Logarithmic W-algebras and Argyres-Douglas theories at higher rank

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ABSTRACT: Families of vertex algebras associated to nilpotent elements of simply-laced Lie algebras are constructed. These algebras are close cousins of logarithmic W-algebras of Feigin and Tipunin and they are also obtained as modifications of semiclassical limits of vertex algebras appearing in the context of S -duality for four-dimensional gauge theories. In the case of type A and principal nilpotent element the character agrees precisely with the Schur-Index formula for corresponding Argyres-Douglas theories with irregular singularities. For other nilpotent elements they are identified with Schur-indices of type IV Argyres-Douglas theories. Further, there is a conformal embedding pattern of these vertex operator algebras that nicely matches the RG-flow of Argyres-Douglas theories as discussed by Buican and Nishinaka.

KEYWORDS: Conformal and W Symmetry, Conformal Field Theory, Supersymmetric Gauge Theory

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1 Introduction

The main point of this work is to construct vertex operator algebras whose characters coincide with Schur indices of corresponding Argyres-Douglas theories as computed by Buican and Nishinaka [1]. Moreover the construction makes the predicted automorphism group $\mathbb{Z}/2\mathbb{Z} \times S_N$ manifest and also gives a nice explanation of RG-flow via conformal embeddings. At the same time these vertex operator algebras are also interesting from the point of view of logarithmic conformal field theories.

The slogan initiated in [2] is that interesting quantities of higher dimensional gauge theories are sometimes described by vertex operator algebras. There are various manifestations of this picture and the one that I am familiar with is S -duality for four-dimensional $\mathcal{N} = 4$ supersymmetric GL-twisted gauge theories. There the two-dimensional intersection of three-dimensional topological boundary conditions is described by certain families of vertex operator algebras, parameterized by the gauge coupling. Moreover categories of line defects ending on the boundary conditions are categories of modules of the corner vertex operator algebra, see [3, 4]. There is no apparent relation known to me between these supersymmetric gauge theories and Argyres-Douglas theories.¹ Nonetheless, one construction of this work is a modification of a semi-classical limit of vertex operator algebras arising naturally in this S -duality picture.

Argyres-Douglas theories are four-dimensional $\mathcal{N} = 2$ supersymmetric field theories [6–8] and they are very interesting from the vertex operator algebra perspective as their Schur indices are believed to coincide with characters of certain vertex operator algebras [9–15]. Moreover indices of line and surface defects are related to characters of modules of the vertex operator algebra [11, 12]. Interestingly also modularity and Verlinde-like formulae have nice interpretations in the gauge theory [16–18]. For some further recent progress see [19–26].

¹Enhancement of supersymmetry can be explained via RG-flow as in [5].

Personally, I find it very interesting that the vertex operator algebras appearing in this gauge theory context are often of logarithmic type, that is they allow for indecomposable but reducible representations. There are various difficult questions in the context of logarithmic vertex operator algebras, see [27] for an introduction, and it is good that they allow for gauge theory interpretations. So far the best studied logarithmic vertex operator algebras are the triplet algebras [28–31], the fractional level WZW theories of $\mathfrak{sl}_2$ [32–34], the logarithmic $\mathcal{B}(p)$ -algebras [35] and some progress is currently made on higher rank cases [36]. All these examples have appearances in higher dimensional super conformal field theories. Especially, I would like to point out the work on the representation theory of the logarithmic $\mathcal{B}(p)$ -algebras [37]. These correspond to Argyres-Douglas theories of type (A_1, A_{2p-3}) and it might serve as a nice example for further investigation of the relation between representation theoretic data of the vertex algebra and properties of the gauge theory. Note that these representation theories usually have uncountable many inequivalent simple objects and in order to go to a conjectural modular tensor category one has to look at the semisimplification, that is the category where negligible morphisms are identified with the zero morphisms. For details and conjectures in the vertex operator algebra context on this please see work with Terry Gannon [38]. Let me now state a few examples of the connection between representation theory data and super conformal field theory data.

1. The paper [16] observes for some examples of type A_1 Argyres-Douglas theories that a limit of wild Hitchin characters can be expressed in terms of the modular data of the semisimplification of the module category of the corresponding vertex operator algebra.
2. In [17] the operator algebra of $\frac{1}{2}$ -BPS line defects was studied and a main claim is that the homomorphism from this operator algebra to the algebra of functions on $U(1)_r$ -invariant vacua of the theory compactified on S^1 factors through the Verlinde algebra of the semisimplification of the category of modules of the chiral algebra.
3. Prototypical examples of negligible modules, i.e. modules on which the identity is a negligible morphism, are the relaxed-highest weight modules of affine vertex algebras, see e.g. [36]. If we restrict to $\mathfrak{sl}_2$ at admissible level or to the $\beta\gamma$ vertex operator algebra, then characters of these modules are formal distributions times certain modular functions [32–34]. It turns out that indices of surface defects can be identified with characters of relaxed-highest weight modules in certain examples [11].²

Results. Buican and Nishinaka give a nice compact formula for Schur-indices of $(A_{N-1}, A_{N(n-1)-1})$ Argyres-Douglas theories [1]. The first statement is a rewriting of this formula that makes comparison with vertex operator algebra characters evident. The resulting expression is the outcome of section 2 and it reads

$$\mathcal{I}(N, n) = \frac{q^{\frac{\ell}{12} + \rho^2}}{\eta(q)^{2\ell}} \sum_{\lambda \in P^+} \sum_{w \in W} \epsilon(w) q^{-(\rho, w(\lambda + \rho))} q^{\frac{n}{2}(\lambda, \lambda + 2\rho)} \sum_{\mu \in Q + \lambda} m_\lambda(\mu) q^{-\frac{n}{2}\mu^2} x^\mu. \quad (1.1)$$

²The authors of [11] denote relaxed-highest weight modules bilateral modules.

