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Basic quantizations of $D = 4$ Euclidean, Lorentz, Kleinian and quaternionic $\mathfrak{o}^{\star}(4)$ symmetries

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Abstract: We construct firstly the complete list of five quantum deformations of $D = 4$ complex homogeneous orthogonal Lie algebra $\mathfrak{o}(4;\mathbb{C}) \cong \mathfrak{o}(3;\mathbb{C}) \oplus \mathfrak{o}(3;\mathbb{C})$, describing quantum rotational symmetries of four-dimensional complex space-time, in particular we provide the corresponding universal quantum R-matrices. Further applying four possible reality conditions we obtain all sixteen Hopf-algebraic quantum deformations for the real forms of $\mathfrak{o}(4;\mathbb{C})$: Euclidean $\mathfrak{o}(4)$, Lorentz $\mathfrak{o}(3,1)$, Kleinian $\mathfrak{o}(2,2)$ and quaternionic $\mathfrak{o}^{\star}(4)$. For $\mathfrak{o}(3,1)$ we only recall well-known results obtained previously by the authors, but for other real Lie algebras (Euclidean, Kleinian, quaternionic) as well as for the complex Lie algebra $\mathfrak{o}(4;\mathbb{C})$ we present new results.

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Contents

1 Introduction

In recent years due to the efforts to construct models of quantum gravity characterized by non-commutative spacetime structures at Planckian distances [1]–[3], the ways in which one can deform the algebras of spacetime coordinates and space-time symmetries became important. In noncommutative description of spacetime the numerical coordinates are replaced by noncommutative algebra, which is consistent with new type of uncertainty relations between the pairs of operator-valued coordinates [2] called further DFR uncertainty relation. This extension of noncommutativity into the spacetime sector describes the limitations on spacetime localization measurements if the quantum gravitational background is present. It appears that high density of energy added by measurement leads to the creation of mini black holes, and one can show that below Planck distance $\lambda_P = 10^{-33}m$ operationally the classical spacetime is not longer applicable. The quantum spacetime is effectively atomized, with lattice structure, and following the derivation in QM of Heisenberg algebra from Heisenberg uncertainty relations, one can deduce from DFR uncertainty relation the noncommutativity of quantum spacetime.

Such noncommutative and/or discrete nature of quantum spacetime follows as well from loop quantum gravity (LQG) approach [4–6], where the discretization is dynamical,[1] based on the existence in LQG framework of minimal lengths, minimal area surfaces or minimal volume quanta. In particular recently by applying LQG techniques to the quantum deformation of $D = 3$ gravity with positive cosmological constant Λ it has been shown [7] that one gets the quantum symmetry of $U_q(\mathfrak{o}(4))$, where $\ln q \sim \Lambda$, in analogy with earlier results of [8] and [9] for Lorentz signature.

In this paper we shall consider the noncommutative structures as linked with the quantum groups, which are described as non-cocommutative Hopf algebras [10, 11]–[15]. The deformed spacetime algebra is described as the irreducible representation (noncommutative Hopf algebra module) of quantum rotations algebra, with semidirect (smash) product structure and build-in covariant action of quantum-deformed symmetry algebra on quantum noncommutative spacetime.

The aim of this paper is to provide the quantum Hopf algebras and universal R-matrices which are obtained by quantization of classical r-matrices classified recently in [20–22]. In this paper we shall describe the Hopf-algebraic deformations of all real four-dimensional rotational algebras given by the real forms of $\mathfrak{o}(4;\mathbb{C})$. Fortunately, all classical r-matrices presented in [20, 22] can be quantized by providing explicit formulae for coproducts, antipodes and universal R-matrices.[2] We see therefore that the present paper provides the completion of the research program which we started in refs. [20–22].

The plan of our paper is the following.

In section 2 we shall present some generalities on quantization of 'infinitesimal' versions of quantum deformations described by complex and real classical r-matrices, which provides the triangular and nontriangular cases and further we present reality conditions for the universal R-matrices. Further, in section 3, we shall illustrate the quantization of classical r-matrices by the explicit presentation, for $\mathfrak{sl}(2;\mathbb{C})$ case, of Jordanian and standard deformations.

In section 4 we shall recall $D = 4$ complex Lie algebra $\mathfrak{o}(4;\mathbb{C})$ and its all four real forms. In particular, besides three real forms $\mathfrak{o}(4-k,k)$ $(k = 0, 1, 2)$ differing by the choice of

[1] We stress that in LQG spacetime lattice has a dynamical origin, in particular it is not a way to regularize neither the QG action nor the QG functional integral in order to perform effectively the numerical calculations.

[2] For classification purposes we listed in [20–22] only the antisymmetric r-matrices. In the case of standard (or Drinfeld-Jimbo [10, 11]) r-matrices one quantizes their symmetric Belavin-Drinfeld form [23, 24], which satisfies CYBE and describes the leading order in the expansion of quantum R-matrix satisfying quantum Yang-Baxter equation [10, 11, 16, 17]. For general formulae describing universal R-matrices see e.g. [25].

