

Probing universalities in $d > 2$ CFTs: from black holes to shockwaves

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ABSTRACT: Gravitational shockwaves are insensitive to higher-curvature corrections in the action. Recent work found that the OPE coefficients of lowest-twist multi-stress-tensor operators, computed holographically in a planar black hole background, are insensitive as well. In this paper, we analyze the relation between these two limits. We explicitly evaluate the two-point function on a shockwave background to all orders in a large central charge expansion. In the geodesic limit, we find that the ANEC exponentiates in the multi-stress-tensor sector. To compare with the black hole limit, we obtain a recursion relation for the lowest-twist products of two stress tensors in a *spherical* black hole background, letting us efficiently compute their OPE coefficients and prove their insensitivity to higher curvature terms. After resumming the lowest-twist stress-tensors and analytically continuing their contributions to the Regge limit, we find a perfect agreement with the shockwave computation. We also discuss the role of double-trace operators, global degenerate states, and multi-stress-tensor conformal blocks. These holographic results suggest the existence of a larger universal structure in higher-dimensional CFTs.

KEYWORDS: AdS-CFT Correspondence, Black Holes, Conformal Field Theory

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1 Introduction and summary

Conformal Field Theories (CFTs) have a relatively rigid structure compared to generic quantum field theories (QFTs), which can be exploited to study some of the behavior of QFT at strong coupling. Our understanding of this structure and its consequences has advanced rapidly in recent years, especially in spacetime dimension $d > 2$, and it is likely that there remains vastly more to learn. One indication of our relative ignorance is that solutions to the constraints of crossing symmetry, as manifested in the bootstrap equation, emerge from numerical analyses as points in parameter space that miraculously survive even as the regions surrounding them are shown to be inconsistent with basic CFT principles. In rational 2d CFTs, such miracles are understood as consequences of shortening conditions of representations of the conformal algebra at special values in the space of conformal dimensions and central charge. The modes of the stress tensor in 2d are the conformal generators and therefore central to such constructions.

In addition to studying general CFTs, one may obtain stronger constraints by making “sparseness” conditions on the dimensions Δ of operators in the theory. Such sparseness conditions are generally statements forbidden certain operators with dimensions below a “gap” dimension chosen by hand. In the mildest cases, they might be simply that a certain Operator Product Expansion (OPE) contains no relevant (i.e. $\Delta < d$) operators, whereas in the strongest cases they might forbid all but a few operators in the theory to have dimensions below a parametrically large gap. These assumptions often lead to interesting consequences, and can drastically simplify the allowed space of CFTs. An important source of intuition and computations about sparseness comes from the Anti de Sitter (AdS)/CFT correspondence, where gaps in the spectrum of operators translate into gaps in the masses of bulk fields. Effective Field Theory (EFT) ideas applied to bulk theories elegantly predict and explain the relative simplicity of CFTs with very sparse spectra [1, 2] as simple consequences of the suppression of irrelevant interactions in the bulk. Some EFT type constraints are not obvious to derive from the CFT, but can be made manifest using unitarity and causality. For instance, the structures of the three-stress tensor coupling can deviate from the prediction of bulk Einstein gravity, but the deviation is perturbatively suppressed by the large gap $\frac{1}{\Delta_{\text{gap}}^{\#}}$. This was first shown in [3] using multiple small shockwaves in the AdS and further proved in the CFT [4–7]. Similar perturbative suppression also happens to the coupling between the stress-tensor sector and other single-trace states in the CFT [8, 9].

The $d > 2$ CFT structures we will be exploring go beyond the perturbative universality described above. They correspond to properties in AdS gravity that do not receive any perturbative $1/\Delta_{\text{gap}}$ corrections. In this paper, we focus on two such properties: the near boundary behavior of the AdS-Schwarzschild solutions and large gravitational shockwaves. Recently, using the blackhole solutions, [10] found that higher-curvature terms do not affect the OPE coefficients of the “lowest twist”¹ products of the CFT stress tensor. On the other hand, it was realized some time ago that higher-curvature terms do not affect the form of gravitational shockwave solutions in AdS [11]. These two results indicate that CFT four-point functions in two different limits are “universal” in the sense that they do not depend on effects that can be absorbed into higher-curvature corrections.² Any corrections to these CFT quantities must be suppressed non-perturbatively as $\sim e^{-\Delta_{\text{gap}}}$, suggesting a greater robustness at large but finite Δ_{gap} . We will explore the relation between the universality of the black-hole and the shockwave, emphasizing where they overlap and where they are complementary.³ Optimistically, their existence hints at a larger, coherent structure which contains them both.

Summary. Our main analysis is as follows. In all cases, we study facets of the heavy-light correlator

$$\langle \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \mathcal{O}_H \rangle \tag{1.1}$$

¹Recall that the twist of an operators is defined as its dimension minus its spin.

²For related recent discussions, see [12–14].

³The analogous connection between leading twist operators and Regge limits in the context of single-trace operators at weak coupling was analyzed in [15].

of two light scalar operators \mathcal{O}_L with dimension Δ_L and two heavy scalar operators \mathcal{O}_H with dimension Δ_H . We use bulk gravity to compute the two-point function of \mathcal{O}_L in the background metric produced by the heavy operator, from which we extract universal pieces of the OPE data of the CFT.

In the black hole regime, the “heavy-light” limit is defined as

$$\text{Heavy-light limit : } C_T \rightarrow \infty \text{ with } \Delta_L, \frac{\Delta_H}{C_T} \text{ fixed.} \quad (1.2)$$

The heavy operator \mathcal{O}_H then creates a black hole geometry that depends on Δ_H as well as all the higher-curvature terms in the gravity action. The two-point function of \mathcal{O}_L in this background can be interpreted as a heavy-light four-point function:⁴

$$G(z, \bar{z}) \equiv \langle \mathcal{O}_L(1) \mathcal{O}_L(1-z, 1-\bar{z}) \rangle_{\text{BH}} \cong \langle \mathcal{O}_H(\infty) \mathcal{O}_L(1) \mathcal{O}_L(1-z, 1-\bar{z}) \mathcal{O}_H(0) \rangle. \quad (1.3)$$

We can extract OPE coefficients by performing a conformal block decomposition of this correlator:

$$G(z, \bar{z}) = \sum_{\mathcal{O}} c_{\mathcal{O}} g_{\Delta_{\mathcal{O}}, J_{\mathcal{O}}}(z, \bar{z}), \quad (1.4)$$

where the sum over \mathcal{O} is a sum over primary operators, $g_{\Delta_{\mathcal{O}}, J_{\mathcal{O}}}(z, \bar{z})$ are conformal blocks, and the coefficient $c_{\mathcal{O}} = c_{LL\mathcal{O}} c_{HH\mathcal{O}}$ is the product of OPE coefficients of \mathcal{O} inside the $\mathcal{O}_L \times \mathcal{O}_L$ OPE and inside the $\mathcal{O}_H \times \mathcal{O}_H$ OPE. At leading order in the heavy-light limit, the operators \mathcal{O} that contribute are “double-trace” operators made from two \mathcal{O}_L s together with derivatives, as well as “ T^n ” operators made from n products of the stress tensor. The dimensions of the double-trace operators are $2\Delta_L$ plus integers, whereas the dimensions of the T^n operators are integers, and therefore can be cleanly separated as long as $\Delta_L \notin \mathbb{Z}$. Moreover, at each n , the lowest possible twist of the T^n operators is $\tau_{\min}(n) = n(d-2)$. For each $n \geq 2$, there are an *infinite* number of such operators with spin increasing from $2n$ to ∞ . In all our calculations so far, we have found that the OPE coefficients of these “lowest-twist” T^n s are universal in the sense that they are fixed by Δ_L, Δ_H and C_T , independently of the higher-curvature terms in the gravity action. In particular, the ratio

$$f_0 = \frac{4\Gamma(d+2)}{(d-1)^2\Gamma^2(\frac{d}{2})} \frac{\Delta_H}{C_T}, \quad (1.5)$$

(in units of the AdS radius of curvature) defined as the coefficient of the first-order correction to the bulk metric in an expansion near the AdS boundary plays an important role for us. In $d = 2$, the resummation of lowest-twist T^n operators is the holomorphic part of the Virasoro vacuum block, which is completely determined by the Virasoro algebra and encapsulates many important aspects of quantum gravity in AdS₃ (e.g. [16–22]). In $d > 2$, the behavior of such operators, even in the heavy-light limit, is much less well

⁴The \mathcal{O}_L two-point function in the black hole background is really a thermal average over the heavy operators in the four-point function. For the composites of stress tensors that we consider at large C_T , the OPE coefficients can be determined solely from a near-boundary expansion of the bulk metric [10], which should be insensitive to this distinction.

understood. We will focus on $d = 4$ for concreteness but our methods should generalize to other dimensions.

In this paper, we give a proof of this universality for $n = 2$ with a *spherical* black hole.⁵ In this case, the lowest-twist T^n dominate over those with higher twists in the limit

$$\text{Lowest-twist limit : } \bar{z} \rightarrow 0 \text{ with } \frac{\Delta_H}{C_T} \bar{z} \text{ fixed.} \quad (1.6)$$

Compared to [10], which mostly considered taking z small as well, we here keep z arbitrary. This limit has the advantage that it retains more information, and allows us to explore the behavior of the T^n contributions in a wider regime. In particular, it allows us to make a connection to shockwaves.