Here $\mathfrak{g} = \mathfrak{su}(N)$ and ℓ is the rank of $\mathfrak{g}$, P^+ its set of dominant weights, W the Weyl group of $\mathfrak{g}$, $\epsilon(w)$ the parity of the Weyl reflection w , ρ is the Weyl vector, Q the root lattice and $m_\lambda(\mu)$ is the dimension of the weight space of weight μ in the irreducible highest-weight representation ρ_λ of highest-weight λ of $\mathfrak{g}$.

In section 3 a family of vertex operator algebras that are denoted $B(n)_Q$ -algebras are constructed. Here Q is the root lattice of a simply-laced Lie algebra and $n > 1$ is a positive integer larger than one. The construction is a generalization of joint work with Ridout and Wood [35] and it is a modification of certain logarithmic W-algebras of Feigin and Tipunin [39]. The main result is then that for $Q = A_{N-1}$

$$\text{ch}[\mathcal{B}(n)_{A_{N-1}}](q, x) = q^{-\frac{c}{24}} \mathcal{I}(N, n)(q, x) \quad (1.2)$$

with the central charge of the $\mathcal{B}(n)_Q$ -algebra

$$c = 2\ell - \frac{\rho^2}{2n}(n-1)^2,$$

and we check that this precisely matches the proposed central charge (see [40]) of the expected chiral algebra for the Argyres Douglas theory.

In section 4 a more general but conjectural construction is given. The idea is to first take a semi-classical limit of vertex operator algebras conjectured by Davide Gaiotto and myself in [3]. The resulting vertex operator algebras are certain objects in a completion of a category of $G \otimes W(\mathfrak{g}, f)$ -modules with $W(\mathfrak{g}, f)$ the quantum Hamiltonian reduction associated to f and $\mathfrak{g}$ at a certain level and G the compact Lie group of the simply-laced Lie algebra $\mathfrak{g}$. The idea is now that one can replace $\text{Rep}(G)$ by a braided equivalent tensor category of modules for the Heisenberg vertex operator algebra of rank the rank of $\mathfrak{g}$. Essentially we forget the G action to view G -modules as T -graded vector spaces with T the maximal torus of G and T -graded vector spaces are equivalent to categories of Fock modules of the Heisenberg vertex operator algebra, see e.g. section 2 of [41]. We then restrict to $\mathfrak{g} = \mathfrak{sl}_N$. For f principal nilpotent we then recover the vertex operator algebra of our previous construction and for general f we get more new vertex operator algebras. Their characters can be identified with type IV Argyres-Douglas theories which are labelled by Young tableaux and as explained in section 4 of [42] super conformal indices change according to the Euler-Poincaré principle for the corresponding quantum Hamiltonian reduction.

One nice outcome of this construction is that all vertex operator algebras have as manifest symmetry the Weyl group W times the automorphism group of the corresponding W-algebra. For example in the case of $\mathfrak{g} = \mathfrak{sl}_N$ and f principal nilpotent this is $\mathbb{Z}/2\mathbb{Z} \times S_N$ as predicted by Buican and Nishinaka [1].

Another nice outcome is that our explicit construction allows to study conformal embeddings of resulting vertex operator algebras, see section 5. The idea is rather simple and is taken from [43]. The new vertex operator algebras are described as intersections of kernels of screening charge on a free field algebra. Removing one such screening, that is only considering the intersection of all screening except for one gives a bigger vertex

operator algebra. This results in the embedding

$$\mathcal{B}(n)_{Q_N} \subset \mathcal{B}(n)_{Q_\nu} \otimes \mathcal{B}(n)_{Q_{N-\nu}} \otimes W_{D_{\Phi'}^0(n)},$$

with the last factor some extension of a rank two Heisenberg vertex operator algebra by Fock modules in a rank one isotropic lattice. This matches nicely the discussion of RG-flow in section 4.2 of [1] and so here we see that RG-flow on the level of vertex operator algebras is described by conformal embeddings.

Questions. The present work gives a vertex operator algebra realization or interpretation of the findings of Buican and Nishinaka [1]. There are a few natural questions that are interesting from physics and mathematics perspective.

1. Is there a domain for the Jacobi variables so that the characters of the vertex operator algebras found here converge in this domain and can be meromorphically continued to meromorphic Jacobi forms? The answer is yes for the case of $\mathfrak{g} = \mathfrak{sl}_2$, see [13]. In this case characters also have a nice product form and so a second question would be if the more general case has a nice product form as well.
2. Setting the Jacobi variable to one, one expects modular forms as characters. This property holds for characters of ordinary modules of quasi-lisse vertex operator algebras [44]. Quasi-lisse means that the associated variety (the maximal spectrum of the C_2 -quotient) is symplectic with only finitely many symplectic leaves. Presumably the new vertex operator algebras of this work are quasi-lisse. Characters and thus Schur indices would then be solutions to certain modular differential equations, see [45, 46] for discussions of relations to Higgs branches.
3. The vertex operator algebras here have been constructed as modifications of semi-classical limits of vertex operator algebras appearing in the context of S-duality for four-dimensional supersymmetric gauge theories [3] and one might wonder if the construction has a physics explanation, i.e. is there a relation between [3] and Argyres-Douglas theories?
4. From the vertex operator algebra perspective generalizations to non simply-laced and even superalgebras is possible, though probably difficult. For example it should be possible to construct a vertex operator algebra associated to F_4 whose character coincides with [1, eq. (6.7)].
5. Level-rank type dualities for the vertex operator algebras appearing here seem to be rather rich and deserve further study. The expectation is that Heisenberg cosets of $\mathcal{B}(n)_{A_{N-1}}$ are isomorphic to certain W -algebras and the two A_1 examples corresponding to $\mathcal{B}(4)_{A_1}$ and $\mathcal{B}(5)_{A_1}$ of this are given in Theorem 10.5 and 6 point (2) of [47].
6. Schur indices of Argyres-Douglas theories of type (A_{n-1}, A_{m-1}) with $\gcd(n, m) \notin \{1, n, m\}$ are unknown. The construction of this paper can however (at least conjectural) be generalized. I plan to report on this generalization as well as its relation to level-rank duality in the future and hope that it will shed light on the understanding of chiral algebras of these unexplored Argyres-Douglas theories.

