RECEIVED: October 3, 2019

REVISED: November 6, 2019

ACCEPTED: November 13, 2019

PUBLISHED: November 28, 2019

Superfield approach to higher derivative $\mathcal{N} = 1$ superconformal mechanics

Ivan Masterov and Boris Merzlikin

*Tomsk State University of Control Systems and Radioelectronics,
634050, Tomsk, Lenin Ave. 40, Russia*

E-mail: iv.masterov@yandex.ru, merzlikin@tspu.edu.ru

ABSTRACT: We formulate the equations which determine a potential function in an $\mathcal{N} = 1$ higher derivative supersymmetric mechanics compatible with the $osp(2|1) \oplus so(d)$ symmetry and provide a few explicit examples.

KEYWORDS: Conformal and W Symmetry, Superspaces, Conformal Field Theory

ARXIV EPRINT: [1909.12574](https://arxiv.org/abs/1909.12574)

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1 Introduction

Superconformal mechanical systems have attracted considerable attention over the last three decades. The very first examples in the literature provided supersymmetric counterparts for the $d = 1$ de Alfaro-Fubini-Furlan conformal mechanics [1]–[3] and the Calogero model [4]. Recent studies of the nonrelativistic version of the AdS/CFT-correspondence¹ and a possible link to the black holes physics [6]–[9] have brought the $d = 1$ superconformal mechanics into focus again.²

In parallel, there has been considerable progress in exploration of superconformal mechanics in $d > 1$ [31]–[48]. In this context, supersymmetric extensions of the so-called l -conformal Galilei algebra [49]–[50] and its dynamical realizations play the key role.

The l -conformal Galilei algebra, where l is a positive (half)integer parameter, represents a finite-dimensional conformal extension of the Galilei algebra [50]. It includes

- the generator of time translations L_{-1} ,
 - the generator of dilatations L_0 ,
 - the generator of special conformal transformations L_1 ,
 - the chain of vector generators ($n = 0, 1, \dots, 2l$) $C_i^{(n)}$,
 - the generators of spatial rotations M_{ij} ,
- (1.1)

where $i, j = 1, \dots, d$ and d is the spatial dimension. The algebra is not semisimple. Its semisimple part is given by $so(1, 2) \oplus so(d)$ while the abelian ideal is formed by the vector

¹See, e.g., ref. [5] and references therein.

²There exists a rather extensive literature on the subject. For a review and further references see, e.g., refs. [10]–[30].

generators $C_i^{(n)}$. Subalgebras $so(1,2)$ and $so(d)$ are spanned by the generators L_n ($n = -1, 0, 1$) and M_{ij} , respectively.

The case of $l = \frac{1}{2}$ is known as the Schrödinger algebra, as it comprises the symmetry of the Schrödinger equation associated with a free massive particle [51]. The instance of $l = 1$ is conventionally referred to as the conformal Galilei algebra. It is regarded as the nonrelativistic counterpart of the relativistic conformal algebra $so(d+1,2)$ [52].

Recently, there has been an upsurge of interest in dynamical realizations of the l -conformal Galilei algebra for $l > 1$ [53]–[71]. In general, the order of differential equations which govern such a system correlates with the value of l . In particular, it was argued in [55] that a free higher derivative particle described by the action functional³

$$S = \frac{1}{2} \int dt \lambda_{ij} x_i x_j^{(2l+1)}, \quad \lambda_{ij} = \begin{cases} \delta_{ij} & i, j = 1, 2, \dots, d, \quad \text{for half-integer } l, \\ \epsilon_{ij} & i, j = 1, 2, \quad \text{for integer } l, \end{cases} \quad (1.2)$$

possesses the l -conformal Galilei symmetry. In (1.2) ϵ_{ij} is the Levi-Civita symbol with $\epsilon_{12} = 1$. Despite the fact that the l -conformal Galilei algebra for integer l is formulated for arbitrary space dimension d , the Lagrangian formulation for the free odd order derivative particle exists only for $d = 2$.

$\mathcal{N} = 1$, $\mathcal{N} = 2$, and $\mathcal{N} = 4$ supersymmetric extensions of the l -conformal Galilei algebra for $l > 1$ have been constructed in refs. [72]–[77]. In contrast to $\mathcal{N} = 2$ and $\mathcal{N} = 4$ cases, there is a unique $\mathcal{N} = 1$ superextension. Apart from the generators (1.1), $\mathcal{N} = 1$ l -conformal Galilei superalgebra includes

- the generator of supersymmetry transformations $Q_{-1/2}$,
- the generator of superconformal transformations $Q_{1/2}$,
- the chain of odd vector generators ($n = 0, 1, \dots, 2l-1$) $L_i^{(n)}$.

$Q_{-1/2}$, $Q_{1/2}$ together with L_{-1} , L_0 , and L_1 form $osp(2|1)$ subalgebra while $C_i^{(n)}$ and $L_i^{(n)}$ enter abelian ideal of the superalgebra.

In ref. [45], $\mathcal{N} = 1$, $l > 1/2$ conformal Galilei superalgebra was identified with the symmetry algebra of a free $\mathcal{N} = 1$ higher derivative superparticle. The goal of the present paper is to construct a superfield formulation for that model and to describe potential functions which preserve $osp(2|1) \oplus so(d)$ symmetry. It is presented in the next section. Some explicit examples of the $osp(2|1) \oplus so(d)$ -invariant models are given in section 3. In the concluding section 4, we summarize our results and discuss further possible developments. Some technical details are given in appendix.

2 $\mathcal{N} = 1$ superconformal mechanics with $osp(2|1) \oplus so(d)$ symmetry

The model of a free $\mathcal{N} = 1$ superparticle of order $(2l+1)$ is described by the action functional

$$S = \frac{1}{2} \int dt \lambda_{ij} (x_i x_j^{(2l+1)} - i \psi_i \psi_j^{(2l)}), \quad (2.1)$$

³Throughout the work we use the notation $f^{(n)} = \frac{d^n f}{dt^n}$. Summation over repeated spatial indices is understood.

with λ_{ij} in (1.2). As was shown in [45], the action holds invariant under the $l > 1/2$ conformal Galilei group (no sum over n below)

$$\begin{aligned}
 L_n : \quad & \delta t = t^{n+1} a_n, \quad \delta x_i = l(n+1)t^n x_i a_n, \quad \delta \psi_i = (l-1/2)(n+1)t^n \psi_i a_n; \\
 Q_r : \quad & \delta x_i = i t^{r+1/2} \psi_i \alpha_r, \quad \delta \psi_i = (t^{r+1/2} \dot{x}_i - 2l(r+1/2)x_i) \alpha_r; \\
 C_i^{(n)} : \quad & \delta x_i = b_i^{(n)} t^n; \\
 L_i^{(n)} : \quad & \delta \psi_i = \beta_i^{(n)} t^n; \\
 M_{ij} : \quad & \delta x_i = \omega_{ij} x_j, \quad \delta \psi_i = \omega_{ij} \psi_j,
 \end{aligned} \tag{2.2}$$

where a_n , $b_i^{(n)}$, α_r , $\beta_i^{(n)}$ and $\omega_{ij} = -\omega_{ji}$ are infinitesimal parameters.

