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Towards the minimal seesaw model via CP violation of neutrinos

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ABSTRACT: We study the minimal seesaw model, where two right-handed Majorana neutrinos are introduced, focusing on the CP violating phase. In addition, we take the trimaximal mixing pattern for the neutrino flavor where the charged lepton mass matrix is diagonal. Owing to this symmetric framework, the 3×2 Dirac neutrino mass matrix is given in terms of a few parameters. It is found that the observation of the CP violating phase determines the flavor structure of the Dirac neutrino mass matrix in the minimal seesaw model. New minimal Dirac neutrino mass matrices are presented in the case of TM_1 , which is given by the additional 2-3 family mixing to the tri-bimaximal mixing basis in the normal hierarchy of neutrino masses. Our model includes the Littlest seesaw model by King et al. as one of the specific cases. Furthermore, it is remarked that our 3×2 Dirac neutrino mass matrix is reproduced by introducing gauge singlet flavons with the specific alignments of the VEV's. These alignments are derived from the residual symmetry of S_4 group.

KEYWORDS: Neutrino Physics, CP violation, Discrete Symmetries

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1 Introduction

The standard model (SM) has been well established by the discovery of the Higgs boson. However, the origin of flavor in quarks and leptons is still unknown in spite of the remarkable success of the SM. The underlying physics for the flavor in quarks and leptons is one of the fundamental problems in particle physics.

On the other hand, the neutrino oscillation experiments are going on a new step to observe the CP violation in the lepton sector. The T2K experiment has confirmed the neutrino oscillation in the $\nu_\mu \rightarrow \nu_e$ appearance events [1], which provides us a new information of the CP violation in the lepton sector. Recent T2K data strongly indicate the CP violation in the neutrino oscillation [2, 3]. The NO ν A data also suggest the CP violation [4], which is consistent with the T2K result. Since the experimental data of the CP violating phase will be available in the near future, one can develop the flavor structure of Yukawa couplings by taking account of it.

We study the flavor structure in the seesaw model [5]–[8–10] in order to find a clue of the underlying physics of flavor. It is advantageous to consider the minimum number of parameters needed for reproducing the neutrino mixing angles and CP violating phases completely [11]. Indeed, there are some attempts toward so-called the minimal seesaw model [12]–[24].

In this work, we investigate the minimal seesaw model via the CP violation, where we assume two right-handed Majorana neutrinos. In addition, we take the trimaximal mixing pattern [25, 26] of the neutrino flavor to reduce the number of free parameters of the Dirac neutrino mass matrix. The trimaximal mixing pattern is derived from the flavor symmetry.

Before the reactor experiments reported the non-zero value of θ_{13} in 2012 [27, 28], there has been a paradigm of the tri-bimaximal (TBM) mixing [29, 30]. Since the TBM mixing pattern is a highly symmetric, the non-Abelian discrete groups have become the center of attention at the flavor symmetry [31]–[38]. The observation of the non-vanishing θ_{13} forces to study the deviation from the TBM mixing. It is remarked that the trimaximal mixing pattern is still available after the observation of θ_{13} . The trimaximal mixing is given by the additional rotation of 2-3 (TM₁) or 1-3 (TM₂) families of neutrinos to the TBM mixing basis [39, 40]. The additional rotation of 1-2 families is called as TM₃, however it still leads to $\theta_{13} = 0$. We do not discuss the additional rotation of 1-2 families, TM₃, since it leads to $\theta_{13} = 0$.

One obtains testable relations among the neutrino mixing angles and the CP violating phase, so-called mixing sum rules in TM₁ and TM₂ [41]–[43]. The prediction for the CP violation of neutrinos has a big impact on the study of the flavor structure [44]–[55] because T2K and NO ν A experiments are expected to confirm the non-zero CP violating phase in the near future. We discuss the CP violation of neutrinos for both TM₁ and TM₂ in the framework of the minimal seesaw model to restrict the structure of the Dirac neutrino mass matrix. It is found that those Dirac neutrino mass matrices are reproduced by introducing gauge singlet flavons with the vacuum expectation value (VEV) in the non-Abelian discrete symmetry, for example, S_4 symmetry. The specific alignments of the VEV's suggest the residual symmetry of S_4 group in our work.

We investigate the Dirac neutrino mass matrix in the diagonal basis of the charged lepton mass matrix and the 2×2 right-handed Majorana neutrino mass matrix focusing the CP violating Dirac phase. Since our results depend on the neutrino mass hierarchies, we discuss both cases of the normal hierarchy (NH) and the inverted hierarchy (IH) for neutrino masses, respectively.

Our analyses include the Littlest seesaw model by King et al. [56]–[58] as the one of the specific cases of our model.

The paper is organized as follows. We present our framework of the minimal seesaw model in section 2, where the structure of the Dirac neutrino mass matrix is discussed to reproduce TM₁ and TM₂ for both cases of NH and IH. The Dirac neutrino mass matrix is also discussed in the view of the flavor symmetry. In section 3, the numerical results are presented. The section 4 is devoted to the summary and discussions. Appendix A gives the detail studies of the Dirac neutrino mass matrix, and appendix B presents the necessary group theory of S_4 .

2 Realization of minimal seesaw model

Let us start with discussing the structure of the lepton mixing matrix, so-called the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix U_{PMNS} [59, 60], where neutrinos are supposed to be Majorana particles. It is parametrized in terms of three mixing angles θ_{ij} ($i, j = 1, 2, 3$; $i < j$), one CP violating Dirac phase δ_{CP} , and two Majorana phases α, β as follows:

$$U_{\text{PMNS}} \equiv \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\text{CP}}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\text{CP}}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{\text{CP}}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{\text{CP}}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{\text{CP}}} & c_{23}c_{13} \end{pmatrix} \times \begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 & 0 \\ 0 & e^{i\frac{\beta}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.1)$$

where c_{ij} and s_{ij} denote $\cos \theta_{ij}$ and $\sin \theta_{ij}$, respectively. The CP violating measure, Jarlskog invariant [61], is defined by the PMNS matrix elements $U_{\alpha i}$, and is written in terms of the mixing angles and the CP violating phase as:

$$J_{\text{CP}} = \text{Im} [U_{e1}U_{\mu 2}U_{e2}^*U_{\mu 1}^*] = s_{23}c_{23}s_{12}c_{12}s_{13}c_{13}^2 \sin \delta_{\text{CP}}. \quad (2.2)$$

For the lepton mixing matrix, Harrison-Perkins-Scott proposed a simple form of the mixing matrix, so-called TBM mixing [29, 30] as follows:

$$V_{\text{TBM}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (2.3)$$

which was a good scheme for the lepton sector before the reactor experiments reported non-zero θ_{13} . Therefore, it is reasonable to take the TBM mixing as the starting point of the lepton mixing. In order to avoid vanishing θ_{13} , we move to the trimaximal mixing basis from the TBM mixing basis.

