SO(32) heterotic standard model vacua in general Calabi-Yau compactifications

Hajime Otsuka and Kenta Takemoto

*Department of Physics, Waseda University,
Tokyo 169-8555, Japan*

E-mail: h.otsuka@aoni.waseda.jp, phys-tm.3000@asagi.waseda.jp

ABSTRACT: We study a direct flux breaking scenario in SO(32) heterotic string theory on general Calabi-Yau threefolds. The direct flux breaking, corresponding to hypercharge flux breaking in the F-theory context, allows us to derive the Standard Model in general Calabi-Yau compactifications. We present a general formula leading to the three generations of quarks and leptons and no chiral exotics in a background-independent way. As a concrete example, we show the three-generation model on a complete intersection Calabi-Yau threefold.

KEYWORDS: Superstring Vacua, Superstrings and Heterotic Strings

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1 Introduction

String theory is an attractive candidate not only for a theory of quantum gravity but for a unified theory of elementary forces. It predicts phenomenologically promising higher-dimensional non-abelian gauge theories which are expected to include the Standard Model (SM) as well as its realistic spectra. Indeed, string theory naturally incorporates non-abelian gauge groups, appearing from stacks of D-branes in type I and II string theories, seven-branes in F-theory and closed string sector in heterotic string theories.

Non-trivial gauge backgrounds such as internal gauge fluxes and Wilson lines play an important role in breaking the higher-rank gauge group down to the SM one [1, 2], but Wilson lines are only allowed for restricted non-simply-connected manifolds [2]. As an example, there exist 195 non-simply-connected complete intersection Calabi-Yau (CY) threefolds (CICYs) among 7820 CICYs [3–5]. Hence, it is interesting to check whether or not gauge fluxes directly lead to the SM gauge group. This approach allows us to open up a new window for the string model building in more general CY compactifications. In the context of F-theory grand unified theories (GUTs), such a direct flux breaking called hypercharge flux breaking is an attractive scenario to break the $SU(5)$ gauge group [6, 7] (also discussed in the dual heterotic string side [8–11]). In contrast, the authors of ref. [12] pointed out that the realization of hypercharge flux scenario is generically difficult to achieve in $E_8 \times E_8$ heterotic string theory on smooth CY threefolds due to the large number of index conditions, corresponding to the three generations of quarks and leptons. It therefore motivates us to search for models with a hypercharge flux in other string theories.

In this paper, we systematically study $\text{SO}(32)$ heterotic line bundle models as a realization of direct flux breaking.¹ In a way similar to ref. [12], we search for the three-generation SM against several branchings of $\text{SO}(32)$ in a background-independent way. After solving a large number of index conditions for each elementary particle, together with the K-theory condition and hypercharge masslessness conditions, it turns out that the SM-like spectrum can be realized in general CY compactifications. Note that supersymmetric and stability conditions are required to be checked for each CY threefold.

The remainder of this paper is as follows. In section 2, we first show the model-building approach to derive heterotic Standard Models on smooth CY threefolds. After discussing the hypercharge direction for several group decompositions of $\text{SO}(32)$ and corresponding spectrum, we next present the general formula satisfying the index conditions, K-theory condition and hypercharge masslessness conditions. The obtained formula is applicable to general CY threefolds. In section 3, the specific MSSM (minimal supersymmetric Standard Model)-like spectrum is shown for a concrete CICY. In appendix A, we present the algorithm to compute the particle spectrum for all the group decompositions discussed in this paper.

2 Direct flux breaking in heterotic string

2.1 Model-building approach

In the context of heterotic string, internal gauge fluxes open up wide possibilities of constructing the SM. Among a lot of possibilities of group decomposition, we assume that $\text{SO}(32)$ gauge group decomposes as follows:

$$\text{SO}(32) \rightarrow \text{SO}(2m) \times \text{SO}(32-2m) \rightarrow \text{SU}(3)_C \times \text{SU}(2)_L \times \prod_{a=1}^{m-3} \text{U}(1)_a \times \text{SO}(32-2m), \quad (2.1)$$

where $\text{SO}(2m)$ gauge group includes the SM one. Now $\text{U}(1)$ s descend from $\text{U}(m) \subset \text{SO}(2m)$ and their number is restricted to be $a = 1, 2, \dots, m-3$, where 3 is the rank of non-abelian gauge groups in the SM. In particular we focus on line bundles L_a each with structure group $\text{U}(1)_a$, that is the internal bundle of the form

$$W = \oplus_a L_a. \quad (2.2)$$

Then, $\text{U}(1)_a$ gauge fluxes are inserted into the Cartan direction of $\text{SO}(2m)$ to realize the SM-like gauge group. We expand $\text{U}(1)_a$ gauge field strengths $\text{tr}(\bar{F}_a)$ in the basis of $H^{1,1}(\text{CY}, \mathbb{R})$ w_i , $i = 1, 2, \dots$, $h^{1,1}$ with $h^{1,1}$ being a hodge number of CY, namely

$$\text{tr}(\bar{F}_a) = 2\pi \sum_{i=1}^{h^{1,1}} \text{tr}(T_a) m_a^{(i)} w_i, \quad (2.3)$$

where T_a are $\text{U}(1)_a$ generators descending from $\text{U}(m) \subset \text{SO}(2m)$. Here and in what follows, “tr” denotes the trace in the fundamental representation and $m_a^{(i)}$ are integers constrained

¹See, refs. [13–15] for $\text{SO}(32)$ heterotic line bundle models using Wilson lines.

by the Dirac quantization condition. Note that the hypercharge is a linear combination of $U(1)_a$, namely

$$U(1)_Y = \sum_a f_a U(1)_a, \quad (2.4)$$

with normalization factors f_a . Throughout this paper, we assume the uncorrelated $U(1)_a$ line bundles, otherwise the $U(1)$ gauge groups are enhanced to be a non-abelian one.

According to the group decomposition (2.1), the adjoint representation of $SO(32)$, corresponding to the massless mode in the heterotic string, decomposes under $SO(2m) \times SO(32-2m)$,

$$496 \rightarrow (\text{Adj}_{SO(2m)}, 1) \oplus (2m, 32-2m) \oplus (1, \text{Adj}_{SO(32-2m)}), \quad (2.5)$$

where the adjoint representation of $SO(2m)$, $\text{Adj}_{SO(2m)}$, is expected to include the SM particles, whereas the vector and adjoint representations of $SO(32-2m)$ correspond to the exotic particles. On this line bundle background, the net number of chiral zero-modes with $U(1)_a$ charges Y_a is counted by the index

$$\chi(\otimes_{a=1}^n L_a^{Y_a}) = \frac{1}{6} \sum_{a,b,c} X_{abc} Y_a Y_b Y_c + \frac{1}{12} \sum_a Z_a Y_a, \quad (2.6)$$

where we consider internal bundles $\otimes_{a=1}^n L_a^{Y_a}$ and for the later purpose, we define

$$Z_a \equiv \sum_{i=1}^{h^{1,1}} c_{2,i} m_a^{(i)}, \quad X_{abc} \equiv \sum_{i,j,k=1}^{h^{1,1}} d_{ijk} m_a^{(i)} m_b^{(j)} m_c^{(k)}. \quad (2.7)$$

Here, d_{ijk} are the intersection numbers of the basis of two-forms w_i and the second Chern class of the tangent bundle of CY threefolds is expanded in their Hodge dual four-forms $\hat{w}^i$, namely $c_2(T\mathcal{M}) = \sum_i c_{2,i} \hat{w}^i$. Variables $\{X_{abc}, Z_a\}$ are written in terms of the internal gauge fluxes $m_a^{(i)}$ along $U(1)_a$ with generators T_a descending from $U(m) \subset SO(2m)$. Since X_{abc} are totally symmetric tensors with respect to a, b, c from the fact that d_{ijk} are totally symmetric tensors with respect to i, j, k , we note that variables $\{X_{abc}, Z_a\}$ consist of $m-3C_3+2(m-3C_2)+m-3C_1=m-1C_3$ and $m-3$ degrees of freedom, totally $m-1C_3+m-3$, determined by the values of gauge fluxes and topological data of CY.

The aim of this work is to search for variables $\{X_{abc}, Z_a\}$ leading to the three-generation SM without specifying the topological data of CY. An advantage of this approach is the possibility to perform the systematic search on general CY manifolds. Before going to the detailed analysis, we remark three consistency conditions in the four-dimensional effective action of heterotic string. (For more details, see, e.g., refs. [8, 9].) First one is the ‘‘K-theory condition’’ to admit the spinorial representation in the first excited mode [16, 17]

$$c_1(W) = \sum_a \text{tr}(T_a) m_a^{(i)} w_i = 2\kappa^{(i)} w_i \in H^2(\text{CY}, 2\mathbb{Z}), \quad (2.8)$$

where $c_1(W)$ is the first Chern class of the total internal bundle W and $\kappa^{(i)}$ denote integers.

Second one is the stability condition of our discussed four-dimensional effective action. Stability of the effective action requires a positive number of heterotic five-branes, constraining the background curvatures through the anomaly cancellation condition,

$$\text{ch}_2(W) + c_2(T\mathcal{M}) = \sum_i N_i \hat{w}^i, \quad (2.9)$$

where $\text{ch}_2(W)$ is the second Chern character of the internal bundle and N_i is the number of heterotic five-branes wrapping the internal holomorphic two-cycles on the CY threefold. We stress that the N_i must be positive for supersymmetry to be preserved. Note that, in the perturbative heterotic string vacua ($N_i = 0$), by multiplying the stability condition (2.9) by $m_a^{(i)}$, we obtain

$$\sum_b \frac{1}{2} \text{tr}(T_b)^2 X_{abb} + Z_a = 0 \quad (2.10)$$

for all $a = 1, 2, \dots, m-3$.

In addition, internal bundles have to satisfy the zero-slope poly-stability conditions, namely D -term conditions associated to $U(1)_a$ gauge symmetries,

$$D_a \propto \frac{1}{2} \sum_{i,j,k} d_{ijk} t_i t_j \text{tr}(T_a) m_a^{(k)} + \frac{e^{2\phi_{10}}}{6} \left(\sum_{i,j,k} \text{tr}(T_a^3) d_{ijk} m_a^{(i)} m_a^{(j)} m_a^{(k)} + \frac{1}{8} \sum_i \text{tr}(T_a) m_a^{(i)} c_{2,i} \right) = 0, \quad (2.11)$$

where ϕ_{10} is the ten-dimensional dilaton and the Kähler form is now expanded as $J = l_s^2 \sum_i t_i w_i$ with t_i being the Kähler moduli in string units $l_s = 2\pi\sqrt{\alpha'} = 1$.² Here, we include the one-loop correction to D -terms [8, 9].

The last one is the hypercharge masslessness conditions. On the non-trivial gauge background, the internal gauge fluxes induce the Stückelberg couplings between string axions and the hypercharge gauge boson through the Green-Schwarz terms [18, 19]. It is known that some of $U(1)$ s are anomalous due to the internal gauge fluxes and their number is counted by the rank of $U(1)$ mass matrix in string units $l_s = 2\pi\sqrt{\alpha'} = 1$ [8, 9],

$$M_{ai} = \begin{cases} 2\pi \text{tr}(T_a^2) m_a^{(i)} & \text{for } i = 1, 2, \dots, h^{1,1} \\ \sum_{b,c,d} \frac{1}{6} \text{tr}(T_a T_b T_c T_d) X_{bcd} + \frac{1}{24} \text{tr}(T_a^2) Z_a & \text{for } i = 0 \end{cases}, \quad (2.12)$$

where the first and second lines are coming from Stückelberg couplings of Kähler axions and dilaton axion, respectively. To ensure the masslessness of the hypercharge direction $U(1)_Y = \sum_a f_a U(1)_a$, we impose two constraints originating from Kähler axions and dilaton axion,

$$\sum_a \text{tr}(T_a^2) f_a m_a^{(i)} = 0, \quad (2.13)$$

$$\sum_{a,b,c,d} \text{tr}(T_a T_b T_c T_d) f_a X_{bcd} = 0, \quad (2.14)$$

²Note that we now use a different notation for the Kähler moduli, $\mathcal{T}_i = \frac{1}{2\pi}(t_i + i b_i^{(0)})$ compared with ref. [9]. Here, $b_i^{(0)}$ denote the model-dependent axions.

where we note that $\sum_a \text{tr}(T_a^2) f_a Z_a = 0$ is satisfied under the constraint (2.13).