In order to construct a superfield formulation for (2.1), the temporal coordinate t is to be accompanied by a real Grassmann variable θ which give rise to a superfield

$$X_i(t, \theta) = x_i(t) + i\theta \psi_i(t) \tag{2.3}$$

and allow one to rewrite the action (2.1) in the form

$$S = \frac{i^{2l+1}}{2} \int dt d\theta \lambda_{ij} X_i(t, \theta) \mathbb{D}^{4l+1} X_j(t, \theta), \tag{2.4}$$

where

$$\mathbb{D} = \frac{\vec{\partial}}{\partial \theta} - i\theta \frac{\partial}{\partial t} \tag{2.5}$$

is the covariant spinor derivative.⁴

Let us consider the superfield action of the form

$$S = \frac{i^{2l+1}}{2} \int dt d\theta \left(\lambda_{ij} X_i(t, \theta) \mathbb{D}^{4l+1} X_j(t, \theta) + V \left(X_i, \mathbb{D} X_i, \dots, \mathbb{D}^{2l} X_i \right) \right), \tag{2.6}$$

where $V = V(X_i, \mathbb{D} X_i, \dots, \mathbb{D}^{2l} X_i)$ is a Grassmann-odd potential function. The action functional is invariant under time translations as well as under supersymmetry transformations for any V . The requirement that (2.6) be invariant under dilatations, special conformal transformations and superconformal transformations yields the constraints on V

$$V + \sum_{n=0}^{2l-1} (2l-n) \mathbb{D}^n X_i \frac{\partial V}{\partial \mathbb{D}^{2n} X_i} = 0, \tag{2.7}$$

$$\sum_{n=0}^{l+\varepsilon-1} (2l-n) \mathbb{D}^{2n} X_i \frac{\partial V}{\partial \mathbb{D}^{2n+1} X_i} - \sum_{n=1}^{l-\varepsilon} n \mathbb{D}^{2n-1} X_i \frac{\partial V}{\partial \mathbb{D}^{2n} X_i} = 0, \tag{2.8}$$

$$\sum_{n=0}^{l-\varepsilon-1} (n+1)(2l-n) \mathbb{D}^{2n} X_i \frac{\partial V}{\partial \mathbb{D}^{2n+2} X_i} + \sum_{n=1}^{l+\varepsilon-1} n(2l-n) \mathbb{D}^{2n-1} X_i \frac{\partial V}{\partial \mathbb{D}^{2n+1} X_i} = 0, \tag{2.9}$$

⁴Some technical details concerning the symmetry transformations of the action functional (2.4) are collected in appendix.

with $\mathbb{D}^0 X_i = X_i$. The parameter ε is given by

$$\varepsilon = \begin{cases} \frac{1}{2}, & \text{for half-integer } l, \\ 0, & \text{for integer } l. \end{cases} \quad (2.10)$$

The formal symbols $\sum_{n=1}^0 f(n)$ and $\sum_{n=0}^{-1} f(n)$, which appear for $l = \frac{1}{2}$ and $l = 1$ cases, are assumed to be equal to zero. Any solution of the equations (2.7) and (2.8) defines an $osp(2|1)$ -invariant superconformal mechanics.

Notice that the equations (2.8) and (2.9) are not independent. It is straightforward to verify that they can be presented in the form

$$\text{Eq. (2.8)} : \Omega V = 0, \quad \text{Eq. (2.9)} : -\Omega^2 V = 0,$$

where Ω is given by

$$\Omega = \sum_{n=0}^{l+\varepsilon-1} (2l-n) \mathbb{D}^{2n} X_i \frac{\partial}{\partial \mathbb{D}^{2n+1} X_i} - \sum_{n=1}^{l-\varepsilon} n \mathbb{D}^{2n-1} X_i \frac{\partial}{\partial \mathbb{D}^{2n} X_i}. \quad (2.11)$$

Let us analyze equation (2.8) in more detail. As the first step, we consider the instance $l = \frac{1}{2}$

$$X_i \frac{\partial V}{\partial \mathbb{D} X_i} = 0. \quad (2.12)$$

When $d = 1$, the general solution of (2.12) is an arbitrary function of the superfield X , i.e.

$$V = V(X).$$

While for $d > 1$, the general solution is constructed by the conventional method of characteristics

$$V = V\left(X_i, P_{ij}^{(1)}\right), \quad P_{ij}^{(1)} = X_i \mathbb{D} X_j - X_j \mathbb{D} X_i. \quad (2.13)$$

As the next step, let us consider the equation (2.8) for the case of $l = 1$

$$2X_i \frac{\partial V}{\partial \mathbb{D} X_i} - \mathbb{D} X_i \frac{\partial V}{\partial \mathbb{D}^2 X_i} = 0. \quad (2.14)$$

The analysis of the characteristic system for this equation yields the following solution

$$V = V(X_i, P_{ij}^{(1)}, P_{ij}^{(2)}), \quad (2.15)$$

where we denoted

$$P_{ij}^{(2)} = X_i \mathbb{D}^2 X_j - X_j \mathbb{D}^2 X_i + \frac{1}{2} \mathbb{D} X_i \mathbb{D} X_j. \quad (2.16)$$

Let us obtain a solution to the equation (2.8) for an arbitrary value of the parameter l . Taking into account the previously considered instances of $l = \frac{1}{2}$ and for $l = 1$, it is natural to chose the following ansatz

$$\begin{aligned} P_{ij}^{(2n-1)} &= \sum_{k=0}^{n-1} \alpha_{k,n} \mathbb{D}^{2k} X_i \mathbb{D}^{2n-2k-1} X_j + \sum_{k=0}^{n-1} \beta_{k,n} \mathbb{D}^{2k+1} X_i \mathbb{D}^{2n-2k-2} X_j, \quad n = 1, 2, \dots, l + \varepsilon; \\ P_{ij}^{(2n)} &= \sum_{k=0}^n \gamma_{k,n} \mathbb{D}^{2k} X_i \mathbb{D}^{2n-2k} X_j + \sum_{k=0}^{n-1} \sigma_{k,n} \mathbb{D}^{2k+1} X_i \mathbb{D}^{2n-2k-1} X_j, \quad n = 1, 2, \dots, l - \varepsilon. \end{aligned} \quad (2.17)$$

Substituting (2.17) into (2.8) one gets the recurrence relations for the coefficients $\alpha_{k,n}$, $\beta_{k,n}$, $\gamma_{k,n}$, and $\sigma_{k,n}$:

$$\begin{aligned} (2l - n + k + 1)\alpha_{k,n} + (2l - k)\beta_{k,n} &= 0, & (k + 1)\alpha_{k+1,n} - (n - k - 1)\beta_{k,n} &= 0, \\ (2l - k)\sigma_{k,n} - (n - k)\gamma_{k,n} &= 0, & (2l - n + k + 1)\sigma_{k,n} + (k + 1)\gamma_{k+1,n} &= 0. \end{aligned} \quad (2.18)$$

A solution to these equations reads

$$\begin{aligned} \alpha_{k,n} &= \frac{(-1)^k (2l - n + k)! (2l - k)! (n - 1)!}{(2l - n)! (2l)! (n - k - 1)! k!}, & \beta_{k,n} &= -\frac{2l - n + k + 1}{2l - k} \alpha_{k,n}, \\ \gamma_{k,n} &= \frac{(-1)^k (2l - n + k)! (2l - k)! n!}{(2l - n)! (2l)! (n - k)! k!}, & \sigma_{k,n} &= \frac{n - k}{2l - k} \gamma_{k,n}. \end{aligned} \quad (2.19)$$

A few comments are in order. Firstly, by construction, the polynomials $P_{ij}^{(k)}$ are Grassmann-odd functions for odd k and Grassmann-even functions otherwise. Secondly, the coefficients (2.19) possess the following properties

$$\begin{aligned} \alpha_{n-k-1,n} &= (-1)^n \beta_{k,n}, & \beta_{n-k-1,n} &= (-1)^n \alpha_{k,n}, \\ \gamma_{n-k,n} &= (-1)^n \gamma_{k,n}, & \sigma_{n-k-1,n} &= (-1)^{n-1} \sigma_{k,n}. \end{aligned} \quad (2.20)$$

Due to these properties, the polynomials $P_{ij}^{(2n-1)}$ and $P_{ij}^{(2n)}$ are antisymmetric for odd n and symmetric for even n , i.e.

$$P_{ij}^{(2n-1)} = (-1)^n P_{ji}^{(2n-1)}, \quad P_{ij}^{(2n)} = (-1)^n P_{ji}^{(2n)}. \quad (2.21)$$

Thirdly, a solution to equation (2.8) can be written as follows

$$V = V \left(X_i, P_{ij}^{(1)}, \dots, P_{ij}^{(2l)} \right). \quad (2.22)$$

In terms of the variables $P_{ij}^{(n)}$, the equation (2.7) takes the form⁵

$$V + 2l X_i \frac{\partial V}{\partial X_i} + \sum_{n=1}^{2l} (4l - n) P_{ij}^{(n)} \frac{\partial V}{\partial P_{ij}^{(n)}} = 0. \quad (2.23)$$

⁵It should be noted that the equation (2.23) can be written in a more compact form

$$V + \sum_{n=0}^{2l} (4l - n) P_{ij}^{(n)} \frac{\partial V}{\partial P_{ij}^{(n)}} = 0,$$

when V depends on the superfield X_i via the combination $P_{ij}^{(0)} = X_i X_j$.