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signature, it should be added fourth quaternionic real form $\mathfrak{o}(2;\mathbb{H}) \equiv \mathfrak{o}(2,1)\oplus\mathfrak{o}(3) \equiv \mathfrak{o}^*(4)$. We shall work mostly with the generators of four-dimensional complex rotations in Cartan-Weyl bases.

In section 5 we quantize the full list of five classical r-matrices from [20, 22], i.e. provide the complete list of all Hopf-algebraic deformations of $U(\mathfrak{o}(4;\mathbb{C}))$: two of them triangular, and remaining three quasitriangular.[3] In order to present the results in detail we shall calculate the coproducts, antipods and universal quantum R-matrices. Further we specify all real forms of the quantum deformations of $\mathfrak{o}(4;\mathbb{C})$ described by $\star$-Hopf algebras. Following the standard recipe (see e.g. [17, 19, 26, 27]) we assume that the $\star$-operation, defining respective real form, acts on tensor product (coproduct) in unflipped way $(a\otimes b)^\star = a^\star\otimes b^\star$. We add that all quantum deformations of Lorentz algebra were already obtained earlier by the present authors [28, 29] and five out of eight Kleinian $D = 4$ real deformations can be obtained from the complex $\mathfrak{o}(4;\mathbb{C})$ deformations listed in section 4 simply by replacing the complex $\mathfrak{sl}(2;\mathbb{C})$ generators by the real ones describing $\mathfrak{sl}(2;\mathbb{R})$ algebra. The Hopf-algebraic deformation of Euclidean $\mathfrak{o}(4)$ algebra as well as $\mathfrak{o}(2,2)$ deformations and quaternionic $\mathfrak{o}(2;\mathbb{H})$ case are the most important because the obtained results are new.

In section 6 we shall present a brief outlook; the paper contains also two appendices.

The quantum deformations of four-dimensional rotational symmetries presented in this paper can be applied at least in the following contexts:

i) The deformed $D = 4$ rotation groups with the signature $(+,+,-,-)$ (Kleinian case) describe the deformed $D = 3$ AdS symmetry, and for Lorentz signature $(+,-,-,-)$ the $D = 3$ dS quantum symmetries. If we introduce (A)dS radius Λ and the rescaling of three rotations $M_{1k} \to \tilde{M}_{1k} = \Lambda P_k$ (P_k describes curved (A)dS momenta, $k = 1,2,3$), by suitable quantum Wigner-Inönü contraction [8, 30] one can get various κ-deformed $D = 3$ Poincaré algebras.

ii) The knowledge of classical r-matrices permits to introduce explicitly the action of deformed (super)string models, described by so-called YB (Yang-Baxter) sigma models [31]–[41]. The quantum deformations presented in this paper can be applied to the description of the YB deformation of principal $\mathfrak{o}(4-k,k)$ $\sigma-$models $(k = 0,1,2)$ as well as to the coset sigma models with noncommutative target space, described by the deformed cosets $\frac{\mathfrak{o}(4-k,k)}{\mathfrak{o}(3-k,k)}$. These deformed $\mathfrak{o}(4-k,k)$ groups or their coset manifolds can appear as parts of internal symmetry target spaces obtained by the reduction to $D = 4$ of deformed $D = 10$ Green-Schwarz superstrings.

iii) The classical r-matrices and their quantizations provide a powerful algebraic tool in description of integrable models and provide effective methods for studying their multihamiltonian systems [42–44]. In particular, the methods of noncommutative geometry permits to consider as well the Hamiltonian theories over the noncommutative rings [45, 46] and their integrability conditions.

[3]Only five $\mathfrak{o}(4;\mathbb{C})$ r-matrices are independent modulo $\mathfrak{o}(4;\mathbb{C})$ automorphism (see [22]). It should be noted that in these papers we have used the reality condition with the flip and therefore the deformation parameter in [20, 22] differs from the present paper by the replacement $\xi \mapsto \imath\xi$.

iv) Eight quantum deformations of $\mathfrak{o}(2,2)$ presented in the paper provide the set of finite $D=2$ quantum conformal algebras, with six generators, which in general case cannot be factorized into a sum of 'left' and 'right' $D=1$ quantum deformed conformal algebras. It is interesting to study which $\mathfrak{o}(2,2)$ deformations presented in the paper can be consistently extended to infinite-dimensional quantum groups, describing new classes of deformed $D=2$ infinite-dimensional conformal Virasoro algebras.

v) For various real forms of quantum-deformed $\mathfrak{o}(4;\mathbb{C})$ groups one can obtain corresponding four-dimensional spacetime with different signatures (see e.g. [47]). With all Hopf-algebraic deformations of $\mathfrak{o}(4;\mathbb{C})$ which will be presented in this paper one can obtain the complete list of quantum spacetimes with signatures $(4,0)$, $(3,1)$ and $(2,2)$.

