Supersymmetric partially massless fields and non-unitary superconformal representations

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ABSTRACT: We find and classify the $\mathcal{N} = 1$ SUSY multiplets on AdS_4 which contain partially massless fields. We do this by studying the non-unitary representations of the $d = 3$ superconformal algebra of the boundary. The simplest super-multiplet which contains a partially massless spin-2 particle also contains a massless photon, a massless spin-3/2 particle and a massive spin-3/2 particle. The gauge parameters form a Wess-Zumino super-multiplet which contains the gauge parameters of the photon, the partially massless graviton, and the massless spin-3/2 particle. We find the AdS_4 action and SUSY transformations for this multiplet. More generally, we classify new types of shortening conditions that can arise for non-unitary representations of the $d = 3$ superconformal algebra.

KEYWORDS: Higher Spin Symmetry, Supergravity Models, Gauge Symmetry

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1 Introduction

De Sitter (dS) and anti de Sitter (AdS) spacetimes allow for exotic irreducible representations which go by the name of partially massless (PM) particles [1–15]. A partially massless particle of spin s possesses a mass and also gauge symmetry, labelled by a depth $t \in \{0, 1, \dots, s - 1\}$. The gauge symmetry eliminates the helicity components $0, 1, \dots, t$

from the particle, leaving a number of degrees of freedom intermediate between that of a massless and a massive field. These gauge invariances emerge at fixed values of the particle mass relative to the (A)dS curvature.

In this work, we consider the supersymmetric (SUSY) extension of partially massless representations on AdS_4 . We do this through the AdS/CFT correspondence by studying the representations of the $d = 3$ superconformal algebra of the boundary. In AdS, the partially massless representations are non-unitary, and so we must study the non-unitary representations of the superconformal algebra. Unitary superconformal representations in three dimensions have been extensively studied [16–18], but the non-unitary case has remained relatively unexplored (see however [19, 20]).

In the boundary CFT, partially massless particles correspond to short multiplets with conformal dimension $\Delta = t + 2$, which have a null descendent at level $s - t$. We will find that partially massless particles appear in superconformal multiplets which contain the standard four conformal primaries for generic massive particles. For instance, the simplest super-multiplet which contains a partially massless spin-2 particle also contains a massless photon, a massless spin-3/2 particle and a massive spin-3/2 particle. The scalar partially massless gauge parameter is a Wess-Zumino super-multiplet which contains the scalar gauge parameters of the photon and the partially massless graviton, as well as the spin-1/2 gauge parameter of the massless spin-3/2 particle. We explicitly find the AdS action and SUSY transformations for this multiplet.

Away from the partially massless values, we classify new types of BPS-like shortening conditions that can arise for non-unitary representations of the $d = 3$ superconformal algebra. For fields that obey the usual “standard quantization” we recover the expected short multiplets whose duals consist of massless fields in AdS_4 . We also find a shortening condition for “alternatively quantized” fields which themselves have finite dimensional Verma modules, including an additional shortening condition for certain alternatively quantized partially massless fields. Finally, for $\Delta = 1$, we find a supersymmetric example of the exotic “extended modules” of [21].

Studies of massive supergravity can be found in [22–26]. Partially massless supergravity is potentially interesting for a number of reasons. While theories of free partially massless particles are straightforward to construct, various studies and no-go theorems seem to forbid consistent theories of a single interacting partially massless spin-2 particle [27–50]. However, it may still be possible to construct theories of partially massless particles interacting with other fields, and there are examples of Vasiliev-like theories with infinite towers of fields [51–55]. A theory of an interacting supersymmetric partially massless multiplet would not contradict any known no-go theorems. Finally, non-unitary conformal theories and representations have found applications in a range of condensed matter systems [56], and in understanding the analytic structure of the conformal blocks in unitary theories [57]. It is thus useful to explore and categorize the non-unitary irreducible representations of the superconformal group.

Conventions. We work in $D = 4$ spacetime dimensions for AdS, and $d = 3$ Euclidean spatial dimensions for CFT. We use the mostly plus metric signature convention. Ten-

sors are symmetrized and anti-symmetrized with unit weight, i.e $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$, $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$. We use $\mu, \nu, \dots$ for AdS spacetime indices, $i, j, \dots$ for flat CFT 3d tensor indices, $a, b, \dots$ for 3d CFT spinor indices. Conventions for 3d spinor quantities are detailed in appendix A, and conventions for 4D spinor quantities are detailed in appendix B. Conventions for the curvature tensors and covariant derivatives are those of Carroll [58]. L the AdS radius. The scalar curvature R and cosmological constant Λ are related as $R = -\frac{12}{L^2}$, $\Lambda = -\frac{3}{L^2}$.

2 Partially massless fields and their dual operators

Through the AdS/CFT correspondence, a spin s field in AdS_4 corresponds to a spin s primary operator in CFT_3 . The mass m of the field and the dimension Δ of the primary are related by

$$m^2 L^2 = \begin{cases} \Delta(\Delta - 3), & s = 0, \\ (\Delta + s - 2)(\Delta - s - 1), & s \geq 1/2. \end{cases} \quad (2.1)$$

For a given mass, there are two different ways to quantize the field in AdS_4 . These correspond to the greater and lesser roots of (2.1), $\Delta_{\pm}$. Δ_+ corresponds to the so-called “standard quantization” which covers the operators with $\Delta > 3/2$, and Δ_- to the “alternate quantization” [59] which covers the operators with $\Delta < 3/2$.

The unitarity bound for primary operators is [60–62]

$$\Delta \geq \begin{cases} 1/2, & s = 0, \\ 1, & s = 1/2, \\ s + 1, & s \geq 1. \end{cases} \quad (2.2)$$

Theories containing primary operators violating these bounds are necessarily non-unitary.

2.1 Bosons

Bosonic fields of spin s on AdS_4 are carried by fully symmetric tensors $\phi_{\mu_1 \dots \mu_s}$ whose equations of motion can be brought to the form:

$$\begin{aligned} \left[\square - \frac{1}{L^2} (s(s-2) - 2) - m^2 \right] \phi_{\mu_1 \dots \mu_s} &= 0, \\ \nabla^\nu \phi_{\nu \mu_2 \dots \mu_s} &= 0, \\ \phi_{\nu \mu_3 \dots \mu_s}^\nu &= 0, \end{aligned} \quad (2.3)$$

i.e. they are transverse and traceless and obey the wave equation. Here $\square \equiv \nabla^\mu \nabla_\mu$ is the bare curved space Laplacian.

For bosons of spin $s \geq 1$, partially massless points occur at the following mass values labelled by t ,

$$m_{s,t}^2 = -\frac{1}{L^2} (s-t-1)(s+t), \quad t = 0, 1, 2, \dots, s-1. \quad (2.4)$$

Here, t is known as the *depth* of partial masslessness of the field. The massless field is the one at depth $t = s - 1$. At these mass values, the equations (2.3) becomes invariant under a partially massless gauge symmetry,

$$\delta\phi_{\mu_1\ldots\mu_s} = \nabla_{(\mu_{t+1}} \cdots \nabla_{\mu_s} \xi_{\mu_1\ldots\mu_t)} + \cdots \quad (2.5)$$

with a symmetric tensor gauge parameter $\xi_{\mu_1\ldots\mu_t}$, which on shell is subject to its own conditions analogous to (2.3). The $\cdots$ in (2.5) stand for lower-derivative terms proportional to factors of the curvature $1/L^2$. Note that the depth t corresponds to the number of indices on the gauge parameter in (2.5).

The partially massless values (2.4) correspond to the conformal dimensions

$$\Delta_{s,t} = t + 2. \quad (2.6)$$

(This is the normal quantization root Δ_+ , the alternate root is $\Delta_- = 1 - t$.) They violate the unitarity bound (2.2) except for the massless case at $t = s - 1$, which saturates it.

A primary state $|\Delta\rangle^{i_1\cdots i_s}$ with the conformal dimension (2.6) gives rise to a short multiplet [63]. There is a null descendent at level $s - t$,

$$P_{i_1} \cdots P_{i_{s-t}} |\Delta\rangle^{i_1\cdots i_s} = 0, \quad (2.7)$$

(this is the type II shortening in the language of [57]). This primary can be called a multiply-conserved current.

2.2 Fermions

A spin s fermion on AdS_4 is a totally symmetric tensor spinor $\Psi_{\mu_1\cdots\mu_{s-1/2}}$, whose equations of motion can be brought to the form:

$$\begin{aligned} (i\not{\nabla} - \tilde{m}) \Psi_{\mu_1\cdots\mu_{s-1/2}} &= 0, \\ \nabla^\nu \Psi_{\nu\mu_2\cdots\mu_{s-1/2}} &= 0, \\ \gamma^\nu \Psi_{\nu\mu_2\cdots\mu_{s-1/2}} &= 0, \end{aligned} \quad (2.8)$$

i.e. they are transverse, gamma traceless (hence traceless) and obey the Dirac equation. Here ∇ is the full AdS covariant derivative, which includes both Christoffel symbols acting on the tensor indices and a spin connection acting on the spinor index. The parameter $\tilde{m}$ in the Dirac equation is related to the mass m^2 appearing in the AdS/CFT formula (2.1) by

$$\tilde{m}^2 = \frac{(2s-1)^2}{4L^2} + m^2. \quad (2.9)$$

For $s \geq 3/2$, partially massless points occur at the following mass values labelled by the depth t ,

$$\tilde{m}_{s,t}^2 = \frac{(2t+1)^2}{4L^2}, \quad m_{s,t}^2 = \frac{(t-s+1)(s+t)}{L^2}, \quad t = 1/2, 3/2, \cdots, s-1. \quad (2.10)$$

The massless field is the one at depth $t = s - 1$. At these mass values, the equations (2.8) becomes invariant under a partially massless gauge symmetry,

$$\delta\Psi_{\mu_1\cdots\mu_{s-1/2}} = \nabla_{(\mu_{t+1/2}} \cdots \nabla_{\mu_{s-1/2}} \Phi_{\mu_1\cdots\mu_{t-1/2})} + \cdots \quad (2.11)$$

with a symmetric tensor spinor gauge parameter $\Phi_{\mu_1\cdots\mu_t}$ which on shell is subject to its own conditions analogous to (2.3). The $\cdots$ in (2.11) stand for lower-derivative terms proportional to factors of the curvature $1/L^2$. Note that the depth t corresponds to the spin of the gauge parameter in (2.11).

The partially massless values (2.10) have $m^2 < 0$, increasingly negative as t decreases, with $m^2 = 0$ the massless value $t = s - 1$. However they have $\tilde{m}^2 > 0$. They correspond to the conformal dimensions

$$\Delta_{s,t} = t + 2. \quad (2.12)$$

(This is the normal quantization root Δ_+ , the alternate root is $\Delta_- = 1 - t$.)

The partially massless fields are dual to short multiplets with tensor spinor conformal primaries $|\Delta\rangle^{i_1\cdots i_{s-1/2} a}$ (a is the spinor index) that have a null descendent at level $s - t$,

$$P_{i_1} \cdots P_{i_{s-t}} |\Delta\rangle^{i_1\cdots i_{s-1/2} a} = 0. \quad (2.13)$$

These are fermionic multiply-conserved current.

3 $d = 3$ superconformal representations

We are interested in seeing how the partially massless bosons and fermions are joined together into supermultiplets. For this we must study the SUSY representations of AdS_4 , which are the same as the superconformal representations of the flat three dimensional boundary. Unitary superconformal representations in three dimensions have been studied in [16, 17], and in other dimensions in [18, 62], and the multiplets on AdS_4 have also been studied [64]. The partially massless values (2.6), (2.12) fall below the unitarity bound (2.2), so we will be interested in non-unitary superconformal representations, which have not been studied as far as we know. Along the way, we will find new kinds of BPS-like shortening conditions for the non-unitary superconformal multiplets which have no analogue among the unitary representations.

