The Schwinger point

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ABSTRACT: The Sommerfield model with a massive vector field coupled to a massless fermion in 1+1 dimensions is an exactly solvable analog of a Bank-Zaks model. The "physics" of the model comprises a massive boson and an unparticle sector that survives at low energy as a conformal field theory (Thirring model). I discuss the "Schwinger point" of the Sommerfield model in which the vector boson mass goes to zero. The limit is singular but gauge invariant quantities should be well-defined. I give a number of examples, both (trivially) with local operators and with nonlocal products connected by Wilson lines (the primary technical accomplishment in this note is the explicit and very pedestrian calculation of correlators involving straight Wilson lines). I hope that this may give some insight into the nature of bosonization in the Schwinger model and its connection with unparticle physics which in this simple case may be thought of as "incomplete bosonization."

Contents

1 Introduction

In [1], with Kats, we explored techniques for studying the effects of self-interactions in the conformal sector of an unparticle model. There, physics is encoded in the higher n-point functions of the conformal theory. We studied inclusive processes and argued that the inclusive production of unparticle stuff in standard model processes due to the unparticle self-interactions can be decomposed using the conformal partial wave expansion and its generalizations into a sum over contributions from the production of various kinds of unparticle stuff, corresponding to different primary conformal operators. Such processes typically involve the production of unparticle stuff associated with operators other than those to which the standard model couples directly. Thus just as interactions between particles allow scattering processes to produce new particles in the final state, so unparticle self-interactions cause the production of various kinds of unparticle stuff. The resulting picture, we believe, was a step towards understanding what unparticle stuff "looks like" because it is somewhat analogous to the way we describe the production and scattering of ordinary particles in quantum field theory, with the primary conformal operators playing the role of particles and the coefficients in the conformal partial wave expansion (and its generalization to include more fields) playing the role of amplitudes. We illustrated our methods in the 2D Sommerfield model [2–6] that we discussed previously [7] in which the Banks-Zaks theory is exactly solvable.

We also discussed explicitly how unparticle interactions at low energies evolve as the energy increases and showed in detail how the underlying physics of the Banks-Zaks model appears at high energy. The unparticle physics is always there, but as the energy increases, more and more massive states in the Banks-Zaks model are produced, mocking up the conventional scaling.

In this modest note, I continue with the study of the Sommerfield model, and make more explicit the connection with the Schwinger model in the limit that the vector boson

mass in the Lagrangian goes to zero. What I hope may be new in this note is the explicit calculation of correlators involving straight Wilson lines which is possible using the operator solution of the Sommerfield model. The literature on the Schwinger model is huge and varied, and I would not be surprised to find that many or all of the calculations in the paper have appeared in some form elsewhere. Because trying to find every example is a hopeless task, I will put a preliminary version of the paper on the arXiv and encourage readers to let me know of connections with this work that should be discussed and/or included in the references. And even if some of the results are familiar, I hope readers will find that I have a different way of talking about them that may be stimulating.

2 Sommerfield and Thirring

We will begin with a review of the Sommerfield model to set notation which will be slightly different from that in Kats.[1] The Sommerfield Lagrangian is

$$\mathcal{L}_S = \overline{\psi}\left(i\,\slashed{\partial} - e\slashed{A}\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m_0^2}{2}A^\mu A_\mu \tag{2.1}$$

It will be useful for comparison to consider the corresponding Lagrangian without the A^μ kinetic energy term.

$$\mathcal{L}_T = \overline{\psi}\left(i\,\slashed{\partial} - e\slashed{A}\right)\psi + \frac{m_0^2}{2}A^\mu A_\mu \tag{2.2}$$

In (2.2), A^μ is an auxiliary field proportional to the vector current

$$A^\mu = \frac{e}{m_0^2}\,\overline{\psi}\,\gamma^\mu\,\psi = \frac{e}{m_0^2}\,j^\mu \tag{2.3}$$

So (2.2) is equivalent to the Thirring model

$$\mathcal{L}_T = i\,\overline{\psi}\,\slashed{\partial}\,\psi - \frac{\lambda}{2}\,j^\mu\,j_\mu \tag{2.4}$$

with

$$\lambda = \frac{e^2}{m_0^2} \tag{2.5}$$

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[1]Our conventions, as in [7], are: $g^{00} = -g^{11} = 1$, $\epsilon^{01} = -\epsilon^{10} = -\epsilon_{01} = \epsilon_{10} = 1$. From the defining properties $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\gamma^5 = -\frac{1}{2}\epsilon_{\mu\nu}\gamma^\mu\gamma^\nu$, it follows that $\gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu$ and $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \epsilon^{\mu\nu}\gamma^5$, and we will use the representation $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then the components ψ_1 and ψ_2 describe a right-moving and left-moving fermion, respectively. Lightcone coordinates are defined by

$$x^\pm = x^0 \pm x^1 \qquad\qquad \partial_\pm = \frac{\partial_0 \pm \partial_1}{2}$$

$$x^+\partial_+ + x^-\partial_- = (x^0 + x^1)\frac{\partial_0 + \partial_1}{2} + (x^0 - x^1)\frac{\partial_0 - \partial_1}{2} = x^0\partial_0 + x^1\partial_1$$

$$A^0 = \partial^0\mathcal{V}/m_0 - \partial^1\mathcal{A}/m \qquad A^1 = \partial^1\mathcal{V}/m_0 - \partial^0\mathcal{A}/m$$

$$A_0 = \partial_0\mathcal{V}/m_0 + \partial_1\mathcal{A}/m \qquad A_1 = \partial_1\mathcal{V}/m_0 + \partial_0\mathcal{A}/m \qquad A_\pm = \partial_\pm\mathcal{V}/m_0 \pm \partial_\pm\mathcal{A}/m\,.$$

To solve these models, we decompose A^μ as

$$A^\mu = \partial^\mu \mathcal{V}/m_0 + \epsilon^{\mu\nu}\partial_\nu \mathcal{A}/m \tag{2.6}$$

where

$$m^2 = m_0^2 + e^2/\pi \tag{2.7}$$

Then we can write

$$\epsilon_{\mu\nu}\partial^\mu A^\nu = \epsilon_{\mu\nu}\partial^\mu \epsilon^{\nu\beta}\partial_\beta \mathcal{A}/m = \partial_\mu \partial^\mu \mathcal{A}/m \qquad \partial_\mu A^\mu = \partial_\mu \partial^\mu \mathcal{V}/m_0 \tag{2.8}$$

and the Sommerfield Lagrangian becomes

$$\mathcal{L}_S = i\overline{\psi}\,\partial\!\!\!/\,\psi - e\overline{\psi}\gamma_\mu\psi\left(\partial^\mu \mathcal{V}/m_0 + \epsilon^{\mu\nu}\partial_\nu \mathcal{A}/m\right) + \frac{1}{2m^2}\mathcal{A}\Box^2\mathcal{A} + \frac{1}{2}\partial_\mu\mathcal{V}\partial^\mu\mathcal{V} - \frac{m_0^2}{2m^2}\partial_\mu\mathcal{A}\partial^\mu\mathcal{A} \tag{2.9}$$

while the Thirring Lagrangian is just missing the $\Box^2$ term

$$\mathcal{L}_T = i\overline{\psi}\,\partial\!\!\!/\,\psi - e\overline{\psi}\gamma_\mu\psi\left(\partial^\mu \mathcal{V}/m_0 + \epsilon^{\mu\nu}\partial_\nu \mathcal{A}/m\right) + \frac{1}{2}\partial_\mu\mathcal{V}\partial^\mu\mathcal{V} - \frac{m_0^2}{2m^2}\partial_\mu\mathcal{A}\partial^\mu\mathcal{A} \tag{2.10}$$