Any solution of this equation defines an $\mathcal{N} = 1$ superconformal mechanics with $osp(2|1) \oplus so(d)$ symmetry provided V transforms as a scalar under space rotations. In the next section we will consider a few examples of such superconformal mechanics.

3 Examples

3.1 The second-order $osp(2|1) \oplus so(d)$ invariant superconformal mechanics ($l = \frac{1}{2}$)

Let us consider a way to obtain $so(d)$ -invariant solutions of the equation (2.23) for the case of $l = \frac{1}{2}$. In accord with (2.13), a rotationally invariant potential function V may depend on any scalars which can be constructed from the superfield X_i and the polynomial

$$P_{ij}^{(1)} = X_i \mathbb{D} X_j - X_j \mathbb{D} X_i. \quad (3.1)$$

Taking into account that the function $P_{ij}^{(1)}$ is Grassmann-odd and antisymmetric, one may construct scalars by making use of X_i only. The simplest rotationally invariant combination is

$$\mathbb{P}_0 = X_i X_i. \quad (3.2)$$

In agreement with eq. (2.23), the potential function $V = V(\mathbb{P}_0)$ satisfies the equation

$$V + 2\mathbb{P}_0 \frac{dV}{d\mathbb{P}_0} = 0. \quad (3.3)$$

The solution of this equation has the form

$$V = \frac{g}{\sqrt{\mathbb{P}_0}}, \quad (3.4)$$

where g is a coupling constant. By construction, the function V must be Grassmann-odd. But this condition can be met only if g is Grassmann-odd.⁶ On physical grounds we discard such potential functions from our consideration.

To obtain a more appropriate rotationally invariant solution to eq. (2.23), we need to have Grassmann-odd scalars which can be constructed from polynomials (3.1). For $d = 2$, a scalar of such a type can be obtained by contracting the Levi-Civita symbol with $P_{ij}^{(1)}$, i.e.

$$\mathbb{P}_1 = \epsilon_{ij} P_{ij}^{(1)} = 2P_{12}^{(1)}. \quad (3.5)$$

The equation (2.23) for the function $V = V(\mathbb{P}_0, \mathbb{P}_1)$ takes the form

$$V + 2\mathbb{P}_0 \frac{\partial V}{\partial \mathbb{P}_0} + \mathbb{P}_1 \frac{\partial V}{\partial \mathbb{P}_1} = 0, \quad (3.6)$$

whose general solution reads

$$V = \frac{g}{\sqrt{\mathbb{P}_0}} + \frac{i\gamma \mathbb{P}_1}{\mathbb{P}_0}. \quad (3.7)$$

⁶See also related discussion in [78].

The first term reproduces the inappropriate potential function (3.4) revealed above, while the requirement for the second term in (3.7) to be Grassmann-odd is satisfied provided the coupling constant γ is Grassmann-even.

Let us consider the superfield action

$$S = -\frac{1}{2} \int dt d\theta \left(X_i(t, \theta) \mathbb{D}^3 X_i(t, \theta) + \frac{i\gamma \mathbb{P}_1}{\mathbb{P}_0} \right). \quad (3.8)$$

Being rewritten in components, it takes the form⁷

$$S = \frac{1}{2} \int dt \left(x_i \ddot{x}_i - i\psi_i \dot{\psi}_i - \frac{2\gamma \epsilon_{ij} x_i \dot{x}_j}{x_k x_k} \right) = \frac{1}{2} \int dt \left(x_i \ddot{x}_i - i\psi_i \dot{\psi}_i - 2\gamma \frac{d}{dt} \left(\arctan \frac{x_2}{x_1} \right) \right). \quad (3.9)$$

Thus, the model (3.8) corresponds to a free $\mathcal{N} = 1$ second-order superparticle.⁸

3.2 The third-order $osp(2|1) \oplus so(2)$ invariant superconformal mechanics ($l = 1$)

When constructing $osp(2|1) \oplus so(2)$ -invariant potentials for the third-order $\mathcal{N} = 1$ superconformal mechanics, we have at our disposal one more polynomial

$$P_{ij}^{(2)} = X_i \mathbb{D}^2 X_j - X_j \mathbb{D}^2 X_i + \frac{1}{2} \mathbb{D} X_i \mathbb{D} X_j,$$

which is Grassmann-even and antisymmetric. As a consequence, the number of possible rotationally invariant combinations increases. For example, one can use

$$\begin{aligned} 1) \quad \mathbb{P}_0 &= X_i X_i && \text{--- even,} \\ 2) \quad \mathbb{P}_1 &= \epsilon_{ij} P_{ij}^{(1)} && \text{--- odd,} \\ 3) \quad \mathbb{P}_2 &= \epsilon_{ij} P_{ij}^{(2)} && \text{--- even,} \\ 4) \quad \mathbb{P}_{12} &= P_{ij}^{(1)} P_{ij}^{(2)} && \text{--- odd,} \\ 5) \quad \mathbb{P}_{22} &= P_{ij}^{(2)} P_{ij}^{(2)} && \text{--- even,} \end{aligned} \quad (3.10)$$

as independent variables on which the potential function V depends. In this case the equation (2.23) can be rewritten as

$$V + 4\mathbb{P}_0 \frac{\partial V}{\partial \mathbb{P}_0} + 3\mathbb{P}_1 \frac{\partial V}{\partial \mathbb{P}_1} + 2\mathbb{P}_2 \frac{\partial V}{\partial \mathbb{P}_2} + 5\mathbb{P}_{12} \frac{\partial V}{\partial \mathbb{P}_{12}} + 4\mathbb{P}_{22} \frac{\partial V}{\partial \mathbb{P}_{22}} = 0. \quad (3.11)$$

The general solution of this equation reads

$$V = \frac{1}{\sqrt[4]{\mathbb{P}_0}} \Lambda \left(\frac{\mathbb{P}_1}{\sqrt[4]{\mathbb{P}_0}^3}, \frac{\mathbb{P}_2}{\sqrt{\mathbb{P}_0}}, \frac{\mathbb{P}_{12}}{\sqrt[4]{\mathbb{P}_0}^5}, \frac{\mathbb{P}_{22}}{\mathbb{P}_0} \right), \quad (3.12)$$

⁷To obtain the component form (3.9) of the superfield action (3.8), the property of the covariant spinor derivative $\mathbb{D}^2 = -i\partial_t$ proves to be helpful.

⁸The same result can be obtained by making the polar decomposition $X_i = \rho Y_i$, where $\rho^2(t, \theta) = X_i X_i$ and $Y_1 = \cos \varphi(t)$, $Y_2 = \sin \varphi(t)$. In terms of new variables the potential function in (3.8) can be written as $2\gamma\theta\dot{\varphi}$. We are grateful to the anonymous referee for drawing our attention to this observation.

where Λ is an arbitrary function. The Taylor expansion of the function (3.12) in the Grassmann-odd variables can be divided into two parts

$$V = U + W, \quad (3.13)$$

where

$$U = \frac{1}{\sqrt[4]{\mathbb{P}_0}} \cdot \Psi_1 \left(\frac{\mathbb{P}_2}{\sqrt{\mathbb{P}_0}}, \frac{\mathbb{P}_{22}}{\mathbb{P}_0} \right) + \frac{\mathbb{P}_1 \mathbb{P}_{12}}{\sqrt[4]{\mathbb{P}_0^9}} \cdot \Psi_2 \left(\frac{\mathbb{P}_2}{\sqrt{\mathbb{P}_0}}, \frac{\mathbb{P}_{22}}{\mathbb{P}_0} \right), \quad (3.14)$$

$$W = \frac{\mathbb{P}_1}{\mathbb{P}_0} \cdot \Phi_1 \left(\frac{\mathbb{P}_2}{\sqrt{\mathbb{P}_0}}, \frac{\mathbb{P}_{22}}{\mathbb{P}_0} \right) + \frac{\mathbb{P}_{12}}{\sqrt{\mathbb{P}_0^3}} \cdot \Phi_2 \left(\frac{\mathbb{P}_2}{\sqrt{\mathbb{P}_0}}, \frac{\mathbb{P}_{22}}{\mathbb{P}_0} \right). \quad (3.15)$$

Here Ψ_1 , Ψ_2 , Φ_1 , and Φ_2 are arbitrary functions.