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2 Schur-indices of type $(A_{N-1}, A_{N(n-1)-1})$

In [1], Buican and Nishinaka found a nice compact formula for Schur-indices of $(A_{N-1}, A_{N(n-1)-1})$ Argyres-Douglas theories. I will rewrite them in a form that makes the comparison to vertex operator algebra characters obvious. They start with a proposal for simple wave functions for certain irregular punctures in $SU(N)$ q -deformed Yang-Mills theory,

$$\tilde{f}_R^{(n)}(q, x) = \prod_{k=1}^{\infty} (1 - q^k)^{1-N} q^{nC_2(R)} \text{tr}_R \left(q^{-\frac{n}{2} F^{i,j} h_i h_j} x \right).$$

Here $F_{i,j}$ denotes the Gram matrix of the weight lattice P of $\mathfrak{su}(N)$. $N - 1$ is the rank ℓ of it. R denotes an irreducible highest-weight representation of $\mathfrak{su}(N)$, so we prefer to denote it by the corresponding highest-weight λ . $C_2(R)$ is the eigenvalue of the quadratic Casimir, which is $(\lambda, \lambda + 2\rho)/2$ with ρ the Weyl vector, i.e. one half the sum of all positive roots. The element x is in the Cartan subalgebra. Let $m_\lambda(\mu)$ be the multiplicity of μ in the representation R of highest-weight λ , then

$$\text{tr}_R \left(q^{-\frac{n}{2} F^{i,j} h_i h_j} x \right) = \sum_{\mu \in Q + \lambda} m_\lambda(\mu) q^{-\frac{n}{2} \mu^2} x^\mu, \quad x^\mu := \mu(x).$$

The root lattice is denoted by Q . So that we have rewritten

$$\tilde{f}_R^{(n)}(q, x) = \frac{q^{\frac{\ell}{24}}}{\eta(q)^\ell} q^{\frac{n}{2}(\lambda, \lambda + 2\rho)} \sum_{\mu \in Q + \lambda} m_\lambda(\mu) q^{-\frac{n}{2} \mu^2} x^\mu$$

for R the highest-weight representation of highest-weight λ .

The second ingredient is the coefficient

$$C_R(q) = \frac{\prod_{k=1}^{N-1} (1 - q^k)^{N-k}}{(q; q)_\infty^{N-1}} \chi_R \left(q^{-\frac{N-1}{2}}, q^{-\frac{N-3}{2}}, \dots, q^{\frac{N-1}{2}} \right).$$

The last term needs explanation. The character is defined to be

$$\chi_\lambda(x) = \text{tr}_\lambda(x) = \sum_{\mu \in Q + \lambda} m_\lambda(\mu) x^\mu$$

and here [1] chose a standard realization of the simple roots of A_{N-1} . Namely consider $\epsilon_1, \dots, \epsilon_N$ with product $\epsilon_i \epsilon_j = \delta_{i,j}$ and then the simple positive roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$.

One identifies weights with the Cartan subalgebra via the pairing so that one can write $x = x_1^{\epsilon_1} \dots x_N^{\epsilon_N}$, with $x_i \in \mathbb{C}$. For x to be in the Cartan subalgebra one actually has to enforce that $x_1 \dots x_N = 1$ so that $\epsilon_1 + \dots + \epsilon_N$ acts trivially. With this notation one has

$$\chi_\lambda(x) = \chi_R(x_1, \dots, x_N).$$

Now, $x_j = q^{j - \frac{N+1}{2}} = e^{2\pi i \tau(j - \frac{N+1}{2})}$, so that

$$x = x_1^{\epsilon_1} \dots x_N^{\epsilon_N} = \prod_{j=1}^N e^{2\pi i \tau(j - \frac{N+1}{2})\epsilon_j} = e^{-2\pi i \tau \rho} = q^{-\rho} \quad (2.1)$$

with the Weyl vector

$$\rho = \sum_{\alpha \in \Delta_+} \alpha = - \sum_{j=1}^N \left(j - \frac{N+1}{2} \right) \epsilon_j.$$

Recall that Δ_+ denotes the set of positive roots which are precisely the $\epsilon_i - \epsilon_j$ for $i < j$. The Weyl denominator is

$$\delta(x) = x^\rho \prod_{\alpha \in \Delta_+} (1 - x^{-\alpha}) = x^\rho \prod_{\substack{i,j=1 \\ i < j}}^N (1 - x_i x_j^{-1})$$

so that

$$\delta(q^{-\rho}) = q^{-\rho^2} \prod_{\substack{i,j=1 \\ i < j}}^N (1 - q^{-(j-i)}) = \prod_{k=1}^{N-1} (1 - q^k)^{N-k}.$$

