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Large N bilocals at the infrared fixed point of the three dimensional O(N) invariant vector theory with a quartic interaction

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Abstract: We study the three dimensional O(N) invariant bosonic vector model with a $\frac{\lambda}{N}(\phi^a\phi^a)^2$ interaction at its infrared fixed point, using a bilocal field approach and in an $1/N$ expansion. We identify a (negative energy squared) bound state in its spectrum about the large N conformal background. At the critical point this is identified with the $\Delta = 2$ state. We further demonstrate that at the critical point the $\Delta = 1$ state disappears from the spectrum.

Keywords: 1/N Expansion, AdS-CFT Correspondence, Higher Spin Symmetry, Nonperturbative Effects

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Contents

1 Introduction

The AdS/CFT correspondence [1–3] finds a very interesting area of application in the context of the higher spin theories/vector model correspondence [4]. Of particular interest to us, in this context, is the AdS_4/CFT_3 correspondence.[1] Although the higher spin degrees of freedom of Fronsdal and Vasiliev are not those of string theory,[2] there are several reasons why this correspondence is of great interest. These include the absence of supersymmetry and the fact that vector models are "solvable" in the large N limit, allowing for a more concrete and detailed study of the workings of the correspondence, and possibly even providing a definition of (gauge fixed) higher spin theories themselves, through their dual vector valued field theories.

We are in particular interested in and motivated by the constructive approach of [17–21]. In this approach, the singlet sector of $O(N)$ invariant field theories is described in terms of equal time bilocals:

$$\psi_{\vec{x_1}\vec{x_2}} = \sum_{a=1}^{N} \phi^a\left(t, \vec{x_1}\right) \phi^a\left(t, \vec{x_2}\right), \tag{1.1}$$

where $\vec{x_1}$ and $\vec{x_2}$ are two dimensional space vectors. These 5 degrees of freedom and their canonical conjugates are mapped, in the free UV fixed point, to $AdS_4 \times S_1$ where the S_1

[1] There is a vast literature on the subject; [5–12] are representative of the work on the subject, but they do not form by any means an exhaustive list.

[2] For attempts to link the two, see for instance [13–16].

encodes the spin degrees of freedom. Originally formulated in a light cone gauge, the map has since been obtained in a temporal gauge [18, 19]. In momentum space it is a point transformation and is given by:

$$E = E_1 + E_2 = |\vec{p_1}| + \vec{p_2}| \tag{1.2}$$

$$\vec{p} = \vec{p_1} + \vec{p_2} \tag{1.3}$$

$$p^z = 2\sqrt{|\vec{p_1}|\,|\vec{p_2}|}\sin\left(\frac{\varphi_2 - \varphi_1}{2}\right) \tag{1.4}$$

$$\theta = \arctan\left(\frac{2\vec{p_2}\times\vec{p_1}}{(|\vec{p_1}| - |\vec{p_2}|)\,p^z}\right) \tag{1.5}$$

with $\varphi_2 - \varphi_1$ being the angle between $\vec{p_1}$ and $\vec{p_2}$ [18, 19].

The holographic coordinate is given by

$$z = \frac{(\vec{x_1} - \vec{x_2})\cdot(\vec{p_1}|p_2| - \vec{p_2}|p_1|)}{p^z(|\vec{p_1}| + |\vec{p_2}|)}. \tag{1.6}$$

The three dimensional $O(N)$ vector theory with a $\frac{\lambda}{N}(\phi^a\phi^a)^2$ interaction has a IR fixed critical point. At this critical point, the theory is expected to contain a state with dimension $\Delta = 2$, a boundary field in the standard AdS/CFT correspondence with the standard positive branch for the expression of the dimension of the operator [4], and no longer the $\Delta = 1$ state of present in the UV critical point. Although general arguments exist relating the two through a Legendre transformation [22], in practice the IR fixed point is described in terms of a non-linear sigma model [23–29]. In this description, the Lagrange multiplier field is naturally identified with the $\Delta = 2$ state, but it is certainly not apparent that the $\Delta = 1$ is no longer present in the theory, or equivalently, that the constraint is enforced beyond the leading large N order.

It is the purpose of this article to elucidate these issues directly in terms of the $\frac{\lambda}{N}(\phi^a\phi^a)^2$ theory, using a bilocal approach that allows for a systematic expansion in $1/N$. Keeping in mind that the AdS_4/CFT_3 constructive approach developed for the UV free fixed point is canonical (ensuring that the correct number of degrees of freedom are matched), it is important to have a single time bilocal field description of the field theory IR fixed point. This is developed in this article. However, it is not the purpose of this article to discuss the map [18, 19] at the IR fixed point. This is left for a later communication.

This paper is organized as follows: in section 2, the collective field theory [31, 32] Hamiltonian, expressed in terms of equal time bilocal fields and their canonical conjugates, is presented. The large N conformal background is obtained. The bilocal fluctuations about this background are obtained for large but finite λ and the equations of motion cast in the form of a highly non-trivial integral equation in momentum space. In sections 3 and 4, we consider the path integral description of the theory. This requires the introduction of (two-time) covariant bilocals. In section 3, we consider the description of the non-linear sigma model in terms of (two-time) bilocal fields plus the dynamical Lagrange multiplier field. We obtain the two point function for the (shifted) bilocals and for the dynamical Lagrange

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multiplier field. The two point function for the bilocal fields takes the same form as that of the free theory, and the two point function of the dynamical Lagrange multiplier is that of a $\Delta = 2$ conformal field. In section 4, we present the (two time) bilocal description of the $\frac{\lambda}{N}(\phi^a\phi^a)^2$ theory. The two point function of the bilocal fields, which is equivalent to the Bethe-Salpeter equation for the underlying fundamental vector fields, consists of a disconnected (free) piece and a connected diagram describing the s-channel scattering of a composite field. For finite λ, it has a pole at[3]

$$E^2 - (\vec{p}_1 + \vec{p}_2)^2 = -\frac{\lambda^2}{48^2}\,. \tag{1.7}$$

At the critical ($\lambda \to \infty$) point, the connected diagram is identical, up to external leg factors, to the two point function of the dynamical Lagrange multiplier field, and hence is identified with the $\Delta = 2$ state. The disconnected piece is the same as that of the free case.

In section 5, we successfully integrate over an intermediate energy variable in the bound state integral equation identified in the connected diagram of the two point function of section 4, and establish that the result is the same as that obtained in the Hamiltonian approach of section 2. It is left then to understand the role of the disconnected (free) piece of the (two time) bilocal propagator and how the corresponding states are solutions of the quadratic Hamiltonian equations of motion. After all, we require bilocals to construct the bulk. The answer, which we discuss in section 6, is that these are scattering states of positive squared energy. The problem can then be formulated as that of potential scattering off a delta function potential in the relative coordinates. An equation for the local $\Delta = 1$ composite is obtained, and is shown to vanish at the critical point as $\lambda \to \infty$. Further evidence of the disappearance of this state from the spectrum is provided at the level of the path integral approach, where we show that the correlator of two $\Delta = 1$ composites vanishes at the critical point. This makes explicit the conjectured cancellation between disconnected and connected diagrams in the original proposal of Klebanov and Polyakov [4].

In summary, we explicitly demonstrate in this article that in a conformal background, the large N spectrum of the three dimensional $O(N)$ vector model with a $\frac{\lambda}{N}(\phi^a\phi^a)^2$ interaction consists of a (negative energy squared) bound state that at the IR critical point becomes a $\Delta = 2$ state and of scattering states, with an energy dispersion the same as that of the free theory, but with a $\Delta = 1$ state that is removed from the spectrum at the IR critical point.

