \frac{|y_R(r)|}{|y_R(s)|} \geq - \int_s^r dt \mu \left(-\mathfrak{F}'_{\text{alt}}(t) \mathfrak{F}_{\text{alt}}^{-1}(t) \right) \geq 0, \quad (\text{B.18})$$

resulting in

$$\left\| \mathcal{P} \exp \left[\int_r^s dt \mathfrak{F}'_{\text{alt}}(t) \mathfrak{F}_{\text{alt}}^{-1}(t) \right] \right\| \leq 1. \quad (\text{B.19})$$

Similarly, we evaluate the left path-ordered exponential in eq. (B.12). From eq. (B.5),

$$\left\| \mathcal{P} \exp \left[\int_s^r dt \left(-\mathfrak{F}'_{\text{alt}}(t) \mathfrak{F}_{\text{alt}}^{-1}(t) - \Lambda(t) \right) \right] \right\| = \max_{|y_L(s)| \neq 0} \frac{|y_L(r)|}{|y_L(s)|}, \quad (\text{B.20})$$

with

$$y'_L(t) = \left(-\mathfrak{F}'_{\text{alt}}(t) \mathfrak{F}_{\text{alt}}^{-1}(t) - \Lambda(t) \right) y_L(t). \quad (\text{B.21})$$

Then, we obtain

$$\begin{aligned} D^+ |y_L(t)| &= \lim_{h \rightarrow 0^+} \frac{|y_L(t+h)| - |y_L(t)|}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{|y_L(t) + h \left(-\mathfrak{F}'_{\text{alt}}(t) \mathfrak{F}_{\text{alt}}^{-1}(t) - \Lambda(t) \right) y_L(t)| - |y_L(t)|}{h} \\ &\leq \mu \left(-\mathfrak{F}'_{\text{alt}}(t) \mathfrak{F}_{\text{alt}}^{-1}(t) - \Lambda(t) \right) |y_L(t)|, \end{aligned} \quad (\text{B.22})$$

where D^+ is the right-hand derivative, based on which we find

$$\ln \frac{|y_L(r)|}{|y_L(s)|} \leq \int_r^s dt \mu \left(-\mathfrak{F}'_{\text{alt}}(t) \mathfrak{F}_{\text{alt}}^{-1}(t) \right) + \int_r^s dt \mu \left(-\Lambda(t) \right) < -k \ln \left(\frac{r}{s} \right). \quad (\text{B.23})$$

Thus, the following inequality holds:

$$\left\| \mathcal{P} \exp \left[\int_s^r dt \left(-\mathfrak{F}'_{\text{alt}}(t) \mathfrak{F}_{\text{alt}}^{-1}(t) - \Lambda(t) \right) \right] \right\| < \left(\frac{s}{r} \right)^k. \quad (\text{B.24})$$

Using eqs. (B.19) and (B.24), we obtain

$$\|\Theta^{(p)}(r)\| < \int_{r_*}^r ds \left(\frac{s}{r} \right)^k \|\Delta^{(p)}(s)\|. \quad (\text{B.25})$$

Now, we show that $||\Theta^{(p)}||$ is bounded from above as

$$||\Theta^{(p)}(r)|| < C^{(p)} \frac{r_*}{r^2}, \quad (\text{B.26})$$

where $C^{(p)}$'s are positive constants. Indeed, it is the case for $p = 1$ as

$$\begin{aligned} ||\Theta^{(1)}(r)|| &< \int_{r_*}^r ds \left(\frac{s}{r}\right)^k ||\Delta^{(1)}(s)|| \\ &< (C_{\delta\Xi} + C_{\delta\Lambda} C_{\mathfrak{F}}) \int_{r_*}^r ds \left(\frac{s}{r}\right)^k \frac{1}{s^3} \\ &= \frac{C_{\delta\Xi} + C_{\delta\Lambda} C_{\mathfrak{F}}}{k-2} \frac{1}{r^2} \left(1 - \frac{r_*^{k-2}}{r^{k-2}}\right) \\ &< C^{(1)} \frac{r_*}{r^2}, \end{aligned} \quad (\text{B.27})$$

where

$$C^{(1)} = \frac{C_{\delta\Xi} + C_{\delta\Lambda} C_{\mathfrak{F}}}{(k-2)r_*}. \quad (\text{B.28})$$