We find that the corresponding lowest-twist T^2 OPE coefficients in $d = 4$ take the form

$$c_{T^2, J} = \Delta f_0^2 \frac{J(J-2)\Gamma(J-3)\Gamma(J+1)}{(J+2)(J+4)(J+6)\Gamma(2J+2)} \frac{9(\Delta+1)(\Delta+2) + \Delta(\Delta+1)x_J + \Delta(\Delta+2)y_J}{\Delta-2}$$

$$x_J = \frac{3}{16}(J-1)(J-3)(J+4)(J+6), \quad y_J = -\frac{1}{8}(J-2)(J-3)(J+5)(J+6), \quad (1.7)$$

for $J = 4, 6, 8, \dots$ ($\Delta \equiv \Delta_L$ for compactness). As z is arbitrary, it can be analytically continued through a branch cut passing from $z = 1$ to ∞ to the second sheet of the correlator. We can then study the ‘‘Regge’’ limit, where

$$\text{Regge limit : } z \in \text{2nd-sheet}, \quad z, \bar{z} \rightarrow 0 \text{ with } \frac{z}{\bar{z}} \text{ fixed.} \quad (1.8)$$

Our present black-hole computation lets us resum only the lowest-twist T^2 s and thus it probes the Regge limit only at $\bar{z} \ll z \ll 1$. Moreover, the black-hole approach does not determine double-trace contributions as they require a horizon boundary condition. However, the full contribution in the Regge limit for general \bar{z}/z , including double-traces, can be computed using instead a shockwave background. Based on an earlier work [23–26], which extended Eikonal methods to AdS/CFT correlators, we will explicitly evaluate $G(z, \bar{z})$ in the Regge limit to *any order* in $\frac{1}{C_T}$; the result in $d = 4$ is

$$G_{\text{Eik}}(z, \bar{z}) = \sum_{n=0}^{\infty} \frac{(3\pi i f_0)^n}{(z\bar{z})^{\Delta+\frac{n}{2}}} \frac{(-1)^n \Gamma(1-n) \Gamma(2\Delta+n) \Gamma(2n+\Delta-1)}{\Gamma(n+1) \Gamma(\Delta-1)^2 \Gamma(\Delta+n) (2\Delta+n-1)} \quad (1.9)$$

$$\times \frac{\eta^{\Delta+\frac{n}{2}}}{\eta-1} \left(F_{\Delta, n, -1}(\eta) + \frac{(\eta-1)(\Delta+2n-1) F_{\Delta, n, 0}(\eta)}{\Delta+n} \right. \\ \left. - \frac{\eta(\Delta+2n-1)(\Delta+2n) F_{\Delta, n, 1}(\eta)}{(\Delta+n)(\Delta+n+1)} \right) + (\eta \rightarrow \eta^{-1}),$$

where $\eta \equiv \frac{\bar{z}}{z}$ and $F_{\Delta, n, a}$ is a hypergeometric function defined in (3.17). To compare with our black-hole background method, we will extract the T^n contributions and take the small \bar{z}/z limit, with the result

$$G_{\text{Eik}}(z, \bar{z})|_{C_T^{-n}} \propto \left(f_0 \frac{\bar{z}}{z^2} \right)^n \frac{\Gamma(\Delta-n) \Gamma(2n+\Delta-1)}{n! \Gamma(\Delta-1) \Gamma(\Delta)} \left(1 + \mathcal{O}\left(\frac{\bar{z}}{z}\right) \right). \quad (1.10)$$

⁵It would be more satisfactory to extend this proof to general n , but we shall not do so in this work. The general proof with a planar black hole was given in [10].

Taking $n = 2$, we find that the result exactly matches the resummation of T^2 operators from the black-hole method. Moreover, we recognize the matching piece in section 4.4 as the contribution of a spin 3 null line operator on the second sheet. From the perspective of null line operators, the $O_H O_H$ OPE and the $O_L O_L$ OPE each contributes an ANEC. The OPE of these two ANEC operators contain a spin-3 operator [27, 28] whose contribution is observed here.

Aside from its relation with the black-hole method analysis, the all-orders result (1.9) is interesting in its own right. As we discuss in section 3, in the geodesic limit $C_T, \Delta_L, \Delta_H \rightarrow \infty, z, \bar{z} \rightarrow 0$ with $\frac{\bar{z}}{z}$ and $\frac{\Delta_L \Delta_H}{C_T z}$ fixed, the result takes the form

$$\ln G_{\text{Eik}} \propto -\frac{\Delta_H \Delta_L \bar{z}}{C_T (z - \bar{z}) z} \propto \left\langle \mathcal{O}_H \left(\int_{-\infty}^{\infty} dx^- T_{--} \right) \mathcal{O}_H \right\rangle. \quad (1.11)$$

The last term above is the averaged null energy (ANEC) operator in the heavy-state background. We may interpret the behavior of G_{Eik} in this limit as the exponentiation of the ANEC.

Another interesting feature of (1.9) is the pattern of zeros and poles in the coefficients of the leading singularity (1.11). The zeros are very similar to those that appear in $d = 2$ [29], and we make some comments about how a simple derivation of these factors in $d = 2$ might generalize to higher dimensions.

Outline. We first discuss basic relations between a shockwave and a black hole in section 2. A more detailed analysis on shockwaves and the explicit Eikonal resummation are presented in section 3. In this section, we also make a couple of remarks for $d > 2$ CFTs. In section 4, we prove the universality of the lowest-twist operators with double stress-tensors with a spherical black hole, focusing on $d = 4$. We will provide a closed form for the corresponding lowest-twist OPE coefficients. After analytically continuing to the second sheet, we resum the contributions and compare with the shockwave computation. We conclude with some future directions. An appendix includes some technical details related to section 4 and several lowest-twist OPE coefficients.

Note. While this paper was in preparation, [14] appeared on arXiv, which also contains the formula (1.7) and discusses the matching between the Eikonal limit and the analytic continuation of the black-hole limit at $\mathcal{O}(\frac{1}{C_T^2})$.

2 Shockwaves and black holes

A gravitational shockwave is created when a massive source is boosted close to the speed of light. This includes, for example, boosting a black hole. A large boost increases the gravitational backreaction of the source, so naively taking the large boost limit might not result in a well-defined geometry. In a limit where the rest mass of source scales inversely with the boost factor, the resulting geometry may still be singular, but it is geodesic complete and all its geodesics will only be shifted a finite distance away from the vacuum geodesics. We shall refer to this as the shockwave limit, and the resulting geometry the shockwave solution.

The shockwave solutions in flat space were discovered in [30]. They were generalized to AdS₄ in [31], and to general dimensions [11, 32]. A salient nature of shockwave solutions is that they are universal in that they are insensitive to higher-order derivatives and/or stringy corrections to gravity [11, 33, 34]. Extracting what this shockwave universality implies for the CFT data and how is it related to the universality in the black-hole background is the main motivations of the present work.

As a warm-up, we briefly review the properties of shockwave solutions and their relations to black hole solutions in this section. We first introduce the embedding space for AdS_{d+1} and show how to create a shockwave by boosting a small black hole, following [31]. We next discuss the universality of the shockwave and make preliminary comments on its relation to the universality of black holes. In particular, we shall see that the leading order in the large boost limit for a shockwave is related to the leading order in the large r limit for a black hole. We leave more detailed discussions to later sections.

2.1 Embedding space

It is convenient to describe the Lorentzian AdS_{d+1} space in terms of an $d + 2$ dimensional embedding space $\mathbb{M}_2 \times \mathbb{M}_d$ with signature $(2, d)$,

$$ds^2 = -dx_+ dx_- - dx d\bar{x} + \sum_{i=1}^{d-2} dx_i^2. \tag{2.1}$$

We denote vectors in this embedding space using capital letters. AdS_{d+1} is represented by $X = (x_+, x_-, x, \bar{x}, x_1, \dots, x_{d-2}) \in \mathbb{M}_2 \times \mathbb{M}_d$ satisfying $X^2 = -1$; the boundary of AdS_{d+1} is represented as $P^2 = 0$ with $P \sim \lambda P$ identified, where λ is any real positive number. In the embedding space, the AdS_{d+1} isometry and the corresponding conformal transformation in the CFT can be realized as the same linear transform in $\mathbb{M}_2 \times \mathbb{M}_d$.

The empty AdS_{d+1} metric is

$$ds_0^2 = -(r^2 + 1) dt^2 + \frac{dr^2}{r^2 + 1} + r^2 d\Omega_{d-1}, \tag{2.2}$$

where $d\Omega_{d-1}$ is the volume form of the $d - 1$ dimensional unit sphere. One can relate the embedding space coordinate to the spherical coordinate via the map

$$X = (x_{-1}, x_0, x_1, \dots, x_d) = \left(\sqrt{1 + r^2} \cos t, \sqrt{1 + r^2} \sin t, r\hat{x} \right), \tag{2.3}$$

where $\hat{x} \in \Omega_{d-1}$ is a d -dimensional unit vector. This transforms the empty AdS metric to

$$ds_0^2 = -dx_{-1}^2 - dx_0^2 + \sum_{i=1}^d dx_i^2. \tag{2.4}$$

It is easy to check that $X^2 = -1$ using this metric. It is related to the lightcone coordinate used in (2.1) by $x_{\pm} = x_0 \pm x_{d-1}$, $x = x_{-1} + x_d$ and $\bar{x} = x_{-1} - x_d$.

Relatedly, another useful coordinate system is the Poincare coordinate, in which the empty AdS metric takes the form

$$ds_0^2 = \frac{1}{y^2} \left(dy^2 - dudv + \sum_{i=1}^{d-2} dw_i^2 \right), \tag{2.5}$$

where $y \in (0, \infty)$ is the radial coordinate with $y = 0$ being the boundary. Here

$$(x_+, x_-, x, \bar{x}, x_1, \dots, x_{d-2}) = \frac{1}{y} \left(y^2 - uv + \sum_{i=1}^{d-2} w_i^2, 1, u, v, w_1, \dots, w_{d-2} \right). \quad (2.6)$$

A boost in AdS along a boundary direction can be represented as $(u, v) \rightarrow (\lambda u, \lambda^{-1}v)$ with other coordinates fixed. This is also a boost in the embedding space with $(x, \bar{x}) \rightarrow (\lambda x, \lambda^{-1}\bar{x})$. Note that the embedding space is symmetric under the exchange of $(x, \bar{x}) \leftrightarrow (x_+, x_-)$ while there is no such a manifest invariance in the Poincare coordinate.

2.2 Creating a shockwave by boosting a small black hole

We here demonstrate that a shockwave metric can be created by boosting a small black hole. See [31] for more details. We focus on $d = 4$ but the method generalizes.