2 Hamiltonian

Our main interest is to investigate the collective large-N spectrum of the critical $O(N)$ vector-model

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi^a\partial^\mu\phi^a - m^2\phi^a\phi^a\right) - \frac{\lambda}{4!N}(\phi^a\phi^a)^2\,. \tag{2.1}$$

[3] Our Minkowski signature is $(+,-,-)$.

As such, the starting point is an Hamiltonian expressed in terms of equal time collective bilocals and their canonical conjugates. The equal time bilocals have been defined earlier:

$$\psi_{\vec{x}\vec{y}} = \sum_{a=1}^{N} \phi^a(t,\vec{x})\,\phi^a(t,\vec{y})\,. \tag{2.2}$$

Changing variables from the ϕ^a $(a = 1, \cdots, N)$ fields to the bilocals introduces a non-trivial Jacobian that, to leading order in N, is given by

$$\log J = \frac{N}{2}\mathrm{Tr}\log\psi. \tag{2.3}$$

The trace is in (spatial) functional space.

It is by now well known that the collective field theory hamiltonian can be written as [18, 19, 33] ($d = 3$ in this article):[4]

$$H = \frac{2}{N}\mathrm{Tr}\left(\Pi\psi\Pi\right) + \frac{N}{8}\mathrm{Tr}\psi^{-1} + N\int d^{d-1}\vec{x}\left(\frac{1}{2}m^2\psi_{\vec{x}\vec{x}} + \frac{1}{2}\lim_{\vec{y}\to\vec{x}} -\partial^2\psi_{\vec{y}\vec{x}} + \frac{\lambda}{4!}\psi^2_{\vec{x}\vec{x}}\right), \tag{2.4}$$

where the conjugate momentum is

$$\Pi_{\vec{x}\vec{y}} = -i\frac{\delta}{\delta\psi_{\vec{x}\vec{y}}}. \tag{2.5}$$

In the large-N limit the kinetic term is subleading and, with the large N translationally invariant ansatz

$$\psi_{\vec{x}\vec{y}} = \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} e^{i\vec{k}(\vec{x}-\vec{y})}\psi_{\vec{k}}, \tag{2.6}$$

the saddle-point equations yields:

$$\psi^0_{\vec{k}} = \frac{1}{2}\left(\vec{k}^2 + m^2 + \frac{\lambda}{6}\int \frac{d^{d-1}\vec{k}'}{(2\pi)^{d-1}}\psi_{\vec{k}'}\right)^{-1/2}. \tag{2.7}$$

Integrating both sides one obtains the standard gap equation:

$$s = \frac{1}{2}\int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{1}{\sqrt{\vec{k}^2 + m^2 + \frac{\lambda}{6}s}}, \tag{2.8}$$

where

$$s = \int \frac{d^{d-1}\vec{k}'}{(2\pi)^{d-1}}\psi_{\vec{k}'}. \tag{2.9}$$

Defining

$$\alpha = m^2 + \frac{\lambda}{6}s,$$

one has

$$\frac{6}{\lambda}(\alpha - m^2) = \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\frac{1}{2\sqrt{\vec{k}^2+\alpha}} = \int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-\alpha} = \int \frac{d^dk_E}{(2\pi)^d}\frac{1}{k_E^2+\alpha}.$$

[4]The fields have been rescaled $\psi \to N\psi$ to evidence explicitly the N dependence.

Our regularization is defined as:

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{k_E^2+\alpha} = \frac{1}{(4\pi)^{d/2}} \Gamma\left(1-\frac{d}{2}\right) \alpha^{\frac{d-2}{2}} . \tag{2.10}$$

Thus, for $d=3$ one obtains the equation $\alpha + \frac{\lambda}{24\pi}\sqrt{\alpha} - m^2 = 0$. The IR fixed point is associated with the root:

$$\sqrt{\alpha} = \frac{24\pi m^2}{\lambda} + O\left(\frac{m^4}{\lambda^3}\right) \tag{2.11}$$

and is approached by keeping m^2 finite and taking $\lambda \to \infty$. At the critical point then, the background propagator takes the conformal form:

$$\psi^0_{\vec{k}} = \frac{1}{2\sqrt{\vec{k}^2}}, \tag{2.12}$$

and is the $O(N)$ invariant two point function of the underlying scalar fields.

For the next $1/N$ correction, we study the spectrum of fluctuations about the large N conformal background. We let

$$\psi = \psi^0 + \frac{1}{\sqrt{N}}\eta, \qquad \Pi = \sqrt{N}p, \tag{2.13}$$

and expand the Hamiltonian up to quadratic order. We find:

$$H^{(2)} = 2\mathrm{Tr}\left(p\psi^0 p\right) + \frac{1}{8}\mathrm{Tr}\left(\psi_0^{-1}\eta\psi_0^{-1}\eta\psi_0^{-1}\right) + \frac{\lambda}{4!}\int d^{d-1}\vec{x}\eta^2_{\vec{x}\vec{x}}. \tag{2.14}$$

The fluctuations satisfy the Hamiltonian equations of motion. We note:[5]

$$\begin{aligned} \dot{\eta}_{\vec{x}\vec{y}} &= \frac{\delta H_2}{\delta p_{\vec{x}\vec{y}}} \\ &= 2\Big((p\psi_0)_{\vec{y}\vec{x}} + (\psi_0 p)_{\vec{y}\vec{x}}\Big) = \dot{\eta}_{\vec{y}\vec{x}}, \end{aligned} \tag{2.15}$$

and obtain

$$\begin{aligned} \ddot{\eta}_{\vec{x}_1\vec{x}_2} = -\frac{1}{4}\Big(\eta\psi^{0^{-1}}\psi^{0^{-1}} + \psi^{0^{-1}}\eta\psi^{0^{-1}} + \psi^{0^{-1}}\eta\psi^{0^{-1}} + \psi^{0^{-1}}\psi^{0^{-1}}\eta\Big)_{\vec{x}_1\vec{x}_2} \\ + \frac{\lambda}{6}\left(\eta\delta\psi^0\right)_{\vec{x}_1\vec{x}_2} + \frac{\lambda}{6}\left(\psi^0\eta\delta\right)_{\vec{x}_1\vec{x}_2}. \end{aligned} \tag{2.16}$$

We look for eigen-frequencies in momentum space,

$$\begin{aligned} \eta_{\vec{x}_1\vec{x}_2}(t) &= e^{-iEt}\eta_{\vec{x}_1\vec{x}_2} \\ \eta_{\vec{x}_1\vec{x}_2} &= \int \frac{d^{d-1}\vec{k}_1}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}\vec{k}_2}{(2\pi)^{\frac{d-1}{2}}} e^{i\vec{k}_1\vec{x}_1 + i\vec{k}_2\vec{x}_2}\eta_{\vec{k}_1\vec{k}_2}. \end{aligned} \tag{2.17}$$

[5]We consistently use the coordinate exchange symmetry of the bilocals.

