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The anisotropic chiral boson

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ABSTRACT: We construct the theory of a chiral boson with anisotropic scaling, characterized by a dynamical exponent z that takes positive odd integer values. The action reduces to that of Floreanini and Jackiw in the isotropic case ($z = 1$). The standard free boson with Lifshitz scaling is recovered when both chiralities are nonlocally combined. Its canonical structure and symmetries are also analyzed. As in the isotropic case, the theory is also endowed with a current algebra. Noteworthy, the standard conformal symmetry is shown to be still present, but realized in a nonlocal way. The exact form of the partition function at finite temperature is obtained from the path integral, as well as from the trace over $\hat{u}(1)$ descendants. It is essentially given by the generating function of the number of partitions of an integer into z -th powers, being a well-known object in number theory. Thus, the asymptotic growth of the number of states at fixed energy, including subleading corrections, can be obtained from the appropriate extension of the renowned result of Hardy and Ramanujan.

KEYWORDS: Conformal and W Symmetry, Space-Time Symmetries, Field Theories in Lower Dimensions

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1 Introduction

The self-dual free boson is a relativistic two-dimensional model describing chiral massless excitations, that evolve according to the field equation

$$\dot{X}^{\pm} = \pm \partial_x X^{\pm}, \quad (1.1)$$

where the sign corresponds to left or right movers. It enjoys a number of interesting properties that appear in a variety of contexts, ranging from string theory [1, 2] to the edge description of the quantum Hall effect [3, 4]. Finding a suitable action principle for bosonic chiral fields requires to deal with certain subtleties (see e.g., [5–8]) that in the case of a scalar were already addressed long ago by Floreanini and Jackiw [9].

More recently, in the search for extensions of the holographic principle (see e.g. [10–13]), there has been interest in analyzing quantum field theories that manifestly break Lorentz invariance, but admit anisotropic scaling transformations of the form

$$t \rightarrow \lambda^z t, \quad x \rightarrow \lambda x, \quad (1.2)$$

where z is the dynamical exponent. The prototypical example of such kind of field theories is the free boson with Lifshitz scaling [14], which realizes the anisotropic symmetry due

to a quadratic term that contains higher spatial derivatives in the action. Field theories enjoying these features are of interest on their own (see e.g. [15–18]), as well as in different contexts related to condensed matter physics [19–24]. Gravitational duals in which the anisotropic scaling symmetry is realized through the (asymptotic) isometries of the so-called Lifshitz spacetimes have been studied in e.g. [10–13, 25–32]. In the context of three-dimensional gravity, it has been shown that the anisotropic scale invariance can also be induced by suitable choices of boundary conditions [33–38]. In two spacetime dimensions, field theories with generic anisotropic scaling have also been discussed in e.g. [35, 39–43].

One of the main purposes of our work is the analysis of a new bosonic quantum field theory in two dimensions that simultaneously combines chirality with anisotropy. In the case of a single free boson, the anisotropic scaling (1.2) implies that the field equation must be of the form

$$\dot{X}^\pm = \pm \sigma^{z-1} \partial_x^z X^\pm, \quad (1.3)$$

where σ is a constant with units of length, so that the speed of light becomes the unity for $z = 1$. From (1.3) it is clear that when z takes even values, the modes can neither suitably oscillate nor carry a single chirality (see below). Therefore, hereafter the dynamical exponent z is assumed to be given by an odd integer.

The paper is organized as follows. In the next section we propose an action principle for the anisotropic chiral boson that reduces to the Floreanini-Jackiw action for $z = 1$, and analyze some of its properties. We also show how to combine both chiralities in order to recover the standard free boson with Lifshitz scaling. In section 3, we study the canonical structure and also explore the global symmetries. It is found that, as in the isotropic case, the theory is also endowed with a $\widehat{u}(1)$ current algebra; and remarkably, the conformal symmetry is still present, but turns out to be non-local for $z > 1$. In section 4, the exact form of the partition function at finite temperature is obtained first from the path integral, and then from the trace over $\widehat{u}(1)$ descendants. It is essentially given by the generating function of the sequence of “power partitions”, i.e., the number of partitions of an integer into z -th powers. Therefore, the asymptotic growth of the number of states at a given energy, including subleading corrections, corresponds to power partitions associated to $\widehat{u}(1)$ descendants, and it can be obtained from the appropriate extension of the famous result of Hardy and Ramanujan [44].