Let us take a closer look at the K-theory condition (2.8) and hypercharge masslessness conditions (2.13) which are summarized as

$$\begin{pmatrix} \text{tr}(T_1) & \text{tr}(T_2) \\ \text{tr}(T_1^2)f_1 & \text{tr}(T_2^2)f_2 \end{pmatrix} \begin{pmatrix} m_1^{(i)} \\ m_2^{(i)} \end{pmatrix} = - \begin{pmatrix} \sum_{A=3}^{m-3} \text{tr}(T_A) m_A^{(i)} \\ \sum_{A=3}^{m-3} \text{tr}(T_A^2) m_A^{(i)} f_A \end{pmatrix} + \begin{pmatrix} 2\kappa^{(i)} \\ 0 \end{pmatrix}. \quad (2.15)$$

Thus two U(1) magnetic fluxes $m_\alpha^{(i)}$ ($\alpha = 1, 2$) can be determined by other U(1) fluxes

$$m_\alpha^{(i)} = \sum_{A=3}^{m-3} K_{\alpha A} m_A^{(i)} + V_\alpha \kappa^{(i)}, \quad (2.16)$$

where

$$\begin{aligned} K_{1A} &= \frac{\text{tr}(T_2)\text{tr}(T_A^2)f_A - \text{tr}(T_2^2)\text{tr}(T_A)f_2}{\text{tr}(T_1)\text{tr}(T_2^2)f_2 - \text{tr}(T_2)\text{tr}(T_1^2)f_1}, & K_{2A} &= \frac{\text{tr}(T_1^2)\text{tr}(T_A)f_1 - \text{tr}(T_1)\text{tr}(T_A^2)f_A}{\text{tr}(T_1)\text{tr}(T_2^2)f_2 - \text{tr}(T_2)\text{tr}(T_1^2)f_1}, \\ V_1 &= \frac{2\text{tr}(T_2^2)f_2}{\text{tr}(T_1)\text{tr}(T_2^2)f_2 - \text{tr}(T_2)\text{tr}(T_1^2)f_1}, & V_2 &= -\frac{2\text{tr}(T_1^2)f_1}{\text{tr}(T_1)\text{tr}(T_2^2)f_2 - \text{tr}(T_2)\text{tr}(T_1^2)f_1}. \end{aligned} \quad (2.17)$$

Here and in what follows, two generators $T_{1,2}$ are chosen such that $\text{tr}(T_1)\text{tr}(T_2^2)f_2 - \text{tr}(T_2)\text{tr}(T_1^2)f_1 \neq 0$. It then allows us to rewrite variables $\{X_{\alpha BC}, X_{\alpha\beta C}, X_{\alpha\beta\gamma}, Z_\alpha\}$ in terms of others, namely

$$\begin{aligned} X_{\alpha BC} &= \sum_A K_{\alpha A} X_{ABC} + V_\alpha X'_{BC}, & X_{\alpha\beta C} &= \sum_A K_{\alpha A} X_{\beta AC} + V_\alpha \left(\sum_B K_{\beta B} X'_{BC} + V_\beta X''_C \right), \\ X_{\alpha\beta\gamma} &= \sum_C K_{\gamma C} X_{\alpha\beta C} + K_{\alpha A} V_\gamma \left(\sum_B K_{\beta B} X'_{BA} + V_\beta X''_A \right) + V_\alpha V_\gamma \left(\sum_B K_{\beta B} X''_B + V_\beta X''' \right), \\ Z_\alpha &= \sum_A K_{\alpha A} Z_A + V_\alpha Z', \end{aligned} \quad (2.18)$$

where

$$\begin{aligned} X'_{AB} &= \sum_{i,j,k} d_{ijk} \kappa^{(i)} m_A^{(j)} m_B^{(k)}, & X''_A &= \sum_{i,j,k} d_{ijk} \kappa^{(i)} \kappa^{(j)} m_A^{(k)}, & X''' &= \sum_{i,j,k} d_{ijk} \kappa^{(i)} \kappa^{(j)} \kappa^{(k)}, \\ Z' &= \sum_i c_{2,i} \kappa^{(i)}. \end{aligned} \quad (2.19)$$

As a result, independent variables are $\{X_{ABC}, X'_{AB}, X''_A, X''', Z_A, Z'\}$ not $\{X_{abc}, Z_a\}$. Thus, the number of variables reduces to $m(m-4)(m-5)/6 + 2(m-4)$ from $_{m-1}C_3 + m - 3$. We now also use the totally symmetric properties of d_{ijk} .

In addition to the theoretical requirements such as K-theory condition (2.8) and hypercharge masslessness conditions (2.13), variables $\{X_{ABC}, X'_{AB}, X''_A, X''', Z_A, Z'\}$ are further constrained by phenomenological requirements. To realize phenomenologically consistent

models, we impose the three generations of quarks and charged leptons³ and no chiral exotics, namely

$$\chi_Q = \chi_L = \chi_{u^c} = \chi_{d^c} = \chi_{e^c} = -3|\Gamma|, \quad \chi_{\text{exotic}}^{(p)} = 0, \quad (2.20)$$

for all $1 \leq p \leq p_{\text{ex}}$, where the number of chiral exotics p_{ex} depends on the branching of $\text{SO}(32)$ but is at least $p_{\text{ex}} \geq m - 3$ appearing from exotic states $(2m, 32 - 2m)$ in eq. (2.5). We have taken into account the order of freely-acting discrete symmetry group of CY threefolds $|\Gamma|$ in order to be applicable to the model building on non-simply-connected CY threefolds. The above phenomenological requirements constrain variables $\{X_{ABC}, X'_{AB}, X''_A, X'''_A, Z_A, Z'\}$ through eq. (2.6). It turns out that in the case with $m = 6$, i.e., $\text{SO}(32) \rightarrow \text{SO}(12) \times \text{SO}(20) \rightarrow \text{SU}(3)_C \times \text{SU}(2)_L \times \Pi_{a=1}^3 \text{U}(1)_a \times \text{SO}(20)$, total six variables are not enough to satisfy at least eight index conditions (2.20) and remaining hypercharge masslessness condition (2.14) in general. Here, we have not considered the other stability conditions (2.9) and (2.11) which depend on the number of five-branes and values of moduli fields.

To simplify our analysis, we focus on $m = 8$ case, namely $\text{SO}(32) \rightarrow \text{SO}(16) \times \text{SO}(16)' \rightarrow \text{SU}(3)_C \times \text{SU}(2)_L \times \Pi_{a=1}^5 \text{U}(1)_a \times \text{SO}(16)'$, including the $m = 7$ case. Note that such a visible $\text{SO}(16)$ gauge group can be also embedded into the T-dual $E_8 \times E_8$ and non-supersymmetric $\text{SO}(16) \times \text{SO}(16)$ heterotic string theories, taking into account the spinor representation of $\text{SO}(16)$ as a massless mode. The following analysis is then applicable to other heterotic string theories.

2.2 Embedding the SM gauge group into $\text{SO}(16)$

Then, let us consider the detailed group decomposition of $\text{SO}(32)$ using the internal gauge fluxes. As a concrete example, we directly embed the SM gauge group into $\text{SO}(16) \subset \text{SO}(32)$ as mentioned before. Among the branching of $\text{SO}(16)$, for illustrative purpose, we focus on the following decomposition of $\text{SO}(32)$.⁴

$$\text{SO}(32) \rightarrow \text{SO}(16) \times \text{SO}(16)' \rightarrow \text{SU}(3)_C \times \text{SU}(2)_L \times \Pi_a \text{U}(1)_a \times \text{SO}(16)', \quad (2.21)$$

where the $\text{U}(1)_a$ directions are chosen such that

$$\begin{aligned} \text{U}(1)_1 &: (1, 1, 1, 0, 0, 0, 0, 0; 0, \dots, 0), \\ \text{U}(1)_2 &: (0, 0, 0, 1, 1, 0, 0, 0; 0, \dots, 0), \\ \text{U}(1)_3 &: (0, 0, 0, 0, 0, 1, 0, 0; 0, \dots, 0), \\ \text{U}(1)_4 &: (0, 0, 0, 0, 0, 0, 1, 0; 0, \dots, 0), \\ \text{U}(1)_5 &: (0, 0, 0, 0, 0, 0, 0, 1; 0, \dots, 0), \end{aligned} \quad (2.22)$$

in the basis of Cartan directions of $\text{SO}(32)$ $H_i (i = 1, 2, \dots, 16)$. The Cartan directions of $\text{SU}(3)_C$ and $\text{SU}(2)_L$ are respectively taken as $H_1 - H_2$, $H_2 - H_3$ and $H_4 - H_5$ in our analysis and $\text{SO}(32)$ roots correspond to $(\pm 1, \pm 1, 0, \dots, 0)$ under H_i , where the underline

³At this stage, we have not distinguished between the Higgsino fields and the charged leptons.

⁴For other gauge branchings, see, appendix A.

denotes the possible permutation. Under the decomposition $\text{SO}(32) \rightarrow \text{SO}(16) \times \text{SO}(16)'$, the adjoint representation of $\text{SO}(32)$ 496 decomposes as follows:

$$496 \rightarrow (120, 1) \oplus (16, 16') \oplus (1, 120'), \quad (2.23)$$

where the adjoint representation of $\text{SO}(16)$ includes the visible sector and others correspond to the hidden sector. In this decomposition, we find that there exist eight options to take the correct hypercharge, but for our purpose, we take the specific hypercharge direction,⁵

$$\text{U}(1)_Y = -\frac{1}{6}\text{U}(1)_1 - \frac{1}{2}\left(\text{U}(1)_3 - \text{U}(1)_4 + \text{U}(1)_5\right), \quad (2.24)$$

under which the spectrum consists of the visible sector:

$$\begin{aligned} Q &: (3, 2)_{-1,1,0,0,0} \oplus (3, 2)_{-1,-1,0,0,0}, \\ L &: (1, 2)_{0,\pm 1,1,0,0} \oplus (1, 2)_{0,\pm 1,0,-1,0} \oplus (1, 2)_{0,\pm 1,0,0,1}, \\ u^c &: (\bar{3}, 1)_{1,0,1,0,0} \oplus (\bar{3}, 1)_{1,0,0,-1,0} \oplus (\bar{3}, 1)_{1,0,0,0,1}, \\ d^c &: (\bar{3}, 1)_{1,0,-1,0,0} \oplus (\bar{3}, 1)_{1,0,0,1,0} \oplus (\bar{3}, 1)_{1,0,0,0,-1}, \\ e^c &: (1, 1)_{0,0,-1,1,0} \oplus (1, 1)_{0,0,-1,0,-1} \oplus (1, 1)_{0,0,0,1,-1}, \\ n^c &: (1, 1)_{0,-2,0,0,0} \oplus (1, 1)_{0,0,-1,-1,0} \oplus (1, 1)_{0,0,-1,0,1} \oplus (1, 1)_{0,0,0,1,1}, \\ \text{Exotic state} &: (\bar{3}, 1)_{-2,0,0,0,0}, \end{aligned} \quad (2.25)$$

and the hidden sector:

$$\begin{aligned} & (3, 1, 16')_{1,0,0,0,0} \oplus (1, 2, 16')_{0,1,0,0,0} \oplus (1, 1, 16')_{0,0,1,0,0} \oplus \\ & (1, 1, 16')_{0,0,0,1,0} \oplus (1, 1, 16')_{0,0,0,0,1} \oplus (1, 1, 120')_{0,0,0,0,0}. \end{aligned} \quad (2.26)$$

Now the subscript indices denote the $\text{U}(1)_a$ charges Y_a and the particle $(\bar{3}, 1)_{-2,0,0,0,0}$ is an exotic particle due to the absence of the Yukawa couplings as indicated by their weights (A.1) in appendix A. Note that we cannot distinguish the particle and anti-particles for SM singlets n^c at this stage. We thus show one of the concrete examples out of eight possibilities.

To obtain the SM-like spectra without chiral exotics, we require

$$\begin{aligned} Q &: \chi(L_1^{-1} \times L_2) + \chi(L_1^{-1} \times L_2^{-1}) = -3|\Gamma|, \\ L &: \sum_{s=\pm} \chi(L_2^s \times L_3^{-1}) + \sum_{s=\pm} \chi(L_2^s \times L_4^1) + \sum_{s=\pm} \chi(L_2^s \times L_5^1) = -3|\Gamma|, \\ u^c &: \chi(L_1 \times L_3) + \chi(L_1 \times L_4^{-1}) + \chi(L_1 \times L_5) = -3|\Gamma|, \\ d^c &: \chi(L_1 \times L_3^{-1}) + \chi(L_1 \times L_4) + \chi(L_1 \times L_5^{-1}) = -3|\Gamma|, \\ e^c &: \chi(L_3^{-1} \times L_4) + \chi(L_3^{-1} \times L_5^{-1}) + \chi(L_4 \times L_5^{-1}) = -3|\Gamma|, \end{aligned} \quad (2.27)$$

⁵It is possible to consider other hypercharge directions: $\text{U}(1)_Y = -\frac{1}{6}\text{U}(1)_1 \pm \frac{1}{2}(\text{U}(1)_3 \pm \text{U}(1)_4 \pm \text{U}(1)_5)$, where the sign is taken in the random order. The algorithm to determine the hypercharge direction is discussed in appendix A.

and

$$\chi(L_1) = \chi(L_2) = \chi(L_3) = \chi(L_4) = \chi(L_5) = \chi(L_1^{-2}) = 0, \quad (2.28)$$

employing the index theorem on CY threefolds. The above index formulae are applicable to both simply- and non-simply-connected CY threefolds with the order of freely-acting discrete symmetry group $|\Gamma|$ and here we have not counted the number of Higgs/Higgsino fields.

Although we have focused on the particular branching of $\text{SO}(32)$ with specific hypercharge direction, we systematically investigate several gauge embeddings with possible hypercharge directions listed in table 1 of appendix A. In the next section 2.3, we solve the eleven index conditions (2.27), (2.28) and remaining hypercharge masslessness condition (2.14) in terms of 24 variables $\{X_{ABC}, X'_{AB}, X''_A, X'''_A, Z_A, Z'\}$ against several branchings of $\text{SO}(32)$.