In summary $C_R(q)$ has been rewritten as

$$\begin{aligned} C_R(q) &= \frac{\prod_{k=1}^{N-1} (1 - q^k)^{N-k}}{(q; q)_\infty^{N-1}} \chi_R \left(q^{-\frac{N-1}{2}}, q^{-\frac{N-3}{2}}, \dots, q^{\frac{N-1}{2}} \right) \\ &= \frac{q^{\frac{\ell}{24}}}{\eta(q)^\ell} q^{\rho^2} \delta(q^{-\rho}) \chi_\lambda(q^{-\rho}), \end{aligned} \quad (2.2)$$

where R is the highest-weight representation of highest-weight λ and $\ell = N - 1$ is the rank. Weyl's character formula says that

$$\chi_\lambda(e^x) = \frac{\sum_{w \in W} \epsilon(w) e^{(x, w(\lambda + \rho))}}{\delta(e^x)}$$

with W the Weyl group and $\epsilon(w)$ the sign of the Weyl reflection w , i.e. minus for an odd reflection and plus for an even one. Thus putting everything together one gets the desired rewriting of the Schur index. The Schur index is

$$\mathcal{I}(N, n) := \mathcal{I}_{A_{N-1}, N(n-1)-1} = \sum_R C_R(q) \tilde{f}_R^{(n)}(q, x),$$

here the sum is over all inequivalent finite dimensional simple modules of $SU(N)$. These are parameterized by the positive Weyl chamber P^+ . Thus

$$\mathcal{I}(N, n) = \frac{q^{\frac{\ell}{12} + \rho^2}}{\eta(q)^{2\ell}} \sum_{\lambda \in P^+} \sum_{w \in W} \epsilon(w) q^{-(\rho, w(\lambda + \rho))} q^{\frac{n}{2}(\lambda, \lambda + 2\rho)} \sum_{\mu \in Q + \lambda} m_\lambda(\mu) q^{-\frac{n}{2}\mu^2} x^\mu. \quad (2.3)$$

The main objective is to construct vertex operator algebras whose graded characters coincide with these Schur indices. Before doing so one can obtain a series of more general Schur indices as discussed in the introduction of [1]. These correspond to type IV Argyres-Douglas theories, see [40, 48]. They are labelled by Young tableaux which can be interpreted as partitions of the integer N with N the number of boxes of the Young tableau. Another interpretation is as decompositions of the standard representation of $SU(N)$ into irreducible representations of $SU(2)$ and thus determining an embedding of $SU(2)$ in $SU(N)$. This latter is exactly what determines the different quantum Hamiltonian reductions of the affine vertex operator algebra of $\mathfrak{sl}_n$ to W -algebras. The example discussed above corresponds to the principal embedding of $SU(2)$ in $SU(N)$ which corresponds to the partition $N = N$ or the Young tableau consisting just out of one column of height N . The simplest example is the partition $N = 1 + 1 + \dots + 1$ corresponding to $SU(N)$ decomposing into $N^2 - 1$ copies of the trivial representation of $SU(2)$. In this case the W -algebra is just the affine vertex operator algebra, i.e. no reduction is performed. The Schur-index is then given by

$$\mathcal{I}_{A_{N-1}, N(n-1)-1, \square \square \dots \square} = \sum_R C_R^{\square \square \dots \square}(q, z) \tilde{f}_R^{(n)}(q, x) \quad (2.4)$$

with $C_R^{\square \square \dots \square}(q, z)$ defined as

$$C_R^{\square \square \dots \square}(q, z) = \text{P.E.} \left(\frac{q}{1-q} \chi_{\text{adj}}(z) \right) \chi_R(z) \quad (2.5)$$

and χ_R the character of the representation R and

$$\text{P.E.}(f(q, z)) = \exp \left(\sum_{n=1}^{\infty} f(q^n, z^n) \right).$$

So that $C_R^{\square \square \dots \square}(q, z)$ is nothing but the $\chi_R(z)$ times the Weyl denominator of $\widehat{\mathfrak{sl}}_n$,

$$\begin{aligned} C_R^{\square \square \dots \square}(q, z) &= \text{P.E.} \left(\frac{q}{1-q} \chi_{\text{adj}}(z) \right) \chi_R(z) \\ &= \exp \left(\sum_{n,m=1}^{\infty} q^{nm} \left(\ell + \sum_{\alpha \in \Delta_+} (z^{n\alpha} + z^{-n\alpha}) \right) \right) \chi_R(z) \\ &= \exp \left(- \sum_{m=1}^{\infty} \ell \ln(1 - q^m) + \sum_{\alpha \in \Delta_+} \ln(1 - z^\alpha) + \ln(1 - z^{-\alpha}) \right) \chi_R(z) \\ &= \prod_{m=1}^{\infty} \frac{\chi_R(z)}{(1 - q^m)^\ell \prod_{\alpha \in \Delta_+} (1 - z^\alpha q^m)(1 - z^{-\alpha} q^m)}. \end{aligned} \quad (2.6)$$

Here $\ell = N - 1$ is the rank of $\mathfrak{g} = \mathfrak{sl}_N$ and Δ_+ is the set of simple roots. Recall that the character of the irreducible highest-weight representation $\mathbb{L}_k(\lambda)$ of highest-weight λ for generic k is given by

$$\text{ch}[\mathbb{L}_k(\lambda)] = q^{\frac{(\lambda, \lambda + 2\rho)}{2(k+h^\vee)} - \frac{c}{24}} \prod_{m=1}^{\infty} \frac{\chi_\lambda(z)}{(1 - q^m)^\ell \prod_{\alpha \in \Delta_+} (1 - z^\alpha q^m)(1 - z^{-\alpha} q^m)} \quad (2.7)$$

so that we recognize $C_R^{\square \square \dots \square}(q, z)$ to be this character up to the prefactor $q^{\frac{(\lambda, \lambda + 2\rho)}{2(k+h^\vee)} - \frac{c}{24}}$. Similarly, if we pass to a non-trivial Young tableau, then as explained in section 4.3 of [42], see also [49], one has to replace this character by the Euler-Poincaré character of the corresponding quantum Hamiltonian reduction. For example $C_R(q, z)$ is obtained from $C_R^{\square \square \dots \square}(q, z)$ by multiplying it with the supercharacter of a fermionic ghost vertex operator algebra and then specializing the Jacobi variable to $z = q^{-\rho}$.