2 Action principle

Let us consider the following action principle

$$S_z^\pm[X^\pm] = \int dt dx \left[\pm \partial_x X^\pm \partial_t X^\pm - \sigma^{(z-1)} \partial_x X^\pm \partial_x^z X^\pm \right], \quad (2.1)$$

which is invariant under the anisotropic scaling in (1.2), provided that the field $X^\pm$ does not transform. Note that the Floreanini-Jackiw theory is recovered for $z = 1$.

It is worth highlighting that the action (2.1) is invariant under the gauge symmetry

$$X^\pm \rightarrow X^\pm + f^\pm(t), \quad (2.2)$$

with arbitrary $f^\pm$ depending only on time. Indeed, the field equation that follows from (2.1) is

$$\partial_x \dot{X}^\pm = \pm \sigma^{z-1} \partial_x^{z+1} X^\pm, \quad (2.3)$$

and it can be readily integrated once, yielding

$$\dot{X}^\pm = \pm \sigma^{z-1} \partial_x^z X^\pm + h^\pm(t), \quad (2.4)$$

with $h^\pm$ arbitrary. Thus, $h^\pm$ can always be gauged away by virtue of the gauge transformation in (2.2) with $h^\pm = \dot{f}^\pm$, so that (2.4) precisely reduces to the field equation of the anisotropic chiral boson in (1.3), i.e.,

$$\dot{X}^\pm = \pm \sigma^{z-1} \partial_x^z X^\pm. \quad (2.5)$$

The canonical analysis of the gauge symmetry (2.2) is discussed in section 3.1.

According to (2.5) the anisotropic chiral boson evolves as a superposition of modes of the form

$$X_k^\pm = e^{i(kx \pm (i\sigma)^{z-1} k^z t)}, \quad (2.6)$$

with a phase velocity given by $c_p^\pm = \pm (i\sigma k)^{z-1}$, so that propagate in the same direction regardless the value of k . As expected, different chiralities propagate in opposite directions.¹ The magnitude of the phase velocity is then generically different for each mode, reflecting the fact that the theory is not Lorentz invariant (unless $z = 1$). A wave packet then propagates in the same direction than its composing modes, since the group velocity obeys $c_g^\pm = z c_p^\pm$.

It is worth pointing out that a parity transformation $\mathcal{P}$, defined by $x \rightarrow -x$, swaps the chirality. Indeed, the action (2.1) fulfills $\mathcal{P}[S_z^\pm] = -S_z^\mp$, regardless the field $X^\pm$ transforms as a (pseudo-)scalar. Analogously, chirality is also swapped by a time reversal transformation $\mathcal{T}$, defined through $t \rightarrow -t$, i.e., $\mathcal{T}[S_z^\pm] = -S_z^\mp$. Therefore, the joint action of parity and time reversal becomes a symmetry of the anisotropic chiral boson action (2.1) because $\mathcal{PT}[S_z^\pm] = S_z^\pm$.

2.1 Recovering the standard free boson with Lifshitz scaling

Here we show that two independent sectors with opposite chiralities can be suitably combined in order to recover the standard free boson with anisotropic scaling.

In two-spacetime dimensions, the action for a free boson with Lifshitz scaling is given by [14] (see also [35])

$$I[\varphi] = \frac{1}{2} \int dt dx \left(\dot{\varphi}^2 - \sigma^{2(z-1)} (\partial_x^z \varphi)^2 \right), \quad (2.7)$$

¹Note that if z were an even integer, the modes would not even propagate since $c_p^\pm$ becomes purely imaginary, and also sensitive to the sign of k . Besides, the second term in the action (2.1) turns out to be a boundary term in this case, so that by virtue of the gauge symmetry, the field equation would actually be $\dot{X}^\pm = 0$, generically solved by $X^\pm = g^\pm(x)$. Thus, although an alternative action principle could be written for even z , we do not consider this possibility, since the free bosonic field has no chance of being simultaneously propagating and chiral.