Any potential in (3.14) is Grassmann-odd only if the corresponding coupling constant is also Grassmann-odd. At the same time the potential functions in (3.15) are more appropriate for our consideration. As an example, let us set $\Phi_1 = 0$, $\Phi_2 = -i\gamma/2$ in (3.15) and consider the model which is described by the superfield action functional

$$S = -\frac{i}{2} \int dt d\theta \left(\epsilon_{ij} X_i \mathbb{D}^5 X_j - \frac{i\gamma \mathbb{P}_{12}}{2\sqrt{\mathbb{P}_0^3}} \right) \quad (3.16)$$

which has the component form

$$S = \frac{1}{2} \int dt \left(\epsilon_{ij} x_i \ddot{x}_j - i\epsilon_{ij} \psi_i \ddot{\psi}_j + \gamma \frac{(\epsilon_{ij} x_i \dot{x}_j)^2 + i(\epsilon_{ij} x_i \dot{\psi}_j)(\epsilon_{kp} x_k \dot{\psi}_p)}{\sqrt{(x_s x_s)^3}} \right). \quad (3.17)$$

The invariance of this action under the $osp(2|1) \oplus so(2)$ transformations from (2.2) yields the following conserved charges

$$\begin{aligned} \mathcal{L}_{-1} &= \epsilon_{ij} \dot{x}_i \ddot{x}_j - \frac{i}{2} \epsilon_{ij} \dot{\psi}_i \ddot{\psi}_j - \frac{\gamma (\epsilon_{ij} x_i \dot{x}_j)^2}{2\sqrt{(x_s x_s)^3}}, \\ \mathcal{L}_0 &= t\mathcal{L}_{-1} - \epsilon_{ij} x_i \ddot{x}_j + \frac{i}{2} \epsilon_{ij} \psi_i \ddot{\psi}_j, \\ \mathcal{L}_1 &= -t^2 \mathcal{L}_{-1} + 2t\mathcal{L}_0 + 2\epsilon_{ij} x_i \dot{x}_j - \frac{i}{2} \epsilon_{ij} \psi_i \dot{\psi}_j, \\ \mathcal{Q}_{-1/2} &= i\epsilon_{ij} \psi_i \ddot{x}_j - i\epsilon_{ij} \dot{\psi}_i \ddot{x}_j - \frac{i\gamma (\epsilon_{ij} x_i \dot{\psi}_j)(\epsilon_{kp} x_k \dot{x}_p)}{\sqrt{(x_s x_s)^3}}, \\ \mathcal{Q}_{1/2} &= t\mathcal{Q}_{-1/2} - i\epsilon_{ij} \psi_i \dot{x}_j + 2i\epsilon_{ij} \dot{\psi}_i x_j, \\ \mathcal{M}_{12} &= \frac{1}{2} \dot{x}_i \dot{x}_i - x_i \ddot{x}_i + i\psi_i \dot{\psi}_i - \frac{\gamma (x_i x_i)(\epsilon_{jk} x_j \dot{x}_k)}{\sqrt{(x_s x_s)^3}} + \frac{i\gamma (x_i \psi_i)(\epsilon_{jk} x_j \dot{\psi}_k)}{2\sqrt{(x_s x_s)^3}}. \end{aligned} \quad (3.18)$$

Here and what follows we denote constants of the motion by the same letters which were used for designating the corresponding symmetry generators, but in a calligraphic style.

3.3 The fourth-order $osp(2|1) \oplus so(d)$ invariant superconformal mechanics ($l = \frac{3}{2}$)

As was mentioned above, in general, the superpotential for the fourth-order $osp(2|1)$ -invariant superconformal mechanics is a function of X_i and $P_{ij}^{(n)}$ with $n = 1, 2, 3$, which obeys the equation (2.23). The polynomial $P_{ij}^{(1)}$ is the same as in (3.1), but others are given by

$$\begin{aligned} P_{ij}^{(2)} &= X_i \mathbb{D}^2 X_j - X_j \mathbb{D}^2 X_i + \frac{1}{3} \mathbb{D} X_i \mathbb{D} X_j, \\ P_{ij}^{(3)} &= X_i \mathbb{D}^3 X_j + X_j \mathbb{D}^3 X_i - \frac{2}{3} (\mathbb{D} X_i \mathbb{D}^2 X_j + \mathbb{D} X_j \mathbb{D}^2 X_i). \end{aligned} \quad (3.19)$$

Note that $P_{ij}^{(3)}$ is Grassmann-odd symmetric polynomial.

Taking into account the analysis in the preceding subsection, let us consider the superpotential which is a linear function of the anticommuting scalars $\mathbb{P}_{12}$ and $\mathbb{P}_{03} = X_i X_j P_{ij}^{(3)}$,

$$V = \mathbb{P}_{12} V_1(\mathbb{P}_0, \mathbb{P}_{22}) + \mathbb{P}_{03} V_2(\mathbb{P}_0, \mathbb{P}_{22}), \quad (3.20)$$

with $\mathbb{P}_0$, $\mathbb{P}_{12}$, and $\mathbb{P}_{22}$ defined in (3.10). The ansatz (3.20) solves (2.23) provided

$$V_1 = \frac{1}{\sqrt[3]{\mathbb{P}_0^5}} \Phi_1 \left(\frac{\mathbb{P}_{22}}{\sqrt[3]{\mathbb{P}_0^4}} \right), \quad V_2 = \frac{1}{\sqrt[3]{\mathbb{P}_0^5}} \Phi_2 \left(\frac{\mathbb{P}_{22}}{\sqrt[3]{\mathbb{P}_0^4}} \right), \quad (3.21)$$

where Φ_1 and Φ_2 are arbitrary functions.

As an example, let us consider the potential function

$$V = -\frac{\gamma \mathbb{P}_{12}}{\sqrt[3]{\mathbb{P}_0^5}}, \quad (3.22)$$

which corresponds to the component action

$$S = \frac{1}{2} \int dt \left(x_i \dot{x}_i^{(4)} - i \psi_i \dot{\psi}_i^{(3)} + \gamma U \right), \quad (3.23)$$

where

$$U = \frac{\dot{x}_i \dot{x}_i + i \dot{\psi}_i \dot{\psi}_i}{\sqrt[3]{(x_k x_k)^2}} - \frac{(x_i \dot{x}_i)^2 + i(x_i \dot{\psi}_i)(x_j \dot{\psi}_j) - \frac{i}{3}(x_i \dot{\psi}_i)(\dot{x}_j \dot{\psi}_j)}{\sqrt[3]{(x_k x_k)^5}}. \quad (3.24)$$

By conventional means one finds the integrals of motion

$$\begin{aligned} \mathcal{L}_{-1} &= \dot{x}_i \ddot{x}_i - \frac{1}{2} \ddot{x}_i \ddot{x}_i - i \dot{\psi}_i \ddot{\psi}_i - \frac{\gamma}{2} \left(\frac{\dot{x}_i \dot{x}_i}{\sqrt[3]{(x_j x_j)^2}} - \frac{(x_i \dot{x}_i)^2}{\sqrt[3]{(x_j x_j)^5}} \right), \\ \mathcal{L}_0 &= t \mathcal{L}_{-1} - \frac{3}{2} x_i \ddot{x}_i + \frac{1}{2} \dot{x}_i \ddot{x}_i + i \dot{\psi}_i \ddot{\psi}_i, \\ \mathcal{L}_1 &= -t^2 \mathcal{L}_{-1} + 2t \mathcal{L}_0 - 2 \dot{x}_i \dot{x}_i + 3 x_i \ddot{x}_i, \\ \mathcal{Q}_{-1/2} &= i \dot{\psi}_i \ddot{x}_i - i \dot{\psi}_i \ddot{x}_i + i \ddot{\psi}_i \dot{x}_i - \frac{i \gamma \dot{\psi}_i \dot{x}_i}{\sqrt[3]{(x_k x_k)^2}} + \frac{i \gamma (x_i \dot{\psi}_i)(x_j \dot{x}_j)}{\sqrt[3]{(x_k x_k)^5}}, \\ \mathcal{Q}_{1/2} &= t \mathcal{Q}_{-1/2} - i \dot{\psi}_i \ddot{x}_i + 2i \dot{\psi}_i \dot{x}_i - 3i \ddot{\psi}_i x_i, \\ \mathcal{M}_{ij} &= x_{[i} \ddot{x}_{j]} - \dot{x}_{[i} \ddot{x}_{j]} - i \dot{\psi}_{[i} \ddot{\psi}_{j]} + i \dot{\psi}_i \dot{\psi}_j - \frac{\gamma x_{[i} \dot{x}_{j]}}{\sqrt[3]{(x_s x_s)^2}} + \frac{i \gamma \dot{\psi}_i \dot{\psi}_j}{\sqrt[3]{(x_s x_s)^2}} + \frac{2i \gamma}{3} \frac{x_{[i} \dot{\psi}_{j]}(x_k \dot{\psi}_k)}{\sqrt[3]{(x_s x_s)^5}}, \end{aligned} \quad (3.25)$$

associated with the $osp(2|1) \oplus so(d)$ symmetry.

