On systematic and GR effects on muon $g - 2$ experiments

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ABSTRACT: We derive in full generality the equations that govern the time dependence of the energy $\mathcal{E}$ of the decay electrons in a muon $g - 2$ experiment. We include both electromagnetic and gravitational effects and we estimate possible systematics on the measurements of $a \equiv (g - 2)/2$, whose experimental uncertainty will soon reach $\Delta a/a \approx 10^{-7}$. In addition to the standard modulation of $\mathcal{E}$ when the motion is orthogonal to a constant magnetic field B , with angular frequency $\omega_a = ea|B|/m$, we study effects due to: (1) a non constant muon γ factor, in presence of electric fields E , (2) a correction due to a component of the muon velocity along B (the “pitch correction”), (3) corrections to the precession rate due to E fields, (4) non-trivial spacetime metrics. Oscillations along the radial and vertical directions of the muon lead to oscillations in $\mathcal{E}$ with a relative size of order 10^{-6} , for the BNL $g - 2$ experiment. We then find a subleading effect in the “pitch” correction, leading to a frequency shift of $\Delta\omega_a/\omega_a \approx \mathcal{O}(10^{-9})$ and subleading effects of about $\Delta\omega_a/\omega_a \approx \text{few} \times \mathcal{O}(10^{-8}-10^{-9})$ due to E fields. Finally we show that GR effects are dominated by the Coriolis force, due to the Earth rotation with angular frequency ω_T , leading to a correction of about $\Delta\omega_a/\omega_a \approx \omega_T/(\gamma\omega_a) \approx \mathcal{O}(10^{-12})$. A similar correction might be more appreciable for future electron $g - 2$ experiments, being of order $\Delta\omega_a/\omega_{a,\text{el}} \approx \omega_T/(\omega_{a,\text{el}}) \approx 7 \times 10^{-13}$, compared to the present experimental uncertainty, $\Delta a_{\text{el}}/a_{\text{el}} \approx 10^{-10}$, and forecasted to reach soon $\Delta a_{\text{el}}/a_{\text{el}} \approx 10^{-11}$.

KEYWORDS: Other experiments

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1 Introduction

A spinning particle orbiting in a plane orthogonal to a constant magnetic field experiences a precession of the spin, relative to its velocity, due to a gyromagnetic factor $g \equiv 2(1+a) \neq 2$. The most recent measurements of a for the muon [1] give $a_{\text{EXP}} = 116592091(63) \cdot 10^{-11}(3)$, and so have a relative precision of about $\Delta a/a \approx 5 \times 10^{-7}$, while ongoing experiments should improve it by a factor of 3 [2].

The Standard Model prediction a_{SM} has been computed [3–5] with an uncertainty of about 3×10^{-7} . Any discrepancy from the experimental measurement could reveal additional contributions beyond the Standard Model [6, 7]. Using the hadronic contributions from [3–5], there is indeed indication of a discrepancy between experiment and the Standard Model prediction, with a statistical significance around 3.5–4 standard deviations.

Some authors [8] have claimed non-negligible effects due to general relativity (GR) on the frequency of the muon spin precession in a magnetic field. This has been questioned by several other authors [9–13]. In [14] it has been shown that, even if the size of the GR corrections claimed in [8] were correct, they would imply anyway a negligible effect, of order $\Delta a/a \approx 10^{-10}$, for the Brookhaven National Laboratory (BNL) E821 experiment [1].

In the present paper we derive a full treatment of the spin precession in GR and we show what is the correct size of such effects in a realistic muon experiment. In order to do this, we need to clarify basic issues: what are the observable quantities and in which frame. What is actually observed in an experiment is the energy $\mathcal{E}$ of the decay products (electrons or positrons) of the muons, as seen in the laboratory frame. This will be shown to be given by the sum of two terms: one is proportional to the muon gamma factor and the second is the scalar product of a generalized spin Σ with the observed velocity v . In the particular case of flat spacetime and of a circular motion of muons of mass m , in a plane orthogonal to a constant magnetic B field, the latter corresponds to the standard anomalous oscillatory term, with frequency $\omega_a = ea|B|/m$ (e.g. see [15]).

It is well known that, even in the flat spacetime case, in a realistic experiment the muon is not on a perfect circular orbit but performs small radial and vertical oscillations. This implies: (1) a correction to the precession frequency, usually called “pitch correction” (e.g. see [16, 17]); (2) corrections, due to electric fields, that are assumed to vanish at a particular value of the muon gamma factor (the *magic* momentum); (3) the fact that the muon gamma factor is not constant due to electric fields, which introduces additional time dependence of the electron energy $\mathcal{E}$. We will analyze thus such corrections in the case of the BNL experiment.

Then, we will analyze the extra terms introduced by the presence of a non-trivial metric, including the particular cases of a rotating metric and of a Schwarzschild metric, representing respectively Earth’s rotation and gravity.

The paper is organized as follows: in section 2 we introduce the general formalism, in section 3 we write in full generality the equations of motion for velocity and spin. In section 4 we define the quantities that are relevant for experiments and study the Minkowski cases with (i) pure magnetic field, (ii) magnetic plus electric fields. In section 5 we study then the effects of non-trivial metrics. In section 6 we analyze the impact on a typical experiment of the various corrections and finally we draw our Conclusions.

2 General setup

We use the spacetime signature $(-, +, +, +)$ and we will work with a generic metric $g_{\mu\nu}$ (unless specified), so that scalar products between any two 4-vectors, A and B , are given by $A \cdot B = A^\mu g_{\mu\nu} B^\nu$. We mostly follow the treatment and notation of [18].

A given frame (F) is defined by the observer 4-velocity $u^{(F)}$. In the rest of this paper we will use the muon rest frame, (F) = (M), and the laboratory frame, (F) = (L), although we will drop the L index in the latter case. The energy per unit mass of a particle with 4-velocity U , measured by $u^{(F)}$, is its gamma factor, given by

$$\gamma^{(F)} = -(U \cdot u^{(F)}) . \quad (2.1)$$

Four-velocities are always normalized, $U \cdot U = u^{(F)} \cdot u^{(F)} = -1$.

The time observed in the frame (F) is given by

$$dT^{(F)} = \gamma^{(F)} d\tau , \quad (2.2)$$

where τ is the proper time of the particle with velocity U .

The observer in a given (F) frame measures a velocity $v^{(F)}$ for the above particle, given by:

$$v^{(F)} = \frac{U}{\gamma^{(F)}} - u^{(F)} . \quad (2.3)$$

By construction $v^{(F)}$ is purely spatial with respect to $u^{(F)}$, i.e. $v^{(F)} \cdot u^{(F)} = 0$, and $|v^{(F)}| = \sqrt{v^{(F)} \cdot v^{(F)}}$ is the modulus of the observed velocity. Indeed the following property holds:

$$\gamma^{(F)} = 1/\sqrt{1 - |v^{(F)}|^2} . \quad (2.4)$$

In general, useful quantities are the projector operators, which define the decomposition of the local frame of a given observer u into orthogonal subspaces. Precisely we define the temporal and spatial projector operators in the following way:

$$\mathcal{T}(u) = -u \otimes u \quad \text{and} \quad \mathcal{P}(u) = I + u \otimes u , \quad (2.5)$$

or in terms of their components:

$$\mathcal{T}(u)^\mu_\nu = -u^\mu u_\nu \quad \text{and} \quad \mathcal{P}(u)^\mu_\nu = \delta^\mu_\nu + u^\mu u_\nu . \quad (2.6)$$

The spin of a particle is described by a 4-vector S , with the property that it is orthogonal to its 4-velocity:

$$S \cdot U = 0 . \quad (2.7)$$

It is also useful to define the spatially projected spin, $\Sigma^{(F)} = \mathcal{P}(u^{(F)})S$, on a given observer. This satisfies then the following equations

$$S = \Sigma^{(F)} - (S \cdot u^{(F)})u^{(F)} = \Sigma^{(F)} + (\Sigma^{(F)} \cdot v^{(F)})u^{(F)} , \quad (2.8)$$

where we have used that

$$\Sigma^{(F)} \cdot v^{(F)} = -S \cdot u^{(F)} . \quad (2.9)$$

Note that indeed $\Sigma^{(F)}$ is spatially projected with respect to $u^{(F)}$, since $\Sigma^{(F)} \cdot u^{(F)} = 0$.

In the particle rest frame (M) the observer velocity is U itself and so we simply have

$$\Sigma^{(M)} = S + (S \cdot U)U = S. \quad (2.10)$$

In this frame we may use, for example, coordinates $\{x_{(M)}^{\hat{\mu}}\}$ defined in the adapted frame $\{e_{(M)}^{\hat{\mu}}\}$, where $U = e_{\hat{0}}^{(M)}$, $e_{\hat{\alpha}}^{(M)} \cdot e_{\hat{\beta}}^{(M)} = \eta_{\hat{\alpha}\hat{\beta}}$ and $e_{\hat{0}}^{(M)} \cdot e_{\hat{\alpha}}^{(M)} = 0$, where $\eta_{\hat{\alpha}\hat{\beta}} \equiv \text{diag}(-1, 1, 1, 1)$. Then we find¹

$$\Sigma_{(M)}^{\hat{\mu}} = S_{(M)}^{\hat{\mu}} = e_{(M)}^{\hat{\mu}} \cdot S \equiv (0, s^{\hat{m}}), \quad \text{with} \quad |S|^2 = S \cdot S = \mathbf{s} \cdot \mathbf{s}. \quad (2.11)$$

We use Latin indices, from 1 to 3, and boldface for three-dimensional vectors and we defined $\mathbf{s} \cdot \mathbf{s} = s^{\hat{m}} s^{\hat{n}} \eta_{\hat{m}\hat{n}}$, where $\eta_{\hat{m}\hat{n}} \equiv e_{\hat{m}}^{(M)} \cdot e_{\hat{n}}^{(M)}$. Greek indices are always from 0 to 4.

3 Equations of motion

Let us now consider a particle (the muon) in an electromagnetic field, described by an antisymmetric field strength F , and in a generic metric. We need the equation of motion for the observed velocity and spin.

We first review the definitions of electric and magnetic fields as observed by the laboratory frame, with 4-velocity $u^{(L)} \equiv u$. In any system of coordinates the fields observed by u are described by 4-vectors E and B ,

$$E^{\alpha} = F^{\alpha\rho} u_{\rho} \quad B^{\alpha} = \frac{1}{2} \eta^{\rho\alpha\mu\nu} F_{\mu\nu} u_{\rho}, \quad \text{with} \quad \eta^{\rho\alpha\mu\nu} \equiv \frac{1}{\sqrt{-g}} \epsilon^{\rho\alpha\mu\nu}, \quad (3.1)$$

where $g = \det(g_{\mu\nu})$ and $\epsilon_{\rho\alpha\mu\nu}$ is the standard totally antisymmetric Levi-Civita symbol, $\epsilon_{0123} = -\epsilon^{0123} = 1$. Note that both E and B are purely spatial w.r.t. u , that is $E \cdot u = B \cdot u = 0$. One can also write the F tensor as:

$$F_{\beta\mu} = u_{\beta} E_{\mu} - u_{\mu} E_{\beta} + \eta(u)_{\beta\mu}{}^{\sigma} B_{\sigma}, \quad (3.2)$$

where

$$\eta(u)_{\alpha\beta\gamma} \equiv u^{\delta} \eta_{\delta\alpha\beta\gamma} \quad \text{with} \quad \eta_{\delta\alpha\beta\gamma} = \sqrt{-g} \epsilon_{\delta\alpha\beta\gamma}.$$

Let us check the flat space case, with an observer at rest in Minkowski coordinates: $u^{\mu} = (1, 0, 0, 0)$. In our signature we have $u_{\mu} = (-1, 0, 0, 0)$, and so:

$$\begin{aligned} E^1 &= -F^{10}, & B^1 &= \frac{1}{2} \epsilon^{1\mu\nu} F_{\mu\nu} u_{\rho} = \frac{1}{2} \epsilon^{1mn} F_{mn} = F_{23}, \\ E^2 &= -F^{20}, & B^2 &= F_{31}, \\ E^3 &= -F^{30}, & B^3 &= F_{12}. \end{aligned} \quad (3.3)$$

Using the above definition of the Levi-Civita symbol, here we have used $\epsilon^{mnp} \equiv -\epsilon^{0mnp}$ and $\epsilon_{mnp} \equiv \epsilon_{0mnp}$.

¹ $S^{\hat{\mu}}$ is briefly analyzed in curved space in appendix A.

3.1 Velocity

We will consider now also the motion of muons as seen by the laboratory frame (L), defined by a 4-velocity u . A muon with 4-velocity U has therefore in this frame an observed velocity $v \equiv v^{(L)}$ and a gamma factor $\gamma \equiv \gamma^{(L)}$.

The equation of motion for the muon velocity is

$$\frac{DU}{D\tau} = \frac{e}{m} F * U, \quad (3.4)$$

where our notation is $(F * U)^\mu \equiv F^{\mu\nu} U_\nu$ in a given coordinate system, e is the electric charge and m is the muon mass. We will use the covariant derivative of a four-vector A always along U , defined as

$$\frac{DA}{D\tau} = \left(\frac{dA}{d\tau} \right)^\alpha + [\Gamma AU]^\alpha, \quad (3.5)$$

where we used the shorthand notation $[\Gamma AB]^\mu = \Gamma_{\nu\rho}^\mu A^\nu B^\rho$, with $\Gamma_{\nu\rho}^\mu$ the Christoffel symbols in a given coordinate system and A and B any two vectors. We can also decompose the equation of motion using the observed E and B fields:²

$$\frac{DU}{D\tau} = \mathcal{T}(u) \frac{DU}{D\tau} + \mathcal{P}(u) \frac{DU}{D\tau},$$

We have thus

$$\mathcal{P}(u) \frac{DU}{D\tau} = \gamma f_{\text{EM}}, \quad (3.6)$$

where f_{EM} is just the Lorentz force:³

$$f_{\text{EM}} = \frac{e}{m} [E + (v \times_u B)]. \quad (3.7)$$

Here we have introduced the spatial cross product $\times_u$ in the following way

$$[A \times_u B]^\alpha \equiv \eta(u)_{\beta\gamma}^\alpha A^\beta B^\gamma. \quad (3.8)$$

So we find

$$\left(\frac{DU}{D\tau} \right)^\alpha = \frac{e}{m} \gamma [E^\alpha + (v \times_u B)^\alpha] - \frac{e}{m} u^\alpha (u * F * U), \quad (3.9)$$

where $u * F * U \equiv u_\mu F^{\mu\nu} U_\nu$.