If we change the fermionic variable to

$$\Psi = e^{ie\left(\mathcal{V}/m_0 + \mathcal{A}\gamma^5/m\right)}\psi \tag{2.11}$$

the fermion becomes free:

$$\mathcal{L}_S = i\overline{\Psi}\,\partial\!\!\!/\,\Psi + \frac{1}{2}\partial_\mu\mathcal{V}\partial^\mu\mathcal{V} + \frac{1}{2m^2}\mathcal{A}\Box^2\mathcal{A} - \frac{1}{2}\partial_\mu\mathcal{A}\partial^\mu\mathcal{A} \tag{2.12}$$

$$\mathcal{L}_T = i\overline{\Psi}\,\partial\!\!\!/\,\Psi + \frac{1}{2}\partial_\mu\mathcal{V}\partial^\mu\mathcal{V} + -\frac{1}{2}\partial_\mu\mathcal{A}\partial^\mu\mathcal{A} \tag{2.13}$$

In the last terms in both (2.12) and (2.13), m_0^2/m^2 has been replaced by 1 in order to account for the fact that the path integral measure is not invariant under the $\mathcal{A}$ part of (2.11) [8].[2]

Focusing on $\mathcal{A}$ in (2.12), we can replace it with somewhat more normal looking fields as follows.

$$\frac{1}{2m^2}\mathcal{A}\Box^2\mathcal{A} - \frac{1}{2}\partial_\mu\mathcal{A}\partial^\mu\mathcal{A} \rightarrow -\frac{m^2}{2}\mathcal{B}^2 + \mathcal{B}\Box\mathcal{A} - \frac{1}{2}\partial_\mu\mathcal{A}\partial^\mu\mathcal{A} \tag{2.14}$$

$$= -\frac{m^2}{2}\mathcal{B}^2 + \frac{1}{2}\partial_\mu\mathcal{B}\partial^\mu\mathcal{B} - \frac{1}{2}\partial_\mu\mathcal{C}\partial^\mu\mathcal{C} \tag{2.15}$$

where $\mathcal{C} = \mathcal{A} + \mathcal{B}$, so $\mathcal{B}$ is a massive field and $\mathcal{C}$ is a massless ghost. In the Thirring Lagrangian, $\mathcal{A}$ is already a ghost, so we can just replace $\mathcal{A} \rightarrow \mathcal{C}$ and the Lagrangians become

$$\mathcal{L}_S = i\overline{\Psi}\,\partial\!\!\!/\,\Psi + \frac{1}{2}\partial_\mu\mathcal{V}\partial^\mu\mathcal{V} - \frac{m^2}{2}\mathcal{B}^2 + \frac{1}{2}\partial_\mu\mathcal{B}\partial^\mu\mathcal{B} - \frac{1}{2}\partial_\mu\mathcal{C}\partial^\mu\mathcal{C} \tag{2.16}$$

$$\mathcal{L}_T = i\overline{\Psi}\,\partial\!\!\!/\,\Psi + \frac{1}{2}\partial_\mu\mathcal{V}\partial^\mu\mathcal{V} - \frac{1}{2}\partial_\mu\mathcal{C}\partial^\mu\mathcal{C} \tag{2.17}$$

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[2]The same effect gives mass $e/\sqrt{\pi}$ to the gauge boson in the Schwinger model. See also [9].

and the original fermion and vector fields can be written in terms of free fields

$$\psi_S = e^{-ie\left(\mathcal{V}/m_0 + (\mathcal{C}-\mathcal{B})\gamma^5/m\right)}\Psi \qquad \psi_T = e^{-ie\left(\mathcal{V}/m_0 + \mathcal{C}\gamma^5/m\right)}\Psi \qquad (2.18)$$

$$A_S^\mu = \partial^\mu \mathcal{V}/m_0 + \epsilon^{\mu\nu}\partial_\nu(\mathcal{C}-\mathcal{B})/m \qquad A_T^\mu = \partial^\mu \mathcal{V}/m_0 + \epsilon^{\mu\nu}\partial_\nu \mathcal{C}/m \qquad (2.19)$$

Thus the Thirring model is just the Sommerfield model without the $\mathcal{B}$ field! This makes sense because it is physically obvious that the Sommerfield model goes to the Thirring model in the limit $m_0 \to \infty$ with e/m_0 fixed, but (2.3), (2.18) and (2.19) make the correspondence very explicit.

We can use (2.18) and (2.19) straightforwardly to write down the Green's functions of both models. This is done in appendix A.

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There a much less trivial limit of the Sommerfield model — the limit $m_0 \to 0$ with m fixed. The $m_0 = 0$ theory is the Schwinger model [10], invariant under gauge transformations:

$$\psi \to e^{i\theta}\,\psi \quad A^\mu \to A^\mu - \frac{\partial\theta}{e} \qquad (3.1)$$

But the limit $m_0 \to 0$ is potentially singular because the formal gauge invariance of the $m_0 = 0$ theory means that there is no physical degree of freedom associated with the A^μ field. This shows up in the factors of $1/m_0$ in the A^μ propagator. However, the singular piece is a pure gauge. As long as we calculate only gauge invariant quantities (including appropriate Wilson lines [11]), nothing will depend on this and the limit should makes sense and go over smoothly to corresponding calculations in the Schwinger model [10]. We should be able to see that the fermions are confined — or "bosonized" [12] — and understand how the unparticle sector disappears and a mass gap appears.

The first comment is that to have any hope of constructing a gauge invariant quantity, we can only look at objects with fermion number zero. For these, it is easy to see how this works for the $\mathcal{V}$ field part of A^μ where the contribution from a Wilson line can completely cancel the $\mathcal{V}$ dependence and get rid of everything that is singular as $m_0 \to 0$, so the limit should be well defined. Conversely, if the fermion number is not zero, there is no way to cancel the $\mathcal{V}$ dependence and this implies that these things will not be well-defined as $m_0 \to 0$.

The simplest interesting things to look at are the correlations of the local "unparticle" operators

$$O_{21}(x) \equiv \psi_2^*(x)\,\psi_1(x) \quad \text{and} \quad O_{12}(x) \equiv \psi_1^*(x)\,\psi_2(x) \qquad (3.2)$$

These are gauge invariant and should make sense in the Schwinger limit. First consider the 2pt function,

$$\langle 0|\mathrm{T}\,O_{12}(x_1)\,O_{21}(x_2)|0\rangle \qquad (3.3)$$

We can read off (3.3) from figure 2 with C_0 set equal to 1 and the result is

$$S_1(x_1-x_2)\,S_2(x_1-x_2)\,C(x_1-x_2)^4 \tag{3.4}$$

$$= \frac{1}{4\pi^2} \exp\left(\frac{2e^2}{\pi m^2}\left(K_0\left(m\sqrt{-(x_1-x_2)^2+i\epsilon}\right)+\ln\left(\xi m\right)\right)\right)\left(\frac{1}{-(x_1-x_2)^2+i\epsilon}\right)^{1-(e^2/\pi)/m^2} \tag{3.5}$$

where

$$\xi \equiv e^{\gamma_E}/2 \tag{3.6}$$

At short distances, $C(x_1-x_2) \to 1$ in (3.4) and the result goes to a product of free fermion propagators. But in (3.5) at long distances you can see clearly the magic result of the Schwinger limit of the Sommerfield model. When $m^2 = e^2/\pi$, the last term in (3.5) goes to 1 and only the massive propagator survives. But for $m^2 > e^2/\pi$, we see the unparticle contribution at long distances.

The magic at $m^2 = e^2/\pi$ is responsible for one of the more confusing features of the Schwinger point. If (3.5) is to satisfy cluster decomposition, the operators must have non-zero vacuum expectation values, because it must be that

$$\langle 0|\mathrm{T}\,O_{12}(x_1)\,O_{21}(x_2)|0\rangle \underset{-(x_1-x_2)^2\to\infty}{\longrightarrow} \langle 0|O_{12}(x_1)|0\rangle\,\langle 0|O_{21}(x_2)|0\rangle \tag{3.7}$$

This means the vacuum at the Schwinger point must be degenerate with

$$\langle 0|O_{12}(x_1)|0\rangle = \frac{\xi m}{2\pi}\,e^{i\theta} \qquad \langle 0|O_{21}(x_2)|0\rangle = \frac{\xi m}{2\pi}\,e^{-i\theta} \tag{3.8}$$

where θ is the parameter that labels the vacuum state. [13–15] One might worry that because these VEVs vanish in the Sommerfield model, there is something discontinuous about the limits that we are studying that will cause problems. But in fact, unless something else is coupled to the unparticle operators, (3.2), such as a mass term, a source, or a more complicated interaction, there is absolutely no physics in these VEVs. They must be there for the theory to be consistent with cluster decomposition, but they have no other consequences.