which is invariant under the anisotropic scaling (1.2), provided that the field transforms as

$$\varphi \rightarrow \lambda^{\frac{z-1}{2}} \varphi. \quad (2.8)$$

The field equation then corresponds to an anisotropic version of the wave equation

$$\ddot{\varphi} - \sigma^{2(z-1)} \partial_x^{2z} \varphi = 0. \quad (2.9)$$

A key feature is that (2.9) can be factorized as

$$D_+ D_- \varphi = 0, \quad (2.10)$$

where $D_{\pm} \equiv \partial_t \pm \sigma^{(z-1)} \partial_x^z$ are linear differential operators, possessing well-defined scaling behavior, since $D_{\pm} \rightarrow \lambda^{-z} D_{\pm}$. As a consequence of the linearity of the equation — and up to zero modes — the local solution to (2.9) is a d'Alembert-like superposition of two non-interacting waves

$$\varphi = \varphi^+ + \varphi^-, \quad (2.11)$$

where

$$D_{\mp} \varphi^{\pm} = 0, \quad (2.12)$$

which agree with the field equations of two anisotropic chiral bosons $X^{\pm}$ with opposite chiralities in (2.5). However, $\varphi^{\pm}$ inherit the scaling of φ in (2.8), while $X^{\pm}$ do not transform under the anisotropic scaling symmetry.

In order to see the precise link between the bosonic field φ and the chiral fields $X^{\pm}$, it is useful to express the action (2.7) in Hamiltonian form, which reads

$$I_H[\varphi, p] = \int dt dx \left[p \dot{\varphi} - \frac{1}{2} p^2 - \frac{\sigma^{2(z-1)}}{2} (\partial_x^z \varphi)^2 \right], \quad (2.13)$$

with p the canonical momentum. Now, instead of using (2.11), we focus on the following non-local field redefinition

$$\varphi = \frac{1}{\sqrt{\sigma^{(z-1)}}} \left(\partial_x^{\frac{1-z}{2}} X^+ + \partial_x^{\frac{1-z}{2}} X^- \right), \quad p = \sqrt{\sigma^{(z-1)}} \left(\partial_x^{\frac{z+1}{2}} X^+ - \partial_x^{\frac{z+1}{2}} X^- \right), \quad (2.14)$$

so that, up to zero modes, the dynamics of (2.13) can be equivalently expressed in terms of the chiral fields $X^{\pm}$. This procedure yields, up to boundary terms, the sum of two decoupled actions with opposite chiralities

$$I_H[X^+, X^-] = (-1)^{\frac{z-1}{2}} \left(S_z^+[X^+] + S_z^-[X^-] \right), \quad (2.15)$$

where $S_z^{\pm}[X^{\pm}]$ are given by (2.1).

Note that the anisotropic scaling transformation of the non-chiral field $\varphi \rightarrow \lambda^{\frac{z-1}{2}} \varphi$ is recovered from those of the chiral fields $X^{\pm} \rightarrow X^{\pm}$, by virtue of the field redefinition (2.14).

It is also worth stressing that the field redefinition of φ in (2.14) is generically non-local, except for the case $z = 1$. Nonetheless, after carefully dealing with the boundary terms, which can be consistently dropped out, the action in (2.15) becomes exclusively written in terms of local variables.

3 Canonical structure and symmetries

3.1 Hamiltonian analysis

Since the actions for chiral and antichiral fields were shown to be connected through the parity operator $\mathcal{P}$ in section 2, afterwards we drop the $\pm$ index. Thus, without loss of generality, we continue with the analysis for a chiral field $X = X^+$. It is also useful to assume that the chiral field is defined on a cylinder of radius l , so that the spatial coordinate is rescaled as $x = l\phi$, with $0 \leq \phi < 2\pi$.

In order to acquire a deeper understanding of the theory, including its local and global symmetries, here we study its canonical structure.

The action of the anisotropic chiral boson in (2.1) possesses the following Hamiltonian form

$$I_H[\Pi, X, v] = \int dt d\phi \left[\Pi \dot{X} - \alpha \partial_\phi X \partial_\phi^z X - v (\Pi - \partial_\phi X) \right], \quad (3.1)$$

where Π stands for the canonical momentum, v is a Lagrange multiplier, and $\alpha \equiv l^{-(z+1)} \sigma^{(z-1)}$. The Hamiltonian equations of motion then read

$$\dot{X} = v, \quad \dot{\Pi} = 2\alpha \partial_\phi^{z+1} X - \partial_\phi v, \quad \Pi - \partial_\phi X = 0, \quad (3.2)$$

being equivalent to (2.3). The latter equality corresponds to a primary constraint that arises from the definition of the momentum, that is included in the Hamiltonian action (3.1) along with the Lagrange multiplier. The canonical Poisson bracket can then be read off from the kinetic term of the action

$$\{X(\phi), \Pi(\phi')\} = \delta(\phi - \phi'). \quad (3.3)$$

In terms of Fourier modes, the fields expand as

$$X(t, \phi) = \sum_{n=-\infty}^{\infty} e^{in\phi} X_n(t), \quad \Pi(t, \phi) = \sum_{n=-\infty}^{\infty} e^{in\phi} \Pi_n(t), \quad v = \sum_{n=-\infty}^{\infty} e^{in\phi} v_n(t), \quad (3.4)$$