The minimal seesaw model consists of two right-handed Majorana neutrinos and three left-handed neutrinos in Type I seesaw [11]. Taking both the charged lepton mass matrix and right-handed Majorana neutrino one M_R to be real diagonal, M_R and the Dirac neutrino mass matrix M_D are generally written as:

$$M_R = -M_0 \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad M_D = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}, \quad (2.4)$$

respectively, where $a \sim f$ are complex parameters and p is the ratio between the two right-handed Majorana neutrino masses. The minus sign in front of M_0 is taken as our sign convention. By using the seesaw mechanism of Type I, the left-handed Majorana neutrino mass matrix M_ν is give by

$$M_\nu = -M_D M_R^{-1} M_D^T = \frac{1}{M_0} \begin{pmatrix} a^2 p + d^2 & abp + de & acp + df \\ abp + de & b^2 p + e^2 & bcp + ef \\ acp + df & bcp + ef & c^2 p + f^2 \end{pmatrix}. \quad (2.5)$$

By turning the neutrino mass matrix M_ν to the TBM mixing basis, the left-handed Majorana neutrino mass matrix is given as

$$\hat{M}_\nu \equiv V_{\text{TBM}}^T M_\nu V_{\text{TBM}} = \frac{1}{M_0} \begin{pmatrix} \frac{A_\nu^2 p + D_\nu^2}{6} & \frac{A_\nu B_\nu p + D_\nu E_\nu}{3\sqrt{2}} & \frac{A_\nu C_\nu p + D_\nu F_\nu}{2\sqrt{3}} \\ \frac{A_\nu B_\nu p + D_\nu E_\nu}{3\sqrt{2}} & \frac{B_\nu^2 p + E_\nu^2}{3} & \frac{B_\nu C_\nu p + E_\nu F_\nu}{\sqrt{6}} \\ \frac{A_\nu C_\nu p + D_\nu F_\nu}{2\sqrt{3}} & \frac{B_\nu C_\nu p + E_\nu F_\nu}{\sqrt{6}} & \frac{C_\nu^2 p + F_\nu^2}{2} \end{pmatrix}, \quad (2.6)$$

where

$$\begin{aligned} A_\nu &\equiv 2a - b - c, & B_\nu &\equiv a + b + c, & C_\nu &\equiv c - b, \\ D_\nu &\equiv 2d - e - f, & E_\nu &\equiv d + e + f, & F_\nu &\equiv f - e. \end{aligned} \quad (2.7)$$

The mass matrix $\hat{M}_\nu$ is the left-handed Majorana neutrino mass matrix in the TBM mixing basis. In appendix A, we have classified this neutrino mass matrix to reproduce the trimaximal mixing TM₁ and TM₂ by the additional rotation of 2-3 or 1-3 families for both cases of NH and IH.

2.1 TM₁: 2-3 family mixing in NH

First of all, we consider the case of TM₁ in NH. The non-vanishing θ_{13} is obtained by an additional 2-3 family mixing to the TBM mixing basis [11, 40]. This case is given by the following Dirac neutrino mass matrix as shown in appendix A.1:

$$M_D = \begin{pmatrix} \frac{b+c}{2} & \frac{e+f}{2} \\ b & e \\ c & f \end{pmatrix}, \quad (2.8)$$

which leads to the neutrino mass matrix

$$\hat{M}_\nu = \frac{f^2}{M_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4}[B^2 e^{2i\phi_B}(1+j)^2 + (k+1)^2] & \frac{1}{2}\sqrt{\frac{3}{2}}[B^2 e^{2i\phi_B}(1-j^2) - k^2 + 1] \\ 0 & \frac{1}{2}\sqrt{\frac{3}{2}}[B^2 e^{2i\phi_B}(1-j^2) - k^2 + 1] & \frac{1}{2}[B^2 e^{2i\phi_B}(1-j)^2 + (k-1)^2] \end{pmatrix}, \quad (2.9)$$

where we put $p = 1$ in eq. (2.6) after rescaling parameters. This matrix is derived from eq. (A.7) in appendix A, where we can take e and f to be real, and b and c to be complex in general by using the freedom of redefinitions of phases in the left-handed lepton fields. We consider the case that the relative phase between b and c is 0 or π in order to reduce the number of parameters of the neutrino mass matrix. Then, the only one phase which leads to the CP violation of neutrinos is the relative phase between the first and the second columns of the Dirac neutrino mass matrix in eq. (2.8), that is the phase of c/f . Since we neglect the relative phase between b and c , the effect of it is discussed in subsection 3.2. Parameters in eq. (2.9) are defined as

$$\frac{e}{f} = k, \quad \frac{b}{c} = j, \quad \frac{c}{f} = B e^{i\phi_B}, \quad (2.10)$$

where k , j , and B are real.

The neutrino mass matrix in eq. (2.9) is diagonalized by the rotation of 2-3 families as

$$V_{23} = \frac{1}{\mathcal{A}} \begin{pmatrix} \mathcal{A} & 0 & 0 \\ 0 & 1 & \mathcal{V} \\ 0 & -\mathcal{V}^* & 1 \end{pmatrix}, \quad \mathcal{A} = \sqrt{1 + |\mathcal{V}|^2}, \quad (2.11)$$

where $\mathcal{V}$ is given in terms of k , j , B , and ϕ_B . Therefore, the PMNS matrix is calculated as follows:

$$U_{\text{PMNS}} = V_{\text{TBM}} V_{23}, \quad (2.12)$$

which gives three mixing angles, one Dirac phase, and one Majorana phase.

On the other hand, three neutrino masses are related to those parameters as follows:

$$\begin{aligned} m_1 &= 0, & m_2^2 m_3^2 &= \frac{9}{4} (j - k)^4 B^4 f^8, \\ m_2^2 + m_3^2 &= \frac{f^4}{16} [B^4 (5j^2 + 2j + 5)^2 + 2B^2 (5jk + j + k + 5)^2 \cos 2\phi_B + (5k^2 + 2k + 5)^2], \end{aligned} \quad (2.13)$$

which indicate that the PMNS matrix elements are correlated with neutrino masses.

At first, we study the specific Dirac neutrino mass matrices where the relative phase of b/c vanishes. These cases lead to one zero textures for the Dirac neutrino mass matrix (see appendix A). For case I, we consider the case of $j = b/c = -1$. The neutrino mass matrix is given as

$$\text{Case I: } \hat{M}_\nu = \frac{f^2}{M_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4}(k+1)^2 & -\frac{1}{2}\sqrt{\frac{3}{2}}(k^2-1) \\ 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}(k^2-1) & 2B^2 e^{2i\phi_B} + \frac{1}{2}(k-1)^2 \end{pmatrix}, \quad (2.14)$$

where the Dirac neutrino mass matrix is

$$M_D = \begin{pmatrix} 0 & \frac{e+f}{2} \\ b & e \\ -b & f \end{pmatrix}. \quad (2.15)$$

For case II where $c = 0$ is put, it is

$$\text{Case II: } \hat{M}_\nu = \frac{f^2}{M_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4}[\hat{B}^2 e^{2i\phi_B} + (k+1)^2] & -\frac{1}{2}\sqrt{\frac{3}{2}}[\hat{B}^2 e^{2i\phi_B} + k^2 - 1] \\ 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}[\hat{B}^2 e^{2i\phi_B} + k^2 - 1] & \frac{1}{2}[\hat{B}^2 e^{2i\phi_B} + (k-1)^2] \end{pmatrix}, \quad (2.16)$$

which corresponds to $j = \infty$ and $B = 0$ with a finite quantity $\hat{B} \equiv Bj$, that is

$$M_D = \begin{pmatrix} \frac{b}{2} & \frac{e+f}{2} \\ b & e \\ 0 & f \end{pmatrix}. \quad (2.17)$$

For case III where $b = 0$, it is

$$\text{Case III: } \hat{M}_\nu = \frac{f^2}{M_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4}[B^2 e^{2i\phi_B} + (k+1)^2] & -\frac{1}{2}\sqrt{\frac{3}{2}}[-B^2 e^{2i\phi_B} + k^2 - 1] \\ 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}[-B^2 e^{2i\phi_B} + k^2 - 1] & \frac{1}{2}[B^2 e^{2i\phi_B} + (k-1)^2] \end{pmatrix}, \quad (2.18)$$

which corresponds to $j = 0$, that is

$$M_D = \begin{pmatrix} \frac{c}{2} & \frac{e+f}{2} \\ 0 & e \\ c & f \end{pmatrix}. \quad (2.19)$$

It is interesting to note the relation between our neutrino mass matrices and the Littlest seesaw model by King et al. [56]–[58]. It corresponds to $k = -3$ in case I, that is,

$$M_D = \begin{pmatrix} 0 & f \\ b & 3f \\ -b & -f \end{pmatrix}. \quad (2.20)$$

We discuss the phenomenological implication for this model.