3.1 $d = 3$ superconformal algebra

The generators of the euclidean $\mathcal{N} = 1$ superconformal algebra are

$$P^i, J_{ij}, D, K^i, Q^a, S_a. \quad (3.1)$$

Here P^i are the translations and J_{ij} the rotations, which together generate the Poincare transformations of 3d euclidean space. D is the dilation and K^i the special conformal generators, which together with the Poincare generators generate the conformal symmetries of 3d euclidean space. Q^a is the supersymmetry, which together with the Poincare generators generate $\mathcal{N} = 1$ SUSY. S_a is the special superconformal generator, which completes the SUSY generators and conformal generators into to the superconformal algebra.

We now list all the non-vanishing (anti)commutators of the 3d euclidean superconformal algebra [62] (see appendix A for our conventions on 3d spinors and representations). First there are the standard commutation relation of the Poincare algebra,

$$\begin{aligned} [J^{ij}, P^k] &= i \left(-\delta^{ki} P^j + \delta^{kj} P^i \right), \\ [J^{ij}, J^{kl}] &= i \left(-\delta^{ik} J^{jl} + \delta^{jk} J^{il} - \delta^{jl} J^{ik} + \delta^{il} J^{jk} \right). \end{aligned} \quad (3.2)$$

Next are the commutators which when taken together with (3.2) form the conformal algebra,

$$\begin{aligned} [D, P^i] &= P^i, \\ [D, K^i] &= -K^i, \\ [K^i, P^j] &= 2(\delta^{ij} D + iJ^{ij}), \\ [J^{ij}, K^k] &= i \left(-\delta^{ki} K^j + \delta^{kj} K^i \right). \end{aligned} \quad (3.3)$$

Then we have the commutators which when taken together with (3.2) form the $\mathcal{N} = 1$ SUSY algebra,

$$\begin{aligned} \{Q^a, Q^b\} &= 2\sigma_i^{ab} P^i, \\ [J_{ij}, Q^a] &= -\frac{i}{2} (\sigma_{ij})^a_b Q^b. \end{aligned} \quad (3.4)$$

The first line of (3.4) is the fundamental anti-commutator of SUSY, the second indicates that Q^a transforms as a spinor.

Finally we have the commutators which when taken with all the above fill out the superconformal algebra,

$$\begin{aligned} [J_{ij}, S^a] &= -\frac{i}{2} (\sigma_{ij})^a_b S^b, \\ [D, Q^a] &= \frac{1}{2} Q^a, \\ [D, S^a] &= -\frac{1}{2} S^a, \\ \{S^a, S^b\} &= -2\sigma_i^{ab} K^i, \\ \{Q^a, S^b\} &= 2\epsilon^{ab} D - i\sigma_{ij}^{ab} J^{ij}, \\ [K_i, Q^a] &= -(\sigma_i)^a_b S^b, \\ [P_i, S^a] &= (\sigma_i)^a_b Q^b. \end{aligned} \quad (3.5)$$

The first line of (3.5) indicates that S^a transforms as a spinor, the second and third lines indicate that Q^a carries dimension 1/2 and S^a carries dimension $-1/2$.

In radial quantization, the generators satisfy the following reality conditions¹

$$P_i^\dagger = K_i, \quad D^\dagger = D, \quad J_{ij}^\dagger = J_{ij}, \quad Q^{a\dagger} = S_a. \quad (3.6)$$

¹Note that with our conventions, outlined in appendix A, there is a subtlety with the spinor indices; the condition $Q^{a\dagger} = S_a$ implies $S_a^\dagger = Q^a$, but when raising and lowering indices we get a minus sign (A.15), $Q_a^\dagger = -S^a$, $S^a^\dagger = -Q_a$.

It is straightforward to check that all the (anti)commutation relations above are consistent with the reality conditions (3.6).

3.2 Conformal representations

Conformal representations are built by starting with a spin s conformal primary operator of dimension Δ . We represent this as a state $|\Delta\rangle^{a_1 \cdots a_{2s}}$ carrying $2s$ completely symmetric spinor indices. It transforms as an irreducible spin s representation under rotations,

$$J_{ij}|\Delta\rangle^{a_1 \cdots a_{2s}} = -\frac{i}{2}(\sigma_{ij})^{a_1}_{b}|\Delta\rangle^{ba_2 \cdots a_{2s}} - \cdots - \frac{i}{2}(\sigma_{ij})^{a_{2s}}_{\phantom{a_{2s}}b}|\Delta\rangle^{a_1 \cdots a_{2s-1}b}, \quad (3.7)$$

$$J_{ij}J^{ij}|\Delta\rangle^{a_1 \cdots a_{2s}} = 2s(s+1)|\Delta\rangle^{a_1 \cdots a_{2s}}, \quad (3.8)$$

and carries eigenvalue Δ under the dilation generator D ,

$$D|\Delta\rangle^{a_1 \cdots a_{2s}} = \Delta|\Delta\rangle^{a_1 \cdots a_{2s}}. \quad (3.9)$$

The primary has the property that it is annihilated by K^i ,

$$K^i|\Delta\rangle^{a_1 \cdots a_{2s}} = 0. \quad (3.10)$$

The rest of the conformal representation is built by acting repeatedly with P^i on the primary. The representation is graded by the level, which is the number of P 's acting on the primary, so that at level l the states are

$$P^{i_1} \cdots P^{i_l}|\Delta\rangle^{a_1 \cdots a_{2s}}, \quad (3.11)$$

and they have dimension $\Delta + l$. Each level is then further decomposed into irreducible representations of J_{ij} . This forms a Verma module where we go up in dimension starting from the primary by acting with P , and down in dimension acting with K .

We take the primary to be normalized to unity in the inner product. Denoting the conjugate $|\Delta\rangle^{a_1 \cdots a_{2s}\dagger} = {}_{a_1 \cdots a_{2s}}\langle\Delta|$, this normalization condition reads

$${}_{b_1 \cdots b_{2s}}\langle\Delta|\Delta\rangle^{a_1 \cdots a_{2s}} = \delta^{(a_1}_{b_1} \cdots \delta^{a_{2s})}_{b_{2s}}. \quad (3.12)$$

The inner product of any of the descendants can then be computed by using the conjugation properties (3.6) and the conformal algebra commutators. At certain values of Δ , some of the descendent states become null. These null states can then be factored out and the result is a shortened multiplet. The values of Δ at which this happens are catalogued in [57]. Of interest to us are the multiply conserved currents (2.7), (2.13) dual to partially massless fields of depth t , which have null states at level $s - t$. We denote the conformal representation built from the primary² $|\Delta\rangle^{a_1 \cdots a_{2s}}$ as $[s]_\Delta$.

²Unitary representations of the conformal group are all of this type [65], however there exist non-unitary representations which cannot be built by the action of P 's on some primary. They contain states which are neither primary nor descendent. An example is the so-called extended modules present in higher order free CFTs, which can be thought of as a kind of gluing together of ordinary modules [21, 66]. The AdS dual is a field which cannot be diagonalized [55].

3.3 Generic superconformal representations

The superconformal algebra is larger than the conformal algebra, and so the superconformal representations are larger than the conformal representations and can be built by joining together the conformal representations using the action of the generators Q and S which are not present in the conformal algebra. The superconformal representation is built by starting with a superconformal primary, which is a conformal primary which in addition is annihilated by S ,

$$S_a |\Delta\rangle^{a_1 \cdots a_{2s}} = 0. \quad (3.13)$$

We get the rest of the conformal primaries present in the superconformal multiplet by acting with Q 's in all possible combinations:

$$Q^a |\Delta\rangle^{a_1 \cdots a_{2s}}, \quad Q^{[a} Q^{b]} |\Delta\rangle^{a_1 \cdots a_{2s}} + (P|\Delta\rangle \text{ terms}). \quad (3.14)$$

The $P|\Delta\rangle$ terms in (3.14) are necessary and fixed by the requirement that the state be a conformal primary (i.e. annihilated by K), and we will be explicit about these terms shortly. Each time we act by Q , Δ increases by $1/2$. We need only consider the action by anti-symmetric combinations of the Q 's because the symmetric ones give descendants, due to the $\{Q, Q\} \sim P$ anti-commutation relation in (3.4). We do not have to act with more than two Q 's because we cannot anti-symmetrize three or more Q 's.

The states $Q^a |\Delta\rangle^{a_1 \cdots a_{2s}}$ with weight $\Delta + \frac{1}{2}$ at the first level break up into two irreducible representations, of spin $s \pm \frac{1}{2}$,

$$Q_a |\Delta\rangle^{aa_2 \cdots a_{2s}}, \quad Q^{(a} |\Delta\rangle^{a_1 \cdots a_{2s})}. \quad (3.15)$$

These are conformal primaries which are not superconformal primaries, i.e. they are annihilated by K but not by S .

Using (A.16) the states at the second level are a spin s state with weight $\Delta + 1$, which we call $|\Delta\rangle_1^{a_1 \cdots a_{2s}}$,

$$|\Delta\rangle_1^{a_1 \cdots a_{2s}} \equiv Q^a Q_a |\Delta\rangle^{a_1 \cdots a_{2s}}, \quad (3.16)$$

as well as the first P descendent of $|\Delta\rangle^{a_1 \cdots a_{2s}}$ which has the same quantum numbers as $|\Delta\rangle_1^{a_1 \cdots a_{2s}}$, which we call $|\Delta\rangle_2^{a_1 \cdots a_{2s}}$,

$$\begin{aligned} |\Delta\rangle_2^{a_1 \cdots a_{2s}} &\equiv 2s P^i \sigma_i^{(a_1} \sigma_{b}^{a_2)} |\Delta\rangle^{a_2 \cdots a_{2s}} b \\ &= P^i \sigma_i^{a_1} \sigma_b^{a_2} |\Delta\rangle^{ba_2 \cdots a_{2s}} + P^i \sigma_i^{a_2} \sigma_b^{a_1} |\Delta\rangle^{a_1 ba_3 \cdots a_{2s}} + \cdots + P^i \sigma_i^{a_{2s}} \sigma_b^{a_{2s-1}} |\Delta\rangle^{ba_2 \cdots a_{2s-1} b}. \end{aligned} \quad (3.17)$$

There is one linear combination of $|\Delta\rangle_1^{a_1 \cdots a_{2s}}$ and $|\Delta\rangle_2^{a_1 \cdots a_{2s}}$ which is annihilated by K , and which is the true second level conformal primary. By acting with K and demanding the result vanish, we find this conformal primary, which we call $|\tilde{\Delta}\rangle^{a_1 \cdots a_{2s}}$,

$$|\tilde{\Delta}\rangle^{a_1 \cdots a_{2s}} = |\Delta\rangle_1^{a_1 \cdots a_{2s}} + \frac{1}{\Delta - 1} |\Delta\rangle_2^{a_1 \cdots a_{2s}}. \quad (3.18)$$

(We will see later what happens when $\Delta = 1$.)

Thus the generic superconformal representation contains four conformal primaries, and the structure of this representation can be illustrated as follows,

$$\begin{array}{c}
 [s]_{\Delta+1} \\
 \swarrow \quad \searrow \\
 \{s\}_{\Delta} \text{ generic multiplet: } \left[s - \frac{1}{2} \right]_{\Delta+\frac{1}{2}} \quad \left[s + \frac{1}{2} \right]_{\Delta+\frac{1}{2}} \\
 \swarrow \quad \searrow \\
 [s]_{\Delta}
 \end{array} \quad (3.19)$$

In this diagram the conformal dimension increases upward and the spin increases towards the right. The superconformal primary is at the bottom. We use the notation $\{s\}_{\Delta}$ to denote the full superconformal multiplet, where s, Δ are the spin and dimension of the superconformal primary. The notation $[s]_{\Delta}$ is used for the individual conformal primaries within the superconformal multiplet.

In AdS_4 , the multiplet (3.19) corresponds to a generic multiplet of massive fields. For example, the massive vector multiplet of AdS_4 corresponds to the case $\{\frac{1}{2}\}_{\Delta}$; in this multiplet there is a massive spin-1 $[1]_{\Delta+\frac{1}{2}}$ with 3 propagating degrees of freedom, a massive spin 0 $[0]_{\Delta+\frac{1}{2}}$ with one propagating degree of freedom, and two massive Weyl (or Majorana) fermions $[\frac{1}{2}]_{\Delta}$, $[\frac{1}{2}]_{\Delta+1}$ of spin-1/2, each with two propagating degrees of freedom, for a total of 4 bosonic and 4 fermionic propagating degrees of freedom.