so that the Poisson bracket (3.3) reads

$$\{X_n, \Pi_m\} = \frac{1}{2\pi} \delta_{n, -m}. \quad (3.5)$$

Analogously, the smeared constraint $\theta[v] \equiv \int d\phi v (\Pi - \partial_\phi X)$, can be written as

$$\theta[v] = \sum_{n=-\infty}^{\infty} v_n \theta_n, \quad (3.6)$$

with

$$\theta_n = 2\pi (\Pi_{-n} + in X_{-n}). \quad (3.7)$$

It is then straightforward to check that

$$\{\theta_n, \theta_m\} = 4\pi in \delta_{n+m, 0}, \quad (3.8)$$

implying that $\theta_0 = 2\pi \Pi_0$ is a first class constraint, while θ_n for $n \neq 0$ are of second class.

Note that the time evolution of the constraints reads

$$\dot{\theta}_n = 4\pi i n [\alpha(in)^z X_{-n} - v_{-n}] . \quad (3.9)$$

Thus, for the mode $n = 0$ the consistency of the constraint does not add any new condition, while for $n \neq 0$ it fixes the Lagrange multiplier v_{-n} . Consequently, the first class constraint $\theta[v_0]$ generates gauge transformations, given by

$$\delta X = \{X, \theta[v_0]\} = v_0(t) , \quad (3.10)$$

corresponding to the gauge symmetry in (2.2), with $v_0(t) = f^+(t)$.

In order to deal with second class constraints, we use the Dirac bracket, defined by

$$\{X_n, \Pi_m\}_D = \{X_n, \Pi_m\} - \sum_{k \neq 0, j \neq 0} \{X_n, \theta_k\} C_{kj} \{\theta_j, \Pi_m\} , \quad (3.11)$$

where C_{kj} is the inverse of the matrix $\{\theta_k, \theta_j\}$ in (3.8), given by

$$C_{kj} = \frac{1}{4\pi i j} \delta_{k, -j} . \quad (3.12)$$

In particular, it is simple to verify that the Dirac bracket of the dynamical fields X_n and Π_m (with $n, m \neq 0$) reads

$$\{X_n, \Pi_m\}_D = \frac{1}{4\pi} \delta_{n, -m} , \quad (3.13)$$

being equivalent to

$$\{X(\phi), \Pi(\phi')\}_D = \frac{1}{2} \delta(\phi - \phi') , \quad (3.14)$$

excluding zero modes. Therefore, the second class constraints can be strongly imposed in a consistent way, so that

$$\{X(\phi), \partial_{\phi'} X(\phi')\}_D = \frac{1}{2} \delta(\phi - \phi') . \quad (3.15)$$

It is worth pointing out that the Dirac bracket in (3.15) precisely agrees with the derivative of the inverse of the symplectic form that can be obtained directly from (2.1) (see e.g. [45]). The latter bracket also helps in order to directly perform the analysis of the global symmetries and their corresponding algebras, that is carried out next.

3.2 Global symmetries

3.2.1 Kinematical symmetries & Lifshitz algebra in $2d$

Apart from the scaling symmetry in (1.2), with $X \rightarrow X$, the action of the anisotropic chiral boson (2.1) is also invariant under displacements in space and time. This set of kinematical symmetries is spanned by the vector field

$$Y = (a^t + bzt)\partial_t + (a^\phi + b\phi)\partial_\phi , \quad (3.16)$$

where a^t , a^ϕ and $b = \log \lambda$ are independent constants. The infinitesimal transformation of the field X is then given by

$$\delta X = Y^\mu \partial_\mu X , \quad (3.17)$$

so that the associated Noether charges can be readily found to be

$$Q[a^t, a^\phi, b] = \int d\phi \left[(a^t + bzt) \alpha \partial_\phi X \partial_\phi^z X + (a^\phi + b\phi) (\partial_\phi X)^2 \right]. \quad (3.18)$$

The energy then agrees with the Hamiltonian in (3.1), i.e.,

$$H = Q[1, 0, 0] = \alpha \int d\phi \partial_\phi X \partial_\phi^z X, \quad (3.19)$$

while the momentum P and the anisotropic scaling generator D are identified as

$$P = Q[0, 1, 0] = \int d\phi (\partial_\phi X)^2, \quad (3.20)$$

$$D = Q[0, 0, 1] = \int d\phi \left[zt \alpha \partial_\phi X \partial_\phi^z X + \phi (\partial_\phi X)^2 \right]. \quad (3.21)$$

The generators of the kinematical symmetries can then be shown to span the Lifshitz algebra in $1 + 1$ dimensions, i.e.,

$$\{H, P\}_D = 0, \quad \{P, D\}_D = P, \quad \{H, D\}_D = zH, \quad (3.22)$$

in agreement with the Lie bracket of the Killing vectors (3.16).