2.2 TM₁: 2-3 family mixing in IH

Next, we discuss the IH case of neutrino masses in TM₁. As shown in appendix A.2, the Dirac and left-handed Majorana neutrino mass matrices are given as:

$$M_D = \begin{pmatrix} -2b & \frac{e+f}{2} \\ b & e \\ b & f \end{pmatrix}, \quad (2.21)$$

and

$$\hat{M}_\nu = \frac{1}{M_0} \begin{pmatrix} 6b^2 & 0 & 0 \\ 0 & \frac{3}{4}(e+f)^2 & -\frac{1}{2}\sqrt{\frac{3}{2}}(e-f)(e+f) \\ 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}(e-f)(e+f) & \frac{1}{2}(e-f)^2 \end{pmatrix}, \quad (2.22)$$

where $m_1 = 6b^2/M_0$ and $m_3 = 0$. By taking the elements in the first column of the Dirac neutrino mass matrix to be real, parameters e and f can be complex in contrast to the case of subsection 2.1. When we redefine the complex parameters e and f in terms of two real parameters k and ϕ_k as

$$\frac{e}{f} = k e^{i\phi_k}, \quad (2.23)$$

the neutrino mass matrix turns to

$$\hat{M}_\nu = \frac{6b^2}{M_0} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{f^2}{M_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4}(k e^{i\phi_k} + 1)^2 & -\frac{1}{2}\sqrt{\frac{3}{2}}(k^2 e^{2i\phi_k} - 1) \\ 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}(k^2 e^{2i\phi_k} - 1) & \frac{1}{2}(k e^{i\phi_k} - 1)^2 \end{pmatrix}. \quad (2.24)$$

Note that the 2-3 family mixing is determined only by k and ϕ_k . If the relative phase between e and f , that is ϕ_k , is zero, the CP symmetry is conserved. It is remarked that the 2-3 family mixing is independent of neutrino masses m_1 and m_2 because the neutrino masses m_1 and m_2 are given in terms of b and f , respectively. This situation makes our numerical analysis simple for the case of IH.

Besides our neutrino mass matrix, the two-zero texture for the Dirac neutrino mass matrix has been discussed in the context of the minimal seesaw model [15, 20, 21, 23]. Especially, for the IH case, it is completely consistent with the experimental data of mixing angles and masses [20]. The prediction of the CP violating phase will be discussed comparing with our result in section 3.

2.3 TM₂: 1-3 family mixing in NH or IH

Let us consider the other case, in which the additional rotation of 1-3 families diagonalizes the neutrino mass matrix $\hat{M}_\nu$. As seen in appendix A.3, we obtain the Dirac neutrino mass matrix and the left-handed Majorana neutrino mass matrix as follows:

$$M_D = \begin{pmatrix} b & -e & -f \\ b & e & \\ b & f & \end{pmatrix}, \quad (2.25)$$

$$\hat{M}_\nu = \frac{1}{M_0} \begin{pmatrix} \frac{3}{2}(e+f)^2 & 0 & \frac{\sqrt{3}}{2}(e^2-f^2) \\ 0 & 3b^2 & 0 \\ \frac{\sqrt{3}}{2}(e^2-f^2) & 0 & \frac{1}{2}(e-f)^2 \end{pmatrix}, \quad (2.26)$$

respectively. By using the same notation in eq. (2.23), $\hat{M}_\nu$ is rewritten as:

$$\hat{M}_\nu = \frac{3b^2}{M_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{f^2}{M_0} \begin{pmatrix} \frac{3}{2}(ke^{i\phi_k} + 1)^2 & 0 & \frac{\sqrt{3}}{2}(k^2e^{2i\phi_k} - 1) \\ 0 & 0 & 0 \\ \frac{\sqrt{3}}{2}(k^2e^{2i\phi_k} - 1) & 0 & \frac{1}{2}(ke^{i\phi_k} - 1)^2 \end{pmatrix}. \quad (2.27)$$

As well as the case of the 2-3 family mixing for IH in subsection 2.2, the 1-3 family mixing is also determined only by k and ϕ_k , namely, it is independent of neutrino masses.

2.4 Dirac neutrino mass matrix and vacuum alignment of flavons

Above specific structures of the 3×2 Dirac neutrino mass matrix are given by introducing flavons with VEV's in the framework of the non-Abelian discrete symmetry S_4 . The Dirac neutrino mass matrices in subsections 2.1, 2.2, and 2.3 can be reproduced by the four types of flavons ϕ_i . These four flavons are represented as triplets of S_4 group and have specific alignments of VEV's:

$$\langle \phi_i \rangle = \begin{pmatrix} \frac{b+c}{2} \\ c \\ b \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -e-f \\ f \\ e \end{pmatrix}. \quad (2.28)$$

The first two VEV's in eq. (2.28) preserve the SU (US) symmetry for $\mathbf{3}'$ and $\mathbf{3}$, respectively since the generator SU (US) is expressed as

$$SU = US = \mp \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}, \quad (2.29)$$

for $\mathbf{3}$ and $\mathbf{3}'$, respectively. These VEV's are broken by S , T , and U (see appendix B). Those flavons have the Z_2 symmetry with elements $\{1, SU\}$. This Z_2 symmetry is a residual one after S_4 is broken. The third VEV in eq. (2.28) holds the S symmetry for both $\mathbf{3}$ and $\mathbf{3}'$, but broken by T . The last VEV in eq. (2.28) is not preserved by S , T , nor U unless $e = f$.

Let us reproduce the Dirac neutrino mass matrix in eq. (2.8) by introducing those flavons. Suppose that the left-handed lepton L and the flavons ϕ_{atm} , ϕ_{sol} are $\mathbf{3}'$ of S_4 , while the right-handed Majorana neutrinos ν_{R1} and ν_{R2} are $\mathbf{1}$ of S_4 . The Higgs field H_u is also $\mathbf{1}$. For the 2-3 family mixing in NH, the Dirac neutrino mass matrix is reproduced by the Yukawa couplings

$$\frac{y_{\text{atm}}}{\Lambda} \phi_{\text{atm}} L H_u \nu_{R1}^c + \frac{y_{\text{sol}}}{\Lambda} \phi_{\text{sol}} L H_u \nu_{R2}^c, \quad (2.30)$$

where y_{atm} and y_{sol} are arbitrary coupling constants, Λ is the cut-off scale of S_4 symmetry. The VEV's of ϕ_{atm} and ϕ_{sol} are

$$\langle \phi_{\text{atm}} \rangle \sim \begin{pmatrix} \frac{b+c}{2} \\ c \\ b \end{pmatrix}, \quad \langle \phi_{\text{sol}} \rangle \sim \begin{pmatrix} \frac{e+f}{2} \\ f \\ e \end{pmatrix}, \quad (2.31)$$

since S_4 singlet contraction $\mathbf{3}' \otimes \mathbf{3}'$ implies that $L(\mathbf{3}')\phi(\mathbf{3}') = L_1\phi_1 + L_2\phi_3 + L_3\phi_2$ as seen in appendix B.

