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Numerical study of multiparticle scattering in $\lambda\phi^4$ theory

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ABSTRACT: We study numerically classical collisions of waves in $\lambda\phi^4$ theory. These processes correspond to multiparticle scattering in the semiclassical regime. Parameterizing initial and final wavepackets by energy E and particle numbers N_i, N_f we find classically allowed region in the parameter space. We describe properties of the scattering solutions at the boundary of the classically allowed region. We comment on the implications of our results for multiparticle production in the quantum regime.

KEYWORDS: Nonperturbative Effects, Scattering Amplitudes

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1 Introduction

Multiparticle production in weakly coupled bosonic field theories has long been of great interest (see e.g. [1, 2] for reviews), which was initially related to instanton-induced processes [3, 4]. Tree level calculations performed in $(3+1)$ -dimensional scalar $\lambda\phi^4$ theory show [5–9] that multiparticle amplitudes of $1 \rightarrow N_f$ processes near the threshold grow factorially with the number of produced particles N_f . This behaviour persists at one-loop level [10] and violates unitarity at sufficiently large multiplicities $N_f \sim 1/\lambda$. It was observed [11], that corresponding multiparticle cross section of $1 \rightarrow N_f$ processes can be conveniently written in the following functional form

$$\sigma_{1 \rightarrow N_f} \sim \exp \left(\frac{1}{\lambda} F(\lambda N_f, \mathcal{E}) \right) \quad (1.1)$$

in the limit $\lambda \rightarrow 0$ with λN_f and $\mathcal{E}$ fixed, where $\mathcal{E}$ is the average kinetic energy of the final particles. Inspired by this exponential behaviour several semiclassical techniques for calculation of the multiparticle cross sections were developed in [12–16]. They involved singular solutions of classical equations of motion and allowed to extend the observations obtained with perturbative calculations to higher energies. Still in spite of those vast efforts the most reliable results were obtained in the regime of small λN_f where corresponding cross sections turn out to be exponentially suppressed. Unitarity bounds [2, 17, 18] indicate that such suppression of the probability (1.1) takes place at all particle numbers. Interesting insights come from studies of an analogue of the scattering amplitudes in $(0+1)$ -dimensional theory, i.e. for the anharmonic oscillator with a quartic potential [19–21], where it was found that the perturbative factorial growth is replaced by the exponential suppression for $\lambda N_f \gtrsim 1$.

Let us note that recently an interest to the problem of multiparticle production has been renewed [22–32]. In particular, calculations of refs. [28, 32] performed in $\lambda\phi^4$ theory

with spontaneously broken $\mathbb{Z}_2$ symmetry, exhibited that, contrary to the common belief, the probability of $2 \rightarrow N_f$ scattering process near the threshold is not suppressed at $\lambda N_f \gg 1$ but exhibits an exponential growth at least near the threshold. This conclusion was used to invite the Higgsplosion scenario for solution of the hierarchy and fine-tuning problems of the Standard Model. Very recently authors of refs. [33, 34] argued that the Higgsplosion scenario and in particular the exponential growth of the multiparticle production probability is not consistent with basic principles of quantum field theory and at least requires some modification. Therefore behaviour of the probability (1.1) in $\lambda\phi^4$ field theory (in broken and unbroken phases) at large λN_f is still an open problem even in the weak coupling regime which will be assumed in the paper.

It was long ago understood that the processes $few \rightarrow N_f$ for large N_f can be studied starting from the processes $N_i \rightarrow N_f$ where both initial N_i and final N_f particle numbers are large, i.e. of order $1/\lambda$. In this case, one can consider classical counterpart of the quantum scattering processes, i.e. collisions of classical wave packets. If the ingoing and outgoing waves are in the linear regime at $t \rightarrow \pm\infty$ corresponding initial and final field configurations can be associated with coherent states having average energy E , initial N_i and final N_f particle numbers. The probability of corresponding quantum scattering in the semiclassical regime is not exponentially suppressed and the whole family of such solutions span a classically allowed region in the space of parameters E, N_i and N_f . At a given energy E_* and final particle number N_f one can try to minimize the initial particle number N_i with respect to initial conditions. If the minimum goes to zero for some energy then the probability of the processes $2 \rightarrow N_f$ is not exponentially suppressed at $E > E_*$ [18]. Otherwise the existence of a nontrivial minimum at $N_i \sim 1/\lambda$ would indicate on an exponential suppression of the $2 \rightarrow N_f$ scattering probability for $E \lesssim E_*$. This idea of exploring the classical region was used previously for studies of several different processes induced by collisions of particles. Among of them are the false vacuum decay [35], baryon number violating processes in the Standard Model [36, 37] and soliton-antisoliton pair production in the $(1+1)$ -dimensional scalar field theory [38].

In this paper we study classical solutions describing scattering of the wave packets in the unbroken $\lambda\phi^4$ theory. In particular, we try to approach the boundary of the corresponding classically allowed region. The motivation for this study is two-fold. First, by identifying classically forbidden and allowed regions we can unambiguously tell which multiparticle scattering processes are suppressed in the semiclassical limit and which are not. In particular, our results give clear indication for the suppression of the probability of $few \rightarrow N_f$ processes even in the regime $\lambda N_f \gtrsim 1$, $\lambda \ll 1$ which is in agreement with unitarity arguments. Note that it would be very interesting to apply the method of classical solutions used in our study to explore the classically allowed region in the (E, N_i, N_f) parameter space in the spontaneously broken $\lambda\phi^4$ theory in view of direct relevance to the Higgs physics. Another part of our motivation is related to the boundary of the classically allowed parameter space and corresponding classical solutions. Using suitable semiclassical methods one can start obtaining solutions of the classical equations of motion which describe multiparticle scattering processes $N_i \rightarrow N_f$ in the classically forbidden region in the semiclassical approximation and calculate the semiclassical exponent similar to those

in eq. (1.1). As one approaches the boundary of the classically allowed region in the (E, N_i, N_f) parameter space the semiclassical exponent is expected to nullify and comparison with the boundary and corresponding classical solutions obtained in the present study would be a valuable check of the semiclassical procedure which is known to plague from multiple branches of solutions.