3.2.2 $\hat{u}(1)$ current algebra

Since the field equation of the anisotropic chiral boson is linear, the superposition principle certainly holds. This simple fact can be realized as a Noetherian symmetry of the action (2.1), whose associated conserved charges satisfy the $\hat{u}(1)$ current algebra. Those are “shift symmetries” given by

$$\delta X = \eta, \quad (3.23)$$

where η is regarded as a parameter that fulfills the field equation. Assuming periodic boundary conditions, η is given by a superposition of modes of the form

$$\eta_n = e^{i \left(n\phi + (-1)^{\frac{z-1}{2}} \alpha n^z t \right)}. \quad (3.24)$$

The corresponding conserved charges that span the shift symmetries (3.23) then read

$$K[\eta] = 2 \int \eta \partial_\phi X d\phi, \quad (3.25)$$

and it is then simple to verify that their modes $K_n \equiv \frac{1}{2} K[\eta_n]$ fulfill the $\hat{u}(1)$ current algebra

$$i\{K_n, K_m\}_D = \pi n \delta_{n+m, 0}. \quad (3.26)$$

It is worth highlighting that, although the modes K_n depend on the dynamical exponent z , the $\hat{u}(1)$ current algebra does not; and hence the algebra coincides with that of the standard (isotropic) chiral boson for $z = 1$. These generators play a leading role in the construction of the Hilbert space for the quantum theory (see section 4).

3.2.3 Conformal algebra from a nonlocal symmetry

Following the common lore (see e.g. [46]) the generators of the standard conformal symmetry can be constructed out from those of the $\widehat{u}(1)$ currents by means of the Sugawara construction

$$L_n = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} K_j K_{n-j}, \quad (3.27)$$

so that their Dirac brackets span a single chiral copy of the Witt algebra

$$i\{L_n, L_m\}_D = (n - m)L_{n+m}. \quad (3.28)$$

In the anisotropic case ($z > 1$), it is amusing to verify that the conformal algebra, spanned by L_n , is still present, but realized as a nonlocal symmetry of the action (2.1). Indeed, the corresponding transformation law of the field is given by

$$\delta X(\phi) = \int d\phi' f(\phi, \phi', t) \partial_{\phi'} X(\phi'), \quad (3.29)$$

where $f(\phi, \phi', t) = f(\phi', \phi, t)$ can be expressed in terms of the modes $\eta_j(\phi, t)$ in (3.24), so that

$$f(\phi, \phi', t) = \frac{1}{2\pi} \sum_{n,j=-\infty}^{\infty} \epsilon_n \eta_j(\phi, t) \eta_{n-j}(\phi', t). \quad (3.30)$$

Therefore, the conserved quantity associated to the nonlocal conformal symmetry (3.29) is given by

$$L[\epsilon_n] = \int d\phi d\phi' f \partial_{\phi} X(\phi) \partial_{\phi'} X(\phi'), \quad (3.31)$$

$$= \sum_{n=-\infty}^{\infty} \epsilon_n L_n. \quad (3.32)$$

It is worth highlighting that the zero mode actually corresponds to a local symmetry, since $L_0 = P$, where P stands for the momentum generator in (3.20) (instead of the energy H). Indeed, for the zero mode, the function in (3.30) reduces to

$$f(\phi, \phi', t) = \epsilon_0 \sum_{j=-\infty}^{\infty} \eta_j(\phi, t) \eta_{-j}(\phi', t) = 2\pi \epsilon_0 \delta(\phi - \phi'), \quad (3.33)$$

and hence $L[\epsilon_0] = 2\pi \epsilon_0 \int d\phi (\partial_{\phi} X)^2$. For $z > 1$, the remaining modes necessarily span a nonlocal symmetry.