For the case of IH of TM_1 , the assignments of the irreducible representations should be changed from the one of NH. We suppose that L and ϕ_{atm} are $\mathbf{3}$, and ϕ_{sol} is $\mathbf{3}'$ of S_4 , while ν_{R1} is $\mathbf{1}$ and ν_{R2} is $\mathbf{1}'$ of S_4 . It is easily seen that the Dirac neutrino mass matrix is reproduced by the Yukawa couplings in eq. (2.30), where the VEV's of ϕ_{atm} and ϕ_{sol} are

$$\langle \phi_{\text{atm}} \rangle \sim \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \phi_{\text{sol}} \rangle \sim \begin{pmatrix} \frac{e+f}{2} \\ f \\ e \end{pmatrix}, \quad (2.32)$$

since $\mathbf{3} \otimes \mathbf{3}'$ also implies that $L(\mathbf{3})\phi(\mathbf{3}') = L_1\phi_1 + L_2\phi_3 + L_3\phi_2$ for $\mathbf{1}'$. Thus, the Dirac neutrino mass matrix is reproduced by the relevant assignments of the flavons. The 2×2 diagonal right-handed Majorana neutrino mass matrix is reproduced by help of the auxiliary Z_2 symmetry in NH while an auxiliary Z_2 is not necessary in IH. The diagonal charged lepton mass matrix is realized as well as the Littlest seesaw model [56] by introducing three $\mathbf{3}$ flavons, which have VEV's:

$$\langle \phi_e \rangle \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \langle \phi_\mu \rangle \sim \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \phi_\tau \rangle \sim \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (2.33)$$

For TM_2 , that is the case of the 1-3 family mixing, the Dirac neutrino mass matrix can be reproduced by the relevant Yukawa couplings in the similar way. However, the situation of symmetry is different from the case of TM_1 because the fourth VEV in eq. (2.28) is not preserved in any subgroups of S_4 . In order to obtain desirable Yukawa couplings, we need an auxiliary Z_2 symmetry.

3 Numerical results

In this section, we discuss the numerical results for the CP violating Dirac phase δ_{CP} as well as neutrino mixing angles. We also discuss the Majorana phases, which contribute the effective mass for the neutrinoless double beta decay ($0\nu\beta\beta$ decay). The neutrino mixing angles are obtained in terms of the PMNS matrix elements $U_{\alpha i}$ of eq. (2.1) as follows:

$$s_{12}^2 \equiv \sin^2 \theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2}, \quad s_{23}^2 \equiv \sin^2 \theta_{23} = \frac{|U_{\mu 3}|^2}{1 - |U_{e3}|^2}, \quad s_{13}^2 \equiv \sin^2 \theta_{13} = |U_{e3}|^2. \quad (3.1)$$

The Dirac CP violating phase δ_{CP} can be calculated by using the Jarlskog invariant in eq. (2.2):

$$\sin \delta_{\text{CP}} = \frac{J_{\text{CP}}}{s_{23} c_{23} s_{12} c_{12} s_{13} c_{13}^2}, \quad (3.2)$$

where $\cos \delta_{\text{CP}}$ is fixed by $|U_{\mu 1}|^2 = 1/6$ for the case of 2-3 family mixing, and $|U_{\mu 2}|^2 = 1/3$ for the case of 1-3 family mixing, respectively.

The effective mass for the $0\nu\beta\beta$ decay is given in terms of the Dirac phase δ_{CP} and Majorana phases α and β as follows:

$$\begin{aligned} |m_{ee}| &= |m_1 U_{e1}^2 + m_2 U_{e2}^2 + m_3 U_{e3}^2| \\ &= \left| m_1 c_{13}^2 c_{12}^2 e^{i\alpha} + m_2 c_{13}^2 s_{12}^2 e^{i\beta} + m_3 s_{13}^2 e^{-2i\delta_{\text{CP}}} \right|, \end{aligned} \quad (3.3)$$

where $m_1 = 0$ or $m_3 = 0$ for NH or IH, respectively. The Majorana phase β is directly related to the expression of the effective mass $|m_{ee}|$ for NH, while $(\alpha - \beta)$ is related to $|m_{ee}|$ for IH.

Let us explain how to obtain our predictions of the CP violation taking the case I of subsection 2.1 (TM_1 in NH) as an example. The result of global analyses is often used as the inputting data to constrain the unknown parameters [62, 63]. In our calculations, we have adopted the result in ref. [63]. At first, by inputting the data of Δm_{13}^2 and Δm_{12}^2 within 3σ (1σ) range in table 1, we remove the two free parameters f^2/M_0 and B in eq. (2.14). The remained free parameter k is scanned in the region of $-20 \sim 20$ by generating random numbers in the linear scale. On the other hand, the phase ϕ_B is also scanned in the full region of $-\pi \sim \pi$ in the linear scale. Then, we calculate three neutrino mixing angles. These calculated values are judged by using the experimental data within 3σ (1σ) range as shown in table 1. If they are allowed for the experimental data, we keep the point, in which the CP violating phases and $|m_{ee}|$ are calculated. Otherwise, we throw the point. We continue this procedure to obtain enough points for plotting allowed region.

In the next subsections, we show the numerical results in the case of TM_1 and TM_2 for both NH and IH.

observable	NH: 1σ	3σ	IH: 1σ	3σ
$ \Delta m_{13}^2 \times 10^3 \text{ [eV}^2\text{]}$	2.55 ± 0.04 ,	$2.43 \sim 2.67$	2.49 ± 0.04 ,	$2.37 \sim 2.61$
$\Delta m_{12}^2 \times 10^5 \text{ [eV}^2\text{]}$	7.56 ± 0.19 ,	$7.05 \sim 8.14$	7.56 ± 0.19 ,	$7.05 \sim 8.14$
$\sin^2 \theta_{23}$	$0.430^{+0.020}_{-0.018}$,	$0.384 \sim 0.635$	$0.596^{+0.017}_{-0.018}$,	$0.388 \sim 0.638$
$\sin^2 \theta_{12}$	$0.321^{+0.018}_{-0.016}$,	$0.273 \sim 0.379$	$0.321^{+0.018}_{-0.016}$,	$0.273 \sim 0.379$
$\sin^2 \theta_{13} (10^{-2})$	$2.155^{+0.090}_{-0.075}$,	$1.89 \sim 2.39$	$2.140^{+0.082}_{-0.085}$,	$1.89 \sim 2.39$

Table 1. 1σ and 3σ ranges of the global analysis of the neutrino oscillation experimental data for NH and IH, respectively [63].

3.1 Three cases of TM_1 (2-3 family mixing) in NH

Let us show numerical results of case I, II, and III in subsection 2.1. The most interesting case is the case I. The case I corresponds to one of the one-zero texture models, where the $(1,1)$ element of the Dirac neutrino mass matrix is zero. There are two free parameters, k and ϕ_B after inputting data of neutrino masses in table 1. As seen in figure 1(a), the prediction of δ_{CP} versus $\sin^2 \theta_{23}$ is same as the previous result of TM_1 [48], where the neutrino mass matrix and the neutrino mass hierarchy have not been specified. Unless the magnitude of the dimensionless parameter k is fixed, the predicted $\sin^2 \theta_{23}$ is allowed to take all values within the 3σ range of the experimental data while δ_{CP} is in the region of $\pm(45^\circ \sim 125^\circ)$ depending on $\sin^2 \theta_{23}$ for the inputting data within 3σ . It is helpful to show the predictions for the inputting data within 1σ . The predicted δ_{CP} is in the region of $\pm(100^\circ \sim 115^\circ)$ for the 1σ range of $\sin^2 \theta_{23} = 0.412 - 0.450$. It is remarked that $|\delta_{\text{CP}}|$ should be larger than 90° if the observed θ_{23} is in the first octant. This prediction is consistent with the recent data of T2K, which present the allowed 2σ range for the CP violating phase, δ_{CP} , such as $(-171^\circ, -34.4^\circ)$ for NH [3].

As well known, the predicted $\sin^2 \theta_{12}$ is $0.317 \sim 0.320$ [48], which is a characteristic one for TM_1 independent of details of the neutrino mass matrix. This predicted value is inside the region of the experimental data of 1σ range.

We also show the result for the Majorana phase β in figure 1(b). For the inputting data within 3σ , $\beta = 0$ is allowed, but $\beta = 0$ is excluded for the inputting data with 1σ . It is found that β is zero when the Dirac CP violating phase δ_{CP} is $\pm\pi/2$. The remaining Majorana phase α is arbitrary because of $m_1 = 0$.