The AdS₅-Schwarzschild metric reads

$$ds^2 = - \left(r^2 + 1 - \frac{2M}{r^2} \right) dt^2 + \frac{1}{r^2 + 1 - \frac{2M}{r^2}} dr^2 + r^2 d\Omega_3^2. \quad (2.7)$$

As explained above, we shall take the small M limit, where

$$ds^2 = ds_0^2 + M ds_1^2 + \mathcal{O}(M^2), \quad ds_1^2 = \frac{2}{r^2} \left(dt^2 + \frac{1}{(1+r^2)^2} dr^2 \right). \quad (2.8)$$

A boost $(x, \bar{x}) \rightarrow (\lambda x, \lambda^{-1}\bar{x})$ leaves ds_0^2 invariant and transforms ds_1^2 . The shockwave solution, which contains a Dirac δ -function, will emerge from the large boost limit of $\lambda \rightarrow 0$. To understand the appearance of this δ -function, consider a generic function $\Omega(x, \lambda)$ satisfying

$$\Omega(x \neq 0, 0) = 0, \quad \text{and} \quad \Omega(\lambda \bar{x}, \lambda) = \frac{1}{\lambda} g(\bar{x}). \quad (2.9)$$

Thus,

$$\lim_{\lambda \rightarrow 0} \Omega(x, \lambda) = \delta(x) \int_{-\infty}^{\infty} g(\bar{x}) d\bar{x}. \quad (2.10)$$

The boosted metric $ds_{1,b}^2$ takes the following form after a further coordinate transformation with $x \rightarrow x - \lambda^2 \bar{x}$:

$$ds_{1,b}^2 = \frac{1}{\lambda} \left(\Omega_1 dx^2 + \lambda \Omega_2 dx d\bar{x} + \lambda^2 \Omega_3 d\bar{x}^2 \right) \quad (2.11)$$

where $\Omega_{1,2,3}$ are functions satisfying (2.9). For instance,

$$\Omega_1 = \frac{32m \left(\frac{(x_0^2+1)x^2}{\lambda^2} + 4(x_0-1)x_0^2(x_0+1) \right)}{\lambda \left(\frac{x^2}{\lambda^2} + 4x_0^2 - 4 \right)^2 \left(\frac{x^2}{\lambda^2} + 4x_0^2 \right)^2}, \quad m = \frac{M}{\lambda}. \quad (2.12)$$

The expressions of Ω_2, Ω_3 become lengthy and we will not list them here. In the large boost, small mass limit where $\lambda \rightarrow 0, m = \frac{M}{\lambda}$ fixed, the dx^2 term is the only surviving piece. We obtain

$$\lim_{\lambda \rightarrow 0, M=\lambda m} ds_{1,b}^2 = - \frac{3\pi m \left(2x_0 \left(\sqrt{x_0^2-1} - x_0 \right) + 1 \right)}{2\sqrt{x_0^2-1}} \delta(x) dx^2 \quad (2.13)$$

With an infinite boost, a point mass generates a singular metric that remains well-behaved as it only causes a finite shift in the geodesic. The prefactor in (2.13) corresponds to a propagator in the transverse hyperbolic space with $x = 0$, which will be given later in (3.8).

More generally, one can consider

$$ds^2 = -(1 + r^2 f(r)) dt^2 + \frac{1}{1 + r^2 h(r)} dr^2 + r^2 d\Omega_3^2 \quad (2.14)$$

as the spherically symmetric vacuum solution of a more general gravitational theory and adopt a near-boundary expansion⁶

$$f(r) = 1 - \frac{f_0}{r^4} + \dots, \quad h(r) = 1 - \frac{h_0}{r^4} + \dots \quad (2.15)$$

with possible higher-order corrections. However, as the metric ds_1^2 defined in (2.8) is not affected by higher-order corrections, the shockwave solution is manifestly universal.

3 Eikonal limit resummation

In this section, starting with the results of [23], we will obtain the Eikonal limit of $d = 4$ heavy-light four-point functions from AdS₅, to all orders in $\frac{1}{C_T}$. We will compare to the computation of the lowest-twist stress tensors from a black hole background in Euclidean space at $\frac{1}{C_T^2}$. At this order $\frac{1}{C_T^2}$, only T^2 operators contribute, but already that will be enough to see explicitly how the lowest-twist black hole and shockwave computations overlap. The generalization to any higher order should be straightforward in principle, though computationally more involved.

3.1 Eikonal AdS/CFT

We begin by briefly reviewing the relevant results of [23]; see there for more details. The main physical point is that the Eikonal approximation for CFT heavy-light four-point correlators can be computed by treating the two heavy operators in the correlator as sources of a shockwave, and solving for the two-point function for the remaining two light operators in the shockwave geometry (see Fig-1). The two heavy operators are almost null separated from each other and generates a shockwave on a null sheet.

In embedding space notation, we can take the two light operators to be at P_{L_1}, P_{L_2} with

$$P_{L_1} = (1, 0, 0, 0, 0, 0), \quad P_{L_2} = (q\bar{q}, -1, \mathbf{q}), \quad \mathbf{q} = -(q, \bar{q}, 0, 0), \quad (3.1)$$

and the two heavy source operators to be at P_{H_1}, P_{H_2} with

$$P_{H_1} = (0, 1, 0, 0, 0, 0), \quad P_{H_2} = (-1, p\bar{p}, \mathbf{p}), \quad \mathbf{p} = -(p, \bar{p}, 0, 0). \quad (3.2)$$

To connect to standard representations of the four-point function in terms of the $d = 4$ positions of the operators, we need the following formulae for the z, \bar{z} coordinates in terms of \mathbf{p}, \mathbf{q} :

$$z\bar{z} = \mathbf{q}^2 \mathbf{p}^2 = p\bar{p}q\bar{q}, \quad z + \bar{z} = 2\mathbf{p} \cdot \mathbf{q} = p\bar{q} + q\bar{p}. \quad (3.3)$$

⁶One can show by performing a conformal block decomposition that $h_0 = f_0$ [10] if the stress tensor is the only dimension ≤ 4 operator in the theory.

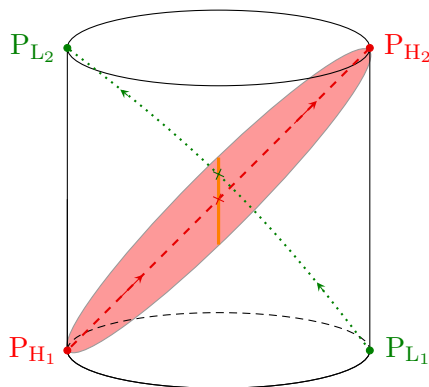


Figure 1. Operators in the shockwave geometry.

Without loss of generality, we take $p = \bar{p} = -1$ to get the simple relations

$$q = z, \quad \bar{q} = \bar{z}. \quad (3.4)$$

The light operator two-point function in the shockwave background follows from matching the standard AdS bulk-to-boundary propagator across the shockwave itself. As a result, the AdS computation of the Eikonal heavy-light correlator $G_{\text{Eik}}(z, \bar{z})$ reduces to an integral over the shockwave surface:

$$G_{\text{Eik}}(z, \bar{z}) = \frac{\Gamma(2\Delta)}{\Gamma^2(\Delta - 1)} \int_{H_3} \frac{d^3\mathbf{x}}{(-2\mathbf{q} \cdot \mathbf{x} + h(\mathbf{x} \cdot \mathbf{p}) + i\epsilon)^{2\Delta}}. \quad (3.5)$$

The coordinate \mathbf{x} parameterizes the shockwave surface, which is described by 3d hyperbolic space H_3 :

$$\mathbf{x} = -(x, \bar{x}, x_1, x_2), \quad \mathbf{x}^2 = -1, \quad \int_{H_3} d^3\mathbf{x} = \int dx d\bar{x} dx_1 dx_2 \delta(\mathbf{x}^2 + 1). \quad (3.6)$$

The function h is the metric deformation in the shockwave background. In the limit where the shockwave is created by heavy operators, it is given by the propagator in hyperbolic space:⁷

$$h(\mathbf{x} \cdot \mathbf{p}) = Az^{-3} {}_2F_1\left(3, \frac{5}{2}, 5, \frac{1}{z}\right), \quad (3.8)$$

where $A \equiv A_4 = \frac{-5i\pi}{2} \frac{\Delta_H}{C_T}$, and z is the chordal distance between \mathbf{x} and \mathbf{p} :

$$z \equiv -\frac{(\mathbf{x} - \mathbf{p})^2}{4} = \frac{1 + \mathbf{x} \cdot \mathbf{p}}{2} = \frac{1}{2} \left(1 + \frac{x + \bar{x}}{2}\right). \quad (3.9)$$

⁷Although we are focusing on $d = 4$, at this point very little changes for general d . Aside from the change in the number of components of vectors, the only difference in general d is the prefactor in G_{Eik} , and the propagator in hyperbolic space:

$$G_{\text{Eik}}(z, \bar{z}) = \frac{\Gamma(2\Delta)}{\Gamma^2(\Delta + 1 - \frac{d}{2})} \int_{H_{d-1}} \frac{d^{d-1}\mathbf{x}}{(2\mathbf{q} \cdot \mathbf{x} + h(\mathbf{x} \cdot \mathbf{p}) + i\epsilon)^{2\Delta}},$$

$$h(\mathbf{x} \cdot \mathbf{p}) = A_d z^{-(d-1)} {}_2F_1\left(d-1, \frac{d+1}{2}, d+1, \frac{1}{z}\right). \quad (3.7)$$

3.2 Integrating over the shockwave surface

Our task now is to explicitly evaluate the integral (3.5) to all orders in an expansion in powers of h . As the only dependence in the integral on the coordinates x_1, x_2 is through the δ -function, we can eliminate them immediately:

$$\int dx_1 dx_2 \delta(-x\bar{x} + x_1^2 + x_2^2 + 1) = \pi \Theta(x\bar{x} - 1). \quad (3.10)$$

We still have to integrate over x and \bar{x} . A convenient choice of coordinates is

$$x + \bar{x} = 2 \cosh \chi, \quad x\bar{x} = \cosh^2 t. \quad (3.11)$$

This set of coordinates has the advantage that the condition $x\bar{x} \geq 1$ is automatically satisfied for t , and furthermore the propagator reduces to

$$h(\mathbf{x} \cdot \mathbf{p}) = 64A \frac{e^{-|\chi|}}{e^{2|\chi|} - 1}. \quad (3.12)$$

Note that the inverse change of coordinates has two branches

$$x, \bar{x} = \cosh \chi \pm \sqrt{\cosh^2 \chi - \cosh^2 t}, \quad (3.13)$$

and each branch covers only half of the original space. To cover the full space, we have to include both branches, which are related by $x \leftrightarrow \bar{x}$. Because the integrand is invariant if we perform $x \leftrightarrow \bar{x}$ together with $z \leftrightarrow \bar{z}$, we can integrate over only a single branch and then symmetrize on $z \leftrightarrow \bar{z}$. It will be convenient to define the ratio

$$\eta \equiv \frac{\bar{z}}{z}, \quad (3.14)$$

and $z \leftrightarrow \bar{z}$ symmetrization means $\eta \leftrightarrow \eta^{-1}$.