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and obtain the following equation for the spectrum of fluctuations:

$$E^2\eta_{\vec{k}_1\vec{k}_2} = \frac{1}{4}\left(\psi^{0^{-1}}_{\vec{k}_1} + \psi^{0^{-1}}_{\vec{k}_2}\right)^2 \eta_{\vec{k}_1\vec{k}_2} + \frac{\lambda}{6}\left(\psi^{0}_{\vec{k}_1} + \psi^{0}_{\vec{k}_2}\right)\int \frac{d^{d-1}\vec{l}}{(2\pi)^{d-1}}\eta_{\vec{k}_1+\vec{k}_2-\vec{l},\vec{l}}, \tag{2.18}$$

Note that for $\lambda = 0$

$$E^2_{\vec{k}_1\vec{k}_2} = \frac{1}{4}\left(\psi^{0^{-1}}_{\vec{k}_1} + \psi^{0^{-1}}_{\vec{k}_2}\right)^2 = \left(|\vec{k}_1| + |\vec{k}_2|\right)^2, \tag{2.19}$$

a result known for some time [33] and at the root of the AdS_4/CFT_3 constructive map [18, 19] in the free UV fixed point. For finite λ, equation (2.18) can be recast in the form:

$$\eta_{\vec{k}_1\vec{k}_2} = \frac{\frac{\lambda}{6}\left(\psi^{0}_{\vec{k}_1} + \psi^{0}_{\vec{k}_2}\right)}{E^2 - \frac{1}{4}\left(\psi^{0^{-1}}_{\vec{k}_1} + \psi^{0^{-1}}_{\vec{k}_2}\right)^2}\int \frac{d^{d-1}\vec{l}}{(2\pi)^{d-1}}\eta_{\vec{k}_1+\vec{k}_2-\vec{l},\vec{l}}. \tag{2.20}$$

A solution only exists for this equation if the following condition is satisfied:

$$1 = \frac{\lambda}{12}\int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{1}{E_p^2 - \left(\left|\vec{k}\right| + \left|\vec{p}-\vec{k}\right|\right)^2}\left(\frac{1}{\left|\vec{k}\right|} + \frac{1}{\left|\vec{p}-\vec{k}\right|}\right). \tag{2.21}$$

This is obtained when we multiply both sides of (2.20) with $\delta\left(\vec{k}_1 + \vec{k}_2 - \vec{p}\right)$ and integrate over $\vec{k}_1$ and $\vec{k}_2$.

We will be able to obtain the solution to (2.20) and (2.21), and show that they correspond to a relativistic bound state with the energy momentum relation given by (1.7). Scattering states with energies given by (2.19) (or (1.2)), are general solutions of (2.18) which can be thought of as a relativistic version of a quantum mechanical potential scattering problem. We will establish that as $\lambda \to \infty$ the $\Delta = 1$ field $\eta_{\vec{x}\vec{x}}$ is no longer present in the spectrum.

In order to do so, in the next two sections we first examine the two-time bilocal formulation of the same problem in the path integral formalism, which is the standard approach to conformal field theories in terms of their correlators, and where one hopes to find a covariant description of the spectrum of the theory by the identification of poles in the appropriate propagators. We will then show how the equations and conditions of this section can be obtained by successfully integrating over an appropriate energy variable.

3 Bilocal description of the non-linear σ model

In this section, we consider the non-linear sigma model in the collective field theory approach. The reason behind this is the argument [23–25] that in the large-N limit, the $O(N)$ vector-model at its infra-red critical point is described by a non-linear sigma model [11, 26–29].

Recall that the action for the non-linear sigma model can be written as[6]

$$S = N\int d^dx\left(\frac{1}{2}\partial_\mu\vec{S}\partial_\mu\vec{S} + \frac{\alpha(x)}{2}\left(\vec{S}^2 - \frac{1}{\lambda}\right)\right) \tag{3.1}$$

with the $\alpha(x)$ field playing the role of a Lagrange multiplier enforcing the constraint $\vec{S}^2 = \frac{1}{\lambda}$.

[6] In this section we use an euclidean signature.

We introduce the covariant bilocals

$$\psi_{xy} = \vec{S}(x) \cdot \vec{S}(y) \,. \tag{3.2}$$

For the log of the Jacobian, we have (e.g. [34]):

$$\log J = \frac{N}{2} \mathrm{Tr} \log \psi \,. \tag{3.3}$$

The trace is now taken in functional (euclidean) space time. In terms of the collective bilocals, the action for the non-linear sigma model reads

$$S_{\text{eff}} = N \left[-\frac{1}{2} \mathrm{Tr} \ln \psi + \int d^d x \left(-\frac{1}{2} \lim_{y \to x} \partial^2 \psi_{xy} + \frac{1}{2} \alpha_x \psi_{xx} - \frac{1}{2\lambda} \alpha_x \right) \right] . \tag{3.4}$$

The large N saddle point equations of motion can be obtained by varying the action above with respect to the bilocals and the Lagrange multiplier.

With a large N translational invariant ansatz

$$\psi_{xy} = \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} \psi_k ,$$

we vary the effective action with respect to the bilocals and find that

$$\psi_k^0 = \frac{1}{k^2 + \alpha} , \tag{3.5}$$

or in coordinate space

$$\psi_{xy} = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik(x-y)}}{k^2 + \alpha} . \tag{3.6}$$

Likewise, varying the effective action with respect to α_x leads us to

$$\psi_{xx} = \frac{1}{\lambda} . \tag{3.7}$$

We then have the gap equation

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + \alpha} = \frac{1}{\lambda} . \tag{3.8}$$

Thus, from (2.10), one has [11, 26–29]

$$\frac{1}{(4\pi)^{d/2}} \Gamma \left(1 - \frac{d}{2} \right) \alpha^{\frac{d-2}{2}} = \frac{1}{\lambda} . \tag{3.9}$$

For $d = 3$,

$$\sqrt{\alpha} = -\frac{4\pi}{\lambda} \tag{3.10}$$

and the critical point is reached as $\lambda \to \infty$, corresponding to the large-N conformal background configuration:

$$\psi_k^0 = \frac{1}{k^2} . \tag{3.11}$$

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To generate $1/N$ corrections, we expand about this large-N background:[7]

$$\alpha = 0 + \frac{1}{\sqrt{N}} \tilde{\alpha}\,(x) \tag{3.12}$$

$$\psi_{xy} = \psi^0_{xy} + \frac{1}{\sqrt{N}} \eta_{xy}\,. \tag{3.13}$$

and insert (3.12) and (3.13) into the effective action, keeping only terms quadratic in the fields. This quadratic effective action reads:

$$S^{(2)}_{\text{eff}} = \frac{1}{4} \text{Tr}\left(\psi_0^{-1} \tilde{\eta} \psi_0^{-1} \tilde{\eta}\right) - \frac{1}{4} \text{Tr}\left(\tilde{\alpha} \psi_0 \tilde{\alpha} \psi_0\right) \tag{3.14}$$

after a shift of the bilocal fields defined by $\tilde{\eta} = \psi_0 \tilde{\alpha} \psi_0 + \eta$ which decouples the η and $\tilde{\alpha}$ fields [35].

We move into momentum space by writing

$$\tilde{\eta}_{xy} = \int \frac{d^d k_1}{(2\pi)^{d/2}} \int \frac{d^d k_2}{(2\pi)^{d/2}} e^{ik_1 x} e^{ik_2 y} \tilde{\eta}_{k_1 k_2} \tag{3.15}$$

$$\tilde{\alpha}_x = \int \frac{d^d k}{(2\pi)^{d/2}} e^{ikx} \tilde{\alpha}_k. \tag{3.16}$$

The quadratic effective action then becomes

$$\begin{aligned} S^{(2)}_{\text{eff}} &= \frac{1}{4} \int d^d k_1 \int d^d k_2 \tilde{\eta}_{k_1 k_2} \left(\psi_0^{-1}\right)_{k_1} \left(\psi_0^{-1}\right)_{k_2} \tilde{\eta}_{-k_2,-k_1} \\ &\quad - \frac{1}{4} \int d^d k_1 \tilde{\alpha}_{k_1} \left(\int \frac{d^d p}{(2\pi)^d} \psi^0_p \psi^0_{k_1+p} \right) \tilde{\alpha}_{-k_1}. \end{aligned} \tag{3.17}$$

Since [17, 26–29]

$$\begin{aligned} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2} \frac{1}{(k-p)^2} &= -\frac{\left(k^2\right)^{\frac{d}{2}-2} \pi \Gamma\left(\frac{d}{2}-1\right)}{(4\pi)^{d/2} \sin\left(\frac{\pi d}{2}\right) \Gamma\left(d-2\right)} \\ &= \frac{1}{8\,|k|}, \qquad d = 3\,. \end{aligned} \tag{3.18}$$

we can then write the quadratic effective action as

$$S^{(2)}_{\text{eff}} = \frac{1}{4} \int d^d k_1 \int d^d k_2 \tilde{\eta}_{k_1 k_2} k_1^2 k_2^2 \tilde{\eta}_{-k_2,-k_1} - \frac{1}{4} \int d^d k_1 \tilde{\alpha}_{k_1} \left(\frac{1}{8\,|k|}\right) \tilde{\alpha}_{-k_1}. \tag{3.19}$$

Therefore, the propagators — which can be read off from the quadratic effective action — are

$$\langle \tilde{\eta}_{k_1 k_2} \tilde{\eta}_{p_1 p_2} \rangle = \frac{2}{k_1^2 k_2^2} \delta\left(k_2 + p_2\right) \delta\left(k_1 + p_1\right) \tag{3.20}$$

[7] In [26–29], Lang and Ruhl introduce an imaginary $\tilde{\alpha}$. Here, we will keep $\tilde{\alpha}$ real, to ensure that the coefficient of the two point function is positive; this is also the choice of Giombi and Yin [11] and the later work of Leonhardt and Ruhl [30].