Further remarks related with the applications of quantum deformations considered in this paper we shall present also in section 6.

2 Quantizations of complex and real Lie algebras: general remarks

2.1 From classical r-matrices to quantum universal R-matrices

It is known that formulated by Drinfeld [10, 11] the quantization problem of Lie bialgebras has been answered by Etingof and Kazhdan [14]: to each Lie bialgebra one can associate a quantized enveloping algebra supplemented with Hopf algebra structure (see also [15] for less technical presentation). Unfortunately, their proof is not constructive and the methods of explicit quantizations are known only in specific situations, as e.g. Drinfeld-Jimbo quantization of semi-simple Lie algebras and twist quantization in the triangular case (when twist tensor can be constructed explicitly). We shall show however that the known quantization techniques are sufficient for finding all explicit non-isomorphic quantizations of the enveloping algebra $\mathfrak{o}(4;\mathbb{C})$ and its real forms.

Principal tool for the classification of quantum deformations is provided by the classical r-matrices [10, 11, 23, 24, 49] which determine coboundary Lie bialgebra[4] structures (see e.g. [10, 11, 15–17]). Quantization procedure of bialgebras leads to the construction of quantum-deformed associative and coassociative Hopf-algebras [12] and determine the corresponding universal (quantum) R-matrices [16, 17, 25].

For semi-simple Lie algebras, due to the classical Whithead lemma, all bialgebras are coboundary. In such a case there is one-to-one correspondence between the Lie bialgebra structure and the corresponding classical r-matrix given as the skew-symmetric element $r\in\mathfrak{g}\wedge\mathfrak{g}$ satisfying the classical (homogenous or inhomogenous) YB equation:

$$[[r,\,r]] = t\,\Omega, \quad t\in\mathbb{C}\,, \tag{2.1}$$

with Schouten bracket

$$[[r,r]] \equiv [r_{12}, r_{13}+r_{23}] + [r_{13}, r_{23}], \tag{2.2}$$

[4]With the cobracket given by the commutator $\delta_r(x) = [x\otimes 1 + 1\otimes x, r]$.

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where $r_{12} = r^{(1)} \otimes r^{(2)} \otimes 1 \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ etc. and the three-form Ω is the $\mathfrak{g}$-invariant element in $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$, i.e.

$$ad_x\Omega \equiv [x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x, \Omega] = 0, \quad x \in \mathfrak{g}. \tag{2.3}$$

For skew-symmetric two-tensor monomials $x \wedge y = x \otimes y - y \otimes x$ and $u \wedge v$ $(x, y, u, v \in \mathfrak{g})$ the explicit formula for Schouten brackets reads[5]

$$\begin{aligned}[][[x \wedge y,\, u \wedge v]] &:= x \wedge \big([y, u] \wedge v + u \wedge [y, v]\big) \\ &\quad - y \wedge \big([x, u] \wedge v + u \wedge [x, v]\big) \\ &= [[u \wedge v,\, x \wedge y]].\end{aligned} \tag{2.4}$$

The complex Lie bialgebra is described by a pair $\mathsf{g} \equiv (\mathfrak{g}, r)$ consisting of complex Lie algebra $\mathfrak{g}$ and skew-symmetric classical r-matrix r satisfying the equation (2.1). One can distinguish two cases (cf. e.g. [15–17]):

A) If in (2.1) $t = 0$, one gets the so-called triangular or non-standard case with vanishing Schouten brackets describing homogenous classical Yang-Baxter equation (denoted as CYBE). In such a case the equation (2.1) is scale invariant. The triangularity is preserved by Lie algebra homomorphisms and can be reduced to non-degenerate case on the Borel subalgebra.

B) If $t \neq 0$, eq. (2.1) describes so-called non-triangular (quasitriangular) classical r-matrix, satisfying inhomogenous or modified classical Yang-Baxter equation (mCYBE). In such case one can introduce $r_{BD} \in \mathfrak{g} \otimes \mathfrak{g}$ called Belavin-Drinfeld form of the r-matrix satisfying CYBE, such that $r = r_{BD} - r^{\tau}_{BD}$ ($(x \otimes y)^{\tau} = y \otimes x$ is the flip operation) and the symmetric element $r^{s}_{BD} \equiv r_{BD} + r^{\tau}_{BD}$ which is ad-invariant.[6] In general, the initial skew-symmetric r-matrix is not scale invariant nor preserved by a Lie algebra homomorphisms. It is remarkable that Belavin-Drinfeld r-matrices for simple Lie algebras has been fully classified by means of so-called Belavin-Drinfel triples in [15, 24].