In the case where the superconformal primary is spin 0, there is only one representation at the first level, $Q^a|\Delta\rangle$, so there are only three conformal primaries in the representation, and the structure of the representation can be illustrated as follows

$$\begin{array}{c}
 [0]_{\Delta+1} \\
 \swarrow \\
 \{0\}_{\Delta} \text{ generic scalar multiplet: } \left[\begin{array}{c} 1 \\ 2 \end{array} \right]_{\Delta+\frac{1}{2}} \\
 \searrow \\
 [0]_{\Delta}
 \end{array} \quad (3.20)$$

The conformal primary $[0]_{\Delta+1}$ at the second level is just $Q^a Q_a |\Delta\rangle$. In AdS_4 , this corresponds to the generic massive Wess-Zumino multiplet; there are two scalars (which on flat space would be of equal mass packaged into a complex scalar, but on AdS have different masses) and one massive fermion, for a total of 2 bosonic and 2 fermionic propagating degrees of freedom.

3.4 Shortening conditions

Assuming that the superconformal primary is normalized to unity, the norms of the remaining conformal primaries within the superconformal multiplet can be computed using the conjugation properties (3.6) and the commutators of the superconformal algebra. When the dimension Δ of the superconformal primary takes on certain values, some of the other conformal primaries may have zero norm. When this happens, the multiplet may shorten. In this section, we classify the various cases for which the conformal primaries within the superconformal multiplet get zero norms.

3.4.1 $s = 0$

We start with the case where the superconformal primary has spin $s = 0$. Calculating the norm of the first descendent $Q^a|\Delta\rangle$, we find

$$|Q^a|\Delta\rangle|^2 = \langle\Delta|S_bQ^a|\Delta\rangle = \langle\Delta|\{S_b, Q^a\}|\Delta\rangle = 2\delta_b^a\langle\Delta|D|\Delta\rangle = 2\delta_b^a\Delta\langle\Delta|\Delta\rangle = 2\delta_b^a\Delta. \quad (3.21)$$

We see that when $\Delta = 0$ this is a zero norm state, and in this case the second descendent $Q^aQ_a|\Delta\rangle$ also has zero norm. Both of these zero norm states have zero inner product with the remaining states and are thus null states. They form their own sub-module which can be factored out. We denote this by,

$$\begin{array}{ccc} & [0]_1 & \\ & \swarrow & \\ \{0\}_0 \text{ vacuum multiplet:} & \left[\begin{array}{c} 1 \\ \frac{1}{2} \end{array} \right]_{\frac{1}{2}} & \\ & \searrow & \\ & [0]_0 & \end{array} \quad (3.22)$$

Here the null states are indicated in green, and they are removed from the multiplet. Once they are removed we have a shortened multiplet consisting of only the state $[0]_0$. A scalar conformal primary with $\Delta = 0$ has no non-null conformal descendants, and so there is only one state in the entire supermultiplet (3.22). This is the trivial vacuum multiplet.

Looking now at the second level, there is another shortening condition which we can find by looking at the norm of the second level descendant $Q^aQ_a|\Delta\rangle$. For this it is useful to first calculate an expression for S^2Q^2 between superconformal primaries (SCPs),

$$S^2Q^2 \equiv S^bS_bQ^aQ_a \Big|_{\text{between SCPs}} = -2 \left\{ Q^a, S^b \right\} \left\{ Q_a, S_b \right\} + \left\{ Q^a, \left[\left\{ Q_a, S_b \right\}, S^b \right] \right\}. \quad (3.23)$$

This can now be evaluated using the (anti)commutation relations of the superconformal algebra,

$$S^2 Q^2 \Big|_{\text{between SCPs}} = -16D^2 + 8J_{ij}J^{ij} + 8D = -16 \left[\Delta \left(\Delta - \frac{1}{2} \right) - s(s+1) \right], \quad (3.24)$$

where we have used (3.9) and (3.8) which say that $D = \Delta$ and $J_{ij}J^{ij} = 2s(s+1)$ when acting on conformal primaries. Thus, between scalar superconformal primaries where $s = 0$, we have

$$|Q^2|\Delta\rangle|^2 = \langle\Delta| \left(-S^b S_b \right) Q^a Q_a |\Delta\rangle = 16\Delta \left(\Delta - \frac{1}{2} \right). \quad (3.25)$$

We see again the vacuum multiplet shortening condition at $\Delta = 0$, but now there is another root at $\Delta = \frac{1}{2}$. This is a shortening condition where the second level descendent goes null without the first one going null, and we have the shortened multiplet

$$\begin{array}{ccc} & [0]_{\frac{3}{2}} & \\ & \cdots & \\ \{0\}_{\frac{1}{2}} \text{ singleton multiplet:} & \left[\frac{1}{2} \right]_1 & \\ & \cdots & \\ & [0]_{\frac{1}{2}} & \end{array} \quad (3.26)$$

This is the supersymmetric Dirac singleton [67].

As Δ crosses the values where zero norm states appear, the norms of various superconformal descendants will switch sign. From (3.21) and (3.25), we can read off the relative signs of the kinetic energy terms expected in the AdS_4 duals. We see that when $\Delta > 1/2$, all the norms are positive, consistent with unitarity. When $\Delta = 1/2$ we reach the first shortening condition where the second level superconformal descendant $[0]_{\Delta+1}$ goes null, and the other state $\left[\frac{1}{2}\right]_{\Delta+\frac{1}{2}}$ retains positive norm, so this multiplet is unitary. When $0 < \Delta < 1/2$, the norm of $\left[\frac{1}{2}\right]_{\Delta+\frac{1}{2}}$ is still positive, while that of $[0]_{\Delta+1}$ becomes negative. At $\Delta = 0$ we hit the second shortening condition where we have the unitary vacuum multiplet. When $\Delta < 0$, the state $\left[\frac{1}{2}\right]_{\Delta+\frac{1}{2}}$ goes to negative norm, whereas $[0]_{\Delta+1}$ turns back to positive. These norms are illustrated in figure 1.

3.4.2 $s \geq 1/2$

We now turn to the case where the superconformal primary has spin $s \geq 1/2$. Starting at the first level, there are two possible shortening conditions, corresponding to when one or the other of the two first level descendants in (3.15) go null. For the norm of $Q_a|\Delta\rangle^{aa_1\cdots a_{2s-1}}$,

Figure 1. Relative norms of the conformal primaries within the $\{0\}_\Delta$ multiplets, for the various values of Δ . Black is positive norm, red is negative norm, and green is zero norm. We see the unitarity bound $\Delta \geq 1/2$. All the cases below this bound have a negative norm conformal primary and are non-unitary, except for the vacuum multiplet at $\Delta = 0$.

we find

$$\begin{aligned}
 |Q_a|\Delta\rangle^{aa_1\cdots a_{2s-1}}|^2 &= bb_1\cdots b_{2s-1}\langle\Delta|(-S^b)Q_a|\Delta\rangle^{aa_1\cdots a_{2s-1}} \\
 &= -bb_1\cdots b_{2s-1}\langle\Delta|\{S^b, Q_a\}|\Delta\rangle^{aa_1\cdots a_{2s-1}} \\
 &= \frac{2s+1}{s}[\Delta - (s+1)]\delta_{b_1}^{(a_1}\cdots\delta_{b_{2s-1}}^{a_{2s-1})}. \tag{3.27}
 \end{aligned}$$

For the norm of $Q^{(a}|\Delta\rangle^{a_1\cdots a_{2s})}$ we find

$$\begin{aligned}
 \left|Q^{(a}|\Delta\rangle^{a_1\cdots a_{2s})}\right|^2 &= (b_1\cdots b_{2s}\langle\Delta|S_b)Q^{(a}|\Delta\rangle^{a_1\cdots a_{2s})} = (b_1\cdots b_{2s}\langle\Delta|\{S_b, Q^{(a}\}|\Delta\rangle^{a_1\cdots a_{2s})} \\
 &= 2(\Delta + s)\delta_b^{(a}\delta_{b_1}^{a_1}\cdots\delta_{b_{2s}}^{a_{2s})}. \tag{3.28}
 \end{aligned}$$

At the second level, it is useful to compute the Graham matrix between the states $|\Delta\rangle_1^{a_1\cdots a_{2s}}$ and $|\Delta\rangle_2^{a_1\cdots a_{2s}}$ defined in (3.16) and (3.17),

$$\begin{aligned}
 &b_1\cdots b_{2s}\begin{pmatrix} {}_1\langle\Delta|\Delta\rangle_1 & {}_1\langle\Delta|\Delta\rangle_2 \\ {}_2\langle\Delta|\Delta\rangle_1 & {}_2\langle\Delta|\Delta\rangle_2 \end{pmatrix}^{a_1\cdots a_{2s}} \\
 &= \begin{pmatrix} 16\left[\Delta\left(\Delta - \frac{1}{2}\right) - s(s+1)\right] & -8s(s+1) \\ -8s(s+1) & 8s(s+1)(\Delta - 1) \end{pmatrix} \delta_{b_1}^{(a_1}\cdots\delta_{b_{2s}}^{a_{2s})}. \tag{3.29}
 \end{aligned}$$

For $\Delta \neq 1$, this hermitian form is diagonalized by the conformal primary $|\tilde{\Delta}\rangle^{a_1\cdots a_{2s}}$ and the descendent $|\Delta\rangle_2^{a_1\cdots a_{2s}}$,

$$b_1\cdots b_{2s}\langle\tilde{\Delta}|\tilde{\Delta}\rangle^{a_1\cdots a_{2s}} = \frac{8(2\Delta - 1)(\Delta - s - 1)(\Delta + s)}{\Delta - 1}\delta_{b_1}^{(a_1}\cdots\delta_{b_{2s}}^{a_{2s})}, \tag{3.30}$$

$$b_1\cdots b_{2s}{}_2\langle\Delta|\Delta\rangle_2^{a_1\cdots a_{2s}} = 8s(s+1)(\Delta - 1)\delta_{b_1}^{(a_1}\cdots\delta_{b_{2s}}^{a_{2s})}, \tag{3.31}$$

$$b_1\cdots b_{2s}\langle\tilde{\Delta}|\Delta\rangle_2^{a_1\cdots a_{2s}} = 0. \tag{3.32}$$

We see from (3.30) that the second level conformal primary becomes null when $\Delta = s + 1$ and when $\Delta = -s$, which are the two values when the first level conformal primaries

become null, as we can see from (3.27), (3.28). There is thus a shortened multiplet when $\Delta = s + 1$ with the following structure,

$$\begin{array}{ccccc}
 & & [s]_{s+2} & & \\
 & \swarrow & & \searrow & \\
 \{s\}_{s+1} \text{ massless short multiplet:} & & \left[s - \frac{1}{2}\right]_{s+\frac{3}{2}} & & \left[s + \frac{1}{2}\right]_{s+\frac{3}{2}} \\
 & \swarrow & & \searrow & \\
 & & [s]_{s+1} & &
 \end{array} \quad (3.33)$$

This is the conserved tensor multiplet, whose dual consists of massless fields in AdS_4 . For example, $\{\frac{3}{2}\}_{\frac{5}{2}}$ is the $\mathcal{N} = 1$ supergravity multiplet on AdS_4 , consisting of a massless spin 3/2 and a massless graviton.