Now let us write an equation for dv/dT , i.e. the evolution of the observed velocity w.r.t. the observed time T . Using eq. (2.3) we have that:

$$\frac{dv}{dT} = \frac{1}{\gamma^2} \frac{dU}{d\tau} - \frac{U}{\gamma^2} \frac{d\gamma}{dT} - \frac{du}{dT}. \quad (3.10)$$

Now we can use the following relation,⁴

$$\dot{\gamma} \equiv d\gamma/dT = v \cdot (f_{\text{EM}} + f_{\text{G}}) = v \cdot \left(\frac{e}{m} E + f_{\text{G}} \right), \quad (3.11)$$

where he have defined the gravitational force as [18]:

$$f_{\text{G}}^\alpha \equiv -P(u)_{\beta}^{\alpha} \left(\frac{Du}{D\tau} \right)^\beta = - \left(\frac{Du}{D\tau} \right)^\alpha = - \left(\frac{du}{d\tau} \right)^\alpha - [\Gamma u U]^\alpha. \quad (3.12)$$

²See [18], eqs. (6.133)–(6.136).

³See eq. (6.75) of ref. [18].

⁴See eq. (6.81) of ref. [18].

3.2 Spin

Let us write now the equation for the spin [19] of a particle with gyromagnetic factor g , adapted to our signature $(-, +, +, +)$, as:

$$\begin{aligned}\frac{DS}{d\tau} &= \frac{ge}{2m} [F * S - (S * F * U)U] + \left(\frac{DU}{d\tau} \cdot S \right) U = \\ &= \frac{ge}{2m} (F * S) - \frac{e}{m} a (S * F * U)U,\end{aligned}\quad (3.13)$$

where $a \equiv (\frac{g}{2} - 1)$. One can check that signs are consistent by evaluating the flat space case with both observer and particle at rest, $u^\mu = U^\mu = (1, 0, 0, 0)$, and with⁵ $S = (0, \mathbf{s})$.

Let us now work in the most general case and let us write an equation for $\Sigma^{(L)} \equiv \Sigma$. Eq. (2.8) becomes

$$S = \Sigma + (\Sigma \cdot v)u = \Sigma - (S \cdot u)u, \quad (3.15)$$

and inserting this in the l.h.s. of eq. (3.13) we get:⁶

$$\frac{DS}{d\tau} = \frac{D\Sigma}{d\tau} + \frac{Du}{d\tau}(v \cdot \Sigma) + u \frac{D}{D\tau}(v \cdot \Sigma). \quad (3.16)$$

Now we can spatially project this w.r.t. u , together with eq. (3.12), so that:⁷

$$P(u) \frac{D\Sigma}{d\tau} = f_G(v \cdot \Sigma) + \frac{ge}{2m} P(u)(F * S) - \frac{e}{m} a \gamma v(S * F * U). \quad (3.17)$$

We may also write it in terms of the observed time T ,

$$P(u) \left(\gamma \frac{d\Sigma}{dT} + [\Gamma \Sigma U] \right) = f_G(v \cdot \Sigma) + \frac{ge}{2m} P(u)(F * S) - \frac{e}{m} a \gamma v(S * F * U). \quad (3.18)$$

We can now use eq. (3.2) and get:

$$\begin{aligned}P(u)(F * S)^\alpha &= P^{\alpha\beta} F_{\beta\mu} S^\mu = P^{\alpha\beta} (u_\beta E_\mu - u_\mu E_\beta + \eta_{\beta\mu}^\sigma B_\sigma)(\Sigma + (v \cdot \Sigma)u)^\mu = \\ &= E^\alpha(v \cdot \Sigma) + P^{\alpha\beta} \eta_{\beta\mu}^\sigma B_\sigma \Sigma^\mu + P^{\alpha\beta} \eta_{\beta\mu}^\sigma B_\sigma u^\mu (v \cdot \Sigma) = \\ &= E^\alpha(v \cdot \Sigma) + (\Sigma \times_u B)^\alpha,\end{aligned}\quad (3.19)$$

where we used

$$E_\beta u^\beta = 0, \quad \eta_{\beta\mu}^\sigma B_\sigma \Sigma^\mu = (\Sigma \times_u B)_\beta, \quad u^\beta \eta_{\beta\mu}^\sigma B_\sigma \Sigma^\mu = 0 \quad \text{and} \quad \eta_{\beta\mu}^\sigma B_\sigma u^\mu = 0.$$

Similarly:

$$\begin{aligned}(S * F * U) &= S^\beta F_{\beta\mu} U^\mu = (\Sigma^\beta + (v \cdot \Sigma)u^\beta)(u_\beta E_\mu - u_\mu E_\beta + \eta_{\beta\mu}^\sigma B_\sigma) \gamma (u^\mu + v^\mu) = \\ &= \gamma [(\Sigma \cdot E) - (v \cdot \Sigma)(E \cdot v)] + \gamma \Sigma \cdot (v \times_u B).\end{aligned}\quad (3.20)$$

⁵In such a case we have that

$$\frac{ds^a}{d\tau} = \frac{ge}{2m} (\mathbf{s} \times \mathbf{B})^a, \quad (3.14)$$

where boldface are used for three dimensional vectors in Minkowski, $\mathbf{B}$ is the spatial part of B and $\times$ is the standard three-dimensional vector product in flat space. We will also use in the following $\cdot$ for the standard three-dimensional vector product in flat space.

⁶See ref. [18], eq. (10.8), or ref. [20].

⁷This coincides with the first line of eq. (10.15) by [18], plus an additional term proportional to g .

4 Experimental setup

In an experimental setup muons are moving on an (almost) circular orbit and the laboratory is at rest with the earth surface. Muons decay then into a neutrino-antineutrino pair and an electron (we use the word electron to represent either the positron or electron in the generic $\mu \rightarrow e\nu\nu$ decay chain) and the observable quantity is the number of electrons whose energy in the (L) frame is above a given threshold. For simplicity we will look only at electrons emitted with maximal energy $\mathcal{E}_{\text{Max}}$ in the (M) frame:

$$\mathcal{E}_{\text{Max}}^{(\text{M})} = \frac{m^2 + m_e^2 - (m_{\bar{\nu}_e} + m_{\nu_\mu})^2}{2m}. \quad (4.1)$$

This happens when a pair of neutrino-antineutrino is emitted in the same direction and the electron is thus emitted in the opposite direction. In this case the neutrino-antineutrino pair carries zero angular momentum and so the electrons inherits the same spin of the muons. In the limit of $m_e \rightarrow 0$ only one helicity can be produced and thus electron emission is zero (or maximal) if its velocity is aligned (or anti-aligned) with its spin, which is equal to the muon spin. Therefore the emission of electrons with maximal energy is peaked when the velocity of the electron is opposite to the muon spin direction.

We can idealize and simplify this situation by assuming that:

- In the muon rest frame (M), where the observer velocity is the muon 4-velocity U , all electrons are emitted with the same energy $\mathcal{E}^{(\text{M})} = m_e \gamma_e^{(\text{M})}$. Here $\gamma_e^{(\text{M})}$ and $v_e^{(\text{M})}$ are the electron gamma factor and velocity, as observed by U . Using eqs. (2.1) and (2.3) they read:

$$\gamma_e^{(\text{M})} = -(U_e \cdot U), \quad (4.2)$$

$$v_e^{(\text{M})} = \frac{U_e}{\gamma_e^{(\text{M})}} - U, \quad (4.3)$$

where U_e is the electron 4-velocity.

- We assume that the electrons in the frame (M) are emitted with velocity $v_e^{(\text{M})}$ opposite to the muon spin direction $\Sigma^{(\text{M})}$:

$$\hat{v}_e^{(\text{M})} = -\hat{\Sigma}^{(\text{M})}, \quad (4.4)$$

where

$$\hat{v}_e^{(\text{M})} \equiv \frac{v_e^{(\text{M})}}{|v_e^{(\text{M})}|}, \quad \hat{\Sigma}^{(\text{M})} \equiv \frac{\Sigma^{(\text{M})}}{|\Sigma^{(\text{M})}|} = \frac{S}{|S|}. \quad (4.5)$$

We will always use a hat for vectors normalized to 1. Now we can express U_e in the two frames as:

$$U_e = \gamma_e^{(\text{L})}(u + v_e^{(\text{L})}) = \gamma_e^{(\text{M})}(U + v_e^{(\text{M})}), \quad \text{where} \quad \gamma_e^{(\text{L})} = -(U_e \cdot u). \quad (4.6)$$

So let us now compute the experimentally observed quantity, $\mathcal{E}/m_e = \gamma_e^{(L)}$, using the above equations and also eq. (3.15):

$$\gamma_e^{(L)} = -(U_e \cdot u) = -\gamma_e^{(M)}(U + v_e^{(M)}) \cdot u = \gamma_e^{(M)} \left[\gamma - \frac{|v_e^{(M)}|}{|S|}(\Sigma \cdot v) \right]. \quad (4.7)$$

Here all quantities are constant in time, except for γ (the muon gamma factor) and $(\Sigma \cdot v)$ (proportional to the alignment between the muon spin and the muon velocity in the (L) frame). In the following we will write the above equation as

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_R, \quad (4.8)$$

where

$$\mathcal{E}_0 \equiv m_e \gamma_e^{(M)} \gamma, \quad \mathcal{E}_R \equiv -\mathcal{E}_1(\Sigma \cdot v) = \mathcal{E}_1(S \cdot u), \quad \mathcal{E}_1 \equiv m_e \gamma_e^{(M)} \frac{|v_e^{(M)}|}{|S|}. \quad (4.9)$$

We will now write an equation for $\dot{\mathcal{E}}$ in various setups: (i) in Minkowski metric, with the observer at rest and with a static magnetic field only, (ii) in Minkowski metric, with the observer at rest and with both electric and magnetic fields, (iii) including a non-trivial metric and motion of the observer.

4.1 Static magnetic field in flat space

Let us first work in a simplified framework, where we assume:

- B constant in time in the (L) frame;
- No electric fields in the (L) frame ($E = 0$);
- Minkowski metric and observer at rest $u^\mu = (1, 0, 0, 0)$.

The γ of the muons is thus constant in this case, from eq. (3.11) and so $\mathcal{E}_0$ and $\mathcal{E}_1$ in eq. (4.9) are constants. In this simplified framework the time-dependence of the observed electron energy is only due to $\mathcal{E}_R$. As we will see, under the condition $v \cdot B = 0$, this turns out to have a periodic behavior with respect to the observer's time T , with a frequency ω_a , the anomalous precession frequency. Let us indeed compute time derivatives of the electron energy as follows:

$$\frac{1}{\mathcal{E}_1} \frac{d\mathcal{E}}{dT} = \frac{dS}{dT} \cdot u = \frac{dS}{\gamma d\tau} \cdot u. \quad (4.10)$$

Now we use eq. (3.13) and get:

$$\frac{dS}{d\tau} \cdot u = \frac{ge}{2m} u \cdot (F * S) - \frac{e}{m} a(S * F * U) U \cdot u = \frac{e}{m} a\gamma(S * F * U), \quad (4.11)$$

since the first term vanishes in absence of electric fields. Using eq. (3.20) and keeping again only the magnetic part we find:

$$\frac{1}{\mathcal{E}_1} \frac{d\mathcal{E}}{dT} = -\frac{e}{m} a\gamma B \cdot (v \times_u \Sigma). \quad (4.12)$$

From here we immediately see that there is a non-trivial time dependence only if $g \neq 2$.

Now, the spin S can be written in the rest frame coordinates as in eq. (2.11), or it can be written also using the (L) coordinates (e.g. see [21], chapter 11), as:

$$S = \left(\gamma \mathbf{v} \cdot \mathbf{s}, \mathbf{s} + \frac{\gamma^2}{\gamma + 1} (\mathbf{v} \cdot \mathbf{s}) \mathbf{v} \right), \quad (4.13)$$

where $\mathbf{v}$ is the spatial part of v in the L coordinates. One can check explicitly that in both coordinates $S \cdot U = 0$ and $S^2 = S \cdot S = \mathbf{s} \cdot \mathbf{s}$.

This implies that the scalar and vector products are equal to

$$\Sigma \cdot v = -S \cdot u = \gamma \mathbf{s} \cdot \mathbf{v}, \quad (4.14)$$

$$\Sigma \times_u v = \mathbf{s} \times \mathbf{v}, \quad (4.15)$$

so that using

$$\frac{\mathcal{E} - \mathcal{E}_0}{\mathcal{E}_1} = -\gamma \mathbf{s} \cdot \mathbf{v}, \quad (4.16)$$

together with eq. (4.12) we have

$$\frac{d(\mathbf{s} \cdot \mathbf{v})}{dT} = \frac{e}{m} a [\mathbf{B} \cdot (\mathbf{v} \times \mathbf{s})]. \quad (4.17)$$

where $\mathbf{B}$ is the spatial part of B . This coincides with the magnetic term of eq. (9) by Bargmann et al. [19] and eq. (11.171) of [21]. Note, though, that such a form of the equations mixes quantities in different frames.