Quantization of (complex) Lie bialgebra leads to quantum groups in Drinfeld sense [10, 11] with the Hopf algebra structure supplementing the complex deformed universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ (in general one needs its topological ξ-adic extension $U_\xi(\mathfrak{g}) \equiv \mathrm{U}(\mathfrak{g})[[\xi]]$ formulated also for multiparameter deformation, i.e. $\xi \to (\xi_1, \ldots, \xi_k)$ (see e.g. [10]–[19]) and also section 3.2.2). Now we would like to focus our attention on the quantum universal R-matrix as an important byproduct of the quantization procedure.[7]

The universal R-matrix is an invertible element in some extension of $U_\xi(\mathfrak{g}) \otimes U_\xi(\mathfrak{g})$ (see [25]) which provides the flip τ of the noncocommutative coproduct $\tau : \Delta_\xi \to \Delta^{\tau}_\xi$ given

[5]For general elements $r_1, r_2 \in \mathfrak{g} \wedge \mathfrak{g}$ one can extend (2.4) by bilinearity.

[6]The symmetric part r_s is $\mathfrak{g}$-invariant and in the case of semi-simple algebra is related to the so-called split Casimir (non-degenerate Cartan-Killing form).

[7]The importance of quantum R-matrices follows from their applications as solutions of qYBE in various branches of theoretical physics e.g. conformal field theory, statistical mechanical models, and in mathematics, e.g. for description of link invariants.

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by the following similarity transformation

$$\Delta_\xi^\tau(\cdot) = R\,\Delta_\xi(\cdot)\,R^{-1}\,. \tag{2.5}$$

The universal $R-$matrix describes quantum group (see [12, 15]–[19]) if it satisfies quasitriangularity conditions

$$(\Delta_\xi \otimes id)R = R_{12}R_{23}\,, \qquad (id \otimes \Delta_\xi)R = R_{13}R_{12}\,, \tag{2.6}$$

where $R = R^{(1)} \otimes R^{(2)}$ and $R_{12} = R^{(1)} \otimes R^{(2)} \otimes 1$, etc.. The properties (2.5), (2.6) imply in the extension of $U_\xi(\mathfrak{g})\otimes U_\xi(\mathfrak{g})\otimes U_\xi(\mathfrak{g})$ the following quantum Yang-Baxter equation (qYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2.7}$$

as well as the normalization conditions

$$(\epsilon \otimes id)R = (id \otimes \epsilon)R = 1, \tag{2.8}$$

where ϵ denotes a counit.

In fact, the same properties (2.5)–(2.8) are satisfied by another universal R-matrix, which is $(R^\tau)^{-1}$. Therefore one can distinguish two case:

i) the element $Q_R = RR^\tau = 1 \otimes 1$ is trivial;

ii) $Q_R \neq 1 \otimes 1$ is non-trivial.

It turns out that the first case corresponds to the triangular or twist quantization case while the second characterizes the non-triangular case. In order to describe their difference let us expand the R-matrix (2.5) in the powers of the deformation parameter ξ, entering linearly in the definition of classical r-matrix[8]

$$R(\xi) = 1 \otimes 1 + \tilde{r} + O(\xi^2)\,. \tag{2.9}$$

From (2.9) and (2.7) it follows that the element $\tilde{r} \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies classical Yang-Baxter equation (CYBE) [10, 11, 16, 17]. In triangular case one has $\tilde{r} + \tilde{r}^\tau = 0$, i.e. $\tilde{r}$ is skew-symmetric and can be identified with the classical r-matrix satisfying (2.1) with $t = 0$. In the second (non-triangular, see ii)) case, $\tilde{r}$ is not skew-symmetric, satisfies CYBE and takes the Belavin-Drinfeld form of r-matrix, i.e. $\tilde{r} = r_{BD}$.

The classical r-matrices describe the infinitesimal version of quantum deformed Lie-algebraic symmetries; the quantum deformation parameterized by an arbitrary (formal) deformation parameter ξ determines Hopf-algebraic quantization and universal R-matrix.

In general case it is not known how to obtain the universal R-matrix from the solutions of (2.2); however for canonical Belavin-Drinfeld nontriangular r-matrices [23] the explicit formula for universal R-matrices is well-known (see e.g. [25]). It is worth noticing that

[8] The parameter ξ in $o(\xi^2)$ should be replaced by $(\xi_1, \ldots, \xi_k)$ in the case of multiparameter deformation. The multiparameter classical r-matrix is linear in ξ_i $(i = 1, \ldots, k)$, the expansion (2.9) is valid up to any quadratic terms in ξ_i.

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in contrast to the triangular case, the non-triangular one provides two different quantum R-matrices: $R(\xi)$ and $R^\tau(\xi)^{-1}$. The element $Q(\xi) \equiv R^\tau(\xi)R(\xi) = 1 + (r + r^\tau) + O(\xi^2)$ is called a quantum Killing form since its first order term, if not degenerate, defines a classical Cartan-Killing form on $\mathfrak{g}$. We would like to add that the skew-symmetric classical r-matrices are sufficient for classification[9] as well as for the description of the correspondence with classical Lie-Poisson groups.