Note that the superpotential (3.22) is proportional to $\mathbb{P}_{12} = P_{ij}^{(1)} P_{ij}^{(2)}$ and consequently it vanishes when $d = 1$ because the polynomial $P_{ij}^{(1)}$ is antisymmetric. Therefore, for one dimensional case, the conserved quantities (3.25) go over to the corresponding expressions for a free $\mathcal{N} = 1$ fourth-order superparticle. However, the potential functions (3.20) with nonzero V_2 are viable in arbitrary dimension including $d = 1$. As an example, one can consider

$$V = \frac{\gamma \mathbb{P}_{03}}{2 \sqrt[3]{\mathbb{P}_0^5}}, \quad (3.26)$$

which corresponds to an $\mathcal{N} = 1$ fourth-order superconformal mechanical system which is governed by the component action (3.23) with

$$U = \frac{\dot{x}_i \dot{x}_i + i \psi_i \dot{\psi}_i}{\sqrt[3]{(x_s x_s)^2}} - \frac{2}{3} \frac{(x_i \dot{x}_i)^2 + i(x_i \psi_i)(x_j \dot{\psi}_j) - i(x_i \psi_i)(\dot{x}_j \psi_j)}{\sqrt[3]{(x_s x_s)^5}}. \quad (3.27)$$

Conserved quantities associated with the $osp(2|1) \oplus so(d)$ symmetry read

$$\begin{aligned} \mathcal{L}_{-1} &= \dot{x}_i \ddot{x}_i - \frac{1}{2} \ddot{x}_i \ddot{x}_i - i \dot{\psi}_i \ddot{\psi}_i - \frac{\gamma}{2} \frac{\dot{x}_i \dot{x}_i}{\sqrt[3]{(x_s x_s)^2}} + \frac{\gamma}{3} \frac{(x_i \dot{x}_i)^2}{\sqrt[3]{(x_s x_s)^5}}, \\ \mathcal{L}_0 &= t \mathcal{L}_{-1} - \frac{3}{2} x_i \ddot{x}_i + \frac{1}{2} \dot{x}_i \dot{x}_i + i \dot{\psi}_i \ddot{\psi}_i + \frac{\gamma}{2} \frac{x_i \dot{x}_i}{\sqrt[3]{(x_s x_s)^2}}, \\ \mathcal{L}_1 &= -t^2 \mathcal{L}_{-1} + 2t \mathcal{L}_0 + 3x_i \ddot{x}_i - 2\dot{x}_i \dot{x}_i - 2i \dot{\psi}_i \dot{\psi}_i - \frac{3\gamma}{2} \sqrt[3]{x_s x_s}, \\ \mathcal{Q}_{-1/2} &= i \psi_i \ddot{x}_i - i \dot{\psi}_i \ddot{x}_i + i \ddot{\psi}_i \dot{x}_i - \frac{i \gamma \psi_i \dot{x}_i}{\sqrt[3]{(x_k x_k)^2}} + \frac{2i \gamma}{3} \frac{(x_i \psi_i)(x_j \dot{x}_j)}{\sqrt[3]{(x_k x_k)^5}}, \\ \mathcal{Q}_{1/2} &= t \mathcal{Q}_{-1/2} - i \psi_i \ddot{x}_i + 2i \dot{\psi}_i \dot{x}_i - 3i \ddot{\psi}_i x_i + \frac{i \gamma x_i \psi_i}{\sqrt[3]{(x_s x_s)^2}}, \\ \mathcal{M}_{ij} &= x_{[i} \ddot{x}_{j]} - \dot{x}_{[i} \ddot{x}_{j]} - i \psi_{[i} \ddot{\psi}_{j]} + i \dot{\psi}_i \dot{\psi}_j - \frac{\gamma x_{[i} \dot{x}_{j]}}{\sqrt[3]{(x_s x_s)^2}} + \frac{i \gamma \psi_i \psi_j}{\sqrt[3]{(x_s x_s)^2}} + \frac{2i \gamma}{3} \frac{x_{[i} \psi_{j]}(x_k \psi_k)}{\sqrt[3]{(x_s x_s)^5}}. \end{aligned} \quad (3.28)$$

4 Conclusion

To summarize, in this work we formulated the equations (2.7), (2.8) which determine a potential function in an $\mathcal{N} = 1$ higher derivative supersymmetric mechanics compatible with the $osp(2|1) \oplus so(d)$ symmetry and provided a few explicit examples. Our strategy to obtain $osp(2|1) \oplus so(d)$ -invariant mechanics included the following steps:

- list all the polynomials $P_{ij}^{(n)}$ in (2.17) which correspond to the given order of a dynamical system;
- construct $so(d)$ -invariant combinations from these polynomials and the superfield X_i ;
- find $so(d)$ -invariant solutions of the equation (2.23) in terms of the rotationally invariant combinations;
- restrict oneself to a subset of the solutions with Grassmann-even coupling constants.

Turning to further possible developments, it would be interesting to construct quantum mechanical counterparts of the models described above. In this context, the method of conformal automorphisms previously developed in ref. [79] might be a reasonable starting point.

A generalization of the present analysis to cover interaction potentials which preserve the full $\mathcal{N} = 1$ l -conformal Galilei supersymmetry is also worth studying. In particular, it is interesting to obtain and analyse additional equations which correspond to invariance of such potential functions under symmetry transformations (2.2) generated by $C_i^{(n)}$ and $L_i^{(n)}$. This issue will be studied elsewhere [80].

Classical stability of higher-derivative systems is an important issue. Recently the concept of the so-called Lagrange anchor [81] was successfully applied to construct and investigate stable higher derivative mechanical systems (see, e.g., [82]–[84]). It is interesting to see how whether that approach can be adopted to the case of higher derivative $\mathcal{N} = 1$ superconformal mechanics.

In general, l -conformal Galilei algebra can be realized in nonrelativistic space-time with universal cosmological attraction or repulsion by means of Niederer's transformations. It would be interesting to obtain a superfield analogue of such transformation as well as to construct Newton-Hooke counterparts of the $\mathcal{N} = 1$ superconformal mechanics described above. A generalization of the analysis obtained in the present paper to the case of the so-called meta-conformal Lie algebras [85] is also worth studying.

Finally, it should be noted that the second term in the Lagrangian of the model (2.1) for $l = \frac{1}{2}$ admits an independent $\mathcal{N} = 1$ supersymmetrization. Indeed, this term can be rewritten as follows

$$-\frac{i}{2}\psi\dot{\psi}_i = -\frac{1}{2}\int d\theta\Psi_i\mathbb{D}\Psi_i, \quad (4.1)$$

where it is assumed that bosonic components x_i of the fermionic superfield $\Psi_i = \psi_i + i\theta x_i$ are auxiliary and can be eliminated. It looks very interesting to generalize this construction to the case of higher l . This may yield new $osp(2|1) \oplus so(d)$ -invariant systems with non-equal numbers of bosonic and fermionic components. We leave this issue for further study.

Acknowledgments

We thank A. Galajinsky and anonymous referees for useful comments on the manuscript. We also grateful S. Fedoruk and M. Henkel for their interest in the work. This work was supported by the Russian Science Foundation, grant No 19-11-00005.