In the following sections these procedures will be realized on the level of vertex operator algebra constructions.

3 The $\mathcal{B}(n)_Q$ -algebra

In [35] a series of W -algebras named $\mathcal{B}_p$ -algebras were constructed. The construction is reviewed in [13] and it goes as follows. The triplet vertex operator algebra is defined as the kernel of a screening charge inside the lattice vertex operator algebra of the rescaled root lattice $\sqrt{p}A_1$. Its $U(1)$ -orbifold is called the singlet algebra [50–52]. The $\mathcal{B}_p$ -algebra is a simple current extension of an $U(1)$ -orbifold of the triplet algebra times a Heisenberg vertex operator algebra, that is a free boson. Namely, consider the isotropic diagonal sublattice inside $\sqrt{p}A'_1 \oplus \sqrt{-p}A'_1$ and the kernel of screening on the lattice vertex operator algebra. This is the $\mathcal{B}_p$ -algebra and it corresponds to the Schur-index of $(A_1, A_{2(p-1)-1})$ -Argyres-Douglas theories.

There are similar higher rank analogues to the triplet and singlet algebras constructed by Feigin and Tipunin [39]. These have been investigated in [53] and I follow that notation. Let $\mathfrak{g}$ be a simply-laced Lie algebra and let Q be its root lattice and P the weight lattice. Moreover let P_+ be the positive Weyl chamber. The higher rank analogue of the triplet vertex operator algebra is denoted by $\mathcal{W}(p)_Q$ with $p \in \mathbb{Z}_{>1}$ and it is defined as the joint intersection of certain screening operators $e_0^{-\alpha_i/\sqrt{p}}$ associated to the simple roots $\alpha_1, \dots, \alpha_\ell$ with ℓ the rank of $\mathfrak{g}$:

$$\mathcal{W}(p)_Q := \bigcap_{j=1}^{\ell} \ker_{V_{\sqrt{p}Q}} e_0^{-\alpha_j/\sqrt{p}}.$$

Here $V_{\sqrt{p}Q}$ denotes the lattice vertex operator algebra associated to the rescaled root lattice $\sqrt{p}Q$. To match notation with the previous section I will now replace p by n . The $\mathcal{B}(n)_Q$ -algebra will be defined as an intersection of the same screenings on a different vertex operator algebra, namely on the tensor product of a certain isotropic lattice vertex operator algebra and a Heisenberg vertex operator algebra. The lattice in question is isotropic, that

is every vector has norm zero. For this let

$$D^\pm = \sqrt{\pm n}P = \mathbb{Z}\beta_1^\pm \oplus \cdots \oplus \mathbb{Z}\beta_\ell^\pm$$

with $\beta_i^\pm \beta_j^\pm = \pm n G_{i,j}$ and G the Gram matrix of P . We then define the isotropic lattice

$$D_P^0(n) := \mathbb{Z}(\beta_1^+ + \beta_1^-) \oplus \cdots \oplus \mathbb{Z}(\beta_\ell^+ + \beta_\ell^-). \quad (3.1)$$

Let $d = \text{rank}(\mathfrak{g})$ and $\mathcal{H}(2d)$ be the Heisenberg vertex operator algebra of rank $2d$ and let $\mathcal{F}_\lambda$ be the Fock module of weight λ , then the corresponding vertex operator algebra is

$$W_{D_P^0(n)} := \bigoplus_{\lambda \in D_P^0(n)} \mathcal{F}_\lambda \cong V_{D_P^0(n)} \otimes \mathcal{H}(d)$$

and

$$\mathcal{B}(n)_Q := \bigcap_{j=1}^{\ell} \ker_{W_{D_P^0(n)}} e_0^{-\alpha_j/\sqrt{n}}.$$

The claim is that the character of $\mathcal{B}(n)_Q$ in the instance of $Q = A_{N-1}$ coincides with the Schur-Index of $\mathcal{I}(N, n)(q, x)$. It remains to compute the character. Firstly,

$$\text{ch}[\mathcal{W}(n, \nu)_Q] = q^{\frac{\rho^2}{2n}} \sum_{\lambda \in P^+ \cap (Q+\nu)} q^{\frac{n}{2}(\lambda+\rho)^2} \chi_\lambda(z) \frac{1}{\eta(q)^\ell} \sum_{w \in W} \epsilon(w) q^{-(\rho, w(\lambda+\rho))}$$

with $\nu \in P$ and

$$\mathcal{W}(n, \nu)_Q := \bigcap_{j=1}^{\ell} \ker_{V_{\sqrt{n}(Q+\nu)}} e_0^{-\alpha_j/\sqrt{n}}.$$

Let $m_\lambda(\mu)$ the multiplicity of μ in the highest-weight representation of highest-weight λ , so that

$$\chi_\lambda(z) = \sum_{\mu \in P} m_\lambda(\mu) z^\mu.$$

The character of the subspace (actually it is a module for the $U(1)^\ell$ -orbifold of $\mathcal{W}(n)_Q$)

$$\mathcal{M}(n, \mu)_Q := \bigcap_{j=1}^{\ell} \ker_{\mathcal{F}_{\sqrt{n}\mu}} e_0^{-\alpha_j/\sqrt{n}}$$

is the coefficient of z^μ in the characer of $\mathcal{W}(n, \nu)_Q$ for $\nu = \mu \pmod{Q}$. We thus have

$$\text{ch}[\mathcal{M}(n, \mu)_Q] = q^{\frac{\rho^2}{2n}} \sum_{\lambda \in P^+ \cap (Q+\mu)} q^{\frac{n}{2}(\lambda+\rho)^2} m_\lambda(\mu) \frac{1}{\eta(q)^\ell} \sum_{w \in W} \epsilon(w) q^{-(\rho, w(\lambda+\rho))}.$$