2.3 General formula

We are now ready to search for 24 variables $\{X_{ABC}, X'_{AB}, X''_A, X'''_A, Z_A, Z'\}$ with $A = 3, 4, 5$, satisfying the eleven index conditions (2.27), (2.28) and the remaining hypercharge masslessness condition (2.14). Since the stability conditions depend on the detail of CY data, for the time being, we focus on only K-theory condition and hypercharge masslessness conditions. We remark that the stability conditions have to be checked for each CY threefold. The details of the algorithm computing the particle spectrum is shown in appendix A.

On simply-connected CY threefolds, our systematic search reveals that a direct flux breaking scenario is available for the following decompositions:

$$\begin{aligned} \text{SO}(16) \rightarrow & \left\{ \begin{array}{l} \text{SO}(6) \times \text{SO}(4) \times \text{SO}(2)^3 \\ \text{SO}(10) \times \text{SO}(6) \rightarrow \text{SU}(5) \times \text{SU}(3) \times \text{U}(1)^2 \\ \text{SO}(8) \times \text{SO}(4) \times \text{SO}(4) \rightarrow \text{SU}(4)_C \times \text{SU}(2)_L \times \text{SU}(2) \times \text{U}(1)^3 \\ \text{SU}(4)_C \times \text{SU}(4) \times \text{U}(1)^2 \rightarrow \text{SU}(4)_C \times \text{SU}(2)_L \times \text{SU}(2) \times \text{U}(1)^3 \\ \text{SO}(6) \times \text{SO}(6) \times \text{SO}(4) \rightarrow \text{SU}(3)_C \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)^3 \\ \text{SO}(8) \times \text{SO}(6) \times \text{SO}(2) \rightarrow \text{SU}(4)_C \times \text{SU}(3) \times \text{U}(1)^3 \\ \text{SO}(8) \times \text{SO}(4) \times \text{SO}(2)^2 \rightarrow \text{SU}(4)_C \times \text{SU}(2)_L \times \text{U}(1)^3 \\ \text{SO}(6) \times \text{SO}(10) \rightarrow \text{SU}(3)_C \times \text{SU}(5) \times \text{U}(1)^2 \\ \text{SO}(6) \times \text{SO}(4) \times \text{SO}(4) \times \text{SO}(2) \rightarrow \text{SU}(3)_C \times \text{SU}(2)_L \times \text{SU}(2) \times \text{U}(1)^3 \\ \text{SO}(6) \times \text{SO}(6) \times \text{SO}(2)^2 \rightarrow \text{SU}(3)_C \times \text{SU}(3) \times \text{U}(1)^4 \end{array} \right. \\ & \rightarrow G_{\text{SM}} \times \text{U}(1)^4, \end{aligned} \quad (2.29)$$

with $G_{\text{SM}} = \text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y$. In the case with the first branch of eq. (2.29) corresponding to the model of section 2.2, the following 24 variables satisfy all the require-

ments:

$$\begin{aligned}
 X_{333} &= p_1, X_{334} = p_2, X_{335} = p_3, X_{344} = p_4, X_{345} = 3, X_{355} = p_5, X_{444} = p_6, \\
 X_{445} &= p_7, X_{455} = -6 - p_2 + p_3 + p_4 + p_5 + p_7, X_{555} = -p_1 + p_6, X'_{33} = p_8, X'_{34} = p_9, X'_{35} = p_{10}, \\
 X'_{44} &= p_{11}, X'_{45} = p_{12}, X'_{55} = p_{13}, X''_3 = p_{14}, X''_4 = p_{15}, \\
 X''_5 &= -3 + 2p_3 - 3p_4 + 2p_5 + 3p_6 - 3p_7 - 2p_8 + 6p_9 - 4p_{10} - 4p_{11} + 6p_{12} - 2p_{13} - p_{14} + p_{15}, \\
 X''' &= p_{16}, Z_3 = -2p_1, Z_4 = -2p_6, Z_5 = 2p_1 - 2p_6, \\
 Z' &= 18 - 6p_3 + 6p_4 - 6p_5 - 6p_6 + 6p_7 + 6p_8 - 12p_9 + 12p_{10} - 12p_{12} + 6p_{13} + 6p_{15} - 2p_{16},
 \end{aligned} \tag{2.30}$$

where $p_1, \dots, p_{16}$ are integers and we now consider the specific hypercharge direction (2.24). For other gauge branchings, see, table 2 of appendix A. As mentioned before, some of U(1)s are anomalous due to the internal gauge fluxes and their number is counted by the rank of U(1) mass matrix (2.12). Remarkably, for all the gauge decompositions of table 2, the dilaton-axion induced mass term in eq. (2.12) turns out to be

$$M_{a0} = \sum_{b,c,d} \frac{1}{6} \text{tr}(T_a T_b T_c T_d) X_{bcd} + \frac{1}{24} \text{tr}(T_a^2) Z_a = -\frac{1}{24} \text{tr}(T_a^2) Z_a, \tag{2.31}$$

due to the correlation between X_{abc} and Z_a .

Let us consider in more detail the $\kappa^{(i)} = 0$ case which means that the right-handed side of K-theory condition (2.8) vanishes. In such a case, independent variables are $\{X_{ABC}, Z_A\}$, since other variables are written by

$$\begin{aligned}
 X_{\alpha BC} &= \sum_A K_{\alpha A} X_{ABC}, \quad X_{\alpha \beta C} = \sum_{A,B} K_{\alpha A} K_{\beta B} X_{ABC}, \quad X_{\alpha \beta \gamma} = \sum_{A,B,C} K_{\alpha A} K_{\beta B} K_{\gamma C} X_{ABC}, \\
 Z_\alpha &= \sum_A K_{\alpha A} Z_A,
 \end{aligned} \tag{2.32}$$

from which two U(1) gauge boson mass terms are provided by the other one, i.e., $M_{\alpha i} = \sum_A K_{\alpha A} M_{Ai}$. Thus, the maximal rank of U(1) gauge boson matrix is 3. One of the massless U(1) directions will be identified with U(1)_{B-L} in addition to U(1)_Y. On the other hand, in the $\kappa^{(i)} \neq 0$ case, the rank of U(1) gauge boson matrix is generically 4 when $h^{1,1} \geq 3$ and the remaining gauge symmetry consists of $\text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y \times \text{SO}(16)'$.

Finally, we comment on the direct flux breaking scenario on non-simply-connected CY threefolds. In contrast to the simply-connected CY cases, some special non-simply-connected CY threefolds, specifically $|\Gamma| = 5\mathbb{Z}_{>0}$, allow intermediate GUT-like models without chiral exotics. The detail of phenomenologically acceptable CYs is shown in table 1 of appendix A.

3 Phenomenological aspects of three-generation models

The obtained general formula in section 2.3 is applicable to general CY threefolds. In this section, we discuss three-generation models on a specific CICY.

3.1 Three-generation models on simply-connected CY threefolds

We consider simply-connected CY threefolds, namely $|\Gamma| = 1$. Among known 7890 CICYs, we consider the following CICY called # 7247 in the list of [20],

$$\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{array}{c} 4,40 \\ \\ \\ -72 \end{array}, \quad (3.1)$$

where the above configuration matrix characterizes how to embed the CY manifold in four projective spaces $\mathbb{P}^2$. The superscript and subscript indices denote the hodge number of CY $(h^{1,1}, h^{2,1}) = (4, 40)$ and the Euler number of CY $\chi_{\text{CY}} = -72$, respectively. (See for details of CICYs, e.g., ref. [21].) The topological data of the above CICY is calculated as

$$\begin{aligned} d_{123} = 6, \quad d_{124} = d_{134} = d_{234} = 5, \quad d_{112} = d_{113} = d_{122} = d_{133} = d_{223} = d_{233} = 3, \\ d_{114} = d_{144} = d_{224} = d_{244} = d_{334} = d_{344} = 2, \quad d_{111} = d_{222} = d_{333} = d_{444} = 0, \end{aligned} \quad (3.2)$$

in the basis of two-forms w_i and

$$c_2(TM) = (36, 36, 36, 36), \quad (3.3)$$

in the basis of dual four-forms $\hat{w}^i$, respectively. Note that this CY has Z_3 and $Z_3 \times Z_3$ discrete symmetries, but we concentrate on the case with $|\Gamma| = 1$ in what follows.

For concreteness, we study compactifications of the $\text{SO}(32)$ heterotic string on the above CICY with line bundles leading to the group decomposition (2.21). Although the model-building approach is classified into two cases: $\kappa^{(i)} \neq 0$ and $\kappa^{(i)} = 0$ appearing in the K-theory condition (2.8), in the following analysis, we focus on the case with $\kappa^{(i)} \neq 0$, providing just the SM gauge group in the visible sector.

We search for the internal $\text{U}(1)$ gauge fluxes within the range $-1 \leq m_A^{(i)} \leq 1$ and $-1 \leq \kappa^{(i)} \leq 1$ where $i = 1, 2, 3, 4$ and $A = 3, 4, 5$, constrained by eqs. (2.27) and (2.28), the stability conditions (2.9) and (2.11) and hypercharge masslessness condition (2.14). Note that other $\text{U}(1)_{1,2}$ fluxes are determined by eq. (2.16). It turns out that some of the fluxes satisfy all the requirements. For example, under the flux ansatz,

$$\begin{aligned} m_1 = (1, 0, 0, -1), \quad m_2 = (1, 0, -1, 3), \quad m_3 = (0, 1, 0, -1), \\ m_4 = (0, 0, 1, -1), \quad m_5 = (-1, -1, 1, 1), \end{aligned} \quad (3.4)$$

the number of five-branes in the basis of $\hat{w}^i$,

$$N_i = (6, 30, 72, 6), \quad (3.5)$$

is positive and at the same time, the D -term conditions (2.11) are satisfied at the physical domain of Kähler moduli and dilaton, namely $t_i, s > 1$ in string units,

$$t_1 = t_2 = t_3 = \frac{3 + 2\sqrt{3}}{3} t_4, \quad s = \frac{2 + \sqrt{3}}{2\pi} t_4, \quad (3.6)$$

where the ten-dimensional dilaton field in eq. (2.11) is now written by $s = e^{-2\phi_{10}}\mathcal{V}/(2\pi)$ with CY volume $\mathcal{V} = \sum_{i,j,k} d_{ijk}t_it_jt_k/6$ in string units. The above fluxes result in the following chiral zero-modes:

$$\begin{aligned}
 Q &: 3(3, 2)_{-1,1,0,0,0}, \\
 L &: 3(1, 2)_{0,1,0,0,1}, \\
 H_u^1 &: 6(1, 2)_{0,-1,-1,0,0}, & H_u^2 &: 12(1, 2)_{0,1,-1,0,0}, & H_u^3 &: 21(1, 2)_{0,1,0,0,-1}, \\
 H_d^1 &: 15(1, 2)_{0,-1,0,-1,0}, & H_d^2 &: 24(1, 2)_{0,1,0,-1,0}, \\
 u^c &: 3(\bar{3}, 1)_{1,0,1,0,0}, \\
 d^c &: 3(\bar{3}, 1)_{1,0,0,1,0}, \\
 e^c &: 3(1, 1)_{0,0,0,1,-1},
 \end{aligned} \tag{3.7}$$

and singlets:

$$n_1^c : 3(1, 1)_{0,0,0,-1,-1}, \quad n_2^c : 3(1, 1)_{0,0,1,1,0}, \quad n_3^c : 54(1, 1)_{0,2,0,0,0}, \tag{3.8}$$

where the particles n_1^c or n_2^c could be identified with right-handed neutrinos. Remarkably, we have phenomenologically interesting perturbative Yukawa couplings,

$$QH_u^1 u^c, \quad QH_d^1 d^c, \quad LH_d^1 e^c, \quad LH_u^1 n_1^c n_2^c, \tag{3.9}$$

and interestingly, the higher-dimensional proton decay operators are prohibited by the massive $U(1)_{B-L}$ gauge symmetry. However, we require the mechanism generating the mass terms for extra Higgs doublets, which will be the subject of future work.

3.2 Gauge coupling unification

So far, we have focused on the number of chiral generations in $SO(32)$ heterotic compactification. In this section, we discuss the unification of gauge couplings at the string scale. The four-dimensional gauge coupling is extracted from the ten-dimensional $SO(32)$ Yang-Mills action,

$$-\frac{1}{2g_{10}^2} \int e^{-2\phi_{10}} \text{tr}_{SO(32)}(F \wedge *_{10} F), \tag{3.10}$$

where $g_{10}^2 = 4\pi(l_s)^6$ is the ten-dimensional gauge coupling. After expanding the ten-dimensional gauge field strength as the four-dimensional part F and internal part $\bar{F}$, namely $F \rightarrow F + \bar{F}$, the kinetic terms of the four-dimensional gauge bosons become

$$-\frac{1}{4} \int s \text{tr}_{SO(32)}(F \wedge *_4 F), \tag{3.11}$$

where s is the dilaton field

$$s = e^{-2\phi_{10}}\mathcal{V}/2\pi, \tag{3.12}$$

with CY volume $\mathcal{V}$ in string units.