It is also customary to define the following quantities:

$$\frac{d\hat{\mathbf{v}}}{dT} \equiv \boldsymbol{\omega}_{\mathbf{v}} \times \hat{\mathbf{v}}, \quad (4.18)$$

$$\frac{d\hat{\mathbf{s}}}{dT} \equiv \boldsymbol{\omega}_{\mathbf{s}} \times \hat{\mathbf{s}}. \quad (4.19)$$

The first quantity can be found combining eqs. (3.9), (3.10), and the second combining eqs. (3.15)–(3.20) and (4.13), leading to:⁸

$$\boldsymbol{\omega}_{\mathbf{v}} = -\frac{e}{m\gamma} \mathbf{B}, \quad (4.20)$$

$$\boldsymbol{\omega}_{\mathbf{s}} = -\frac{e}{m\gamma} \mathbf{B} - \frac{e}{m} a \left[\mathbf{B} - \frac{\gamma}{\gamma + 1} (\mathbf{v} \cdot \mathbf{B}) \mathbf{v} \right], \quad (4.21)$$

in agreement with [21]. The above equations also imply that

$$\frac{d(\hat{\mathbf{s}} \cdot \hat{\mathbf{v}})}{dT} = (\boldsymbol{\omega}_{\mathbf{v}} - \boldsymbol{\omega}_{\mathbf{s}}) \cdot (\hat{\mathbf{v}} \times \hat{\mathbf{s}}), \quad (4.22)$$

⁸Here we have also used the following identity

$$A \times_u [B \times_u (C \times_u D)] = B[A \cdot_u (C \times_u D)] - (A \cdot_u B)(C \times_u D).$$

We can also find eqs. (4.20) and (4.21) in [21].

One can indeed define a quantity

$$\tilde{\omega}_a \equiv \omega_{\mathbf{v}} - \omega_{\mathbf{s}} = \frac{e}{m} a \left[\mathbf{B} - \frac{\gamma}{\gamma + 1} (\mathbf{v} \cdot \mathbf{B}) \mathbf{v} \right], \quad (4.23)$$

where the second term is the so-called “pitch correction”,

$$\omega_{\mathbf{P}} \equiv -\frac{e}{m} a \frac{\gamma}{\gamma + 1} (\mathbf{v} \cdot \mathbf{B}) \mathbf{v}. \quad (4.24)$$

One should be aware, though, that the quantity $\tilde{\omega}_a$ does not have a clean physical interpretation, since it is a difference of two vectors defined in different frames. Indeed we will see in subsection 4.1.2 that $|\tilde{\omega}_a|$ does *not* represent the oscillation frequency in a fully correct way.

Note also that only the component of $\tilde{\omega}_a$ orthogonal to the plane defined by $\hat{\mathbf{s}}$ and $\hat{\mathbf{v}}$ enter in eq. (4.22), so that it reduces to

$$\frac{d(\hat{\mathbf{s}} \cdot \hat{\mathbf{v}})}{dT} = \omega_a \cdot (\hat{\mathbf{v}} \times \hat{\mathbf{s}}), \quad (4.25)$$

where

$$\omega_a \equiv \frac{e}{m} a \mathbf{B}. \quad (4.26)$$

4.1.1 Solving the equations of motion

If the constant magnetic field is homogenous in space, one can write it as $\mathbf{B} = (0, 0, B_{\perp})$. The velocity rotates thus only in the x - y plane, so it has the form $\mathbf{v} = (v_x(t), v_y(t), v_z)$, where $v_z = \text{const.}$, because of eq. (4.18) and eq. (4.20).

Let us study first the case of a particle with an initial velocity orthogonal to the magnetic field, $v \cdot B = 0$. In this case $v_z = 0$ at all times and so the spin $\mathbf{s}$ also rotates only in the x - y plane, as $\mathbf{s} = (s_x(t), s_y(t), s_z)$, with $s_z = \text{const.}$, because of eq. (4.19) and eq. (4.21).

From eq. (4.17) one can show that in this case the angular frequency of oscillation of $\hat{\mathbf{v}} \cdot \hat{\mathbf{s}}$ is $\omega_a \equiv |\omega_a|$. Indeed one can define the following decomposition:

$$\hat{\mathbf{s}} = |s_{\parallel}| (\cos(\omega_a T) \hat{\mathbf{v}} + \sin(\omega_a T) \hat{\mathbf{n}}) + s_z \hat{\mathbf{z}}, \quad (4.27)$$

where $\hat{\mathbf{n}}$ is a unit vector orthogonal to $\hat{\mathbf{v}}$ lying in the x - y plane and $s_{\parallel} = \text{constant}$, with $S^2 = s_z^2 + s_{\parallel}^2$. As a consequence we will have an oscillatory behavior of $\hat{\mathbf{s}} \cdot \hat{\mathbf{v}}$ and of $\mathcal{E}(T)$:

$$\frac{\mathcal{E}(T)}{\gamma |v| |S| \mathcal{E}_1} = \hat{\mathbf{s}} \cdot \hat{\mathbf{v}} = \cos(\omega_a T), \quad (4.28)$$

$$\frac{\dot{\mathcal{E}}(T)}{\gamma |v| |S| \mathcal{E}_1} = -\omega_a \sin(\omega_a T). \quad (4.29)$$

If instead $v_z = v \cdot B \neq 0$ then the evolution of s is more complicated, due to the last term of eq. (4.21), the pitch correction, and the motion is not simply harmonic.

One can also infer the frequency from the following definition:

$$r_{v_z=0, \mathbf{E}=0} \equiv \frac{1}{\mathcal{E}_1 \gamma [\hat{\mathbf{B}} \cdot (\Sigma \times_u v)]} \frac{d\mathcal{E}}{dT} = \frac{1}{\mathcal{E}_1 \gamma [\hat{\mathbf{B}} \cdot (\mathbf{s} \times \mathbf{v})]} \frac{d\mathcal{E}}{dT} = \frac{e}{m} a B_\perp = \omega_a. \quad (4.30)$$

Note that Bargmann et al. [19] defined an angle ϕ as follows:

$$S/|S| = \cos(\phi) e_l + \sin(\phi) e_t, \quad (4.31)$$

$$e_t = (0, \hat{\mathbf{n}}), \quad e_l = \gamma(|v|, \hat{\mathbf{v}}), \quad (4.32)$$

$$e_t \cdot e_t = 1, \quad e_l \cdot e_l = 1, \quad e_l \cdot e_t = 0, \quad (4.33)$$

where L coordinates have been used to define e_l and e_t . Using eqs. (4.13), (4.14) and (4.15) we have that

$$|S| \sin(\phi) = S \cdot e_t = \mathbf{s} \cdot \hat{\mathbf{n}} = \frac{[\hat{\mathbf{B}} \cdot (\Sigma \times_u v)]}{|v|}, \quad (4.34)$$

$$|S| \cos(\phi) = S \cdot e_l = \mathbf{s} \cdot \hat{\mathbf{v}} = \frac{(\Sigma \cdot v)}{|v| \gamma}. \quad (4.35)$$

They used, then, its time derivative as a definition of the anomalous precession frequency:

$$\dot{\phi} = -\frac{1}{\sin \phi} \frac{d \cos \phi}{dT} = -\frac{1}{\mathbf{s} \cdot \hat{\mathbf{n}}} \frac{d}{dT} (\mathbf{s} \cdot \hat{\mathbf{v}}). \quad (4.36)$$

Note that, crucially the quantity $\dot{\phi}$ coincides with our definition r only when γ is constant. As we will see in the next subsection $\dot{\phi}$ and r do *not* coincide in the presence of electric fields aligned with the velocity since $\dot{\gamma} = (e/m)v \cdot E$.

4.1.2 General treatment

The oscillation frequency ω can be found in a more general way, by computing the second derivative of $\mathcal{E}$, starting from eq. (4.12):

$$-\omega^2 \equiv \frac{1}{\mathcal{E}} \frac{d^2 \mathcal{E}}{dT^2} = \frac{e}{m(\Sigma \cdot v)} a \gamma B \cdot \left(\frac{dv}{dT} \times_u \Sigma + v \times_u \frac{d\Sigma}{dT} \right). \quad (4.37)$$

Using eq. (3.9) and eq. (3.10) we get:

$$\frac{dv}{dT} \times_u \Sigma = \frac{e}{m\gamma} (v \times_u B) \times_u \Sigma = \frac{e}{m\gamma} (B(\Sigma \cdot v) - v(\Sigma \cdot B)), \quad (4.38)$$

and using eq. (3.13), together with eq. (3.15) and with the property that $v \cdot u = v \times_u u = 0$, we get:

$$v \times_u \frac{d\Sigma}{dT} = \frac{ge}{2m\gamma} v \times_u (\Sigma \times_u B) = \frac{ge}{2m\gamma} [\Sigma(v \cdot B) - B(\Sigma \cdot v)], \quad (4.39)$$

arriving thus at:

$$\omega^2 = \frac{e^2}{m^2} a^2 B^2 \left[1 - \frac{(\hat{B} \cdot \hat{\Sigma})(\hat{v} \cdot \hat{B})}{\hat{\Sigma} \cdot \hat{v}} \right]. \quad (4.40)$$

Note that this result has been computed in a completely coordinate-free way.

Now, the above quantity ω is not simply a number, but a function of time T . If $B \cdot v = 0$ it reduces to a number, meaning that we have an exact harmonic motion with a constant frequency, in agreement with eq. (4.28) and (4.29). In the case $B \cdot v \neq 0$ we get instead a correction. Note that this differs from the so-called “pitch correction” of eq. (4.23). Indeed using eq. (4.13) the above equation becomes:

$$\omega^2 = \frac{e^2}{m^2} a^2 B^2 \left\{ 1 - \left[\frac{\gamma}{\gamma+1} (\mathbf{v} \cdot \hat{\mathbf{B}})^2 + \frac{(\hat{\mathbf{B}} \cdot \hat{\mathbf{s}})(\hat{\mathbf{v}} \cdot \hat{\mathbf{B}})}{\gamma(\hat{\mathbf{v}} \cdot \hat{\mathbf{s}})} \right] \right\}, \quad (4.41)$$

while from eq. (4.23) we have that

$$|\tilde{\omega}_a|^2 = \tilde{\omega}_a \cdot \tilde{\omega}_a = \frac{e^2}{m^2} a^2 B^2 \left[1 - (v \cdot \hat{B})^2 \right]. \quad (4.42)$$

At large γ the two expressions are very similar to each other, but we will check in section 6 that eq. (4.41) better reproduces numerical results.

4.2 Electric and magnetic field in flat space

In this section we still keep Minkowski metric and observer at rest, but we reintroduce the electric fields. In this case, when taking the derivative of eq. (4.8), γ is not constant, from eq. (3.11), leading to

$$\frac{1}{\mathcal{E}_1} \frac{d\mathcal{E}}{dT} \Big|_{\text{e.m.}} = \frac{e}{m} \frac{|S|}{|v_e^{(M)}|} (E \cdot v) + u \cdot \frac{dS}{\gamma d\tau}. \quad (4.43)$$

Now using eq. (3.13) we obtain the full electromagnetic contributions:

$$\begin{aligned} \frac{1}{\mathcal{E}_1} \frac{d\mathcal{E}}{dT} \Big|_{\text{e.m.}} &= \frac{e}{m} \frac{|S|}{|v_e^{(M)}|} (E \cdot v) - \frac{e}{m\gamma} \left(\frac{g}{2} u \cdot (F * S) + a\gamma (S * F * U) \right) = \\ &= \frac{e}{m} \frac{|S|}{|v_e^{(M)}|} (E \cdot v) + \frac{e}{m\gamma} \left(\frac{g}{2} \Sigma \cdot E - a\gamma^2 [(\Sigma \cdot E) - (v \cdot \Sigma)(E \cdot v) + \Sigma \cdot (v \times_u B)] \right) = \\ &= \frac{e}{m} \left[\frac{|S|}{|v_e^{(M)}|} (E \cdot v) - a\gamma (v \cdot \Sigma)(E \cdot v) + a\gamma B \cdot (\Sigma \times_u v) - \frac{(1-a(\gamma^2-1))}{\gamma} (\Sigma \cdot E) \right]. \end{aligned} \quad (4.44)$$

Now, a general definition of the parameter r , which can be used also in this situation, is the following

$$r \equiv \frac{1}{\mathcal{E}_1 \gamma |v| (S \cdot e_t)} \frac{d\mathcal{E}|_{\text{e.m.}}}{dT}. \quad (4.45)$$

This leads to

$$r = \frac{e}{m} a \frac{B \cdot (\Sigma \times_u v)}{|v|(S \cdot e_t)} + \frac{e}{m\gamma} \frac{|S|}{|v_e^{(M)}|} \frac{(E \cdot v)}{|v|(S \cdot e_t)} - \frac{e}{m} a \frac{(v \cdot \Sigma)(E \cdot v)}{|v|(S \cdot e_t)} - \frac{e}{m\gamma^2} \frac{\Sigma \cdot E}{|v|(S \cdot e_t)} [1 - a(\gamma^2 - 1)]. \quad (4.46)$$

The first term is the one we analyzed in the previous subsection. The last term vanishes at the so-called *magic* momentum $\gamma = (1 + a^{-1})^{1/2}$. The second and third terms are the ones due to $\dot{\gamma}$. We will analyze the impact of such terms in experiments in section 6.

Note that if we used instead $\dot{\phi}$, as defined in eq. (4.36), we would get:

$$\dot{\phi} = \frac{e}{m} a \frac{B \cdot (\Sigma \times_u v)}{|v|(S \cdot e_t)} + \frac{e}{m} \frac{(v \cdot \Sigma)(E \cdot v)}{|v|(S \cdot e_t)} \left(a - \frac{1}{\gamma^2 - 1} \right) - \frac{e}{m\gamma^2} \frac{\Sigma \cdot E}{|v|(S \cdot e_t)} [1 - a(\gamma^2 - 1)], \quad (4.47)$$

so that also the second and the third term would vanish at the magic momentum. However the quantity directly observable in an experiment is the period of variation of the energy, which is set by r , not $\dot{\phi}$. Therefore the choice of the magic momentum does not guarantee that effects due to electric fields can be completely neglected.