As expected, in the isotropic case ($z = 1$) the conformal transformation (3.29) becomes local. In fact, in this case the function in (3.30) reads

$$f(\phi, \phi', t) = \sum_{n=-\infty}^{\infty} \epsilon_n e^{in(\phi' + \alpha t)} \delta(\phi - \phi') = \epsilon(\phi' + \alpha t) \delta(\phi - \phi'), \quad (3.34)$$

and hence (3.29) and (3.31) reduce to the transformation law and the generators of the conformal symmetry of the Floreanini-Jackiw action [9], given by

$$\delta X(\phi) = \epsilon(\phi + \alpha t) \partial_\phi X(\phi), \quad (3.35)$$

$$L[\epsilon] = \int d\phi \epsilon(\phi + \alpha t) (\partial_\phi X(\phi))^2, \quad (3.36)$$

respectively.

4 Quantum aspects

In order to make the passage to the quantum theory, here we represent the vacuum-to-vacuum transition amplitude in terms of a path integral in the Hamiltonian formulation. This approach is particularly well suited to deal with constrained systems, as it is the case of the theory under discussion. Among its advantages, one has a more precise control of the integration measure. We begin working in Lorentzian signature and then perform a Wick rotation in order to obtain the partition function at finite temperature.

In presence of first class constraints $\psi_a \approx 0$ with gauge fixing conditions $\chi_a \approx 0$, together with second class constraints $\theta_n \approx 0$, the vacuum-to-vacuum transition amplitude W for a Hamiltonian system is given by [47–49]

$$W = \int \mathcal{D}X \mathcal{D}\Pi \exp \left[i \int dt \left(\Pi_i \dot{X}_i - H \right) \right], \quad (4.1)$$

where the integration measure reads

$$\mathcal{D}X \mathcal{D}\Pi = \prod_a \delta(\chi_a) \delta(\psi_a) \det\{\chi_a, \psi_b\} \prod_k \delta(\theta_k) (\det\{\theta_n, \theta_m\})^{1/2} \prod_i \mathcal{D}X_i \mathcal{D}\Pi_i. \quad (4.2)$$

In order to adapt (4.1) and (4.2) to our case, the first step is decomposing the Hamiltonian action in (3.1) (without the constraints) in terms of Fourier modes,

$$I_H[\Pi, X, 0] = 2\pi \int dt \Pi_0 \dot{X}_0 + 2\pi \sum_{n \neq 0} \int dt \left(\Pi_n \dot{X}_{-n} - \frac{1}{l} n^{z+1} X_n X_{-n} \right), \quad (4.3)$$

where the zero mode has been manifestly isolated from the remaining modes and we have made the choice $\alpha = (-1)^{(z-1)/2} l^{-1}$. As explained in section 3.1, there is a single first class constraint given by $\psi = 2\pi \Pi_0$, which can be gauge fixed according to $\chi = X_0$. Thus, replacing these conditions on the first part of (4.2) and integrating over Π_0 and X_0 , we obtain that the zero mode sector completely disappears from the amplitude. Furthermore, integrating over Π_n yields

$$W = \int \prod_{n \neq 0} \mathcal{D}X_n (\det\{\theta_k, \theta_l\})^{1/2} e^{iS[X]}, \quad (4.4)$$

where $S[X]$ corresponds to strongly imposing the second class constraints $\Pi_n = inX_n$ in what remains of the action (4.3), i.e.,

$$S[X] = 2\pi \sum_{n \neq 0} \int dt \left(inX_n \dot{X}_{-n} - \frac{1}{l} n^{z+1} X_n X_{-n} \right). \quad (4.5)$$

Note that the integration measure in (4.4) manifestly carries a factor $(\det\{\theta_k, \theta_l\})^{1/2}$, due to the presence of second class constraints, which can be readily evaluated by virtue of (3.8) (see below).

4.1 Partition function

At finite temperature $T = \beta^{-1}$, the partition function $Z[\beta]$ can be obtained performing a Wick rotation of W , with $t = -iy$ and summing over periodic configurations with period β , i.e.,²

$$X_n(y) = \sum_m X_{n,m} e^{\frac{2\pi i m y}{\beta}}. \quad (4.6)$$

In terms of these variables, the Euclidean continuation of the action (4.5) reads

$$iS_E[X] = -\frac{1}{2} \sum_{n \neq 0, m, k, s} (-4\pi i n \delta_{n+m, 0}) \left(2\pi s + i \frac{\beta}{l} n^z \right) \delta_{k+s, 0} X_{m,k} X_{n,s}. \quad (4.7)$$

Performing the remaining Gaussian integrals, it can be explicitly seen that the contribution coming from the term $4\pi i n \delta_{n+m, 0}$ in the Euclidean action (4.7), and the factor $(\det\{\theta_k, \theta_l\})^{1/2}$ that comes from the integration measure in (4.4), precisely cancel out. Thus, in terms of the modular parameter of the torus $\tau = i \frac{\beta}{2\pi l}$, the partition function is given by

$$Z[\tau] = \mathcal{N} \prod_{n \neq 0} \prod_{s=-\infty}^{\infty} (s + n^z \tau)^{-1/2}, \quad (4.8)$$

where the normalization factor $\mathcal{N}$ does not depend on the temperature.