Classical scattering of waves in relation to multiparticle production was previously studied to some extent in the $(1+3)$ -dimensional ϕ^4 model [39], $(1+1)$ -dimensional abelian Higgs model [40] and non-abelian gauge theories [41–43]. In particular, authors of ref. [39] constructed an initial wave packet consisting of a few high frequency free field modes and examine its evolution. They found no significant energy transfer to low frequency modes which would indicate to the production of many “quanta”. In our study we use stochastic sampling technique to scan numerically over classical solutions describing the multiparticle scattering. We limit ourselves to spherically symmetric solutions which reduces the system to a $(1+1)$ -dimensional model and makes the problem numerically feasible. We find it is technically more convenient to fix not final N_f but initial particle number N_i and energy E and find the minimal and maximal values of the particle number N_f of outgoing wave packets which is obtained by solving classical equations of motion. Both approaches are equivalent due to the time reversal symmetry. With our numerical methods we find a non-trivial classically allowed region for the multiparticle scattering processes. Namely, we are able to determine the upper and lower boundaries $N_f^{\min}$ and $N_f^{\max}$ as functions of energy E for a set of fixed values of N_i . We observe that the change in particle number in the scattering reaches values up to 22% for the studied energy range. We examine properties of the classical solutions corresponding to the boundaries.

The rest of the paper is organized as follows. Section 2 is devoted to the description of the model and introduction of notations and most relevant quantities. In section 3 we describe numerical method which we utilize to study solutions to the classical equations of motion describing scattering of wave packets with a particular interest to those “providing” maximal change in their particle numbers. In section 4 we present our numerical results. Section 5 is reserved for a discussion and conclusions.

2 The model

We consider $(3+1)$ -dimensional model of a scalar field with the action

$$S[\phi] = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \right]. \quad (2.1)$$

The parameter m^2 is taken to be positive which corresponds to the unbroken phase of this theory. By making rescaling $x^\mu \rightarrow m^{-1}x^\mu$ and $\phi \rightarrow \sqrt{\frac{m^2}{\lambda}}\phi$ this action can be cast into

$$S[\phi] = \frac{1}{\lambda} \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4 \right]. \quad (2.2)$$

In this form λ is a semiclassical parameter which does not enter equations of motion. We are interested in a particular type of classical solutions, which describe collisions of wave

packets. In what follows we limit ourselves to spherically symmetric field configurations. In this case it is useful to make the following redefinition

$$\phi(t, r) = \frac{1}{r} \chi(t, r) \quad (2.3)$$

and we obtain the action

$$S = \frac{4\pi}{\lambda} \int dt dr \left[\frac{1}{2} \left(\frac{\partial \chi}{\partial t} \right)^2 - \frac{1}{2} \left(\frac{\partial \chi}{\partial r} \right)^2 - \frac{\chi^2}{2} - \frac{\chi^4}{4r^2} \right] \quad (2.4)$$

of $(1+1)$ -dimensional theory on a half-line with the boundary condition $\chi(t, 0) = 0$ and the position dependent interaction. The corresponding equation of motion reads

$$\frac{d^2 \chi}{dt^2} - \frac{d^2 \chi}{dr^2} + \chi + \frac{1}{r^2} \chi^3 = 0 \quad (2.5)$$

and the energy related to the field configuration $\chi(t, r)$ is

$$E = \frac{4\pi}{\lambda} \int_0^\infty dr \left[\frac{1}{2} \left(\frac{\partial \chi}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial \chi}{\partial r} \right)^2 + \frac{1}{2} \chi^2 + \frac{1}{4r^2} \chi^4 \right]. \quad (2.6)$$

In what follows we solve the classical e.o.m. (2.5) numerically. To do this we restrict our solutions to a finite space interval $[0, R]$ where R is sufficiently large. At $r = R$ we impose Neumann boundary condition $\partial_r \chi = 0$. This restriction allows us to expand the field configuration as follows

$$\chi(t, r) = \sum_{n=0}^{\infty} c_n(t) \sqrt{\frac{2}{R}} \sin k_n r, \quad \text{where } k_n = \frac{\pi}{2R} (2n+1), \quad n = 0, 1, \dots \quad (2.7)$$

Inserting this expansion in the action (2.4) we obtain the following equations of motion

$$\ddot{c}_n + \omega_n^2 c_n + I_n = 0, \quad n = 0, 1, \dots \quad (2.8)$$

where $\omega_n^2 = k_n^2 + 1$ and the interaction term reads

$$I_n = \sqrt{\frac{2}{R}} \int_0^R dr \frac{\chi^3(t, r)}{r^2} \sin k_n r, \quad n = 0, 1, \dots \quad (2.9)$$

Total energy of the solution expressed via $c_n(t)$ has the form

$$E = \frac{4\pi}{\lambda} \sum_n \left[\frac{1}{2} \dot{c}_n^2 + \frac{1}{2} \omega_n^2 c_n^2 + \frac{1}{4} c_n I_n \right]. \quad (2.10)$$

Below we consider solutions which linearize at initial and final times. In this case they can be interpreted as describing a multiparticle scattering. In the linear regime the time-dependent Fourier components $c_n(t)$ can be conveniently written via positive and negative frequency components as follows

$$c_n(t) \rightarrow \frac{1}{\sqrt{2\omega_n}} (a_n e^{-i\omega_n t} + a_n^* e^{i\omega_n t}) \quad \text{as } t \rightarrow -\infty \quad (2.11)$$

and

$$c_n(t) \rightarrow \frac{1}{\sqrt{2\omega_n}} (b_n e^{-i\omega_n t} + b_n^* e^{i\omega_n t}) \quad \text{as } t \rightarrow +\infty \quad (2.12)$$

Using these representations one can compute the energy of colliding wave packets as

$$E = \frac{4\pi}{\lambda} \sum_n \omega_n |a_n|^2 = \frac{4\pi}{\lambda} \sum_n \omega_n |b_n|^2 . \quad (2.13)$$

Here $|a_n|^2$ and $|b_n|^2$ can be thought as occupation numbers for initial and final states and their sums

$$N_i = \frac{4\pi}{\lambda} \sum_n |a_n|^2 , \quad N_f = \frac{4\pi}{\lambda} \sum_n |b_n|^2 \quad (2.14)$$

are initial and final particle numbers, respectively. For convenience we introduce the following shorthand notations

$$\tilde{E} = \frac{\lambda}{4\pi} E , \quad \tilde{N}_i = \frac{\lambda}{4\pi} N_i , \quad \tilde{N}_f = \frac{\lambda}{4\pi} N_f . \quad (2.15)$$

For the numerical implementation we truncate the expansion (2.7) at $n = N_r$ and solve the evolution equations (2.8) using Bulirsch-Stoer method, see e.g. [44]. Some of the results have been verified with the 4-th order Runge-Kutta method with a very small time step. For convenience, we introduce the uniform spacial lattice r_i , $i = 0, \dots, N_r$ with $r_0 = 0$ and $r_{N_r} = R$. In what follows we use several values of $R = 20, 30$ and 50 and $N_r = 400, 600$ and 1000 to study the dependence of our results on the lattice. The time interval is taken to be somewhat smaller than $\sim 2R$. We use FFTW implementation of discrete Fourier transformation [45] to compute the field configuration $\chi(t, r)$ as (2.7) and the interaction term (2.9).