The shortened multiplet when $\Delta = -s$ has the following structure,

$$\begin{array}{ccccc}
 & & [s]_{-s+1} & & \\
 & \swarrow & & \searrow & \\
 \{s\}_{-s} \text{ finite short multiplet:} & & \left[s - \frac{1}{2}\right]_{-s+\frac{1}{2}} & & \left[s + \frac{1}{2}\right]_{-s+\frac{1}{2}} \\
 & \swarrow & & \searrow & \\
 & & [s]_{-s} & &
 \end{array} \quad (3.34)$$

There are conformal multiplets which themselves have null descends which when factored out make the Verma module finite dimensional. This is the type I shortening in the language of [57], which occurs for the conformal representations $[s]_{-s-k}$, $k = 0, 1, 2, \dots$. These finite multiplets have dimension $\Delta < 3/2$ and so occur for alternately quantized fields which have a finite number of modes on AdS [55, 68]. The superconformal multiplets $\{s\}_{-s-1-k}$, $k = 0, 1, 2, \dots$ are long SUSY multiplets which link together these finite dimensional multiplets. These are thus finite dimensional supermultiplets, which can be thought of as supersymmetric spherical harmonics on AdS_4 . The short multiplets (3.34) are the boundary case, linking together two finite dimensional conformal multiplets on the boundary of the region of finite dimensional multiplets, thus they can be thought of as short supersymmetric spherical harmonics on AdS_4 .

We see also from (3.30) that the second level conformal primary becomes null when $\Delta = 1/2$. At this value, neither of the two first level conformal primaries are null, so we have the shortened multiplet:

$$\begin{array}{ccc}
 & [s]_{\frac{3}{2}} & \\
 & \swarrow \quad \searrow & \\
 \{s\}_{\frac{1}{2}} \text{ PM shadow short multiplet:} & \left[s - \frac{1}{2}\right]_1 & \left[s + \frac{1}{2}\right]_1 \\
 & \swarrow \quad \searrow & \\
 & [s]_{\frac{1}{2}} &
 \end{array} \quad (3.35)$$

This multiplet also involves only $\Delta < 3/2$ and so corresponds to alternately quantized fields. In fact, the fields here are all alternately quantized partially massless fields of depth $t = 0, 1/2$ and so these are the shadow operators of multiply conserved currents. These shadow operators themselves have shortening conditions corresponding to the vanishing of the partially massless field strengths [45] (this is the type IV shortening in the appendix of [57]). The short multiplet (3.35) sits at the boundary of the region of multiplets of partially massless shadow operators.

Finally, we must treat the case $\Delta = 1$. In this case, there is a degeneration because the conformal primary $|\tilde{\Delta}\rangle^{a_1 \dots a_{2s}}$ degenerates with the descendent $|\Delta\rangle_2^{a_1 \dots a_{2s}}$, and this state has zero norm: $b_1 \dots b_{2s} 2 \langle \Delta | \Delta \rangle_2^{a_1 \dots a_{2s}} = 0$. However, as we can tell from the Graham matrix (3.29), this state is not null because it does not have zero inner product with the other states, and so it does not decouple. In fact there is a second null state $|\Delta_N\rangle^{a_1 \dots a_{2s}}$, linearly independent of $|\Delta\rangle_2^{a_1 \dots a_{2s}}$, which is not orthogonal to $|\Delta\rangle_2^{a_1 \dots a_{2s}}$, and is neither primary nor descendent,

$$|\Delta_N\rangle^{a_1 \dots a_{2s}} \equiv |\Delta\rangle_1^{a_1 \dots a_{2s}} - \frac{2s(s+1)-1}{2s(s+1)} |\Delta\rangle_2^{a_1 \dots a_{2s}}, \quad (3.36)$$

$$b_1 \dots b_{2s} \langle \Delta_N | \Delta_N \rangle^{a_1 \dots a_{2s}} = 0, \quad (3.37)$$

$$b_1 \dots b_{2s} \langle \Delta_N | \Delta \rangle_2^{a_1 \dots a_{2s}} = -8s(s+1) \delta_{b_1}^{(a_1} \dots \delta_{b_{2s}}^{a_{2s})}. \quad (3.38)$$

The Graham matrix is Lorentzian. Diagonalizing it would produce a state with positive norm and a state with negative norm, though these would not have nice actions under the superconformal generators. This phenomenon can only happen in the non-unitary case where some states have negative norms. The structure of the module is that of an “extended module” rather than a shortening. This is a supersymmetric version of the extended modules found in the Hilbert space of higher order free CFTs [21]. Note that this case bridges the $\Delta = 3/2$ divide between the alternately quantized fields and the standardly quantized fields.

Figure 2. Relative norms of the conformal primaries within the $\{s\}_\Delta$ multiplets with $s \geq 1/2$, for the various values of Δ . Black is positive norm, red is negative norm, and green is zero norm. We see the unitarity bound $\Delta \geq s + 1$; all the multiplets below this bound have at least one negative norm conformal primary and are thus non-unitary.

Finally, we can read off from (3.27), (3.28), (3.30) the relative norms for various values of Δ , which correspond to the signs of the kinetic energy terms expected in AdS_4 . These norms are illustrated in figure 2.

4 Partially massless multiplets

With the representations in hand, we can now look for those that contain the partially massless fields. In what follows, we will restrict to those multiplets which contain only $\Delta \geq 3/2$ states, so that we are dealing only with standardly quantized fields with a clean AdS_4 interpretation.

The partially massless multiplets are shown in figure 3. Other than the massless fields, which appear in the shortened multiplet (3.33), the multiplets with standard quantization containing partially massless fields are not short multiplets. The PM fields all appear in the generic long multiplet (3.19). For example, the simplest multiplet containing a partially

massless spin-2 is the following:

$$\begin{array}{ccccc}
 & & \left[\begin{array}{c} 3 \\ 2 \end{array} \right]_{\frac{3}{2}} & \text{massless } 3/2 & \\
 & \swarrow & & \searrow & \\
 \left\{ \begin{array}{c} 3 \\ 2 \end{array} \right\}_{\frac{3}{2}} & \text{PM graviton multiplet:} & [1]_2 & \text{massless photon} & [2]_2 \quad \text{PM spin 2} \\
 & & \swarrow & & \searrow \\
 & & \left[\begin{array}{c} 3 \\ 2 \end{array} \right]_{\frac{3}{2}} & \text{massive } 3/2 & \\
 & & & m^2 L^2 = -1 &
 \end{array} \tag{4.1}$$

The propagating bosonic and fermionic degrees of freedom of the corresponding AdS_4 fields match: the PM spin-2 has 4 degrees of freedom and the photon has 2, whereas the massive spin-3/2 has 4 degrees of freedom and the massless spin-3/2 has 2. The general pattern of PM field multiplets can be visualized from figure 3. In all the multiplets, it can be checked explicitly that the number of bosonic and fermionic propagating degrees of freedom in AdS_4 match.

The relative norms of the states within the partially massless multiplets can be read from figure 2. The massless multiplets are unitary. The partially massless multiplets we are considering all lie in the range $1 < \Delta < s + 1$, so as indicated in the figure, the states $[s - \frac{1}{2}]_{\Delta + \frac{1}{2}}$ and $[s]_{\Delta + 1}$ have negative norm, whereas $[s + \frac{1}{2}]_{\Delta + \frac{1}{2}}$ and the superconformal primary $[s]_{\Delta}$ have positive norm. Thus on AdS_4 , the two bosonic fields within the multiplet will have opposite sign kinetic terms, and the two fermionic fields will have opposite signs. We will see in section 5.3 how these opposite signs are necessary in order to be consistent with the reality properties of the fields and transformations on AdS_4 .

4.1 Branching rules

In the partially massless multiplets, the partially massless gauge parameters for all the fields themselves form a supermultiplet. For example, for the partially massless spin-2 multiplet in (4.1), the gauge parameters form the following scalar multiplet:

$$\begin{array}{ccc}
 & & [0]_4 \quad \begin{array}{l} \text{PM gauge parameter} \\ m^2 L^2 = 4 \end{array} \\
 & \swarrow & \\
 \{0\}_3 & \text{PM graviton gauge} & \left[\begin{array}{c} 1 \\ 2 \end{array} \right]_{\frac{7}{2}} \quad \begin{array}{l} \text{massless spin } 3/2 \text{ gauge parameter} \\ m^2 L^2 = 4 \end{array} \\
 \text{parameter multiplet:} & & \\
 & \searrow & \\
 & & [0]_3 \quad \begin{array}{l} \text{photon gauge parameter} \\ m^2 = 0 \end{array}
 \end{array} \tag{4.2}$$

Figure 3. The $\mathcal{N} = 1$ partially massless multiplets. The filled circles are bosons, the hollow circles are fermions. The blue circles are the partially massless points, the grey open circles on the bottom row at $\Delta = 3/2$ are massive fermions that participate in some of the multiplets. The multiplets are all diamonds containing four fields, like the example diamonds illustrated which show the $\left\{\frac{3}{2}\right\}_{\frac{3}{2}}$ partially massless spin-2 representation (4.1) as well as the $\{4\}_3$ representation which contains only partially massless fields. The diamonds degenerate at the top due to the shortening condition (3.33). These are the familiar unitary massless multiplets, and in these short multiplets there are only two massless particles, like the $\{4\}_5$ example shown.

As the dimension Δ of a generic massive graviton multiplet approaches the value $\Delta = 3/2$ of the partially massless multiplet, conformal descendants of the spin-2, spin-1 and spin 3/2 conformal primaries become null. On the AdS_4 side, the fields are becoming massless and partially massless, and so they are developing gauge symmetries. The gauge parameters corresponding to the null descendent states can be thought of as Stückelberg fields which are decoupling in the limit (see [69] for a recent discussion of this in context of the massless and partially massless spin 2).

This describes a branching rule: as the generic massive multiplet approaches one of the partially massless values, it breaks up into the partially massless multiplet plus the multiplet corresponding to the emerging gauge parameters,

$$\left\{\frac{3}{2}\right\}_{\Delta} \xrightarrow{\Delta \rightarrow \frac{3}{2}} \left\{\frac{3}{2}\right\}_{\frac{3}{2}} \oplus \{0\}_3. \quad (4.3)$$

In terms of all the fields, the branching rule is as follows,

$$(4.4)$$

As the partially massless point $\Delta = 3/2$ is approached, the massive $\left\{\frac{3}{2}\right\}_{\Delta}$ representation shortens into the $\left\{\frac{3}{2}\right\}_{\frac{3}{2}}$ representation, and the null states split off into the massive $\{0\}_3$ representation. Within this massive $\{0\}_3$ representation, the $[0]_4$ is the longitudinal mode of the $[2]_{\Delta+\frac{1}{2}}$ spin-2 which becomes partially massless, the $[0]_3$ is the longitudinal mode of the $[1]_{\Delta+\frac{1}{2}}$ spin-1 which becomes massless, and the $\left[\frac{1}{2}\right]_{\frac{7}{2}}$ is the longitudinal mode of the $\left[\frac{3}{2}\right]_{\Delta+\frac{1}{2}}$ spin-3/2 which becomes massless.

There is also a branching rule in the massless limit $\Delta \rightarrow 5/2$,

$$\left\{\frac{3}{2}\right\}_{\Delta} \xrightarrow{\Delta \rightarrow \frac{5}{2}} \left\{\frac{3}{2}\right\}_{\frac{5}{2}} \oplus \{1\}_3. \quad (4.5)$$

In terms of the fields, this branching rule is as follows,

$$(4.6)$$

As the massless point $\Delta = 5/2$ is approached, the massive $\left\{\frac{3}{2}\right\}_{\Delta}$ representation shortens into the $\left\{\frac{3}{2}\right\}_{\frac{5}{2}}$ representation, and the null states split off into the massive $\{1\}_3$ representation. Within this massive $\{1\}_3$ representation, the $[1]_4$ is the longitudinal mode of the

massive spin-2 which becomes massless, the $[\frac{1}{2}]_{\frac{7}{2}}$ is the longitudinal mode of the massive spin-3/2 which becomes massless, and the $[1]_3$ and $[\frac{3}{2}]_{\frac{7}{2}}$ are the two conformal primaries of the $\{\frac{3}{2}\}_{\Delta}$ which go null. In section 5.2, we will see that in the AdS_4 action there will be fields which capture these null states, in addition to the Stückelberg fields for the emerging gauge parameters, and all these fields together will split off into their own massive multiplet.