Since the generator of hypercharge direction listed in all the gauge decompositions of table 1 is of the form

$$U(1)_Y = \frac{1}{6} \text{diag}(-1, -1, -1, 0, 0, \pm 3, \pm 3, \pm 3), \tag{3.13}$$

where the sign is taken in the random order, the four-dimensional gauge field strength of the SM gauge group is expanded as

$$F = \begin{pmatrix} \sum_{A=1}^8 F_{\text{SU}(3)_C}^A \lambda^A & 0 & 0 & 0 & 0 \\ 0 & \sum_{\alpha=1}^3 F_{\text{SU}(2)_L}^\alpha \sigma^\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \frac{F_Y}{6} \begin{pmatrix} -1_{3 \times 3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pm 3 & 0 & 0 \\ 0 & 0 & 0 & \pm 3 & 0 \\ 0 & 0 & 0 & 0 & \pm 3 \end{pmatrix}, \quad (3.14)$$

using the Gell-Mann matrices λ^A and Pauli matrices σ^α . In our case, the non-abelian part is normalized as $\text{tr}_{\text{SO}(32)}(T_{\text{SU}(3)_C}^A T_{\text{SU}(3)_C}^B) = 2\delta^{AB}$, $\text{tr}_{\text{SO}(32)}(T_{\text{SU}(2)_L}^\alpha T_{\text{SU}(2)_L}^\beta) = 2\delta^{\alpha\beta}$ due to the fact that the level of Kac-Moody algebra is one. The trace of gauge field strength then reads

$$\text{tr}_{\text{SO}(32)}(F \wedge *_4 F) = 2F_{\text{SU}(3)_C}^a \wedge *_4 F_{\text{SU}(3)_C}^a + 2F_{\text{SU}(2)_L}^m \wedge *_4 F_{\text{SU}(2)_L}^m + \frac{5}{3}F_{\text{U}(1)_Y} \wedge *_4 F_{\text{U}(1)_Y}. \quad (3.15)$$

Note that off-diagonal U(1) gauge couplings are absent in contrast to the $E_8 \times E_8$ heterotic string case [9]. Taking into account the normalization of $\text{U}(1)_Y$ generator, the gauge couplings satisfy

$$g_{\text{SU}(3)_C}^2 = g_{\text{SU}(2)_L}^2 = \frac{5}{6}g_{\text{U}(1)_Y}^2, \quad (3.16)$$

at tree level.

Thus, we cannot arrive at phenomenologically interesting so-called GUT-relation. However, the unification of gauge couplings actually depends on the number of Higgs doublets, radiative corrections to the gauge couplings and supersymmetry-breaking scale. Indeed, the radiative corrections to the non-abelian gauge groups are non-universal in contrast to the $E_8 \times E_8$ heterotic case [8, 9], which allows us to obtain the realistic values of gauge couplings at the string scale as demonstrated on toroidal background [22]. To evaluate their precise values, we have to discuss the stabilization mechanism of moduli fields and supersymmetry-breaking sector. We will postpone the concrete model building for a future analysis.

4 Conclusion

We have searched for $\text{SO}(32)$ heterotic SM vacua directly with the SM gauge group from smooth CY threefolds. The non-trivial internal gauge background allows us to directly embed the SM gauge group G_{SM} into $\text{SO}(32)$ one. In this paper, we focus on the branching $G_{\text{SM}} \subset \text{SO}(16) \subset \text{SO}(32)$ as listed in table 1. Against several branchings of $\text{SO}(16)$, we have derived a general formula leading to the three generations of quarks and leptons and no chiral exotics, taking into account the K-theory condition and the hypercharge masslessness conditions. Since the obtained formula is independent of the topological data of CY, it is applicable in general CY compactifications in contrast to models using Wilson lines. Such a direct flux breaking is attractive scenario not only in F-theory GUTs [6, 7], but also in heterotic string theory.

For concreteness, we have discussed phenomenological aspects of direct flux breaking scenario on a specific CICY, where the spectrum consists of MSSM particles, singlets, extra Higgs doublets and vector-like particles whose number cannot be captured by the index theorem. In our setup, the normalization of hypercharge direction is different from conventional SU(5) GUT normalization, which also be stated in $E_8 \times E_8$ heterotic compactification with a hypercharge flux [12]. However, one-loop threshold corrections to gauge couplings are non-universal for non-abelian gauge groups compared with $E_8 \times E_8$ heterotic models. It thus opens up the way for phenomenologically acceptable models.

It is interesting to check whether or not one can achieve direct flux breaking (i.e., hypercharge flux breaking) in the dual global F-theory compactifications.

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A Group decompositions

In this appendix, we present the algorithm to compute the particle spectrum for all the group decompositions of $SO(16) \subset SO(32)$ treated in this paper. First we decide which weights correspond to particles or anti-particles without determining the hypercharge direction. In our analysis, the matter weights belonging to $SO(16)$ adjoint representation are chosen as

$$\begin{aligned}
 Q : \left\{ \begin{array}{l} Q_1 = (\underline{0}, \underline{0}, \underline{-1}, \underline{1}, 0, 0, 0, 0) \\ Q_2 = (\underline{0}, \underline{0}, \underline{-1}, \underline{0}, \underline{-1}, 0, 0, 0) \end{array} \right. , \quad L : \left\{ \begin{array}{l} L_{1,\pm} = (0, 0, 0, \underline{\pm 1}, \underline{0}, \pm 1, 0, 0) \\ L_{2,\pm} = (0, 0, 0, \underline{\pm 1}, \underline{0}, \mp 1, 0, 0) \\ L_{3,\pm} = (0, 0, 0, \underline{\pm 1}, \underline{0}, 0, \pm 1, 0) \\ L_{4,\pm} = (0, 0, 0, \underline{\pm 1}, \underline{0}, 0, \mp 1, 0) \\ L_{5,\pm} = (0, 0, 0, \underline{\pm 1}, \underline{0}, 0, 0, \pm 1) \\ L_{6,\pm} = (0, 0, 0, \underline{\pm 1}, \underline{0}, 0, 0, \mp 1) \end{array} \right. , \\
 u^c, d^c : \left\{ \begin{array}{l} u_1^c = (\underline{1}, \underline{0}, \underline{0}, 0, 0, 1, 0, 0) \\ u_2^c = (\underline{1}, \underline{0}, \underline{0}, 0, 0, -1, 0, 0) \\ u_3^c = (\underline{1}, \underline{0}, \underline{0}, 0, 0, 0, 1, 0) \\ u_4^c = (\underline{1}, \underline{0}, \underline{0}, 0, 0, 0, -1, 0) \\ u_5^c = (\underline{1}, \underline{0}, \underline{0}, 0, 0, 0, 0, 1) \\ u_6^c = (\underline{1}, \underline{0}, \underline{0}, 0, 0, 0, 0, -1) \end{array} \right. , \quad e^c, \text{singlets} : \left\{ \begin{array}{l} e_{1,\pm}^c = (0, 0, 0, \underline{\pm 1}, \underline{\pm 1}, 0, 0, 0) \\ e_{2,\pm}^c = (0, 0, 0, 0, 0, \underline{\pm 1}, \underline{\pm 1}, 0) \\ e_{3,\pm}^c = (0, 0, 0, 0, 0, \underline{\mp 1}, \underline{\pm 1}, 0) \\ e_{4,\pm}^c = (0, 0, 0, 0, 0, \underline{\pm 1}, 0, \underline{\pm 1}) , \\ e_{5,\pm}^c = (0, 0, 0, 0, 0, \underline{\mp 1}, 0, \underline{\pm 1}) \\ e_{6,\pm}^c = (0, 0, 0, 0, 0, 0, \underline{\pm 1}, \underline{\pm 1}) \\ e_{7,\pm}^c = (0, 0, 0, 0, 0, 0, \underline{\mp 1}, \underline{\pm 1}) \end{array} \right. \\
 \text{Exotic state} : (\underline{0}, \underline{-1}, \underline{-1}, 0, 0, 0, 0, 0) , \tag{A.1}
 \end{aligned}$$

(double sign in the same order) under the Cartan direction of $SO(16)$, where the underline represents the possible permutations. It is now assumed that there are no exotic charged particles except for $(\underline{0}, -1, -1, 0, 0, 0, 0, 0)$. Note that when $L_{i,+}$ with positive sign is a left-handed lepton, which has $(1, 2)_{-1/2}$ representation in the SM, $L_{i,-}$ belonging to $(1, 2)_{+1/2}$ representation is an anti left-handed lepton. It holds for e^c and singlets. At this stage, we cannot distinguish between u^c and d^c because the hypercharge direction is not specified.

To extract the SM particles from eq. (A.1) including particles and anti-particles, we define the following maps:

$$\begin{aligned} q_L &: \{1, 2, \dots, 6\} \rightarrow \{\pm 1\}, \\ q_{u^c} &: \{1, 2, \dots, 6\} \rightarrow \{\pm 1\}, \\ q_{e^c} &: \{1, 2, \dots, 7\} \rightarrow \{\pm 1, \pm 2\}, \end{aligned} \quad (\text{A.2})$$

where

$$\begin{aligned} q_L(i) &= \begin{cases} +1 & \text{when } L_{i,+} \text{ (} L_{i,-} \text{) is the left-handed (anti-) lepton} \\ -1 & \text{when } L_{i,-} \text{ (} L_{i,+} \text{) is the left-handed (anti-) lepton} \end{cases}, \\ q_{u^c}(i) &= \begin{cases} +1 & \text{when } u_i^c \text{ is the conjugate of right-handed up-type quark} \\ -1 & \text{when } u_i^c \text{ is the conjugate of right-handed down-type quark} \end{cases}, \\ q_{e^c}(i) &= \begin{cases} +1 & \text{when } e_{i,+}^c \text{ (} e_{i,-}^c \text{) is the conjugate of right-handed (anti-) electron} \\ -1 & \text{when } e_{i,-}^c \text{ (} e_{i,+}^c \text{) is the conjugate of right-handed (anti-) electron} \\ +2 & \text{when } e_{i,+}^c \text{ (} e_{i,-}^c \text{) is the conjugate of right-handed (anti-) singlet} \\ -2 & \text{when } e_{i,-}^c \text{ (} e_{i,+}^c \text{) is the conjugate of right-handed (anti-) singlet} \end{cases}. \end{aligned} \quad (\text{A.3})$$

Now, we have not distinguished between the Higgsino fields and the charged leptons.

For example, $(q_{u^c}(1), q_{u^c}(2), q_{u^c}(3), q_{u^c}(4), q_{u^c}(5), q_{u^c}(6)) = (1, -1, 1, -1, -1, 1)$ means that u_1^c, u_3^c, u_6^c are the conjugates of the right-handed up-type quarks $(\bar{3}, 1)_{-2/3}$ and u_2^c, u_4^c, u_5^c are the conjugates of the right-handed down-type quarks $(\bar{3}, 1)_{1/3}$.

Next, we searched for all $2^6 \cdot 2^6 \cdot 4^7 \approx 6.71 \times 10^7$ possibilities of $\{q_L(i), q_{u^c}(i), q_{e^c}(i)\}$ against each branching of $SO(16)$ listed in table 1. Then we determine the coefficients of hypercharge direction f_a to realize the correct hypercharge of quarks and leptons. It is remarkable that, in our analysis, f_a (if exist) are determined uniquely against each set of $\{q_L(i), q_{u^c}(i), q_{e^c}(i), T_a\}$ with $U(1)$ generators T_a . Finally we solve equations for $\{X_{ABC}, X'_{AB}, X''_A, X''', Z_A, Z'\}$ (like eqs. (2.27) and (2.28)) to obtain the three generation of quarks and leptons without chiral exotics, taking into account the hypercharge masslessness conditions and the K-theory condition. In table 1, we show the possible order of freely-acting discrete symmetry group Γ . In particular, on simply-connected CY threefolds ($|\Gamma| = 1$), the list of 24 variables $\{X_{ABC}, X'_{AB}, X''_A, X''', Z_A, Z'\}$ in table 2 lead to the MSSM-like models, where we choose a specific hypercharge direction for simplicity.

In the search of concrete models in section 3, we use the brute force attack to solve eqs. (2.27) and (2.28) within $m_A^{(i)} \in [-1, 1]$ for all the possibilities, some of which have

solutions of $\{X_{ABC}, X'_{AB}, X''_A, X'''_A, Z_A, Z'\}$. We check the D -term conditions after finding the solution of chiral zero-modes.