In order to deal with the divergent infinite product in (4.8), it is useful to consider the average energy of the system (see e.g., [50]), given by

$$\langle E \rangle = -\frac{i}{2\pi l} \frac{\partial}{\partial \tau} \log Z = \frac{i}{2l} \sum_{n=1}^{\infty} n^z \cot(\pi n^z \tau), \quad (4.9)$$

which can be evaluated by virtue of the ζ -function regularization. Indeed,

$$\frac{\partial}{\partial \tau} \log Z = -\pi \sum_{n=1}^{\infty} n^z [\cot(\pi n^z \tau) + i] + i\pi \sum_{n=1}^{\infty} n^z, \quad (4.10)$$

$$= \frac{\partial}{\partial \tau} \left[\sum_{n=1}^{\infty} \log \left(\frac{1}{1 - q^{n^z}} \right) \right] + i\pi \zeta(-z), \quad (4.11)$$

where $q = \exp(2\pi i \tau)$, and $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function.

²For simplicity, we do not consider winding modes on both inequivalent cycles of the torus.

Therefore, the partition function is given by

$$Z[\tau] = q^{\frac{1}{2}\zeta(-z)} \prod_{n=1}^{\infty} \frac{1}{1 - q^{n^z}}, \quad (4.12)$$

where $E_0[z] = -\frac{1}{2l}\zeta(-z)$ corresponds to the energy of the ground state (see below).

It is worth mentioning that eq. (4.12) can be seen as a chiral copy of the partition function of the (nonchiral) anisotropic free boson found in [35], once the zero modes of the latter are discarded.

4.2 Microscopic counting of states and number theory

The partition function $Z[\beta]$ can be alternatively obtained from the trace over the Hilbert space of $\exp(-\beta H)$, where H is the Hamiltonian operator, i.e.,

$$Z[\beta] = \text{Tr} [\exp(-\beta H)] . \quad (4.13)$$

The Hilbert space can be constructed out from the $\widehat{u}(1)$ current operators K_n associated to (3.25), whose algebra is given by the quantum version of (3.26)

$$[K_n, K_m] = \pi n \delta_{n+m, 0} . \quad (4.14)$$

The occupation number states can then be defined as

$$|n_1, n_2, \dots\rangle = K_{-1}^{n_1} K_{-2}^{n_2} \dots |0\rangle , \quad (4.15)$$

where $|0\rangle$ stands for the vacuum state that is annihilated by K_n with $n > 0$.

In order to diagonalize the Hamiltonian operator that comes from (3.19), we express it in the basis of normal ordered currents K_n , which reads

$$H = \frac{1}{\pi l} \sum_{n=1}^{\infty} n^{z-1} K_{-n} K_n + \frac{1}{2l} \zeta(-z) . \quad (4.16)$$

The $\widehat{u}(1)$ descendants (4.15) are then eigenstates of the Hamiltonian (4.16), since

$$H|n_1, n_2, \dots\rangle = \left(\sum_{k=1}^{\infty} n_k E_k - E_0[z] \right) |n_1, n_2, \dots\rangle , \quad (4.17)$$

with

$$E_k = k^z , \quad (4.18)$$

and as aforementioned, $E_0[z] = -\frac{1}{2l}\zeta(-z)$ stands for the ground state energy.

Therefore, the partition function (4.13) can be explicitly computed as

$$Z = \sum_{n_1, n_2, \dots} \langle n_1, n_2, \dots | \exp(-\beta H) | n_1, n_2, \dots \rangle , \quad (4.19)$$

$$= q^{\frac{1}{2}\zeta(-z)} \prod_{k=1}^{\infty} \sum_{n_k} (q^{E_k})^{n_k} , \quad (4.20)$$

$$= q^{\frac{1}{2}\zeta(-z)} \prod_{k=1}^{\infty} \frac{1}{1 - q^{k^z}} , \quad (4.21)$$

in full agreement with the result from the path integral in (4.12), as expected. The equivalence of both ways of computing the partition function is reassuring, since it means that the states have been well identified and well counted.