Changing variables in the integral and expanding in powers of h , we find, at a given n ,

$$\begin{aligned} G_{\text{Eik}}(z, \bar{z})|_{h^n} &= \frac{1}{(z\bar{z})^{\Delta + \frac{n}{2}}} \frac{(-1)^n \Gamma(n+2\Delta)}{\Gamma(n+1)\Gamma^2(\Delta-1)} \quad (3.15) \\ &\times \int_0^\infty d\chi \int_0^\chi dt \frac{\sinh 2t \sinh \chi \left(64A \frac{e^{-|\chi|}}{e^{2|\chi|} - 1}\right)^n}{\sqrt{\cosh^2 \chi - \cosh^2 t} \left(2 \cosh\left(\frac{\ln \eta}{2}\right) \sqrt{\cosh^2 \chi - \cosh^2 t} + 2 \cosh \chi \sinh\left(\frac{\ln \eta}{2}\right)\right)^{2\Delta+n}} \\ &+ (\eta \leftrightarrow \eta^{-1}). \end{aligned}$$

Integrating over t and χ gives

$$\begin{aligned} G_{\text{Eik}}(z, \bar{z})|_{h^n} &= \sum_{n=0}^\infty \frac{(64A)^n}{(z\bar{z})^{\Delta + \frac{n}{2}}} \frac{(-1)^n \Gamma(1-n) \Gamma(2\Delta+n) \Gamma(2n+\Delta-1)}{\Gamma(n+1) \Gamma(\Delta-1)^2 \Gamma(\Delta+n) (2\Delta+n-1)} \quad (3.16) \\ &\times \frac{\eta^{\Delta + \frac{n}{2}}}{\eta - 1} \left(F_{\Delta, n, -1}(\eta) + \frac{(\eta - 1)(\Delta + 2n - 1) F_{\Delta, n, 0}(\eta)}{\Delta + n} \right. \\ &\quad \left. - \frac{\eta(\Delta + 2n - 1)(\Delta + 2n) F_{\Delta, n, 1}(\eta)}{(\Delta + n)(\Delta + n + 1)} \right) + (\eta \rightarrow \eta^{-1}), \end{aligned}$$

where

$$F_{\Delta,n,a}(\eta) \equiv {}_2F_1(a + 2n + \Delta, n + 2\Delta; a + n + \Delta + 1; -\eta). \quad (3.17)$$

The result (3.16) is singular on the physical values $n \in \mathbb{N}$ due to the $\Gamma(1 - n)$ factor. This corresponds to the singularity in the propagator at $\chi \sim 0$, which is a UV divergence. We shall regulate this divergence by analytic continuation in n , i.e. by subtracting off the pole at integer values of n and keeping the finite piece. However, this regularization is not necessary, as we will see, if we restrict our attention to the contribution from T^2 operators: the singularity arises only in the OPE coefficients of the “ \mathcal{O}_L^2 ” double-trace operators.

3.3 Separating double-traces and T^n s

The expression (3.16) contains contributions from the exchange of double-trace operators made from \mathcal{O}_L as well as from stress tensors. One can separate out these two types of contributions in a small η expansion based on whether the powers of η are integers or Δ_L plus integers.⁸

We see that all the stress-tensor contributions come from the $\eta \leftrightarrow \eta^{-1}$ piece in the last line of (3.16), as the series expansion of the other pieces in (3.16) manifestly contains only powers of η of the form $\Delta + m$ for m a certain integer because of the prefactor $\eta^{\Delta + \frac{n}{2}}$ and the fact that the hypergeometric functions have regular series expansions.

Let us convert the result into a form where the T^n pieces can be read off easily. Use the following identity:

$$\begin{aligned} {}_2F_1(a, b, c, -z) &= \frac{\Gamma(b - c)\Gamma(c)}{\Gamma(c - a)\Gamma(b)} z^{-a} {}_2F_1(a, a - c + 1, a - b + 1, -z^{-1}) \\ &\quad + \frac{\Gamma(a - b)\Gamma(c)}{\Gamma(c - b)\Gamma(a)} z^{-b} {}_2F_1(b, b - c + 1, b - a + 1, -z^{-1}). \end{aligned} \quad (3.18)$$

The stress-tensor pieces can be extracted by taking the terms without η^Δ factors. We are most interested in the leading-twist contribution at each order in h^n :

$$\frac{G_{\text{Eik}}(z, \bar{z})|_{h^n}}{G_{\text{Eik}}(z, \bar{z})|_{h^0}} = (3i\pi f_0)^n \left(\frac{\bar{z}}{z^2}\right)^n \frac{\Gamma(\Delta - n)\Gamma(2n + \Delta - 1)}{n!\Gamma(\Delta - 1)\Gamma(\Delta)} \left(1 + \mathcal{O}\left(\frac{\bar{z}}{z}\right)\right). \quad (3.19)$$

We have divided by the $n = 0$ term, since this term corresponds to the identity exchange and therefore just sets the overall normalization of the external operators.

The expression (3.19) is the leading contribution of the lowest-twist T^n operators in the Regge limit with $\bar{z} \ll z$. In the next section, we will see how to reproduce the $n = 2$ term by resummation and analytic continuation of an all-orders (in spin) computation of the lowest-twist T^2 operators in a black hole background, using the methods of [10].

3.4 Cancellation of poles

Although we focus in this work on the contributions from T^n operators rather than from double-traces, they are clearly connected and it is interesting to understand when contributions from one can be used to determine contributions from the other. One relatively

⁸This distinction breaks down when Δ is itself an integer.

simple connection between them is that the poles $\sim \frac{1}{\Delta-m}$ in the T^n operator contributions at positive integer values m must also be present with the opposite residue in the double-trace contributions. The reason for this relation is simply that the full correlator should not have such poles and therefore they must cancel between the two types of contributions. This relation was predicted in [10], but it was difficult to check explicitly in that context, because in a black hole background the double-trace operator coefficients depend on a boundary condition at the black hole horizon. In the shockwave case, however, we have both the T^n contributions and the double-trace contributions. In principle, the necessary cancelation is already manifest in the expression for the four-point function by virtue of the fact that it does not have any poles at $\Delta = 2, 3, 4, \dots$, so any poles in the T^n piece obviously must cancel in the full correlator. Let us here see this cancelation explicitly.

We can expand (3.16) at small η and observe the following two terms:

$$(z\bar{z})^\Delta G_{\text{Eik}} \supset \left(\frac{-2^n \sin^2(\pi\Delta) \Gamma(1-\Delta)}{n! \pi^2} \right) \left[\eta^\Delta \Gamma(2-n) \Gamma(n-\Delta) \Gamma(n+2\Delta-1) \right. \\ \left. + \eta^n \Gamma(2-\Delta) \Gamma(\Delta-n) \Gamma(\Delta+2n-1) \right]. \quad (3.20)$$

The powers of η identify the contribution in the first line as coming from a double-trace, and in the second line as coming from a T^n . At $n = 2$, the double-trace contribution has a pole $\sim \frac{1}{n-2}$, which we discard as it is due to a UV divergence as mentioned earlier. Keeping the finite piece at $n = 2$, the result has a pole at $\Delta = 2$ of the form

$$(z\bar{z})^\Delta G_{\text{Eik}}(z, \bar{z}) \supset -\frac{48\eta^2}{\Delta-2} \quad (\text{double-trace}). \quad (3.21)$$

In addition, the second line has a pole at $\Delta = 2$ of the form

$$(z\bar{z})^\Delta G_{\text{Eik}}(z, \bar{z}) \supset \frac{48\eta^2}{\Delta-2} \quad (T^2), \quad (3.22)$$

so the two poles can be seen explicitly to cancel, as required.

3.5 Zeros and degenerate states

Observe that the Γ function factors for the first few terms of (3.19) are

$$\begin{aligned} n = 0 : & \quad 1, \\ n = 1 : & \quad \Delta, \\ n = 2 : & \quad \frac{\Delta(\Delta+1)(\Delta+2)}{2(\Delta-2)}, \\ n = 3 : & \quad \frac{\Delta(\Delta+1)(\Delta+2)(\Delta+3)(\Delta+4)}{6(\Delta-3)(\Delta-2)}. \end{aligned} \quad (3.23)$$

Clearly, the entire dependence of these factors on $\Delta \equiv \Delta_L$ is fixed by a simple sequence of poles and zeros.⁹ We do not have a simple derivation of the poles, but they are natural in the

⁹The location of the poles and zeros does not fix the overall constant prefactor, but this can be fixed from knowledge of the geodesic limit computed in section 3.7.

sense that they occur for values where double-trace operators have the same dimension and spin as lowest-twist T^n s, and are indistinguishable from them in a conformal block decomposition. We will see that these poles arise as a common factor in all the OPE coefficients of leading twist T^n operators, and therefore will be easier to obtain in our black-hole calculation on the first sheet without having to do any resummation of operators. By contrast, extracting the zeros at $\Delta = -m$ for $m = 0, 1, \dots, 2(n-1)$ from a first sheet calculation requires resumming the infinite series of the lowest-twist operators, and therefore is more difficult.

Suggestively, the zeros occur exactly at values of Δ such that the primary operator \mathcal{O}_L has degenerate descendants under the global conformal algebra. In [29], it was shown how arguments based on null descendants can be used to completely fix the analogue of equation (3.19) in $d = 2$. Here we make some speculative comments about such an argument might work in $d > 2$.

In $d = 2$, it is well-known that at certain values of Δ , the primary operator has null descendants that require its correlators to obey certain differential equations. These equations restrict the allowed operators that can appear in the $\mathcal{O}_L \times \mathcal{O}_L$ OPE, and constrain the position-dependence of correlators of \mathcal{O}_L . To identify the necessary values of Δ and the corresponding differential equations, one has to look at the full Gram matrix of inner products of descendants of Δ at some level m and make sure that not only does the descendant in question have vanishing norm but moreover its overlap with all other descendants at that level vanish. Crucially, the resulting differential equations are satisfied on all sheets of the correlators, not only on the first sheet, so one can impose that only certain operators are allowed in the $\mathcal{O}_L \times \mathcal{O}_L$ OPE directly on the second sheet. The conformal weights of allowed operators in the OPE can be written explicitly at any value of C_T , but we only need them at infinite C_T where they are especially simple and are controlled by the global conformal group. For $h \equiv \Delta_L|_{d=2} = -\frac{n}{2}$, the allowed intermediate operators have weights $h_I = 0, -1, \dots, -n$. This is easy to see from the fact that at $C_T = \infty$, one of the degeneracy conditions is that the four-point function must satisfy

$$d = 2 : \quad \partial_z^{n+1} \langle \mathcal{O}_L(0) \mathcal{O}_L(z) \mathcal{O}_H(1) \mathcal{O}_H(\infty) \rangle = 0. \tag{3.24}$$

Since the leading small z behavior of a block with weight h_I is $\sim z^{h_I-2h}$, one immediately sees that (3.24) can be satisfied only for the values of h_I stated above. Therefore, a singularity on the second sheet with z^{-m-1} or greater is forbidden when¹⁰

$$d = 2 : \quad h = 0, -\frac{1}{2}, -1, -\frac{3}{2}, \dots, -\frac{m}{2}, \tag{3.25}$$

and thus its coefficient must vanish for all these values of h . Conveniently, we did not need to use the form of the finite C_T degeneracy conditions for this argument.