GUT-like decompositions	
(i) $\text{SO}(8) \times \text{U}(1) \rightarrow \text{SU}(5) \times \text{U}(1)^4 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
U(1) generators T_a	Available $ \Gamma $
$\text{U}(1)_1 : \text{diag}(1, 1, 1, 1, 1, 1, 1, 1),$ $\text{U}(1)_2 : \text{diag}(2, 2, 2, -3, -3, 0, 0, 0),$ $\text{U}(1)_3 : \text{diag}(1, 1, 1, 1, 1, 1, 1, -7),$ $\text{U}(1)_4 : \text{diag}(1, 1, 1, 1, 1, 1, 1, -6, 0),$ $\text{U}(1)_5 : \text{diag}(1, 1, 1, 1, 1, 1, -5, 0, 0),$	$ \Gamma = 10\mathbb{Z}_{>0}$
(ii) $\text{SO}(14) \times \text{SO}(2) \rightarrow \text{SU}(7) \times \text{U}(1)^2 \rightarrow \text{SU}(5) \times \text{U}(1)^4 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
U(1) generators T_a	Available $ \Gamma $
$\text{U}(1)_1 : \text{diag}(1, 1, 1, 1, 1, 1, 1, 0),$ $\text{U}(1)_2 : \text{diag}(1, 1, 1, 1, 1, 1, -6, 0)$ $\text{U}(1)_3 : \text{diag}(1, 1, 1, 1, 1, -5, 0, 0),$ $\text{U}(1)_4 : \text{diag}(0, 0, 0, 0, 0, 0, 0, 1)$ $\text{U}(1)_5 : \text{diag}(2, 2, 2, -3, -3, 0, 0, 0)$	$ \Gamma = 5\mathbb{Z}_{>0}$
(iii) $\text{SO}(12) \times \text{SO}(2)^2 \rightarrow \text{SU}(6) \times \text{U}(1)^3 \rightarrow \text{SU}(5) \times \text{U}(1)^4 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
U(1) generators T_a	Available $ \Gamma $
$\text{U}(1)_1 : \text{diag}(0, 0, 0, 0, 0, 0, 1, 0),$ $\text{U}(1)_2 : \text{diag}(2, 2, 2, -3, -3, 0, 0, 0)$ (a) $\text{U}(1)_3 : \text{diag}(1, 1, 1, 1, 1, 1, 0, 0),$ $\text{U}(1)_4 : \text{diag}(1, 1, 1, 1, 1, -5, 0, 0)$ $\text{U}(1)_5 : \text{diag}(0, 0, 0, 0, 0, 0, 0, 1)$	$ \Gamma = 5\mathbb{Z}_{>0}$
$\text{U}(1)_1 : \text{diag}(0, 0, 0, 0, 0, 0, 1, 1),$ $\text{U}(1)_2 : \text{diag}(2, 2, 2, -3, -3, 0, 0, 0)$ (b) $\text{U}(1)_3 : \text{diag}(1, 1, 1, 1, 1, 1, 0, 0),$ $\text{U}(1)_4 : \text{diag}(1, 1, 1, 1, 1, -5, 0, 0)$ $\text{U}(1)_5 : \text{diag}(0, 0, 0, 0, 0, 0, 1, -1)$	$ \Gamma = 5\mathbb{Z}_{>0}$
(iv) $\text{SO}(10) \times \text{SO}(2)^3 \rightarrow \text{SU}(5) \times \text{U}(1)^4 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
U(1) generators T_a	Available $ \Gamma $
$\text{U}(1)_1 : \text{diag}(1, 1, 1, 1, 1, 1, 1, 1),$ $\text{U}(1)_2 : \text{diag}(0, 0, 0, 0, 0, 0, 0, 1)$ $\text{U}(1)_3 : \text{diag}(2, 2, 2, -3, -3, 0, 0, 0),$ $\text{U}(1)_4 : \text{diag}(0, 0, 0, 0, 0, 0, 1, 0)$ $\text{U}(1)_5 : \text{diag}(0, 0, 0, 0, 0, 0, 1, 0, 0)$	$ \Gamma = 5\mathbb{Z}_{>0}$
(v) $\text{SO}(10) \times \text{SO}(2) \times \text{SO}(4) \rightarrow \text{SU}(5) \times \text{U}(1)^4 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
U(1) generators T_a	Available $ \Gamma $
$\text{U}(1)_1 : \text{diag}(1, 1, 1, 1, 1, 0, 0, 0),$ $\text{U}(1)_2 : \text{diag}(0, 0, 0, 0, 0, 0, 0, 1)$ $\text{U}(1)_3 : \text{diag}(2, 2, 2, -3, -3, 0, 0, 0),$ $\text{U}(1)_4 : \text{diag}(0, 0, 0, 0, 0, 1, 1, 0)$ $\text{U}(1)_5 : \text{diag}(0, 0, 0, 0, 0, 1, -1, 0)$	$ \Gamma = 5\mathbb{Z}_{>0}$
(vi) $\text{SO}(10) \times \text{SO}(6) \rightarrow \text{SU}(5) \times \text{SU}(3) \times \text{U}(1)^2 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
U(1) generators T_a	Available $ \Gamma $
$\text{U}(1)_1 : \text{diag}(1, 1, 1, 1, 1, 0, 0, 0),$ $\text{U}(1)_2 : \text{diag}(2, 2, 2, -3, -3, 0, 0, 0)$ (a) $\text{U}(1)_3 : \text{diag}(0, 0, 0, 0, 0, 1, 1, 1),$ $\text{U}(1)_4 : \text{diag}(0, 0, 0, 0, 0, 1, 1, -2)$ $\text{U}(1)_5 : \text{diag}(0, 0, 0, 0, 0, 1, -1, 0)$	$ \Gamma = \mathbb{Z}_{>0}$

$U(1)_1 : \text{diag}(1, 1, 1, 0, 0, 0, 1, 1), \quad U(1)_2 : \text{diag}(0, 0, 0, 1, 1, 1, 0, 0)$ (b) $U(1)_3 : \text{diag}(0, 0, 0, 1, 1, -2, 0, 0), \quad U(1)_4 : \text{diag}(1, 1, 1, 0, 0, 0, 1, -4)$ $U(1)_5 : \text{diag}(1, 1, 1, 0, 0, 0, -3, 0)$	$ \Gamma = \mathbb{Z}_{>0}$
(vii) $SO(14) \times SO(2) \rightarrow SU(5) \times SU(2)_L \times U(1)^3 \rightarrow G_{\text{SM}} \times U(1)^4$	
U(1) generators T_a	Available $ \Gamma $
$U(1)_1 : \text{diag}(1, 1, 1, 1, 1, 1, 0, 0), \quad U(1)_2 : \text{diag}(2, 2, 2, -3, -3, 0, 0, 0)$ $U(1)_3 : \text{diag}(1, 1, 1, 1, 1, -\frac{5}{2}, -\frac{5}{2}, 0), \quad U(1)_4 : \text{diag}(0, 0, 0, 0, 0, 0, 0, 1)$ $U(1)_5 : \text{diag}(0, 0, 0, 0, 0, 1, -1, 0)$	$ \Gamma = 5\mathbb{Z}_{>0}$
(viii) $SU(8) \times U(1) \rightarrow SU(7) \times U(1)^2 \rightarrow SU(5) \times SU(2)_L \times U(1)^3 \rightarrow G_{\text{SM}} \times U(1)^4$	
U(1) generators T_a	Available $ \Gamma $
$U(1)_1 : \text{diag}(1, 1, 1, 1, 1, 1, 1, 1), \quad U(1)_2 : \text{diag}(2, 2, 2, -3, -3, 0, 0, 0)$ (a) $U(1)_3 : \text{diag}(1, 1, 1, 1, 1, -\frac{5}{2}, -\frac{5}{2}, 0), \quad U(1)_4 : \text{diag}(1, 1, 1, 1, 1, 1, 1, -7)$ $U(1)_5 : \text{diag}(0, 0, 0, 0, 0, 1, -1, 0)$	$ \Gamma = 5\mathbb{Z}_{>0}$
$U(1)_1 : \text{diag}(1, 1, 1, 1, 1, 1, 1, 1), \quad U(1)_2 : \text{diag}(2, 2, 2, -3, -3, 0, 0, 0)$ (b) $U(1)_3 : \text{diag}(1, 1, 1, 1, 1, -3, -3), \quad U(1)_4 : \text{diag}(1, 1, 1, 1, 1, -5, 0, 0)$ $U(1)_5 : \text{diag}(0, 0, 0, 0, 0, 0, -1, 1)$	$ \Gamma = 5\mathbb{Z}_{>0}$
$U(1)_1 : \text{diag}(1, 1, 1, 1, 1, 1, 1, 1), \quad U(1)_2 : \text{diag}(2, 2, 2, -3, -3, 0, 0, 0)$ (c) $U(1)_3 : \text{diag}(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}), \quad U(1)_4 : \text{diag}(0, 0, 0, 0, 0, 1, 1, -2)$ $U(1)_5 : \text{diag}(0, 0, 0, 0, 0, 1, -1, 0)$	$ \Gamma = 5\mathbb{Z}_{>0}$
(ix) $SU(8) \times U(1) \rightarrow SU(6) \times SU(2) \times U(1)^2 \rightarrow SU(5) \times SU(2)_L \times U(1)^3 \rightarrow G_{\text{SM}} \times U(1)^4$	
U(1) generators T_a	Available $ \Gamma $
$U(1)_1 : \text{diag}(1, 1, 1, 1, 1, 1, 1, 1), \quad U(1)_2 : \text{diag}(1, 1, 1, 1, 1, 1, -3, -3)$ $U(1)_3 : \text{diag}(1, 1, 1, 1, 1, -5, 0, 0), \quad U(1)_4 : \text{diag}(0, 0, 0, 0, 0, 0, 1, -1)$ $U(1)_5 : \text{diag}(2, 2, 2, -3, -3, 0, 0, 0)$	$ \Gamma = 5\mathbb{Z}_{>0}$
Pati-Salam-like decompositions	
(x) $SO(8) \times SO(4) \times SO(4) \rightarrow SU(4)_C \times SU(2)_L \times SU(2) \times U(1)^3 \rightarrow G_{\text{SM}} \times U(1)^4$	
U(1) generators T_a	Available $ \Gamma $
$U(1)_1 : \text{diag}(1, 1, 1, 0, 0, 0, 0, 1), \quad U(1)_2 : \text{diag}(1, 1, 1, 0, 0, 0, 0, -3)$ $U(1)_3 : \text{diag}(0, 0, 0, 1, 1, 0, 0, 0), \quad U(1)_4 : \text{diag}(0, 0, 0, 0, 0, 1, 1, 0)$ $U(1)_5 : \text{diag}(0, 0, 0, 0, 0, 1, -1, 0)$	$ \Gamma = \mathbb{Z}_{>0}$
(xi) $SU(4) \times SU(4) \times U(1)^2 \rightarrow SU(4)_C \times SU(2)_L \times SU(2) \times U(1)^3 \rightarrow G_{\text{SM}} \times U(1)^4$	
U(1) generators T_a	Available $ \Gamma $
$U(1)_1 : \text{diag}(1, 1, 1, 0, 0, 0, 0, 1), \quad U(1)_2 : \text{diag}(1, 1, 1, 0, 0, 0, 0, -3)$ $U(1)_3 : \text{diag}(0, 0, 0, 1, 1, -1, -1, 0), \quad U(1)_4 : \text{diag}(0, 0, 0, 1, 1, 1, 1, 0)$ $U(1)_5 : \text{diag}(0, 0, 0, 0, 0, 1, -1, 0)$	$ \Gamma = \mathbb{Z}_{>0}$
Others	
(xii) $SO(6) \times SO(6) \times SO(4) \rightarrow SU(3)_C \times SU(3) \times SU(2) \times U(1)^3 \rightarrow G_{\text{SM}} \times U(1)^4$	
U(1) generators T_a	Available $ \Gamma $