The tools in the appendix (and [1]) can be used to show that the behavior we see in (3.5) persists in correlation functions involving more than two of the local unparticle operators, (3.2). In the free-field description of section 1, the local unparticle operators are

$$O_{21}(x) = \psi_2^*(x)\,\psi_1(x) = \Psi_2^*(x)\,\Psi_1(x)\,e^{-2ie\mathcal{A}/m} = \Psi_2^*(x)\,\Psi_1(x)\,e^{-2ie(\mathcal{C}-\mathcal{B})/m} \tag{3.9}$$

$$O_{12}(x) \equiv \psi_1^*(x)\,\psi_2(x) = \Psi_1^*(x)\,\Psi_2(x)\,e^{2ie\mathcal{A}/m} = \Psi_1^*(x)\,\Psi_2(x)\,e^{2ie(\mathcal{C}-\mathcal{B})/m} \tag{3.10}$$

In the Schwinger limit, the Ψ and $\mathcal{C}$ contributions conspire to give constant contribution to all long-distance correlators of these objects, so that all the physics (except the VEVs, (3.8)), is in the exponentials involving the massive field, $\mathcal{B}$,

$$e^{\pm 2ie\mathcal{B}/m} \tag{3.11}$$

How does the magic result in the Schwinger model fit in with bosonization? It seems that we can create perfectly well-defined operators out of the local fields in which the massless degrees of freedom show up at short distances. In the local limit, there is nothing fermionic about it, but the short distance limit of (3.5) looks like it arises from a pair of massless fermions. Where does this come from in a theory with a mass gap? Clearly, it is a large energy phenomenon. The large momentum behavior of the Källén-Lehman representation is obtained asymptotically because the exponentials, (3.11), produce more and more massive vector states as the energy increases.[3]

In more detail, what is happening is that the exponential of the unparticle ghost exactly compensates for the bi-fermion contribution in (3.4). At smaller e^2, the compensation is not exact. The fermion wins and one has an anomalous dimension for the unparticle operator. For larger e^2, the ghost wins and the theory is not unitary.

Going in the other direction, from the Schwinger model to the Sommerfield model, this discussion suggests that we might regard the unparticle sector as the result of "incomplete bosonization." In the Sommerfield model, for $e^2 < \pi m^2$, the ghost fields do not couple strongly enough to completely eliminate the long-distance physics of the massless fermion fields. The fermions are not confined into particle bound states. But neither do their propagators have poles like normal particles. They are unparticles.

Although it is not the primary thrust of this paper, it is worth mentioning what happens to this discussion of the local unparticle operators if we generalize the Schwinger model to include n massless flavors (see [16]). This model has a classical chiral $\mathrm{U}(n) \times \mathrm{U}(n)$ symmetry which is presumably broken by the chiral anomaly down to $\mathrm{SU}(n) \times \mathrm{SU}(n) \times \mathrm{U}(1)$. At the Schwinger point, because the vector boson mass gets contributions from each of the n flavors, e^2/m^2 is $1/n$ times what it is in the 1-flavor Schwinger model. The ghost contributions to the anomalous dimensions of the $(\overline{n}, n)$ of unparticle operators (where the first subscript on ψ is the fermion label and the second subscript indicates the chirality),

$$O^j_{k12} \equiv \psi^*_{j1}\psi_{k2} \quad \text{and} \quad O^j_{k21} \equiv \psi^*_{j2}\psi_{k1} \tag{3.12}$$

are down by $1/n$ compared to what they are in the Schwinger model and so do not cancel the free fermion contributions to the 2-point functions. But the cancellation does take place in the 2-point function of the chiral $\mathrm{SU}(n) \times \mathrm{SU}(n)$ singlet operators

$$O^{n-\text{flavor}}_{12} \equiv \prod_{\ell=1}^{n} \psi^*_{\ell 1}\psi_{\ell 2} \quad \text{and} \quad O^{n-\text{flavor}}_{21} \equiv \prod_{\ell=1}^{n} \psi^*_{\ell 2}\psi_{\ell 1} \tag{3.13}$$

for which

$$\langle 0|\mathrm{T}\, O^{n-\text{flavor}}_{12}(x_1)\, O^{n-\text{flavor}}_{21}(x_2)|0\rangle = \left(\frac{\xi m}{4\pi^2}\right)^{2n} \exp\left(2n\pi K_0\left(m\sqrt{-(x_1-x_2)^2+i\epsilon}\right)\right) \tag{3.14}$$

with $\xi = \frac{e^{\gamma_E}}{2}$ as in (3.6). Thus cluster decomposition requires that these operators have VEVs,

$$\langle 0|O^{n-\text{flavor}}_{12}(x_1)|0\rangle = e^{in\theta}\left(\frac{\xi m}{4\pi^2}\right)^n \qquad \langle 0|O^{n-\text{flavor}}_{21}(x_2)|0\rangle = e^{-in\theta}\left(\frac{\xi m}{4\pi^2}\right)^n \tag{3.15}$$

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[3]There are lots of less trivial examples worked out in [1].

4 Wilson lines

If the fermions and antifermions in our operators are separated in space-time, we need Wilson lines [11] to make things gauge invariant. Thus, for example, we should be able to look at the VEV

$$\langle 0 | T \, O_{11}(y,x) | 0 \rangle \tag{4.1}$$

$$O_{11}(y,x) \equiv T \, \psi_1^*(y) \exp\left(-ie \int_x^y A_\mu(z) \, dz^\mu \right) \psi_1(x) \tag{4.2}$$

I am particularly interested in space-like Wilson lines because they are the simplest thing to look at, so (4.1) could be all at one time, but if we want to think about anything but a straight path in 1+1, we need the time dimension as well, so we do the calculation in general. Under the gauge transformation this goes to

$$\langle 0 | T \, \psi_1^*(y) \, e^{-i\theta(y)} \exp\left(-ie \int_x^y A_\mu(z) \, dz^\mu + i \int_x^y \partial_\mu \theta(z) \, dz^\mu \right) e^{i\theta(x)} \, \psi_1(x) | 0 \rangle \tag{4.3}$$

$$\langle 0 | T \, \psi_1^*(y) \, e^{-i\theta(y)} \exp\left(-ie \int_x^y A_\mu(z) \, dz^\mu + i\theta(y) - i\theta(x) \right) e^{i\theta(x)} \, \psi_1(x) | 0 \rangle \tag{4.4}$$

The VEV in (4.1) is gauge invariant for any path in the Wilson line, but the value may depend on the path. For simplicity, we will calculate it for a straight path from x to y, which should give a Lorentz covariant quantity:

$$z(\alpha)^\mu = (1-\alpha)x^\mu + \alpha y^\mu \qquad\qquad dz^\mu = (y^\mu - x^\mu) \, d\alpha \tag{4.5}$$

$$z(\alpha)^\mu - x^\mu = \alpha(y^\mu - x^\mu) \qquad\qquad z(\alpha)^\mu - y^\mu = (1-\alpha)(x^\mu - y^\mu) \tag{4.6}$$

and we can use

$$\psi_1(x) = e^{-ie\left(\mathcal{V}(x)/m_0 + \mathcal{A}(x)/m \right)} \Psi_1(x) \tag{4.7}$$

and (2.6) to calculate the contribution of the Wilson line. The general argument above shows that the Wilson line simply cancels the $1/m_0$ dependence in the anomalous dimension that come from the $\mathcal{V}$ fields. So we set these to zero in calculating (4.1). Thus the Wilson line is

$$\exp\left(-ie \int_{z(0)}^{z(1)} A_\mu(z(\alpha)) \, dz(\alpha)^\mu \right) \Bigg|_{\mathcal{V}=0} = \exp\left(-i\frac{e}{m} \int_{z(0)}^{z(1)} \epsilon_{\mu\nu} \, \partial_{z(\alpha)}^\nu \mathcal{A}(z(\alpha)) \, dz(\alpha)^\mu \right) \tag{4.8}$$