In $d > 2$ (here $d = 4$), let us assume that, in certain effective limits, one can naturally expand the contribution of T^n operators on the second sheet in some kind of “generalized” primary operators, which may be called “higher- d Virasoro”. At infinite C_T , such generalized conformal blocks would reduce to $d = 4$ global conformal blocks, which behave on the

¹⁰We have multiplied by the prefactor $(z\bar{z})^{\Delta_L} = (z\bar{z})^{-m}$.

second sheet at small z, \bar{z} like

$$d = 4 : \quad \propto \frac{1}{(z\bar{z})^\Delta} \frac{z^a \bar{z}^b}{z - \bar{z}} \tag{3.26}$$

for values of a, b that depend on the dimension and spin of the exchanged operator. For $\Delta = -n$, the blocks again are annihilated by ∂_z^{n+1} . Taking the small z limit of an individual block, which is necessary to separate out the contributions from different blocks, and imposing $\partial_z^{n+1} = 0$, we see that the allowed values of a are $a = 0, -1, \dots, -n$. To connect to our leading-twist limit, we consider the blocks in the limit $\bar{z} \ll z$, where the power of z is z^{a-1} . Therefore, the singularities z^{-2m} must have zeros at

$$d = 4 : \quad \Delta = 0, -1, \dots, -2(m - 1), \tag{3.27}$$

which is exactly what we see in our explicit computation!

These observations are encouraging, but a potential problem could be that, in the first limit, $z \ll \bar{z}$, there is no obvious reason why there could *not* be a cancellation between the leading-twist and subleading-twist operators, yet we are attempting to constrain the contribution from the leading-twist operators alone. Nevertheless, it might be possible that the leading twists can be thought of as a special kind of conformal block.

3.6 Factorization assumption and light-light limit

So far, we have been working in the heavy-light limit, where the heavy operators \mathcal{O}_H in the 4-point function have dimensions Δ_H that scale like the central charge of the theory, whereas the light operators are parametrically smaller than the central charge. As a result, although our expression (3.19) has complicated dependence on Δ_L , all the dependence on Δ_H comes in simple powers together with G_N in the combination $\Delta_H G_N \sim \frac{\Delta_H}{C_T} \sim f_0$. If Δ_H is taken to scale to infinity less quickly than C_T , then the true expression will have subleading corrections suppressed by powers of $\frac{1}{\Delta_H}$. However, we can try to infer these corrections from the large Δ_H limit that we already know. The intuition is that the leading singularities on the second sheet can be thought of as coming from a single operator, and therefore their coefficients should factor into a product of a function of Δ_L only and a function of Δ_H only. In $d = 2$, the second sheet OPE of null operators is simple and under rigorous control, and it is easy to see in that case that this “single operator” assumption is valid. In $d > 2$, the results of [35] suggest that the leading singularity (3.19) should be thought of as coming from a single “light-ray” operator,¹¹ so that factorization should hold here as well.

Let us assume the coefficients do factorize. One can then obtain the exact expression (at each order in a $1/C_T$ expansion but with $\Delta_H, \Delta_L \sim \mathcal{O}(1)$) by factorization together with the fact that the result must be symmetric in $\Delta_H \leftrightarrow \Delta_L$. Therefore, the full Δ_H dependence is identical to the Δ_L dependence we have already computed, and the resulting “light-light” correlator in the Regge limit at $\bar{z} \ll z$ in $d = 4$ is

$$\frac{G_{\text{Eik}}(z, \bar{z})|_{h^n}}{G_{\text{Eik}}(z, \bar{z})|_{h^0}} = (160i\pi)^n \left(\frac{\bar{z}}{z^2 C_T} \right)^n \frac{\Gamma(\Delta_L - n) \Gamma(\Delta_L + 2n - 1)}{n! \Gamma(\Delta_L - 1) \Gamma(\Delta_L)} \frac{\Gamma(\Delta_H - n) \Gamma(\Delta_H + 2n - 1)}{\Gamma(\Delta_H - 1) \Gamma(\Delta_H)}. \tag{3.28}$$

We have used the fact that (3.19) must be reproduced in the limit $\Delta_H \rightarrow \infty$.

¹¹In particular, as we will discuss in section 4.4, the leading T^2 singularity is due to an isolated pole at $J = 3$ in the analytic continuation of the lowest-twist T^2 OPE data.

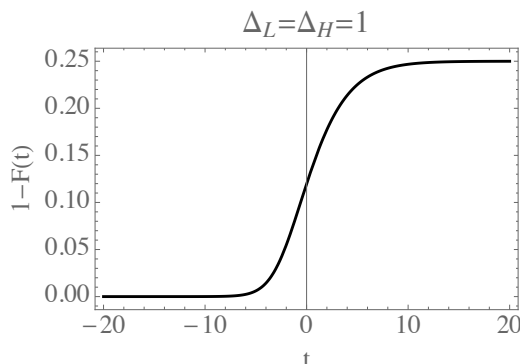


Figure 2. $F(t) \equiv \frac{3}{4} + \sqrt{\frac{x}{8\pi}} e^x K_0(x) |_{x=e^{-t}}$.

We would like to sum the above expression on n in order to obtain a formula for the Regge limit at large C_T with

$$\frac{\bar{z}}{z^2 C_T} \quad \text{fixed.} \quad (3.29)$$

The naive sum has zero radius of convergence, but its Borel transform can be written in closed form, allowing us to obtain the following expression that interpolates between the region of large and small $\frac{\bar{z}}{z^2 C_T}$:

$$\frac{G_{\text{Eik}}(z, \bar{z})|_{h^n}}{G_{\text{Eik}}(z, \bar{z})|_{h^0}} = \int_0^\infty dt e^{-t} {}_4F_3 \left(\frac{\Delta_H}{2} - \frac{1}{2}, \frac{\Delta_H}{2}, \frac{\Delta_L}{2} - \frac{1}{2}, \frac{\Delta_L}{2}; 1, 1 - \Delta_H, 1 - \Delta_L; \frac{2560i\pi t \bar{z}}{C_T z^2} \right). \quad (3.30)$$

The above formula simplifies in the limit $\Delta_L = \Delta_H = 1$ to be¹²

$$\frac{G_{\text{Eik}}(z, \bar{z})|_{h^n}}{G_{\text{Eik}}(z, \bar{z})|_{h^0}} = \frac{3}{4} + \sqrt{\frac{x}{8\pi}} e^x K_0(x), \quad x \equiv \frac{iC_T z^2}{5120\pi \bar{z}}, \quad (3.31)$$

where K_0 is a Bessel function. The correlator is shown in this limit in figure 2, as a function of $t = -\log(x)$. In $d = 2$, the coordinate transformation $z = e^{2i\pi t/\beta}$ maps the theory to finite temperature, and the analogous plot to figure 2 shows the onset of “chaos” through the growth of out-of-time correlators at finite temperature [29]. In $d = 4$, the relation between zero and nonzero temperature is not just a coordinate transformation, and we do not have a corresponding simple physical interpretation of the behavior in figure 2.

3.7 Geodesic and exponentiation of ANEC

Let us here take the large Δ_L limit after taking other limits, so that the light operator travels along a geodesic. This limit automatically includes only the multi-stress-tensor contributions and we no longer need to eliminate the contribution from the double-traces made from the external operators.

To see this from (3.5), we take the leading order in the saddle point approximation. Denote

$$k(x, z, \bar{z}) \equiv 2\mathbf{q} \cdot \mathbf{x} + h(\mathbf{p} \cdot \mathbf{x}). \quad (3.32)$$

¹²This limit is taken after fixing n and before performing the resummation.

The saddle point of the integral is given by $\partial_x k = 0$. Plugging in the solution x_* , we find

$$k(x_*, z > \bar{z}) = -\sqrt{z\bar{z}} \left(1 - \frac{64A\bar{z}}{(z-\bar{z})z} + \mathcal{O}(A^2) \right), \quad (3.33)$$

$$k(x_*, z < \bar{z}) = k(x_*, z > \bar{z}) \Big|_{z \leftrightarrow \bar{z}}. \quad (3.34)$$

Kinematically, the difference between $z > \bar{z}$ and $z < \bar{z}$ maps to the fact that the probe geodesic can pass the source geodesic from the left or the right side.

Let us check that our result agrees with the single stress-tensor conformal block at linear order. The single stress tensor block is

$$g_T^{4d} = \frac{30}{z\bar{z}(z-\bar{z})} \left(-(z-6)z-6\bar{z}^2 \ln(1-z) + z(6(z-\bar{z})\bar{z} + z((\bar{z}-6)\bar{z} + 6) \ln(1-\bar{z})) \right). \quad (3.35)$$

On the second sheet,

$$\lim_{z \rightarrow 0, \frac{z}{\bar{z}} \text{ fixed}} g_T^{4d} \Big|_{(1-z) \rightarrow (1-z)e^{-2\pi i}} = -\frac{360i\pi\bar{z}}{(\bar{z}-z)z}, \quad (3.36)$$

$$\lim_{z \rightarrow 0, \frac{z}{\bar{z}} \text{ fixed}} g_T^{4d} \Big|_{(1-\bar{z}) \rightarrow (1-\bar{z})e^{-2\pi i}} = -\frac{360i\pi z}{(z-\bar{z})\bar{z}}. \quad (3.37)$$

The z, \bar{z} dependence that appears at linear order in A indeed agrees with the Regge limit of the stress-tensor block on the second sheet. In particular, the two kinematic configurations of the scattering correspond to continuing the 4-point function in z or \bar{z} .