It is remarked that our predictions depend on the parameter k . As seen in figure 1(c), it is allowed in two separated region, $k \simeq -12 \sim -2$ and $k \simeq -0.1 \sim -0.5$ for the inputting data within 3σ . These regions are related to inverse values each other. The predicted value of $\sin^2 \theta_{23}$ crucially depends on k as seen in figure 1(c). Indeed, for the inputting data within 1σ , k is severely constrained around -2.5 and -0.4 .

We also present k dependence of δ_{CP} in figure 1(d). If the observed $|\delta_{\text{CP}}|$ is larger than 90° , k is larger than -3 . If δ_{CP} is observed accurately as well as $\sin^2 \theta_{23}$, k is determined with two-fold ambiguity. The k dependence is important because textures of neutrino mass matrix are given by fixing k as discussing later.

Figure 1. Blue (green) dots denote predictions for case I within 3σ (1σ) range. The red lines for $\sin^2 \theta_{23}$ and δ_{CP} denote the experimental bounds of 3σ (global analyses) and 2σ (T2K) ranges, respectively: (a) δ_{CP} versus $\sin^2 \theta_{23}$, (b) δ_{CP} versus β , (c) $\sin^2 \theta_{23}$ versus k , (d) δ_{CP} versus k , (e) δ_{CP} versus $\sin 2\phi_B$, and (f) $|m_{ee}|$ versus δ_{CP} . The magenta dots in (a), (b), (e) and (f) denote the predictions of the Littlest seesaw model. The vertical magenta lines of $k = -3$ in (c) and (d) also denote its predictions.

We present the $\sin 2\phi_B$ dependence of δ_{CP} in figure 1(e), where the prediction is classified by k . It is remarked that the sign of δ_{CP} depends on both $\sin 2\phi_B$ and k . The sign of δ_{CP} is determined by the sign of $(\sin 2\phi_B)$ for $k > -1$ while by the sign of $(-\sin 2\phi_B)$ for $k < -1$. This result is important for model-buildings.

We show the effective mass for the $0\nu\beta\beta$ decay $|m_{ee}|$ versus δ_{CP} in figure 1(f), where $|m_{ee}|$ is around $1.5 \sim 3.0$ meV. It is minimal if the CP violating phase δ_{CP} is $\pm\pi/2$.

Figure 2. Predictions for case II, where the red lines for $\sin^2 \theta_{23}$ and δ_{CP} denote the experimental bounds of 3σ (global analyses) and 2σ (T2K) ranges, respectively: (a) δ_{CP} versus $\sin^2 \theta_{23}$ and (b) δ_{CP} versus k .

The correlation between $|m_{ee}|$ and δ_{CP} is helpful to test our model for the Dirac neutrino mass matrix.

Finally, we discuss the Littlest seesaw model [56]–[58] in eq. (2.20) of subsection 2.1. This corresponds to $k = -3$ in the case I. As already presented in ref. [56], this model leads to the predictions around $\sin^2 \theta_{23} = 1/2$ and the maximal $\delta_{\text{CP}} = \pm\pi/2$. We show the results of the Littlest seesaw model by the color of magenta in figures 1(a)–(f). As seen in figure 1(a), the predicted $\sin^2 \theta_{23}$ and δ_{CP} are $0.451 \sim 0.544$ and $\pm(80^\circ \sim 105^\circ)$, respectively. As seen in figure 1(b), the Majorana phase β is in the range from -60° to 60° . In both figures 1(c) and (d), the vertical magenta lines of $k = -3$ denote the predictions of the Littlest seesaw model. The effective mass $|m_{ee}|$ is in the minimal regions in figure 1(f).

We propose new Littlest seesaw models by observing the result of figure 1. For example, $k = -5$ or -2 (-0.2 or -0.5) give different predictions for $\sin^2 \theta_{23}$ and δ_{CP} . The Dirac neutrino mass matrices are

$$M_D = \begin{pmatrix} 0 & 2f \\ b & 5f \\ -b & -f \end{pmatrix}, \quad \begin{pmatrix} 0 & 2f \\ b & -f \\ -b & 5f \end{pmatrix}, \quad \begin{pmatrix} 0 & f \\ b & 4f \\ -b & -2f \end{pmatrix}, \quad \begin{pmatrix} 0 & f \\ b & -2f \\ -b & 4f \end{pmatrix}, \quad (3.4)$$

which will be testable in the future experiments. Thus, our analyses present new Dirac neutrino mass matrices, which suggest new models of the lepton flavors.

Next, we discuss the numerical results for the case II, where the $(2, 1)$ element of the Dirac neutrino mass matrix is zero. In figure 2(a), the predicted $\sin^2 \theta_{23}$ is restricted near the upper bound of the 3σ range of the experimental data and the CP violating phase is $\delta_{\text{CP}} \simeq \pm 50^\circ$. The case II may be excluded if the more precise data of $\sin^2 \theta_{23}$ are available in the near future.

Although the prediction of δ_{CP} in figure 2(a) seems to be in the partial regions of the one in the case I, the parameter k of the case II is much different from the range of k in case I, that is, $k = -1.65 \sim -1.10$ as seen in figure 2(b). We find that the sign of δ_{CP} is determined by the sign of $(\sin 2\phi_B)$ in the region of $k = -1.65 \sim -1.10$.

Figure 3. Predictions for case III, where the red lines for $\sin^2 \theta_{23}$ and δ_{CP} denote the experimental bounds of 3σ (global analyses) and 2σ (T2K) ranges, respectively: (a) δ_{CP} versus $\sin^2 \theta_{23}$ and (b) δ_{CP} versus k .

Figure 4. Allowed regions in B - ϕ_B plane for TM_1 in NH. The blue, red, green, and magenta dots denote the case I, II, III, and the Littlest seesaw model, respectively.

We note the Majorana phase and the effective mass for the $0\nu\beta\beta$ decay. The Majorana phase β is restricted to the region around $\pm 5^\circ$ and $\pm(140^\circ \sim 170^\circ)$. The effective mass $|m_{ee}|$ is predicted to be $2.6 \sim 3.0$ meV.

In the case III, the $(3, 1)$ element of the Dirac neutrino mass matrix is zero. In contrast to the case II, the mixing angle $\sin^2 \theta_{23}$ is strongly restricted to the lower bound of the 3σ range as seen in figure 3(a). The case III is almost excluded in the present status of the neutrino oscillation experiments. We find $k = -0.86 \sim -0.71$. In this region, the Dirac CP violating phase δ_{CP} is predicted in the region around $\pm 125^\circ$. The Majorana phase β is predicted in the region around $\pm 5^\circ$ and $\pm 140^\circ$. It is also remarked that the sign of δ_{CP} is determined by the sign of $(-\sin 2\phi_B)$ in the region of $k = -0.86 \sim -0.71$. The predicted effective mass $|m_{ee}|$ is around 2.7 meV.

Finally in this subsection, we discuss allowed region of the parameters B and ϕ_B in eq. (2.10). We show the combined results for cases I, II, III, and the Littlest seesaw model in figure 4, where the blue, red, green, and magenta dots denote the case I, II, III, and the Littlest seesaw model, respectively. Thus, the allowed regions of those parameters are clearly different each other.

Figure 5. Blue (green) dots denote predictions for arbitrary j in $TM_1(NH)$ within 3σ (1σ). The red lines for $\sin^2 \theta_{23}$ and δ_{CP} denote the experimental bounds of 3σ (global analyses) and 2σ (T2K) ranges, respectively: (a) δ_{CP} versus j , (b) $\sin^2 \theta_{23}$ versus j , (c) $|m_{ee}|$ versus j , and (d) k versus j .

3.2 General case for TM_1 (2-3 family mixing) in NH

In the previous subsection, $j \equiv b/c$ has been fixed. The cases I, II, and III are specific cases for j , which correspond to $j = -1$, $-\infty$, and 0 , respectively. In order to show j dependence of our predictions in detail, we discuss the case in which j is not specified as in eq. (2.9).