$U(1)_1 : \text{diag}(0,0,0,0,0,0,1,1), \quad U(1)_2 : \text{diag}(0,0,0,1,1,1,0,0)$ (a) $U(1)_3 : \text{diag}(0,0,0,1,1,-2,0,0), \quad U(1)_4 : \text{diag}(1,1,1,0,0,0,0,0)$ $U(1)_5 : \text{diag}(0,0,0,0,0,0,1,-1)$	$ \Gamma = \mathbb{Z}_{>0}$
$U(1)_1 : \text{diag}(1,1,1,0,0,0,0,0), \quad U(1)_2 : \text{diag}(0,0,0,1,1,0,0,0)$ (b) $U(1)_3 : \text{diag}(0,0,0,0,0,1,1,-2), \quad U(1)_4 : \text{diag}(0,0,0,0,0,0,1,-1,0)$ $U(1)_5 : \text{diag}(0,0,0,0,0,1,1,1)$	$ \Gamma = \mathbb{Z}_{>0}$
(xiii) $SO(6) \times SO(4) \times SO(2)^3 \rightarrow G_{\text{SM}} \times U(1)^4$	
$U(1)$ generators T_a	Available $ \Gamma $
$U(1)_1 : \text{diag}(1,1,1,0,0,0,0,1), \quad U(1)_2 : \text{diag}(0,0,0,1,1,0,0,0)$ $U(1)_3 : \text{diag}(0,0,0,0,0,1,0,0), \quad U(1)_4 : \text{diag}(0,0,0,0,0,0,1,0)$ $U(1)_5 : \text{diag}(0,0,0,0,0,0,0,1)$	$ \Gamma = \mathbb{Z}_{>0}$
(xiv) $SO(8) \times SO(6) \times SO(2) \rightarrow SU(4)_C \times SU(3) \times U(1)^3 \rightarrow G_{\text{SM}} \times U(1)^4$	
$U(1)$ generators T_a	Available $ \Gamma $
$U(1)_1 : \text{diag}(1,1,1,0,0,0,0,1), \quad U(1)_2 : \text{diag}(0,0,0,1,1,1,0,0)$ (a) $U(1)_3 : \text{diag}(0,0,0,1,1,-2,0,0), \quad U(1)_4 : \text{diag}(0,0,0,0,0,0,1,0)$ $U(1)_5 : \text{diag}(1,1,1,0,0,0,0,-3)$	$ \Gamma = \mathbb{Z}_{>0}$
$U(1)_1 : \text{diag}(1,1,1,0,0,0,0,0), \quad U(1)_2 : \text{diag}(0,0,0,1,1,1,1,0)$ (b) $U(1)_3 : \text{diag}(0,0,0,1,1,1,-3,0), \quad U(1)_4 : \text{diag}(0,0,0,1,1,-2,0,0)$ $U(1)_5 : \text{diag}(0,0,0,0,0,0,0,1)$	$ \Gamma = \mathbb{Z}_{>0}$
$U(1)_1 : \text{diag}(1,1,1,0,0,0,0,0), \quad U(1)_2 : \text{diag}(0,0,0,1,1,1,1,0)$ (c) $U(1)_3 : \text{diag}(0,0,0,1,1,-1,-1,0), \quad U(1)_4 : \text{diag}(0,0,0,0,0,1,-1,0)$ $U(1)_5 : \text{diag}(0,0,0,0,0,0,0,1)$	$ \Gamma = \mathbb{Z}_{>0}$
(xv) $SO(8) \times SO(4) \times SO(2)^2 \rightarrow SU(4)_C \times SU(2)_L \times U(1)^3 \rightarrow G_{\text{SM}} \times U(1)^4$	
$U(1)$ generators T_a	Available $ \Gamma $
$U(1)_1 : \text{diag}(1,1,1,0,0,0,0,1), \quad U(1)_2 : \text{diag}(0,0,0,0,0,0,1,0)$ $U(1)_3 : \text{diag}(0,0,0,1,1,0,0,0), \quad U(1)_4 : \text{diag}(0,0,0,0,0,1,0,0)$ $U(1)_5 : \text{diag}(1,1,1,0,0,0,0,-3)$	$ \Gamma = \mathbb{Z}_{>0}$
(xvi) $SO(6) \times SO(10) \rightarrow SU(3)_C \times SU(5) \times U(1)^2 \rightarrow G_{\text{SM}} \times U(1)^4$	
$U(1)$ generators T_a	Available $ \Gamma $
$U(1)_1 : \text{diag}(1,1,1,0,0,0,0,0), \quad U(1)_2 : \text{diag}(0,0,0,1,1,1,1,1)$ (a) $U(1)_3 : \text{diag}(0,0,0,1,1,1,1,-4), \quad U(1)_4 : \text{diag}(0,0,0,1,1,1,-3,0)$ $U(1)_5 : \text{diag}(0,0,0,1,1,-2,0,0)$	$ \Gamma = \mathbb{Z}_{>0}$
$U(1)_1 : \text{diag}(1,1,1,0,0,0,0,0), \quad U(1)_2 : \text{diag}(0,0,0,1,1,1,1,1)$ (b) $U(1)_3 : \text{diag}(0,0,0,3,3,-2,-2,-2), \quad U(1)_4 : \text{diag}(0,0,0,0,0,1,1,-2)$ $U(1)_5 : \text{diag}(0,0,0,0,0,1,-1,0)$	$ \Gamma = \mathbb{Z}_{>0}$
$U(1)_1 : \text{diag}(1,1,1,0,0,0,0,0), \quad U(1)_2 : \text{diag}(0,0,0,1,1,1,1,1)$ (c) $U(1)_3 : \text{diag}(0,0,0,2,2,2,-3,-3), \quad U(1)_4 : \text{diag}(0,0,0,1,1,-2,0,0)$ $U(1)_5 : \text{diag}(0,0,0,0,0,0,1,-1)$	$ \Gamma = \mathbb{Z}_{>0}$

(xvii) $\text{SO}(6) \times \text{SO}(4) \times \text{SO}(4) \times \text{SO}(2) \rightarrow \text{SU}(3)_C \times \text{SU}(2)_L \times \text{SU}(2) \times \text{U}(1)^4 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
U(1) generators T_a	Available $ \Gamma $
$\text{U}(1)_1 : \text{diag}(1, 1, 1, 0, 0, 0, 0, 0), \quad \text{U}(1)_2 : \text{diag}(0, 0, 0, 1, 1, 0, 0, 0)$ $\text{U}(1)_3 : \text{diag}(0, 0, 0, 0, 0, 1, 1, 0), \quad \text{U}(1)_4 : \text{diag}(0, 0, 0, 0, 0, 0, 0, 1)$ $\text{U}(1)_5 : \text{diag}(0, 0, 0, 0, 0, 1, -1, 0)$	$ \Gamma = \mathbb{Z}_{>0}$
(xviii) $\text{SO}(6) \times \text{SO}(6) \times \text{SO}(2)^2 \rightarrow \text{SU}(3)_C \times \text{SU}(3) \times \text{U}(1)^4 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
U(1) generators T_a	Available $ \Gamma $
$\text{U}(1)_1 : \text{diag}(1, 1, 1, 0, 0, 0, 0, 0), \quad \text{U}(1)_2 : \text{diag}(0, 0, 0, 0, 0, 0, 1, 0)$ $\text{U}(1)_3 : \text{diag}(0, 0, 0, 1, 1, -2, 0, 0), \quad \text{U}(1)_4 : \text{diag}(0, 0, 0, 1, 1, 1, 0, 0)$ $\text{U}(1)_5 : \text{diag}(0, 0, 0, 0, 0, 0, 0, 1)$	$ \Gamma = \mathbb{Z}_{>0}$

Table 1: Group decomposition of $\text{SO}(16)$ discussed in this paper. In all cases, the hypercharge direction is $\text{U}(1)_Y = \frac{1}{6} \text{diag}(-1, -1, -1, 0, 0, \pm 3, \pm 3, \pm 3)$ where the sign is taken in the random order. Γ denotes the possible order of freely-acting discrete symmetry group of CY threefold, satisfying the phenomenological requirements (2.27) and (2.28).

GUT-like decompositions
(vi)-(a) $\text{SO}(10) \times \text{SO}(6) \rightarrow \text{SU}(5) \times \text{SU}(3) \times \text{U}(1)^2 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$ Hypercharge direction: $\text{U}(1)_Y = (-3\text{U}(1)_1 - \text{U}(1)_2 + 15\text{U}(1)_3) / 30$ $X_{333} = 0, X_{334} = p_1, X_{335} = p_2, X_{344} = p_3, X_{345} = p_4, X_{355} = -1 - 3p_3, X_{444} = p_5, X_{445} = p_6,$ $X_{455} = 1 + p_5, X_{555} = p_7, X'_{33} = p_8, X'_{34} = p_{10}, X'_{35} = p_{11}, X'_{44} = p_{12}, X'_{45} = p_{13}, X'_{55} = p_{14}, X''_3 = 1 + 12p_8,$ $X''_4 = p_{15}, X''_5 = p_{16}, X''' = p_9, Z_3 = 0, Z_4 = -6p_1 + 12p_3 - 8p_5, Z_5 = -6p_2 - 12p_4 - 6p_6 - 2p_7,$ $Z' = 36 + 216p_8 - 2p_9,$
(vi)-(b) $\text{SO}(10) \times \text{SO}(6) \rightarrow \text{SU}(5) \times \text{SU}(3) \times \text{U}(1)^2 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$ Hypercharge direction: $\text{U}(1)_Y = (-3\text{U}(1)_1 + 5\text{U}(1)_2 - 5\text{U}(1)_3 + 3\text{U}(1)_4 - 5\text{U}(1)_5) / 30$ $X_{333} = p_1, X_{334} = p_2, X_{335} = p_3, X_{344} = p_4, X_{345} = p_5, X_{355} = p_6, X_{444} = p_7, X_{445} = p_8, X_{455} = p_9,$ $X_{555} = 5p_{10}, X'_{33} = p_1 + 2p_{11}, X'_{34} = p_{12}, X'_{35} = p_1 + p_{11} + p_{12} + 2p_{13}, X'_{44} = p_{14},$ $X'_{45} = 3p_4 + 3p_5 + 6p_7 + 6p_8 + 10p_9 + 6p_{10} + 3p_{14} + 12p_{15},$ $X'_{55} = 5p_4 + 3p_6 + 10p_7 + 10p_8 + 10p_9 + 30p_{10} + 5p_{14} + 20p_{15},$ $X''_3 = 144 + p_1 + 100p_2 + 36p_3 + 100p_4 + 40p_5 + 20p_6 + 80p_8 + 80p_{10} + 152p_{11} + 152p_{12} + 144p_{13} + 160p_{16},$ $X''_4 = 24 + 15p_2 + 48p_4 + 24p_5 + 36p_7 + 8p_8 + 84p_9 + 96p_{10} + 24p_{11} + 30p_{12} + 24p_{13} + 8p_{14} + 72p_{15} + 24p_{16},$ $X''_5 = -96 + 6p_1 - 120p_2 - 9p_3 - 120p_5 - 60p_8 + 20p_{10} - 114p_{11} - 114p_{12} - 108p_{13} - 120p_{16},$ $X''' = 9p_1 - 540p_2 + 324p_3 + 660p_4 - 1080p_5 + 180p_6 - 1480p_7 + 1320p_8 - 1560p_9 - 1640p_{10} - 126p_{11}$ $- 72p_{12} - 144p_{13} - 240p_{14} - 960p_{15},$ $Z_3 = -54 - 18p_1 - 15p_2 - 36p_3 - 75p_4 + 60p_5 - 45p_6 - 30p_8 - 30p_{10} - 69p_{11} - 57p_{12} - 69p_{13} - 60p_{16},$ $Z_4 = -9 - 9p_2 - 9p_4 - 9p_5 - 8p_7 + 15p_8 - 18p_9 + 3p_{10} - 9p_{11} - 9p_{12} - 9p_{13} + 6p_{14} + 9p_{15} - 9p_{16},$ $Z_5 = 36 + 45p_2 + 15p_4 + 45p_5 + 30p_7 + 15p_8 + 75p_9 - 55p_{10} + 45p_{11} + 45p_{12} + 45p_{13} + 15p_{14} + 60p_{15} + 45p_{16},$ $Z' = -54p_1 + 135p_2 - 108p_3 - 105p_4 + 360p_5 - 135p_6 + 140p_7 - 210p_8 + 330p_9 - 230p_{10} - 27p_{11} + 9p_{12}$ $- 27p_{13} - 30p_{14} + 30p_{15},$