Next, let us assume $||\Theta^{(q)}(r)|| < C^{(q)} \frac{r_*}{r^2}$ for all $q < p$, and show that the inequality (B.26) holds. Indeed,

$$\begin{aligned} ||\Theta^{(p)}(r)|| &< \int_{r_*}^r ds \left(\frac{s}{r}\right)^k ||\Delta^{(p)}(s)|| \\ &< \int_{r_*}^r ds \left(\frac{s}{r}\right)^k \left(r_* \frac{C_{\delta\Lambda} C^{(p-1)}}{s^5} + \sum_{i=1}^{p-1} r_*^2 \frac{C^{(i)} C^{(p-i)}}{s^4} \right) \\ &= \frac{C_{\delta\Lambda} C^{(p-1)}}{k-4} r_* \left(\frac{1}{r^4} - \frac{r_*^{k-4}}{r^k} \right) + \sum_{i=1}^{p-1} \frac{C^{(i)} C^{(p-i)}}{k-3} r_*^2 \left(\frac{1}{r^3} - \frac{r_*^{k-3}}{r^k} \right) \\ &< C^{(p)} \frac{r_*}{r^2}, \end{aligned} \quad (\text{B.29})$$

where

$$C^{(p)} = \left(\frac{C_{\delta\Lambda} C^{(p-1)}}{r_*^2 |k-4|} + \sum_{i=1}^{p-1} \frac{C^{(i)} C^{(p-i)}}{|k-3|} \right). \quad (\text{B.30})$$

Thus, the inequality (B.26) is valid for all p .

Finally, we evaluate an upper bound on $||\Theta(r)||$. Let us define

$$C^{\text{tot}} = \sum_{p=1}^{\infty} C^{(p)}. \quad (\text{B.31})$$

Then, if it converges, we get

$$||\Theta(r)|| < C^{\text{tot}} \frac{r_*}{r^2}. \quad (\text{B.32})$$

In figure 1, we show the contours of constant C^{tot} , taking $k = 2.5$. As we can see from the figure, C^{tot} is actually convergent for small enough $C^{(1)}$ and $C_{\delta\Lambda}/r_*^2$, and C^{tot} approaches zero as $C^{(1)}$ and $C_{\delta\Lambda}/r_*^2$ go to zero.

Figure 1. Contours of constant C^{tot} , taking $k = 2.5$. The dashed lines indicate $C^{\text{tot}} = 0.1, 0.01$, and 0.001 . In the gray region, C^{tot} diverges.

Thus, if C^{tot} converges, we obtain

$$\lim_{r \rightarrow \infty} \left| \ln \frac{\det \mathfrak{F}(r)}{\det \mathfrak{F}_{\text{alt}}(r)} \right| = \left| \int_{r_*}^{\infty} ds \text{Tr } \Theta(s) \right| \leq n \int_{r_*}^{\infty} ds \|\Theta(s)\| = n C^{\text{tot}}. \quad (\text{B.33})$$

B.2 Alternative fluctuation operators

Let us consider a one-parameter family of differential equations with different transition point, r_* , and we denote the solution as $\mathfrak{F}_{\text{alt}}(r_*; r)$. Then, $C_{\delta\Xi}, C_{\delta\Lambda}$ and $C_{\mathfrak{F}}$ are dependent on r_* . From the above discussion, sufficient conditions for

$$\lim_{r_* \rightarrow \infty} \lim_{r \rightarrow \infty} \left| \ln \frac{\det \mathfrak{F}(r)}{\det \mathfrak{F}_{\text{alt}}(r_*; r)} \right| = 1, \quad (\text{B.34})$$

are given by

$$\lim_{r_* \rightarrow \infty} \frac{C_{\delta\Xi}(r_*) + C_{\delta\Lambda}(r_*) C_{\mathfrak{F}}(r_*)}{r_*} = 0, \quad (\text{B.35})$$

$$\lim_{r_* \rightarrow \infty} \frac{C_{\delta\Lambda}(r_*)}{r_*^2} = 0. \quad (\text{B.36})$$

In this subsection, we give several general remarks. Then, in the following subsections, for $(SL\varphi)$ modes, $(c\bar{c})$ modes and $(\eta\lambda)$ modes, we construct $\mathfrak{F}_{\text{alt}}(r_*; r)$ explicitly and show that the above conditions as well as eq. (B.6) are satisfied.

B.2.1 Extended fluctuation operators

Since the conditions (B.4) are sensitive to the behavior of the fluctuation operators around $r \sim r_*$, we need to specify their deformation explicitly.

In particular, the differential equations for $\Psi_{\text{alt}}^{(\eta\lambda)}$ should be well-defined for all r , which requires that the decomposition of solutions should be exact for all r . The decomposition with the alternative fluctuation operators becomes exact when the following relations are satisfied;

$$(M_{\text{alt}}^T)' M_{\text{alt}} = M_{\text{alt}}^T M_{\text{alt}}', \quad (\text{B.37})$$

$$\Omega_{\text{alt}}^T = \Omega_{\text{alt}}, \quad (\text{B.38})$$

$$P_{H,\text{alt}}(-\Delta_0 M_{\text{alt}} + \Omega_{\text{alt}} M_{\text{alt}}) = 0. \quad (\text{B.39})$$

However, it is generally difficult to deform the fluctuation operators keeping these relations. In the following, we show a way to overcome it by extending the fluctuation operators.