To calculate the contribution of the $\mathcal{A}$ fields to the Wilson line, we will do the Wick expansion of all the $\mathcal{A}$ fields in (4.1). This is much easier than it looks for the straight paths, because the $\epsilon^{\mu\nu}$ in (2.6) causes many terms to vanish. For example, all the terms in which an $\mathcal{A}$ in the Wilson line is contracted with an $\mathcal{A}$ in ψ or ψ^* vanish because all the coordinate dependence is in the same direction, proportional to $y^\mu - x^\mu$. For example, if the $\mathcal{A}(z(\alpha))$ in (4.8) is contracted with $\mathcal{A}(x)$, the result is a function of $(z(\alpha) - x)^2$ and the derivative with respect to $z(\alpha)_\nu$ is porportional to $z(\alpha)^\nu - x^\nu = \alpha(y^\nu - x^\nu)$ which is orthogonal to $\epsilon_{\mu\nu} \, dz(\alpha)^\mu$. Thus for the straight path (4.5), the VEV (4.1) is simply the

usual contribution to the 2-pt function with the $1/m_0$ terms removed multiplied by the vacuum value of the Wilson line.

We will now evaluate the Wilson line contribution explicitly. Lorentz invariance is crucial here and I want to look at space-like Wilson lines so we will use $F(-x^2) = F(-x^\mu x_\mu)$ for the $\mathcal{A}$ 2-pt function which is

$$F(-x^2) = \frac{1}{2\pi}\left[K_0\left(m\sqrt{-x^2 + i\epsilon}\right) + \ln\left(e^{\gamma_E}m\sqrt{-x^2 + i\epsilon}/2\right)\right] \tag{4.9}$$

Now look at the Wick contractions of the Wilson line, which is

$$W(x-y) = \exp\left(-\frac{e^2}{m^2}Y(x-y)\right) \tag{4.10}$$

where

$$Y(x-y) = \frac{1}{2}\int \epsilon_{\mu_1\nu_1}\,dz(\alpha_1)^{\mu_1}\epsilon_{\mu_2\nu_2}\,dz(\alpha_2)^{\mu_2}\partial^{\nu_1}_{z(\alpha_1)}\partial^{\nu_2}_{z(\alpha_2)}F\left(-(z(\alpha_1) - z(\alpha_2))^2\right) \tag{4.11}$$

To evade annihilation by the ϵs, the partial derivatives must both act on the same factor of $(z(\alpha_1) - z(\alpha_2))^2$, so this is

$$= \int \epsilon_{\mu_1\nu_1}\,dz(\alpha_1)^{\mu_1}\epsilon_{\mu_2\nu_2}\,dz(\alpha_2)^{\mu_2}g^{\nu_1\nu_2}\,F'\left(-(z(\alpha_1) - z(\alpha_2))^2\right) \tag{4.12}$$

$$= -(y-x)^2\int d\alpha_1\,d\alpha_2\,F'\left(-(\alpha_1 - \alpha_2)^2(y-x)^2\right) \tag{4.13}$$

where

$$F'\left(-x^2\right) = -\frac{1 - m\sqrt{-x^2}K_1\left(m\sqrt{-x^2}\right)}{4\pi x^2} \tag{4.14}$$

For a space-like Wilson line, (4.13) is

$$Y(x-y) = \ell^2\int d\alpha_1\,d\alpha_2\,F'\left((\alpha_1 - \alpha_2)^2\ell^2\right) \tag{4.15}$$

where ℓ is the invariant length, $\sqrt{-(x-y)^2}$. Now do the α_2 integration for fixed $\alpha = \alpha_1 - \alpha_2$
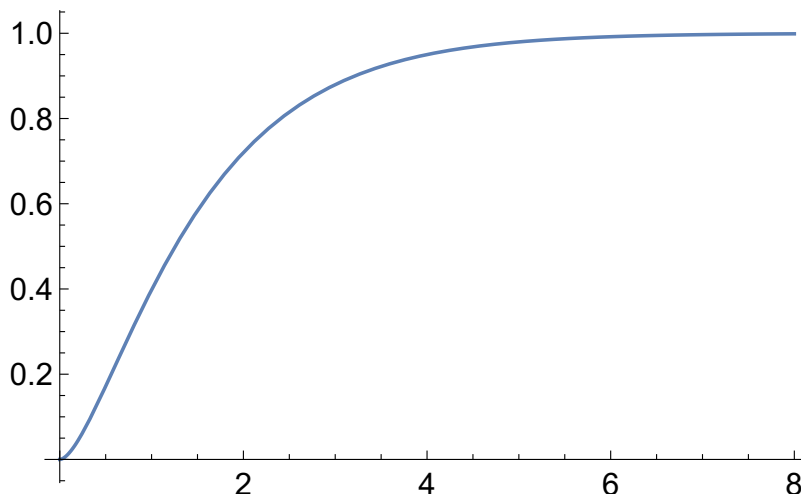
$$\alpha_2 = \alpha_1 - \alpha\begin{cases} \geq \max\left(-\alpha, 0\right) \\ \leq \min\left(1-\alpha, 1\right) \end{cases} \tag{4.16}$$

- If $\alpha > 0$, this is $[0, 1-|\alpha|]$.

- If $\alpha < 0$, this is $[|\alpha|, 1]$.

So the integral is always $1 - |\alpha|$ so it is

$$= \ell^2\int_{-1}^{1}d\alpha\,(1 - |\alpha|)\,F'\left(\alpha^2\ell^2\right) \tag{4.17}$$

$$= 2\ell^2\int_{0}^{1}d\alpha\,(1-\alpha)\,F'\left(\alpha^2\ell^2\right) = \int_{0}^{1}(1-\alpha)\frac{1 - \alpha l K_1(l\alpha)}{2\pi\alpha^2}\,d\alpha \tag{4.18}$$

Figure 1. $1 - \alpha K_1(\alpha)$ versus α.

It is straightforward to get a qualitative understanding of the large ℓ behavior of (4.18) from the graph in figure 1. For $\alpha \ll 1$, the numerator factor $1 - \alpha K_1(\alpha)$ goes to zero because of the cancellation between the ghost and the massive gauge boson contribution. For $\alpha \gg 1$ it goes to 1 because the massive gauge boson contribution vanishes exponentially. The leading term from the 1 in the $(1 - \alpha)$ factor grows linearly with ℓ because the linear divergence from the denominator at small α is cut off for $\alpha \approx 1/\ell$. The integral can be done explicitly for the α term in $1 - \alpha$. It grows more negative like a log at long distances because the log divergence at small α is again cut off for $\alpha \approx 1/\ell$. Putting the two together and putting the factors of m back gives for the large ℓ behavior of the integral

$$\frac{m\ell}{4} - \frac{1}{2\pi} \left(\log(e\xi m\ell) \right) + \cdots \tag{4.19}$$

which means that (4.1) goes to zero exponentially for large ℓ, like

$$\exp\left(-\frac{e^2 \ell}{4m} \right) \ell^{e^2/(2\pi m^2)} (e\xi m)^{e^2/(2\pi m^2)}$$

$$= \exp\left(-\frac{\pi(m^2 - m_0^2)\ell}{4m} \right) \ell^{e^2/(2\pi m^2)} (e\xi m)^{e^2/(2\pi m^2)} \tag{4.20}$$

so

$$W(x - y) = \exp\left(-\frac{e^2 \sqrt{-(x-y)^2}}{4m} \right) \left(-(x-y)^2 \right)^{e^2/(4\pi m^2)} (e\xi m)^{e^2/(2\pi m^2)} \tag{4.21}$$