The general partially massless branching rule is

$$\{s\}_{\Delta} \xrightarrow{\Delta \rightarrow t+2} \{s\}_{t+2} \oplus \left\{ t + \frac{1}{2} \right\}_{s+\frac{3}{2}}. \quad (4.7)$$

Here s, t are the spin and depth of the PM point in figure 3 corresponding to the superconformal primary (with $t = -\frac{1}{2}$ corresponding to the cases where the superconformal primary is one of the $\Delta = 3/2$ fermions). In the PM multiplets without massless fields, the gauge parameter multiplet contains precisely the gauge parameters for the partially massless fields. The gauge parameter is always the reflection of the field about the line $\Delta = s+2$ in the s, Δ plane depicted in figure 3, and so the gauge parameter multiplet is the reflection of the PM multiplet. In the massless multiplets, the Stückelberg multiplet contains, in addition to the gauge parameters for the two massless fields, Stückelberg fields for the two conformal primaries in the massless multiplet (3.33) which are going null. Note that in all cases, the gauge parameter multiplets are always unitary un-shortened multiplets.

5 AdS_4 action and SUSY transformations

In this section we construct the explicit action and SUSY transformation rules for the partially massless multiplet shown in (4.1). This is the simplest multiplet that includes a partially massless spin-2, as all others containing a partially massless spin-2 will also contain higher-spin particles. For the sake of generality, and to study how the degrees of freedom behave as one approaches the partially massless limit, it is useful to begin by constructing the full AdS_4 massive gravity multiplet with spins $(2, 3/2, 3/2, 1)$ and a generic graviton mass, described by the generic multiplet (3.19) with $s = 3/2$.³

The free action is simply given by the sum of the free actions of a massive graviton $h_{\mu\nu}$, two massive Majorana gravitini Ψ_{μ} and Φ_{μ} , and a massive pseudo-vector⁴ B_{μ} ,

$$\begin{aligned} S = \int d^4x \sqrt{-g} & \left[-\frac{1}{2} \nabla^{\rho} h^{\mu\nu} \nabla_{\rho} h_{\mu\nu} + \nabla_{\rho} h^{\mu\nu} \nabla_{\mu} h^{\rho}_{\nu} - \nabla_{\mu} h \nabla_{\nu} h^{\mu\nu} + \frac{1}{2} \nabla^{\mu} h \nabla_{\mu} h \right. \\ & - \frac{3}{L^2} \left(h^{\mu\nu} h_{\mu\nu} - \frac{1}{2} h^2 \right) - \frac{m_2^2}{2} (h^{\mu\nu} h_{\mu\nu} - h^2) \\ & - \frac{1}{2} \bar{\Psi}_{\mu} \gamma^{\mu\nu\rho} \nabla_{\nu} \Psi_{\rho} + \frac{\mu}{2} \bar{\Psi}_{\mu} \gamma^{\mu\nu} \Psi_{\nu} - \frac{1}{2} \bar{\Phi}_{\mu} \gamma^{\mu\nu\rho} \nabla_{\nu} \Phi_{\rho} + \frac{\mu'}{2} \bar{\Phi}_{\mu} \gamma^{\mu\nu} \Phi_{\nu} \\ & \left. - \frac{1}{4} G^{\mu\nu} G_{\mu\nu} - \frac{m_1^2}{2} B^{\mu} B_{\mu} \right]. \end{aligned} \quad (5.1)$$

³The flat-space version of this super-multiplet was studied in [70].

⁴That B_{μ} must be a pseudo-vector for the fields to make a super-multiplet can be seen by analyzing the massless limit, in which the vector's longitudinal mode becomes the pseudo-scalar of a Wess-Zumino multiplet. We will make this explicit below.

Here $g_{\mu\nu}$ is the metric of the AdS_4 background space of radius L , and covariant derivatives and gamma matrices are defined with respect to this background (see appendix B for our conventions on spinors and covariant derivatives in AdS_4). We use the notations $h \equiv g^{\mu\nu}h_{\mu\nu}$ and $G_{\mu\nu} \equiv \nabla_\mu B_\nu - \nabla_\nu B_\mu$. There are four mass scales appearing in this action: m_1 , m_2 , μ and μ' . The scales m_1 and m_2 correspond to the vector and graviton masses. The physical gravitino masses are related to the scales μ and μ' via

$$m_{3/2}^2 = \mu^2 - \frac{1}{L^2}, \quad m_{3/2}'^2 = \mu'^2 - \frac{1}{L^2}. \quad (5.2)$$

We determine the SUSY transformation rules by writing down a fully generic ansatz for the symmetry transformation depending on a single infinitesimal Majorana fermion parameter ϵ and demanding invariance of the action. The SUSY parameter ϵ must be an AdS Killing spinor satisfying

$$\nabla_\mu \epsilon = \frac{1}{2L} \gamma_\mu \epsilon. \quad (5.3)$$

This procedure fixes the vector and gravitino masses in terms of m_2 and the AdS radius as

$$m_1^2 = m_2^2 + \frac{2}{L^2}, \quad \mu = \frac{1}{2L} - \sqrt{m_2^2 + \frac{9}{4L^2}}, \quad \mu' = \frac{1}{2L} + \sqrt{m_2^2 + \frac{9}{4L^2}}. \quad (5.4)$$

Note that $\mu' = -\mu$ in the flat space limit, yielding a Dirac-type mass term for the gravitini as required by the R -symmetry of Poincaré SUSY [71]. Notice also that, given that the theory is symmetric upon interchanging the two gravitini and their masses, we are free to exchange the above solutions for μ and μ' ; we settle this ambiguity by choosing the field Ψ_μ to be the super-partner of the graviton in the limit $m_2 \rightarrow 0$ when Ψ_μ is also massless according to (5.2).

On the other hand, the SUSY transformation is not uniquely determined solely from the invariance of the free action. This is not surprising as free theories are not subject to the Haag–Łopuzsański–Sohnius theorem [72], and hence they may exhibit fermionic symmetries that are not supersymmetries. To find the true SUSY transformation law we must further impose the supersymmetry algebra, that is, we demand that two symmetry transformations should close to an AdS isometry,⁵

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \mathcal{L}_\xi. \quad (5.5)$$

Here $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the AdS Killing vector ξ^μ , itself a function of the Majorana parameters $\epsilon_{1,2}$ of the symmetry transformations. This requirement fixes

⁵As usual in on-shell formulations of SUSY, when applied to the spinors the algebra closes only upon use of the equations of motion.

all the remaining coefficients in our general ansatz for the symmetry. The final result is

$$\begin{aligned}
 \delta_\epsilon h_{\mu\nu} &= \left[\bar{\Psi}_{(\mu} \gamma_{\nu)} + a_1 \bar{\Phi}_{(\mu} \gamma_{\nu)} \right] \epsilon + 2L \nabla_{(\mu} \left[a_2 \bar{\Psi}_{\nu)} \epsilon + a_3 \bar{\Phi}_{\nu)} \epsilon \right], \\
 \delta_\epsilon \Psi_\mu &= \left[\gamma^{\rho\sigma} \nabla_\rho h_{\sigma\mu} + b_1 G_{\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu i\gamma_5 + b_2 \gamma^\nu i\gamma_5 (\nabla_\mu B_\nu + \nabla_\nu B_\mu) \right] \epsilon \\
 &\quad + L \nabla_\mu \left[b_3 G_{\nu\rho} \gamma^{\nu\rho} i\gamma_5 \epsilon \right] + \frac{1}{L} \left[b_4 h_{\mu\nu} \gamma^\nu + b_5 B_\mu i\gamma_5 + b_6 B^\nu \gamma_{\mu\nu} i\gamma_5 \right] \epsilon, \\
 \delta_\epsilon \Phi_\mu &= \left[a_1 \gamma^{\rho\sigma} \nabla_\rho h_{\sigma\mu} + c_1 G_{\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu i\gamma_5 + c_2 \gamma^\nu i\gamma_5 (\nabla_\mu B_\nu + \nabla_\nu B_\mu) \right] \epsilon \\
 &\quad + L \nabla_\mu \left[c_3 G_{\nu\rho} \gamma^{\nu\rho} i\gamma_5 \epsilon \right] + \frac{1}{L} \left[c_4 h_{\mu\nu} \gamma^\nu + c_5 B_\mu i\gamma_5 + c_6 B^\nu \gamma_{\mu\nu} i\gamma_5 \right] \epsilon, \\
 \delta_\epsilon B_\mu &= \left[d_1 \bar{\Psi}_\mu i\gamma_5 + d_2 \bar{\Phi}_\mu i\gamma_5 \right] \epsilon.
 \end{aligned} \tag{5.6}$$

The coefficients a , b , c and d that appear in these expressions are given explicitly by

$$\begin{aligned}
 a_1 &= \frac{\sqrt{9+4x^2}-3}{2x}, & a_2 &= \frac{\sqrt{9+4x^2}-3}{4x^2}, & a_3 &= -\frac{1}{2x}, \\
 b_1 &= -\frac{\sqrt{9+4x^2}-4}{4\sqrt{6}x}, & b_2 &= -\frac{\sqrt{9+4x^2}-1}{2\sqrt{6}x}, & b_3 &= -\frac{1}{2\sqrt{6}x}, \\
 b_4 &= \frac{\sqrt{9+4x^2}-1}{2}, & b_5 &= \frac{3\sqrt{9+4x^2}-(7+2x^2)}{\sqrt{6}x}, & b_6 &= -\frac{\sqrt{9+4x^2}-(3+x^2)}{\sqrt{6}x}, \\
 c_1 &= \frac{\sqrt{9+4x^2}-(3-4x^2)}{8\sqrt{6}x^2}, & c_2 &= -\frac{\sqrt{9+4x^2}-(3+2x^2)}{2\sqrt{6}x^2}, & c_3 &= -\frac{\sqrt{9+4x^2}-3}{4\sqrt{6}x^2}, \\
 c_4 &= \frac{\sqrt{9+4x^2}-(3+2x^2)}{2x}, & c_5 &= \frac{(1-x^2)\sqrt{9+4x^2}-3(1+x^2)}{\sqrt{6}x^2}, & c_6 &= \frac{\sqrt{9+4x^2}+1}{2\sqrt{6}}, \\
 d_1 &= -\sqrt{\frac{3}{2}} \frac{\sqrt{9+4x^2}-3}{2x}, & d_2 &= \sqrt{\frac{3}{2}},
 \end{aligned}$$

where $x \equiv m_2 L$.

With the normalization chosen in (5.6) for the symmetry parameter ϵ , the SUSY algebra is realized as in (5.5) with

$$\xi^\mu(\epsilon_1, \epsilon_2) = \frac{\sqrt{9+4L^2m_2^2} \left(3 - \sqrt{9+4L^2m_2^2} \right)}{L^2m_2^2} (\bar{\epsilon}_1 \gamma^\mu \epsilon_2). \tag{5.7}$$

That ξ^μ is an AdS Killing vector follows from the fact that $\epsilon_{1,2}$ are Killing spinors satisfying (5.3).