By construction

$$\mathcal{B}(n)_Q \cong \bigoplus_{\mu \in P} \mathcal{M}(n, \mu)_Q \otimes \mathcal{F}_{\sqrt{-n}\mu}$$

and so

$$\begin{aligned}
 \text{ch}[\mathcal{B}(n)_Q](q, z) &= \sum_{\mu \in P} \text{ch}[\mathcal{M}(n, \mu)_Q] \text{ch}[\mathcal{F}_{\sqrt{-n}\mu}] \\
 &= \frac{q^{\frac{\rho^2}{2n}}}{\eta(q)^{2\ell}} \sum_{\mu \in P} \sum_{\lambda \in P^+ \cap (Q+\mu)} q^{\frac{n}{2}(\lambda+\rho)^2} m_\lambda(\mu) \sum_{w \in W} \epsilon(w) q^{-(\rho, w(\lambda+\rho))} q^{-\frac{n}{2}\mu^2} z^\mu \\
 &= \frac{q^{\frac{\rho^2}{2n}(1+n^2)}}{\eta(q)^{2\ell}} \sum_{\lambda \in P^+} q^{\frac{n}{2}(\lambda, \lambda+2\rho)} \sum_{\mu \in Q+\lambda} m_\lambda(\mu) q^{-\frac{n}{2}\mu^2} z^\mu \sum_{w \in W} \epsilon(w) q^{-(\rho, w(\lambda+\rho))}.
 \end{aligned}$$

The central charge of the $\mathcal{B}(n)_Q$ -algebra is

$$c = 2\ell - \frac{\rho^2}{2n}(n-1)^2,$$

and we check that this precisely matches the proposed central charge (see [40]) of the expected chiral algebra for the Argyres Douglas theory in the instance that $Q = A_{N-1}$. We see that

$$\text{ch}[\mathcal{B}(n)_{A_{N-1}}](q, x) = q^{-\frac{c}{24}} \mathcal{I}(N, n)(q, x) \quad (3.2)$$

or put differently

$$\mathcal{I}(N, n)(q, x) = \text{tr}_{\mathcal{B}(n)_{A_{N-1}}} (q^{L_0}). \quad (3.3)$$

4 Another construction and more vertex algebras

There is a second point of view and this one relates to S-duality for vertex operator algebras as explored by Davide Gaiotto and myself [3]. For this let $\mathfrak{g}$ be a simply-laced Lie algebra, Q its root lattice and n a positive integer. Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$ and let k, ℓ be complex numbers, such that $\psi = k + h^\vee$ and $\psi' = \ell + h^\vee$ and such that

$$\frac{1}{\psi} + \frac{1}{\psi'} = n.$$

Then one of our basic proposals is that

$$A^n[G, \psi] := \bigoplus_{\lambda \in P^+ \cap nQ} \mathbb{L}_k(\lambda) \otimes \mathbb{L}_\ell(\lambda)$$

can be given the structure of a simple vertex operator algebra for generic ψ . We can generalize this by including W -algebras. For this let f be a nilpotent element of $\mathfrak{g}$ and denote by $H_{\text{DS}, f}$ the corresponding quantum Hamiltonian reduction functor with resulting W -algebra $W_k(\mathfrak{g}, f) = H_{\text{DS}, f}(\mathbb{L}_k(\mathfrak{g}))$. Then we can for example apply this functor to the second factor of $A^n[G, \psi]$ to get

$$A^n[G, f, \psi] := \bigoplus_{\lambda \in P^+ \cap nQ} \mathbb{L}_k(\lambda) \otimes H_{\text{DS}, f}(\mathbb{L}_\ell(\lambda)).$$

We again expect the existence of the structure of a simple vertex operator algebra on this object, but presently this is only known for $n = 1$ and f principal nilpotent by the main

Theorem of [54]. Let us now take the limit ψ to infinity. Then we expect that we can scale fields in such a way that the limit exists and that $\mathbb{L}_k(\mathfrak{g})$ becomes in the limit just a large commutative Heisenberg vertex operator algebra Z but the action of the zero-modes survives and integrates to an action of the compact Lie group G of $\mathfrak{g}$ so that we obtain

$$\lim_{\psi \rightarrow \infty} A^n[G, f, \psi] = Z \otimes \bigoplus_{\lambda \in P^+ \cap Q} \rho_\lambda \otimes H_{\text{DS},f}(\mathbb{L}_\ell(\lambda))$$

with ρ_λ the irreducible highest-weight representation of highest weight λ of G . In the case of f the principal nilpotent element the right-hand side coincides as a $G \otimes W_\ell(\mathfrak{g}, f_{\text{prin}})$ -module with $Z \otimes W(p)_Q$. This construction is conjectural but we are working on understanding it and for example the cases of $\mathfrak{g} = \mathfrak{sl}_2$ and $n = 1, 2$ are completely understood in [3, 55].