2.2 Reality conditions providing the quantizations of real bialgebras

We remind that a real Lie algebra structure $(\mathfrak{g}, \circledast)$ can be introduced by adding an antilinear involutive (Lie algebra) anti-automorphism $\circledast : \mathfrak{g} \to \mathfrak{g}$ ($\circledast$-operation, conjugation) acting on the complex Lie algebra $\mathfrak{g}$. It relies on finding Lie algebra basis with real structure constants for which $\circledast$-operation is anti-Hermitian (i.e. $x^\circledast = -x$).[10] Subsequently, the real coboundary Lie bialgebra can be considered as a triple $(\mathfrak{g}, \circledast) \equiv (\mathfrak{g}, \circledast, r)$, where the skew-symmetric element r is assumed to be anti-Hermitian, i.e.

$$r^{\circledast\otimes\circledast} = -r = r^\tau. \tag{2.10}$$

Such condition comes from (2.11), (2.12) and leads to the suitable reality conditions for parameters involved in the form of r due to the following facts presented below.

The $\circledast$-operation extends, by the property $(ab)^\circledast = b^\circledast a^\circledast$ (i.e. as an antilinear antiautomorphism), to the enveloping algebra $U(\mathfrak{g})$, as well as to quantized enveloping algebra, making both of them associative $\circledast$-algebras. The real Hopf-algebraic structure represented on quantized enveloping algebra $U_\xi(\mathfrak{g})$ by $\circledast$-involution is defined by the conditions for coproducts

$$\Delta_\xi(a^\circledast) = (\Delta_\xi(a))^\circledast \qquad \forall a \in U_\xi(\mathfrak{g}), \tag{2.11}$$

where the $\circledast$-involution acts on the tensor product in a natural way[11]

$$(a \otimes b)^\circledast = a^\circledast \otimes b^\circledast, \tag{2.12}$$

implying as well (see e.g. [16]–[19, 27]) the following conditions

$$S_\xi((S_\xi(a^\circledast))^\circledast) = a, \qquad \epsilon(a^\circledast) = \epsilon(a)^*. \tag{2.13}$$

One can get (2.5) and (2.11) compatible and consistently defining quasitriangular $\circledast$-Hopf algebras by imposing two distinct reality constraints on the universal R-matrix (see e.g. [17]):

a) $R^{\circledast\otimes\circledast} = R^\tau$ (R is called real);

b) $R^{\circledast\otimes\circledast} = R^{-1}$ and the corresponding quantum R-matrix is $\circledast$-unitary (R is called antireal).

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[9] They are more tractable, since $dim\,(\mathfrak{g} \wedge \mathfrak{g}) < dim\,(\mathfrak{g} \otimes \mathfrak{g})$.

[10] In a case of Hilbert space realization this condition leads to operators with imaginary spectrum. For this reason some authors do prefer instead Hermitian generators and imaginary structure constants as representing real Lie algebras.

[11] The real Hopf algebra is identified with a $\star$-Hopf algebra which is a complex Hopf algebra equipped with antilinear involutive anti-homomorphism (star operation) rendering the algebraic sector into $\star$-algebra and coalgebraic sector satisfying (2.11). In this case one can introduce real Lie algebra structure by imposing the following reality condition $X^\star = -X$ on generators.

Particularly, in the triangular case, due to the identity $R^\tau = R^{-1}$, the conditions a) and b) are the same. In non-triangular case ($R^\tau \neq R^{-1}$), the second universal R-matrix $(R^\tau)^{-1}$ satisfies the same reality constraints.[12]

It should be noted that for any element $\tilde{r} \in \mathfrak{g}\otimes\mathfrak{g}$ satisfying CYBE, $\tilde{r}^{\circledast\otimes\circledast}$ satisfies again CYBE. Therefore, one can distinguish two cases:

i) the classical r-matrix $r = \tilde{r}$ corresponding to the universal R-matrix (cf. (2.9)) is skew-symmetric; then r should be anti-Hermitian and satisfy the relation (2.10).

ii) if the element $\tilde{r}$ is not skew-symmetric this corresponds to the non-triangular case; then $\tilde{r}^{\circledast\otimes\circledast} = \tilde{r}^\tau$,[13] for R real and $\tilde{r}^{\circledast\otimes\circledast} = -\tilde{r}$ for R antireal. It is easy to check that in any case the skew-symmetric part of $\tilde{r}$ remains anti-Hermitian, i.e. $(\tilde{r} - \tilde{r}^\tau)^{\circledast\otimes\circledast} = -(\tilde{r} - \tilde{r}^\tau)$.