The contribution of the Wilson line simply gets multiplied by the usual contribution from the fermion 2-point function, without the C_0 terms. This can be read off from figure 2 and (A.11)–(A.14) with C_0 set equal to 1 and the result is

$$S_1(x-y)\, C(x-y) \tag{4.22}$$

$$= \frac{x^+ - y^+}{2\pi} \exp\left(\frac{e^2}{2\pi m^2} \left(K_0\left(m\sqrt{-(x-y)^2 + i\epsilon} \right) + \ln(\xi m) \right) \right) \left(\frac{1}{-(x-y)^2 + i\epsilon} \right)^{1 - e^2/(4\pi m^2)} \tag{4.23}$$

Putting this all together with $\ell = \sqrt{-(x-y)^2}$ gives

$$\frac{x^+ - y^+}{2\pi} \exp\left(-\frac{e^2 \ell}{4m}\right) (\ell)^{-2+e^2/(\pi m^2)} (e\xi^2 m^2)^{e^2/(2\pi m^2)} + \cdots \tag{4.24}$$

or

$$\frac{1}{2\pi} \sqrt{-\frac{x^+ - y^+}{x^- - y^-}} \exp\left(-\frac{e^2 \sqrt{-(x-y)^2}}{4m}\right) (-(x-y)^2)^{-1/2+e^2/(2\pi m^2)} (e\xi^2 m^2)^{e^2/(2\pi m^2)} + \cdots \tag{4.25}$$

where the unwritten terms have additional exponential suppression at large ℓ. In the Schwinger limit, $e^2 = \pi m^2$, there is only exponential scaling at long distances. Thus the Wilson line effectively screens the fermion charges in the Schwinger limit. But as for the correlation functions of the unparticle operators, O_{12} and O_{21}, for $e^2 < \pi m^2$, there is power-law dependence with anomalous dimensions at long distances.

Now for something more complicated.

$$O_{21}(y,x) \equiv \mathrm{T}\, \psi_2^*(y) \exp\left(-ie \int_x^y A_\mu(z)\, dz^\mu\right) \psi_1(x) \tag{4.26}$$

$$O_{12}(y,x) \equiv \mathrm{T}\, \psi_1^*(y) \exp\left(-ie \int_x^y A_\mu(z)\, dz^\mu\right) \psi_2(x) \tag{4.27}$$

$$O_{22}(y,x) \equiv \mathrm{T}\, \psi_2^*(y) \exp\left(-ie \int_x^y A_\mu(z)\, dz^\mu\right) \psi_2(x) \tag{4.28}$$

The interesting case is the object

$$\langle 0|\mathrm{T}\, O_{12}(x_1, y_1)\, O_{21}(x_2, y_2)|0\rangle \tag{4.29}$$

Now we have two Wilson lines on two straight paths,

$$z_j(\alpha)^\mu = (1-\alpha_j) x_j^\mu + \alpha y_j^\mu \qquad dz_j^\mu(\alpha_j) = (y_j^\mu - x_j^\mu)\, d\alpha_j \tag{4.30}$$

$$z_j(\alpha_j)^\mu - x_j^\mu = \alpha_j(y_j^\mu - x_j^\mu) \qquad z_j(\alpha_j)^\mu - y_j^\mu = (1-\alpha_j)(x_j^\mu - y_j^\mu) \quad \text{for } j = 1 \text{ or } 2 \tag{4.31}$$

The two Wilson lines are the same as before, but now there is a contraction between the two and between each of them and the ψs on the other operator.

Before tackling this in general, let's look at the simpler situation in which we keep $y_2 = x_2$. Now there is only one Wilson line, and to avoid subscripts I will take

$$x_1 = x \quad y_1 = y \quad x_2 = y_2 = z \tag{4.32}$$

$$\langle 0|\mathrm{T}\, O_{12}(y,x)\, O_{21}(z,z)|0\rangle \tag{4.33}$$

Without the Wilson line we have

$$\Psi_1^*(y)\, e^{i\mathcal{A}(y)/m} \Psi_2(x)\, e^{i\mathcal{A}(x)/m} \Psi_2^*(z)\, \Psi_1(z)\, e^{-2i\mathcal{A}(z)/m} \tag{4.34}$$

which gives

$$C(x-y)^{-1}\, S_1(x-z)\, C(x-z)^2\, S_2(z-y)\, C(y-z)^2 \tag{4.35}$$

Now we can define the Wilson line exactly as in (4.5)–(4.8). Again the contractions of an $\mathcal{A}$ in the Wilson line with $\mathcal{A}(x)$ or $\mathcal{A}(y)$ give no contribution, and the contractions within the Wilson line give $W(x - y)$, given by (4.21), (4.11), and (4.18). The new piece is the contribution of contractions from the Wilson line to $\mathcal{A}(z)$, which is $Z(x, y, z)^2$ where

$$Z(x, y, z) = \exp\left(\frac{e^2}{m^2}\, X(x, y, z)\right) \tag{4.36}$$

with F given by (4.9) as usual this is

$$X(x,y,z) = \int \epsilon_{\mu\nu}\, dz(\alpha)^\mu \partial_{z(\alpha)}^\nu F\left(-(z(\alpha)-z)^2\right) \tag{4.37}$$

$$= 2\,\epsilon_{\mu\nu}\,(y^\mu - x^\mu)\,(z^\nu - x^\nu) \int d\alpha\, F'\left(-(z(\alpha)-z)^2\right) \tag{4.38}$$

$$= \frac{1}{2\pi}\,\epsilon_{\mu\nu}\,(y^\mu - x^\mu)\,(z^\nu - x^\nu) \int d\alpha\, \frac{1 - m\sqrt{-(z(\alpha)-z)^2}\, K_1\left(-m\,(z(\alpha)-z)^2\right)}{(z(\alpha)-z)^2} \tag{4.39}$$

The denominator of (4.39) is

$$(z(\alpha) - z)^2 = (1-\alpha)^2(x-z)^2 + \alpha^2(y-x)^2 - \alpha(1-\alpha)\Big((x-y)^2 - (x-z)^2 - (y-z)^2\Big) \tag{4.40}$$

$$= (1-\alpha)(x-z)^2 + \alpha(y-z)^2 - \alpha(1-\alpha)(x-y)^2 \tag{4.41}$$

In this case, if we take all the distances large compared to $1/m$, the numerator of the integrand goes to 1 and the denominator is integrable and gives

$$\frac{\log\left(\frac{(x-y)^2-(x-z)^2-(y-z)^2+\sqrt{((x-y)^2)^2-2(x-y)^2((x-z)^2+(y-z)^2)+((x-z)^2-(y-z)^2)^2}}{(x-y)^2-(x-z)^2-(y-z)^2-\sqrt{((x-y)^2)^2-2(x-y)^2((x-z)^2+(y-z)^2)+((x-z)^2-(y-z)^2)^2}}\right)}{\sqrt{((x-y)^2)^2 - 2(x-y)^2((x-z)^2+(y-z)^2) + ((x-z)^2-(y-z)^2)^2}} \tag{4.42}$$

Note that the square root in the denominator of (4.42) is the absolute value of the numerator factor, $\epsilon_{\mu\nu}\,(y^\mu - x^\mu)\,(z^\nu - x^\nu)$,[4] so for large distances, (4.38) can be written as

$$\frac{1}{2\pi} \log\left(\frac{(x-y)^2 - (x-z)^2 - (y-z)^2 + \epsilon_{\mu\nu}\,(y^\mu - x^\mu)\,(z^\nu - x^\nu)}{(x-y)^2 - (x-z)^2 - (y-z)^2 - \epsilon_{\mu\nu}\,(y^\mu - x^\mu)\,(z^\nu - x^\nu)}\right) \tag{4.43}$$

and therefore for large distance

$$Z(x,y,z) = \left(\frac{(x-y)^2 - (x-z)^2 - (y-z)^2 + \epsilon_{\mu\nu}\,(y^\mu - x^\mu)\,(z^\nu - x^\nu)}{(x-y)^2 - (x-z)^2 - (y-z)^2 - \epsilon_{\mu\nu}\,(y^\mu - x^\mu)\,(z^\nu - x^\nu)}\right)^{e^2/(2\pi m^2)} \tag{4.44}$$

Note that a parity transformation interchanges Z and Z^{-1}. Clearly, while there is some dependence on the directions of the 2-vectors, the result is constant as we go to long distance for fixed angles.