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and

$$\langle\tilde{\alpha}_{k_1}\tilde{\alpha}_{k_2}\rangle = -16\,|k_1|\,\delta\left(k_1+k_2\right)\,. \tag{3.21}$$

In coordinate space, the above propagators are

$$\begin{aligned}\langle\eta_{x_1x_2}\eta_{x_3x_4}\rangle &= \left(\frac{2^{d-2}}{(4\pi)^{d/2}}\Gamma\left(\frac{d}{2}-1\right)\right)^2\left(\left(x_{13}^2\right)^{1-\frac{d}{2}}\left(x_{24}^2\right)^{1-\frac{d}{2}}+\left(x_{14}^2\right)^{1-\frac{d}{2}}\left(x_{23}^2\right)^{1-\frac{d}{2}}\right)\\ &\to\left(\frac{1}{4\pi}\right)^2\left(\frac{1}{\left(x_{13}^2\right)^{1/2}}\frac{1}{\left(x_{24}^2\right)^{1/2}}+\frac{1}{\left(x_{14}^2\right)^{1/2}}\frac{1}{\left(x_{23}^2\right)^{1/2}}\right)\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\langle\tilde{\alpha}_{x_1}\tilde{\alpha}_{x_2}\rangle &= 2\left[\frac{(4\pi)^{d/2}\sin\left(\frac{\pi d}{2}\right)\Gamma\left(d-2\right)}{\pi\Gamma\left(\frac{d}{2}-1\right)}\right]\int\frac{d^dk}{(2\pi)^d}e^{ik(x_1-x_2)}\left(k^2\right)^{2-\frac{d}{2}}\\ &= 2^5\left(\frac{\sin\left(\frac{\pi d}{2}\right)}{\pi}\right)\frac{\Gamma\left(d-2\right)}{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}-2\right)}\frac{1}{\left(x_{12}^2\right)^2}\\ &\to\frac{16}{\pi^2}\frac{1}{\left(x_{12}^2\right)^2}\qquad d=3.\end{aligned} \tag{3.23}$$

It follows that the conformal scaling dimension of the Lagrange multiplier field is, as expected, $\Delta=2$. However, it is clear that the scaling dimension of the local field η_{xx} is given by $\Delta=1$ — this can be seen by setting $x_1=x_2$ and $x_3=x_4$ in (3.22).

In conclusion, the non linear sigma model points to two types of excitations at the IR critical point: a $\Delta=2$ state, and bilocal excitations identical to the free theory which contain a local $\Delta=1$ state. We will evidence an entirely similar structure in the $\frac{\lambda}{N}(\phi^a\phi^a)^2$ theory propagator in the next section.

4 $(\phi^2)^2$ in two-time bilocal approach

We now consider the path integral formulation of the $\left(\phi^2\right)^2$ theory (2.1), in terms of the two-time (covariant) collective $O(N)$ invariant bilocals:[8]

$$\psi_{xy}=\sum_{a=1}^{N}\phi^a\left(x\right)\phi^a\left(y\right). \tag{4.1}$$

Including the Jacobian (3.3) resulting from the change of variables to bilocal fields, the effective action for the $O(N)$ $\left(\phi^2\right)^2$ vector theory in Minkowski spacetime, is

$$S_{\text{eff}}=N\int d^dx\left[\frac{1}{2}\left(-\lim_{y\to x}\partial_y^2\psi_{xy}\right)-\frac{1}{2}m^2\psi_{xx}-\frac{\lambda}{4!}\left(\psi_{xx}\right)^2\right]-\frac{Ni}{2}\text{Tr}\ln\psi\,. \tag{4.2}$$

[8] Recall the notation $x\equiv x^\mu=(t,\vec{x})$, and similarly for momentum, $k\equiv k^\mu=(E,\vec{k})$. Our signature is $(+,-,-)$.

The large-N background is now a saddle point solution with the translational invariant ansatz

$$\psi_{xy} = \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} \psi_k,$$

and it yields

$$\psi_k^0 = \frac{i}{k^2 - m^2 - \frac{\lambda}{6} \int \frac{d^d k'}{(2\pi)^d} \psi_{k'}^0}. \tag{4.3}$$

The solution to the ensuing gap equation has been described in detail in section 2, as well as the approach and identification of the IR critical point. At the critical point, the large N background takes the conformal form:

$$\psi_k^0 = \frac{i}{k^2}. \tag{4.4}$$

We expand about this large-N background and write

$$\psi_{xy} = \psi_{xy}^0 + \frac{1}{\sqrt{N}} \eta_{xy}\,. \tag{4.5}$$

The quadratic effective action can be written as

$$S_{\rm eff}^{(2)} = \frac{i}{4} \mathrm{Tr}\left(\psi_0^{-1} \eta \psi_0^{-1} \eta\right) - \frac{\lambda}{4!} \int d^d x \eta_{xx}^2 \tag{4.6}$$

or

$$iS_{\rm eff}^{(2)} = -\frac{1}{2} \int d^d k_1 \int d^d k_2 \int d^d k_3 \int d^d k_4 \eta_{k_1 k_2} \hat{O}_{k_1 k_2; k_3 k_4} \eta_{k_3 k_4}, \tag{4.7}$$

with

$$\hat{O}_{k_1 k_2; k_3 k_4} = \frac{1}{2} \psi_{k_3}^{0^{-1}} \psi_{k_4}^{0^{-1}} \delta\left(k_2 + k_3\right) \delta\left(k_1 + k_4\right) + \frac{2i\lambda}{4!} \frac{1}{(2\pi)^d} \delta\left(k_1 + k_2 + k_3 + k_4\right). \tag{4.8}$$

The Fourier transformation has been defined as:

$$\eta_{xy} = \int \frac{d^d k_1}{(2\pi)^{d/2}} \int \frac{d^d k_2}{(2\pi)^{d/2}} e^{-ik_1 x} e^{-ik_2 y} \eta_{k_1 k_2}. \tag{4.9}$$

The inversion of this operator to yield the (collective field bilocal) propagator has been described in [34]. It corresponds to a Bethe-Salpeter equation for quartic correlators of the underlying vector theory. The answer is:[9]

$$\begin{aligned} \hat{O}^{-1}_{k_1 k_2; p_1 p_2} &= 2 \psi_{p_1}^0 \psi_{p_2}^0 \delta\left(k_1 + p_2\right) \delta\left(k_2 + p_1\right) \\ &+ \frac{-\frac{i\lambda}{3} \frac{1}{(2\pi)^d} \psi_{k_1}^0 \psi_{k_2}^0 \psi_{p_1}^0 \psi_{p_2}^0}{1 + \frac{i\lambda}{6} \frac{1}{(2\pi)^d} \int d^d k_1 \int d^d k_2 \delta\left(k_1 + k_2 - p_1 - p_2\right) \psi_{k_1}^0 \psi_{k_2}^0} \delta\left(k_1 + k_2 + p_1 + p_2\right). \end{aligned} \tag{4.10}$$

[9]We freely use the property that $\psi_k^0 = \psi_{-k}^0$.