We introduce n_G spectator scalars and extend M and Ω as

$$M^{\text{ext}} = \underbrace{\begin{pmatrix} M \\ 0 \end{pmatrix}}_{n_G} \Bigg\} \begin{matrix} n_G + n_H \\ n_G \end{matrix}, \quad (\text{B.40})$$

$$\Omega^{\text{ext}} = \underbrace{\begin{pmatrix} \Omega & 0 \\ 0 & \Omega_{\text{sp}} \end{pmatrix}}_{n_G + n_H} \underbrace{\Bigg\}}_{n_G} \begin{matrix} n_G + n_H \\ n_G \end{matrix}, \quad (\text{B.41})$$

where Ω_{sp} is an $n_G \times n_G$ matrix, which will be determined later. We take the same Ω_{sp} for the false vacuum. Thus, the spectator scalars do not affect the ratio of the determinants. We also introduce $P_{G,\text{alt}}^{\text{ext}}$ and $P_{H,\text{alt}}^{\text{ext}}$, which are defined as

$$P_{G,\text{alt}}^{\text{ext}} = M_{\text{alt}}^{\text{ext}} (M_{\text{alt}}^{\text{ext}T} M_{\text{alt}}^{\text{ext}})^2 M_{\text{alt}}^{\text{ext}T}, \quad (\text{B.42})$$

$$P_{H,\text{alt}}^{\text{ext}} = I_{\varphi}^{\text{ext}} - P_{G,\text{alt}}^{\text{ext}}, \quad (\text{B.43})$$

where I_{φ}^{ext} is the $(n_{\varphi} + n_G) \times (n_{\varphi} + n_G)$ identity matrix.

We assume that the deformation begins at $r = r_* = r_1$ and ends at $r = r_5$ with $r_5 - r_1 \ll r_*$. We construct the alternative matrices, $M_{\text{alt}}^{\text{ext}}$ and $\Omega_{\text{alt}}^{\text{ext}}$, which satisfy $M_{\text{alt}}^{\text{ext}}(r) = M^{\text{ext}}(r)$ and $\Omega_{\text{alt}}^{\text{ext}}(r) = \Omega^{\text{ext}}(r)$ for $r < r_1$. For $r_1 < r < r_5$, we deform them in the following way.

1. We express $M_{\text{alt}}^{\text{ext}}$ as

$$M_{\text{alt}}^{\text{ext}}(r) = \begin{pmatrix} M_{\text{alt}}(r) \\ \omega(r) \end{pmatrix}, \quad (\text{B.44})$$

where ω expresses the elements for the spectator scalars and $\omega(r_1) = 0$. Keeping the following relations:

$$\begin{aligned} \Omega_{\text{alt}}^{\text{ext}} &= \Omega^{\text{ext}} - P_{H,\text{alt}}^{\text{ext}} \left[-\Delta_0 M_{\text{alt}}^{\text{ext}} + \Omega^{\text{ext}} M_{\text{alt}}^{\text{ext}} \right] (M_{\text{alt}}^{\text{ext}T} M_{\text{alt}}^{\text{ext}})^{-1} M_{\text{alt}}^{\text{ext}T} \\ &\quad - M_{\text{alt}}^{\text{ext}} (M_{\text{alt}}^{\text{ext}T} M_{\text{alt}}^{\text{ext}})^{-1} \left[-\Delta_0 M_{\text{alt}}^{\text{ext}T} + M_{\text{alt}}^{\text{ext}T} \Omega^{\text{ext}} \right] P_{H,\text{alt}}^{\text{ext}}, \end{aligned} \quad (\text{B.45})$$

$$\Omega_{\text{sp}} = (\Delta_0 \omega) \omega^{-1}, \quad (\text{B.46})$$

we deform $M_{\text{alt}}^{\text{ext}}$ in the following steps.

- (a) $r_1 < r < r_2$: Turn on ω keeping $\omega \propto \omega(r_2)$.
- (b) $r_2 < r < r_3$: Rotate M_{alt} so that $M_{\text{alt}}^T(r_3) = (W, 0)$, where W is a diagonal matrix. During the rotation, we change ω so that

$$\omega^T \omega' - \omega'^T \omega = M_{\text{alt}}'^T M_{\text{alt}} - M_{\text{alt}}^T M_{\text{alt}}'. \quad (\text{B.47})$$

- (c) $r_3 < r < r_4$: Turn off ω keeping $\omega \propto \omega(r_3)$.