Note also that the first additive term on the l.h.s. of eq. (4.47) can be written in a different way, i.e., taking into account that

$$\frac{B \cdot (\Sigma \times_u v)}{|v|(S \cdot e_t)} = \frac{|S| \sin(\phi) B \cdot (e_t \times_u v)}{|v|(S \cdot e_t)} = \mathbf{v} \cdot (\mathbf{B} \times \hat{\mathbf{n}}),$$

where we used that $\Sigma \times_u v = S \times_u v$ and $S \cdot e_t = |S| \sin(\phi)$. In this case, using the above relation, eq. (4.47) exactly coincides with the result obtained in Bargmann et al. [19].

Finally, also in this case, one may compute the second derivative of the energy $\mathcal{E}$, leading to a lengthy result:

$$\begin{aligned} -\omega^2 \equiv \frac{1}{\mathcal{E}_1} \frac{d^2 \mathcal{E}}{dT^2} = \frac{e^2}{m^2} \Bigg\{ & \frac{|S|}{|v_e^{(M)}|} \left(\frac{E^2 + B \cdot (E \times v) - (E \cdot v)^2}{\gamma} + \dot{E} \cdot v \right) + \\ & + a^2 B^2 (\Sigma \cdot v) - a^2 (B \cdot v)(B \cdot \Sigma) \\ & - (E \cdot v)^2 (\Sigma \cdot v) a(1 + a) + a^2 B \cdot (E \times v) (\Sigma \cdot v) \\ & + a(1 + a) B \cdot (\Sigma \times v) (E \cdot v) + \frac{B \cdot (E \times \Sigma) (-1 - 2a + a^2(\gamma^2 - 1))}{\gamma^2} \\ & + \frac{E^2 (\Sigma \cdot v) (-1 - 2a + a^2(\gamma^2 - 1))}{\gamma^2} + \frac{(E \cdot \Sigma) (E \cdot v) (1 + a + a\gamma^2)}{\gamma^2} \\ & \left. - a\gamma (\Sigma \cdot v) (\dot{E} \cdot v) - \frac{1 - a(\gamma^2 - 1)}{\gamma} (\Sigma \cdot \dot{E}) + a\gamma \dot{B} \cdot (\Sigma \times v) \right\}. \quad (4.48) \end{aligned}$$

Here the time derivatives of the E and B field have to be computed along the trajectory as

$$\dot{E}^i = \partial_j E^i v^j, \quad \dot{B}^i = \partial_j B^i v^j. \quad (4.49)$$

In the BNL experiment [1] there is an electric field (of quadrupolar form) leading to a non-trivial to $\dot{E}$. If the magnetic field can be considered perfectly homogeneous then $\dot{B} = 0$. As we have stressed, not all the terms vanish at the magic momentum. We will comment on the impact of such terms in section 6.

5 Effects due to a generic GR metric

The full result for the derivative of the electron energy, in presence of a generic motion of u in a generic metric is:

$$\begin{aligned} \frac{1}{\mathcal{E}_1} \frac{d\mathcal{E}}{dT} &= \frac{m_e \gamma_e^{(M)}}{\mathcal{E}_1} \dot{\gamma} + \left(u \cdot \frac{DS}{dT} + \frac{Du}{dT} \cdot S \right) = \\ &= \frac{|S|}{|v_e^{(M)}|} \left(\frac{e}{m} E + f_G \right) \cdot v + u \cdot \frac{DS}{dT} - \frac{1}{\gamma} (f_G \cdot S) = \end{aligned} \quad (5.1)$$

$$= \frac{1}{\mathcal{E}_1} \frac{d\mathcal{E}}{dT} \Big|_{\text{e.m.}} + \frac{|S|}{|v_e^{(M)}|} f_G \cdot v - \frac{1}{\gamma} (f_G \cdot S) . \quad (5.2)$$

where we used eq. (3.11) and (3.12). The first term $(1/\mathcal{E}_1)d\mathcal{E}/dT|_{\text{e.m.}}$ is exactly as in eq. (4.44), where all the scalar products have to be taken with the full metric and all the vector products as $\times_u$, defined in eq. (3.8). Then, there are purely gravitational terms: one arising from $\dot{\gamma}$, proportional to $f_G \cdot v$, and a new term proportional to $f_G \cdot S$.

We will study the impact of such terms in two specific cases: a rotating observer and a Schwarzschild metric.

Before doing that, we will also write separate equations for $\hat{\Sigma}$ and for $\hat{v}$ in the following subsections.

5.1 Equations for $\hat{\Sigma}$ and $\hat{v}$

We are going to write here the time derivative of the electron energy, in presence of a generic motion of u in a generic metric, in terms of differential equations for $\hat{\Sigma}$ and $\hat{v}$. Here we define $\hat{\Sigma}$ and $\hat{v}$ in the following way:

$$\hat{\Sigma} = \frac{\Sigma}{|\Sigma|} \quad \text{and} \quad \hat{v} = \frac{v}{|v|} , \quad (5.3)$$

where

$$|\Sigma| = [\Sigma \cdot \Sigma]^{1/2} \quad \text{and} \quad |v| = [v \cdot v]^{1/2} . \quad (5.4)$$

Starting again with the energy of the electron in the laboratory frame

$$\mathcal{E} = m_e \gamma_e^{(M)} = m_e \gamma_e^{(M)} \left[\gamma - \frac{|v_e^{(M)}|}{|S|} |\Sigma| |v| \left(\hat{\Sigma} \cdot \hat{v} \right) \right] . \quad (5.5)$$

Defining the spatial momentum per unit mass of the electron as $p \equiv \gamma v$, we have that

$$\frac{d|p|}{dT} = (f_{\text{EM}} + f_G) \cdot v = \gamma \frac{d|v|}{dT} + \dot{\gamma} |v| = \gamma \frac{d|v|}{dT} + v \cdot (f_{\text{EM}} + f_G) |v| , \quad (5.6)$$

and so

$$\frac{d|v|}{dT} = \frac{(f_{\text{EM}} + f_G) \cdot \hat{v}}{\gamma^3} . \quad (5.7)$$

Multiplying eq. (3.17) by $\hat{\Sigma}$ and using eq. (5.3,a), after a little algebra we can also compute the time derivative of $|\Sigma|$,

$$\frac{d|\Sigma|}{dT} = (\Sigma \cdot v) \left[\mathcal{F} \cdot \hat{\Sigma} + \frac{ea\gamma}{m} |v| B \cdot (\hat{v} \times_u \hat{\Sigma}) \right] , \quad (5.8)$$

where the four vector $\mathcal{F}$ is defined as

$$\mathcal{F} = \frac{(1+a)e}{m\gamma} E + \frac{1}{\gamma} f_G - \frac{ea\gamma}{m} [E - (E \cdot v)v] . \quad (5.9)$$

We can also rewrite eq. (5.8) in different way. Indeed, taking into account that⁹

$$(\Sigma \cdot v) (\mathcal{F} \cdot \hat{\Sigma}) = |v||\Sigma| \left[\left(\hat{\Sigma} \times_u \mathcal{F} \right) \cdot \left(\hat{v} \times_u \hat{\Sigma} \right) + \mathcal{F} \cdot \hat{v} \right] , \quad (5.10)$$

eq. (5.8) becomes

$$\frac{d|\Sigma|}{dT} = |v||\Sigma| \left\{ \left[\left(\hat{\Sigma} \times_u \mathcal{F} \right) + \frac{ea\gamma}{m} |v| \left(\hat{\Sigma} \cdot \hat{v} \right) B \right] \cdot \left(\hat{v} \times_u \hat{\Sigma} \right) + \mathcal{F} \cdot \hat{v} \right\} . \quad (5.11)$$

Another important ingredient is

$$\frac{d|\hat{v} \cdot \hat{\Sigma}|}{dT} = \Omega_a \cdot \left(\hat{v} \times_u \hat{\Sigma} \right) , \quad (5.12)$$

with $\Omega_a \equiv \Omega_v - \Omega_\Sigma$, where Ω_v and Ω_Σ are respectively the angular velocity of v and Σ , which will be defined and computed explicitly in the next subsections. Adding all the above relations we find

$$\begin{aligned} \frac{d\mathcal{E}}{dT} = & \gamma_e^{(M)} [v \cdot (f_{EM} + f_G)] - \gamma_e^{(M)} \frac{|v_e^{(M)}|}{|S|} \left\{ |v|^2 |\Sigma| \left(\hat{v} \cdot \hat{\Sigma} \right) (\mathcal{F} \cdot \hat{v}) \right. \\ & \left. + \frac{|\Sigma|}{\gamma^3} [\hat{v} \cdot (f_{EM} + f_G)] \left(\hat{v} \cdot \hat{\Sigma} \right) + |v||\Sigma| \left[\Omega_{\text{tot}} \cdot \left(\hat{v} \times_u \hat{\Sigma} \right) \right] \right\} , \end{aligned} \quad (5.13)$$

where

$$\Omega_{\text{tot}} \equiv \Omega_a + \Omega_{\mathcal{E}} = \Omega_v - \Omega_\Sigma + \Omega_{\mathcal{E}} , \quad (5.14)$$

with

$$\Omega_{\mathcal{E}} \equiv |v| \left(\hat{\Sigma} \cdot \hat{v} \right) \left[\left(\hat{\Sigma} \times_u \mathcal{F} \right) + \frac{ea\gamma}{m} |v| \left(\hat{\Sigma} \cdot \hat{v} \right) B \right] . \quad (5.15)$$

In order to obtain Ω_{tot} , let us note that if we consider a generic four-vector X orthogonal to u , i.e. $X \cdot u = 0$, we can write

$$P(u) \frac{DX}{dT} = \frac{D_{(\text{fw}, U, u)} X}{dT} = \left(\frac{D_{(\text{fw}, U, u)} X}{dT} \right)^{\hat{\mu}} e_{\hat{\mu}} , \quad (5.16)$$

where, still for $X \cdot u = 0$, we have defined the “Fermi-Walker total spatial covariant derivative” in the following way

$$\gamma \frac{D_{(\text{fw}, U, u)} X}{dT} = \frac{D_{(\text{fw}, U, u)} X}{d\tau} \equiv P(u) \frac{DX}{d\tau} . \quad (5.17)$$

⁹We have used the following identity

$$(A \times_u B) (C \times_u D) = (A \cdot_u C) (B \cdot_u D) - (A \cdot_u D) (B \cdot_u C) ,$$

where $\cdot_u = P(u)$.

Now, we are considering the frame $\{e_{\hat{\mu}}\}$ which is the orthonormal frame adapted to the observer u , so that $X \cdot u = X \cdot e_{\hat{0}} = X_{\hat{0}} = 0$. Applying $D_{(\text{fw}, U, u)}/dT$ to $\{e_{\hat{a}}\}$, we have¹⁰ [18, 20]

$$\frac{D_{(\text{fw}, U, u)} e_{\hat{a}}}{dT} = [\zeta_{(\text{fw})} + \zeta_{(\text{sc})}] \times_u e_{\hat{a}}, \quad (5.18)$$

where $\zeta_{(\text{fw})}$ is the so-called Fermi-Walker angular velocity vector

$$\zeta_{(\text{fw})}^{\hat{a}} = -\frac{1}{2} \eta(u)^{\hat{a}\hat{b}\hat{c}} C_{(\text{fw})\hat{b}\hat{c}}, \quad (5.19)$$

$C_{(\text{fw})\hat{a}\hat{b}}$ are termed Fermi-Walker structure functions,

$$C_{(\text{fw})\hat{b}\hat{a}} \equiv e_{\hat{b}} \cdot \left[P(u) \frac{D e_{\hat{a}}}{dT} \right],$$

and $\zeta_{(\text{sc})}$ is the spatial curvature angular velocity,¹¹

$$\zeta_{(\text{sc})}^{\hat{a}} \equiv -\frac{1}{2} \eta(u)^{\hat{a}\hat{b}\hat{c}} \Gamma_{[\hat{c}|\hat{d}]\hat{b}} v^{\hat{d}}. \quad (5.20)$$

As we will see in the next subsections this adapted frame is useful in order to get quickly Ω_v and Ω_{Σ} , which are measured in the observed frame (see also the discussion in refs. [18, 20]).

5.1.1 Analytical expression for Ω_v

Applying $D_{(\text{fw}, U, u)}/dT$ on p we have

$$\frac{D_{(\text{fw}, U, u)} p}{dT} = (f_{\text{EM}} + f_G) = \frac{d|p|}{dT} \hat{v} + \frac{D_{(\text{fw}, U, u)} \hat{v}}{dT} |p|, \quad (5.21)$$

and using

$$\frac{d|p|}{dT} = [(f_{\text{EM}} + f_G) \cdot \hat{v}], \quad (5.22)$$

we get

$$\frac{D_{(\text{fw}, U, u)} \hat{v}}{dT} = \frac{(f_{\text{EM}} + f_G)}{|p|} - \frac{d \ln |p|}{dT} \hat{v} = \frac{(f_{\text{EM}} + f_G) - [(f_{\text{EM}} + f_G) \cdot \hat{v}] \hat{v}}{|p|}, \quad (5.23)$$

¹⁰From the definition of $\{e_{\hat{a}}\}$, we immediately note that

$$\frac{D_{(\text{fw}, u)} e_{\hat{a}}}{dT} = 0.$$

This is the reason why this tetrad frame is also called Fermi-Walker. Here we have defined the Fermi-Walker derivative $D_{(\text{fw}, Z)} Y/d\tau_Z$ on a generic four-vector Y along the time-like vector Z (i.e. $Z \cdot Z = -1$) in the following way

$$\frac{D_{(\text{fw}, Z)} Y}{d\tau_Z} \equiv \frac{DY}{d\tau_Z} + a(Z)(Z \cdot Y) - Z(a(Z) \cdot Y),$$

where

$$a(Z) \equiv \frac{DZ}{d\tau_Z}.$$

Obviously, in our case, if $Z = \{U, u\}$ then $\tau_Z = \{\tau, T\}$.