We show the prediction of δ_{CP} versus j in figure 5(a), where j is allowed in negative value. As $|j|$ increases, δ_{CP} reaches $\pm 45^\circ$ asymptotically. We show the prediction of $\sin^2 \theta_{23}$ versus j in figure 5(b). As far as j is larger than -3 , $\sin^2 \theta_{23}$ is allowed to take all values within the 3σ range of the experimental data. However, it is predicted to be larger than 0.5 when j is smaller than -3 . It is also easily seen that the cases II ($j = -\infty$) and III ($j = 0$) are marginal for the prediction of $\sin^2 \theta_{23}$ as seen in figure 5(b). We show the prediction of $|m_{ee}|$ versus j in figure 5(c). We find that the predicted region of $|m_{ee}|$ never enlarge compared with those predictions of cases I, II, and III even if j is taken to be arbitrary. Finally, we show the allowed region in k and j plane in figure 5(d). It is verified numerically that j and k are symmetric for their exchange. The allowed regions within 1σ are rather narrow as seen in figure 5. If the error-bar of the inputting data are reduced in the future, both k and j are expected to be constrained considerably.

Figure 6. Blue (green) dots denote predictions for TM_1 in IH with 3σ (1σ). The red lines for $\sin^2 \theta_{23}$ and δ_{CP} denote the experimental bounds of 3σ (global analyses) and 2σ (T2K) ranges, respectively: (a) δ_{CP} versus $\sin^2 \theta_{23}$, (b) δ_{CP} versus difference of two Majorana phases $\beta - \alpha$, (c) δ_{CP} versus k , and (d) δ_{CP} versus ϕ_k . The cyan dots in (a) denote the prediction of the two-zero texture [20].

In these results, we take $j = b/c$ to be real although the relative phase b/c does not vanish in general as discussed below eq. (2.9). We have checked numerically that an extra phase of b/c does not contribute the results in figure 5 while it affects the ϕ_B dependences of δ_{CP} and $|m_{ee}|$ drastically. This situation is easily understood because the experimental data of mixing angles and mass square differences are input.

3.3 TM_1 : 2-3 family mixing in IH

Although the case of the NH is rather favored in the experiments of T2K and NO ν A [2]–[4], the inverted hierarchy of neutrino masses is still allowed by the experimental data of the neutrino oscillations. Let us show the numerical result of the IH case ($m_3 = 0$) for the 2-3 family mixing in eq. (2.24) since the Dirac neutrino mass matrix is completely different from the one of NH. In this case, the first column of the Dirac neutrino mass matrix is uniquely fixed as $(-2, 1, 1)^T$. There are two free parameters, k and ϕ_k , as seen in eq. (2.23). In contrast to the NH case, the ratio e/f is not allowed to be real in order to obtain the CP violation.

We show the predicted δ_{CP} versus $\sin^2 \theta_{23}$ in figure 6(a). The prediction is same as the case I of NH in figure 1(a), that is $\pm(45^\circ \sim 125^\circ)$. The recent T2K data present δ_{CP}

to be in the range of $(-88^\circ, -68^\circ)$ at 2σ range for IH [3]. We also show this range by red lines in the figures as an eye guide. We plot δ_{CP} versus the difference of the two Majorana phases $\beta - \alpha$ in figure 6(b). The difference of two Majorana phases $\beta - \alpha$ is almost zero. That is $\alpha = \beta$.

We also show the k and ϕ_k dependencies of δ_{CP} in figures 6(c) and (d). The parameters are allowed in the narrow ranges $k = 0.65 \sim 1.40$ and $\phi_k = \pm(25^\circ \sim 38^\circ)$. Thus, the structure of the Dirac neutrino mass matrix is restricted considerably.

The effective mass $|m_{ee}|$ is predicted to be around 50 meV since $\beta - \alpha$ is almost zero and $|m_{ee}|$ does not depend on δ_{CP} because of $m_3 = 0$.

In the context of the minimal seesaw model, the two-zero texture for the Dirac neutrino mass matrix has been examined in some works [15, 20, 21, 23]. This texture is different from our ones. However, this model is completely consistent with the experimental data of mixing angles and masses for IH [20] although this texture is disfavored for NH without some corrections. Since it is interesting to compare the predictions of two models, we add its prediction of δ_{CP} in figures 6(a), where (1, 1) and (2, 2) elements of the 3×2 Dirac neutrino mass matrix are zero. The predicted δ_{CP} is $\pm(88 - 93)^\circ$ in the whole range of allowed $\sin^2 \theta_{23}$. The case of vanishing (1, 1) and (3, 2) elements is also available, but the numerical prediction is almost unchanged. The prediction of the two-zero texture is a distinctive one from our ones.

3.4 TM₂: 1-3 family mixing in NH or IH

In this subsection, we discuss the case of the 1-3 family mixing. As well known, the predicted $\sin^2 \theta_{12}$ is close to 0.34 [48], which is a characteristic one for TM₂. The precise data of $\sin^2 \theta_{12}$ provides a crucial test for TM₂ as well as TM₁.

In figures 7 and 8, we show the numerical results for NH and IH, respectively. For both NH and IH cases, δ_{CP} is allowed to take all range in $(-\pi, \pi)$ as seen in figures 7(a) and 8(a). It is found that the Dirac CP violating phase becomes maximal $\pm\pi/2$ around the maximal mixing angle of $\theta_{23} = \pi/4$. For NH, the Majorana phase β is allowed to take all values from $-\pi$ to π as seen in figure 7(b). On the other hand, for IH, the difference of two Majorana phases $\beta - \alpha$ is almost zero as seen in figure 8(b).

The predicted value of δ_{CP} is sensitive to the parameter k which behaves differently for NH and IH. In the case of NH, δ_{CP} is 0 or $\pm\pi$ at the lower bound of k , 0.80 or at the upper bound, 1.25, respectively, as seen in figure 7(c). In the region of $k = 0.80 \sim 1.25$, the predicted δ_{CP} is changed drastically. On the other hand, for the case of IH, δ_{CP} is 0 or $\pm\pi$ at the lower bound $k = 0.50$ or at the upper bound $k = 2.00$, respectively, as seen in figure 8(c). In the region of $k = 0.50 \sim 2.00$, δ_{CP} is predicted for each k . The phase ϕ_k is restricted in the narrow region of $\pm(165^\circ \sim 180^\circ)$ for NH as seen in figure 7(d). On the other hand, for IH, the phase ϕ_k is in the region of $-37^\circ \sim 37^\circ$ as seen in figure 8(d). In both cases, δ_{CP} and θ_{23} become maximal simultaneously if $k = 1$ is taken.

The effective mass $|m_{ee}|$ is predicted in the region around 1.6 $\sim$ 4.2 meV and 50 meV for NH and IH, respectively.

Figure 7. Predictions for TM_2 in NH, where the red lines for $\sin^2 \theta_{23}$ and δ_{CP} denote the experimental bounds of 3σ (global analyses) and 2σ (T2K) ranges, respectively: (a) δ_{CP} versus $\sin^2 \theta_{23}$, (b) δ_{CP} versus Majorana phase β , (c) δ_{CP} versus k , and (d) δ_{CP} versus ϕ_k .

4 Summary and discussions

We have studied the minimal seesaw model where only two right-handed Majorana neutrinos are assumed, focusing on the CP violating phase δ_{CP} . In addition, we have taken the trimaximal mixing pattern for the neutrino flavor (TM_1 or TM_2) where the charged lepton mass matrix is diagonal. Owing to this symmetric framework, the flavor structure of the 3×2 Dirac neutrino mass matrix is given in terms of a few parameters.