2. $r_4 < r < r_5$: Diagonalize $P_{H,\text{alt}}^{\text{ext}} \Omega_{\text{alt}}^{\text{ext}} P_{H,\text{alt}}^{\text{ext}}$.

Here, we take $r_{p+1} - r_p$ independently of r_* . One can easily check that eqs. (B.37), (B.38) and (B.39) hold at each step.

B.2.2 Linear approximation

To evaluate the effect of the deformation on the functional determinants, we construct each step of the deformation more concretely. Since the difference between the original matrices and the alternative matrices are expected to be very small, we can work in the linear approximation. As in the previous sections, we choose the basis of the fields so that

$$\lim_{r \rightarrow \infty} M(r) W^{-1}(r) = \begin{pmatrix} I_G \\ 0 \end{pmatrix}, \quad (\text{B.48})$$

$$\lim_{r \rightarrow \infty} \Omega(r) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{m}^2 \end{pmatrix}. \quad (\text{B.49})$$

Notice that the first condition is somewhat stronger than eq. (A.6) since some of the elements of $W^{-1}(r)$ diverge as $r \rightarrow \infty$. The existence of such a field basis is guaranteed by the one-to-one correspondence between the massive gauge bosons and the NG bosons.

Since O_φ and O_G are very close to the identity matrices for a large enough r_* , we approximate them as

$$O_\varphi(r_*) \simeq I_\varphi + \delta_\varphi, \quad (\text{B.50})$$

$$O_G(r_*) \simeq I_G + \delta_G, \quad (\text{B.51})$$

where δ_φ and δ_G are constant anti-symmetric matrices and satisfy

$$\|\delta_\varphi\|_{\text{max}} < \delta_{\text{max}}^2, \quad \|\delta_G\|_{\text{max}} < \delta_{\text{max}}^2, \quad (\text{B.52})$$

with $\|\cdots\|_{\text{max}}$ being the max norm and $\delta_{\text{max}} \ll 1$.

We also approximate that $M(r)$ and $\Omega(r)$ are almost constant during the deformation and express them as

$$M(r) \simeq M_* = (I_\varphi + \delta_\varphi) \begin{pmatrix} W_* \\ 0 \end{pmatrix} (I_G - \delta_G), \quad (\text{B.53})$$

$$\Omega(r) \simeq (I_\varphi + \delta_\varphi) \begin{pmatrix} 0 & 0 \\ 0 & m^2(r_*) \end{pmatrix} (I_\varphi - \delta_\varphi), \quad (\text{B.54})$$

where $W_* = W(r_*)$. From eq. (B.48), δ_G should satisfy

$$||W_* \delta_G W_*^{-1}||_{\max} \ll 1, \quad (\text{B.55})$$

for a large enough r_* .

We define sigmoid functions that have the following features:

$$\varsigma_p(r) \simeq \begin{cases} 0 & r \lesssim r_p \\ 1 & r \gtrsim r_{p+1} \end{cases}, \quad (\text{B.56})$$

$$\max_r |\partial_r^n \varsigma_p(r)| \sim (r_{p+1} - r_p)^{-n}. \quad (\text{B.57})$$

Then, we construct the deformation as follows.

- $r_1 < r < r_2$: We turn on ω as

$$\omega(r) = \varsigma_1(r) \delta_{\max}^{1/2} W_*. \quad (\text{B.58})$$

In this step, M_{alt} and $\Omega_{\text{alt}}^{\text{ext}}$ are constant.

- $r_2 < r < r_3$: We rotate M_{alt} and ω as

$$M_{\text{alt}}(r) = (I_\varphi - \varsigma_2(r) \delta_\varphi) M_* (I_G + \varsigma_2(r) \delta_G), \quad (\text{B.59})$$

$$\omega(r) = \sqrt{\delta_{\max}} (I_\varphi + \varepsilon_\varphi(r)) W_* (1 - \varepsilon_G(r)), \quad (\text{B.60})$$

where

$$\varepsilon_\varphi(r) = \frac{\varsigma_2(r)}{\delta_{\max}} \begin{pmatrix} I_G & 0 \end{pmatrix} \delta_\varphi \begin{pmatrix} I_G \\ 0 \end{pmatrix}, \quad (\text{B.61})$$

$$\varepsilon_G(r) = \frac{\varsigma_2(r)}{\delta_{\max}} \delta_G. \quad (\text{B.62})$$

Notice that the elements of ε_φ and ε_G are much smaller than one due to eq. (B.52). In this step, $\Omega_{\text{alt}}^{\text{ext}}$ is changed according to eq. (B.45).