We now replace the integral on the shockwave surface by the saddle point value of its integrand. We focus on the $z > \bar{z}$ case; the result for $z < \bar{z}$ can be obtained by $z \leftrightarrow \bar{z}$. We have

$$G_{\text{Eik}} \propto \frac{1}{\left(1 - \frac{64A\bar{z}}{(z-\bar{z})z} + \mathcal{O}(A^2) \right)^{2\Delta_L}}. \quad (3.38)$$

Now take the geodesic limit with

$$\Delta_L, \Delta_H, C_T \rightarrow \infty, \quad z, \bar{z} \rightarrow 0, \quad \text{with } \frac{\bar{z}}{z}, \frac{\Delta_H \Delta_L}{C_T z} \text{ fixed}. \quad (3.39)$$

Then

$$G_{\text{Eik}} \propto e^{-\frac{128A\bar{z}}{(z-\bar{z})z}\Delta_L} + \mathcal{O}\left(\frac{1}{\Delta_L}\right). \quad (3.40)$$

As the piece that is exponentiated arise from the single stress-tensor block, which becomes the ANEC on the second sheet, we may write an operator form

$$\ln G_{\text{Eik}} \propto \int dx^- T_{--}. \quad (3.41)$$

We have therefore shown that 1) there exists a subset of OPE data in the multi-stress-tensor sector that exponentiates the ANEC on the second sheet; 2) the exponentiation of ANEC gives the leading contribution in the geodesic Regge limit.

4 Spherical black holes and lowest-twist

In this section, we consider solving the scalar field equation in a spherical black hole background with a general higher-order derivatives gravity action. After describing the field equation and the change of variables, we discuss the general structure of the perturbative solution in a near-boundary expansion. An all-orders proof of the universal lowest-twist operators involving two stress tensors will be given. We will derive recursion relations which lead to a closed-form prediction for the corresponding lowest-twist OPE coefficients. Finally, we resum and perform an analytic continuation to the 2nd sheet to discuss the relationships with the shockwaves. In this section, we will use Δ without a subscript to denote Δ_L , for compactness.

4.1 Field equation

Consider the bulk Euclidean action

$$S_{\text{tot}} = \int d^4x \sqrt{g} \left(\mathcal{L}_\phi + \mathcal{L}_{\text{grav}} \right), \tag{4.1}$$

$$\mathcal{L}_\phi = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2, \quad \mathcal{L}_{\text{grav}} = R + \Lambda + \sum_i \alpha_i \mathcal{O}(R^2) + \sum_j \beta_j \mathcal{O}(R^3) + \sum_k \gamma_k \mathcal{O}(R^4) + \dots$$

where $\mathcal{L}_{\text{grav}}$ is the most general higher-derivative gravity theory. $\mathcal{O}(R^\#)$ with powers $\#$ fixed denote all possible Lorentz invariants constructed out of Riemann tensor and metric; indices i, j, k represent independent invariants. We focus on a spherical black hole with metric

$$ds^2 = (1 + r^2 f(r)) dt^2 + \frac{dr^2}{1 + r^2 h(r)} + r^2 \sum_{i=1}^3 d\Omega_i^2, \tag{4.2}$$

where $d\Omega_i^2$ with coordinates $(\theta_1, \theta_2, \theta_3)$ represents a unit 3-sphere. Using the rotation symmetry, we will remove θ_2, θ_3 dependence in the scalar and rename $\theta_1 = \theta$ in the following. The asymptotic AdS boundary conditions imply [36, 37]

$$f(r) = \frac{1}{\ell^2} - \frac{f_0}{r^4} + \dots, \quad h(r) = \frac{1}{\ell^2} - \frac{h_0}{r^4} + \dots \tag{4.3}$$

We will set the AdS radius, ℓ , to 1. In general, the higher-order corrections in $1/r$ depend sensitively on the higher-derivative terms in the gravity action. Conformal invariance in the boundary limit requires $h_0 = f_0$ [10]. The factor f_0 is give by

$$f_0 = \frac{160}{3} \frac{\Delta_H}{C_T}, \tag{4.4}$$

at large Δ_H and C_T but with the ratio fixed. A priori, the higher-order corrections to the functions $f(r)$ and $h(r)$ are generally independent. These higher-order terms, however, represent non-universal contributions. Since our final result will manifestly depend only on the coefficient f_0 of r^{-4} , and since $h_0 = f_0$, we will simply set $f(r) = h(r)$ in what follows to keep the expressions simpler.¹³

¹³We shall take the next-order term in $f(r)$ to be $\frac{f_4}{r^8}$ as there is no non-trivial solution consistent with the conformal block decomposition if $f(r)$ or $h(r)$ has $\frac{1}{r^5}$, $\frac{1}{r^6}$, or $\frac{1}{r^7}$ structure [10].

To compute the two-point function of the light operator, we solve for the bulk-to-boundary propagator in the metric (4.2). The bulk field equation is

$$(-\nabla^2 + m^2) \Phi = 0, \quad m^2 = \Delta(\Delta - d), \quad (4.5)$$

with the identification

$$\Phi(r, x_1, x_2) \equiv \langle \mathcal{O}_L(x_1) \phi_L(r, x_2) \rangle_{\text{BH}}. \quad (4.6)$$

That is, we are interested in a two-point function of the light operator in a black hole geometry created by the heavy operator. We will mainly focus on the two-point function in the OPE limit, where two operators are close and the correlator depends only on the near-boundary behavior of the bulk fields and thus justifies a large r expansion. As the OPE is a short distance expansion, i.e., an expansion in small z and \bar{z} , it can be taken regardless of the size of f_0 . The expansion is therefore the conformal block decomposition of the four-point function, which is unambiguous, and this is how we will read off the OPE coefficients.

In [10], two conjectures were made related to the spherical black hole:

- *Weak conjecture:* for each spin, the lowest-twist multi-stress-tensor operator for that spin is universal.
- *Strong conjecture:* for each number n of stress tensors, all the multi-stress-tensor operators with the lowest twist for that n are universal.

In this section, we will prove the weak version, where the non-trivial cases with $J > 2$ have the lowest-twist operators with *two* stress tensors and $J - 4$ derivatives with suitable anti-symmetrization. We expect that a direct generalization on the present work will arrive at a general proof of the strong conjecture, but we will focus on two stress tensors here.

To analyze a partial differential equation (PDE) such as the scalar field equation (4.5) in a black hole background, an important initial step often is to identify a suitable change of variables. Let us first introduce

$$A = (\sinh t)^2, \quad B = (\sin \theta)^2. \quad (4.7)$$

The scalar field equation can be written as

$$\left(C_1 \partial_r^2 + C_2 \partial_A^2 + C_3 \partial_B^2 + C_4 \partial_r + C_5 \partial_A + C_6 \partial_B - \Delta(\Delta - 4) \right) \Phi(r, A, B) = 0, \quad (4.8)$$

where the coefficients are

$$C_1 = 1 + r^2 f, \quad C_2 = \frac{4A(1+A)}{1+r^2 f}, \quad C_3 = \frac{4B(1-B)}{r^2}, \quad (4.9)$$

$$C_4 = \frac{3}{r} + 5r f + r^2 f', \quad C_5 = \frac{2+4A}{1+r^2 f}, \quad C_6 = \frac{6-8B}{r^2}. \quad (4.10)$$

The pure AdS solution is

$$\Phi_{\text{AdS}} = \left(\frac{1}{2\sqrt{(1+r^2)(1+A)} - 2r\sqrt{1-B}} \right)^\Delta. \quad (4.11)$$

We next consider the following change of variables:

$$(r, t, \theta) \rightarrow (r, A, B) \rightarrow (r, w, y), \quad (4.12)$$

where

$$w = r \left(2\sqrt{(1+r^2)(1+A)} - 2r\sqrt{1-B} \right), \quad y = 2r^2 \left(1 - \sqrt{1-B} \right). \quad (4.13)$$

The solution (4.11) has a simple form

$$\Phi_{\text{AdS}} = \left(\frac{r}{w} \right)^\Delta. \quad (4.14)$$

Although the field equation written in new variables (w, y) becomes cumbersome and we shall not list it explicitly here, these new variables better organize the structure of the perturbative solutions discussed below. In particular, these new variables will help us isolate the lowest-twist contributions.

4.2 Universal lowest-twist of T^2

In general, we can write the scalar solution as

$$\Phi = \Phi_{\text{AdS}} G(r, w, y), \quad G(r, w, y) = 1 + G^{\text{T}}(r, w, y) + G^\phi(r, w, y), \quad (4.15)$$

where the stress-tensor contributions, G^{T} , is our main focus. The double-traces, G^ϕ , on the other hand, require certain interior boundary condition and thus we mostly ignore them in the following. As observed in [10], the multi-stress-tensor conformal blocks are insensitive to the horizon boundary condition with either a planar black hole or a spherical black hole. Our goal is to show the OPE coefficients of the lowest-twist operators with two stress tensors are universal, i.e. they depend on $f(r)$ only through f_0 , in the spherical black hole case. The UV boundary conditions we impose are 1) the standard δ -function boundary condition; 2) the regularity at $w = 1$. The resulting stress-tensor contributions admit polynomial forms:

$$G^{\text{T}}(r, w, y) = \frac{1}{r^4} \sum_{i=0,2,4,6,\dots} \frac{G_i^{\text{T}}(w, y)}{r^i}, \quad (4.16)$$

with, for instance,

$$G_0^{\text{T}} = \sum_{i=-1}^2 a_i^{(0)} w^i + \sum_{j=-1}^1 b_j^{(0)} w^j y, \quad G_2^{\text{T}} = \sum_{i=-1}^3 a_i^{(2)} w^i + \sum_{j=-1}^2 b_j^{(2)} w^j y + \sum_{k=-1}^1 c_k^{(2)} w^k y^2, \quad (4.17)$$

where a, b, c are constant coefficients. It is straightforward to obtain similar polynomial forms at higher orders.¹⁴

The proof of the universal lowest-twist OPE coefficients in the planar black-hole case [10] relies on a certain “decoupling” limit, imposed on the scalar field equation (in

¹⁴The variables adopted here are different from that in [10], but the polynomial structure of the solution is formally the same.

suitable variables), which leads a bulk reduced field equation that isolates the lowest-twist operators. We shall adopt a similar strategy with a spherical black hole. Before deriving the corresponding reduced field equation, let us first discuss some general structure.