It is easy to show that twisting of real form of quasitriangular Hopf algebra by unitary twist leads again to real quasitriangular Hopf algebra. More precisely, if $(\mathcal{H}, \Delta, S, \epsilon, R, \star)$ is a quasitriangular $\star$-Hopf algebra with R being real (resp. antireal) universal R-matrix, then for any unitary, normalized 2-cocycle twist $F = (F^{-1})^{\star\otimes\star} \in \mathcal{H}\otimes\mathcal{H}$ the quantized algebra $(\mathcal{H}, \Delta_F, S_F, \epsilon, R_F, \star)$ is a quasitriangular $\star$-Hopf algebra such that $R_F = F^\tau R F^{-1}$ is real (resp. antireal). This property will be used in the consideration of some cases of chain quantization of $\mathfrak{o}(4;\mathbb{C})$ (see section 5).

3 The basic $\mathfrak{sl}(2;\mathbb{C})$ example: complex and real Lie bialgebra and their quantizations

3.1 Complex and real bialgebras

We recall that classical r-matrices, providing Lie bialgebra structure of a given Lie algebra as well as quantum deformations of the corresponding enveloping algebra, are classified up to the isomorphisms; in particular for real Lie algebras one should use the isomorphisms preserving respective reality condition. These are $\circledast$-isomorphisms which for Lie algebra with fixed structure constants are given by real Lie algebra automorphisms. The important example of such classification of classical r-matrices for real forms of $\mathfrak{o}(4;\mathbb{C})$ has been investigated in [20–22].

It is well known that for the complex Lie algebra $\mathfrak{sl}(2;\mathbb{C}) \cong \mathfrak{o}(3;\mathbb{C})$ there exists up to $\mathfrak{sl}(2,\mathbb{C})$ automorphisms two solutions of mCYBE,[14] namely Jordanian r_J (triangular, called also non-standard) and the standard one r_{st} (non-triangular):

$$r_J(\chi) = \chi\, E_+ \wedge H, \qquad [[r_J(\chi), r_J(\chi)]] = 0\,, \tag{3.1}$$

$$r_{st}(\gamma) = \gamma E_+ \wedge E_-, \quad [[r_{st}(\gamma), r_{st}(\gamma)]] = \gamma^2 \Omega\,, \tag{3.2}$$

[12]It should be observed that the presence of these two universal R-matrices may help to obtain finite contraction limits (see e.g. [48]).

[13]This condition can be rewritten as $\tilde{r}^{\tau(\circledast\otimes\circledast)} = \tilde{r}$, where $(a\otimes b)^{\tau(\circledast\otimes\circledast)} = b^\circledast \otimes a^\circledast$ denotes so-called flipped conjugation (cf. [26, 29] and formula (3.22) below).

[14]There are only two orbit types under the action of $\mathfrak{o}(3;\mathbb{C})$ in $\mathbb{C}^3$: null and non-null, see appendix A.

where we use the Cartan-Weyl (CW) basis

$$[H, E_{\pm}] = \pm E_{\pm}, \qquad [E_+, E_-] = 2H. \tag{3.3}$$

In (3.1) the parmeter χ can be replaced by $\chi = 1$ due to the scale invariance of CYBE. In the standard case (3.2) the non skew-symmetric counterpart of $r_{st}(\gamma)$ satisfies CYBE if it takes the Belavin-Drinfeld form

$$r_{BD}(\gamma) = \gamma\big(E_+ \otimes E_- + H \otimes H\big)\,. \tag{3.4}$$

Its symmetric part, described by $\mathfrak{sl}(2;\mathbb{C})$ bilinear split Casimir

$$E_+ \otimes E_- + E_- \otimes E_+ + 2H \otimes H$$

is an invariant element in $\mathfrak{sl}(2,\mathbb{C}) \otimes \mathfrak{sl}(2,\mathbb{C})$ and determines the Cartan-Killing form.

We recall that the (complex) simple Lie algebra $\mathfrak{o}(3;\mathbb{C}) \cong \mathfrak{sl}(2;\mathbb{C})$ has two real forms: compact $\mathfrak{o}(3) \cong \mathfrak{su}(2)$ and noncompact $\mathfrak{o}(2,1) \cong \mathfrak{su}(1,1) \cong \mathfrak{sl}(2;\mathbb{R})$.[15] It is known, see e.g. [21], that with these two real forms there are linked four real Lie bialgebras, one compact and three noncompact ones, which can be expressed in $\mathfrak{su}(1,1) \cong \mathfrak{sl}(2;\mathbb{R})$ or $\mathfrak{o}(2,1)$ bases (see also appendix A).