- $r_3 < r < r_4$: We turn off ω as

$$\omega(r) = (1 - \varsigma_3(r)) \omega(r_3). \quad (\text{B.63})$$

In this step, M_{alt} and $\Omega_{\text{alt}}^{\text{ext}}$ are constant.

- $r_4 < r < r_5$: Finally, we rotate $P_{H,\text{alt}}^{\text{ext}} \Omega_{\text{alt}}^{\text{ext}} P_{H,\text{alt}}^{\text{ext}}$ as

$$P_{H,\text{alt}}^{\text{ext}} \Omega_{\text{alt}}^{\text{ext}} P_{H,\text{alt}}^{\text{ext}} = P_{H,\text{alt}}^{\text{ext}} \begin{pmatrix} I_\varphi - \varsigma_4(r) \delta_\varphi & 0 \\ 0 & I_G \end{pmatrix} \Omega_{\text{alt}}^{\text{ext}} \begin{pmatrix} I_\varphi + \varsigma_4(r) \delta_\varphi & 0 \\ 0 & I_G \end{pmatrix} P_{H,\text{alt}}^{\text{ext}}. \quad (\text{B.64})$$

Notice that we can still use δ_φ to diagonalize Ω since

$$P_{H,\text{alt}}^{\text{ext}}(r_4) \Omega_{\text{alt}}^{\text{ext}}(r_4) P_{H,\text{alt}}^{\text{ext}}(r_4) = P_{H,\text{alt}}^{\text{ext}}(r_4) \Omega^{\text{ext}}(r_4) P_{H,\text{alt}}^{\text{ext}}(r_4). \quad (\text{B.65})$$

In this step, $M_{\text{alt}}^{\text{ext}}$ is constant.

Finally, let us discuss the deformation of V_H^{ext} , which is defined as

$$V_H^{\text{ext}} = \begin{pmatrix} V_H & 0 \\ 0 & I_G \end{pmatrix} \equiv \begin{pmatrix} u_1^{\text{ext}} & \cdots & u_{n_H+n_G}^{\text{ext}} \end{pmatrix}. \quad (\text{B.66})$$

Its alternative can be defined in the following way. Since the difference between M^{ext} and $M_{\text{alt}}^{\text{ext}}$ is small, we can construct $(n_H + n_G)$ independent vectors as

$$\tilde{u}_{p,\text{alt}}^{\text{ext}} = \frac{P_{H,\text{alt}}^{\text{ext}} u_p^{\text{ext}}}{|P_{H,\text{alt}}^{\text{ext}} u_p^{\text{ext}}|}. \quad (\text{B.67})$$

Using Gram-Schmidt orthogonalization, we get an orthonormal basis, $u_{p,\text{alt}}^{\text{ext}}$, and define

$$V_{H,\text{alt}}^{\text{ext}} = \begin{pmatrix} u_{1,\text{alt}}^{\text{ext}} & \cdots & u_{n_H+n_G,\text{alt}}^{\text{ext}} \end{pmatrix}. \quad (\text{B.68})$$

B.3 $(SL\varphi)$ modes

In the rest of this appendix, we confirm eqs. (B.35) and (B.36) for each mode. We start with the $(SL\varphi)$ modes.

We consider the functions $\mathfrak{F}^{(SL\varphi)}$ and $\mathfrak{F}_{\text{alt}}^{(SL\varphi)}$ which obey

$$\begin{pmatrix} \sqrt{\xi} I_G & 0 & 0 \\ 0 & I_G & 0 \\ 0 & 0 & I_\varphi^{\text{ext}} \end{pmatrix} \mathcal{M}_\ell^{(SL\varphi)\text{ext}} \begin{pmatrix} \sqrt{\xi} I_G & 0 & 0 \\ 0 & I_G & 0 \\ 0 & 0 & I_\varphi^{\text{ext}} \end{pmatrix} \mathfrak{F}^{(SL\varphi)} = 0, \quad (\text{B.69})$$

$$\begin{pmatrix} \sqrt{\xi} I_G & 0 & 0 \\ 0 & I_G & 0 \\ 0 & 0 & I_\varphi^{\text{ext}} \end{pmatrix} \mathcal{M}_{\ell,\text{alt}}^{(SL\varphi)\text{ext}} \begin{pmatrix} \sqrt{\xi} I_G & 0 & 0 \\ 0 & I_G & 0 \\ 0 & 0 & I_\varphi^{\text{ext}} \end{pmatrix} \mathfrak{F}_{\text{alt}}^{(SL\varphi)} = 0, \quad (\text{B.70})$$

where the superscript “ext” indicates that the n_G spectator scalars are included. Then,

$$\lim_{r \rightarrow \infty} \left| \ln \frac{\det \Psi_\ell^{(SL\varphi)}(r)}{\det \Psi_{\ell,\text{alt}}^{(SL\varphi)}(r)} \right| = \lim_{r \rightarrow \infty} \left| \ln \frac{\det \mathfrak{F}^{(SL\varphi)}(r)}{\det \mathfrak{F}_{\text{alt}}^{(SL\varphi)}(r)} \right|. \quad (\text{B.71})$$

Here, we have ignored ω , which is non-vanishing only around $r \sim r_*$. One can show that its effect disappears as $r_* \rightarrow \infty$.