Now, I want to employ the picture that has been developped with Kanade and McRae, namely that such vertex operator algebra extensions are possible if and only if one has a braid-reversed equivalence of involved module categories [56, 57]. Let me explain this in a few words. Assume that we have two vertex operator algebras V, W and sets V_i, W_i of inequivalent simple $V = V_0, W = W_0$ modules such that

$$A = \bigoplus_i V_i \otimes W_i$$

is a vertex operator algebra extension of $V \otimes W$. Then under certain assumptions on the underlying categories $\mathcal{C}_V, \mathcal{C}_W$ of V and W -modules this implies that there is a braid-reversed equivalence between $\mathcal{C}_V$ and $\mathcal{C}_W$ mapping V_i to the dual of W_i . The large ψ -limit is a semiclassical limit and there we have to replace one vertex operator algebra by the compact Lie group G . Nonetheless we still expect a braid-reversed equivalence to hold [58]. On the other hand again under certain assumptions on the involved module categories

$$A = \bigoplus_i V_i \otimes W_i$$

can be given the structure of a simple vertex operator algebra only if there is a braid-reversed equivalence between $\mathcal{C}_V$ and $\mathcal{C}_W$ mapping V_i to the dual of W_i . In other words, we can now take $\text{Rep}(G)$ and replace it by a category $\mathcal{C}$ equivalent to it and consider the extension

$$\bigoplus_{\lambda \in P^+ \cap Q} \mathcal{F}(\rho_\lambda) \otimes H_{\text{DS},f}(\mathbb{L}_\ell(\lambda))$$

where $\mathcal{F}$ is the equivalence from $\text{Rep}(G)$ to $\mathcal{C}$. So let us construct a category equivalent to $\text{Rep}(G)$. Actually we only construct one that is equivalent to the subcategory of $\text{Rep}(G)$ whose objects have weights lying in the root lattice and we do this in such a way that we can recover certain subalgebras of $\mathcal{B}(n)_Q$. If we forget about the G -action and just keep the action of the maximal torus T of G , then this forgetful functor embeds $\text{Rep}(G)$ into the category of T -graded vector spaces. But there is a vertex operator algebra that has tensor categories inside T -graded vector spaces, namely the Heisenberg vertex operator algebra of rank the rank of $\mathfrak{g}$. The identification goes as follows. Let n be a positive integer as before

and let $m_\lambda(\mu)$ be the multiplicity of μ in ρ_λ , then we define the module

$$\mathcal{F}(\rho_\lambda) := \bigoplus_{\mu} \mathbb{C}^{m_\lambda(\mu)} \otimes_{\mathbb{C}} \mathcal{F}_{\sqrt{-n}\mu}.$$

The braiding on the $\mathcal{F}(\rho_\lambda)$ is trivial for weights that lie in the root lattice Q and otherwise only depend on the coset of the weight lattice P in the root lattice Q . We immediately see that in the case that f is principal nilpotent that

$$\bigoplus_{\lambda \in P^+ \cap Q} \mathcal{F}(\rho_\lambda) \otimes H_{\text{DS},f}(\mathbb{L}_\ell(\lambda)) \subset \mathcal{B}(n)_Q.$$

We conjecture that

$$\mathcal{B}(n)_{f,Q} := \bigoplus_{\lambda \in P^+} \mathcal{F}(\rho_\lambda) \otimes H_{\text{DS},f}(\mathbb{L}_\ell(\lambda))$$

can be given the structure of a simple vertex operator algebra for any nilpotent element f . This can be viewed as a natural generalization of our construction of $\mathcal{B}(n)_Q$. Proving this conjecture would amount to proving a generalization of the ideas of Feigin and Tipunin [39] to any quantum Hamiltonian reduction. This is difficult.

Let me summarize this discussion. Let n be a positive integer and $\ell = -h^\vee + \frac{1}{n}$ with $h^\vee$ the dual Coxeter number of the simply-laced Lie algebra $\mathfrak{g}$ with root lattice Q and weight lattice P . Let f be a nilpotent element of $\mathfrak{g}$ and let $H_{\text{DS},f}$ be the corresponding quantum Hamiltonian reduction functor. Then conjecturally the following $G \otimes H_{\text{DS},f}(\mathbb{L}_\ell(g))$ -modules have the structure of a simple vertex operator algebra

$$\mathcal{W}(n)_{f,Q} := \bigoplus_{\lambda \in P^+ \cap Q} \rho_\lambda \otimes H_{\text{DS},f}(\mathbb{L}_\ell(\lambda)) \quad (4.1)$$

and the $\mathcal{H} \otimes H_{\text{DS},f}(\mathbb{L}_\ell(g))$ -modules have the structure of a simple vertex operator algebra

$$\mathcal{B}(n)_{f,Q} := \bigoplus_{\lambda \in P^+} \mathcal{F}(\rho_\lambda) \otimes H_{\text{DS},f}(\mathbb{L}_\ell(\lambda)) \quad (4.2)$$

as well. Here $\mathcal{H}$ denotes the Heisenberg vertex operator algebra of rank the rank of $\mathfrak{g}$. We see that by construction the autoorphism group of $\mathcal{W}(n)_{f,Q}$ contains G , while the one of $\mathcal{B}(n)_{f,Q}$ contains the Weyl group W of G as group of automorphisms as these permute the Fock modules appearing in each $\mathcal{F}(\rho_\lambda)$.