¹¹Here we have defined

$$\Gamma_{[\hat{c}|\hat{d}]\hat{b}} \equiv \frac{1}{2} (\Gamma_{\hat{c}\hat{d}\hat{b}} - \Gamma_{\hat{b}\hat{d}\hat{c}}).$$

or, equivalently,

$$\left(\frac{d\hat{v}}{dT}\right)^{\hat{a}} = [\Omega_v \times_u \hat{v}]^{\hat{a}}, \quad (5.24)$$

where

$$\Omega_v = -\frac{(f_{\text{EM}} + f_G) \times_u \hat{v}}{|p|} - (\zeta_{(\text{fw})} + \zeta_{(\text{sc})}). \quad (5.25)$$

Here we have used eq. (5.18).

5.1.2 Analytical expression for Ω_Σ

Let us rewrite eq. (3.17) as

$$\begin{aligned} \frac{D_{(\text{fw}, U, u)} \Sigma}{d\tau} &= P(u) \frac{D\Sigma}{d\tau} = \frac{(1+a)e}{m} [(v \cdot \Sigma)E + \Sigma \times_u B] + (v \cdot \Sigma)f_G - a\gamma^2(f_{\text{EM}} \cdot \Sigma)v \\ &\quad + \frac{ae}{m}\gamma^2(v \cdot \Sigma)(E \cdot v)v. \end{aligned} \quad (5.26)$$

Now, using the definition of $\hat{\Sigma}$, this leads to

$$\begin{aligned} \frac{D_{(\text{fw}, U, u)} \hat{\Sigma}}{d\tau} &= -\frac{d \ln |\Sigma|}{d\tau} \hat{\Sigma} + \frac{(1+a)e}{m} [(v \cdot \Sigma)E + \Sigma \times_u B] + (v \cdot \Sigma)f_G - a\gamma^2(f_{\text{EM}} \cdot \Sigma)v \\ &\quad + a\gamma^2(v \cdot \Sigma)(f_{\text{EM}} \cdot v)v. \end{aligned} \quad (5.27)$$

From eq. (5.11) and taking into account of eq. (5.18), the above equation can be rewritten in the following way

$$\left(\frac{d\hat{\Sigma}}{dT}\right)^{\hat{a}} = [\Omega_\Sigma \times_u \hat{v}]^{\hat{a}}, \quad (5.28)$$

where

$$\begin{aligned} \Omega_\Sigma &= \hat{\Sigma} \times_u \left\{ \frac{(1+a)e}{m\gamma} [(v \cdot \hat{\Sigma})E + (\hat{\Sigma} \times_u B)] + \frac{1}{\gamma}(v \cdot \hat{\Sigma})f_G \right. \\ &\quad \left. + \gamma a [(f_{\text{EM}} \cdot \hat{\Sigma}) - (v \cdot \hat{\Sigma})(f_{\text{EM}} \cdot v)]v \right\} - (\zeta_{(\text{fw})} + \zeta_{(\text{sc})}). \end{aligned} \quad (5.29)$$

Then, in order to have a complete analysis of the problem, in appendix A, we will generalize the three-vector $\mathbf{s}$ in a general GR frame.

5.2 Gravitational terms

We study quantitatively here the size of the gravitational terms, focusing on two particular cases of interest: (i) Minkowski spacetime with a rotating observer, with angular frequency ω_T ; (ii) Schwarzschild metric, with observer at rest at some radial coordinate r . This allows us to analyze separately the two effects of rotation and gravity of the Earth. Clearly the leading effects will be order $\mathcal{O}(\omega_T)$ and of order $\mathcal{O}(g)$ respectively, where $g \equiv MG/r^2$ is the gravitational acceleration felt at a fixed coordinate r , M is the Earth's mass and G is Newton's constant. We will disregard any cross terms between these two effects.

The only ingredients that we need are the metric inside the scalar and vector products and an expression for f_G , which can be readily evaluated using eq. (3.12), inserting the Christoffel symbols in a chosen coordinate system.

Finally we will also study, as a simple extension, the case of a gravitational wave metric.

5.2.1 Rotating metric

In this subsection we neglect Earth's gravity and consider only effects due to its rotation, at an angular frequency $\omega_T \simeq 7 \times 10^{-5}$ Hz. The dominant effect will be a Coriolis acceleration, which is of order $|\omega_T \times v| \approx \omega_T$ (we work in units $c = 1$). This has to be compared with $\omega_a \equiv a \cdot (eB/m) \approx 10^6$ Hz, giving a relative error of order $\omega_T/\omega_a \approx 5 \times 10^{-11}$. This is about 4 orders of magnitude smaller than the present experimental uncertainty for muon experiments, but we compute it anyway, since it represents the leading effect due to a nontrivial metric and it might be important for future experiments. Note also that factors of γ here have to be computed, as they may change the above rough estimate, given that $\gamma \approx 29.3$.

It is also clear that the Coriolis acceleration is much larger than the acceleration due to gravity on the surface of the Earth, since $|\omega_T \times v|/g \approx 2 \times 10^3$, which is the reason why Earth's gravity can be completely neglected.

We start from an inertial reference system, described by coordinates $x_I^\mu = (t, x^i)$, whose metric is simply

$$ds^2 = -dt^2 + dx_i dx^i = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2,$$

where we employ also cylindrical coordinates. We consider then a system that rotates at an angular frequency ω_T , i.e., described by $\varphi \equiv \phi - \omega_T t = \text{const.}$ The metric can thus be written as

$$ds^2 = -dt^2(1 - \omega_T^2 r^2) + dr^2 + r^2 d\varphi^2 + 2\omega_T r^2 d\varphi dt + dz^2. \quad (5.30)$$

We will use from now on such coordinates $x^\mu = (t, r, \varphi, z)$. An observer $\mathcal{O}$ at rest in the rotating system has a four velocity $u^\mu = (u^0, 0, 0, 0)$, where $u^0 = (1 - \omega_T^2 r^2)^{-1/2}$ in order to have $u^\mu u_\mu = -1$. The observed velocity of a particle is given by $v \equiv (v^0, v^r, v^\varphi, v^z)$, where v^0 can be found by imposing that $v \cdot u = 0$.

We will work at linear order in ω_T , i.e., considering only Coriolis forces and neglecting centrifugal terms of $\mathcal{O}(\omega_T^2 R)$, where R is the Earth radius. The latter are indeed smaller, at relativistic velocities, by about $|\omega_T^2 R|/|\omega_T \times v| \approx 0.5 \times 10^{-6}$.

By computing the Christoffel symbols from the above metric one finds

$$f_G \simeq \omega_T \gamma(0, r v^\varphi, -v^r/r, 0) = -\gamma(\Omega_T \times_u v), \quad (5.31)$$

where $\Omega_T \equiv (0, 0, 0, \omega_T)$. The only non-trivial quantity at linear order in ω_T is a Coriolis-type effect

$$f_G \cdot S \simeq -\gamma \Omega_T \cdot (v \times_u \Sigma). \quad (5.32)$$

Using eq. (5.2) and comparing with eqs. (4.12), (4.26), we obtain the Coriolis contribution ω_a^{Cor} (at linear order in ω_T) in analogy to the anomalous precession vector ω_a :

$$\omega_a^{\text{Cor}} \equiv -\frac{\Omega_{\mathbf{T}}}{\gamma}. \quad (5.33)$$

(In the Newtonian limit e.g. see [22].) The same result can be obtained directly using eq. (5.13). Indeed, considering $\Omega_{\text{tot}} = \Omega_v - \Omega_{\Sigma} + \Omega_{\mathcal{E}}$ and making explicit Ω_v , Ω_{Σ} and $\Omega_{\mathcal{E}}$, for the case $E = B = 0$, at linear order in ω_T (and thus ignoring terms in $\dot{\gamma}$, which are quadratic in ω_T) we recover the above result.

5.2.2 Equations of motion and numerical solution

We double checked the previous result by solving numerically the exact equations of motion, adding also the presence of a constant magnetic field. The equations of motion become

$$\frac{Dv}{DT} = -\frac{v \cdot f_G}{\gamma}(u + v) + \frac{f_G}{\gamma} + \frac{e}{m\gamma}(v \times_u B), \quad (5.34)$$

and

$$\frac{D\Sigma}{DT} = -\frac{\Sigma \cdot f_G}{\gamma}u + (\Sigma \cdot v)\frac{f_G}{\gamma} + \frac{ge}{2m\gamma}(\Sigma \times_u B) - \frac{e}{m}av\gamma\Sigma \cdot (v \times_u B). \quad (5.35)$$

We then need to evaluate the quantities $\Sigma \cdot v$ and $\gamma = 1/\sqrt{1-v^2}$ inside the electron energy $\mathcal{E}$, given by eqs. (4.8), (4.9). In the rotating metric we have that:

$$v^2 = (v^r)^2 + \frac{r^2(v^\varphi)^2}{1-r^2\omega_T^2} + (v^z)^2, \quad \Sigma \cdot v = v^r\Sigma^r + \frac{r^2v^\varphi\Sigma^\varphi}{1-r^2\omega_T^2} + v^z\Sigma^z. \quad (5.36)$$

We only need therefore to solve for the r, φ, z components of eqs. (5.34) and (5.35). One can work with the above full equations numerically. We have indeed solved them in the particular case of $B = (0, 0, 0, |B|)$, where B and the Earth rotation are aligned. The results in figure 1 show that the leading effect is captured by the vector ω_a^{Cor} of eq. (5.33). We infer thus that, at linear order in ω_T , the total precession is given by

$$\omega_a \rightarrow \omega_a + \omega_a^{\text{Cor}} \simeq \frac{e}{m}a\mathbf{B} - \frac{\Omega_{\mathbf{T}}}{\gamma}. \quad (5.37)$$

This amounts to a relative correction of order $\Delta\omega_a/\omega_a \approx \omega_T/(\gamma\omega_a) \approx 1.7 \cdot 10^{-12}$, at the magic momentum. Note that also γ can be taken to be constant at linear order in ω_T .

This behavior can be studied also at linear order in ω_T , leading to a more explicit form, which has also been used as a numerical test, giving the same leading order results. The first equation becomes

$$\frac{Dv^i}{DT} = \frac{f_G^i}{\gamma} + \frac{e}{m\gamma}(v \times B)^i. \quad (5.38)$$

The spin equation becomes instead

$$\frac{D\Sigma^i}{DT} = (\Sigma \cdot v)\frac{f_G^i}{\gamma} + \frac{ge}{2m\gamma}(\Sigma \times B)^i - \frac{e}{m}av^i\gamma\Sigma \cdot (v \times B). \quad (5.39)$$

Figure 1. We have solved numerically the equations of motion eqs. (5.34)–(5.35) and then evaluated $\Sigma \cdot v$, from eq. (5.36), on the solution. We show that the sinusoidal behavior with frequency $\omega_a - \omega_T/\gamma_0$ has the correct behavior, modulo tiny higher order effects. We have set realistic values of a and ω_T and an initial condition $\gamma(T=0) \equiv \gamma_0 = 29.3$. Here $T_a \equiv 2\pi/\omega_a$. Note that also γ is constant at linear order in ω_T , and so the only time-dependent part of $\mathcal{E}$ in eq. (4.8) is due to $\Sigma \cdot v$.

In both equations all the scalar and vector products do *not* contain ω_T and the linearized f_G is the only term proportional to ω_T , given by eq. (5.31). The covariant derivatives also contains terms proportional to ω_T , since:

$$\begin{aligned}
 \frac{Dv^r}{DT} &= \frac{dv^r}{dT} - r(v^\varphi)^2 - rv^\varphi\omega_T, \\
 \frac{Dv^\varphi}{DT} &= \frac{dv^\varphi}{dT} + \frac{2v^r v^\varphi}{r} + \omega_T \frac{v^r}{r}, \\
 \frac{Dv^z}{DT} &= \frac{dv^z}{dT}, \\
 \frac{D\Sigma^r}{DT} &= \frac{d\Sigma^r}{dT} - rv^\varphi\Sigma^\varphi - r\Sigma^\varphi\omega_T, \\
 \frac{D\Sigma^\varphi}{DT} &= \frac{d\Sigma^\varphi}{dT} + \frac{\Sigma^\varphi v^r + \Sigma^r v^\varphi}{r} + \omega_T \frac{\Sigma^r}{r}, \\
 \frac{D\Sigma^z}{DT} &= \frac{d\Sigma^z}{dT}.
 \end{aligned} \tag{5.40}$$

Another potentially important issue is that in the case of the Brookhaven E821 experiment stationary protons are used to calibrate the frequencies¹² and this measurement may also be affected by the Coriolis effect. Indeed in the case of particles with negligible velocity one has to look simply at the spin equation, eq. (5.39). In this case setting all velocity terms to zero on the r.h.s. only the standard $\frac{g_e}{2m}(\Sigma \times B)^i$ term survives. However the left-hand side of the equation contains non-trivial Christoffel symbols which survive even at zero velocity, as can be seen from the last terms in the 4th and 5th equations in

¹²We thank the anonymous referee for pointing out this.

eqs. (5.40), and so indeed there will be corrections to the frequency of the spin, linear in ω_T . The relative size of this effect can be estimated simply by comparing such terms with $\frac{ge|B|}{2m} \equiv \tilde{\omega}_p$ of eq. (5.39). Using the fact that $\tilde{\omega}_p \approx 4 \times 10^8$ Hz in the Brookhaven E821 experiment [14], one gets a relative correction of order $\omega_T/\tilde{\omega}_p \approx 1.8 \times 10^{-13}$. This does not affect the results of the experiment, but it might represent another source of uncertainty for more precise future experiments.