Using the variables (r, w, y) defined above and focusing on stress tensors, we find that the perturbative solution can be recast into the following form:

$$\Phi = \Phi_{\text{AdS}} \left(1 + \frac{q_2 w^2 y^2 + \mathcal{O}(y)}{r^8} + \frac{q_3 w^2 y^3 + \mathcal{O}(y^2)}{r^{10}} + \frac{q_4 w^2 y^4 + \mathcal{O}(y^3)}{r^{12}} + \dots + \mathcal{O}(f_0) \right). \quad (4.18)$$

The $\mathcal{O}(f_0)$ piece represents single stress-tensor contribution, which includes the $\mathcal{O}(\frac{1}{r^4})$ and $\mathcal{O}(\frac{1}{r^6})$ terms and also the *leading-y* term at each order in $\frac{1}{r}$. (For instance, $G_4^T \sim \mathcal{O}(\frac{1}{r^8})$ has the leading- y contribution $\sim y^3$, which is absorbed into $\mathcal{O}(f_0)$ part above.) In other words, in the spherical black hole case, the single stress-tensor contribution “contaminates” the leading- y part of the solution at each order in $\frac{1}{r}$. As we are interested in the two stress-tensor contributions $\sim f_0^2$, we will not focus on the $\mathcal{O}(f_0)$ piece in the following.

Without explicitly solving the field equation, we can first consider the boundary limit of (4.18) and perform the conformal block decomposition to find relations between q_2, q_3, q_4, \dots etc and the OPE coefficients. By the boundary limit, we mean $r \rightarrow \infty$ with t, θ fixed and identify

$$G(z, \bar{z}) = \langle \mathcal{O}_L(1) \mathcal{O}_L(1-z) \rangle_{\text{BH}} = \lim_{r \rightarrow \infty} r^\Delta \Phi(r, t, \theta). \quad (4.19)$$

Recall the map between coordinates

$$(1-z) = e^{t+i\theta}, \quad (1-\bar{z}) = e^{t-i\theta}, \quad (4.20)$$

and the scalar 4-point function [38]

$$\langle \mathcal{O}_H(0) \mathcal{O}_L(z, \bar{z}) \mathcal{O}_L(1) \mathcal{O}_H(\infty) \rangle = \sum_{\Delta_T, J} c_{\text{OPE}}(\Delta_T, J) g_{\Delta_T, J}(z, \bar{z}). \quad (4.21)$$

By substituting (4.18) into (4.19) and expanding in terms of the blocks $g_{\Delta_T, J}$, one can verify that the coefficients q_i determine the lowest-twist OPE coefficients associated with two stress tensors. Some explicit low-order relations are shown in appendix A. In the following, we will derive a reduced field equation whose solutions determine the coefficients q_i to all orders. The reduced field equation will depend on $f(r)$ through f_0 only, which implies q_i are protected. Moreover, the map between q_i and lowest-twist coefficients means that the lowest-twist coefficients are also universal.

To extract the lowest-twist coefficients, we have to look at the large y limit of solution (4.18), keeping subleading (in y) contributions. It is convenient to write

$$\frac{\Phi(r, w, u)}{\Phi_{\text{AdS}}} = 1 + \frac{1}{r^2} \left(P(w, u) + \frac{Q(w, u)}{r^2} + \dots \right), \quad u = \frac{y}{r^2}, \quad (4.22)$$

where $P(w, u)$ represents the leading- y solution containing only one stress tensor; the function $Q(w, u)$ contains all the information of the lowest-twist operators with two stress tensors. Note that one can not simply set $P(w, u) = 0$ by saying that one is only interested

in two stress tensors as the subleading-order reduced field equation can have a term $\sim f_0 P$, which is of order f_0^2 . At the field equation level, P and Q are generally entangled and we shall first obtain a leading-order reduced equation before going to the next order.

With u fixed, plugging (4.22) into the full scalar field equation gives the following reduced bulk equation for P :

$$\left(K_1 \partial_w^2 + K_2 \partial_u \partial_w + K_3 \partial_u^2 + K_4 \partial_w + K_5 \partial_u + K_6\right) P = -f_0 \Delta(\Delta + 1)u(u - 4), \quad (4.23)$$

and the next-order reduced bulk equation for P and Q reads

$$\begin{aligned} & \left(M_1 \partial_w^2 + M_2 \partial_u \partial_w + M_3 \partial_u^2 + M_4 \partial_w + M_5 \partial_u + M_6\right) P \\ & + \left(N_1 \partial_w^2 + N_2 \partial_u \partial_w + N_3 \partial_u^2 + N_4 \partial_w + N_5 \partial_u + N_6\right) Q \\ & = f_0 \Delta \left(8 - 6w + 3u(u + w - 4) + 8\Delta + (3(u - 4)u + 4(u - 2)w + w^2)\Delta\right). \end{aligned} \quad (4.24)$$

The coefficients K_i, M_i and N_i are simple but we relegate them to appendix A to avoid clutter.

As multi-stress tensors are fixed by UV boundary conditions and the above reduced field equations depend on $f(r)$ through f_0 only, we arrive at an all-order proof of the lowest-twist universality with two stress tensors in the spherical black hole case. We emphasize that these computations go beyond the geodesic approximation.

Given the reduced field equations and the general structure of the perturbative solution (4.22), we next derive recursion relations. Write

$$P(u, w) = a_{n,m} u^n w^m, \quad Q(u, w) = b_{n,m} u^n w^m. \quad (4.25)$$

We find, after matching powers,

$$a_{n,m} = \frac{(m - n - 1)a_{n-1,m} + 2(m - 2)a_{n,m-1}}{2(m - 2n - 2)}, \quad (4.26)$$

$$\begin{aligned} b_{n,m} &= \left[f_0 \kappa(\kappa + 1)(a_{n-2,m+1} - 4a_{n-1,m+1}) \right. \\ & \quad \left. + 2\kappa(m - n - 3)b_{n-1,m} + 4(\kappa - 1)(m - 3)b_{n,m-1} \right] \\ & \quad \times \frac{1}{4\kappa(m - 2n - 4)}, \quad \kappa \equiv m - \Delta, \end{aligned} \quad (4.27)$$

with initial values

$$a_{1,1} = -\frac{\Delta}{30}f_0, \quad a_{1,0} = -\frac{\Delta}{10}f_0, \quad a_{1,-1} = -\frac{\Delta}{5}f_0, \quad a_{2,-1} = -\frac{3\Delta}{140}f_0, \quad (4.28)$$

and $a_{n,m} = 0$ if $n < 1$, $m < -1$ or $m > 1$; $b_{n,m} = 0$ if $n < 0$, $m < -2$ or $m > 2$. The recursion relations allow us to compute the OPE coefficients effectively; coefficients up to $J = 18$ are listed in appendix A.

4.3 Resummation and continue to the 2nd sheet

A formula that agrees with all the lowest-twist OPE coefficients we have computed is

$$c_{T^2,J} = \gamma_J \frac{9(\Delta + 2)(\Delta + 1) + \Delta(\Delta + 1)x_J + \Delta(\Delta + 2)y_J}{\Delta - 2} \Delta f_0^2, \quad (4.29)$$

where

$$\gamma_J = \frac{J(J - 2)\Gamma(J - 3)\Gamma(J + 1)}{(J + 2)(J + 4)(J + 6)\Gamma(2J + 2)}, \quad (4.30)$$

$$x_J = \frac{3}{16}(J - 1)(J - 3)(J + 4)(J + 6), \quad (4.31)$$

$$y_J = -\frac{1}{8}(J - 2)(J - 3)(J + 5)(J + 6). \quad (4.32)$$

We next resum the contribution from the infinite set of such operators and analytically continue to the Regge limit. In the lightcone limit $\bar{z} \ll 1$, the conformal blocks take the form [38]

$$(z\bar{z})^\Delta g_{\tau,J}(z, \bar{z}) = (z\bar{z})^{\frac{\tau}{2}} z^J F\left(J + \frac{\tau}{2}, J + \frac{\tau}{2}, 2J + \tau, z\right). \quad (4.33)$$

The hypergeometric function has a branch cut from $z = 1$ to $z = \infty$. Moving z through this cut and then taking $z \sim 0$, and using the fact that $\tau = 4$ for the minimum twist T^2 operators we are considering, we have, at small z on the second sheet,

$$(z\bar{z})^\Delta g_{4,J} \sim 2\pi i \frac{\bar{z}^2}{z^{J+1}} \frac{\Gamma(2J + 4)\Gamma(2J + 3)}{\Gamma^4(J + 2)} (1 + \mathcal{O}(z)). \quad (4.34)$$

Putting this together with the OPE coefficients and keeping the most singular term (in $1/z$) for each conformal block, we find that the T^2 contribution to the four-point function is

$$(z\bar{z})^\Delta G_{T^2} \stackrel{z \sim 0}{\sim} - \left(\frac{\bar{z}}{z^2}\right)^2 \frac{9\pi^2 \Delta f_0^2}{2(\Delta - 2)} \left((\Delta + 1)(\Delta + 2) + \mathcal{O}(z) \right). \quad (4.35)$$

Note that the most singular term z^{-2} is softer than the singularity z^{-3} of even the first T^2 operator, $J = 4$. The leading term at small z agrees exactly with (3.19) at $n = 2$, completing the check that the two methods agree in their region of overlap.

4.4 T^2 Regge pole

In the above derivation, we used the explicit knowledge of the lowest-twist T^2 OPE coefficients for all J . Was it necessary to know this entire functional form in order to deduce the leading singularity at $\bar{z} \ll z \ll 1$ in the Regge limit? If it were, then one might in principle be able to work backwards and deduce the function $c_{T^2,J}$ OPE coefficients purely from the shockwave analysis. However, this does not appear to be possible. Instead, the part of the OPE coefficient that is fixed by the shockwave limit is the residue of the pole at $J = 3$.¹⁵ To see this explicitly, write the sum over the T^2 operators as an integral as follows:

$$G_{T^2}(z, \bar{z}) = \sum_{J=4}^{\infty} c_{T^2,J} g_{\tau,J}(z, \bar{z}) \sim \int \frac{dJ}{2\pi i} \frac{\pi}{2 \sin \pi J} c_{T^2,J} \left[g_{\tau,J}(z, \bar{z}) + g_{\tau,J}\left(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}\right) \right], \quad (4.36)$$

¹⁵We thank S. Caron-Huot for pointing this out to us.