The partition function (4.21) is defined in the canonical ensemble, and it is useful to express it in terms of the density of states at fixed energy

$$E = \sum_i E_{n_i} = \sum_i n_i^z = N. \quad (4.22)$$

In order to count only along indistinguishable configurations, the ordering $n_1 \geq n_2 \geq \dots \geq 0$ can be assumed. Hence, following [35], the number of states with a fixed energy E is given by the “power partitions” $p_z(N)$, defined as the number of partitions of an integer N into z -th powers; i.e., partitions of the form $N = \sum_i n_i^z = E$.

Consequently, the sum over states can be rearranged, so that the partition function can also be written as

$$Z = q^{\frac{1}{2}\zeta(-z)} \sum_N p_z(N) q^N. \quad (4.23)$$

Indeed, the equivalence between the different ways of expressing the partition function, as in (4.21) and (4.23), holds due to an old and well-known identity in number theory, which asserts that the sequence of power partitions, for a generic value of z , has the following generating function (see e.g. [44, 51])

$$\sum_N p_z(N) q^N = \prod_{n=1}^{\infty} \frac{1}{1 - q^{n^z}}. \quad (4.24)$$

4.3 Asymptotic growth of the number of states

According to (4.23), the entropy of a gas of non-interacting anisotropic chiral bosons in the microcanonical ensemble is then exactly given by

$$S = \log p_z(N). \quad (4.25)$$

Thus, for high temperatures, which corresponds to energies much greater than the ground state, $E \gg E_0$, the entropy in (4.25) can be expressed in a closed form by virtue of the asymptotic growth of the power partitions $p_z(N)$ with $N \gg 1$.

The renowned formula for the asymptotic growth of the partitions $p_1(N)$ was found long ago by Hardy and Ramanujan in [44] where they provided a thorough proof. Noteworthy, by the end of the same work, they also conjectured a very precise formula for the asymptotic growth of the power partitions, which reads

$$p_z(N) \approx (2\pi)^{-\frac{1}{2}(1+z)} \sqrt{\left(\frac{z}{1+z}\right)} k_z N^{\frac{1}{1+z} - \frac{3}{2}} \exp \left[(1+z) k_z N^{\frac{1}{1+z}} \right], \quad (4.26)$$

with

$$k_z = \left\{ \frac{1}{z} \Gamma \left(1 + \frac{1}{z} \right) \zeta \left(1 + \frac{1}{z} \right) \right\}^{\frac{z}{1+z}}, \quad (4.27)$$

whose accuracy was rigorously proved later in [52].³

Therefore, at high temperature, the entropy in (4.25), including subleading corrections acquires the form

$$S = (1+z)k_z N^{\frac{1}{1+z}} - \frac{1}{2} (1+3z) \log N^{\frac{1}{1+z}} + \dots \quad (4.28)$$

5 Ending remarks

We have constructed the quantum field theory of a chiral free boson with anisotropic scaling which, despite its simplicity, exhibits many interesting features in common with those found in two-dimensional conformal field theories. In this sense, it is worth highlighting that, even though the anisotropic scaling manifestly breaks the relativistic Lorentz symmetry, the standard conformal symmetry is still present, but realized in a nonlocal way. Indeed, in the quantum theory, the corresponding generators fulfill the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}m(m^2-1)\delta_{m+n,0}, \quad (5.1)$$

independently of the dynamical exponent z , and so it agrees with the standard result in the isotropic case.

It is also worth mentioning that the leading term of the entropy, which is obtained from solid results in number theory, exactly agrees with an extension of the Cardy formula that relies on modular invariance for generic systems with anisotropic scaling [29, 33, 35]. This fact suggests that our partition function might correspond to a modular form, at least for a suitable high/low temperature regime.

It would also be interesting to analyze the properties of the fermionic version of the anisotropic chiral field, which fulfills the same field equation, but it is described by the following action principle

$$S_f[\chi] = \int dt d\phi \chi (\partial_t \chi + \partial_\phi^z \chi), \quad (5.2)$$

being invariant under the anisotropic scaling in (1.2), provided that the Grassmann-valued field χ scales as $\chi \rightarrow \lambda^{-\frac{1}{2}} \chi$.

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³Simplified proofs have been recently found in [53] for $z = 2$, and in [51, 54] for $z \geq 2$. Extensions for non-integer values of z have also been recently addressed in [55, 56] (see also [29, 35]).

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