Then, we can show the following.

- All of the eigenvalues of $(\Lambda + \Lambda^T)/2$ are $3/r$ and hence $\mu(-\Lambda) = -3/r$.
- The fluctuation operator at $r \rightarrow \infty$ has the same number of increasing solutions and decreasing solutions. If there are no zero modes, $\mathfrak{F}_{\text{alt}}^{(SL\varphi)}$ contains all of the increasing solutions and $\mu(-\mathfrak{F}_{\text{alt}}^{(SL\varphi)'} \mathfrak{F}_{\text{alt}}^{(SL\varphi)-1}) < 0$. When there are zero modes, $\mathfrak{F}_{\text{alt}}^{(SL\varphi)}$ fails to contain some of the increasing solutions and $\mu(-\mathfrak{F}_{\text{alt}}^{(SL\varphi)'} \mathfrak{F}_{\text{alt}}^{(SL\varphi)-1})$ becomes positive. However, it is harmless since we use the fluctuation operator with regulator, with which all the solutions are increasing.
- Since the growth of $\mathfrak{F}_{\text{alt}}^{(SL\varphi)}$ is either exponential or power at $r \rightarrow \infty$, $C_{\mathfrak{F}}$ is finite.

- Since we can take a smaller $\delta_{\max}$ for a larger r_* , it is easy to find $C_{\delta\Lambda}$ that satisfies eq. (B.35). Notice that the max norm and the spectral norm of an $n \times n$ matrix, A , are related through

$$\|A\| \leq n\|A\|_{\max}. \quad (\text{B.72})$$

- As for $\delta\Xi$, we need a careful treatment since it includes $(W^T W)^{-1}$. It appears in $\Omega^{\text{ext}} - \Omega_{\text{alt}}^{\text{ext}}$ and

$$\delta\Omega^{\text{ext}} = -M_{\text{alt}}^{\text{ext}}(M_{\text{alt}}^{\text{ext},T} M_{\text{alt}}^{\text{ext}})^{-1} \left[-\Delta_0 M_{\text{alt}}^{\text{ext},T} + M_{\text{alt}}^{\text{ext},T} \Omega_{\text{alt}}^{\text{ext}} \right]. \quad (\text{B.73})$$

From eq. (B.45), it can be rewritten as

$$\delta\Omega^{\text{ext}} = -M_{\text{alt}}^{\text{ext}}(M_{\text{alt}}^{\text{ext},T} M_{\text{alt}}^{\text{ext}})^{-1} \left[-\Delta_0 M_{\text{alt}}^{\text{ext},T} + M_{\text{alt}}^{\text{ext},T} \Omega_{\text{alt}}^{\text{ext}} \right] P_{G,\text{alt}}, \quad (\text{B.74})$$

for $r_1 < r < r_4$. It is non-vanishing only when $r_2 < r < r_3$, where

$$\left[-\Delta_0 M_{\text{alt}}^{\text{ext}} + \Omega_{\text{alt}}^{\text{ext}} M_{\text{alt}}^{\text{ext}} \right] (M_{\text{alt}}^{\text{ext},T} M_{\text{alt}}^{\text{ext}})^{-1} M_{\text{alt}}^{\text{ext},T} \simeq \begin{pmatrix} \zeta_2''(r) [-\delta_\varphi + W \delta_G W^{-1}] \\ 0 \end{pmatrix}, \quad (\text{B.75})$$

and $\Omega^{\text{ext}} - \Omega_{\text{alt}}^{\text{ext}}$ and $\delta\Omega^{\text{ext}}$ are obtained by appropriate projections. Since its elements decrease as $\delta_{\max}$ becomes smaller, we can find $C_{\delta\Xi}$ that satisfies eq. (B.36).

The same discussion applies to $\ell = 0$.