3 The method

In this section we describe a method used to explore the classical transitions $\tilde{N}_i \rightarrow \tilde{N}_f$ at fixed energy $\tilde{E}$. In particular, we are interested in solutions of the classical e.o.m. which maximize $|\tilde{N}_f - \tilde{N}_i|$ at fixed values of $\tilde{N}_i$ and $\tilde{E}$.

3.1 Initial conditions

We take initial conditions which correspond to a wave packet which is localized well away from the interaction region and which is propagating towards $r = 0$. Technically, we choose an interval $[r_1, r_2]$, where r_1, r_2 are chosen points belonging to the spacial lattice, i.e. $r_1 = r_{i_1}$ and $r_2 = r_{i_2}$ for some $i_1 < i_2$, and construct following function

$$\chi(r) = \begin{cases} \sum_{n=1}^{i_2-i_1} \frac{1}{\sqrt{2\tilde{\omega}_n}} f_n \sin(\tilde{k}_n(r-r_1)) , & r \in [r_1, r_2] \\ 0, & r \notin [r_1, r_2] \end{cases} \quad (3.1)$$

Here $\tilde{k}_n = \frac{\pi n}{r_2-r_1}$, $\tilde{\omega}_n = \sqrt{\tilde{k}_n^2 + 1}$, $n = 1, \dots, i_2 - i_1$ and f_n are initial Fourier amplitudes. By construction the configuration (3.1) nullifies at the boundaries $r = r_{i_1}$ and $r = r_{i_2}$.

Next, to make the initial configuration smooth enough we introduce an artificial smearing multiplying (3.1) by the function

$$\left(e^{\frac{r_{i_1}+d-r}{a}} + 1\right)^{-1} \cdot \left(e^{-\frac{r_{i_2}-d-r}{a}} + 1\right)^{-1}, \quad (3.2)$$

where d and a are chosen parameters. We checked that our results have a very mild dependence on the precise way of the smearing. In what follows we fix $d = 0.2$ and $a = 0.1$ for concreteness. Next, we obtain Fourier amplitudes $\tilde{f}_n$ of the smeared configuration by inverting (3.1) and take the initial condition for the e.o.m. (2.8) (i.e. the field configuration and its first time derivative at initial time $t = t_i$) from

$$\chi_i(t, r) = \begin{cases} \sum_{n=1}^{i_2-i_1} \frac{1}{\sqrt{2\tilde{\omega}_n}} \tilde{f}_n \sin\left(\tilde{k}_n(r-r_1) + \tilde{\omega}_n(t-t_i)\right), & r \in [r_1, r_2] \\ 0, & r \notin [r_1, r_2]. \end{cases} \quad (3.3)$$

This corresponds to the initial wave packet propagating towards $r = 0$. In what follows we set $t_i = 0$. We consider only solutions whose evolution is linear at initial (and final) times.

3.2 Going to the classical boundary

Using the initial field configuration we calculate its energy $\tilde{E}$ and initial particle number $\tilde{N}_i$ by using eqs. (2.10), (2.11), (2.13) and (2.14). Our aim is to find the classical solutions describing scattering of wave packets in which the particle number changes as much as possible at a given energy $\tilde{E}$. Since we solve initial value problem it is more convenient to fix the initial particle number $\tilde{N}_i$ and find the absolute minimum (or maximum) of $\tilde{N}_f$ with respect to chosen set of initial conditions, i.e. f_n entering (3.1). We fix the initial particle number by the respectful normalization of the Fourier amplitudes $\tilde{f}_n$. The final particle number $\tilde{N}_f$ is a highly non-linear function of initial data $\tilde{f}_n$ and it may have several local extrema. To find its absolute minimum (maximum) we use the stochastic sampling technique in combination with the simulated annealing method, see [44, 46]. Namely, we generate an ensemble of the classical solutions with a fixed initial particle number $\tilde{N}_i$ weighted by the probability

$$p \sim e^{-F}, \quad (3.4)$$

where

$$F = \beta \left(\tilde{N}_f + \xi(\tilde{E} - \tilde{E}_*)^2 \right). \quad (3.5)$$

If large positive β and ξ are taken the ensemble will be dominated by solutions having small F thus driving their distribution towards the lower boundary $\tilde{N}_f^{\min}(\tilde{E}_*)$ of the classically allowed region. To reach the upper boundary $\tilde{N}_f^{\max}(\tilde{E}_*)$ signs of β and ξ should be negative.

To generate the ensemble (3.4) we use the Metropolis Monte Carlo algorithm. We start with a solution specifying by a randomly chosen set of f_n , see (3.1), which have a fixed initial particle number $\tilde{N}_i$. This solution has an energy $\tilde{E}$ and final particle number $\tilde{N}_f$ which is found after solving the classical equations of motion. Next, we choose a few randomly chosen amplitude numbers and change corresponding amplitudes f_n by small amounts,

$f_n \rightarrow f'_n = f_n + \Delta f_n$. We used to take three amplitudes at a time which shows a good performance of our algorithm. The quantities Δf_n are chosen to be normally distributed with the standard deviation $a/\sqrt{\tilde{\omega}_n}$, where a is a small number. In our calculations its value is varied from 10^{-4} to 0.1. The modified set of amplitudes f'_n is rescaled in a proper way and used to construct an initial wave packet having the same initial particle number $\tilde{N}_i$ and some value of energy $\tilde{E}'$, see section 3.1. Then we evolve the system forward in time far enough until it reaches the linear regime and we can find the final particle number $\tilde{N}'_f$ of the new solution. We compute $\Delta F \equiv F' - F$ using (3.5). The new set of amplitudes f'_n is accepted with probability

$$p_{accept} = \min(1, e^{-\Delta F}) \quad (3.6)$$

and used for the next iteration. Typically, we fix $\tilde{N}_i$ and $\tilde{E}_*$ and perform several runs starting with different β and ξ . Each run spans of order $10^3 - 10^4$ iterations. The value of ξ remains a constant during a single run, while β which is analogue of inverse temperature is gradually increased from its initial value β_0 according [47] to

$$\beta_i = \beta_0 \log(1 + i), \quad (3.7)$$

where i is the iteration number. Values of β_0 and ξ are temporarily increased from 10 to 10^6 and from 10^{-4} to about 1, respectively. We stop the procedure when relative change in the value of F during a run becomes smaller than 10^{-3} for the same ξ and it does not increases with the variation of a and β_0 .