The unique compact real bialgebra one can write in $\mathfrak{su}(2)$ basis (cf. [17, 19]),[16] satisfying reality conditions ($\circledast = \dagger$)[17]

$$H^\dagger = H, \qquad E_{\pm}^\dagger = E_{\mp}, \qquad r_{st}(\gamma)\,,\, \gamma \in \mathbb{R} \qquad (\mathsf{su}_\gamma(2)\ \text{ standard bialgebra})\,. \tag{3.5}$$

From three noncompact inequivalent real bialgebras we choose to write one in $\mathfrak{su}(1,1)$ basis ($\circledast = \#$)

$$H^\# = H, \qquad E_{\pm}^\# = -E_{\mp}, \qquad r_{st}(\gamma)\,,\, \gamma \in \mathbb{R} \qquad (\mathsf{su}_\gamma(1,1)\ \text{ standard bialgebra}) \tag{3.6}$$

and remaining two in $\mathfrak{sl}(2;\mathbb{R})$ basis ($\circledast = \star$)

$$H^\star = -H, \quad E_{\pm}^\star = -E_{\pm}, \quad r_{st}(\gamma)\,,\, \gamma \in \imath\mathbb{R} \quad (\mathsf{sl}_\gamma(2;\mathbb{R})\ \text{ standard bialgebra})\,, \tag{3.7}$$

$$H^\star = -H, \quad E_{\pm}^\star = -E_{\pm}, \quad r_J \quad (\mathsf{sl}_J(2;\mathbb{R})\ \text{nonstandard bialgebra})\,. \tag{3.8}$$

First three r-matrices (3.5)–(3.7) are standard (non-triangular) while the last (3.8) is Jordanian (triangular) without multiplicative parameter because it has been rescaled to 1 by suitable $\mathfrak{sl}(2;\mathbb{R})$-automorphism. The first and second bialgebra depends on real parameter, and the third one is multiplied by purely imaginary parameter (it is antireal). We stress that however $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ are isomorphic with real Lie algebra $\mathfrak{o}(2,1)$, the corresponding bialgebras (3.5), (3.6) and (3.7) are not isomorphic (cf. [21]).

More exactly, in formulae (3.5)–(3.7) there are used for the complex CW basis (3.3) three different reality conditions defining $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2)$ real algebras. Because

[15] From now on all Lie algebras and bialgebras are real if not indicated otherwise.

[16] We are working in the CW basis and different reality conditions, cf. [21].

[17] Further we shall use specific notation in order to distinguish between real Lie algebras and bialgebras, e.g. $\mathsf{su}_\gamma(2)$ denotes the triple $\mathsf{su}_\gamma(2) = (\mathfrak{sl}(2;\mathbb{C}), \dagger, r_{st}(\gamma))$, $\gamma \in \mathbb{R}$.

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$\mathfrak{o}(2,1) \cong \mathfrak{su}(1,1) \cong \mathfrak{sl}(2)$, the involutions $\#$ and $\star$ (see (3.6)–(3.7)) can be identified and related with the reality condition defining $\mathfrak{o}(2,1)$ as real form of $\mathfrak{o}(3;\mathbb{C})$. Indeed, the $\mathfrak{su}(1,1)$ real basis $(H', E'_\pm)$ and $\mathfrak{sl}(2;\mathbb{R})$ real bases $(H, E_\pm)$ can be related by the following linear complex $\mathfrak{sl}(2;\mathbb{C}) \cong \mathfrak{o}(3;\mathbb{C})$ automorphism

$$H' = -\frac{\imath}{2}\big(E_+ - E_-\big), \quad E'_\pm = \mp\imath H + \frac{1}{2}\big(E_+ + E_-\big)\,. \tag{3.9}$$

One can use the complex Cartesian basis $I_k \in \mathfrak{o}(3;\mathbb{C})$ $(k = 1,2,3)$

$$[I_i,\, I_j] = \varepsilon_{ijk} I_k\,. \tag{3.10}$$

which is antireal for the real compact form $\mathfrak{o}(3;\mathbb{R}) \cong \mathfrak{su}(2)$

$$I_i^\dagger = -\,I_i \quad (i = 1,2,3) \quad \text{for } \ \mathfrak{o}(3;\mathbb{R})\,. \tag{3.11}$$

For both cases $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ the reality condition in Cartesian basis takes the same $\mathfrak{o}(2,1)$ form

$$I_i^\star = (-1)^{i-1} I_i \quad (i = 1,2,3) \quad \text{for } \ \mathfrak{o}(2,1)\,. \tag{3.12}$$

One can relate the $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ bases with the Cartesian $\mathfrak{o}(2,1)$ generators satisfying the reality condition (3.12) by the following formulae

$$\begin{aligned} H' &:= \imath I_2, \qquad & E'_\pm &:= \imath I_1 \pm I_3\,, \qquad & &\text{for } \ \mathfrak{su}(1,1)\,,\\ H &:= \imath I_3, \qquad & E_\pm &:= \imath I_1 \mp I_2\,, \qquad & &\text{for } \ \mathfrak{sl}(2,\mathbb{R})\,. \end{aligned} \tag{3.13}$$