The integral in the denominator follows from its euclidean version (3.18) with result (in $3d$):

$$\frac{1}{(2\pi)^3}\int d^3k_1\int d^3k_2\delta\left(k_1+k_2-p_1-p_2\right)\psi^0_{k_1}\psi^0_{k_2}=-\frac{i}{8}\frac{1}{|p_1+p_2|_E}\,. \tag{4.11}$$

As a result, the $3d$ two-time collective bilocal propagator is

$$\hat{O}^{-1}_{k_1k_2;p_1p_2}=2\psi^0_{p_1}\psi^0_{p_2}\delta\left(k_1+p_2\right)\delta\left(k_2+p_1\right)+\frac{-\frac{i\lambda}{3}\frac{1}{(2\pi)^3}\psi^0_{k_1}\psi^0_{k_2}\psi^0_{p_1}\psi^0_{p_2}}{1+\frac{\lambda}{48}\frac{1}{|p_1+p_2|_E}}\delta\left(k_1+k_2+p_1+p_2\right)\,. \tag{4.12}$$

The bilocal propagator consists of a free, disconnected piece (identical to the UV critical point) associated with the free propagation of two underlying scalars, and of a s-channel scattering of a composite state. The mass shell condition is as usual obtained by identifying the pole of the propagator, after removal of external legs. The pole condition in (4.12), is

$$1=-\frac{\lambda}{48}\frac{1}{|p_1+p_2|_E}, \tag{4.13}$$

or

$$E^2-\left(\vec{p}_1+\vec{p}_2\right)^2=\left(E_1+E_2\right)^2-\left(\vec{p}_1+\vec{p}_2\right)^2=-\frac{\lambda^2}{48^2}\,. \tag{4.14}$$

It may be of concern that (4.13) was obtained with background large N exact massless propagators whereas for the pole condition λ was kept large, but finite. This is unfounded, as the result (4.11) is finite and does not require regularization. As a matter of fact, it has been obtained in closed form with massive propagators in the first of [26–29], from which one can obtain:

$$|p_1+p_2|_E=-\frac{\lambda}{48}-\frac{4}{\pi}\frac{m^2}{\lambda}+\dots \tag{4.15}$$

The limit $\lambda\to\infty$ results in the finite (independent of λ) propagator presented below.

We also remark that equation (4.13) requires $\lambda<0$, i.e., an attractive quartic potential. This is the case if to agree with the non linear sigma model approach, as it can be seen by comparing (2.11) with (3.10). It is also "natural" in the standard auxiliary field formulation of the quartic theory. A close examination of, for instance, [23–25], shows that for the associated gaussian integral to be well defined, either $\lambda<0$ or the auxiliary field is imaginary. This is related to the discussion in the footnote just before equation (3.12). The discussion of the sign of the coupling is reminiscent of the Gross-Neveu model [36], although in this case it applies to fermionic theories.

Returning to (4.12), we note that at the infra-red fixed point ($\lambda\to\infty$) the propagator takes the finite form:

$$\hat{O}^{-1}_{k_1k_2;p_1p_2}=2\frac{i}{p_1^2}\frac{i}{p_2^2}\delta\left(k_1+p_2\right)\delta\left(k_2+p_1\right)-\frac{i}{k_1^2}\frac{i}{k_2^2}\frac{16i\,|p_1+p_2|_E}{(2\pi)^3}\frac{i}{p_1^2}\frac{i}{p_2^2}\delta\left(k_1+k_2+p_1+p_2\right)\,. \tag{4.16}$$

This result for the bilocal propagator is in direct agreement with the non-linear sigma model results (3.20) and (3.21), up to leg-factors in the $\Delta=2$ channel, confirming the identification of the intermediate state as a $\Delta = 2$ state at criticality. Conformal invariance and dimensional analysis dictates for such a state an infinite "pole" in the two point function. Equation (4.14) makes precise how this limit is approached.[10]

5 From covariant bilocals to equal time bilocals

It is of interest to consider the equations of motion satisfied by the covariant bilocals fluctuations $\hat{O}\eta = 0$. With $\hat{O}$ defined in (4.8), one has:

$$\psi^{0^{-1}}_{k_1}\psi^{0^{-1}}_{k_2}\eta_{k_1k_2} = -\frac{4i\lambda}{4!}\frac{1}{(2\pi)^d}\int d^dk\eta_{k,k_1+k_2-k}\,, \tag{5.1}$$

or

$$k_1{}^2k_2{}^2\eta_{k_1k_2} = \frac{i\lambda}{6}\frac{1}{(2\pi)^d}\int d^dk\eta_{k,k_1+k_2-k}\,. \tag{5.2}$$

In coordinate space,

$$\partial_x^2\partial_y^2\eta_{xy} = \frac{i\lambda}{6}\delta(x-y)\eta_{xx}. \tag{5.3}$$

Rewriting (5.2) as

$$\eta_{k_1k_2} = \frac{i\lambda}{6}\frac{1}{(2\pi)^d}\frac{1}{k_1^2}\frac{1}{k_2^2}\int d^dk\eta_{k,k_1+k_2-k}\,. \tag{5.4}$$

One can show that

$$\eta_{k_1k_2} = \frac{\alpha_{k_1+k_2}}{k_1^2k_2^2}\,, \tag{5.5}$$

is a solution of (5.4) for arbitrary α_k, provided

$$1 = -\frac{4i\lambda}{4!}\frac{1}{(2\pi)^d}\int d^dk_1\psi^0_{k_1}\psi^0_{k_1-p_1-p_2} \tag{5.6}$$

is satisfied. We recognize this equation as the pole condition of the previous section. The $\eta_{k_1k_2}$ of the form (5.5) are nothing but the (momentum space) bound state eigenfunctions.

We have so far two descriptions viz. one in terms of the covariant bilocals and the other one in terms of the single time bilocals. The results in the two descriptions look superficially different. However, the single time equations (2.20) and (2.21) should correspond to equations (5.4) and (5.6) respectively. In the following we show that they are indeed equivalent.

Recall that

$$\begin{aligned}\eta_{xy} = \eta_{t_x,\vec{x};t_y,\vec{y}} &= \int\frac{d^dk_1}{(2\pi)^{d/2}}\int\frac{d^dk_2}{(2\pi)^{d/2}}e^{-ik_1x}e^{-ik_2y}\eta_{k_1k_2}\\ &= \int\frac{dE_1d^{d-1}\vec{k_1}}{(2\pi)^{d/2}}\int\frac{dE_2d^{d-1}\vec{k_2}}{(2\pi)^{d/2}}e^{-iE_1t_x+i\vec{k_1}\vec{x}}e^{-iE_2t_y+i\vec{k_2}\vec{y}}\eta_{E_1\vec{k_1};E_2\vec{k_2}}.\end{aligned} \tag{5.7}$$

[10] In conformal field theories, mass states are not in their Cartan subalgebras, but it is legitimate to discuss the approach to conformal criticality.

Equal time bilocals are obtained from covariant bilocals by setting $t_x = t_y = t$ or, equivalently, they only depend on $E = E_1 + E_2$ and as such can be obtained by integration of an intermediate energy variable (in this case, $E_1 - E_2$).