Pati-Salam-like decompositions	
(x) $\text{SO}(8) \times \text{SO}(4) \times \text{SO}(4) \rightarrow \text{SU}(4)_C \times \text{SU}(2)_L \times \text{SU}(2) \times \text{U}(1)^3 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
Hypercharge direction: $\text{U}(1)_Y = (-\text{U}(1)_2 + 3\text{U}(1)_4)/6$	
$X_{333} = p_1, X_{334} = p_2, X_{335} = p_{12}, X_{344} = p_3, X_{345} = p_{13}, X_{355} = p_4, X_{444} = p_5, X_{445} = p_6, X_{455} = p_7,$ $X_{555} = p_8, X'_{33} = 6 + p_1, X'_{34} = p_9, X'_{35} = p_{14}, X'_{44} = p_{10}, X'_{45} = p_{15},$ $X''_{55} = -6 - p_3 + p_4 - 4p_5 + 4p_7 + p_{10}, X''_3 = p_{11}, X''_4 = -p_2 - 4p_3 - 4p_5 + 4p_7 + 2p_9 + 4p_{10},$ $X''_5 = p_{16}, X''' = -18 - 2p_1 + 3p_{11}, Z_3 = -2p_1, Z_4 = -2p_5 - 6p_7, Z_5 = -6p_6 - 2p_8, Z' = -2p_1,$	
(xi) $\text{SU}(4) \times \text{SU}(4) \times \text{U}(1)^2 \rightarrow \text{SU}(4) \times \text{SU}(2) \times \text{SU}(2) \times \text{U}(1)^3 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
Hypercharge direction: $\text{U}(1)_Y = (-2\text{U}(1)_2 - 3\text{U}(1)_3 + 3\text{U}(1)_4)/12$	
$X_{333} = p_1, X_{334} = p_2, X_{335} = p_3, X_{344} = p_4, X_{345} = p_5, X_{355} = p_6, X_{444} = p_7, X_{445} = p_8, X_{455} = p_9,$ $X_{555} = p_{10}, X'_{33} = p_{11}, X'_{34} = p_{12}, X'_{35} = p_{14}, X'_{44} = 6 + p_1 + 3p_2 + 3p_4 + p_7 - p_{11} - 2p_{12}, X'_{45} = p_{15},$ $X''_{55} = 4p_1 - 8p_2 + 16p_4 - 3p_6 - 4p_7 + 5p_9 - 4p_{12}, X''_3 = p_{13},$ $X''_4 = 36 + 7p_1 + 11p_2 + 33p_4 - 4p_6 - 3p_7 + 4p_9 - 4p_{11} - 20p_{12} + p_{13}, X''_5 = p_{16},$ $X''' = 90 + 19p_1 + 27p_2 + 93p_4 - 12p_6 - 11p_7 + 12p_9 - 12p_{11} - 60p_{12} + 6p_{13},$ $Z_3 = -2p_1 - 6p_4 - 3p_6 + 3p_9, Z_4 = -6p_2 + 3p_6 - 2p_7 - 3p_9,$ $Z_5 = -6p_3 + 12p_5 - 6p_8 - 2p_{10}, Z' = -2p_1 - 6p_2 - 6p_4 - 2p_7,$	
Others	
(xii)-(a) $\text{SO}(6) \times \text{SO}(6) \times \text{SO}(4) \rightarrow \text{SU}(3)_C \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)^3 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
Hypercharge direction: $\text{U}(1)_Y = (3\text{U}(1)_1 + \text{U}(1)_2 - \text{U}(1)_3 - \text{U}(1)_4)/6$	
$X_{333} = p_1, X_{334} = 1 + p_2, X_{335} = p_3, X_{344} = p_4, X_{345} = p_5, X_{355} = p_6, X_{444} = 0, X_{445} = p_7, X_{455} = p_8,$ $X_{555} = p_9, X'_{33} = p_{10}, X'_{34} = p_{11}, X'_{35} = p_{12}, X'_{44} = 24 + 27p_2 + 21p_4 + 2p_8 - 9p_{11}, X'_{45} = p_{13}, X'_{55} = p_{14},$ $X''_3 = p_{15}, X''_4 = 99 + 108p_2 + 84p_4 + 8p_8 - 36p_{11}, X''_5 = p_3 + p_7 + 2p_{16},$ $X''' = 372 + 27p_1 + 441p_2 + 294p_4 - 12p_6 + 14p_8 - 27p_{10} - 147p_{11} + 4p_{14} + 9p_{15},$ $Z_3 = -36 - 18p_1 - 72p_2 - 24p_4 + 18p_{10} + 24p_{11} - 6p_{15}, Z_4 = 0,$ $Z_5 = -15p_3 - 27p_5 - 15p_7 - 2p_9 + 9p_{12} + 9p_{13} - 3p_{16},$ $Z' = -132 - 54p_1 - 234p_2 - 84p_4 + 24p_6 + 20p_8 + 54p_{10} + 78p_{11} - 8p_{14} - 18p_{15},$	
(xii)-(b) $\text{SO}(6) \times \text{SO}(6) \times \text{SO}(4) \rightarrow \text{SU}(3)_C \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)^3 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
Hypercharge direction: $\text{U}(1)_Y = (-\text{U}(1)_1 + 3\text{U}(1)_5)/6$	
$X_{333} = p_1, X_{334} = p_2, X_{335} = p_3, X_{344} = 1 + p_1, X_{345} = p_4, X_{355} = p_5, X_{444} = p_6, X_{445} = -1 - 3p_3,$ $X_{455} = p_7, X_{555} = 0, X'_{33} = p_{10}, X'_{34} = p_{11}, X'_{35} = p_{12}, X'_{44} = p_{13}, X'_{45} = p_{14}, X'_{55} = p_8, X''_3 = p_{15},$ $X''_4 = p_{16}, X''_5 = 1 + 12p_8, X''' = p_9, Z_3 = -8p_1 + 12p_3 - 6p_5,$ $Z_4 = -6p_2 - 12p_4 - 2p_6 - 6p_7, Z_5 = 0, Z' = 36 + 216p_8 - 2p_9,$	
(xiii) $\text{SO}(6) \times \text{SO}(4) \times \text{SO}(2)^3 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
Hypercharge direction: $\text{U}(1)_Y = (-\text{U}(1)_1 - 3\text{U}(1)_3 + 3\text{U}(1)_4 - 3\text{U}(1)_5)/6$	
See, eq. (2.30).	
(xiv)-(a) $\text{SO}(8) \times \text{SO}(6) \times \text{SO}(2) \rightarrow \text{SU}(4)_C \times \text{SU}(3) \times \text{U}(1)^3 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$	
Hypercharge direction: $\text{U}(1)_Y = (\text{U}(1)_2 - \text{U}(1)_3 + 3\text{U}(1)_4 - \text{U}(1)_5)/6$	
$X_{333} = p_1, X_{334} = p_2, X_{335} = p_3, X_{344} = p_4, X_{345} = p_5, X_{355} = p_6, X_{444} = p_7, X_{445} = p_8, X_{455} = p_9,$ $X_{555} = p_{10}, X'_{33} = p_1 + p_2 + 2p_{11}, X'_{34} = p_1 + p_4 + p_{11} + 2p_{12}, X'_{35} = p_{13},$ $X'_{44} = 14 + 8p_1 + 4p_2 + 4p_3 + 7p_4 + 15p_7 + 4p_8 + 12p_{11} + 4p_{12} + 8p_{13} + 16p_{14},$ $X'_{45} = 5 + 2p_1 + p_2 + p_3 + p_4 + 3p_5 + 4p_7 - 3p_8 + 12p_9 + 3p_{11} + p_{12} + 2p_{13} + 4p_{14},$	

$X'_{55} = 2 + 2p_1 - 2p_2 + 7p_3 + p_4 - 3p_5 + 9p_6 + p_7 - p_8 + 3p_9 + 4p_{10} + p_{12} - p_{13} + p_{14},$ $X''_3 = p_{15}, X''_4 = p_{16},$ $X''_5 = 22 + 28p_1 - 34p_2 + 101p_3 + 14p_4 - 48p_5 + 96p_6 + 8p_7 - 7p_8 + 24p_9 + 16p_{10} - 6p_{11} + 14p_{12} - 14p_{13} + 8p_{14},$ $X''' = 246 + 282p_1 - 354p_2 + 996p_3 + 150p_4 - 432p_5 + 864p_6 + 98p_7 - 108p_8 + 384p_9 + 64p_{10} - 72p_{11} + 168p_{12} - 168p_{13} + 96p_{14} + 9p_{15} - 3p_{16},$ $Z_3 = -18p_1 + 18p_2 - 72p_3 - 6p_4 + 48p_5 - 96p_6, Z_4 = -2p_7, Z_5 = -6p_8 + 24p_9 - 32p_{10},$ $Z' = -54p_1 + 54p_2 - 216p_3 - 18p_4 + 144p_5 - 288p_6 + 2p_7 - 24p_8 + 96p_9 - 128p_{10},$
<p>(xiv)-(b) $\text{SO}(8) \times \text{SO}(6) \times \text{SO}(2) \rightarrow \text{SU}(4)_C \times \text{SU}(3) \times \text{U}(1)^3 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$</p> <p>Hypercharge direction: $\text{U}(1)_Y = (-2\text{U}(1)_1 + 3\text{U}(1)_2 - \text{U}(1)_3 - 2\text{U}(1)_4 + 6\text{U}(1)_5)/12$</p> $X_{333} = p_1, X_{334} = p_2, X_{335} = p_3, X_{344} = p_4, X_{345} = p_5, X_{355} = p_6, X_{444} = p_7, X_{445} = 2p_8, X_{455} = p_9,$ $X_{555} = 2p_{10}, X'_{33} = p_{11}, X'_{34} = p_{12}, X'_{35} = p_{13}, X'_{44} = p_8 + p_{10} + 2p_{14},$ $X'_{45} = 2p_1 + 8p_2 + 4p_3 + p_4 + 5p_5 + 2p_6 + 5p_7 + 9p_8 + 2p_9 + 4p_{11} + 2p_{12} + 9p_{13} + 5p_{14} + 10p_{15},$ $X'_{55} = -30 - 32p_1 - 72p_3 + 24p_4 + 72p_5 + 2p_6 - 8p_7 - 32p_8 + 2p_9 + 13p_{10} + 16p_{11} - 16p_{12} + 8p_{14},$ $X''_3 = 21p_1 + 12p_2 + 17p_3 + 21p_4 + 12p_5 + 21p_6 + 5p_7 + 20p_8 + 10p_{10} + 4p_{11} + 19p_{12} + 4p_{13} + 20p_{14} + 25p_{16},$ $X''_4 = -42p_1 + 18p_2 - 59p_3 + 21p_4 + 27p_5 - 17p_6 + p_7 - 36p_8 + 4p_9 - p_{10} + 16p_{11} - 33p_{12} + 11p_{13} - 7p_{14} + 15p_{15} - 25p_{16},$ $X''_5 = -45 - 120p_1 + 32p_2 - 176p_3 + 100p_4 + 200p_5 + 8p_6 - 12p_7 - 50p_8 + 8p_9 + 44p_{10} + 80p_{11} - 56p_{12} + 40p_{13} + 52p_{14} + 40p_{15},$ $X''' = 90 + 98p_1 + 204p_2 + 84p_3 + 114p_4 - 102p_5 + 24p_6 + 98p_7 + 33p_8 + 24p_9 - 34p_{10} + 90p_{13} + 90p_{15},$ $Z_3 = -14p_1 - 6p_3 - 12p_4 - 6p_6 - 2p_7 - 3p_8 - 3p_{10} - 6p_{12} - 6p_{14} - 6p_{16},$ $Z_4 = 12p_1 - 12p_2 + 18p_3 - 6p_4 + 6p_5 + 6p_6 + 18p_8 + 6p_{12} + 6p_{13} + 6p_{14} + 6p_{15} + 6p_{16}, Z_5 = -4p_{10},$ $Z' = -4p_1 - 24p_2 + 24p_3 - 36p_4 + 12p_5 - 4p_7 + 30p_8 - 4p_{10} + 12p_{13} + 12p_{15},$
<p>(xiv)-(c) $\text{SO}(8) \times \text{SO}(6) \times \text{SO}(2) \rightarrow \text{SU}(4)_C \times \text{SU}(3) \times \text{U}(1)^3 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$</p> <p>Hypercharge direction: $\text{U}(1)_Y = (-2\text{U}(1)_1 + 3\text{U}(1)_2 - 3\text{U}(1)_3 + 6\text{U}(1)_5)/12$</p> $X_{333} = p_1, X_{334} = p_2, X_{335} = p_3, X_{344} = p_4, X_{345} = p_5, X_{355} = p_6, X_{444} = p_7, X_{445} = 2p_8, X_{455} = p_9,$ $X_{555} = 2p_{10}, X'_{33} = p_{11}, X'_{34} = p_{12}, X'_{35} = p_{13},$ $X'_{44} = 8p_1 + 14p_3 + 2p_4 + 16p_6 + 19p_8 + 5p_{10} + 16p_{11} + 2p_{13} + 20p_{14}, X'_{45} = p_{15},$ $X'_{55} = -30 + 32p_1 + 56p_3 + 66p_6 + 40p_8 + 29p_{10} + 64p_{11} + 8p_{13} + 80p_{14},$ $X''_3 = 36p_1 + 47p_3 + 61p_6 + 50p_8 + 25p_{10} + 59p_{11} + 9p_{13} + 75p_{14},$ $X''_4 = 21p_2 + 17p_5 + 21p_9 + 4p_{12} + 4p_{15} + 25p_{16},$ $X''_5 = -45 + 128p_1 + 220p_3 + 256p_6 + 210p_8 + 108p_{10} + 256p_{11} + 36p_{13} + 320p_{14},$ $X''' = 90 + 8p_1 - 96p_3 - 66p_6 - 45p_8 - 34p_{10} - 90p_{11} - 90p_{14},$ $Z_3 = -8p_1 - 6p_6 - 3p_8 - 3p_{10} - 6p_{11} - 6p_{14}, Z_4 = -6p_2 - 6p_5 - 2p_7 - 6p_9 - 6p_{16},$ $Z_5 = -4p_{10}, Z' = -16p_1 - 12p_6 - 6p_8 - 4p_{10} - 12p_{11} - 12p_{14},$
<p>(xv) $\text{SO}(8) \times \text{SO}(4) \times \text{SO}(2)^2 \rightarrow \text{SU}(4) \times \text{SU}(2) \times \text{U}(1)^3 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$</p> <p>Hypercharge direction: $\text{U}(1)_Y = (3\text{U}(1)_2 + 3\text{U}(1)_4 - \text{U}(1)_5)/6$</p> $X_{333} = p_1, X_{334} = p_{14}, X_{335} = p_2, X_{344} = p_3, X_{345} = p_4, X_{355} = p_5, X_{444} = p_6, X_{445} = p_7, X_{455} = p_8,$ $X_{555} = p_9, X'_{33} = 6 + p_1, X'_{34} = p_{15}, X'_{35} = p_{10}, X'_{44} = 2 + p_3 + 4p_{11}, X'_{45} = 2 + p_4 - 2p_7 + 8p_8 + p_{11},$ $X'_{55} = p_{12}, X''_3 = p_{13}, X''_4 = p_{16}, X''_5 = -p_2 - 8p_5 + 4p_7 - 16p_8 + 2p_{10} + 8p_{12}, X''' = -18 - 2p_1 + 3p_{13},$ $Z_3 = -2p_1, Z_4 = -2p_6, Z_5 = -6p_7 + 24p_8 - 32p_9, Z' = -2p_1,$