We have examined three cases of the Dirac neutrino mass matrix for TM_1 in NH. It is emphasized that the observation of the CP violating phase determines the flavor structure of the Dirac neutrino mass matrix in the minimal seesaw model. New minimal Dirac neutrino mass matrices have been presented as the result of the numerical study for case I. Our model includes the Littlest seesaw model by King et al. as the one of the specific cases. We have also discussed the case of TM_1 in IH. The structure of the Dirac neutrino mass matrix is restricted considerably. The parameters are determined in the narrow ranges, $k = 0.65 \sim 1.40$ and $\phi_k = \pm(25^\circ \sim 38^\circ)$. We have studied the case of TM_2 in NH and IH. The predicted δ_{CP} is allowed to take all range in $(-\pi, \pi)$, but it is sensitive to the parameter k , which is around 1. In both cases, δ_{CP} and the mixing angle θ_{23} become maximal simultaneously for $k = 1$.

Figure 8. Predictions for TM_2 in IH, where the red lines for $\sin^2 \theta_{23}$ and δ_{CP} denote the experimental bounds of 3σ (global analyses) and 2σ (T2K), respectively: (a) δ_{CP} versus $\sin^2 \theta_{23}$, (b) δ_{CP} versus difference of two Majorana phases $\beta - \alpha$, (c) δ_{CP} versus k , and (d) δ_{CP} versus ϕ_k .

Our 3×2 Dirac neutrino mass matrix is reproduced by introducing gauge singlet flavons with the VEV's in S_4 flavor symmetry. The specific alignments of the VEV's are derived from the residual symmetry of S_4 group. It is interesting to consider the underlying mechanism which dynamically realizes the required vacuum alignments.

Finally, we add comments. If our Dirac neutrino mass matrices are given at the high energy scale, for example, the GUT scale, one should examine the renormalization group correction for the neutrino mixing matrix. However, it is very small since the lightest neutrino mass vanishes in our framework as seen in ref. [64].

The sign of the CP violating phase δ_{CP} is not determined in our framework. As well known, CP violating phases in the neutrino mass matrix are related to CP violating phases at the high energy. Since one can discuss the baryon asymmetry of the universe assuming the leptogenesis [65], the sign of δ_{CP} can be predicted in our framework. This work will appear elsewhere.

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A Minimal seesaw mass matrix

We can take the 2×2 right-handed Majorana neutrino mass matrix M_R to be real diagonal in general:

$$M_R = -M_0 \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad (\text{A.1})$$

where M_0 is the mass scale of the right-handed Majorana neutrino and p is the ratio between the two right-handed Majorana neutrino masses. The minus sign in front of M_0 is taken as our sign convention. On the other hand, the relevant Dirac neutrino mass matrix M_D is defined as

$$M_D = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}, \quad (\text{A.2})$$

where $a \sim f$ are complex parameters. By using the seesaw mechanism, the left-handed Majorana neutrino mass matrix M_ν is given by

$$M_\nu = M_D M_R^{-1} M_D^T = \frac{1}{M_0} \begin{pmatrix} a^2 p + d^2 & abp + de & acp + df \\ abp + de & b^2 p + e^2 & bcp + ef \\ acp + df & bcp + ef & c^2 p + f^2 \end{pmatrix}. \quad (\text{A.3})$$

By turning the neutrino mass matrix M_ν to the TBM mixing basis, M_ν is given as

$$\hat{M}_\nu \equiv V_{\text{TBM}}^T M_\nu V_{\text{TBM}} = \frac{1}{M_0} \begin{pmatrix} \frac{A_\nu^2 p + D_\nu^2}{6} & \frac{A_\nu B_\nu p + D_\nu E_\nu}{3\sqrt{2}} & \frac{A_\nu C_\nu p + D_\nu F_\nu}{2\sqrt{3}} \\ \frac{A_\nu B_\nu p + D_\nu E_\nu}{3\sqrt{2}} & \frac{B_\nu^2 p + E_\nu^2}{3} & \frac{B_\nu C_\nu p + E_\nu F_\nu}{\sqrt{6}} \\ \frac{A_\nu C_\nu p + D_\nu F_\nu}{2\sqrt{3}} & \frac{B_\nu C_\nu p + E_\nu F_\nu}{\sqrt{6}} & \frac{C_\nu^2 p + F_\nu^2}{2} \end{pmatrix}, \quad (\text{A.4})$$

where

$$\begin{aligned} A_\nu &\equiv 2a - b - c, & B_\nu &\equiv a + b + c, & C_\nu &\equiv c - b, \\ D_\nu &\equiv 2d - e - f, & E_\nu &\equiv d + e + f, & F_\nu &\equiv f - e. \end{aligned} \quad (\text{A.5})$$

We discuss neutrino mass structures to realize the additional 2-3 family rotation (TM₁) and the additional 1-3 one (TM₂) to the TBM mixing basis for both NH and IH.

A.1 TM₁: additional 2-3 family rotation in NH

At first, we consider the case of NH in TM₁. Since (1, 1), (1, 2), (2, 1), (1, 3), and (3, 1) entries of the matrix must be zero, conditions for the additional 2-3 rotation to the TBM mixing are

$$A_\nu = 2a - b - c = 0, \quad D_\nu = 2d - e - f = 0, \quad (\text{A.6})$$

where (a, b, c) are supposed to be independent of (d, e, f) . After imposing these conditions on the eq. (A.4), the mass matrix is rewritten as

$$\hat{M}_\nu = \frac{1}{M_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4}((b+c)^2 p + (e+f)^2) & \frac{1}{2}\sqrt{\frac{3}{2}}((c^2 - b^2)p - e^2 + f^2) \\ 0 & \frac{1}{2}\sqrt{\frac{3}{2}}((c^2 - b^2)p - e^2 + f^2) & \frac{1}{2}((b-c)^2 p + (e-f)^2) \end{pmatrix}, \quad (\text{A.7})$$

where the lightest neutrino mass m_1 is zero. Since the Majorana neutrino mass can be rescaled in the seesaw formula, the right-handed Majorana and Dirac neutrino mass matrices are written by putting $p = 1$ as

$$M_R = -M_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_D = \begin{pmatrix} \frac{b+c}{2} & \frac{e+f}{2} \\ b & e \\ c & f \end{pmatrix}, \quad (\text{A.8})$$

respectively. Starting from these textures, we discuss the specific cases which are attractive in the standpoint of the flavor model.

We consider one zero textures leading to the additional 2-3 family rotation to the TBM mixing basis. There are three possible patterns of one zero texture as follows:

$$(\text{I}) \ b + c = 0, \quad (\text{II}) \ c = 0, \quad (\text{III}) \ b = 0, \quad (\text{A.9})$$

in eq. (A.8). Corresponding Dirac neutrino mass matrices can be obtained as

$$M_D = \begin{cases} \begin{pmatrix} 0 & \frac{e+f}{2} \\ b & e \\ -b & f \end{pmatrix} & \text{for (I) } b + c = 0 \\ \begin{pmatrix} \frac{b}{2} & \frac{e+f}{2} \\ b & e \\ 0 & f \end{pmatrix} & \text{for (II) } c = 0 \\ \begin{pmatrix} 0 & \frac{e+f}{2} \\ \frac{c}{2} & \frac{e+f}{2} \\ 0 & e \\ c & f \end{pmatrix} & \text{for (III) } b = 0 \end{cases}. \quad (\text{A.10})$$

One can get another set by exchanging between the first column and second one in the Dirac neutrino mass matrix of eq. (A.10). However, the neutrino mass matrix $\hat{M}_\nu$ is invariant by this exchange. Therefore, we consider only three cases in eq. (A.10).