A The symmetry transformations for the action functional (2.4)

It is straightforward to verify that the differential operators

$$\begin{aligned} L_n &= t^{n+1}\frac{\partial}{\partial t} + \frac{1}{2}(n+1)t^n\theta\frac{\vec{\partial}}{\partial\theta} + l(n+1)t^nX_i\frac{\partial}{\partial X_i}, \\ Q_r &= i\theta t^{r+1/2}\frac{\partial}{\partial t} + t^{r+1/2}\frac{\vec{\partial}}{\partial\theta} + 2l(r+1/2)\theta X_i\frac{\partial}{\partial X_i}, \\ C_i^{(n)} &= t^n\frac{\partial}{\partial X_i}, \quad L_i^{(n)} = i\theta t^n\frac{\partial}{\partial X_i}, \quad M_{ij} = X_i\frac{\partial}{\partial X_j} - X_j\frac{\partial}{\partial X_i}, \end{aligned} \quad (A.1)$$

obey the (anti)commutation relations of the $\mathcal{N} = 1$ l -conformal Galilei superalgebra

$$\begin{aligned}
[L_n, L_m] &= (m - n)L_{m+n}, & [L_n, C_i^{(m)}] &= (m - l(n + 1))C_i^{(n+m)}, \\
\{Q_r, Q_s\} &= 2iL_{r+s}, & [L_n, L_i^{(m)}] &= (m - (l - 1/2)(p + 1))L_i^{(n+m)}, \\
[L_n, Q_r] &= (r - n/2)Q_{n+r}, & [Q_r, C_i^{(n)}] &= (n - 2l(r + 1/2))L_i^{(r+n-1/2)}, \\
\{Q_r, L_i^{(n)}\} &= iC_i^{(n+r+1/2)}, & [M_{ij}, L_k^{(n)}] &= \delta_{ik}L_j^{(n)} - \delta_{jk}L_i^{(n)}, \\
[M_{ij}, C_k^{(n)}] &= \delta_{ik}C_j^{(n)} - \delta_{jk}C_i^{(n)}, & [M_{ij}, M_{kl}] &= \delta_{ik}M_{jl} + \delta_{jl}M_{ik} - \delta_{il}M_{jk} - \delta_{jk}M_{il}.
\end{aligned} \tag{A.2}$$

and generate the symmetry transformations for the action functional (2.4). As an example, let us demonstrate how one obtains the superconformal transformations by making use of $Q_{\frac{1}{2}}$ in (A.1). The symmetry transformations, which correspond to this generator, read

$$t' = t + it\alpha_{\frac{1}{2}}\theta, \quad \theta' = \theta + t\alpha_{\frac{1}{2}}, \quad X'_i(t', \theta') = X_i(t, \theta) + 2il\alpha_{\frac{1}{2}}\theta X_i(t, \theta). \tag{A.3}$$

On the other hand, we have

$$X'_i(t', \theta') = X'_i(t, \theta) + it\alpha_{\frac{1}{2}}\theta \frac{\partial X_i}{\partial t} + it\alpha_{\frac{1}{2}} \frac{\vec{\partial} X_i}{\partial \theta}. \tag{A.4}$$

By comparing (A.3) and (A.4), one obtains

$$\delta^* X_i(t, \theta) \equiv X'_i(t, \theta) - X_i(t, \theta) = it\psi_i\alpha_{\frac{1}{2}} + i\theta(t\dot{x}_i - 2lx_i)\alpha_{\frac{1}{2}}. \tag{A.5}$$

By taking into account that $\delta^* X_i(t, \theta) = \delta x_i + i\theta\delta\psi_i$, we reproduce the superconformal transformations in (2.2).

If the action functional (2.4) holds invariant under the transformations

$$t' = t + \delta t, \quad \theta' = \theta + \delta\theta, \quad X'_i(t', \theta') = X_i(t, \theta) + \delta X_i(t, \theta) \tag{A.6}$$

up to a partial time derivative of some function $F = F(t, \theta)$, i.e. $\delta S = \int dt d\theta \frac{\partial F(t, \theta)}{\partial t}$, then the corresponding conserved quantity can be derived from the expression⁹

$$K = \frac{\vec{\partial}}{\partial \theta} \left[L\delta t + \sum_{n=0}^{2l} \frac{\partial^n}{\partial t^n} \left(\delta X_i - \delta t \partial_t X_i - \delta\theta \vec{\partial}_\theta X_i \right) \sum_{k=0}^{2l-n} (-1)^k \frac{\partial^k}{\partial t^k} \frac{\partial L}{\partial (\partial_t^{n+k+1} X_i)} - F \right] \tag{A.7}$$

by discarding parameter of the transformation. Conserved charges obtained in such a way are the superfield analogues of the integrals of motion associated with the symmetries (2.2) of the component action (2.1).

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⁹Here we use the notations $\partial_t X_i = \frac{\partial X_i(t, \theta)}{\partial t}$, $\vec{\partial}_\theta X_i = \frac{\vec{\partial} X_i(t, \theta)}{\partial \theta}$. L is the Lagrangian of the model (2.4).

References

- [1] V.P. Akulov and A.I. Pashnev, *Quantum superconformal model in $(1, 2)$ space*, *Theor. Math. Phys.* **56** (1983) 862 [[INSPIRE](#)].
- [2] S. Fubini and E. Rabinovici, *Superconformal quantum mechanics*, *Nucl. Phys. B* **245** (1984) 17 [[INSPIRE](#)].
- [3] E.A. Ivanov, S.O. Krivonos and V.M. Leviant, *Geometric superfield approach to superconformal mechanics*, *J. Phys. A* **22** (1989) 4201 [[INSPIRE](#)].
- [4] D.Z. Freedman and P.F. Mende, *An exactly solvable N particle system in supersymmetric quantum mechanics*, *Nucl. Phys. B* **344** (1990) 317 [[INSPIRE](#)].
- [5] V.K. Dobrev, *Non-relativistic holography — A group-theoretical perspective*, *Int. J. Mod. Phys. A* **29** (2014) 1430001 [[arXiv:1312.0219](#)] [[INSPIRE](#)].
- [6] P. Claus et al., *Black holes and superconformal mechanics*, *Phys. Rev. Lett.* **81** (1998) 4553 [[hep-th/9804177](#)] [[INSPIRE](#)].
- [7] G.W. Gibbons and P.K. Townsend, *Black holes and Calogero models*, *Phys. Lett. B* **454** (1999) 187 [[hep-th/9812034](#)] [[INSPIRE](#)].
- [8] J. Michelson and A. Strominger, *Superconformal multiblack hole quantum mechanics*, *JHEP* **09** (1999) 005 [[hep-th/9908044](#)] [[INSPIRE](#)].
- [9] A. Maloney, M. Spradlin and A. Strominger, *Superconformal multiblack hole moduli spaces in four-dimensions*, *JHEP* **04** (2002) 003 [[hep-th/9911001](#)] [[INSPIRE](#)].
- [10] A.V. Galajinsky, *Comments on $N = 4$ superconformal extension of the Calogero model*, *Mod. Phys. Lett. A* **18** (2003) 1493 [[hep-th/0302156](#)] [[INSPIRE](#)].
- [11] S. Fedoruk, E. Ivanov and O. Lechtenfeld, *Superconformal Mechanics*, *J. Phys. A* **45** (2012) 173001 [[arXiv:1112.1947](#)] [[INSPIRE](#)].
- [12] A. Galajinsky, *$\mathcal{N} = 4$ superconformal mechanics from the $SU(2)$ perspective*, *JHEP* **02** (2015) 091 [[arXiv:1412.4467](#)] [[INSPIRE](#)].
- [13] E. Ivanov, S. Sidorov and F. Toppan, *Superconformal mechanics in $SU(2|1)$ superspace*, *Phys. Rev. D* **91** (2015) 085032 [[arXiv:1501.05622](#)] [[INSPIRE](#)].
- [14] S. Fedoruk and E. Ivanov, *New realizations of the supergroup $D(2, 1; \alpha)$ in $\mathcal{N} = 4$ superconformal mechanics*, *JHEP* **10** (2015) 087 [[arXiv:1507.08584](#)] [[INSPIRE](#)].
- [15] A. Galajinsky and O. Lechtenfeld, *Superconformal $SU(1, 1|n)$ mechanics*, *JHEP* **09** (2016) 114 [[arXiv:1606.05230](#)] [[INSPIRE](#)].
- [16] E. Ivanov, O. Lechtenfeld and S. Sidorov, *$SU(2|2)$ supersymmetric mechanics*, *JHEP* **11** (2016) 031 [[arXiv:1609.00490](#)] [[INSPIRE](#)].
- [17] A. Galajinsky, *Couplings in $D(2, 1; \alpha)$ superconformal mechanics from the $SU(2)$ perspective*, *JHEP* **03** (2017) 054 [[arXiv:1702.01955](#)] [[INSPIRE](#)].
- [18] D. Chernyavsky, *Super 0-brane action on the coset space of $D(2, 1; \alpha)$ supergroup*, *JHEP* **09** (2017) 054 [[arXiv:1707.00437](#)] [[INSPIRE](#)].
- [19] N. Kozyrev et al., *Curved Witten-Dijkgraaf-Verlinde-Verlinde equation and $\mathcal{N} = 4$ mechanics*, *Phys. Rev. D* **96** (2017) 101702 [[arXiv:1710.00884](#)] [[INSPIRE](#)].
- [20] N. Kozyrev et al., *$\mathcal{N} = 4$ supersymmetric mechanics on curved spaces*, *Phys. Rev. D* **97** (2018) 085015 [[arXiv:1711.08734](#)] [[INSPIRE](#)].