As a final comment note that, while the Coriolis effect is very small compared to the experimental sensitivity for muon $g - 2$ experiments, it could be slightly more important for *electron* $g - 2$ experiments, which are much more precise. In the electron experiments velocities are again non-relativistic and so the Coriolis frequency vector is just $\mathbf{\Omega}_T$. This should be compared with the Larmor frequency of about $\omega_L = 100$ GHz, for an electron in a Penning trap experiment [23, 24], which gives a relative shift $\Delta\omega/\omega_L \approx 7 \times 10^{-16}$, to be compared with the experimental relative error on g , at present of order $\mathcal{O}(10^{-13})$, and forecasted to reach $\mathcal{O}(10^{-14})$ [25] within one or two years from now. In terms of $a_{\text{el}} \equiv g/2 - 1$ the effect amounts to a relative shift $\Delta a_{\text{el}}/a_{\text{el}} \approx 7 \times 10^{-13}$, compared to the experimental relative error $\Delta a_{\text{el}}/a_{\text{el}} \approx \mathcal{O}(10^{-10})$.

5.2.3 Schwarzschild metric

In this subsection we consider the even more negligible effect of Earth's gravity. As argued above by a rough power counting we expect indeed a suppression of about three orders of magnitude compared to the Coriolis effect.

We consider here a standard Schwarzschild metric, given by

$$ds^2 = -dt^2 \left(1 - 2\frac{GM}{r}\right) + \frac{dr^2}{1 - 2\frac{GM}{r}} + r^2 d\theta^2 + r^2 \sin(\theta) d\varphi^2. \quad (5.41)$$

In this case for an observer at fixed coordinates, with $u = (u^0, 0, 0, 0)$, we get:

$$f_G = \left(0, -\frac{\gamma GM}{r^2}, 0, 0\right), \quad (5.42)$$

leading to

$$f_G \cdot v = \frac{\gamma GM v^r}{2GM r - r^2}, \quad f_G \cdot S = \frac{\gamma GM S^r}{2GM r - r^2}, \quad (5.43)$$

where $S = (S^0, S^r, S^\theta, S^\varphi)$. At leading order in GM/r we get

$$f_G \cdot v = -g_T \gamma v^r, \quad f_G \cdot S = -g_T \gamma S^r, \quad (5.44)$$

where at the Earth's surface $g_T \equiv GM/r^2$. The first effect is simply due to the radial acceleration of a free falling muon and it is thus suppressed by the smallness of g_T and furthermore by the smallness of the radial velocities, $v_r \ll 1$, in a realistic experiment and so, comparing eq. (5.2) with eq. (4.12), it leads to a relative error $\Delta\omega_a/\omega_a \approx g_T v^r/(\omega_a) \approx 10^{-15} v^r$ and thus totally negligible. The second effect is less suppressed and comparing again with eq. (4.12) one gets $\Delta\omega_a/\omega_a \approx g_T/(\gamma\omega_a) \approx 8 \times 10^{-16}$, taking into account that, using eq. (4.13), one has $S^r \approx s^r$ in a realistic experiment, since $v^r \ll 1$.

One can further check this more explicitly by writing a system of equations in a simple setup, where the magnetic field is along r , $B = (0, B^r, 0, 0)$. In this case the observed magnitude of the magnetic field is given by

$$|B| = \sqrt{B \cdot B} = B^r / \sqrt{1 - 2\Phi}, \quad \text{where} \quad \Phi \equiv g_T r. \quad (5.45)$$

Here we see that if we use the system of coordinates of the observer u , then $B^r / \sqrt{1 - 2\Phi} = B^{\hat{r}}$. Moreover in a realistic laboratory setup we can also neglect terms in v^r . The explicit equations of motion eqs. (5.34) and (5.35) get additional terms only in the two following equations

$$\frac{dv^r}{dT} = \left. \frac{dv^r}{dT} \right|_{\text{flatspace}} - g_T, \quad (5.46)$$

$$\frac{d\Sigma^r}{dT} = \left. \frac{d\Sigma^r}{dT} \right|_{\text{flatspace}} - 3g_T r^2 (s^\theta v^\theta + s^\varphi v^\varphi \sin^2 \theta), \quad (5.47)$$

where $d/dT|_{\text{flatspace}}$ contain the usual flat space terms, with $|B|$ defined in the above eq. (5.45). The solution may be plugged then into

$$\Sigma \cdot v = r^2 (s^\theta v^\theta + s^\varphi v^\varphi \sin^2 \theta), \quad (5.48)$$

while terms in $\dot{\gamma}$ are neglected. By comparing with eqs. (5.38) and (5.39) an order of magnitude estimate of the effects is obtained, for relativistic velocities, simply by replacing ω_T with g_T , in agreement with the above $\Delta\omega_a/\omega_a \approx g_T/(\gamma\omega_a) \approx 8 \times 10^{-16}$.

As a side comment, we note that our formalism might be useful to study the case of a spinning body close to a system with strong gravity, such as a black hole. From eq. (5.43) one can see that the effects become very large close to the horizon, when r approaches the Schwarzschild radius, $r_s \equiv 2GM$.

5.2.4 Gravitational wave

We also discuss here, as a very simple application of our formalism, the case of a gravitational wave (GW) metric. This is negligible for a realistic $g - 2$ experiment, due to the smallness of the GW's amplitude on Earth, but it might have applications in other setups, such as the case of a spinning body close to a strong source of GW's, e.g. a black hole or neutron star binary system. We may consider thus a GW metric, as a perturbation of flat spacetime in Cartesian coordinates, using Lorenz and Transverse Traceless gauge conditions, which completely fix the coordinates. For instance the metric for a GW propagating in the $\hat{z}$ direction with $+$ and $\times$ polarizations read

$$ds^2 = -dt^2 + dx^2(1 + h_+(t, z)) + dy^2(1 + h_+(t, z)) + 2h_\times dx dy + dz^2. \quad (5.49)$$

Assuming an observer at rest $u = (1, 0, 0, 0)$, one finds, at linear order in h_+ and $h_\times$,

$$f_G = -\frac{\gamma}{2} \left(0, v^x \frac{dh_+}{dt} + v^y \frac{dh_\times}{dt}, v^x \frac{dh_\times}{dt} - v^y \frac{dh_+}{dt}, 0 \right), \quad (5.50)$$

leading to

$$\begin{aligned} f_G \cdot v &= \gamma \left(\frac{dh_+}{dt} \frac{(v^y)^2 - (v^x)^2}{2} - \frac{dh_\times}{dt} v^x v^y \right), \\ f_G \cdot S &= \frac{\gamma}{2} \left(\frac{dh_+}{dt} (S^y v^y - S^x v^x) - \frac{dh_\times}{dt} (S^y v^x + S^x v^y) \right), \end{aligned} \quad (5.51)$$

where $S = (S^0, S^x, S^y, S^z)$ and $v = (v^0, v^x, v^y, v^z)$. For a gravitational wave of amplitude h and angular frequency ω_{GW} , the effects on a relativistic particle are of order $f_G \cdot v \approx f_G \cdot S/\gamma \approx \gamma h \omega_{\text{GW}}$, using also eq. (4.13). Comparing again eq. (5.2) with eq. (4.12), these represent corrections to the precession frequency of order $\Delta\omega_a/\omega_a \approx h \omega_{\text{GW}}/\omega_a$. For GW's detected on Earth, such as the ones coming from coalescing binaries, with $h \sim 10^{-21}$ and $\omega_{\text{GW}} \sim 10^2$ Hz, this is about 10^{-25} .

6 Impact on experiments

6.1 Possible systematic effects due to E

From the previous sections we have found that the leading effect of a non-trivial metric, the Coriolis force, leads to a shift $\Delta\omega/\omega_a \approx 10^{-12}$, much smaller than the present sensitivity. However we show here that effects purely due to electromagnetism have to be taken into account:

- (1) a proper treatment of the “pitch” correction;
- (2) a proper treatment of the presence of electric fields;
- (3) the fact that the muon γ factor is not constant, also because of electric fields, which has given rise to the first two terms in eq. (4.44) that do not vanish at the magic momentum and might not be small in present experiments.

We estimate carefully such effects here, using a Minkowski metric, and considering a realistic experimental setup.

The experimentally observable quantity in eqs. (4.7) and (4.8) is

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_R = m_e \gamma_e^{(\text{M})} \gamma - \mathcal{E}_1 \gamma |v| |S| (\hat{s} \cdot \hat{v}) = m_e \gamma_e^{(\text{M})} \gamma - \mathcal{E}_1 |S| \sqrt{\gamma^2 - 1} (\hat{s} \cdot \hat{v}). \quad (6.1)$$

As we will see, in a realistic experiment there are non-trivial effects both in $\hat{s} \cdot \hat{v}$, and in γ , leading to deviations from the simple sinusoidal time dependence with frequency ω_a .

We follow here ref. [14]. At zero order we consider a perfectly circular motion, with constant gamma factor, chosen to be equal to the “magic” gamma factor, $\gamma_0 = 1/\sqrt{1 - v_0^2} = \gamma_M$, in a constant magnetic field of magnitude $|B|$, at the cyclotron frequency

$$\omega_c \equiv \frac{e}{m} \frac{|B|}{\gamma_M}, \quad (6.2)$$

and at the equilibrium radius

$$R_0 = \frac{m}{e} \frac{\gamma_M v_0}{|B|}. \quad (6.3)$$

Eq. (6.1) becomes

$$\mathcal{E} = \mathcal{E}_0 - \mathcal{E}_1 \gamma_0 |v_0| |S| \sin(\omega_a T) , \quad (6.4)$$

with constant $\mathcal{E}_0$ and where we have used as an initial condition a spin orthogonal to both B and the velocity.

We evaluate then the size of the corrections relative to the above leading behavior. We model the electric field from a quadrupole potential in cylindrical coordinates (r , φ and z are respectively the radial, azimuthal and vertical coordinate):

$$V(r, z) = -\frac{1}{2} \kappa ((r - R_0)^2 - z^2) . \quad (6.5)$$

This leads to the following electric field:

$$\mathbf{E} = -\kappa z \hat{\mathbf{z}} + \kappa (r - R_0) \hat{\mathbf{r}} . \quad (6.6)$$

A muon can be displaced from the horizontal plane or it can be displaced radially from the equilibrium orbit, leading respectively to vertical or radial oscillations. We analyze them separately.

6.1.1 Vertical oscillations

A muon displaced from the horizontal plane will perform small oscillations due to the electric field in the $\hat{z}$ direction (see appendix III in ref. [14]),

$$\begin{aligned} z(T) &= A_z \cos(\omega_z T + \varphi_{0,z}) , \\ v_z(T) &= -A_z \omega_z \sin(\omega_z T + \varphi_{0,z}) , \end{aligned} \quad (6.7)$$

where

$$\omega_z = \omega_c \sqrt{n} , \quad \text{with} \quad n \equiv \frac{\kappa R_0}{|v| |B|} . \quad (6.8)$$

We have first checked such behavior by numerically solving the differential equations for velocity and spin using the following values, appropriate for the experiment [1]:

$$A_z = 22.5 \text{ mm} , \quad (6.9)$$

$$\kappa = 2 \cdot 10^7 \frac{\text{V}}{\text{m}^2} , \quad (6.10)$$

$$B = 1.5 \text{ T} , \quad (6.11)$$

$$a = 0.00116585 , \quad (6.12)$$

which implies $R_0 \approx 7.112 \text{ m}$ and $\omega_a = 2\pi \times 0.23 \text{ MHz}$. We have set also, as an initial condition, the magic momentum $\gamma_0 = \gamma_M \equiv \sqrt{\frac{1+a}{a}} \approx 29.3$ and we have considered an initial displacement along the $\hat{z}$ axis equal to A_z . The value for A_z has been chosen using half of the radius of the storage volume, $r_{\text{max}} = 45 \text{ mm}$. Note also that in the experiment [14] the effective value for κ is actually reduced by a factor of about 0.43, due to the partial

Figure 2. We have solved numerically the equations of motion and then evaluated $\hat{s} \cdot \hat{v} = (\Sigma \cdot v)/(\gamma|v||S|)$ on the solution. The left plot shows that a better fit is achieved by adding the subleading correction $\delta\omega_{SL}$ of eq. (6.14) to the traditional “pitch correction”, contained in $\tilde{\omega}_a$. Here $T_a \equiv 2\pi/\omega_a$. The residual is well fit (right plot) by the functional form $\epsilon \sin(2\omega_z T) \cos((\tilde{\omega}_a + \delta\omega_{SL}))$, with $\epsilon = 2.436 \times 10^{-8}$.

coverage of the ring with quadrupole plates. We will use in our estimates the value quoted in the above eq. (6.12), but it is trivial to rescale κ by this factor in all our results.

Our numerical results reproduce the above vertical oscillations, except for a small frequency shift $\omega_z \rightarrow \omega_z + \delta\omega_z$, where $\delta\omega_z \approx 2.9072 \times 10^{-4} \omega_z$.

We study now the impact of such oscillations on eq. (6.1), namely on the time dependence of the precession, described by $\hat{s} \cdot \hat{v}$, and of the energy γ .

The vertical oscillations are known to modify the precession frequency by the so-called “pitch correction”. This is usually evaluated by expanding at lowest order eq. (4.42) and then taking a time average $\langle \dots \rangle$ of its modulus over the fast vertical oscillations. This would lead (modulo a phase) to

$$\hat{s} \cdot \hat{v} = \sin(\tilde{\omega}_a T), \quad \text{with} \quad \tilde{\omega}_a \equiv \langle |\tilde{\omega}_a| \rangle \approx \omega_a - \left\langle \frac{(\mathbf{v} \cdot \hat{\mathbf{B}})^2}{2} \right\rangle = \omega_a \left(1 - \frac{(A_z \omega_z)^2}{4} \right) \equiv \omega_a + \delta\omega_P. \quad (6.13)$$

With the above values $\delta\omega_P \approx -8 \cdot 10^{-7} \omega_a$, which is taken into account by the experimental collaborations [1, 2]. However we point out that the true frequency should be given instead by eqs. (4.40) and (4.41), which depends also on the time evolution of the spin $s^z(T)$.