Let us note, that the stochastic sampling technique was previously used to study classically allowed sphaleron transitions [36, 37] and classical soliton-antisoliton pair production in particle collisions in (1+1)-dimensional scalar field theory [38], where its efficiency in determination of the boundary of the classically allowed region was confirmed.

4 Numerical results

Below we describe regions of the classical solutions for several different values of $\tilde{N}_i = 0.1, 1.0, 10.0$ and 30.0. In figure 1 we plot numerical results of the stochastic sampling in the $(\tilde{E}, \tilde{N}_f)$ plane for $\tilde{N}_i = 1$, $N_r = 400$ and $R = 20$. The initial field configurations are set in the interval $[r_1, r_2] \equiv [5.0, 19.0]$. Each dot in the figure represents a classical solution accepted by our numerical procedure on the way to the boundaries $\tilde{N}_f^{\max}(\tilde{E})$ and $\tilde{N}_f^{\min}(\tilde{E})$. To reach the boundaries we run the numerical procedure described in the previous section for a chosen set of $\tilde{E}_*$

$$\frac{\tilde{E}_*}{\tilde{N}_i} = 1.5, 2.0, \dots, 9.5, 10.0, 11.0, \dots, 15.0, \quad (4.1)$$

which determines the relevant energy interval for our study. We comment on this choice below. The solutions which seed our Monte Carlo search are randomly chosen with the only condition that they have $\tilde{N}_i = 1$ and linearize at the initial and final times. These solutions typically have very small change in the particle number and are situated near

Figure 1. The results of stochastic sampling for $\tilde{N}_i = 1$, $N_r = 400$ and $R = 20$. Each dot in the figure represents a classical solution with an energy $\tilde{E}$ and final particle number $\tilde{N}_f$ obtained using the Monte-Carlo technique described in the text. Red (dark) and green (gray) points correspond to positive and negative values of β and ξ , respectively. The dashed line marks the threshold energy $\tilde{E} = \tilde{N}_f = \tilde{N}_i \equiv 1$. The upper and lower boundaries, $\tilde{N}_f^{\max}(\tilde{E})$ and $\tilde{N}_f^{\min}(\tilde{E})$ are shown by solid lines.

the line $\tilde{N}_f = 1$ on the $(\tilde{E}, \tilde{N}_f)$ plane. Initially, we take it relatively large value of $a = 0.1$ and start our stochastic sampling with small β_0 about 10.0. During single run its value is changed according to (3.7). We repeat the runs with a larger starting value of β . The value of a which determines size of the Fourier amplitude changes Δf_n is also decreased after several runs. At the same time the value of ξ is increased to fix the energy of the solution to be near $\tilde{E}_*$. The obtained domain of the classical solutions in $(\tilde{E}, \tilde{N}_f)$ plane, see figure 1, has smooth envelopes, $\tilde{N}_f^{\min}(E)$ and $\tilde{N}_f^{\max}(E)$, which represent the boundary of the classically allowed region of the lattice version of the continuous model. We see that the maximal possible change in the particle number $|\tilde{N}_f - \tilde{N}_i|$ during the classical evolution increases with the energy of solutions but does not exceed 12% in the chosen energy range. Let us stress that due to the time reversal symmetry one can always interchange the initial $\tilde{N}_i$ and final $\tilde{N}_f$ particle numbers in the discussion of the classically allowed region in the figure 1.

Let us study properties of the classical solutions near the boundaries $\tilde{N}_f^{\min}(E)$ and $\tilde{N}_f^{\max}(E)$. Although the solutions which seed our Monte Carlo runs for each value of $\tilde{E}_*$ are chosen at random it is remarkable that the boundary solutions have similar forms for different $\tilde{E}_*$. At the panels of figure 2 we plot initial wave packets of the classical solutions at the upper boundary $\tilde{N}_f = \tilde{N}_f(\tilde{E})$ for several values of energy $\tilde{E} \approx 3.0, 4.5, 8.0$ and 11.0. The interval $[r_1, r_2]$ is marked by the dashed lines. Corresponding final particle number for these solutions can be found in the figure 1. The evolution in time of the upper boundary solution with $\tilde{E} \approx 6.1$ is presented in figure 3 where $\chi(t, r) = \frac{\phi(t, r)}{r}$ is shown for convenience.

Figure 2. The field configurations $\phi(t = 0, r)$ of the solutions at the upper boundary with $\tilde{N}_f = \tilde{N}_f^{\max}(\tilde{E})$ for $\tilde{E} \approx 3.0, 4.5, 8.0$ and 11.0 . The space lattice parameters are $N_r = 400, R = 30$. The dashed lines mark the space interval for the initial wave packet.

Figure 3. Time evolution of the field $\chi(t, r) = \frac{\phi(t, r)}{r}$ for the upper boundary solution with $\tilde{E} \approx 6.1$; $N_r = 400, R = 20$.

For all boundary solutions which we have found with $\tilde{N}_i = 1$ the initial wave packet has a relatively sharp part near its end. This part becomes more pronounced with increase of its energy as seen in figure 3. The initial wave packet propagates towards the interaction region and its spiky parts increases considerably near the point $r = 0$. The reflected wave packet have the form very similar to the incoming one. The spiky part of the incoming wave packet is the last to arrive into the interaction region and the first to leave it.