In the appendix we establish the result:

$$\int \frac{dE}{2\pi} \frac{1}{E^2 - \vec{k}^2 + i\epsilon} \frac{1}{(E - E_p)^2 - \left(\vec{k} - \vec{p}\right)^2 + i\epsilon} = \\ - \frac{i}{2} \frac{1}{E_p^2 - \left(\left|\vec{k}\right| + \left|\vec{k} - \vec{p}\right|\right)^2} \left(\frac{1}{\left|\vec{k}\right|} + \frac{1}{\left|\vec{k} - \vec{p}\right|}\right). \tag{5.8}$$

In other words,

$$\int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{1}{E_p^2 - \left(\left|\vec{k}\right| + \left|\vec{p} - \vec{k}\right|\right)^2} \left(\frac{1}{\left|\vec{k}\right|} + \frac{1}{\left|\vec{p} - \vec{k}\right|}\right) = 2i \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{k^2} \frac{1}{(k - p)^2}. \tag{5.9}$$

Note that $p = p^\mu = (E_p, \vec{p})$, with E_p otherwise arbitrary. These results establish the equivalence of (2.20), (2.21) and equations (5.4), (5.6) respectively. Explicitly, it follows immediately that the pole condition (5.6)

$$1 = \frac{i\lambda}{6} \frac{1}{(2\pi)^d} \int d^d k_1 \frac{1}{k^2} \frac{1}{(k - p_1 - p_2)^2}$$

is equivalent to

$$1 = \frac{\lambda}{12} \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{1}{E_{p_1+p_2}^2 - \left(\left|\vec{k}\right| + \left|\vec{p_1} + \vec{p_2} - \vec{k}\right|\right)^2} \left(\frac{1}{\left|\vec{k}\right|} + \frac{1}{\left|\vec{p_1} + \vec{p_2} - \vec{k}\right|}\right), \tag{5.10}$$

which is nothing but the Hamiltonian pole condition (2.21). The solution is the bound state with dispersion (1.7). At the critical point $\lambda \to \infty$ this state has been identified in the path integral as a $\Delta = 2$ conformal field, both directly in the non-linear sigma model treatment of the IR critical point, and for the $\frac{\lambda}{N}(\phi^a\phi^a)^2$ theory.

6 Potential scattering, the fate of the $\Delta = 1$ state and the $\Delta = 2$ bound state

There is a puzzle when one considers the results of the previous sections. Both bilocal propagators (3.20)–(3.21) and (4.12) display disconnected diagrams identical to those of the (free) UV critical point, in addition to the s-channel $\Delta = 2$ bound state. One would expect the states associated with these disconnected diagrams to include a $\Delta = 1$ boundary state, which should not be present at the IR critical point. On the other hand, it is not clear how these free states are solutions of the Hamiltonian equations of motion (2.18), certainly when written in the form (2.20). But these are the states needed to "build" the bulk.

The answer is that the most general scattering state solution to the equation

$$E^2\eta_{\vec{k}_1\vec{k}_2} = \frac{1}{4}\left(\psi^{0^{-1}}_{\vec{k}_1} + \psi^{0^{-1}}_{\vec{k}_2}\right)^2 \eta_{\vec{k}_1\vec{k}_2} + \frac{\lambda}{6}\left(\psi^{0}_{\vec{k}_1} + \psi^{0}_{\vec{k}_2}\right)\int \frac{d^{d-1}\vec{l}}{(2\pi)^{d-1}}\eta_{\vec{k}_1+\vec{k}_2-\vec{l},\vec{l}}, \tag{6.1}$$

best regarded as a (relativistic) potential scattering problem, is given by[11]

$$\eta_{\vec{k}_1,\vec{p}-\vec{k}_1} = \varphi_{\vec{k}_1,\vec{p}-\vec{k}_1} + \frac{\frac{\lambda}{12}\left(\frac{1}{|\vec{k}_1|} + \frac{1}{|\vec{p}-\vec{k}_1|}\right)}{E^2 - \left(\left|\vec{k}_1\right| + \left|\vec{p}-\vec{k}_1\right|\right)^2}\int \frac{d^2\vec{l}}{(2\pi)^2}\eta_{\vec{l},\vec{p}-\vec{l}}, \tag{6.2}$$

where $\varphi_{\vec{k}_1\vec{k}_2}$ solves the free equation of motion, i.e.,

$$E^2 = (|\vec{p}_1| + |\vec{p}_2|)^2, \qquad \varphi_{\vec{k}_1\vec{k}_2} \sim \delta\left(\vec{k}_1 - \vec{p}_1\right)\delta\left(\vec{k}_2 - \vec{p}_2\right). \tag{6.3}$$

Integrating both sides of (6.2) leads to

$$\int \frac{d^2\vec{k}}{(2\pi)^2}\eta_{\vec{k},\vec{p}-\vec{k}} = \int \frac{d^2\vec{k}}{(2\pi)^2}\varphi_{\vec{k},\vec{p}-\vec{k}} - \frac{\lambda}{48\,|p|_E}\int \frac{d^2\vec{k}}{(2\pi)^2}\eta_{\vec{k},\vec{p}-\vec{k}}, \tag{6.4}$$

where in order to arrive at the final expression we have made use of (4.11) and (5.9). Note that

$$p^\mu = (|\vec{p}_1| + |\vec{p}_2|, \vec{p}_1 + \vec{p}_2)\,, \tag{6.5}$$

with p_E the corresponding euclidean 3-vector. Thus,

$$\int \frac{d^2\vec{k}}{(2\pi)^2}\eta_{\vec{k},\vec{p}-\vec{k}} = \frac{1}{1+\frac{\lambda}{48|p|_E}}\int \frac{d^2\vec{k}}{(2\pi)^2}\varphi_{\vec{k},\vec{p}-\vec{k}}\,. \tag{6.6}$$

This can be substituted back into (6.2), resulting in the equivalent expression

$$\int \frac{d^2\vec{k}}{(2\pi)^2}\eta_{\vec{k},\vec{p}-\vec{k}} = \left(1 - \frac{\lambda}{48\,|p|_E}\frac{1}{1+\frac{\lambda}{48|p|_E}}\right)\int \frac{d^2\vec{k}}{(2\pi)^2}\varphi_{\vec{k},\vec{p}-\vec{k}}\,. \tag{6.7}$$

Either way, at the infra-red conformal point $\lambda \to \infty$,

$$\eta_{xx} \sim \int \frac{d^2\vec{k}}{(2\pi)^2}\eta_{\vec{k},\vec{p}-\vec{k}} \to 0\,, \tag{6.8}$$

and this state with $\Delta = 1$ is removed from the spectrum.

[11]Here, $\vec{p} = \vec{k}_1 + \vec{k}_2$.

We provide a further check by examining the propagator $\langle \eta_{xx}\eta_{yy}\rangle$, in the path integral approach:

$$\begin{aligned}\langle \eta_{xx}\eta_{yy}\rangle &= \int \frac{d^3k_1}{(2\pi)^{3/2}}\frac{d^3k_2}{(2\pi)^{3/2}}\frac{d^3p_1}{(2\pi)^{3/2}}\frac{d^3p_2}{(2\pi)^{3/2}} e^{-ix(k_1+k_2)}e^{-iy(p_1+p_2)}\hat{O}^{-1}_{k_1k_2;p_1p_2} \\ &= \int \frac{d^3k_1}{(2\pi)^{3/2}}\frac{d^3k_2}{(2\pi)^{3/2}}\frac{d^3p_1}{(2\pi)^{3/2}}\frac{d^3p_2}{(2\pi)^{3/2}} e^{-ix(k_1+k_2)}e^{-iy(p_1+p_2)} \\ &\quad\times\left(-2\frac{1}{p_1^2}\frac{1}{p_2^2}\delta\left(k_1{+}p_2\right)\delta\left(k_2{+}p_1\right) - \frac{16i\left|p_1{+}p_2\right|_E}{(2\pi)^3}\frac{1}{k_1^2}\frac{1}{k_2^2}\frac{1}{p_1^2}\frac{1}{p_2^2}\delta\left(k_1{+}k_2{+}p_1{+}p_2\right)\right) \\ &= -2\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{e^{ik_1(y-x)}}{k_1^2}\frac{e^{ik_2(y-x)}}{k_2^2} \\ &\quad -16i\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\left(\int \frac{d^3p_1}{(2\pi)^3}\frac{1}{p_1^2}\frac{1}{(p_1{+}k_1{+}k_2)^2}\right)\frac{e^{ik_1(y-x)}}{k_1^2}\frac{e^{ik_2(y-x)}}{k_2^2}\left|k_1{+}k_2\right|_E .\end{aligned} \tag{6.9}$$