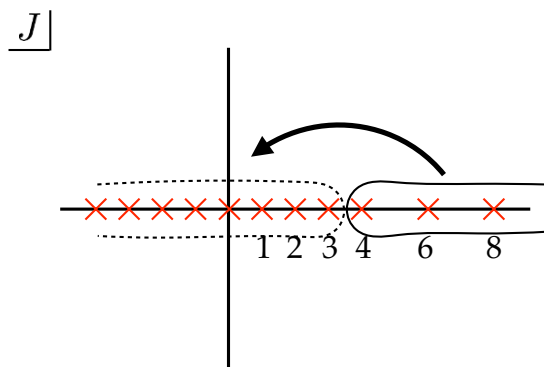


Figure 3. Deforming the contour in the complex J -plane.

where the integration contour is over the solid curve in figure 3. For integer spin, the second term in brackets is $(-1)^J$ times the first term and therefore cancels it for even spin, see e.g. [39, 40]. After analytic continuation in z , on the second sheet the blocks behave like

$$(z\bar{z})^\Delta g_{\tau,J}(z, \bar{z}) \sim 2\pi i \frac{z^{2-J}}{z-\bar{z}} \left(\frac{\bar{z}}{z}\right)^{\frac{\tau}{2}} \frac{\Gamma(\tau+2J)\Gamma(\tau+2J-1)}{\Gamma^4(\frac{\tau}{2}+J)}, \quad (4.37)$$

so that the dominant contributions at small z, \bar{z} with z/\bar{z} fixed are from the largest values of spin. It is therefore advantageous to analytically continue the contour in J to the left, as shown in figure 3. The dominant contribution can then be read off from the pole with the largest value of J contained within the contour, in this case $J = 3$:

$$(i\pi)^2 \frac{\bar{z}^2}{z^{1+J}} \frac{\Gamma(4+2J)\Gamma(3+2J)}{\Gamma^4(2+J)} c_{T^2,J} \stackrel{J \sim 3}{\sim} -\frac{1}{J-3} \left(\frac{\bar{z}}{z^2}\right)^2 \frac{9\pi^2 f_0^2 \Delta(\Delta+1)(\Delta+2)}{2(\Delta-2)}, \quad (4.38)$$

from the formula (4.29). One can think of this pole and its residue as the “intersection” of the shockwave and black hole methods for the T^2 operators. That is, it can be read off from the explicit form of $c_{T^2,J}$ from the black hole computation and then analytically continuing,¹⁶ or it can be read off from the shockwave computation by inspection of the leading singularity in the $\bar{z} \ll z \ll 1$ regime of the Regge limit at $\mathcal{O}(\frac{1}{C_T^2})$ from (3.16). The $O_H O_H$ and the $O_L O_L$ OPE each contributes an ANEC at order T^2 on the second sheet with $\bar{z} \ll z \ll 1$. These two ANEC operators contain a spin-3 operator [27, 28] and we find its contribution here.

5 Discussion

In this work, we discuss the connection between the universality in the lowest-twist limit (with a black hole) and the universality in the Regge limit (with a shockwave), as distinct pieces of a connected universality region in $d > 2$ CFTs.¹⁷ Our results do not rely on

¹⁶In order to deform the contour in the J plane as described, the OPE coefficients must be analytically continued from real integer values to complex values in such a way that they decay sufficiently rapidly at infinity. The form (1.7) has this behavior.

¹⁷While we focus on $d = 4$, we do not expect any major change in other dimensions, but it is possible that the structure in odd dimensions is more involved compared to that in even dimensions.

unitarity or supersymmetry, and they are insensitive to the higher-curvature terms in the gravity action. Evidently, in the large space of holographic CFTs above two dimensions, there is a rather special region where the CFT data are protected in the sense that they do not depend on the additional model-dependent parameters represented by such terms.

What is the “boundary” of this space of universality? The analysis of the present work provides additional hints but a complete answer to this question is still lacking. In the planar black-hole case, it was known that the sub-leading twist OPE coefficients are not universal [10]. The structure of a spherical black hole is richer and a more detailed classification is still needed. In particular, we are able to prove the weak conjecture, which states that, for each spin, the lowest-twist multi-stress-tensor operator for that spin is universal. It would be interesting to generalize our computations to prove the strong conjecture, which states that all the multi-stress-tensor operators with the lowest twist for n stress tensors are universal. It is possible that an even stronger version of universality exists in $d > 2$ CFTs and so far we have only seen the tip of the iceberg. (See also the appendix A of [10] for related remarks.) It will be interesting to carve out more precisely the region where the universality holds.¹⁸

For instance, in both shockwave and black-hole computations, matter fields in the bulk are not included. To our knowledge, the shockwave solution in the presence of bulk matter fields has not been considered in the literature. Heavy matter fields can be integrated out and their effects can be absorbed into coefficients of irrelevant operators, so we would expect any corrections to be at least nonperturbatively suppressed in a limit that the masses of all such fields becomes large. It would be useful to explicitly compute their effects as a function of mass and test this expectation, and understand in detail the behavior of such corrections in both the shockwave and black hole regimes.¹⁹

We would also like to understand from the CFT perspective *why* these universalities exist at all. In $d = 2$, the Virasoro algebra essentially determines the related structures. Above two dimensions, the stress tensors generally do not form a universal algebra. However, the results in $d > 2$ in the lowest-twist limit and also in the Regge limit share striking similarities with $d = 2$ CFTs. It is natural to ask, in the region where these universalities hold, if one could give a CFT derivation. A recent work [41] has shown that a Virasoro-like stress-tensor commutator structure effectively emerges near the lightcone in $d > 2$ CFTs. It would be interesting to search for a connection and provide an algebraic approach to these universalities. Relatedly, it will be also interesting to see if these universal results in $d > 2$ CFTs can link to recent works on ANEC and lightray operators [28, 35, 42–46].

Another potential way of trying to synthesize the “lowest-twist black hole” and “shockwave” limits of the T^n conformal blocks is to focus on how they could be embedded within a larger structure that contain both of them as limits. One option for such a structure is the contribution of all T^n operators in the limit where the bulk theory simply is Einstein gravity. In the heavy-light limit, these contributions can equivalently be defined as the parts of the multi-stress-tensor partial wave that are proportional to powers of f_0 . One may

¹⁸We have adopted heavy operators and a large central charge. It would be very interesting to study the universalities away from these limits.

¹⁹See [13] for recent work in this direction.

therefore define a special “Einstein block” that includes all T^n operators with f_n s in the metric set to zero except f_0 . By definition, it is insensitive to higher-curvature corrections. Nevertheless, we emphasize that the spirit of our present paper has been to understand results which hold beyond Einstein gravity, and thus such a special conformal block appears somewhat ad hoc — why should we set the f_n s with $n > 0$ to vanish, when any other choice for their values would also define a larger structure that includes the lowest-twist black hole and shockwave T^n s as limits? It would be nice to understand if a particular choice follows from purely CFT reasoning, without imposing a large gap in dimensions.

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A OPE and bulk coefficients

In this appendix, we provide explicit expressions for some technical details in section 4.2.

First, we list the lowest-twist coefficients with two stress tensors from $J = 4$ up to $J = 18$,

$$\begin{aligned}
 c_{\text{OPE}}(8, 4) &= \frac{\Delta(\Delta(7\Delta + 6) + 4)f_0^2}{201600(\Delta - 2)}, & c_{\text{OPE}}(10, 6) &= \frac{\Delta(\Delta(33\Delta - 7) + 4)f_0^2}{38438400(\Delta - 2)}, \\
 c_{\text{OPE}}(12, 8) &= \frac{\Delta(286\Delta^2 - 157\Delta + 12)f_0^2}{8576568000(\Delta - 2)}, & c_{\text{OPE}}(14, 10) &= \frac{\Delta(325\Delta^2 - 229\Delta + 6)f_0^2}{219011240448(\Delta - 2)}, \\
 c_{\text{OPE}}(16, 12) &= \frac{\Delta(425\Delta^2 - 336\Delta + 4)f_0^2}{5996736345600(\Delta - 2)}, & c_{\text{OPE}}(18, 14) &= \frac{\Delta(\Delta(2261\Delta - 1907) + 12)f_0^2}{638835994368000(\Delta - 2)}, \\
 c_{\text{OPE}}(20, 16) &= \frac{\Delta(\Delta(3724\Delta - 3271) + 12)f_0^2}{20422788194952000(\Delta - 2)}, & c_{\text{OPE}}(22, 18) &= \frac{\Delta(\Delta(483\Delta - 436) + 1)f_0^2}{50232688912560000(\Delta - 2)}.
 \end{aligned}$$

The relation between the first several coefficients q_i in section 4.2 and the lowest-twist T^2 coefficients are

$$\begin{aligned}
 q_2 &= 16c_{\text{OPE}}(8, 4) + \mathcal{O}(f_0), \\
 q_3 &= \frac{184}{13}c_{\text{OPE}}(8, 4) - 64c_{\text{OPE}}(10, 6) + \mathcal{O}(f_0), \\
 q_4 &= \frac{521}{65}c_{\text{OPE}}(8, 4) - \frac{1232}{17}c_{\text{OPE}}(10, 6) + 256c_{\text{OPE}}(12, 8) + \mathcal{O}(f_0), \\
 q_5 &= \frac{4078}{1105}c_{\text{OPE}}(8, 4) - \frac{16164}{323}c_{\text{OPE}}(10, 6) + \frac{7424}{21}c_{\text{OPE}}(12, 8) - 1024c_{\text{OPE}}(14, 10) + \mathcal{O}(f_0),
 \end{aligned} \tag{A.1}$$

and so on. The $\mathcal{O}(f_0)$ piece on the right-hand side of the relations represent the single stress-tensor contribution which is fixed by the Ward identity and depends only on Δ and f_0 . More precisely, $c_{\text{OPE}}(4, 2) = \frac{\Delta}{120} f_0$.

Finally, the coefficients K_i, M_i and N_i from the bulk reduced field equations are

$$\begin{aligned} K_1 &= 2w^3(u + 2w - 2), & K_4 &= 2w^2\left((2 - u)(\Delta + 1) - 2w(\Delta - 1)\right), \\ K_2 &= 2uw^2(4 - u), & K_5 &= 2uw\Delta(u - 4), \\ K_3 &= 0, & K_6 &= 4w\left(w(\Delta - 1) + (u - 2)\Delta\right), \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} M_1 &= w^2(w^2 - f_0 u(4 - u)), & N_1 &= 2w^3(u + 2w - 2), \\ M_2 &= 0, & N_2 &= 2(4 - u)uw^2, \\ M_3 &= (4 - u)uw^2, & N_3 &= 0, \\ M_4 &= -w(w^2 - 2f_0(4 - u)u\Delta), & N_4 &= -2w^2\left(u(\Delta + 3) + 2(w - \Delta + w\Delta - 3)\right), \\ M_5 &= 3(2 - u)w^2, & N_5 &= 2(u - 4)uw\Delta, \\ M_6 &= -f_0(4 - u)u\Delta(\Delta + 1), & N_6 &= 8w(u + w - 2)\Delta. \end{aligned} \tag{A.3}$$

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