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[4]The combination is cyclic. $\epsilon_{\mu\nu}\,(y^\mu - x^\mu)\,(z^\nu - x^\nu) = \epsilon_{\mu\nu}\,(x^\mu - z^\mu)\,(y^\nu - z^\nu) = \epsilon_{\mu\nu}\,(z^\mu - y^\mu)\,(x^\nu - y^\nu)$.

Thus the long-distance behavior of (4.33) is (4.44) times (4.35) multplied by the Wilson line, (4.10) —

$$\propto Z(x,y,z)^2 \, W(x-y) \, C(x-y)^{-1} \, S_1(x-z) \, C(x-z)^2 \, S_2(z-y) \, C(y-z)^2 \qquad (4.45)$$

If we go to the Schwinger point, $m^2 = e^2/\pi$, we can use cluster decomposition as we did in (3.7) to find the VEV of $O_{12(y,x)}$

$$\langle 0|\mathrm{T}\, O_{12}(y,x)\, O_{21}(z,z)|0\rangle \underset{\substack{-(x-z)^2 \to \infty \\ (x-y)^2 \text{ fixed}}}{\longrightarrow} \langle 0|O_{12}(x,y)|0\rangle \, \langle 0|O_{21}(z)|0\rangle \qquad (4.46)$$

Comparing (4.45) with (3.4) and (3.8) and noting that

$$Z(x,y,z) \underset{\substack{-(x-z)^2 \to \infty \\ (x-y)^2 \text{ fixed}}}{\longrightarrow} 1 \qquad (4.47)$$

we see that

$$\langle 0|O_{12}(x,y)|0\rangle = W(x-y) \, C(x-y)^{-1} \frac{\xi m}{2\pi} \, e^{i\theta} \qquad (4.48)$$

As $x \to y$, this goes to (3.8) (as it must) and for large distances, this is

$$\sqrt{e} \, \exp\left(-\frac{e^2}{4m}\sqrt{-(x-y)^2}\right) \frac{\xi m}{2\pi} \, e^{i\theta} \qquad (4.49)$$

which at the Schwinger point goes to

$$\sqrt{e} \, \exp\left(-\pi m \sqrt{-(x-y)^2}/4\right) \frac{\xi m}{2\pi} \, e^{i\theta} \qquad (4.50)$$

Now back to the fully non-local situation, (4.29). The contribution without the Wilson lines is

$$C(x_1-y_1)^{-1} \, C(x_2-y_2)^{-1} \, C(x_1-x_2) \, C(y_1-y_2) \, S_2(x_1-y_2) \, C(x_1-y_2) \, S_1(x_2-y_1) \, C(x_2-y_1) \qquad (4.51)$$

Now we have two Wilson lines, which give a factor of

$$W(x_1 - y_1) \, W(x_2 - y_2) \qquad (4.52)$$

There are four contractions in which an $\mathcal{A}$ in one of the Wilson lines gets contracted with an $\mathcal{A}$ associated with one of the fermions in the other operator. This is the calculation we just did, so there are two Zs and two Z^{-1},

$$Z(x_1,y_1,x_2) \, Z(x_1,y_1,y_2) \, Z(x_2,y_2,x_1)^{-1} \, Z(x_2,y_2,y_1)^{-1} \qquad (4.53)$$

The new piece is the contraction of an $\mathcal{A}$ in the Wilson line from x_1 to y_1 with an $\mathcal{A}$ in the Wilson line from x_2 to y_2. This gives a contribution that looks familiar in terms of the function F of (4.9):

$$H(x_1,y_1;x_2,y_2) = \exp\left(-\frac{e^2}{m^2} Y_{12}\right) \qquad (4.54)$$

$$Y_{12} = \int \epsilon_{\mu_1\nu_1} \, dz_1(\alpha_1)^{\mu_1} \epsilon_{\mu_2\nu_2} \, dz_2(\alpha_2)^{\mu_2} \partial^{\nu_1}_{z_1(\alpha_1)} \partial^{\nu_2}_{z_2(\alpha_2)} F\left(-(z_1(\alpha_1) - z_2(\alpha_2))^2\right) \qquad (4.55)$$

where

$$(z_1(\alpha_1) - z_2(\alpha_2))^\mu = \Big((x_1 - x_2) + \alpha_1(y_1 - x_1) - \alpha_2(y_2 - x_2) \Big)^\mu \qquad (4.56)$$

For simplicity, we will consider the case in which $z_1(\alpha_1) - z_2(\alpha_2)$ is space-like for all α_1 and α_2.

$$(z_1(\alpha_1) - z_2(\alpha_2))^\mu \, (z_1(\alpha_1) - z_2(\alpha_2))_\mu \quad \text{for } 0 \le \alpha_1, \alpha_2 \le 1 \qquad (4.57)$$

This is a rather restrictive condition in 1+1 dimensions, as we will see.

$$Y_{12} = 2 \int \epsilon_{\mu_1\nu_1} \, dz_1(\alpha_1)^{\mu_1} \epsilon_{\mu_2\nu_2} \, dz_2(\alpha_2)^{\mu_2} \partial^{\nu_1}_{z_1(\alpha_1)} (z_1(\alpha_1) - z_2(\alpha_2))^{\nu_2} F'\Big(-(z_1(\alpha_1) - z_2(\alpha_2))^2 \Big) \qquad (4.58)$$

$$= 2 \int \epsilon_{\mu_1\nu_1} \, dz_1(\alpha_1)^{\mu_1} \epsilon_{\mu_2\nu_2} \, dz_2(\alpha_2)^{\mu_2} \Bigg(g^{\nu_1\nu_2} F'\Big(-(z_1(\alpha_1) - z_2(\alpha_2))^2 \Big)$$
$$- 2(z_1(\alpha_1) - z_2(\alpha_2))^{\nu_1} (z_1(\alpha_1) - z_2(\alpha_2))^{\nu_2} F''\Big(-(z_1(\alpha_1) - z_2(\alpha_2))^2 \Big) \Bigg) \qquad (4.59)$$

$$= 2 \int \epsilon_{\mu_1\nu_1} (y_1 - x_1)^{\mu_1} \, d\alpha_1 \, \epsilon_{\mu_2\nu_2} (y_2 - x_2)^{\mu_2} \, d\alpha_2 \Bigg(g^{\nu_1\nu_2} F'\Big(-(z_1(\alpha_1) - z_2(\alpha_2))^2 \Big)$$
$$- 2(z_1(\alpha_1) - z_2(\alpha_2))^{\nu_1} (z_1(\alpha_1) - z_2(\alpha_2))^{\nu_2} F''\Big(-(z_1(\alpha_1) - z_2(\alpha_2))^2 \Big) \Bigg) \qquad (4.60)$$

$$= -2 \int d\alpha_1 \, d\alpha_2 \Bigg((y_1 - x_1)_\mu (y_2 - x_2)^\mu F'\Big(-(z_1(\alpha_1) - z_2(\alpha_2))^2 \Big)$$
$$- 2(g_{\mu_1\mu_2} g_{\nu_1\nu_2} - g_{\mu_1\nu_2} g_{\nu_1\mu_2})(y_1 - x_1)^{\mu_1} (y_2 - x_2)^{\mu_2}$$
$$(z_1(\alpha_1) - z_2(\alpha_2))^{\nu_1} (z_1(\alpha_1) - z_2(\alpha_2))^{\nu_2} F''\Big(-(z_1(\alpha_1) - z_2(\alpha_2))^2 \Big) \Bigg) \qquad (4.61)$$