From (4.11), we have

$$\int \frac{d^3p_1}{(2\pi)^3}\frac{1}{p_1^2}\frac{1}{(p_1+k_1+k_2)^2} = \frac{i}{8\left|k_1+k_2\right|_E}. \tag{6.10}$$

Accordingly,

$$\begin{aligned}&-16i\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\left(\int \frac{d^3p_1}{(2\pi)^3}\frac{1}{p_1^2}\frac{1}{(p_1+k_1+k_2)^2}\right)\frac{e^{ik_1(y-x)}}{k_1^2}\frac{e^{ik_2(y-x)}}{k_2^2}\left|k_1+k_2\right|_E \\ &\quad= -16i\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\left(\frac{i}{8\left|k_1+k_2\right|_E}\right)\frac{e^{ik_1(y-x)}}{k_1^2}\frac{e^{ik_2(y-x)}}{k_2^2}\left|k_1+k_2\right|_E \\ &\quad= -2\left(-1\right)\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{e^{ik_1(y-x)}}{k_1^2}\frac{e^{ik_2(y-x)}}{k_2^2}.\end{aligned} \tag{6.11}$$

Thus,

$$\begin{aligned}\langle \eta_{xx}\eta_{yy}\rangle &= -2\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{e^{ik_1(y-x)}}{k_1^2}\frac{e^{ik_2(y-x)}}{k_2^2} - 2\left(-1\right)\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{e^{ik_1(y-x)}}{k_1^2}\frac{e^{ik_2(y-x)}}{k_2^2} \\ &= 0 .\end{aligned} \tag{6.12}$$

How is one able to extract $\Delta = 2$ correlators from the bilocal fields? This is suggested from the discussion after equation (4.16) and the eigenfunctions (5.5):

$$\eta_{xy} = (\psi^0\alpha\psi^0)_{xy}, \tag{6.13}$$

where α is the $\Delta = 2$ field. Inverting,

$$\alpha(x)\delta(x-y) = (\psi^{0^{-1}}\eta\psi^{0^{-1}})_{xy} . \tag{6.14}$$

Note that there is no a priori guarantee from this definition that correlators calculated with the right hand side of the above equation will always appear multiplied by a delta function, allowing one to extract α correlators. We will show this to be the case.

For convenience, we change to an euclidean signature (with the Jacobian remaining unchanged, this simply requires $\lambda \to -i\lambda$ and $\psi^0_k = 1/k^2$) so that the critical bilocal propagator (4.16) takes the form:

$$\hat{O^E}^{-1}_{k_1k_2;p_1p_2} = 2\frac{1}{p_1^2}\frac{1}{p_2^2}\delta\left(k_1+p_2\right)\delta\left(k_2+p_1\right) - \frac{1}{k_1^2}\frac{1}{k_2^2}\frac{16\left|p_1+p_2\right|_E}{(2\pi)^3}\frac{1}{p_1^2}\frac{1}{p_2^2}\delta\left(k_1+k_2+p_1+p_2\right). \tag{6.15}$$

One has

$$\begin{aligned}&\left\langle ({\psi^0}^{-1}\eta{\psi^0}^{-1})_{x_1y_1}({\psi^0}^{-1}\eta{\psi^0}^{-1})_{x_2y_2}\right\rangle\\&\quad= \int \frac{d^3k_1}{(2\pi)^{3/2}}\frac{d^3k_2}{(2\pi)^{3/2}}\frac{d^3p_1}{(2\pi)^{3/2}}\frac{d^3p_2}{(2\pi)^{3/2}}e^{ik_1x_1}e^{ik_2y_1}e^{ip_1x_2}e^{ip_2y_2}k_1^2k_2^2p_1^2p_2^2\left\langle \eta_{k_1k_2}\eta_{p_1p_2}\right\rangle .\end{aligned} \tag{6.16}$$

The contribution from the connected piece of the bilocal propagator is

$$\begin{aligned}&\left\langle ({\psi^0}^{-1}\eta{\psi^0}^{-1})_{x_1y_1}({\psi^0}^{-1}\eta{\psi^0}^{-1})_{x_2y_2}\right\rangle\\&\quad= \delta(x_1-y_1)\delta(x_2-y_2)\int\frac{d^3p}{(2\pi)^3}e^{ip(y_2-y_1)}(-16|p|) = \delta(x_1-y_1)\delta(x_2-y_2)\left\langle\alpha(x_1)\alpha(x_2)\right\rangle\end{aligned} \tag{6.17}$$

in agreement with (3.21). For the contribution from the disconnected piece of the bilocal propagator, recall that we consistently use:

$$\int d^dp(p^2)^\alpha e^{ipx} = \pi^{d/2}2^{2\alpha+d}\frac{\Gamma(\alpha+d/2)}{\Gamma(-\alpha)}\left(x^2\right)^{-d/2-\alpha}. \tag{6.18}$$

With this definition, it is straightforward to check that the disconnected contributions are proportional to $1/\Gamma(-1)$ and hence vanish.[12]

7 Summary and outlook

The $O(N)$ invariant $\lambda\phi^4$ theory in 3 dimensions has been studied systematically in a $1/N$ expansion at its infrared critical point, both in the Hamiltonian as well as in the path integral formalism. This systematic $1/N$ expansion is generated through $O(N)$ invariant bilocals following the collective field theory method of Jevicki and Sakita [31, 32]. The presence of bilocal scattering states in the spectrum of the theory with free dispersion

[12]There is also a concept of orthogonality. One can easily show that $\left\langle \eta_{xx}({\psi^0}^{-1}\eta{\psi^0}^{-1})_{x_1y_1}\right\rangle \sim 0 \quad \delta(x_1-y_1)$, as it should be the case for two fields with different conformal dimensions. This follows from a cancellation, again, between the contributions of the connected and disconnected pieces of the critical bilocal propagator.

relations needed to generate the bulk according to the map of [18, 19] was established, and the nature of the $\Delta = 1$ and $\Delta = 2$ fields has been elucidated. These fields have different origins. The $\Delta = 1$ field is part of the bilocal scattering states, and it has been explicitly demonstrated in this article that it is absent from the spectrum at criticality. Marginally away from criticality and in the large N conformal background, the $\Delta = 2$ state is identified with a negative energy squared s-channel bound state with a finite (independent of λ) two point function at criticality.

A related simpler model that can be used to provide physical intuition to the features identified in this article is the non-relativistic one dimensional quantum mechanics with an attractive delta function potential $V = v_0\delta(x)$,$v_0 < 0$. As is well known, this system has scattering states with $E > 0$ and one bound state. In the limit that $v_0 \to -\infty$, an argument entirely similar to the one leading to (6.8) shows that $\psi(0) = 0$. In other words, the particles are prevented from "falling into the (infinitely deep) well". Despite the bound state having infinite negative energy in this limit, the second quantized two point function is finite and independent of v_0, in analogy with the critical bilocal propagator presented above.

