Interface entropy in four dimensions as Calabi's diastasis on the conformal manifold

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ABSTRACT: We conjecture an equality between (1) the entropy associated with a Janus interface in a 4d $\mathcal{N} = 2$ superconformal field theory and (2) Calabi's diastasis, a particular combination of analytically continued Kähler potentials, on the conformal manifold (moduli space) of the 4d theory.

A Janus interface is a codimension-one object across which coupling constants take different values. The original construction in 4d [1, 2] did not preserve supersymmetry. This paper concerns half BPS Janus interfaces in 4d $\mathcal{N} = 4$ [3] and $\mathcal{N} = 2$ [4] superconformal field theories.

It is known [5] that the interface entropy, or the g-factor [6], of the Janus interface in 2d $\mathcal{N} = (2, 2)$ theories is given by a specific combination of analytically continued Kähler potentials, namely, Calabi's diastasis [7]. It is natural to ask if a similar statement holds in other dimensions.

Recall that the interface entropy in 2d admits an interpretation as a contribution of the interface to the entanglement entropy for the entangling region containing the interface [8].

The authors of [9] defined the contribution of an interface (called "defect" by them) to the entanglement entropy as follows. Let us consider the 4d Minkowski space with coordinates x^μ ($\mu = 0, \ldots, 3$), together with a conformal interface located at $x^3 = 0$. We have CFT$_+$ (CFT$_-$) in the region $x^3 > 0$ ($x^3 < 0$). Take as the entangling region the round two-sphere of radius R centered at the origin, and denote the resulting entanglement entropy by S. Let $S_\pm$ be the entanglement entropy defined by the same geometry, but without an interface, for CFT$_\pm$ respectively. They define

$$S_{\text{interface}} = S - \frac{S_+ + S_-}{2} \,. \tag{1}$$

In this combination contributions from the bulk region are canceled. Let ϵ be a UV cut-off. By a holographic analysis they found the behavior

$$S_{\text{interface}} = D_1 \frac{R}{\epsilon} + D_0 + \mathcal{O}(\epsilon) \,. \tag{2}$$

They showed D_1 to be scheme dependent, but D_0 scheme independent and thus universal.

For the Janus interface, the quantity D_0 is a non-holomorphic function $D_0(\tau, \overline{\tau}; \tau', \overline{\tau}')$ that depends on two points τ and τ' on the conformal manifold. These points correspond to the values of the couplings on the two sides of the interface. The function is subject to some constraints. First, it has to vanish when the two points coincide, i.e., when the interface is trivial. Second, it has to be invariant under S-duality transformations.

Another intriguing fact is that the sphere partition functions of 2d $\mathcal{N} = (2, 2)$ and 4d $\mathcal{N} = 2$ SCFTs both compute the Kähler potential K of the respective conformal manifold [10–12]. With a suitable normalization of the metric, the relation in the 4d case reads

$$Z_{S^4} = \left(\frac{r}{r_0} \right)^{-4a} e^{K/12} \,, \tag{3}$$

where a is an anomaly coefficient, r is the radius, and r_0 is a renormalization scale.

Any scheme-independent physical quantity associated with a Janus interface should be invariant under duality transformations when there is no globally defined Kähler potential [13]. Calabi's diastasis (the expression in the bracket of the formula below) possesses

this property because it is invariant under Kähler transformations. This makes it a very natural function to enter the universal part of the interface entropy.

These considerations motivate us to make the following conjecture.

There exists an appropriate notion of interface entropy D_0 generalizing the definition above such that for a half BPS Janus interface in a 4d $\mathcal{N} = 2$ superconformal field theory D_0 is proportional to Calabi's diastasis on the conformal manifold:

$$D_0 = c_0 \left[K(\tau, \overline{\tau}) + K(\tau', \overline{\tau}') - K(\tau, \overline{\tau}') - K(\tau', \overline{\tau}) \right], \tag{4}$$

where c_0 is a constant.

The paper [9] applied the holographic formula [14] for entanglement entropy to the supergravity background of [15] dual to the half BPS Janus interface in 4d $\mathcal{N} = 4$ SU(N) super Yang-Mills theory. Their result for $\theta^+ = \theta^-$, found in equation (3.72) of their paper, is that

$$D_0 = -\frac{N^2}{2} \log \left(1 + \frac{(g_{\mathrm{YM}}^+ - g_{\mathrm{YM}}^-)^2}{2 g_{\mathrm{YM}}^+ g_{\mathrm{YM}}^-} \right), \tag{5}$$

where $g_{\mathrm{YM}}^\pm$ and $\theta^\pm$ are the values of the Yang-Mills coupling and the theta parameter on the two sides of the interface. Although this formula is written for the special case $\theta^+ = \theta^-$, the result for the general case where both g_{YM} and θ vary across the interface can be obtained by an action of SL($2, \mathbb{R}$). This group is a symmetry of type IIB supergravity and transforms the dilaton and the RR scalar that are related to CFT parameters as $C_{(0)} = \theta/2\pi$, $e^{-2\phi} = 4\pi/g_{\mathrm{YM}}^2$. Then the result (5) of [9] can be summarized by saying that $c_0 = -1/24$ with

$$K = -6N^2 \log i(\overline{\tau} - \tau), \qquad \tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g_{\mathrm{YM}}^2} \tag{6}$$

in the large N limit. The Kähler potential (6) gives the usual metric of constant negative curvature on the upper half plane, which is known to be the Zamolodchikov metric of the theory [16]. The normalization of K is determined by the relation (3) and the coupling dependence $Z_{S^4} \propto g_{\mathrm{YM}}^{N^2-1}$ [17]. This provides a modest check of our conjecture.

The appearance of N^2 ($\sim$ central charge) in (5) is similar to the situation of [18], where the holographic computation of the interface entropy for certain 2d CFT's yielded the product of the central charge and Calabi's diastasis on the moduli space of the dual supergravity.

More study is desired to check, prove, or generalize the conjecture. For a further check one may construct the holographic dual of the Janus interface for class S theories by deforming supergravity solutions of [19]. Does a relation similar to (4) hold for other quantities such as those characterizing reflection/transmission? We leave these matters for the future.

Acknowledgments

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