According to the used extremization procedure which involves the stochastic sampling, the found classical solutions describe the scattering of waves with $\tilde{N}_i = 1$ and a maximum

Figure 4. Time evolution of the linearized and exact energy (left panel) as well as the instantaneous particle number (right panel) for the exemplary solution in figure 3.

value of $|\tilde{N}_f - \tilde{N}_i|$ at a given energy $\tilde{E}$. It is interesting to study how the particle number changes during the time evolution. In section 2 we introduced the particle numbers for the initial and final wave packets. However, we can calculate an instantaneous particle number $N(t)$ by using eq. (2.11) (or (2.12)) as a definition of time-dependent positive and negative frequency components at arbitrary time t and applying formulas similar to (2.14). The quantity $\tilde{N}(t)$ coincides with $\tilde{N}_i$ and $\tilde{N}_f$ at initial and final times where the evolution of the field is linear. On the right panel of the figure 4 we plot the instantaneous particle number for the boundary solution shown in figure 3. Comparing it with the time evolution of the field presented on figure 3 we see that the actual change of the particle number occurs precisely when the sharpest part of the initial wave packet reaches the interaction region. For instance, the deviations of $\tilde{N}$ at time stamps $t = 12.4$ and $t = 21.3$ (see figure 3) from its asymptotic values $\tilde{N}_i$ and $\tilde{N}_f$ are less than 0.0005, which are much smaller than $\tilde{N}_f - \tilde{N}_i \approx 0.072$. Such behaviour of $\tilde{N}(t)$ give us support that found extrema of F are independent of choice of the interval $[r_1, r_2]$ which is used to generate initial field configurations as far as it is sufficiently large and include the part of the wave packet responsible for the most of the change in $\tilde{N}(t)$. Also we verified that numerical error in $|\tilde{N}_f - \tilde{N}_i|$ related to the smearing is less than 10^{-3} . In a way similar to $\tilde{N}(t)$ we define a linearized energy $\tilde{E}_{lin}(t)$ which is used to check the linearity of the solution by comparing it with the exact energy (2.10). Such comparison is presented in figure 4 (left panel) for the solution from figure 3. The total energy is conserved on the entire solution with the accuracy less than 10^{-3} , while it coincides with the linearized energy better than 10^{-4} at initial and final times.

A nontrivial change in particle number implies a redistribution of the energy between low and high frequency modes. In figure 5 we show energy distributions ϵ_k per wave number unit for the initial and the final wave packets for the same upper boundary solutions in figure 2. These distributions are defined as

$$\epsilon_k = \begin{cases} \frac{1}{\Delta k} \omega_n |a_n|^2, & \text{for initial wave packet,} \\ \frac{1}{\Delta k} \omega_n |b_n|^2, & \text{for final wave packet,} \end{cases} \quad (4.2)$$

Figure 5. The distributions of energy over the modes ϵ_k per wave number for the initial (red, thick line) and final (blue, thin line) wave packet for the boundary solutions from figure 2.

where $\Delta k = \frac{\pi}{R}$. We observe expected softening of the energy distributions for the final wave packet as compared to the initial one. However, the effect is quite small even for boundary solutions which describe processes with maximal change of the particle number. Also we see that with increase of the collision energy $\tilde{E}$ the modes with larger k_n (i.e. higher frequencies) become filled in and at $\tilde{E} \gtrsim 11.0$ the tail of energy distributions reaches the largest value of k_n for the chosen lattice size and spacing. This is a clear indication that a finer lattice is required to obtain reliable results at these and higher energies. At the same time solutions with energies not far from the threshold energy $\tilde{E}_{th} = \tilde{N}_i$ consist mostly of nonrelativistic modes and a larger space interval R is needed to reach linear regime at initial and final times. For this reason we do not consider solutions with energies lower than 1.5.

Next, we study the dependence of the obtained results for the boundary of the classically allowed region in the $(\tilde{E}, \tilde{N}_f)$ plane for $\tilde{N}_i = 1$ and for the boundary solutions on parameters of the lattice, namely on the spacial cutoff R and on the number of modes N_r . For comparison to the case with $R = 20$ and $N_r = 400$ we repeat the same numerical procedure to find the boundary of the classically allowed region for the case $R = 20$, $N_r = 600$ with a smaller lattice spacing and for the case $R = 30$, $N_r = 600$ with a larger space interval but the same lattice spacing. The results are presented in figure 6, where we show lower and upper boundaries $\tilde{N}_f^{\min}(E)$ and $\tilde{N}_f^{\max}(E)$ for the different cases. The space intervals for selection of the initial wave packets are taken to be $[6.7, 19.2]$ in the former case and $[8.5, 29.8]$ in the latter. We observe that different boundaries coincide with accuracy better than $4 \cdot 10^{-3}$. A deviation is seen for the case $N_r = 600$, $R = 30$ for large energies $\tilde{E} \gtrsim 11$ which reflects the appearance of new high frequency modes. Examples of the upper boundary solutions for the same set of energies as in figure 2 but for the case $N_r = 600$, $R = 30$ are shown in figure 7. Comparing them with the solutions presented in figure 2 we see that the

Figure 6. The classically allowed region in $(\tilde{E}, \tilde{N}_f)$ plane for $\tilde{N}_i = 1$ calculated for three cases: 1) $N_r = 400, R = 20$; 2) $N_r = 600, R = 20$; 3) $N_r = 600, R = 30$; 4) $N_r = 400, R = 30$, time-shifted solutions (see main text for details). The space intervals chosen for selection of the initial wave packets are presented in the main text.

Figure 7. The same as in figure 2 but for $N_r = 600, R = 30$.

difference lies in the enlarged soft oscillating part of the solution. We found that the time evolution of the particle number $\tilde{N}(t)$ for these solution is similar to that of presented in figure 4. Namely, the actual change in $\tilde{N}$ starts with arrival of the sharp tail of the incoming wave train into the interaction region. Energy distributions over Fourier modes for the solutions with the same energy obtained on different lattices are almost indistinguishable as seen from figure 8.

Figure 8. The distribution of energy over the modes ϵ_k per wave number for the upper boundary solutions for $N_r = 400, R = 20$ (red, thick line) and $N_r = 600, R = 30$ (blue, thin line) figures 2 and 7.

Our numerical results clearly show existence of the classically allowed region of a finite size in the $(\tilde{E}, \tilde{N}_f)$ plane for fixed value of $\tilde{N}_i$ in the continuous version of the model with the ϕ^4 potential. We suggest existence of the boundary solutions in the continuous limit of the model. The classical equations of motion (2.5) and (2.8) are invariant with respect to time translations. This symmetry is explicitly broken in our numerical setup by the choice of the initial configuration (3.1) and (3.3) at $t = t_i \equiv 0$, which in particular implies

$$\chi(0, r_1) = \chi(0, r_2) = 0. \quad (4.3)$$

Still the oscillating form of the initial wave packets (see figures 2 and 7) and the time evolution of the particle number indicate that an approximate discrete time-shift symmetry should exist in the lattice version of the model. In terms of initial wave packets this symmetry connects the configurations which are related by time evolution and which satisfy conditions (4.3) as long as the spiky part of the initial wave train lies inside the interval $[r_1, r_2]$. Indeed, with our numerical technique (making several runs at the same energy $\tilde{E}_*$ with different seeds) we actually find several branches of the boundary solutions which are related with each other by such shifts in time. In fact, for constructing the boundaries in figure 1 we selected the solutions whose initial wave packets have the longest soft oscillating part while their sharp edges are placed as far as possible (for given R and initial space interval $[r_1, r_2]$) from the origin. In figure 9 we plot the initial field configurations (red solid lines) corresponding to another branch of the upper boundary solutions at the same values of energy as in figure 2. These wavepackets have spiky parts shifted closer to the origin as compared to those in figure 2. Also in the figure 9 in thin (blue) lines we show the initial field configurations from figure 2 evolved forward in time until the positions of the spiky parts of the different solutions coincide. We observe perfect coincidence of the form of that part. As the spiky part of the wavepackets is responsible for the change of