- [21] S. Fedoruk, E. Ivanov, O. Lechtenfeld and S. Sidorov, *Quantum $SU(2|1)$ supersymmetric Calogero-Moser spinning systems*, *JHEP* **04** (2018) 043 [[arXiv:1801.00206](#)] [[INSPIRE](#)].
- [22] D. Chernyavsky, *$SU(1,1|N)$ superconformal mechanics with fermionic gauge symmetry*, *JHEP* **04** (2018) 009 [[arXiv:1803.00343](#)] [[INSPIRE](#)].
- [23] S. Krivonos, O. Lechtenfeld and A. Sutulin, *N -extended supersymmetric Calogero models*, *Phys. Lett. B* **784** (2018) 137 [[arXiv:1804.10825](#)] [[INSPIRE](#)].
- [24] J. Mateos Guilarte and M.S. Plyushchay, *Nonlinear symmetries of perfectly invisible PT -regularized conformal and superconformal mechanics systems*, *JHEP* **01** (2019) 194 [[arXiv:1806.08740](#)] [[INSPIRE](#)].
- [25] L. Inzunza and M.S. Plyushchay, *Hidden symmetries of rationally deformed superconformal mechanics*, *Phys. Rev. D* **99** (2019) 025001 [[arXiv:1809.08527](#)] [[INSPIRE](#)].
- [26] D. Chernyavsky, *On $OSp(N|2)$ superconformal mechanics*, *JHEP* **02** (2019) 170 [[arXiv:1810.01626](#)] [[INSPIRE](#)].
- [27] E. Ivanov, O. Lechtenfeld and S. Sidorov, *Deformed $N = 8$ supersymmetric mechanics*, *Symmetry* **11** (2019) 135.
- [28] S. Krivonos, O. Lechtenfeld, A. Provorov and A. Sutulin, *Extended supersymmetric Calogero model*, *Phys. Lett. B* **791** (2019) 385 [[arXiv:1812.10168](#)] [[INSPIRE](#)].
- [29] S. Krivonos, O. Lechtenfeld and A. Sutulin, *Supersymmetric many-body Euler-Calogero-Moser model*, *Phys. Lett. B* **790** (2019) 191 [[arXiv:1812.03530](#)] [[INSPIRE](#)].
- [30] A. Galažinsky and O. Lechtenfeld, *Spinning extensions of $D(2,1;\alpha)$ superconformal mechanics*, *JHEP* **03** (2019) 069 [[arXiv:1902.06851](#)] [[INSPIRE](#)].
- [31] J.P. Gauntlett, J. Gomis and P.K. Townsend, *Supersymmetry and the physical phase space formulation of spinning particles*, *Phys. Lett. B* **248** (1990) 288 [[INSPIRE](#)].
- [32] M. Leblanc, G. Lozano and H. Min, *Extended superconformal Galilean symmetry in Chern-Simons matter systems*, *Annals Phys.* **219** (1992) 328 [[hep-th/9206039](#)] [[INSPIRE](#)].
- [33] P.A. Horvathy, *Non-relativistic conformal and supersymmetries*, *Int. J. Mod. Phys. A* **3** (1993) 339 [[arXiv:0807.0513](#)] [[INSPIRE](#)].
- [34] C. Duval and P.A. Horvathy, *On Schrödinger superalgebras*, *J. Math. Phys.* **35** (1994) 2516 [[hep-th/0508079](#)] [[INSPIRE](#)].
- [35] M. Henkel and J. Unterberger, *Supersymmetric extensions of Schrödinger-invariance*, *Nucl. Phys. B* **746** (2006) 155 [[math-ph/0512024](#)] [[INSPIRE](#)].
- [36] S. Fedoruk, E. Ivanov and O. Lechtenfeld, *Supersymmetric Calogero models by gauging*, *Phys. Rev. D* **79** (2009) 105015 [[arXiv:0812.4276](#)] [[INSPIRE](#)].
- [37] P.D. Alvarez, J.L. Cortes, P.A. Horvathy and M.S. Plyushchay, *Super-extended noncommutative Landau problem and conformal symmetry*, *JHEP* **03** (2009) 034 [[arXiv:0901.1021](#)] [[INSPIRE](#)].
- [38] A. Galažinsky and I. Masterov, *Remark on quantum mechanics with $N = 2$ Schrödinger supersymmetry*, *Phys. Lett. B* **675** (2009) 116 [[arXiv:0902.2910](#)] [[INSPIRE](#)].
- [39] A. Galažinsky, *$N = 2$ superconformal Newton-Hooke algebra and many-body mechanics*, *Phys. Lett. B* **680** (2009) 510 [[arXiv:0906.5509](#)] [[INSPIRE](#)].

- [40] A. Galaĭinsky and O. Lechtenfeld, *Harmonic $N = 2$ mechanics*, *Phys. Rev. D* **80** (2009) 065012 [[arXiv:0907.2242](#)] [[INSPIRE](#)].
- [41] A. Galaĭinsky, *Conformal mechanics in Newton-Hooke spacetime*, *Nucl. Phys. B* **832** (2010) 586 [[arXiv:1002.2290](#)] [[INSPIRE](#)].
- [42] P.A. Horvathy, M.S. Plyushchay and M. Valenzuela, *Supersymmetry of the planar Dirac-Deser-Jackiw-Templeton system and of its non-relativistic limit*, *J. Math. Phys.* **51** (2010) 092108 [[arXiv:1002.4729](#)] [[INSPIRE](#)].
- [43] S. Fedoruk and J. Lukierski, *The algebraic structure of Galilean superconformal symmetries*, *Phys. Rev. D* **84** (2011) 065002 [[arXiv:1105.3444](#)] [[INSPIRE](#)].
- [44] N. Kozyrev, S. Krivonos, O. Lechtenfeld and A. Nersessian, *Higher-derivative $N = 4$ superparticle in three-dimensional spacetime*, *Phys. Rev. D* **89** (2014) 045013 [[arXiv:1311.4540](#)] [[INSPIRE](#)].
- [45] I. Masterov, *Dynamical realizations of $N = 1$ l -conformal Galilei superalgebra*, *J. Math. Phys.* **55** (2014) 102901 [[arXiv:1407.1438](#)] [[INSPIRE](#)].
- [46] I. Masterov, *Higher-derivative mechanics with $N = 2$ l -conformal Galilei supersymmetry*, *J. Math. Phys.* **56** (2015) 022902 [[arXiv:1410.5335](#)] [[INSPIRE](#)].
- [47] I. Masterov, *New realizations of $\mathcal{N} = 2l$ -conformal Newton-Hooke superalgebra*, *Mod. Phys. Lett. A* **30** (2015) 1550073 [[arXiv:1412.1751](#)] [[INSPIRE](#)].
- [48] S. Fedoruk, E. Ivanov and J. Lukierski, *From $\mathcal{N} = 4$ Galilean superparticle to three-dimensional non-relativistic $\mathcal{N} = 4$ superfields*, *JHEP* **05** (2018) 019 [[arXiv:1803.03159](#)] [[INSPIRE](#)].
- [49] M. Henkel, *Local scale invariance and strongly anisotropic equilibrium critical systems*, *Phys. Rev. Lett.* **78** (1997) 1940 [[cond-mat/9610174](#)] [[INSPIRE](#)].
- [50] J. Negro, M.A. del Olmo and A. Rodriguez-Marco, *Nonrelativistic conformal groups*, *J. Math. Phys.* **38** (1997) 3786.
- [51] U. Niederer, *The maximal kinematical invariance group of the free Schrödinger equation*, *Helv. Phys. Acta* **45** (1972) 802 [[INSPIRE](#)].
- [52] J. Lukierski, P.C. Stichel and W.J. Zakrzewski, *Exotic Galilean conformal symmetry and its dynamical realisations*, *Phys. Lett. A* **357** (2006) 1 [[hep-th/0511259](#)] [[INSPIRE](#)].
- [53] S. Fedoruk, E. Ivanov and J. Lukierski, *Galilean Conformal mechanics from nonlinear realizations*, *Phys. Rev. D* **83** (2011) 085013 [[arXiv:1101.1658](#)] [[INSPIRE](#)].
- [54] C. Duval and P. Horvathy, *Conformal Galilei groups, Veronese curves and Newton-Hooke spacetimes*, *J. Phys. A* **44** (2011) 335203 [[arXiv:1104.1502](#)] [[INSPIRE](#)].
- [55] J. Gomis and K. Kamimura, *Schrödinger equations for higher order non-relativistic particles and N -galilean conformal symmetry*, *Phys. Rev. D* **85** (2012) 045023 [[arXiv:1109.3773](#)] [[INSPIRE](#)].
- [56] K. Andrzejewski, J. Gonera and P. Maślanka, *Nonrelativistic conformal groups and their dynamical realizations*, *Phys. Rev. D* **86** (2012) 065009 [[arXiv:1204.5950](#)] [[INSPIRE](#)].
- [57] A. Galaĭinsky and I. Masterov, *Dynamical realization of l -conformal Galilei algebra and oscillators*, *Nucl. Phys. B* **866** (2013) 212 [[arXiv:1208.1403](#)] [[INSPIRE](#)].
- [58] K. Andrzejewski, J. Gonera and A. Kijanka-Dec, *Nonrelativistic conformal transformations in Lagrangian formalism*, *Phys. Rev. D* **87** (2013) 065012 [[arXiv:1301.1531](#)] [[INSPIRE](#)].