The general picture that emerges is then clear, following from the general properties of the map [18, 19], and particularly from (1.6). Of the states in the bulk, the $\Delta = 1$ state $\eta_{\vec{x}_1\vec{x}_1}$ (it follows from (1.6) that $\vec{x}_2 \to \vec{x}_1$ corresponds to the boundary) is not present at the boundary. In terms of the bilocal description, these scattering states are prevented from reaching the boundary. At the boundary, as it follows from the identification (6.14), a decoupled $\Delta = 2$ state is present which originates from a bound state in the three dimensional field theory.

Of immediate future interest is to include this state in the map of [18, 19]. Furthermore, as a spin 2, $\Delta = 2$ state with exponential real time dependence, it deserves further study.

A Integrating out the intermediate energy variable

In this appendix, we derive the result in (5.8), viz.

$$\int \frac{dE}{2\pi} \frac{1}{E^2 - \vec{k}^2 + i\epsilon} \frac{1}{(E - E_p)^2 - \left(\vec{k} - \vec{p}\right)^2 + i\epsilon}$$
$$= -\frac{i}{2} \times \frac{1}{E_p^2 - \left(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)^2} \left(\frac{1}{\left|\vec{k}_1\right|} + \frac{1}{\left|\vec{k}_2\right|}\right). \tag{A.1}$$

Define

$$f(E) = \frac{1}{E^2 - \vec{k}^2 + i\epsilon} \frac{1}{(E - E_p)^2 - \left(\vec{k} - \vec{p}\right)^2 + i\epsilon}. \tag{A.2}$$

The poles of the integrand are at $E = \pm\left(\left|\vec{k}\right| - i\epsilon\right)$ and $E = E_p \pm \left(\left|\vec{k} - \vec{p}\right| - i\epsilon\right)$. We will choose to close the contour along the LHP. As a result, we need to compute the residues at $E = \left|\vec{k}\right| - i\epsilon$ and $E = E_p + \left|\vec{k} - \vec{p}\right| - i\epsilon$.

The residue at $E = \left|\vec{k}\right| - i\epsilon$ is

$$\mathrm{Res}\left[f\left(\left|\vec{k}\right|\right)\right] = \frac{1}{2\left|\vec{k}\right|}\frac{1}{\left|\vec{k}\right| - E_p - \left|\vec{k}-\vec{p}\right|}\frac{1}{\left|\vec{k}\right| - E_p + \vec{k}-\vec{p}} \tag{A.3}$$

and the one at $E = E_p + \left|\vec{k}-\vec{p}\right| - i\epsilon$ yields

$$\mathrm{Res}\left[f\left(E_p + \left|\vec{k}-\vec{p}\right|\right)\right] = \frac{1}{2\left|\vec{k}-\vec{p}\right|}\frac{1}{E_p + \left|\vec{k}-\vec{p}\right| - \left|\vec{k}\right|}\frac{1}{E_p + \left|\vec{k}-\vec{p}\right| + \left|\vec{k}\right|}. \tag{A.4}$$

Using the residue theorem, we have

$$\begin{aligned}\int \frac{dE}{(2\pi)} f(E) = -i \Bigg[&\frac{1}{2\left|\vec{k}\right|}\frac{1}{\left|\vec{k}\right| - E_p - \left|\vec{k}-\vec{p}\right|}\frac{1}{\left|\vec{k}\right| - E_p + \left|\vec{k}-\vec{p}\right|} \\ &+ \frac{1}{2\left|\vec{k}-\vec{p}\right|}\frac{1}{E_p + \left|\vec{k}-\vec{p}\right| - \left|\vec{k}\right|}\frac{1}{E_p + \left|\vec{k}-\vec{p}\right| + \left|\vec{k}\right|}\Bigg].\end{aligned} \tag{A.5}$$

We define $\vec{k} = \vec{k_1}$ and $\vec{p} - \vec{k} = \vec{k_2}$. and symmetrize the r.h.s. of (A.5). This leads us to the result

$$\begin{aligned}&\int \frac{dE}{(2\pi)} f(E) = \\ &-\frac{i}{4}\Bigg[\frac{1}{\left|\vec{k}_1\right|}\frac{1}{\left|\vec{k}_1\right| - E_p - \left|\vec{k}_2\right|}\frac{1}{\left|\vec{k}_1\right| - E_p + \left|\vec{k}_2\right|} + \frac{1}{\left|\vec{k}_2\right|}\frac{1}{\left|\vec{k}_2\right| + E_p - \left|\vec{k}_1\right|}\frac{1}{\left|\vec{k}_2\right| + E_p + \left|\vec{k}_1\right|} \\ &+ \frac{1}{\left|\vec{k}_2\right|}\frac{1}{\left|\vec{k}_2\right| - E_p - \left|\vec{k}_1\right|}\frac{1}{\left|\vec{k}_2\right| - E_p + \left|\vec{k}_1\right|} + \frac{1}{\left|\vec{k}_1\right|}\frac{1}{\left|\vec{k}_1\right| + E_p - \left|\vec{k}_2\right|}\frac{1}{\left|\vec{k}_1\right| + E_p + \left|\vec{k}_2\right|}\Bigg].\end{aligned} \tag{A.6}$$

After some trivial but tedious manipulations, we obtain

$$\begin{aligned}&\int \frac{dE}{(2\pi)} f(E) \\ &= -\frac{i}{4}\frac{1}{\left(E_p^2 - \left(\left|\vec{k}_1\right| - \left|\vec{k}_2\right|\right)^2\right)\left(E_p^2 - \left(\left|\vec{k}_1\right| + \left|\vec{k}_2\right|\right)^2\right)} \\ &\times \frac{2}{\left|\vec{k}_1\right|}\left[\left(E_p^2 + \left|\vec{k}_1\right|^2 - \left|\vec{k}_2\right|^2\right) + \frac{2}{\left|\vec{k}_2\right|}\left(E_p^2 + \left|\vec{k}_2\right|^2 - \left|\vec{k}_1\right|^2\right)\right]\end{aligned}$$

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$$
\begin{aligned}
&= -\frac{i}{2}\frac{1}{\left(E_p^2-\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)^2\right)\left(E_p^2+\left(\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)^2\right)}\left[E_p^2\left(\frac{1}{\left|\vec{k}_1\right|}+\frac{1}{\left|\vec{k}_2\right|}\right)\right.\\
&\qquad \left.-\left(\left|\vec{k}_2\right|^2-\left|\vec{k}_1\right|^2\right)\left(\frac{1}{\left|\vec{k}_1\right|}-\frac{1}{\left|\vec{k}_2\right|}\right)\right] = -\frac{i}{2}\frac{1}{\left(E_p^2-\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)^2\right)\left(E_p^2-\left(\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)^2\right)}\\
&= -\frac{i}{2}\frac{1}{\left(E_p^2-\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)^2\right)\left(E_p^2-\left(\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)^2\right)}\left[\frac{E_p^2}{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\left(\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)\right.\\
&\qquad \left.-\frac{\left(\left|\vec{k}_2\right|^2-\left|\vec{k}_1\right|^2\right)\left(\left|\vec{k}_2\right|-\left|\vec{k}_1\right|\right)}{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\right]\\
&= -\frac{i}{2}\frac{1}{\left(E_p^2-\left(\left|\vec{k}_1\right|-\left|\vec{k}_2\right|\right)^2\right)\left(E_p^2-\left(\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)^2\right)}\frac{\left(\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)}{\left|\vec{k}_1\right|\left|\vec{k}_2\right|}\left[E_p^2-\left(\left|\vec{k}_2\right|-\left|\vec{k}_1\right|\right)^2\right]\\
&= -\frac{i}{2}\frac{1}{E_p^2-\left(\left|\vec{k}_1\right|+\left|\vec{k}_2\right|\right)^2}\left(\frac{1}{\left|\vec{k}_1\right|}+\frac{1}{\left|\vec{k}_2\right|}\right) \qquad \text{(A.7)}
\end{aligned}
$$

which is what we set out to prove in the beginning.

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