Figure 9. The upper boundary initial field configurations of the solutions related by time-shift symmetry for $\tilde{E} \approx 3.0, 4.5, 8.0$ and 11.0 . The space lattice parameters are $N_r = 400$, $R = 20$. The dashed lines mark the space interval for the initial wave packet.

Figure 10. The same as in figure 7 but for lower boundary solutions.

the particle number we expect that time-shifted solutions would have similar values of $\tilde{N}_f$. This is indeed the case and in figure 6 we show the upper boundary $\tilde{N}_t^{\max}$ obtained for $N_r = 400$ and $R = 20$ with the branch of the time shifted solutions. It coincides with that of for the upper boundary solutions with a longer soft oscillating part with an accuracy better than $2 \cdot 10^{-3}$.

So far we described the classical solutions at the upper part $\tilde{N}_f = \tilde{N}_f^{\max}(E)$ of the boundary. Examples of solutions corresponding to the lower part $\tilde{N}_f = \tilde{N}_f^{\min}(E)$ of the boundary are shown in figure 10 for the case $N_r = 600$, $R = 30$. We observe that they have

Figure 11. The classically allowed region in $(\tilde{E}, \tilde{N}_f)$ plane for $\tilde{N}_i = 10.0$; $N_r = 600$, $R = 30$.

very similar properties. Let us note that the time reversal symmetry translates a lower boundary solution with initial $\tilde{N}_i$ and final $\tilde{N}_f$ particle numbers into an upper boundary solution with the initial and final particle numbers interchanged.

Now we turn to discussion of numerical results for another values of the initial particle number $\tilde{N}_i$. In figure 11 we show the upper $\tilde{N}_f(E)$ and lower $\tilde{N}_f(E)$ boundaries of the classically allowed region for $\tilde{N}_i = 0.1$ and the relevant energy interval $[1.5\tilde{N}_i, 15.0\tilde{N}_i]$ with the values of $\tilde{E}_*$ defined by (4.1). We see that the maximum relative difference of the initial and final particle numbers, i.e. $\frac{|\tilde{N}_f - \tilde{N}_i|}{\tilde{N}_i}$, is more than two order of magnitude smaller than for the case $\tilde{N}_i = 1$ for energies $\tilde{E} \lesssim 15\tilde{N}_i$. Numerically, the particle number changes no more than 0.33% for the chosen energy range. In figure 12 we show examples of the initial wave packets and instantaneous particle number evolution for the upper and lower boundary solutions for $\tilde{E} \approx 0.6$. The space interval for the initial wavepacket $[r_1, r_2]$ is shown by the dashed lines. Qualitative properties of these solutions are very similar to those obtained for $\tilde{N}_i = 1$. Namely, the solutions have a spiky part whose transition through the interaction region results in actual change in particle number and a soft oscillating part. As in the previous case we obtain several branches of the boundary solutions which differ by corresponding shifts in time.

Let us now turn to the case $\tilde{N}_i = 10$ in which we obtain numerical results concerning boundary solutions which are qualitatively different from those obtained at smaller values of $\tilde{N}_i = 0.1, 1.0$. In figure 13 boundary of the classically allowed interval is shown with the thick solid (red) and thick dashed (blue) lines for the energy interval $[1.5\tilde{N}_i, 15.0\tilde{N}_i]$ with the same set (4.1) of $\tilde{E}_*$. We find that although the form of the boundary is similar to that of for $\tilde{N}_i = 0.1, 1.0$, its upper and lower parts, i.e. $\tilde{N}_f^{\max}(E)$ and $\tilde{N}_f^{\min}(E)$, consist of two different branches of the classical solutions. At energies lower than about 70 the boundary solutions have two distinct spiky parts. An example of the initial field configuration $\phi(t = 0, r)$

Figure 12. The initial wave packets (left panels) and evolution of the instantaneous particle number (right panels) for boundary solutions with $\tilde{N}_i = 0.1$ and $\tilde{E} \approx 0.6$. Upper and lower panels correspond to upper and lower boundary solutions respectively.

Figure 13. The classically allowed region in $(\tilde{E}, \tilde{N}_f)$ plane for $\tilde{N}_i = 10.0$; $N_r = 600$, $R = 30$. The envelopes $\tilde{N}_f^{\min}$ and $\tilde{N}_f^{\max}$ are composed of the classical solutions with two spiky parts (thick red line) and with three spiky parts (dashed blue line). The dots show other branches of the solutions which deliver some local minima to F (or $\tilde{N}_f$).

corresponding to the lower boundary is presented in figure 14 (upper left panel) for $\tilde{E} \approx 65$. The evolution of the instantaneous particle number $\tilde{N}(t)$ as well as the distribution of energy over the modes ϵ_k for initial and final field configurations are also shown in this figure

Figure 14. From top to bottom: a) initial wave packet; b) evolution of the instantaneous particle number $\tilde{N}(t)$; c) distribution of energy over the modes ϵ_k per wave number for the initial field configuration; d) the same as in c) but for the final field configuration. Left and right panels correspond to the lower boundary solutions with $\tilde{E} \approx 65$ and 80 , respectively.