5.1 Flat limit

Before considering the partially massless case it is instructive to first look at the special limits $L \rightarrow \infty$ and $m_2 \rightarrow 0$ which, in addition to being physically interesting, are simpler to understand. The flat limit is straightforward as nothing special occurs to the degrees of freedom and no divergences happen at the level of the SUSY transformation. When

$L \rightarrow \infty$ eq. (5.6) becomes

$$\begin{aligned}
 \delta_\epsilon h_{\mu\nu} &= [\bar{\Psi}_{(\mu} \gamma_{\nu)} + \bar{\Phi}_{(\mu} \gamma_{\nu)}] \epsilon + \frac{1}{m_2} \partial_{(\mu} [\bar{\Psi}_{\nu)} - \bar{\Phi}_{\nu)}] \epsilon, \\
 \delta_\epsilon \Psi_\mu &= \left[\gamma^{\rho\sigma} \partial_\rho h_{\sigma\mu} - \frac{1}{2\sqrt{6}} G_{\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu i\gamma_5 - \frac{1}{\sqrt{6}} \gamma^\nu i\gamma_5 (\partial_\mu B_\nu + \partial_\nu B_\mu) \right] \epsilon \\
 &\quad - \frac{1}{2\sqrt{6} m_2} \partial_\mu G_{\nu\rho} \gamma^{\nu\rho} i\gamma_5 \epsilon + m_2 \left[h_{\mu\nu} \gamma^\nu - \frac{2}{\sqrt{6}} B_\mu i\gamma_5 + \frac{1}{\sqrt{6}} B^\nu \gamma_{\mu\nu} i\gamma_5 \right] \epsilon, \\
 \delta_\epsilon \Phi_\mu &= \left[\gamma^{\rho\sigma} \partial_\rho h_{\sigma\mu} + \frac{1}{2\sqrt{6}} G_{\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu i\gamma_5 + \frac{1}{\sqrt{6}} \gamma^\nu i\gamma_5 (\partial_\mu B_\nu + \partial_\nu B_\mu) \right] \epsilon \\
 &\quad - \frac{1}{2\sqrt{6} m_2} \partial_\mu G_{\nu\rho} \gamma^{\nu\rho} i\gamma_5 \epsilon - m_2 \left[h_{\mu\nu} \gamma^\nu + \frac{2}{\sqrt{6}} B_\mu i\gamma_5 - \frac{1}{\sqrt{6}} B^\nu \gamma_{\mu\nu} i\gamma_5 \right] \epsilon, \\
 \delta_\epsilon B_\mu &= -\sqrt{\frac{3}{2}} [\bar{\Psi}_\mu i\gamma_5 - \bar{\Phi}_\mu i\gamma_5] \epsilon.
 \end{aligned} \tag{5.8}$$

This is the unitary gauge version of the supersymmetry of the flat-space massive gravity multiplet constructed in [71] using the Stückelberg formulation. From (5.2) and (5.4) we see that the masses all become equal in this limit, as expected, while the Killing spinor condition (5.3) now simply implies that the SUSY parameter ϵ is a constant Majorana spinor.

5.2 Massless limit

More care is needed to treat the massless graviton limit since the symmetry rule (5.6) is singular at $m_2 = 0$. This is a manifestation of the fact that the number of degrees of freedom is discontinuous in this limit as both $h_{\mu\nu}$ and Ψ_μ become massless. We can handle this by means of a Stückelberg replacement, introducing a vector A_μ that restores the linearized diff invariance of a massless spin-2 on AdS_4 , and a Majorana spinor λ that restores the gauge symmetry of a massless spin-3/2 on AdS_4 ,

$$\begin{aligned}
 h_{\mu\nu} &\longrightarrow h_{\mu\nu} + \frac{1}{\sqrt{2} m_2} (\nabla_\mu A_\nu + \nabla_\nu A_\mu), \\
 \Psi_\mu &\longrightarrow \Psi_\mu + \frac{1}{m_2} \left(\nabla_\mu \lambda - \frac{1}{2L} \gamma_\mu \lambda \right).
 \end{aligned} \tag{5.9}$$

The prefactors are chosen so that the fields A_μ and λ become canonically normalized upon taking the limit $m_2 \rightarrow 0$. The final action reads

$$\begin{aligned}
 S_{(m_2=0)} &= \int d^4x \sqrt{-g} \left[-\frac{1}{2} \nabla^\rho h^{\mu\nu} \nabla_\rho h_{\mu\nu} + \nabla_\rho h^{\mu\nu} \nabla_\mu h^\rho{}_\nu - \nabla_\mu h \nabla_\nu h^{\mu\nu} + \frac{1}{2} \nabla^\mu h \nabla_\mu h \right. \\
 &\quad \left. - \frac{3}{L^2} \left(h^{\mu\nu} h_{\mu\nu} - \frac{h^2}{2} \right) \right. \\
 &\quad \left. - \frac{1}{2} \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \Psi_\rho - \frac{1}{2L} \bar{\Psi}_\mu \gamma^{\mu\nu} \Psi_\nu - \frac{1}{2} \bar{\Phi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \Phi_\rho + \frac{1}{L} \bar{\Phi}_\mu \gamma^{\mu\nu} \Phi_\nu \right. \\
 &\quad \left. - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{3}{L^2} A^\mu A_\mu - \frac{1}{4} G^{\mu\nu} G_{\mu\nu} - \frac{1}{L^2} B^\mu B_\mu - \frac{1}{2} \bar{\lambda} \gamma^\mu \nabla_\mu \lambda + \frac{1}{L} \bar{\lambda} \lambda \right],
 \end{aligned} \tag{5.10}$$

where $F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu$. This action has the expected gauge symmetries,

$$\delta_\xi h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \quad \delta_\eta \Psi_\mu = \nabla_\mu \eta - \frac{1}{2L} \gamma_\mu \eta, \tag{5.11}$$

of massless spin-2 and massless spin-3/2 fields in AdS_4 , which now form an independent massless short super-multiplet $(2, 3/2)$, as depicted in figure (3.33) with $s = 3/2$.

The rest of the fields have masses given by

$$m_\Phi^2 = \frac{3}{L^2}, \quad m_A^2 = \frac{6}{L^2}, \quad m_B^2 = \frac{2}{L^2}, \quad m_\lambda^2 = \frac{4}{L^2}, \quad (5.12)$$

and make their own massive AdS gravitino multiplet $\{1\}_3$, with the massive spins $(3/2, 1, 1, 1/2)$. One can check this by comparing with the mass spectrum of a generic gravitino multiplet, which we describe in appendix C.2. We can confirm this by performing the Stückelberg replacement (5.9) in the full SUSY transformation (5.6), and then take the limit $m_2 \rightarrow 0$. One then finds that the two sets of fields, $(h_{\mu\nu}, \Psi_\mu)$ and $(\Phi_\mu, A_\mu, B_\mu, \lambda)$, indeed transform independently of each other under supersymmetry:

$$\begin{aligned} \delta_\epsilon h_{\mu\nu} &= \bar{\Psi}_{(\mu} \gamma_{\nu)} \epsilon, \\ \delta_\epsilon \Psi_\mu &= \gamma^{\rho\sigma} \nabla_\rho h_{\sigma\mu} \epsilon + \frac{1}{L} h_{\mu\nu} \gamma^\nu \epsilon, \end{aligned} \quad (5.13)$$

$$\begin{aligned} \delta_{\epsilon'} \Phi_\mu &= \left[-\frac{1}{12} F_{\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu - \frac{2}{3} \gamma^\nu (\nabla_\mu A_\nu + \nabla_\nu A_\mu) \right] \epsilon' + \frac{L}{6} \nabla_\mu \left[F_{\nu\rho} \gamma^{\nu\rho} \epsilon' \right] - \frac{2}{3L} A^\nu \gamma_{\mu\nu} \epsilon', \\ &\quad + \left[\frac{7}{12\sqrt{3}} G_{\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu i\gamma_5 + \frac{2}{3\sqrt{3}} \gamma^\nu i\gamma_5 (\nabla_\mu B_\nu + \nabla_\nu B_\mu) \right] \epsilon' \\ &\quad - \frac{L}{6\sqrt{3}} \nabla_\mu \left[G_{\nu\rho} \gamma^{\nu\rho} i\gamma_5 \epsilon' \right] + \frac{1}{L} \left[-\frac{16}{3\sqrt{3}} B_\mu i\gamma_5 + \frac{2}{\sqrt{3}} B^\nu \gamma_{\mu\nu} i\gamma_5 \right] \epsilon', \\ \delta_{\epsilon'} A_\mu &= -\bar{\Phi}_\mu \epsilon' + \bar{\lambda} \gamma_\mu \epsilon' + \frac{L}{3} \nabla_\mu [\bar{\lambda} \epsilon'], \\ \delta_{\epsilon'} B_\mu &= \sqrt{3} \bar{\Phi}_\mu i\gamma_5 \epsilon' - \frac{1}{\sqrt{3}} \bar{\lambda} \gamma_\mu i\gamma_5 \epsilon' - \frac{L}{\sqrt{3}} \nabla_\mu [\bar{\lambda} i\gamma_5 \epsilon'], \\ \delta_{\epsilon'} \lambda &= \frac{1}{2} \left[F_{\mu\nu} \gamma^{\mu\nu} + \frac{4}{L} A^\mu \gamma_\mu \right] \epsilon' - \frac{1}{2\sqrt{3}} \left[G_{\mu\nu} \gamma^{\mu\nu} i\gamma_5 + \frac{4}{L} B^\mu \gamma_\mu i\gamma_5 \right] \epsilon'. \end{aligned} \quad (5.14)$$

We have written the Majorana parameter as ϵ' in the transformation of the gravitino multiplet to emphasize that it is now independent from that of the massless graviton multiplet. Comparing eqs. (5.14) and (C.6) one can verify that they indeed match for the value $m_{3/2}^2 = 3/L^2$ of the gravitino mass quoted above, thus confirming that it is an AdS SUSY multiplet.

In terms of the SUSY representations, what is described here is the branching rule $\left\{ \frac{3}{2} \right\}_\Delta \xrightarrow{\Delta \rightarrow \frac{5}{2}} \left\{ \frac{3}{2} \right\}_{\frac{5}{2}} \oplus \{1\}_3$, as described in section 4.1.

5.3 Partially massless limit

We now examine the massive gravity multiplet when the spin-2 is partially massless, by taking the mass $m_2^2 = -2/L^2$. From (5.4) we get the other masses,

$$m_{3/2}^2 = -\frac{1}{L^2}, \quad m_{3/2}^{\prime 2} = 0, \quad m_1^2 = 0, \quad (5.15)$$

and hence the vector and one of the gravitini are massless at the PM point. This implies that if we take the PM mass value as a limit starting with the generic theory then the number degrees of freedom will be discontinuous, just like in the massless case considered

above. However, what is different here is that the number of bosonic and fermionic degrees of freedom would still match if one were to ignore the Stückelberg fields. This means that it is consistent, from the point of view of the supersymmetry, to directly set the PM value $m_2^2 = -2/L^2$ and ignore the lost degrees of freedom (which are therefore expected to form their own independent SUSY multiplet, as we will check below).

This is also confirmed by noting that the coefficients of the full SUSY transformation law, eq. (5.6), are all finite at the PM point. However, because of the appearance of $m_2 = i\sqrt{2}/L$, some coefficients turn out to be imaginary. This manifests the fact that some of the fields in the multiplet carry negative norm, whereas in writing the original action as in (5.1) we implicitly assumed that all norms were positive. The relative signs of the norms can be inferred by looking back at eqs. (3.27), (3.28), (3.30) and (3.31), where we take $s = 3/2$ and $\Delta = 3/2$ for the PM multiplet. The field that corresponds to the conformal primary is Ψ_μ , which we take to have positive norm at the level of the action, whose overall sign is irrelevant. It follows that B_μ and Φ_μ must have negative norm, while $h_{\mu\nu}$ must have positive norm. We therefore perform the replacement $B_\mu \rightarrow iB_\mu$ and $\Phi_\mu \rightarrow i\Phi_\mu$ both in the action and in the SUSY transformation. For the final PM spin-2 multiplet action we get

$$\begin{aligned}
 S_{\text{PM}} = \int d^4x \sqrt{-g} & \left[-\frac{1}{2} \nabla^\rho h^{\mu\nu} \nabla_\rho h_{\mu\nu} + \nabla_\rho h^{\mu\nu} \nabla_\mu h^\rho{}_\nu - \nabla_\mu h \nabla_\nu h^{\mu\nu} + \frac{1}{2} \nabla^\mu h \nabla_\mu h \right. \\
 & - \frac{2}{L^2} \left(h^{\mu\nu} h_{\mu\nu} - \frac{1}{4} h^2 \right) \\
 & \left. + \frac{1}{4} G^{\mu\nu} G_{\mu\nu} - \frac{1}{2} \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \Psi_\rho + \frac{1}{2} \bar{\Phi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \Phi_\rho - \frac{1}{2L} \bar{\Phi}_\mu \gamma^{\mu\nu} \Phi_\nu \right].
 \end{aligned} \tag{5.16}$$