Let now $\mathfrak{g} = \mathfrak{sl}_n$ and Y_f be the Young tableau corresponding to the nilpotent element f . Comparing with equation (2.4)–(2.7) and for the trivial nilpotent element 0 one obtains

$$\begin{aligned} \text{ch}[\mathcal{B}(n)_{0,Q}](z, q) &:= \sum_{\lambda \in P^+} \sum_{\mu \in Q + \lambda} m_\lambda(\mu) q^{-\frac{n}{2}\mu^2} x^\mu \text{ch}[\mathbb{L}_\ell(\lambda)](z, q) \\ &= q^{-\frac{c}{24}} \sum_{\lambda \in P^+} \tilde{f}_R^{(n)}(q, x) C_{\rho_\lambda}^{\square \square \dots \square}(z, q) \\ &= q^{-\frac{c}{24}} \mathcal{I}_{A_{N-1}, N(n-1)-1, \square \square \dots \square}(z, q) \end{aligned} \quad (4.3)$$

with $c = (N^2 - 1)(1 - nN) + N - 1$ the central charge of $\mathcal{B}(n)_{0,Q}$. According to the principle of [42] that the super conformal index associated to the Young tableay Y_f differs from

the one of the trivial one Y_0 by replacing characters by the corresponding Euler-Poincaré characters we immediately get the identification

$$\begin{aligned}
 \text{ch}[\mathcal{B}(n)_{f,Q}](z, q) &:= \sum_{\lambda \in P^+} \sum_{\mu \in Q + \lambda} m_\lambda(\mu) q^{-\frac{n}{2}\mu^2} x^\mu \text{ch}[H_{\text{DS},f}(\mathbb{L}_\ell(\lambda))](z, q) \\
 &= q^{-\frac{c_f}{24}} \sum_{\lambda \in P^+} \tilde{f}_R^{(n)}(q, x) C_{\rho_\lambda}^{Y_f}(z, q) \\
 &= q^{-\frac{c_f}{24}} \mathcal{I}_{A_{N-1}, N(n-1)-1, Y_f}(z, q).
 \end{aligned} \tag{4.4}$$

with c_f the central charge of $\mathcal{B}(n)_{f,Q}$.

5 Conformal embeddings versus RG flow

Buican and Nishinaka also discuss RG-flow of Argyres-Douglas theories [1, section 4]. They find that RG-flow interpolates between the Argyres-Douglas theories as

$$(A_{N-1}, A_{N(n-1)-1}) \rightarrow (A_{\nu-1}, A_{\nu(n-1)-1}) \oplus (A_{N-\nu-1}, A_{(N-\nu)(n-1)-1}) \oplus (A_1, A_1). \tag{5.1}$$

I want to conclude this work with the observation that the chiral algebra of the left-hand side almost embeds conformally in the one of the right-hand side. Here almost means that we have to replace the $\beta\gamma$ vertex operator algebra, that is the chiral algebra of the (A_1, A_1) -theory by a larger vertex operator algebra.

Let $Q_N = A_{N-1}$ denote the root lattice of $\mathfrak{sl}_N$ and $P_N = A'_{N-1}$ its weight lattice. We then have the embedding

$$Q_\nu \oplus Q_{N-\nu} \subset Q_N$$

and the orthogonal complement is denoted by

$$\Phi = \mathbb{Z}x, \quad x = (N - \nu)(\epsilon_1 + \cdots + \epsilon_\nu) - \nu(\epsilon_{\nu+1} + \cdots + \epsilon_N).$$

We compute that $x^2 = N\nu(N - \nu)$. So that the dual lattice is $\Phi' = \frac{x}{N\nu(N-\nu)}\mathbb{Z}$ and we have the embedding of dual lattices

$$P_N \subset P_\nu \oplus P_{N-\nu} \oplus \Phi'.$$

This induces embeddings of

$$\sqrt{n}P_N \oplus \sqrt{-n}P_N \subset \sqrt{n}P_\nu \oplus \sqrt{-n}P_\nu \oplus \sqrt{n}P_{N-\nu} \oplus \sqrt{-n}P_{N-\nu} \oplus \sqrt{n}\Phi' \oplus \sqrt{-n}\Phi'$$

and via restriction to the isotropic sublattices defined in (3.1)

$$D_{P_N}^0(n) \subset D_{P_\nu}^0(n) \oplus D_{P_{N-\nu}}^0(n) \oplus D_{\Phi'}^0(n).$$

We thus get the following chain of conformal embeddings

$$\begin{aligned}
\mathcal{B}(n)_{Q_N} &= \bigcap_{j=1}^{N-1} \ker_{W_{D_{P_N}^0(n)}} \left(e_0^{-\alpha_j/\sqrt{n}} \right) \\
&\subset \bigcap_{\substack{j=1 \\ j \neq \nu}}^{N-1} \ker_{W_{D_{P_N}^0(n)}} e_0^{-\alpha_j/\sqrt{n}} \\
&\subset \bigcap_{\substack{j=1 \\ j \neq \nu}}^{N-1} \ker_{W_{D_{P_\nu}^0(n)} \otimes W_{D_{P_{N-\nu}}^0(n)} \otimes W_{D_{\Phi'}^0(n)}} \left(e_0^{-\alpha_j/\sqrt{n}} \right) \\
&= \bigcap_{j=1}^{\nu-1} \ker_{W_{D_{P_\nu}^0(n)}} \left(e_0^{-\alpha_j/\sqrt{n}} \right) \otimes \bigcap_{j=\nu+1}^{N-1} \ker_{W_{D_{P_{N-\nu}}^0(n)}} \left(e_0^{-\alpha_j/\sqrt{n}} \right) \otimes W_{D_{\Phi'}^0(n)} \\
&= \mathcal{B}(n)_{Q_\nu} \otimes \mathcal{B}(n)_{Q_{N-\nu}} \otimes W_{D_{\Phi'}^0(n)}.
\end{aligned} \tag{5.2}$$

In the last line, we have the chiral algebras of $(A_{\nu-1}, A_{\nu(n-1)-1})$ and $(A_{N-\nu-1}, A_{(N-\nu)(n-1)-1})$ Argyres-Douglas theories. But the last factor is different than the $\beta\gamma$ vertex operator algebra. Note however that $\beta\gamma \subset W_{D_{\Phi'}^0(n)}$ by the construction of [35].

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