B.4 $(c\bar{c})$ modes

For the ghosts, we introduce $\mathfrak{F}^{(c\bar{c})}$ and $\mathfrak{F}_{\text{alt}}^{(c\bar{c})}$, obeying

$$\mathcal{M}_\ell^{(c\bar{c})\text{ext}} \mathfrak{F}^{(c\bar{c})} = 0, \quad (\text{B.76})$$

$$\mathcal{M}_{\ell,\text{alt}}^{(c\bar{c})\text{ext}} \mathfrak{F}_{\text{alt}}^{(c\bar{c})} = 0. \quad (\text{B.77})$$

The discussion is almost the same as in the case of the $(SL\varphi)$ modes, but is much simpler because, for the ghosts, $\delta\Lambda = 0$ and

$$\delta\Xi = \xi(M^{\text{ext},T} M^{\text{ext}} - M_{\text{alt}}^{\text{ext},T} M_{\text{alt}}^{\text{ext}}). \quad (\text{B.78})$$

B.5 $(\eta\lambda)$ modes

Let us define

$$\mathcal{M}_\ell^{(\eta\lambda)} = \begin{pmatrix} W^{-1} \left[-\Delta_\ell^{(M^T M)} + 2M^T M' (M^T M)^{-1} \frac{1}{r^2} \partial_r r^2 \right] W & -\frac{2L}{r} W^{-1} M'^T V_H \\ \frac{2L}{r} V_H^T M' (M^T M)^{-1} W & V_H^T \left[-\Delta_\ell^{(\Omega)} - 4M' (M^T M)^{-1} M'^T \right] V_H \end{pmatrix}, \quad (\text{B.79})$$

where $\Delta_\ell^{(M^T M)} \equiv \Delta_\ell - M^T M$ and $\Delta_\ell^{(\Omega)} \equiv \Delta_\ell - \Omega$. Then, we consider the solution of the following equation:

$$\mathcal{M}_\ell^{(\eta\lambda)} \tilde{\Psi}_\ell^{(\eta\lambda)} = 0. \quad (\text{B.80})$$

Here, the solution, $\tilde{\Psi}_\ell^{(\eta\lambda)}$, with appropriate boundary conditions is related to $\Psi_\ell^{(\eta\lambda)}$ as

$$\begin{pmatrix} W & 0 \\ 0 & I_H \end{pmatrix} \tilde{\Psi}_\ell^{(\eta\lambda)} = \Psi_\ell^{(\eta\lambda)}. \quad (\text{B.81})$$

We also define its alternative version and its extension to include the spectator scalars. We introduce $\mathfrak{F}^{(\eta\lambda)}$ and $\mathfrak{F}_{\text{alt}}^{(\eta\lambda)}$ which obey

$$\mathcal{M}_\ell^{(\eta\lambda)\text{ext}} \mathfrak{F}^{(\eta\lambda)} = 0, \quad (\text{B.82})$$

$$\mathcal{M}_{\ell,\text{alt}}^{(\eta\lambda)\text{ext}} \mathfrak{F}_{\text{alt}}^{(\eta\lambda)} = 0. \quad (\text{B.83})$$

Then,

$$\lim_{r \rightarrow \infty} \left| \ln \frac{\det \Psi_\ell^{(\eta\lambda)}(r)}{\det \Psi_{\ell,\text{alt}}^{(\eta\lambda)}(r)} \right| = \lim_{r \rightarrow \infty} \left| \ln \frac{\det \mathfrak{F}^{(\eta\lambda)}(r)}{\det \mathfrak{F}_{\text{alt}}^{(\eta\lambda)}(r)} \right|. \quad (\text{B.84})$$

We can show the following.

- Since Λ is diagonal with the diagonal components $3/r$, $\mu(-\Lambda) = -3/r$.
- From eq. (A.20) and the divergent behavior of W_U^{-1} , $\mathfrak{F}_{\text{alt}}^{(\eta\lambda)}$ contains only the increasing solutions. Thus, $\mu(-\mathfrak{F}_{\text{alt}}^{(\eta\lambda)'} \mathfrak{F}_{\text{alt}}^{(\eta\lambda)-1}) < 0$.
- Since $\mathfrak{F}_{\text{alt}}^{(\eta\lambda)}$ grows exponentially, $C_{\mathfrak{F}}$ is finite.
- Just like the $(SL\varphi)$ modes, substituting the formulas of the linear approximation, we can find $C_{\delta\Lambda}$ and $C_{\delta\Xi}$ that satisfy eqs. (B.35) and (B.36).

The same discussion applies to $\ell = 0$.

C $\overline{\text{MS}}$ Counterterms

In this appendix, we give the explicit form of the counterterms in the $\overline{\text{MS}}$ scheme.