Define

$$L(\alpha_1, \alpha_2) \equiv -(z_1(\alpha_1) - z_2(\alpha_2))^2$$
$$= -\Big((1 - \alpha_1)(1 - \alpha_2)(x_1 - x_2)^2 - \alpha_1(1 - \alpha_1)(x_1 - y_1)^2 - \alpha_2(1 - \alpha_2)(x_2 - y_2)^2$$
$$+ \alpha_1\alpha_2(y_1 - y_2)^2 + \alpha_1(1 - \alpha_2)(y_1 - x_2)^2 + \alpha_2(1 - \alpha_1)(x_1 - y_2)^2 \Big) \qquad (4.62)$$

$$\frac{\partial L}{\partial \alpha_1} = -2 \frac{\partial z_1(\alpha_1)^\mu}{\partial \alpha_1} (z_1(\alpha_1) - z_2(\alpha_2))_\mu = -2(y_1 - x_1)^\mu (z_1(\alpha_1) - z_2(\alpha_2))_\mu \qquad (4.63)$$

$$\frac{\partial L}{\partial \alpha_2} = 2 \frac{\partial z_2(\alpha_2)^\mu}{\partial \alpha_2} (z_1(\alpha_1) - z_2(\alpha_2))_\mu = 2(y_2 - x_2)^\mu (z_1(\alpha_1) - z_2(\alpha_2))_\mu \qquad (4.64)$$

$$\frac{\partial^2 L}{\partial \alpha_1 \partial \alpha_2} = 2(y_1 - x_1)^\mu (y_2 - x_2)_\mu \qquad (4.65)$$

$$\frac{\partial^2 F}{\partial \alpha_1 \partial \alpha_2} = \frac{\partial}{\partial \alpha_1} \left(\frac{\partial L}{\partial \alpha_2} F'(L) \right) = \frac{\partial L}{\partial \alpha_1} \frac{\partial L}{\partial \alpha_2} F''(L) + 2(y_1 - x_1)^\mu (y_2 - x_2)_\mu F'(L) \qquad (4.66)$$

Then we can rewrite (4.61) as

$$Y_{12} = \int \left(-2(y_1-x_1)_\mu (y_2-x_2)^\mu \left(F'(L) + 2L\,F''(L) \right) + \frac{\partial L}{\partial \alpha_1} \frac{\partial L}{\partial \alpha_2} F''(L) \right) d\alpha_1\, d\alpha_2 \quad (4.67)$$

$$= \int \left(-4(y_1-x_1)_\mu (y_2-x_2)^\mu \left(F'(L) + L\,F''(L) \right) + \frac{\partial^2 F}{\partial \alpha_1 \partial \alpha_2} \right) d\alpha_1\, d\alpha_2 \quad (4.68)$$

The structure of (4.68) is remarkably simple, and the consequences of this simple form are even simpler and more remarkable. As long as L is bounded away from zero in the integral (4.67) (which follows from (4.57)), the term proportional to $(y_1 - x_1)_\mu (y_2 - x_2)^\mu$ is exponentially suppressed for distances larger than $1/m$ because the two terms cancel the log term in F, leaving only the Bessel function term. For the second term, the integral can done trivially (and, in fact, is independent of the path), and the final result is

$$Y_{12} = \Big(F(L(0,0)) + F(L(1,1)) - F(L(1,0)) - F(L(0,1)) \Big) + \cdots$$
$$= \Big(F(-(x_1-x_2)^2) + F(-(y_1-y_2)^2) - F(-(x_1-y_2)^2) - F(-(y_1-x_2)^2) \Big) + \cdots \quad (4.69)$$

If all the distances are large and space-like, this is

$$\frac{1}{4\pi} \left(\log \frac{(x_1 - x_2)^2 (y_1 - y_2)^2}{(x_1 - y_2)^2 (y_1 - x_2)^2} \right) + \cdots \quad (4.70)$$

where the unwritten terms come from the Bessel function and are exponentially supressed if (4.57) is satisfied and thus at long distances

$$H(x_1, y_1; x_2, y_2) = \left(\frac{(x_1 - y_2)^2 (y_1 - x_2)^2}{(x_1 - x_2)^2 (y_1 - y_2)^2} \right)^{e^2/(4\pi m^2)} \quad (4.71)$$

But it is useful to remember (4.69) in its general form, which gives

$$H(x_1, y_1; x_2, y_2) = \left(\frac{e^{F(-(x_1-y_2)^2)}\, e^{F(-(y_1-x_2)^2)}}{e^{F(-(x_1-x_2)^2)}\, e^{F(-(y_1-y_2)^2)}} \right)^{e^2/m^2} + \cdots \quad (4.72)$$

because we can use this form to calculate this contribution even if we put n Wilson lines together end-to-end. The n Wilson lines are

$$W(x_j - y_j) = W(z_j - z_{j+1}) \quad (4.73)$$

where we have labeled $x_j = z_j$, $y_j = z_{j+1}$ for $j = 1$ to n. In addition to the n Wilson lines, we have $n(n-1)/2$ H factors — one for each pair of Wilson lines. So the result should be

$$\left(\prod_{j=1}^{n} W(z_j - z_{j+1}) \right) \left(\prod_{j<k} H(z_j, z_{j+1}; z_k, z_{k+1}) \right) \quad (4.74)$$

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Naively substituting this into (4.71) would give factors of $(-(z_j - z_j)^2)^{e^2/(4\pi m^2)}$. However, from (4.72) we see that these factors should all be replaced by 1, because they arise from the exponential of $F(0) = 0$. Interestingly, when this is done, the result is rather simple

$$\exp\left(e^2 F(-(z_1 - z_{n+1})^2)/m^2\right) \prod_{j=1}^{n} \left(\frac{W(z_j - z_{j+1})^{e^2/(4\pi m^2}}{\exp\left(e^2 F(-(z_j - z_{j+1})^2)/m^2\right)} \right) \qquad (4.75)$$

If each of the segments is very long compared to $1/m$, this becomes

$$\exp\left(-\frac{e^2}{4m} \sum_{j=1}^{n} \sqrt{-(z_j - z_{j+1})^2} \right) \left(-(z_1 - z_{n+1})^2\right)^{e^2/(4\pi m^2)} (e^n \xi m)^{e^2/(2\pi m^2)} \qquad (4.76)$$

The factors are suggestive when compared to the result for a straight Wilson line, (4.21). The first factor is just the exponential of minus the (now jagged) path length times $e^2/(4m)$, as in the single Wilson line. The power-law factor in the middle also appears in the Wilson line at long distances, (4.21). There are some issues however. The last factor of (4.76) differs from the corresponding factor in (4.21) by $e^{(n-1)e^2/(2\pi m^2)}$. This difference is related, I believe, to failure of the condition (4.57) at the $n-1$ points where there Wilson lines are joined together. When $y_1 = x_2$ in $H(x_1, y_1; x_2, y_2)$, L vanishes in the corner of the integration region, for $\alpha_1 = 1$ and $\alpha_2 = 0$. Thus we cannot conclude that the contribution from the first term in (4.68) is exponentially suppressed. And in fact, if all the Wilson lines are parallel, it is easy to see analytically that this provides the missing factors (as it must in this case because we could have calculated the result for the straight Wilson line by breaking it up in pieces). In general, the last factor gets replaced by

$$(e\xi m) \prod_{j=1}^{n-1} \exp\left(\frac{e^2}{2\pi m^2}(1 - \theta_j \coth \theta_j) \right) \qquad (4.77)$$

where

$$\theta_j = \text{ArcCosh}\left(\frac{-(z_{j+2} - z_{j+1})^\mu (z_{j+1} - z_j)_\mu}{\sqrt{(z_{j+2} - z_{j+1})^\mu (z_{j+2} - z_{j+1})_\mu (z_{j+1} - z_j)^\nu (z_{j+1} - z_j)_\nu}} \right) \qquad (4.78)$$

is a measure of the change of direction in 1+1D between the jth and $(j+1)$st Wilson lines. Thus one may think of this as some kind of "curvature correction." The θ_j dependence cancels the $n-1$ extra factors of $e^{e^2/(2\pi m^2)}$ in (4.76) when all the θ_j vanish, and gives additional suppression for non-zero θ_j.