We show the neutrino mass matrix $\hat{M}_\nu$ for three cases:

$$\text{Case I: } \hat{M}_\nu = \frac{1}{M_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4}(e+f)^2 & -\frac{1}{2}\sqrt{\frac{3}{2}}(e-f)(e+f) \\ 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}(e-f)(e+f) & 2b^2 + \frac{1}{2}(e-f)^2 \end{pmatrix}, \quad (\text{A.11})$$

$$\text{Case II: } \hat{M}_\nu = \frac{1}{M_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4}[b^2 + (e+f)^2] & -\frac{1}{2}\sqrt{\frac{3}{2}}[b^2 + (e-f)(e+f)] \\ 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}[b^2 + (e-f)(e+f)] & \frac{1}{2}[b^2 + (e-f)^2] \end{pmatrix}, \quad (\text{A.12})$$

$$\text{Case III: } \hat{M}_\nu = \frac{1}{M_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4}[c^2 + (e+f)^2] & -\frac{1}{2}\sqrt{\frac{3}{2}}[-c^2 + (e-f)(e+f)] \\ 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}[-c^2 + (e-f)(e+f)] & \frac{1}{2}[c^2 + (e-f)^2] \end{pmatrix}. \quad (\text{A.13})$$

A.2 TM₁: additional 2-3 rotation in IH

Let us discuss the case of IH in TM₁. In order to give the additional 2-3 family rotation to the TBM mixing, the (1, 2), (1, 3), (2, 1), and (3, 1) elements in eq. (A.4) should vanish. These conditions are given as

$$A_\nu = a + b + c = 0, \quad C_\nu = c - b = 0, \quad D_\nu = 2d - e - f = 0. \quad (\text{A.14})$$

By setting $p = 1$, we have

$$M_R = -M_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_D = \begin{pmatrix} -2b & \frac{e+f}{2} \\ b & e \\ b & f \end{pmatrix}. \quad (\text{A.15})$$

The neutrino mass matrix $\hat{M}_\nu$ is given as

$$\hat{M}_\nu = \frac{1}{M_0} \begin{pmatrix} 6b^2 & 0 & 0 \\ 0 & \frac{3}{4}(e+f)^2 & -\frac{1}{2}\sqrt{\frac{3}{2}}(e-f)(e+f) \\ 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}(e-f)(e+f) & \frac{1}{2}(e-f)^2 \end{pmatrix}, \quad (\text{A.16})$$

where the neutrino mass m_3 vanishes.

A.3 TM₂: additional 1-3 rotation in NH or IH

Let us consider the case of the additional 1-3 family rotation to the TBM mixing basis. This case is called as TM₂. Then, (1, 2), (2, 3), (2, 1), and (3, 2) elements in eq. (A.4) should vanish. These conditions are given as

$$A_\nu = 2a - b - c = 0, \quad C_\nu = c - b = 0, \quad E_\nu = d + e + f = 0. \quad (\text{A.17})$$

The neutrino mass matrix $\hat{M}_\nu$ is written by

$$\hat{M}_\nu = \frac{1}{M_0} \begin{pmatrix} \frac{3}{2}(e+f)^2 & 0 & \frac{\sqrt{3}}{2}(e^2 - f^2) \\ 0 & 3b^2 & 0 \\ \frac{\sqrt{3}}{2}(e^2 - f^2) & 0 & \frac{1}{2}(e-f)^2 \end{pmatrix}, \quad (\text{A.18})$$

where $p = 1$ is set. The right-handed Majorana and the Dirac neutrino mass matrices are

$$M_R = -M_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_D = \begin{pmatrix} b & -e - f \\ b & e \\ b & f \end{pmatrix}, \quad (\text{A.19})$$

respectively. There is another solution

$$B_\nu = a + b + c = 0, \quad D_\nu = 2d - e - f = 0, \quad F_\nu = f - e = 0. \quad (\text{A.20})$$

However, this set leads to same structure of the neutrino mass matrix as in eq. (A.18). The mass eigenvalue m_1 or m_3 vanishes for NH or IH, respectively.

B Representations of S_4 group

We show the representations of S_4 group in this appendix. All elements of S_4 group are expressed by products of the generators s and t , which satisfy

$$s^4 = t^3 = e, \quad st^2s = t, \quad sts = ts^2t, \quad (\text{B.1})$$

where e is an identity element. These generators are represented on $\mathbf{1}_1$, $\mathbf{1}_2$, $\mathbf{2}$, $\mathbf{3}_1$, and $\mathbf{3}_2$ of S_4 group as follows [36, 37]:

$$\begin{aligned} \mathbf{1}_1 : \quad s &= 1, & t &= 1, \\ \mathbf{1}_2 : \quad s &= -1, & t &= 1, \\ \mathbf{2} : \quad s &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & t &= \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \\ \mathbf{3}_1 : \quad s &= \frac{1}{3} \begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix}, & t &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \\ \mathbf{3}_2 : \quad s &= -\frac{1}{3} \begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix}, & t &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \end{aligned} \quad (\text{B.2})$$

On the other hand, in refs. [66, 67], all elements of S_4 group are also expressed by products of the three generators S , T , and U , which satisfy

$$S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = (STU)^4 = e. \quad (\text{B.3})$$

Note that the minimal number of S_4 generators is only two. However, in order to compare generators of S_4 group with that of A_4 group, it is convenient to express elements of the group in terms of S , T , and U . These generators are also represented on $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{2}$, $\mathbf{3}$, and $\mathbf{3}'$ of S_4 group as follows:

$$\begin{aligned} \mathbf{1} : \quad S &= 1, & T &= 1, & U &= 1, \\ \mathbf{1}' : \quad S &= 1, & T &= 1, & U &= -1, \\ \mathbf{2} : \quad S &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & T &= \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, & U &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \mathbf{3} : \quad S &= \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, & T &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, & U &= -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \mathbf{3}' : \quad S &= \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, & T &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, & U &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \end{aligned} \quad (\text{B.4})$$

where $\mathbf{1}$ corresponds to $\mathbf{1}_1$, $\mathbf{1}'$ corresponds to $\mathbf{1}_2$, $\mathbf{3}$ corresponds to $\mathbf{3}_2$, and $\mathbf{3}'$ corresponds to $\mathbf{3}_1$ in eq. (B.2), respectively. It is remarked that s and t generators are related with S , T , and U as follows:

$$s = STUST, \quad t = T. \quad (\text{B.5})$$

The tensor products of S_4 group are independent of the basis of representations and written as follows:

$$\begin{aligned} \mathbf{1} \otimes \mathbf{r} &= \mathbf{r} \otimes \mathbf{1} = \mathbf{r}, & \mathbf{1}' \otimes \mathbf{1}' &= \mathbf{1}, & \mathbf{1}' \otimes \mathbf{2} &= \mathbf{2} \otimes \mathbf{1}' = \mathbf{2}, \\ \mathbf{1}' \otimes \mathbf{3} &= \mathbf{3} \otimes \mathbf{1}' = \mathbf{3}', & \mathbf{1}' \otimes \mathbf{3}' &= \mathbf{3}' \otimes \mathbf{1}' = \mathbf{3}, \\ \mathbf{2} \otimes \mathbf{2} &= \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2}, & \mathbf{2} \otimes \mathbf{3} &= \mathbf{3} \otimes \mathbf{2} = \mathbf{2} \otimes \mathbf{3}' = \mathbf{3}' \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{3}', \\ \mathbf{3} \otimes \mathbf{3} &= \mathbf{3}' \otimes \mathbf{3}' = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}', & \mathbf{3} \otimes \mathbf{3}' &= \mathbf{3}' \otimes \mathbf{3} = \mathbf{1}' \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}', \end{aligned} \quad (\text{B.6})$$

where $\mathbf{r}$ is an arbitrary representation of S_4 group.

For the following tensor products,

$$\mathbf{3} \otimes \mathbf{3} \rightarrow \mathbf{1}, \quad \mathbf{3}' \otimes \mathbf{3}' \rightarrow \mathbf{1}, \quad \mathbf{3} \otimes \mathbf{3}' \rightarrow \mathbf{1}', \quad (\text{B.7})$$

the Clebsch-Gordan coefficient is given as

$$\alpha_1 \beta_1 + \alpha_2 \beta_3 + \alpha_3 \beta_2, \quad (\text{B.8})$$

where α_i and β_i are elements of $\mathbf{3}^{(\prime)}$.

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