We have first checked the true value of the “pitch correction” numerically. We find that $\delta\omega_P$ gives a reasonably good approximation, but we also find a subleading correction $\delta\omega_{SL}$, so that the true frequency is

$$\omega_T = \tilde{\omega}_a + \delta\omega_{SL}, \quad \delta\omega_{SL} \approx 1.75 \cdot 10^{-3} \delta\omega_P. \quad (6.14)$$

This is shown in figure 2.

It turns out that the quantity ω defined in our eqs. (4.40) and (4.41) capture such additional terms since we find that $\langle \omega \rangle = \tilde{\omega}_a + \delta\omega_{SL}$, with very good precision. This can

Figure 3. We have solved numerically the equations of motion and then evaluated

$$\omega \equiv \omega_a \left[1 - \frac{(\hat{B} \cdot \Sigma)(\hat{\Sigma} \cdot \hat{B})}{(\hat{\Sigma} \cdot v)} \right]^{1/2},$$

from eq. (4.40) and $|\tilde{\omega}_a|$ (which contains the traditional “pitch” correction) from eq. (4.42), on the solution. We also show the very good agreement with our analytical result of eq. (6.18). Taking a time average over the fast oscillations we get precisely the subleading correction $\delta\omega_{\text{SL}}$ of eq. (6.14), which also coincides with eq. (6.19) (green line). Here $T_a \equiv 2\pi/\omega_a$ and we used $\Sigma_0^z = 0$.

be checked numerically, evaluating eqs. (4.40) and (4.41) on the full solutions, as can be seen in figure 3. We can also evaluate it analytically, solving the differential equations for Σ^z . The full equation is

$$\frac{d\Sigma^z}{dT} = \frac{e}{m} \left[\frac{1+a}{\gamma} E^z (\Sigma \cdot v) - a\gamma v^z (B \cdot (\Sigma \times v) + (\Sigma \cdot E) - (v \cdot \Sigma)(E \cdot v)) \right], \quad (6.15)$$

but we have checked numerically that at leading order we may: *i*) neglect the last two terms, *ii*) consider $\gamma = \gamma_0$, *iii*) use the zeroth order solution for $\Sigma \cdot v = \gamma_0 v_0 |S| \sin(\omega_a T)$ in the r.h.s., *iv*) use eqs. (4.15) and (4.25) for $B \cdot (\Sigma \times v)$. This leads to a simpler equation

$$\frac{d\Sigma^z}{dT} = v_0 |S| \left[\omega_a \gamma_0 v^z \cos(\omega_a t) + (1+a) \frac{e}{m} \kappa z(t) \sin(\omega_a t) \right]. \quad (6.16)$$

One can now use now eqs. (6.7) and integrate in time, finding

$$\begin{aligned} \Sigma^z(T) = & -A_z \frac{\omega_z \sin(T\omega_a) \sin(T\omega_z) [(a+1)e\kappa + \gamma_0 m \omega_a^2] + \omega_a \cos(T\omega_a) \cos(T\omega_z) [(a+1)e\kappa + \gamma_0 m \omega_z^2]}{m(\omega_a^2 - \omega_z^2)} \\ & + A_z \frac{\omega_a [(a+1)e\kappa + \gamma_0 m \omega_z^2]}{m(\omega_a^2 - \omega_z^2)} + \Sigma_0^z, \end{aligned} \quad (6.17)$$

where we used the initial condition $\Sigma^z(t=0) = \Sigma_0^z$. As a final step we plug this into

eq. (4.40) and compare with eq. (4.42), leading, at linear order, to

$$\begin{aligned} \frac{\omega - \tilde{\omega}_a}{\omega_a} &= \frac{A_z^2}{\gamma_0 \omega_z} \sin(\omega_z T) \left[\frac{e}{m} \kappa(1+a) + \gamma_0 \omega_z^2 \right] \\ &\times [\omega_a \cos(\omega_z T) \cot(\omega_a T) + \tilde{\omega}_z \sin(\omega_z T) - \omega_a \csc(\omega_a T)] - \Sigma_0^z \frac{A_z \omega_z \csc(\omega_a t) \sin(\omega_z t)}{\gamma_0}, \end{aligned} \quad (6.18)$$

we have set for simplicity $\varphi_{0,z} = 0$ and we have also used the approximations $\omega_a \ll \omega_z$ and $v_0 \approx 1$. As can be seen from figure 3 this formula reproduces in a very accurate way the numerical behavior of $\omega(T)$, obtained evaluating eq. (4.40) on the full numerical solution. Performing now an average over the fast vertical oscillations only the term quadratic in $\sin(\omega_z T)$ survives, leading to:

$$\frac{\langle \omega \rangle - \tilde{\omega}_a}{\omega_a} = \frac{A_z^2 \left[\frac{e}{m} \kappa(1+a) + \gamma_0 \omega_z^2 \right]}{4\gamma_0}. \quad (6.19)$$

This formula reproduces with excellent approximation the subleading correction $\delta\omega_{\text{SL}}$.

We may then look at the behavior of the γ factor. Using eq. (3.11) and, for simplicity $\varphi_{0,z} = 0$, we get

$$\dot{\gamma} = \frac{e}{m} E_z v_z = \frac{e}{2m} \kappa A_z^2 \omega_z \sin(2\omega_z T) \implies \gamma = \gamma_0 + \frac{e}{2m} \kappa A_z^2 \sin^2(\omega_z T), \quad (6.20)$$

where we have used the initial condition $\gamma(T=0) = \gamma_0$.

This leads to the following correction for the $\mathcal{E}_0$ term of the electron energy in eq. (6.1),

$$\frac{\Delta \mathcal{E}_0}{\mathcal{E}_0} \equiv \frac{\gamma - \gamma_0}{\gamma_0} = \frac{e}{2m\gamma_0} \kappa A_z^2 \sin^2(\omega_z T) \approx 1.6 \cdot 10^{-6} \sin^2(\omega_z T). \quad (6.21)$$

The $\mathcal{E}_R$ term is corrected in the following way

$$\begin{aligned} \frac{\Delta \mathcal{E}_R}{\mathcal{E}_1 |S| \sqrt{\gamma_0^2 - 1}} &= \frac{\sqrt{\gamma^2 - 1}(\hat{s} \cdot \hat{v}) - \sqrt{\gamma_0^2 - 1}(\hat{s} \cdot \hat{v})}{\sqrt{\gamma_0^2 - 1}} \approx \frac{e\gamma_0}{2m(\gamma_0^2 - 1)} \kappa A_z^2 \sin^2(\omega_z T) \sin(\omega_T T) \\ &\approx 1.6 \cdot 10^{-6} \sin^2(\omega_z T) \sin(\omega_T T). \end{aligned} \quad (6.22)$$

The comparison with numerical results is shown in figure 4.

6.1.2 Radial oscillations

We consider here the radial motion assuming that each muon has an equilibrium radius $R_0 + x_e$, where x_e depends on the muon average gamma factor, γ_0 ; note that here in general γ_0 is not equal to γ_M . If a muon is displaced from this orbit it will perform small radial oscillations, due to the combined effect of the centrifugal term and the radial electric field:¹³

$$r(T) = R_0 + x_e + A_r \cos(\omega_r T + \varphi_{0,r}), \quad \text{where} \quad \omega_r \equiv \omega_c \sqrt{1 - n}, \quad (6.23)$$

¹³See eq. (15) and (16) in ref. [1].

Figure 4. We have solved numerically the equations of motion, with a vertical displacement, and then evaluated the energy $\mathcal{E}$ on the solution and compared to a simple sinusoidal behavior $E_0 + E_1 \sin((\tilde{\omega}_a + \delta\omega_{\text{SL}})T)$, where E_0 and E_1 are constants. The upper left plot shows the deviation from such a simple behavior and the analytical form of the residual of eqs. (6.21), (6.22). The upper right plot shows the residual for $\mathcal{E}_0$, eq. (6.28). The bottom plot shows an analytical fit of the residual, for $\mathcal{E}_R$, given by eq. (6.29), plus higher order corrections that we have found numerically. Here $T_a \equiv 2\pi/\omega_a$.

so that:

$$v_r = -\omega_r A_r \sin(\omega_r T + \varphi_{0,r}), \quad (6.24)$$

$$E_r = \kappa[A_r \cos(\omega_r T + \varphi_{0,r}) + x_e]. \quad (6.25)$$

We will use the same numerical values given above in eq. (6.12), with A_r of the same size of A_z .

The gamma factor is determined by

$$\dot{\gamma} = \frac{e}{m} E_r v_r = -\frac{e}{2m} \kappa A_r \omega_r [A_r \sin(2\omega_r T) + 2x_e \sin(\omega_r T)], \quad (6.26)$$

where $\varphi_{0,r}$ has been taken equal to zero for simplicity. This implies

$$\gamma = \gamma_0 - \frac{e}{2m} \kappa A_r \left[A_r \sin^2(\omega_r T) + 4x_e \sin^2\left(\frac{\omega_r T}{2}\right) \right]. \quad (6.27)$$

This leads to the following corrections:

$$\begin{aligned}\frac{\Delta\mathcal{E}_0}{\mathcal{E}_0} &= -\frac{e}{2m\gamma_0}\kappa A_r \left[A_r \sin^2(\omega_r T) + 4x_e \sin^2\left(\frac{\omega_r T}{2}\right) \right] \\ &\approx -1.6 \cdot 10^{-6} \left[\sin^2(\omega_r T) + \left(\frac{x_e}{5.625\text{mm}}\right) \sin^2\left(\frac{\omega_r T}{2}\right) \right],\end{aligned}\quad (6.28)$$

and

$$\begin{aligned}\frac{\Delta\mathcal{E}_R}{\mathcal{E}_1|S|\sqrt{\gamma_0^2-1}} &= \frac{\sqrt{\gamma^2-1}(\hat{s} \cdot \hat{v}) - \sqrt{\gamma_0^2-1}(\hat{s} \cdot \hat{v})}{\sqrt{\gamma_0^2-1}} \\ &\approx -\frac{e\gamma_0}{2m(\gamma_0^2-1)}\kappa A_r \left[A_r \sin^2(\omega_r T) + 4x_e \sin^2\left(\frac{\omega_r T}{2}\right) \right] \sin(\omega_T T) \\ &\approx -1.6 \cdot 10^{-6} \left[\sin^2(\omega_r T) + \left(\frac{x_e}{5.625\text{mm}}\right) \sin^2\left(\frac{\omega_r T}{2}\right) \right] \sin(\omega_T T).\end{aligned}\quad (6.29)$$

where x_e is typically comparable to the half radius of the storage volume.

Also in this case our numerical results can be seen in figure 5, for the case $x_e = 0$. We confirm the existence of such effects, modulo a small frequency shift $\omega_r \rightarrow \omega_r + \delta\omega_r$, where $\delta\omega_r \approx -8 \cdot 10^{-5}\omega_r$.

Summing all contributions, in the particular case $\varphi_{0,r} = \varphi_{0,z} = 0$, we get

$$\begin{aligned}\frac{\Delta\mathcal{E}_0}{\mathcal{E}_0} &= \frac{e\kappa}{2m\gamma_0} \left[A_z^2 \sin((\omega_z + \omega_r)T) \sin((\omega_z - \omega_r)T) + (A_z^2 - A_r^2) \sin^2(\omega_r T) + \right. \\ &\quad \left. -4x_e A_r \sin^2\left(\frac{\omega_r T}{2}\right) \right],\end{aligned}\quad (6.30)$$

and

$$\begin{aligned}\frac{\Delta\mathcal{E}_R}{\mathcal{E}_1|S|\sqrt{\gamma_0^2-1}} &= \frac{e\gamma_0}{2m(1+\gamma_0^2)} \left[A_z^2 \sin((\omega_z + \omega_r)T) \sin((\omega_z - \omega_r)T) + (A_z^2 - A_r^2) \sin^2(\omega_r T) + \right. \\ &\quad \left. -4x_e A_r \sin^2\left(\frac{\omega_r T}{2}\right) \right] \sin(\omega_a T).\end{aligned}\quad (6.31)$$

Such corrections due to a non-constant γ should be taken into account in a fit and could significantly affect the χ^2 determination. Nonetheless, since they oscillate rapidly, one expects them not to contaminate much the measurement of a . However one should also analyze if, in a given the experimental setup, other typical frequencies might play a role in eq. (6.1). If they are close to ω_a , as in the case of the so-called Coherent Betatron Oscillations (CBO), one should check if this could introduce possible systematics, which should be evaluated properly in an experiment. The CBO oscillations are low-frequency oscillations of the beam, in which the width and centroid of the beam oscillate at frequency

$$\omega_{\text{CBO}} \approx 2\pi \times 0.476 \text{ MHz}.$$

Figure 5. We have solved numerically the equations of motion, with a radial displacement, and then evaluated $\hat{s} \cdot \hat{v} = (\Sigma \cdot v)/(\gamma |v| |S|)$ and the electron energy $\mathcal{E}$ on the solution. We have used here $x_e = 0$. The upper left plot shows $\hat{s} \cdot \hat{v}$, compared to a simple sinusoidal behavior: the best fit shows a tiny offset $\delta\omega_M \approx 10^{-11}\omega_a$. The upper right plot shows the energy $\mathcal{E}$, compared to a simple sinusoidal behavior $E_0 + E_1 \sin(\omega_a T)$, where E_0 and E_1 are constants, and its residual. The bottom plots show the residuals, for $\mathcal{E}_0$ and $\mathcal{E}_R$. A very small higher order residual is present for $\mathcal{E}_R$, which grows in time, signaling a tiny correction $\delta\omega$ to the frequency, since $\cos((\omega_a + \delta\omega)T) \approx \cos(\omega_a T) - T\delta\omega \sin(\omega_a T)$. Here $T_a \equiv 2\pi/\omega_a$.

If we assume an amplitude for such oscillations similar to the previous amplitudes A_z or A_r , and a sinusoidal behavior as in the previous case, the oscillations in $\mathcal{E}$, due to $\dot{\gamma} \neq 0$, might be potentially relevant. Such a detailed analysis goes beyond the scope of the present paper and we leave this for future study.