Finally, one may be tempted to take $z_{n+1} = z_1$ in (4.74) and create a gauge invariant Wilson loop. Unfortunately, in 1+1D, this is not consistent with (4.57), which requires that

$$-(z_{j+2} - z_{j+1})^\mu (z_{j+1} - z_j)_\mu > 0 \quad \text{for all } j = 1 \text{ to } n-1. \qquad (4.79)$$

Thus because a loop in 1+1 requires a change in the spacial direction and/or time-like Wilson lines, there is always a region in the α integration for some of the Wilson lines in which L changes sign, so the result and the calculation become complex (in different ways). The explicit calculation of entire Wilson loops in this model have been studied in a very different way by Falomir, Gamboa Saravi, and Schaposnik in [17]

5 Comments

I hope that focusing on the relationship between the Sommerfield model and the Schwinger model as I have in this paper may provide a slightly different approach to some of the fascinating physics of these models. I hope also that the simple, explicit calculations done here may find applications in other areas.

Acknowledgments

A Correlation functions

From [1], we find the non-zero fermion correlators (which must have equal numbers, n_1, of ψ_1 and ψ_1^* and equal numbers, n_2, of ψ_2 and ψ_2^*)

$$\left\langle 0 \left| T \left(\prod_{j=1}^{n_1} \psi_1(x_{1j}) \psi_1(y_{1j})^* \right) \left(\prod_{j=1}^{n_2} \psi_2(x_{2j}) \psi_2(y_{2j})^* \right) \right| 0 \right\rangle \tag{A.1}$$

$$= \left(\prod_{j,k}^{n_1} C_0(x_{1j}-y_{1k}) \, C(x_{1j}-y_{1k}) \, S_1(x_{1j}-y_{1k}) \right) \tag{A.2}$$

$$\times \left(\prod_{j,k}^{n_2} C_0(x_{2j}-y_{2k}) \, C(x_{2j}-y_{2k}) \, S_2(x_{2j}-y_{2k}) \right) \tag{A.3}$$

$$\times \left(\prod_{j<k}^{n_1} C_0(x_{1j}-x_{1k})^{-1} \, C(x_{1j}-x_{1k})^{-1} \, S_1(x_{1j}-x_{1k})^{-1} \right) \tag{A.4}$$

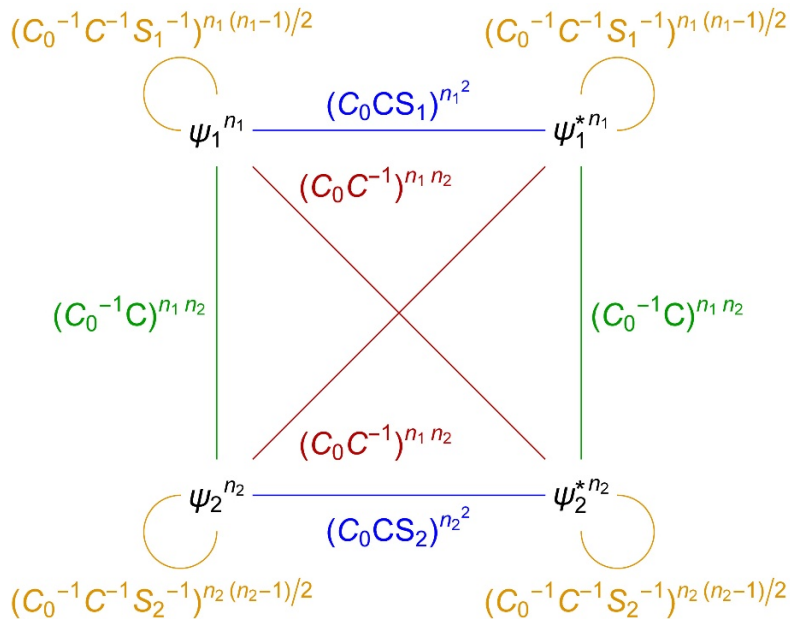$$\times \left(\prod_{j<k}^{n_2} C_0(x_{2j}-x_{2k})^{-1} \, C(x_{2j}-x_{2k})^{-1} \, S_2(x_{2j}-x_{2k})^{-1} \right) \tag{A.5}$$

$$\times \left(\prod_{j<k}^{n_1} C_0(y_{1j}-y_{1k})^{-1} \, C(y_{1j}-y_{1k})^{-1} \, S_1(y_{1j}-y_{1k})^{-1} \right) \tag{A.6}$$

$$\times \left(\prod_{j<k}^{n_2} C_0(y_{2j}-y_{2k})^{-1} \, C(y_{2j}-y_{2k})^{-1} \, S_2(y_{2j}-y_{2k})^{-1} \right) \tag{A.7}$$

$$\times \left(\prod_{j,k}^{n_1,n_2} C_0(x_{1j}-y_{2k}) \, C(x_{1j}-y_{2k})^{-1} \right) \left(\prod_{j,k}^{n_2,n_1} C_0(x_{2j}-y_{1k}) \, C(x_{2j}-y_{1k})^{-1} \right) \tag{A.8}$$

$$\times \left(\prod_{j,k}^{n_1,n_2} C_0(x_{1j}-x_{2k})^{-1} \, C(x_{1j}-x_{2k}) \right) \left(\prod_{j,k}^{n_1,n_2} C_0(y_{1j}-y_{2k})^{-1} \, C(y_{1j}-y_{2k}) \right) \tag{A.9}$$

Figure 2. Pictorial representation of the fermion correlation functions.

where for the Sommerfield model

$$C_0(x) = \exp\left[i\frac{e^2}{m_0^2}\left[D(x) - D(0)\right]\right] \propto \left(-x^2 + i\epsilon\right)^{-e^2/4\pi m_0^2} \tag{A.10}$$

$$C(x) = \exp\left[i\frac{e^2}{m^2}\left[(\Delta(x) - \Delta(0)) - (D(x) - D(0))\right]\right]$$

$$= \exp\left[\frac{e^2}{2\pi m^2}\left[K_0\left(m\sqrt{-x^2 + i\epsilon}\right) + \ln\left(\xi m\sqrt{-x^2 + i\epsilon}\right)\right]\right] \tag{A.11}$$

with $\xi = \frac{e^{\gamma_E}}{2}$ as defined in (3.6)

$$S_0^\alpha(x) = \int \frac{d^2p}{(2\pi)^2} \, e^{-ipx} \frac{p^0 - (-1)^\alpha p^1}{p^2 + i\epsilon} = -\frac{1}{2\pi} \frac{x^0 - (-1)^\alpha x^1}{x^2 - i\epsilon} \tag{A.12}$$

$$S_1(x) = \int \frac{d^2p}{(2\pi)^2} \, e^{-ipx} \frac{p^0 + p^1}{p^2 + i\epsilon} = -\frac{1}{2\pi} \frac{x^0 + x^1}{x^2 - i\epsilon} \tag{A.13}$$

$$S_2(x) = \int \frac{d^2p}{(2\pi)^2} \, e^{-ipx} \frac{p^0 - p^1}{p^2 + i\epsilon} = -\frac{1}{2\pi} \frac{x^0 - x^1}{x^2 - i\epsilon} \tag{A.14}$$

and for the Thirring model, the massive Δ propagator is absent.

Many 1+1 miracles go into making this work. The most miraculous is that we can write the sum of all the ways of contracting the fermions as a single term

$$\left(\sum_P (-1)^{s(P)} \prod_{j=1}^n S_\ell(x_j - y_{P(j)})\right) = (-1)^{n(n-1)/2} \prod_{j,k=1}^n S_\ell(x_j - y_k) / \prod_{j<k} S_\ell(x_j - x_k) / \prod_{j<k} S_\ell(y_j - y_k) \tag{A.15}$$

for $\ell = 1$ or 2. The factors in (A.1)–(A.9) are summarized in the diagram in figure 2.

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