The theory is invariant under the gauge symmetries

$$\begin{aligned}
 \delta_\xi h_{\mu\nu} &= \nabla_\mu \nabla_\nu \xi - \frac{1}{L^2} g_{\mu\nu} \xi, \\
 \delta_\eta \Phi_\mu &= \nabla_\mu \eta + \frac{1}{2L} \gamma_\mu \eta, \\
 \delta_\alpha B_\mu &= \nabla_\mu \alpha,
 \end{aligned} \tag{5.17}$$

and under the supersymmetry

$$\begin{aligned}
 \delta_\epsilon h_{\mu\nu} &= \left[\bar{\Psi}_{(\mu} \gamma_{\nu)} + \frac{1}{\sqrt{2}} \bar{\Phi}_{(\mu} \gamma_{\nu)} \right] \epsilon + L \nabla_{(\mu} \left[\frac{1}{2} \bar{\Psi}_{\nu)} \epsilon + \frac{1}{\sqrt{2}} \bar{\Phi}_{\nu)} \epsilon \right], \\
 \delta_\epsilon \Psi_\mu &= \left[\gamma^{\rho\sigma} \nabla_\rho h_{\sigma\mu} - \frac{\sqrt{3}}{8} G_{\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu i\gamma_5 \right] \epsilon + L \nabla_\mu \left[\frac{1}{4\sqrt{3}} G_{\nu\rho} \gamma^{\nu\rho} i\gamma_5 \epsilon \right], \\
 \delta_\epsilon \Phi_\mu &= \left[-\frac{1}{\sqrt{2}} \gamma^{\rho\sigma} \nabla_\rho h_{\sigma\mu} + \frac{5}{8\sqrt{6}} G_{\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu i\gamma_5 + \frac{1}{2\sqrt{6}} \gamma^\nu i\gamma_5 (\nabla_\mu B_\nu + \nabla_\nu B_\mu) \right] \epsilon \\
 & \quad + L \nabla_\mu \left[-\frac{1}{4\sqrt{6}} G_{\nu\rho} \gamma^{\nu\rho} i\gamma_5 \epsilon \right] + \frac{1}{L} \left[\frac{1}{\sqrt{2}} h_{\mu\nu} \gamma^\nu - \frac{\sqrt{3}}{\sqrt{2}} B_\mu i\gamma_5 + \frac{1}{\sqrt{6}} B^\nu \gamma_{\mu\nu} i\gamma_5 \right] \epsilon, \\
 \delta_\epsilon B_\mu &= \left[\frac{\sqrt{3}}{2} \bar{\Psi}_\mu i\gamma_5 + \frac{\sqrt{3}}{\sqrt{2}} \bar{\Phi}_\mu i\gamma_5 \right] \epsilon.
 \end{aligned} \tag{5.18}$$

This SUSY transformation is now consistent with the reality properties of the fields (which can be seen explicitly for instance by using a “really real” representation where the gamma matrices and Majorana fields are all purely real, so that γ_5 is purely imaginary [73]).

We have thus succeeded in performing an explicit construction of the simplest super-multiplet containing a partially massless spin-2. The result is fully consistent with the group theory analysis of the previous section. From the action (5.16) we see that the relative signs of the norms agree with those of the conformal primary state and its descendants, while the final result for the SUSY transformation (5.18) confirms that no shortening occurs at the PM point. Finally, it is straightforward to also check that the mass spectrum agrees with the AdS/CFT relation (2.1) with the scaling dimensions as given in (4.1).

We remarked above that taking the PM limit starting from the generic massive multiplet does not require the introduction of Stückelberg fields as long as one is only interested in the PM multiplet and its symmetries. It is however instructive to anyway perform a Stückelberg replacement as a further consistency check, which will also serve us to justify the claim that the gauge parameters of the theory themselves form an AdS SUSY multiplet. Following the pattern in eq. (5.17) we do the replacements

$$\begin{aligned} h_{\mu\nu} &\longrightarrow h_{\mu\nu} + \frac{1}{\sqrt{3}} \frac{L^2}{\sqrt{2+L^2m_2^2}} \left(\nabla_\mu \nabla_\nu \varphi - \frac{1}{L^2} g_{\mu\nu} \varphi \right), \\ \Phi_\mu &\longrightarrow \Phi_\mu + \frac{1}{\sqrt{3}} \frac{L}{\sqrt{2+L^2m_2^2}} \left(\nabla_\mu \lambda + \frac{1}{2L} \gamma_\mu \lambda \right), \\ B_\mu &\longrightarrow B_\mu + \frac{L}{\sqrt{2+L^2m_2^2}} \nabla_\mu \pi. \end{aligned} \quad (5.19)$$

Again the prefactors are such that the Stückelbergs φ , λ and π come out canonically normalized after one takes the limit $m_2^2 \rightarrow -2/L^2$ from above. All three fields φ , π and λ have negative norm in this limit, as one can straightforwardly check by performing the replacement in the action. (Alternatively, one may approach the PM point from below and then all the Stückelbergs will have positive norm.) From the result one can read off the masses as

$$m_\lambda^2 = \frac{4}{L^2}, \quad m_\varphi^2 = \frac{4}{L^2}, \quad m_\pi^2 = 0. \quad (5.20)$$

Replacing the Stückelberg fields in the SUSY transformation one finds, in the limit $m_2^2 \rightarrow -2/L^2$, that they can indeed be isolated and satisfy an independent transformation rule,

$$\begin{aligned} \delta_\epsilon \varphi &= \bar{\lambda} \epsilon, & \delta_\epsilon \pi &= \bar{\lambda} i \gamma_5 \epsilon, \\ \delta_\epsilon \lambda &= \gamma^\mu (\nabla_\mu \varphi + i \gamma_5 \nabla_\mu \pi) \epsilon - \frac{1}{L} \varphi \epsilon - \frac{3}{L} \pi i \gamma_5 \epsilon. \end{aligned} \quad (5.21)$$

In appendix C we give the generic AdS Wess-Zumino multiplet. Comparison of (5.20) and (5.21) with (C.2) and (C.3) then verifies that the Stückelbergs φ , λ and π form a super-multiplet.

In terms of the SUSY representations, what is described here is the branching rule $\left\{ \frac{3}{2} \right\}_\Delta \xrightarrow{\Delta \rightarrow \frac{3}{2}} \left\{ \frac{3}{2} \right\}_{\frac{3}{2}} \oplus \{0\}_3$, as described in section 4.1. In the action, the Stückelberg fields

carry these null states, and hence they split off into their own massive multiplet. Note that, unlike in the massless limit, none of the conformal primaries themselves are becoming null, only descendants are. This is why there was no obstruction to directly taking the limit in the action.

6 Conclusions

We have studied non-unitary representations of the $\mathcal{N} = 1$ 3d superconformal algebra. These correspond to non-unitary $\mathcal{N} = 1$ supersymmetric multiplets on AdS_4 . Among these, we have identified the representations containing partially massless fields. In addition, we have found supersymmetric BPS-like shortening conditions that occur among the non-unitary representations. These shortening conditions have no analogue among the unitary representations, and include supersymmetric examples of exotic “extended modules.”

We have identified the simplest multiplet which contains the partially massless spin 2. This multiplet contains a massless vector, a massless spin 3/2 and a massive spin 3/2 in addition to the partially massless spin 2. We have written the free AdS_4 action and supersymmetry transformations for this multiplet, and studied how it arises from the partially massless limit of the full massive supermultiplet. We worked with a component formulation; it would also be interesting to see how these multiplet emerge from an AdS superspace approach [74, 75]. It should also emerge from conformal supergravity [76, 77] linearized around an AdS background.

There is still a possibility that an interacting theory of this partially massless SUSY multiplet exists. This would be a partially massless supergravity, and it would have a consistent truncation to a bosonic part consisting of an interacting partially massless spin-2 and a photon. This possibility is not yet ruled out by any of the no-go theorems against interacting partially massless spin-2 fields which are known so far, so it would be interesting to further study such a possibility.

We restricted our study of partially massless fields to those with standard quantization conditions, which meant that the new exotic non-unitary shortening conditions (3.34), (3.35) and the extended multiplet at $\Delta = 1$ did not play a role. It would be interesting to understand how these other multiplets play out in AdS_4 . Finally, it would be interesting to extend this study to $\mathcal{N} = 2$ and higher and classify the resulting partially massless multiplets of extended SUSY, as well as to go to higher dimensions where there are more possibilities including mixed symmetry partially massless representations.

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A $d = 3$ spinor conventions

Here we detail our conventions for $d = 3$ euclidean spinors used in the superconformal algebra and its representations. The space indices are $i, j, \dots$ which range over $1, 2, 3$. The metric is

$$\delta_{ij} = \text{diag}(1, 1, 1), \quad (\text{A.1})$$

and is used to raise and lower space indices. The fundamental spinor representation in $d = 3$ euclidean space is two dimensional and is irreducible and complex (there is no Majorana or Weyl condition). We use $a, b, c, \dots$ ranging over $1, 2$ for spinor indices.

The gamma matrices are the standard Pauli matrices σ^i ,

$$\sigma^1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \quad (\text{A.2})$$

As determined by their transformation laws, they come with the index structure

$$(\sigma^i)^a_b. \quad (\text{A.3})$$

They span the space $su(2)$ of traceless hermitian matrices. They satisfy the standard gamma matrix anti-commutation rule,

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}. \quad (\text{A.4})$$

The Lorentz generators are given by

$$M_{ij} = \frac{1}{2}\sigma_{ij}, \quad \sigma^{ij} \equiv \sigma^{[i}\sigma^{j]}, \quad (\text{A.5})$$

which satisfy

$$[M_{ij}, \sigma_k] = -\delta_{ik}\sigma_j + \delta_{jk}\sigma_i. \quad (\text{A.6})$$

We have the following relation,

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ij}_k\sigma^k, \quad (\text{A.7})$$

where ϵ_{ijk} is the $d = 3$ anti-symmetric symbol (with the convention $\epsilon_{123} = 1$). This implies

$$\sigma^{ij} = i\epsilon^{ij}_k\sigma^k, \quad (\text{A.8})$$

so the Lorentz generators are the same as the gamma matrices (this is the Lie algebra isomorphism $su(2) \simeq so(3)$).

Useful traces of the gammas are

$$\text{Tr} [\sigma^i\sigma^j] = 2\delta^{ij}, \quad (\text{A.9})$$

$$\text{Tr} [\sigma^i\sigma^j\sigma^k] = 2i\epsilon^{ijk}, \quad (\text{A.10})$$

and they satisfy a completeness relation

$$(\sigma^i)^a_b (\sigma_i)^c_d = 2\delta^a_d \delta^c_b - \delta^a_b \delta^c_d, \quad (\text{A.11})$$

$$(\sigma^{ij})^a_b (\sigma_{ij})^c_d = -4\delta^a_d \delta^c_b + 2\delta^a_b \delta^c_d. \quad (\text{A.12})$$

Indices on spinors are raised and lowered with the invariant tensor ϵ ,

$$\epsilon_{ab} = \epsilon^{ab} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad \epsilon^{ac} \epsilon_{bc} = \delta^a_b. \quad (\text{A.13})$$

For a spinor ψ we raise and lower spinor indices using the convention

$$\psi_a = \epsilon_{ab} \psi^b, \quad \psi^a = \psi_b \epsilon^{ba}. \quad (\text{A.14})$$

Note that we defined the ϵ 's so that we have the property $\epsilon^{ab} = \epsilon_{cd} \epsilon^{ca} \epsilon^{db}$ i.e. ϵ^{ab} is in fact the same as ϵ_{ab} with raised indices. We only have to be careful that δ has indices in the right position; the Kronecker delta is $\delta^a_b = \epsilon^a_b$, and the raised and lowered version is $\delta_b^a = -\delta^a_b$. With these rules the raising and lowering of indices on ϵ 's and δ 's is consistent. For any tensor T we have

$$T^a{}_a \dots = -T_a{}^a \dots. \quad (\text{A.15})$$

There are identities that follow from the fact that we are on a two dimensional space: any pair of anti-symmetric indices can be replaced by an ϵ ,

$$T^{[ab]} \dots = \frac{1}{2} \epsilon^{ab} T^c{}_{\dots c}, \quad (\text{A.16})$$

and the anti-symmetrization of any three indices vanishes,

$$T^{[abc]} \dots = 0. \quad (\text{A.17})$$

Raising an index on the gamma matrices (A.3), the result is symmetric:⁶

$$(\sigma^i)^{ab} = (\sigma^i)^a{}_c \epsilon^{cb}, \quad (\sigma^i)^{ab} = (\sigma^i)^{ba}. \quad (\text{A.19})$$