Finally, as it is well known, the existence of a radial displacement x_e induces also a radial electric field correction to the precession frequency. This is usually taken into account by considering $\tilde{\omega}_a = \omega_v - \omega_s$, as in eq. (4.23), with the addition of electric fields.¹⁴ This leads to a term ω_{EL} of the form

$$\omega_{EL} = \frac{e}{m} \left[\left(a - \frac{1}{\gamma^2 - 1} \right) \mathbf{v} \times \mathbf{E} \right], \quad (6.32)$$

which vanishes at the magic momentum γ_R . In the BNL experiment [1] the distribution of momenta has a spread around γ_R : the r.m.s. of the energy distribution is of about 0.15%,

¹⁴See eqs. (11.168) and (11.170), in [21].

while the maximal possible deviations from the central value is of about 0.5%. This induces a correction, which is usually estimated, after averaging the radial oscillations over time, as [1]

$$\frac{\langle |\omega_{\text{EL}}| \rangle}{\omega_a} \approx -2n(1-n)v_0^2 \frac{x_e^2}{R_0^2}. \quad (6.33)$$

We confirm numerically the existence of such a correction in figure 6, but we find also a small subleading correction $\delta\omega_{\text{EL},2}$ between $10^{-9}\omega_a$ and $\text{few} \times 10^{-8}\omega_a$, depending on the value of γ_0 , taken within 0.15%–0.5% around the magic momentum. Moreover the subleading correction turns out to be asymmetric whether γ_0 is larger or smaller than the magic momentum γ_M . Such a numerical subleading correction should be then averaged over the distribution of momenta in a realistic experiment and could lead to a non-trivial overall correction, presumably leading to a systematic error of at most $\Delta\omega/\omega_a \approx 10^{-8}$ – 10^{-9} .

The existence of such subleading corrections could be expected, as we have learned from the case of vertical oscillations that the approach of considering the second derivative of the energy $\mathcal{E}$ and taking a time average better reproduces the behavior of the system. In presence of electric fields this is given by eq. (4.48), which could be averaged over time. At the central radius of the experiment R_0 the electric field is vanishing and so all corrections (except for the “pitch” correction, if there are vertical oscillations) vanish. As soon as the muons are displaced from R_0 there will be several corrections, which, as we have already noted, do not vanish at $\gamma = \gamma_M$.

7 Conclusions

We have derived a full set of equations for the energy $\mathcal{E}$ of the decay electrons, in a muon $g-2$ experiment. We have considered the equations for velocity and spin of the muon in an electromagnetic field, in a generic metric and seen by a generic observer. The energy $\mathcal{E}$ is modulated mainly by the scalar product of the observed spin Σ and the observed velocity v . As it is well known, in a cyclotron in the horizontal plane and in flat spacetime, with a constant strong magnetic field of amplitude $|B|$ along the vertical axis, we have that $\mathcal{E}$ oscillates in time, at the standard anomalous frequency $\omega_a = ea|B|/m$. We have studied then several corrections to this leading behavior.

First, we have analyzed the frequency shift $\delta\omega_P$, due to the so-called “pitch correction”, that arises when the muon trajectory is not purely orthogonal to the magnetic field. In a realistic cyclotron, indeed, muons perform small oscillations of amplitude A_z and angular frequency ω_z , along the vertical direction of the cyclotron $\hat{z}$. The commonly used estimate of the pitch correction is determined by the time average of the velocity $\langle (v^z)^2 \rangle / 2$, leading to $\delta\omega_P/\omega_a = A_z^2\omega_z^2/4 \approx 10^{-6}$. We find a good numerical agreement with such an estimate, but we find a subleading correction $\delta\omega_{\text{SL}}/\omega_a \approx \mathcal{O}(10^{-9})$. The Brookhaven E821 experiment treatment of such correction is thus appropriate for its experimental uncertainty, but the subleading effect that we found might be relevant in future experiments. We demonstrate the existence of such a subleading correction analytically, by defining the frequency through the second derivative of the observed energy $\mathcal{E}$ and showing that it depends on the time

Figure 6. We have solved numerically the equations of motion on equilibrium orbits with $\gamma \neq \gamma_M$ and so with $x_e \neq 0$. We have then evaluated $\hat{s} \cdot \hat{v} = (\Sigma \cdot v)/(\gamma|v||S|)$ on the solution and compared with a sinusoidal behavior. The plot shows that a better fit is achieved by adding the subleading correction $\delta\omega_{EL,2}$ to the radial electric field correction of eq. (6.33). The correction turns out to vary with γ_0 between $10^{-9}\omega_a$ and $\text{few} \times 10^{-8}\omega_a$ and in a non-symmetric way for values of γ_0 around the magic momentum γ_M .

evolution of the spin Σ^z . After averaging over time, we find good agreement between the numerical and analytical results.

Then, we have analyzed another well-known correction due to radial electric fields: such a correction is also commonly estimated with the term $\frac{e}{m} \left[\left(a - \frac{1}{\gamma^2 - 1} \right) \mathbf{v} \times \mathbf{E} \right]$, which vanishes at the so-called magic momentum. Also in this case we find that this commonly used treatment is enough for the Brookhaven E821 experimental uncertainty, but we find numerically subleading corrections $\delta\omega_{SL}/\omega_a \approx \text{few} \times 10^{-8} \sim 10^{-9}$, which should be related to the existence of many correction terms in the second derivative of $\mathcal{E}$, which do not all vanish at the magic momentum.

Then we have analyzed rapidly oscillating terms in $\mathcal{E}$, due to the time dependence of the muon γ factor, in the presence of electric fields. In a cyclotron such corrections arise due to oscillations of the muons along both the radial and vertical direction and are in principle non-negligible, being of relative size $\Delta\mathcal{E}/\mathcal{E} \approx 10^{-6}$ in a realistic experiment. It remains to be checked whether such terms, which oscillate at high frequency, might have an impact in a fit of a realistic experiment, such as the Brookhaven E821 experiment. In

particular one should extend the analysis including the so-called lower frequency Coherent Betatron Oscillations, which are potentially dangerous.

Finally, we have analyzed the extra terms introduced by the presence of a non-trivial metric, finding that they depend on the gravitational “force” f_G . We have considered two specific cases: a Schwarzschild metric and a rotating metric, representing the Earth’s gravity and rotation respectively.

For the Schwarzschild case the effects are tiny: the leading term in f_G is due to the effect of the standard surface gravity, $g_T = 9.8 \text{ m/s}^2$, on the spin vector, leading to a relative correction of order $g_T/(\gamma\omega_a) \approx 8 \cdot 10^{-16}$, at the so-called magic momentum.

The leading effects due to a non-trivial metric are instead due to rotation. We worked at linear order in the angular frequency ω_T , finding that f_G contains a Coriolis force, which amounts to an additional frequency vector $\mathbf{\Omega}_T/\gamma$, with $|\mathbf{\Omega}_T| = \omega_T$, aligned with the Earth’s rotation. This corresponds to an angular frequency of $\Delta\omega \approx 2\pi \times 4 \times 10^{-7} \text{ Hz}$, which represents a relative correction of about $\Delta\omega/\omega_a \approx 1.2 \times 10^{-12}$. Another correction is also due to the Coriolis effect on the spin frequency $\tilde{\omega}_p$ of the stationary protons that are used for calibration in [14], which represents a relative correction of order $\omega_T/\tilde{\omega}_p \approx 1.8 \times 10^{-13}$. Such effects are very small for muon $g - 2$ experiments, but they could be more important for *electron* $g - 2$ experiments. In the latter case the Coriolis frequency vector is just $\mathbf{\Omega}_T$, to be compared with the Larmor frequency of about $\omega_L = 100 \text{ GHz}$, for an electron in a Penning trap experiment [23, 24]. This gives a relative effect $\Delta\omega/\omega_L \approx 7 \times 10^{-16}$ to be compared with the experimental relative error on g , at present of order $\mathcal{O}(10^{-13})$ and forecasted to reach $\mathcal{O}(10^{-14})$ [25] within one or two years.

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A Precession rate of the spin S in the muon rest frame

In this appendix we want to examine the precession rate of the spin S w.r.t. the adapted frame $\{e_{(M)}^{\hat{a}}\}$ orthogonal to $U = e_0^{(M)}$. Following [18] (see also [20]), this is equivalent to the orientation w.r.t. the observer u , described with the adapted frame $\{e^{\hat{\mu}}\}$, of a boosted spin vector $\mathcal{S}$, which is obtained with the following relation

$$\mathcal{S} = \Sigma - \frac{\gamma}{1 + \gamma}(v \cdot \Sigma)v. \quad (\text{A.1})$$

We note immediately that eq. (A.1) is a generalisation (in GR) of the inverse function of eq. (4.13), e.g. see [21]. The boosted spin vector $\mathcal{S}$ has the following properties

$$u \cdot \mathcal{S} = 0, \quad P(u)\mathcal{S} = \mathcal{S}, \quad v \cdot \Sigma = \gamma(v \cdot \mathcal{S}). \quad (\text{A.2})$$

Using

$$\Sigma = S - (v \cdot \Sigma)u = S - \gamma(v \cdot \mathcal{S})u, \quad (\text{A.3})$$

and eq. (A.1), we can write the map between $\mathcal{S}$ and S , i.e.

$$S = \mathcal{S} + \frac{\gamma}{1+\gamma}(v \cdot \mathcal{S})(U + u). \quad (\text{A.4})$$

Differentiating both sides of the above relation along U and taking into account eq. (3.13) we are able to write the following equation

$$\begin{aligned} \frac{D\mathcal{S}}{d\tau} &= \frac{e(1+a)}{m} F * [\mathcal{S} + \mathcal{C}_S(U + u)] - a\gamma [(f_{\text{EM}} \cdot \mathcal{S}) - \mathcal{C}_S(v \cdot f_{\text{EM}})] U - (U + u) \frac{d\mathcal{C}_S}{d\tau} \\ &\quad + \left(\frac{DU}{d\tau} - f_G \right) \mathcal{C}_S, \end{aligned} \quad (\text{A.5})$$

where we have defined

$$\mathcal{C}_S \equiv \frac{\gamma}{1+\gamma}(v \cdot \mathcal{S}),$$

and used the following relations

$$u \cdot \frac{DU}{d\tau} = -\gamma(v \cdot f_{\text{EM}}) \quad \text{and} \quad P(u) \frac{DU}{d\tau} = \gamma f_{\text{EM}}.$$

Now, projecting eq. (A.5) with $P(u)$, we have

$$\begin{aligned} P(u) \frac{D\mathcal{S}}{d\tau} &= \frac{D_{(\text{fw}, U, u)} \mathcal{S}}{d\tau} = \frac{e(1+a)}{m} [P(u)F * \mathcal{S}] + \frac{e(1+a)}{m} \mathcal{C}_S [P(u)F * (U + u)] \\ &\quad - a\gamma^2 [(f_{\text{EM}} \cdot \mathcal{S}) - \mathcal{C}_S(v \cdot f_{\text{EM}})] v - \gamma \frac{d\mathcal{C}_S}{d\tau} v + \mathcal{C}_S(-\gamma f_{\text{EM}} + f_G); \end{aligned} \quad (\text{A.6})$$

instead, projecting eq. (A.5) along u , we find

$$\begin{aligned} u \cdot \frac{D\mathcal{S}}{d\tau} &= -\frac{Du}{d\tau} \cdot \mathcal{S} = f_G \cdot \mathcal{S} = \frac{e(1+a)}{m} [u * F * \mathcal{S}] + \frac{e(1+a)}{m} \mathcal{C}_S (u * F * U) \\ &\quad + a\gamma^2 [(f_{\text{EM}} \cdot \mathcal{S}) - \mathcal{C}_S(v \cdot f_{\text{EM}})] + (1+\gamma) \frac{d\mathcal{C}_S}{d\tau} + \gamma \mathcal{C}_S(v \cdot f_{\text{EM}}). \end{aligned} \quad (\text{A.7})$$

In this way, using eq. (A.7) we can remove $d\mathcal{C}_S/d\tau$ within (A.6) and we obtain the final result

$$\begin{aligned} \frac{D_{(\text{fw}, U, u)} \mathcal{S}}{d\tau} &= \frac{e(1+a)}{m} [P(u)F * \mathcal{S}] + \frac{e(1+a)}{m} \mathcal{C}_S [P(u)F * (U + u)] \\ &\quad - a\gamma^2 [(f_{\text{EM}} \cdot \mathcal{S}) - \mathcal{C}_S(v \cdot f_{\text{EM}})] v + \mathcal{C}_S(-\gamma f_{\text{EM}} + f_G) \\ &\quad - \frac{\gamma}{\gamma+1} \left\{ -\frac{e(1+a)}{m} [u * F * \mathcal{S}] - \frac{e(1+a)}{m} \mathcal{C}_S (u * F * U) \right. \\ &\quad \left. + [-a\gamma^2 f_{\text{EM}} + f_G] \cdot \mathcal{S} - \gamma \mathcal{C}_S(v \cdot f_{\text{EM}}) [-a\gamma + 1] \right\}, \end{aligned} \quad (\text{A.8})$$

or, equivalently,

$$\left(\frac{d\mathcal{S}}{dT}\right)^{\hat{a}} = [\Omega_{\mathcal{S}} \times_u \mathcal{S}]^{\hat{a}}, \quad (\text{A.9})$$

where

$$\Omega_{\mathcal{S}} = \frac{1}{\gamma + 1} v \times_u (f_G - \gamma f_{\text{EM}}) + \frac{e(1 + a)}{m} \left\{ v \times_u \left[E + \frac{1}{\gamma + 1} (v \times_u B) \right] - \frac{B}{\gamma} \right\} - (\zeta_{(\text{fw})} + \zeta_{(\text{sc})}). \quad (\text{A.10})$$

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