on left panels. The oscillating pattern in ϵ_k indicates that the incoming (and outgoing) field configuration looks as a sum of two separated in space wavetrains each having a similar smooth Fourier image. In particular, the distance δr between the spikes in the initial configuration and the oscillation period δk in the initial energy distribution are related approximately by $\delta r \cdot \delta k \approx 2\pi$ for all solutions of this branch. For example, the solution with $\tilde{E} \approx 65$ presented on left panels of figure 14 has $\delta r \approx 2.8$ and $\delta k \approx 2.3$. Comparing the energy distributions ϵ_k for the field configurations at initial and final times one observes a small energy transfer from low to high frequency modes. At energies larger than about 70 our numerical results show that the absolute minimum (and maximum) of $\tilde{N}_f$ is delivered by a different branch of the solutions whose initial (and final) space field configuration contains already three spikes. On the right panels in figure 14 we present the initial field configuration $\phi(t=0, r)$, time evolution of $\tilde{N}(t)$, and energy distributions over the Fourier modes ϵ_k for the initial and final field configurations for a boundary solution with $\tilde{E} \approx 80$ having three spikes. On this branch of solutions the initial and final field configurations looks as a sum of three separated in space wavetrains with similar Fourier image. As in the cases of smaller $\tilde{N}_i$, the instantaneous particle number $\tilde{N}(t)$ undergoes the most dramatic changes when spiky parts of the field configurations reach the interaction region. We find that the two-spike solutions at $\tilde{E} \gtrsim 70$ and the three-spike solutions at $\tilde{E} \lesssim 70$ still represent local minima of $\tilde{N}_f$. To figure that out we take a two-spike lower boundary solution at $\tilde{E} \approx 65$ as a seed for our Monte-Carlo procedure with $E_* > 70$. To ensure that the field configuration does not jump to another (three-spike) branch we take a relatively small value of a which governs the size of changes in the amplitudes f_n . In this way we find that two-spike branch of solutions at $\tilde{E} \gtrsim 70$ which is shown in figure 13 by thin red (solid) line. In similar way we find a continuation of the three-spike branch of the solutions at $\tilde{E} \lesssim 70$ which is shown by this blue (dashed) line in figure 13. Upper boundary solutions have very similar properties.¹ Let us note that in the chosen energy range the maximal changes in particle number reach values around 20%.

There have been also found another branches of solutions which deliver only local extrema to $\tilde{N}_f$. In the linear regime these solutions look as a sum of two or three single-spike wavetrains which differ from those already described by somewhat different distances between the wavetrains. In figure 15 we show two examples of such solutions. Left panels show a solution of this type near the upper boundary with two spikes and $\tilde{N}_f > \tilde{N}_i$. The distance between the wavetrains in the initial field configuration is larger than that of for the boundary two-spike solution. Similarly, the right panels show an example of the solution near the lower boundary delivering a local minima to $\tilde{N}_f$ with three spike. Again we observe somewhat larger distance between the wavetrains in the initial field configuration, cf. figure 14 (upper right panel). These two branches of solutions are shown in figure 13 by dots. Let us note, that for $\tilde{N}_i = 10$ we do not find any field configuration which would deliver (even local) extrema to the functional $\tilde{N}_f(E)$ and whose initial (final) wave packet would have a single spike only. We tried to find it on purpose by taking as a

¹As previously, for construction of the boundaries $\tilde{N}_f^{\min}(E)$ and $\tilde{N}_f^{\max}(E)$ we select solutions with the longest oscillating tale, although we obtained also time-shifted field configurations with the same number of the spiky parts.

Figure 15. The initial wave packets (left panels) and evolution of the instantaneous particle number (right panels) for the solutions with $\tilde{N}_i = 10$ and $\tilde{E} \approx 80$ corresponding to local minima of $\tilde{N}_f$. Upper and lower panels correspond to $\tilde{N}_f > \tilde{N}_i$ and $\tilde{N}_f < \tilde{N}_i$.

seed some initial field configuration which was obtained for $\tilde{N}_i = 1$ as a boundary solution which is of single-spike type and increasing $\tilde{N}_i$ by small steps to $\tilde{N}_i = 10$. We find that at intermediate values of $\tilde{N}_i$ around 5–6 the initial field configurations of the boundary configurations prefer to be splitted into more than one wavetrains to produce larger change in particle number $|\tilde{N}_f - \tilde{N}_i|$.

The picture becomes even more complicated at larger $\tilde{N}_i$. In figure 16 we show our numerical results for the boundary of the classically allowed region for $\tilde{N}_i = 30$ taking $R = 30$, $N_r = 600$ (solid blue line). We observe that the difference $|\tilde{N}_f - \tilde{N}_i|$ does not exceed 22% in this case in the chosen energy interval. We find that the corresponding boundary solutions contain already 4–7 spikes in their initial (and final) wavetrains. Examples of the initial field configurations of the boundary solutions are shown in figure 17. At the same time we find that the number of local extrema of $\tilde{N}_f$ increases dramatically at large $\tilde{N}_i$. Corresponding solutions have similar number of wavetrains but with different distances between them. This greatly complicates the task of finding the boundary of the classically allowed region because, on the one hand, many of our numerical runs get stuck in such local minima and, on the other, the values of $\tilde{N}_f$ at the same $\tilde{E}$ for different branches of near-boundary solutions are at some cases seem to be close to our numerical accuracy. For these reasons at present we are unable to classify in details the boundary solutions as it has been done for the cases with smaller $\tilde{N}_i$. In the figure 16 we present only the envelopes $\tilde{N}_f^{\min}(\tilde{E})$ and $\tilde{N}_f^{\max}(\tilde{E})$ of the classically allowed region. We performed an additional

Figure 16. The classically allowed region in the $(\tilde{E}, \tilde{N}_f)$ plane for $\tilde{N}_i = 30.0$. The lattice parameters are $R = 30, N_r = 600$ (solid blue line) and $R = 50, N_r = 1000$ (dashed red line).

Figure 17. The initial wave packets (left panels) and evolution of the instantaneous particle number (right panels) for the boundary solutions with $\tilde{N}_i = 30$ and $\tilde{E} \approx 200$. Upper and lower panels correspond to $\tilde{N}_f > \tilde{N}_i$ and $\tilde{N}_f < \tilde{N}_i$.

check by finding $\tilde{N}_f^{\max}$ and $\tilde{N}_f^{\min}$ for larger space interval $R = 50$ and taking $N_r = 1000$. Corresponding boundary is shown in figure 17 by dashed red line. From our numerical results we expect that the number of closely separated spiky parts in the wavepackets of the boundary solutions continue to grow with the increase of $\tilde{N}_i$.

Figure 18. Left panel: the classically allowed regions in $(\tilde{E}, \tilde{N}_f)$ plane for different $\tilde{N}_i$. Right panel: the difference $|\tilde{N}_f^{\min} - \tilde{N}_i|$ as a function of $\tilde{N}_i$.