<p>(xvi)-(a) $\text{SO}(6) \times \text{SO}(10) \rightarrow \text{SU}(3)_C \times \text{SU}(5) \times \text{U}(1)^2 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$</p> <p>Hypercharge direction: $\text{U}(1)_Y = (-10\text{U}(1)_1 + 6\text{U}(1)_2 + 9\text{U}(1)_3 - 5\text{U}(1)_4 - 10\text{U}(1)_5) / 60$</p> <p> $X_{333} = p_1, X_{334} = p_2, X_{335} = p_3, X_{344} = p_4, X_{345} = p_5, X_{355} = p_6, X_{444} = p_7,$ $X_{445} = 1 + p_1 + p_2 + p_3 + p_4 + p_7 + 2p_8, X_{455} = 2 + 2p_1 + 2p_2 + 2p_3 + 2p_4 + 4p_8 + 5p_9, X_{555} = p_{10},$ $X'_{33} = p_{11}, X'_{34} = p_{12}, X'_{35} = p_1 + p_2 + p_3 + p_4 + 2p_5 + p_6 + p_{11} + 2p_{12} + 3p_{13},$ $X'_{44} = 1 + p_{11} + 2p_{14}, X'_{45} = 3 + 25p_1 + 5p_5 + 5p_8 + 5p_9 - 2p_{11} + 5p_{12} + 5p_{13} + p_{14},$ $X'_{55} = 38 + 30p_1 + 30p_2 + 30p_3 + 30p_4 + 20p_5 + 5p_6 + 20p_7 + 20p_8 + 5p_9 + 5p_{10} + 8p_{11} + 36p_{14} + 40p_{15},$ $X''_3 = 24 + p_1 + 56p_2 + 56p_3 + 33p_4 - 78p_5 - 15p_6 + 72p_8 + 72p_9 - 4p_{11} + 6p_{12} + 6p_{13} - 24p_{14} - 24p_{15},$ $X''_4 = 4 + 6p_1 + 3p_2 + 4p_3 + 2p_4 + 5p_7 + 2p_8 + 5p_9 + 6p_{11} + 4p_{12} + 2p_{13} + 6p_{14} + 8p_{16},$ $X''_5 = 269 + 459p_1 + 137p_2 + 136p_3 + 193p_4 - 60p_5 - 30p_6 + 95p_7 + 328p_8 + 250p_9 + 25p_{10} + 14p_{11}$ $- 34p_{12} - 32p_{13} + 114p_{14} + 120p_{15} - 8p_{16},$ $X''' = 765 - 870p_1 + 2280p_2 + 2280p_3 - 360p_4 - 990p_5 - 825p_6 + 500p_7 + 1710p_8 + 1335p_9 + 125p_{10}$ $- 120p_{11} - 120p_{12} - 120p_{13} - 120p_{14} - 120p_{15},$ $Z_3 = -36 - 122p_1 - 12p_2 - 12p_3 - 36p_4 + 9p_5 - 12p_7 - 54p_8 - 45p_9 - 3p_{10} + 6p_{11} - 9p_{12} - 9p_{13}$ $- 9p_{14} - 9p_{15},$ $Z_4 = -51 - 96p_1 - 48p_2 - 39p_3 - 42p_4 - 15p_5 - 23p_7 - 57p_8 - 60p_9 - 5p_{10} - 3p_{11} - 9p_{12} - 12p_{13}$ $- 30p_{14} - 30p_{15} - 3p_{16},$ $Z_5 = -174 - 351p_1 - 99p_2 - 108p_3 - 120p_4 + 15p_5 + 15p_6 - 69p_7 - 222p_8 - 150p_9 - 18p_{10} + 3p_{11}$ $- 6p_{12} - 3p_{13} - 75p_{14} - 75p_{15} + 3p_{16},$ $Z' = -585 - 405p_1 - 555p_2 - 555p_3 - 270p_4 - 135p_5 + 75p_6 - 280p_7 - 585p_8 - 375p_9 - 70p_{10} - 90p_{11}$ $+ 60p_{12} + 60p_{13} - 390p_{14} - 390p_{15},$ </p>
<p>(xvi)-(b) $\text{SO}(6) \times \text{SO}(10) \rightarrow \text{SU}(3)_C \times \text{SU}(5) \times \text{U}(1)^2 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$</p> <p>Hypercharge direction: $\text{U}(1)_Y = (-10\text{U}(1)_1 + 6\text{U}(1)_2 + 9\text{U}(1)_3 - 5\text{U}(1)_4 - 10\text{U}(1)_5) / 60$</p> <p> $X_{333} = p_1, X_{334} = p_2, X_{335} = p_3, X_{344} = p_4, X_{345} = p_5, X_{355} = p_6, X_{444} = p_7, X_{445} = p_8, X_{455} = 4p_7 + 5p_9,$ $X_{555} = p_{10}, X'_{33} = p_1 + 2p_2 + 4p_{11}, X'_{34} = p_{12}, X'_{35} = p_{13}, X'_{44} = 5 + 5p_4 + 5p_7 + 5p_9 + 10p_{14}, X'_{45} = p_{15},$ $X'_{55} = 45 + 5p_6 + 35p_7 + 55p_9 + 90p_{14}, X''_3 = 24 + 25p_1 - 80p_2 + 60p_4 + 32p_7 + 32p_9 - 4p_{12} + 32p_{14},$ $X''_4 = -36 - 25p_2 - 40p_7 - 20p_9 + 10p_{12} - 80p_{14}, X''_5 = 7p_3 + 4p_5 + 4p_8 + 2p_{13} + 4p_{15} + 8p_{16},$ $X''' = -60 + 425p_1 - 1350p_2 + 900p_4 - 820p_7 - 420p_9 - 300p_{11} - 60p_{12} - 120p_{14},$ $Z_3 = 3 - 35p_1 + 45p_2 - 36p_4 - 8p_7 - 15p_{11} + 6p_{12} + 12p_{14}, Z_4 = 21 + 3p_7 + 15p_9 + 45p_{14},$ $Z_5 = -12p_3 - 9p_5 - 3p_8 - 2p_{10} + 3p_{13} - 3p_{16},$ $Z' = -195 - 175p_1 + 225p_2 - 180p_4 - 70p_7 - 150p_9 - 75p_{11} + 30p_{12} - 390p_{14},$ </p>
<p>(xvi)-(c) $\text{SO}(6) \times \text{SO}(10) \rightarrow \text{SU}(3)_C \times \text{SU}(5) \times \text{U}(1)^2 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$</p> <p>Hypercharge direction: $\text{U}(1)_Y = (-5\text{U}(1)_1 + 3\text{U}(1)_2 + \text{U}(1)_3 - 5\text{U}(1)_4 + 15\text{U}(1)_5) / 60$</p> <p> $X_{333} = p_1, X_{334} = p_2, X_{335} = p_3, X_{344} = p_4, X_{345} = p_5, X_{355} = p_6, X_{444} = p_7, X_{445} = p_8, X_{455} = p_9,$ $X_{555} = 2p_8 + p_9 + 5p_{10}, X'_{33} = p_{11}, X'_{34} = p_{12}, X'_{35} = p_1 + p_2 + p_5 + p_6 + p_{11} + 2p_{13},$ $X'_{44} = 3p_4 + p_7 + 3p_8 + 4p_{14},$ $X'_{45} = 5p_1 + 8p_4 + 5p_5 + 5p_6 + 6p_7 + 3p_8 + 5p_{10} + 5p_{11} + 5p_{13} + 4p_{14} + 10p_{15},$ $X'_{55} = 5p_1 - 5p_2 + 25p_3 + 16p_4 + 15p_6 + 12p_7 + 6p_8 + 15p_{10} + 5p_{11} + 8p_{14} + 20p_{15},$ $X''_3 = 5p_1 + 6p_3 + 7p_4 + 5p_6 + 4p_{11} + 2p_{12} + 4p_{13} + 8p_{16},$ $X''_4 = 24 + 30p_1 - 113p_2 + 88p_3 + 32p_4 - 84p_5 + 22p_6 + 13p_7 - 14p_8 + 15p_9 + 22p_{10} - 18p_{11} - 10p_{12}$ $- 26p_{13} + 8p_{14} - 4p_{15} - 16p_{16},$ </p>

$X_5'' = 24 + 70p_1 - 112p_2 - 43p_3 + 8p_4 - 6p_5 - 32p_6 + 12p_7 - 41p_8 + 25p_9 - 17p_{10} - 2p_{11} + 6p_{13} - 8p_{14} + 4p_{15} - 24p_{16},$ $X''' = -575p_1 + 585p_2 - 375p_3 - 345p_4 + 180p_5 - 165p_6 + 65p_7 - 445p_8 + 100p_9 - 415p_{10} - 150p_{11} - 30p_{12} - 90p_{13} - 60p_{14} - 120p_{15},$ $Z_3 = -20p_1 + 12p_2 - 6p_3 - 6p_4 + 9p_5 - 3p_6 + 2p_7 + 6p_{11} - 3p_{12} + 3p_{13} + 6p_{15} - 3p_{16},$ $Z_4 = -9 + 33p_2 - 33p_3 + 54p_5 + 3p_6 + 3p_7 + 12p_8 - 9p_9 + 3p_{10} + 18p_{11} + 21p_{13} + 3p_{14} + 24p_{15} + 6p_{16},$ $Z_5 = -9 - 15p_1 + 42p_2 - 12p_3 + 15p_4 + 36p_5 + 12p_6 + 9p_7 + 8p_8 - 8p_9 + 2p_{10} + 12p_{11} + 9p_{13} + 12p_{14} + 21p_{15} + 9p_{16},$ $Z' = 175p_1 - 105p_2 - 75p_3 - 45p_4 + 45p_5 - 30p_6 - 40p_7 + 20p_8 - 5p_9 + 5p_{10} + 15p_{12} + 45p_{13} - 45p_{14} - 15p_{15},$
<p>(xvii) $\text{SO}(6) \times \text{SO}(4) \times \text{SO}(4) \times \text{SO}(2) \rightarrow \text{SU}(3)_C \times \text{SU}(2)_L \times \text{SU}(2) \times \text{U}(1)^4 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$</p> <p>Hypercharge direction: $\text{U}(1)_Y = (-\text{U}(1)_1 + 3\text{U}(1)_3 + 3\text{U}(1)_4) / 6$</p> $X_{333} = p_1, X_{334} = p_2, X_{335} = p_3, X_{344} = p_4, X_{345} = p_{11}, X_{355} = p_1 + 2p_2 + p_4, X_{444} = -8p_1 - 12p_2 - 6p_4,$ $X_{445} = -3 + p_2,$ $X_{455} = p_{12}, X_{555} = p_5, X'_{33} = p_6, X'_{34} = p_7, X'_{35} = p_{13}, X'_{44} = p_8, X'_{45} = p_{14}, X'_{55} = p_{15}, X''_3 = p_9,$ $X''_4 = 3 + 16p_6 + 16p_7 + 4p_8 - 2p_9, X''_5 = p_{16}, X''' = p_{10},$ $Z_3 = -8p_1 - 12p_2 - 6p_4, Z_4 = 16p_1 + 24p_2 + 12p_4, Z_5 = -6p_3 - 2p_5, Z' = 36 + 96p_6 + 96p_7 + 24p_8 - 2p_{10},$
<p>(xviii) $\text{SO}(6) \times \text{SO}(6) \times \text{SO}(2)^2 \rightarrow \text{SU}(3)_C \times \text{SU}(3) \times \text{U}(1)^4 \rightarrow G_{\text{SM}} \times \text{U}(1)^4$</p> <p>Hypercharge direction: $\text{U}(1)_Y = (-\text{U}(1)_1 + 3\text{U}(1)_2 - \text{U}(1)_3 + \text{U}(1)_4 + 3\text{U}(1)_5) / 6$</p> $X_{333} = p_1, X_{334} = p_2, X_{335} = p_3, X_{344} = p_4, X_{345} = p_5, X_{355} = p_6, X_{444} = p_7, X_{445} = p_8, X_{455} = p_9,$ $X_{555} = p_{10}, X'_{33} = p_{11}, X'_{34} = p_{12}, X'_{35} = p_{13}, X'_{44} = 6 + p_1 + 3p_2 + 3p_4 + p_7 - p_{11} - 2p_{12},$ $X'_{45} = 6 + 6p_3 - 9p_5 - 4p_6 + 3p_8 + 2p_9 + 2p_{13}, X'_{55} = p_{14}, X''_3 = p_{15}, X''_4 = p_{16},$ $X''_5 = 60 - 2p_1 + 15p_2 + 27p_3 + 18p_4 - 36p_5 - 18p_6 + p_7 + 9p_8 + 6p_9 - 18p_{12} + 6p_{13} + 2p_{14} + 2p_{15} - p_{16},$ $X''' = -18 - 2p_1 - 6p_2 - 6p_4 - 2p_7 + 3p_{15} + 3p_{16}, Z_3 = -6p_1 + 6p_2 - 6p_4,$ $Z_4 = 4p_1 - 12p_2 - 2p_7, Z_5 = -2p_{10}, Z' = -2p_1 - 6p_2 - 6p_4 - 2p_7,$

Table 2: The list of 24 variables $\{X_{ABC}, X'_{AB}, X''_A, X''', Z_A, Z'\}$ with integers p_m , leading to three-generation models on simply-connected CY threefolds. We now choose a specific hypercharge direction for simplicity.

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