- [59] K. Andrzejewski, J. Gonera, P. Kosiński and P. Maślanka, *On dynamical realizations of l -conformal Galilei groups*, *Nucl. Phys. B* **876** (2013) 309 [[arXiv:1305.6805](#)] [[INSPIRE](#)].
- [60] K. Andrzejewski and J. Gonera, *Dynamical interpretation of nonrelativistic conformal groups*, *Phys. Lett. B* **721** (2013) 319 [[INSPIRE](#)].
- [61] A. Galajinsky and I. Masterov, *On dynamical realizations of l -conformal Galilei and Newton-Hooke algebras*, *Nucl. Phys. B* **896** (2015) 244 [[arXiv:1503.08633](#)] [[INSPIRE](#)].
- [62] D. Chernyavsky and A. Galajinsky, *Ricci-flat spacetimes with l -conformal Galilei symmetry*, *Phys. Lett. B* **754** (2016) 249 [[arXiv:1512.06226](#)] [[INSPIRE](#)].
- [63] D. Chernyavsky, *Coset spaces and Einstein manifolds with l -conformal Galilei symmetry*, *Nucl. Phys. B* **911** (2016) 471 [[arXiv:1606.08224](#)] [[INSPIRE](#)].
- [64] I. Masterov, *Remark on higher-derivative mechanics with l -conformal Galilei symmetry*, *J. Math. Phys.* **57** (2016) 092901 [[arXiv:1607.02693](#)] [[INSPIRE](#)].
- [65] S. Krivonos, O. Lechtenfeld and A. Sorin, *Minimal realization of ℓ -conformal Galilei algebra, Pais-Uhlenbeck oscillators and their deformation*, *JHEP* **10** (2016) 078 [[arXiv:1607.03756](#)] [[INSPIRE](#)].
- [66] O. Baranovsky, *Higher-derivative generalization of conformal mechanics*, *J. Math. Phys.* **58** (2017) 082903 [[arXiv:1704.04880](#)] [[INSPIRE](#)].
- [67] D. Chernyavsky and D. Sorokin, *Three-dimensional (higher-spin) gravities with extended Schrödinger and l -conformal Galilean symmetries*, *JHEP* **07** (2019) 156 [[arXiv:1905.13154](#)] [[INSPIRE](#)].
- [68] A. Galajinsky and I. Masterov, *Dynamical realizations of l -conformal Newton-Hooke group*, *Phys. Lett. B* **723** (2013) 190 [[arXiv:1303.3419](#)] [[INSPIRE](#)].
- [69] K. Andrzejewski, A. Galajinsky, J. Gonera and I. Masterov, *Conformal Newton-Hooke symmetry of Pais-Uhlenbeck oscillator*, *Nucl. Phys. B* **885** (2014) 150 [[arXiv:1402.1297](#)] [[INSPIRE](#)].
- [70] K. Andrzejewski, *Hamiltonian formalisms and symmetries of the Pais-Uhlenbeck oscillator*, *Nucl. Phys. B* **889** (2014) 333 [[arXiv:1410.0479](#)] [[INSPIRE](#)].
- [71] K. Andrzejewski, *Conformal Newton-Hooke algebras, Niederer's transformation and Pais-Uhlenbeck oscillator*, *Phys. Lett. B* **738** (2014) 405 [[arXiv:1409.3926](#)] [[INSPIRE](#)].
- [72] I. Masterov, *$N = 2$ supersymmetric extension of l -conformal Galilei algebra*, *J. Math. Phys.* **53** (2012) 072904 [[arXiv:1112.4924](#)] [[INSPIRE](#)].
- [73] N. Aizawa, *$N = 2$ Galilean superconformal algebras with central extension*, *J. Phys. A* **45** (2012) 475203 [[arXiv:1206.2708](#)] [[INSPIRE](#)].
- [74] N. Aizawa, Z. Kuznetsova and F. Toppan, *Chiral and real $N = 2$ supersymmetric l -conformal Galilei algebras*, *J. Math. Phys.* **54** (2013) 093506 [[arXiv:1307.5259](#)] [[INSPIRE](#)].
- [75] A. Galajinsky and I. Masterov, *$N = 4l$ -conformal Galilei superalgebra*, *Phys. Lett. B* **771** (2017) 401 [[arXiv:1705.02814](#)] [[INSPIRE](#)].
- [76] A. Galajinsky and S. Krivonos, *$N = 4$ ℓ -conformal Galilei superalgebras inspired by $D(2,1;\alpha)$ supermultiplets*, *JHEP* **09** (2017) 131 [[arXiv:1706.08300](#)] [[INSPIRE](#)].
- [77] N. Aizawa, P.S. Isaac and J. Segar, *$\mathbb{Z}_2 \times \mathbb{Z}_2$ generalizations of $N = 1$ superconformal Galilei algebras and their representations*, *J. Math. Phys.* **60** (2019) 023507 [[arXiv:1808.09112](#)] [[INSPIRE](#)].

- [78] R. de Lima Rodrigues, W. Pires de Almeida and I. Fonseca Neto, *Supersymmetric classical mechanics: Free case*, [hep-th/0201242](#) [[INSPIRE](#)].
- [79] A. Galajinsky, O. Lechtenfeld and K. Polovnikov, *Calogero models and nonlocal conformal transformations*, *Phys. Lett. B* **643** (2006) 221 [[hep-th/0607215](#)] [[INSPIRE](#)].
- [80] I. Masterov, B. Merzlikin, work in progress.
- [81] P.O. Kazinski, S.L. Lyakhovich and A.A. Sharapov, *Lagrange structure and quantization*, *JHEP* **07** (2005) 076 [[hep-th/0506093](#)] [[INSPIRE](#)].
- [82] D.S. Kaparulin, S.L. Lyakhovich and A.A. Sharapov, *Classical and quantum stability of higher-derivative dynamics*, *Eur. Phys. J. C* **74** (2014) 3072 [[arXiv:1407.8481](#)] [[INSPIRE](#)].
- [83] D.S. Kaparulin and S.L. Lyakhovich, *On the stability of a nonlinear oscillator with higher derivatives*, *Russ. Phys. J.* **57** (2015) 1561 [[INSPIRE](#)].
- [84] D.S. Kaparulin, S.L. Lyakhovich and A.A. Sharapov, *Stable interactions via proper deformations*, *J. Phys. A* **49** (2016) 155204 [[arXiv:1510.08365](#)] [[INSPIRE](#)].
- [85] M. Henkel and S. Stoimenov, *Infinite-dimensional meta-conformal Lie algebras in one and two spatial dimensions*, *J. Stat. Mech.* (2019) 084009.