We have the complex conjugation properties

$$(\sigma_i)^a{}_b{}^* = (\sigma_i)^b{}_a, \quad (\sigma_i)^{ab}{}^* = -(\sigma_i)_{ab}, \quad (\sigma_i)_{ab}{}^* = -(\sigma_i)^{ab}, \quad (\sigma_i)_a{}^b{}^* = (\sigma_i)^a{}_b, \quad (\text{A.20})$$

$$(\sigma_{ij})^a{}_b{}^* = -(\sigma_{ij})^b{}_a, \quad (\sigma_{ij})^{ab}{}^* = (\sigma_{ij})_{ab}, \quad (\sigma_{ij})_{ab}{}^* = (\sigma_{ij})^{ab}, \quad (\sigma_{ij})_a{}^b{}^* = -(\sigma_{ij})^a{}_b. \quad (\text{A.21})$$

A.1 Representations

The irreducible representations of $SU(2)$ can be packaged as symmetric spinors,

$$T^{a_1 a_2 \dots a_{2s}}, \quad (\text{A.22})$$

where $s = 0, 1/2, 1, 3/2, \dots$ is a half integer which corresponds to the spin. These are the only irreducible representations; any lowered indices are equivalent to upper ones by raising with ϵ^{ab} , and any non-symmetric components can be reduced to fully symmetric ones by

⁶Explicitly:

$$\sigma^1 \epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^2 \epsilon = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \sigma^3 \epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (\text{A.18})$$

using (A.16) to remove non-symmetric parts. Note that there is no tracelessness condition imposed on these tensors, since there is no invariant trace operation.

The irreducible representations of $SO(3)$, on the other hand, are symmetric *traceless* tensors

$$T^{i_1 \dots i_s}, \quad (\text{A.23})$$

with $s = 0, 1, 2, \dots$ an integer. All other symmetry types can be reduced to this by using the epsilon symbol ϵ_{ijk} , and traces removed using δ_{ij} .

The $SO(3)$ tensor representations of rank s are equivalent to the integer spin $SU(2)$ spinor representations of rank $2s$. We can use the gamma matrices (A.19) to pass between them,

$$T^{a_1 b_1 \dots a_s b_s} = \sigma_{i_1}^{a_1 b_1} \dots \sigma_{i_s}^{a_s b_s} T^{i_1 \dots i_s}, \quad (\text{A.24})$$

$$T^{i_1 \dots i_s} = \frac{1}{2^s} \sigma_{a_1 b_1}^{i_1} \dots \sigma_{a_s b_s}^{i_s} T^{a_1 b_1 \dots a_s b_s}. \quad (\text{A.25})$$

The odd rank spinor representations have no tensor counterpart.

B $D = 4$ AdS spinor conventions

We follow the conventions of [73] for spinors in $D = 4$ AdS space. The covariant gamma matrices are given by $\gamma_\mu \equiv e^{\hat{\mu}}_\mu \gamma_{\hat{\mu}}$, where the $\gamma_{\hat{\mu}}$ are the standard constant gamma matrices and $e^{\hat{\mu}}_\mu$ is the background AdS vierbein. The anti-commutator is then

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad (\text{B.1})$$

with $g_{\mu\nu}$ the background AdS metric. We also define

$$\gamma^{\mu_1 \dots \mu_n} \equiv \gamma^{[\mu_1} \dots \gamma^{\mu_n]}, \quad (\text{B.2})$$

and the matrix $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$, which satisfies

$$\{\gamma^\mu, \gamma_5\} = 0, \quad (\gamma_5)^2 = 1. \quad (\text{B.3})$$

We use 4-component notation for spinors and omit the spinor indices throughout. The Majorana conjugate of a spinor χ is given by $\bar{\chi} = \chi^T C$, where the charge conjugation matrix C satisfies $C^T = C^\dagger = C^{-1} = -C$. The charge conjugate is defined as $\chi^C = B^{-1}\chi^*$, where the matrix B can be defined by the relation $(\gamma^\mu)^* = B\gamma^\mu B^{-1}$. A Majorana spinor then satisfies the constraint $\chi^C = \chi$. In four dimensions there exists a so-called “really real” representation in which $B = 1$, so that the gamma matrices are all real and the Majorana constraint simply states that the spinor is real.

When acting on spinors, the commutator of two AdS covariant derivatives yields an extra piece coming from the spin connection acting on the suppressed spinor index, for example

$$\begin{aligned} [\nabla_\mu, \nabla_\nu] \chi &= -\frac{1}{2L^2} \gamma_{\mu\nu} \chi, \\ [\nabla_\mu, \nabla_\nu] \Psi_\rho &= -\frac{2}{L^2} g_{\rho[\mu} \Psi_{\nu]} - \frac{1}{2L^2} \gamma_{\mu\nu} \Psi_\rho. \end{aligned} \quad (\text{B.4})$$

The same is true for the Lie derivative, for instance

$$\begin{aligned}\mathcal{L}_\xi \chi &= \xi^\mu \nabla_\mu \chi + \frac{1}{4} (\nabla_\mu \xi_\nu) \gamma^{\mu\nu} \chi, \\ \mathcal{L}_\xi \Psi_\mu &= \xi^\rho \nabla_\rho \Psi_\mu + \nabla_\mu \xi^\rho \Psi_\rho + \frac{1}{4} (\nabla_\rho \xi_\sigma) \gamma^{\rho\sigma} \Psi_\mu.\end{aligned}\tag{B.5}$$

C Other explicit AdS super-multiplets

C.1 AdS Wess–Zumino multiplet

The free AdS Wess–Zumino multiplet $\{0\}_\Delta$, containing massive spins $(1/2, 0, 0)$ [64, 78] has the action

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2} (\nabla \varphi)^2 - \frac{1}{2} (\nabla \pi)^2 - \frac{1}{2} \bar{\lambda} \gamma^\mu \nabla_\mu \lambda - \frac{m_0^2}{2} \varphi^2 - \frac{m_0'^2}{2} \pi^2 + \frac{m_{1/2}}{2} \bar{\lambda} \lambda \right], \tag{C.1}$$

with π a pseudo-scalar. The scalar masses are given in terms of the spinor mass as

$$m_0^2 = m_{1/2}^2 - \frac{m_{1/2}}{L} - \frac{2}{L^2}, \quad m_0'^2 = m_{1/2}^2 + \frac{m_{1/2}}{L} - \frac{2}{L^2}, \tag{C.2}$$

and the SUSY transformation rule reads

$$\begin{aligned}\delta_\epsilon \varphi &= \bar{\lambda} \epsilon, & \delta_\epsilon \pi &= \bar{\lambda} i \gamma_5 \epsilon, \\ \delta_\epsilon \lambda &= \gamma^\mu (\nabla_\mu \varphi + i \gamma_5 \nabla_\mu \pi) \epsilon + \left(m_{1/2} + \frac{1}{L} \right) \varphi \epsilon + \left(m_{1/2} - \frac{1}{L} \right) \pi i \gamma_5 \epsilon.\end{aligned}\tag{C.3}$$

C.2 AdS gravitino multiplet

The free AdS gravitino multiplet $\{1\}_\Delta$, containing massive spins $(3/2, 1, 1, 1/2)$, was investigated in [79] using a full Stückelberg formulation. Here we provide the result in unitary gauge. The action is

$$\begin{aligned}S &= \int d^4x \sqrt{-g} \left[-\frac{1}{2} \bar{\Phi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \Phi_\rho - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4} G^{\mu\nu} G_{\mu\nu} - \frac{1}{2} \bar{\lambda} \gamma^\mu \nabla_\mu \lambda \right. \\ &\quad \left. + \frac{\mu}{2} \bar{\Phi}_\mu \gamma^{\mu\nu} \Phi_\nu - \frac{m_1^2}{2} A^\mu A_\mu - \frac{m_1'^2}{2} B^\mu B_\mu + \frac{m_{1/2}}{2} \bar{\lambda} \lambda \right],\end{aligned}\tag{C.4}$$

where B_μ is a pseudo-vector and the physical gravitino mass is given by $m_{3/2}^2 = \mu^2 - 1/L^2$. The spinor and vector masses are determined in terms of the scale μ as

$$m_{1/2} = \mu, \quad m_1^2 = \mu^2 + \frac{\mu}{L}, \quad m_1'^2 = \mu^2 - \frac{\mu}{L}, \tag{C.5}$$

and the action is invariant under the supersymmetry

$$\begin{aligned}
\delta_\epsilon \Phi_\mu &= \left[-\frac{2\tilde{\mu}-3}{12} F_{\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu - \frac{\tilde{\mu}}{3} \gamma^\nu (\nabla_\mu A_\nu + \nabla_\nu A_\mu) \right] \epsilon \\
&\quad + \frac{L}{6} \nabla_\mu [F_{\nu\rho} \gamma^{\nu\rho} \epsilon] + \frac{1}{L} \left[\frac{2\tilde{\mu}(\tilde{\mu}-2)}{3} A_\mu - \frac{\tilde{\mu}(\tilde{\mu}-1)}{3} A^\nu \gamma_{\mu\nu} \right] \epsilon, \\
&\quad + \sqrt{\frac{\tilde{\mu}-1}{\tilde{\mu}+1}} \left\{ \left[\frac{2\tilde{\mu}+3}{12} G_{\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu i\gamma_5 + \frac{\tilde{\mu}}{3} \gamma^\nu i\gamma_5 (\nabla_\mu B_\nu + \nabla_\nu B_\mu) \right] \epsilon \right. \\
&\quad \left. - \frac{L}{6} \nabla_\mu [G_{\nu\rho} \gamma^{\nu\rho} i\gamma_5 \epsilon] + \frac{1}{L} \left[-\frac{2\tilde{\mu}(\tilde{\mu}+2)}{3} B_\mu i\gamma_5 + \frac{\tilde{\mu}(\tilde{\mu}+1)}{3} B^\nu \gamma_{\mu\nu} i\gamma_5 \right] \epsilon \right\}, \quad (\text{C.6}) \\
\delta_\epsilon A_\mu &= (1-\tilde{\mu}) \bar{\Phi}_\mu \epsilon + \frac{\sqrt{\tilde{\mu}^2-1}}{\sqrt{3}} \bar{\lambda} \gamma_\mu \epsilon + \frac{L}{\sqrt{3}} \sqrt{\frac{\tilde{\mu}-1}{\tilde{\mu}+1}} \nabla_\mu [\bar{\lambda} \epsilon], \\
\delta_\epsilon B_\mu &= \sqrt{\tilde{\mu}^2-1} \bar{\Phi}_\mu i\gamma_5 \epsilon - \frac{|\tilde{\mu}-1|}{\sqrt{3}} \bar{\lambda} \gamma_\mu i\gamma_5 \epsilon - \frac{L}{\sqrt{3}} \frac{|\tilde{\mu}-1|}{\tilde{\mu}-1} \nabla_\mu [\bar{\lambda} i\gamma_5 \epsilon], \\
\delta_\epsilon \lambda &= \frac{\sqrt{\tilde{\mu}^2-1}}{2\sqrt{3}} \left[F_{\mu\nu} \gamma^{\mu\nu} + \frac{2\tilde{\mu}}{L} A^\mu \gamma_\mu \right] \epsilon - \frac{|\tilde{\mu}-1|}{2\sqrt{3}} \left[G_{\mu\nu} \gamma^{\mu\nu} i\gamma_5 + \frac{2\tilde{\mu}}{L} B^\mu \gamma_\mu i\gamma_5 \right] \epsilon,
\end{aligned}$$

with the shorthand notation $\tilde{\mu} \equiv \mu L$.

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