We define mass squared matrices of gauge bosons and scalars in the false vacuum as

$$\widehat{m}_A^2 \equiv \widehat{M}^T \widehat{M}, \quad (\text{C.1})$$

$$\widehat{m}_\varphi^2 \equiv \widehat{\Omega} + \widehat{M} \widehat{M}^T, \quad (\text{C.2})$$

both of which are diagonal, and the Fourier transformation of mass squared difference

$$\widetilde{\delta m_A^2}(k) = \int d^4x [M^T(r) M(r) - \widehat{m}_A^2] e^{-ikx}, \quad (\text{C.3})$$

$$\widetilde{\delta m_\varphi^2}(k) = \int d^4x [\Omega(r) + M(r) M^T(r) - m_\varphi^2] e^{-ikx}, \quad (\text{C.4})$$

$$\widetilde{M}(k) = \int d^4x M(r) e^{-ikx}. \quad (\text{C.5})$$

Note that M and Ω only depend on $r = \sqrt{x_\mu x_\mu}$ due to the spherical symmetry of the bounce. Accordingly, all of the Fourier transformation given above are functions only of $k \equiv \sqrt{k_\mu k_\mu}$. Furthermore, $\widehat{\delta m_A^2}$ and δm_φ^2 are symmetric matrices.

After a straightforward calculation, we obtain

$$s_{1,\overline{\text{MS}}}^{(A_\mu\varphi)} = 2\mathcal{I}_1(\widehat{m}_A^2, \widehat{\delta m_A^2}) + \frac{1}{2}\mathcal{I}_1(\widehat{m}_\varphi^2, \widehat{\delta m_\varphi^2}) + \frac{1}{16\pi^2}\text{Tr} \left[\widehat{m}_A^2 \widehat{\delta m_A^2}(0) \right], \quad (\text{C.6})$$

$$s_{1,\overline{\text{MS}}}^{(c\bar{c})} = -\mathcal{I}_1(\widehat{m}_A^2, \widehat{\delta m_A^2}), \quad (\text{C.7})$$

$$\begin{aligned} s_{2,\overline{\text{MS}}}^{(A_\mu\varphi)} &= -\mathcal{I}_2(\widehat{m}_A^2, \widehat{m}_A^2, \widehat{\delta m_A^2}) - 2\mathcal{I}_2(\widehat{m}_A^2, \widehat{m}_\varphi^2, k\widetilde{M}) - \frac{1}{4}\mathcal{I}_2(\widehat{m}_\varphi^2, \widehat{m}_\varphi^2, \widehat{\delta m_\varphi^2}) \\ &\quad + \frac{1}{256\pi^4} \int_0^\infty dk k^3 \text{Tr} \left[\widehat{\delta m_A^2}(k) \widehat{\delta m_A^2}(k) \right], \end{aligned} \quad (\text{C.8})$$

$$s_{2,\overline{\text{MS}}}^{(c\bar{c})} = \frac{1}{2}\mathcal{I}_2(\widehat{m}_A^2, \widehat{m}_A^2, \widehat{\delta m_A^2}), \quad (\text{C.9})$$

where

$$\mathcal{I}_1(m^2, \widehat{\delta m^2}) = -\sum_i [\widehat{\delta m^2}(0)]_{ii} \frac{1}{16\pi^2} A_{0,\overline{\text{MS}}}([m^2]_{ii}), \quad (\text{C.10})$$

$$\mathcal{I}_2(m_1^2, m_2^2, \widehat{\delta m^2}) = \sum_{ij} \frac{1}{128\pi^4} \int_0^\infty dk k^3 \left[\widehat{\delta m^2}(k) \right]_{ij} \left[\widehat{\delta m^2}(k) \right]_{ij} B_{0,\overline{\text{MS}}}(k, [m_1^2]_{ii}, [m_2^2]_{jj}). \quad (\text{C.11})$$

Renormalizing the divergence in the $\overline{\text{MS}}$ scheme, we can evaluate the one-point function $A_{0,\overline{\text{MS}}}$ and the two-point function $B_{0,\overline{\text{MS}}}$ as [62]

$$A_{0,\overline{\text{MS}}}(m^2) = \lim_{\delta \rightarrow +0} m^2 \left(1 + \ln \frac{\mu^2}{m^2 - i\delta} \right), \quad (\text{C.12})$$

$$\begin{aligned} B_{0,\overline{\text{MS}}}(k, m^2, M^2) &= \lim_{\delta \rightarrow +0} \left[2 + x_1(\delta) \ln \frac{x_1(\delta) - 1}{x_1(\delta)} + x_2(\delta) \ln \frac{x_2(\delta) - 1}{x_2(\delta)} \right. \\ &\quad \left. - \ln(1 - x_1(\delta)) - \ln(x_2(\delta) - 1) + \ln \left(\frac{\mu^2}{k^2} \right) \right], \end{aligned} \quad (\text{C.13})$$

where μ is the renormalization scale while $x_1(\delta)$ and $x_2(\delta)$ are solutions of the following quadratic equation:

$$-k^2 x^2 + (k^2 - m^2 + M^2)x + m^2 - i\delta = 0. \quad (\text{C.14})$$

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