It is interesting to compare the classically allowed regions at different $\tilde{N}_i$, see figure 18 (left panel). Firstly, let us see that if two classical solutions describing scattering of waves with different sets of parameters, i.e. $\tilde{N}_i^{(1)}, \tilde{N}_f^{(1)}, \tilde{E}^{(1)}$ and $\tilde{N}_i^{(2)}, \tilde{N}_f^{(2)}, \tilde{E}^{(2)}$, exists then there should also exist a solution which has energy and particle numbers equal to the following sums

$$N_i = N_i^{(1)} + N_i^{(2)}, \quad N_f = N_f^{(1)} + N_f^{(2)}, \quad E = E^{(1)} + E^{(2)}. \quad (4.4)$$

Such a solution can be constructed explicitly by taking it as a sum of the individual solutions sufficiently separated in space-time. This observation, in particular, means that the width of the classically allowed region, i.e. $|\tilde{N}_f^{\max} - \tilde{N}_i|$ or $|\tilde{N}_f^{\min} - \tilde{N}_i|$ at fixed ratio $\tilde{E}/\tilde{N}_i$ should grow faster than a linear function with increase of initial particle number $\tilde{N}_i$. On the figure 18 we show² the dependence of the quantity $|\tilde{N}_f^{\min} - \tilde{N}_i|$ as a function of $\tilde{N}_i$ for two values of $\tilde{E}/\tilde{N}_i$. For comparison a linear function is shown in thin (blue) line. One can see that the energy dependence of $|\tilde{N}_f^{\min} - \tilde{N}_i|$ tends to be linear at large $\tilde{N}_i$. We found that $|\tilde{N}_f^{\max} - \tilde{N}_i|$ has similar behaviour. This can be considered as an indication on the existence of some limiting boundary of the classically allowed region in the plane $(\tilde{E}/\tilde{N}_i, \tilde{N}_f/\tilde{N}_i)$ at large $\tilde{N}_i$ or in the plane $(\tilde{E}/\tilde{N}_f, \tilde{N}_i/\tilde{N}_f)$ at large $\tilde{N}_f$ due to the symmetry $\tilde{N}_i \leftrightarrow \tilde{N}_f$.

5 Discussions and conclusions

In this paper we considered the classical scattering of wavepackets in the unbroken $\lambda\phi^4$ theory. Initial and final field configurations were characterized by energy E , initial N_i and final N_f particle numbers. We found that although the maximal change of particle number in the scattering grows with energy at fixed value of N_i (or N_f), the value of $\frac{|N_f - N_i|}{N_i}$ does not exceed 22% for the considered energy range $E \lesssim 10N_fm$. In the quantum counterpart of the problem the initial and final wavepackets correspond to initial and final coherent states with given average values of energy and particle number. Our classical

²To plot this dependence we additionally found several extrema at the lower boundary for $\tilde{N}_i = 0.01$.

approach to the multiparticle scattering is valid when occupation numbers are large, i.e. $E, N_i, N_f \sim \frac{1}{\lambda}$ and $\lambda \ll 1$. In this study we numerically obtained the classically allowed regions of these processes in the (E, N_f) plane at several values of N_i (or, equivalently, in the (E, N_i) plane at several values of N_f). They are shown in figures 1, 11, 13 and 16. Here we considered spherically symmetric field configurations only. We believe that this simplified consideration has much in common with most general setup.

The scattering processes with the parameters E, N_i and N_f outside the classically allowed regions are classically forbidden. The common lore here is that their probability in quantum theory is expected to be exponentially suppressed, i.e. have the form $\mathcal{A}e^{\frac{1}{\lambda}F}$ with some negative suppression exponent F and a prefactor $\mathcal{A}$. At the same time the probability of the classically allowed processes is not exponentially suppressed. In the most interesting case $2 \rightarrow N_f$ production, with $\lambda N_f \gtrsim 1$, the initial state contains semiclassically small number of particles. Our results show that such processes (as well as $N_f \rightarrow 2$ scattering) lie deeply in the classically forbidden region and therefore their probabilities are expected to be exponentially suppressed at least for $\frac{4\pi}{\lambda}N_f \lesssim 30$ and $E \lesssim 10N_f$. Limitations of our numerical procedure do not permit us to obtain the boundary of the classically allowed region for arbitrarily large values of N_f and E . However, we obtain an evidence for existence of a limiting boundary of the classically allowed region in $(E/N_i, N_f/N_i)$ plane at large N_i .

In summary, our results imply that the probability of the multiparticle production processes $few \rightarrow N_f$ in the unbroken weakly coupled $\lambda\phi^4$ theory is suppressed even at large $\lambda N_f \gtrsim 1$. Therefore, factorial growth of multiparticle amplitudes observed at $\lambda N_f \ll 1$ should stop at its larger values leaving the probability consistent with unitarity. Based on the perturbative results at small λN_f and on the experience with other non-perturbative processes such as instanton-like transitions, false vacuum decay, soliton pair production in particle collisions we expect this suppression most likely to be exponential in the sense of eq. (1.1). Let us note that the analysis described in this study can be (at a cost of a considerably larger CPU time) extended to the case of field configurations with axial symmetry which would be more appropriate for discussion of the limit of two-particle collisions. Also, it would be interesting to apply the method of classical solutions to the scalar theory with the spontaneously broken symmetry which we leave for future study.

Although our findings indicate that $few \rightarrow N_f$ scattering processes lie in the classically forbidden region, they tell nothing about actual value of the probability of these processes. As we already mentioned in the introduction there exist semiclassical methods to calculate the suppression exponent of the probability which, however, are difficult to apply due to singular nature of corresponding classical solutions. On the one hand, one can start, as we did in this paper, with processes $N_i \rightarrow N_f$ where both initial and final states contain large occupation numbers and apply the same semiclassical techniques. In this case based on the results of the semiclassical studies of baryon number violating processes [48, 49] and soliton pair production [50, 51] in particle collisions one can expect that corresponding classical solutions will be nonsingular. The classical solutions at the boundary of the classically allowed region obtained in the present study are expected to be close to the solutions describing transitions in the forbidden region. Using the Rubakov-Son-Tinyakov conjecture [52] the probability of $2 \rightarrow N_f$ scattering in the leading semiclassical approxi-

mation can be obtained from that of $N_i \rightarrow N_f$ process by taking the limit $\lambda N_i \rightarrow 0$ if it exists.³ This procedure can be viewed as a regularization to the semiclassical method of singular classical solutions. We are going to pursue this idea in future work [59].

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³This conjecture was checked in several models [53–56] and was used in refs. [50, 51, 57, 58] for semiclassical calculations of the probability of soliton pair production and false vacuum decay induced by particle collisions.

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