Both CW bases $\{E'_\pm, H'\}$ and $\{E_\pm, H\}$ have different reality properties which follow however from the same reality condition (3.12)

$$\begin{aligned} H'^\star &= H', \qquad & {E'_\pm}^\star &= -E'_\mp \qquad & &\text{for } \ \mathfrak{su}(1,1)\,,\\ H^\star &= -H, \qquad & {E_\pm}^\star &= -E_\pm \qquad & &\text{for } \ \mathfrak{sl}(2;\mathbb{R})\,. \end{aligned} \tag{3.14}$$

It should be noted that in the case of $\mathfrak{su}(1,1)$ the Cartan generator H' is compact while for the case $\mathfrak{sl}(2;\mathbb{R})$ the generator H is noncompact, what also explains the difference between $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ CW basis. In this way the involutions (3.12) and (3.6)–(3.7) are identified (it can be checked that the relations (3.9) and (3.13) are consistent). Concluding, it is sufficient for $\mathfrak{sl}(2;\mathbb{C})$ to introduce only two involutions: defining $\mathfrak{o}(3;\mathbb{R}) \cong \mathfrak{su}(2)$ and $\mathfrak{o}(2,1) \cong \mathfrak{su}(1,1) \cong \mathfrak{sl}(2;\mathbb{R})$. In fact the formulae (3.13) can be used for the introduction of Cartesian basis in all classical $\mathfrak{o}(2,2)$ and $\mathfrak{o}^*(4)$ r-matrices containing the $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$ sectors.

In the next subsection we shall describe explicitly the quantization of the complex bialgebras (3.1) and (3.2). In order to obtain the quantization of the bialgebras listed in (3.5)–(3.8) one should insert the generators $(H, E_\pm)$ satisfying the respective reality condition and impose the suitable restriction on the parameter γ. Standard deformation of arbitrary simple Lie algebras is given by the explicit algorithm introduced firstly by Drinfeld and Jimbo. Non-standard quantum deformation of $\mathsf{g} \equiv (\mathfrak{g}, r)$, where r a skew symmetric solution of CYBE, is obtained by employing the 2-cocycle Drinfeld twist element $F \in \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$ which does not change the algebra but modifies the coproduct Δ and antipode S as follows (see e.g. [10, 11, 16, 17]):

$$\Delta \longrightarrow \Delta_F = F\,\Delta\,F^{-1}\,, \qquad S \longrightarrow S_F = u\,S\,u^{-1}\,, \tag{3.15}$$

where

$$F = \sum_i f_i^{(1)} \otimes f_i^{(2)}\,, \qquad u = \sum_i f_i^{(1)}\,S\,(f_i^{(2)})\,. \tag{3.16}$$

If classical enveloping Lie algebra $\mathrm{U}(\mathfrak{g})$ is considered as a Hopf algebra $\mathcal{H}^{(0)} = (U(\mathfrak{g}), m, \Delta^{(0)}, S^{(0)}, \epsilon)$ then

$$\Delta^{(0)}(x) = x \otimes 1 + 1 \otimes x, \qquad S^{(0)}(x) = -x, \qquad \forall x \in \mathfrak{g}\,.$$

In order to get the coassociative coproduct one should postulate the normalized 2-cocycle condition for the invertible twist element $\mathcal{F}$ (see [10, 11])

$$F^{12}(\Delta \otimes id)\,(F) = F^{23}(id \otimes \Delta)\,(F)\,, \qquad (\epsilon \otimes id)\,(F) = 1 = (id \otimes \epsilon)(F)\,. \tag{3.17}$$

In $\mathcal{H}$ one can introduce the universal (quantum) R-matrix by the formula

$$R_F = F^\tau\,F^{-1} = (R_F^\tau)^{-1} \sim 1 \otimes 1 + r + o(\chi^2) \tag{3.18}$$

which under the reality conditions (2.10) becomes unitary. More generally, twist deformation of quasitriangular Hopf algebra $(\mathcal{H}, R)$ give rise to quasitriangular Hopf algebra with new universal R-matrix $R \longrightarrow R_F = F^\tau\,R\,F^{-1}$.

The simplest case one can deal is an Abelian twist

$$F_{A,1/2} = \exp\Big(-\frac{\chi}{2} X \wedge Y\Big), \qquad R_A = \exp\left(\chi X \wedge Y\right), \tag{3.19}$$

where two primitive commuting elements $[X, Y] = 0$ determines $r_A = \chi X \wedge Y$ the skew-symmetric solutions of CYBE. In fact, the same Abelian quantum R-matrix R_A can be implemented by the one-parameter family of Abelian twists

$$F_{A,s} = \exp\xi(s\,X \otimes Y - (1-s)\,Y \otimes X), \qquad s \in [0,1] \tag{3.20}$$

which are related with each other by a trivial (coboundary) twists. For example

$$F_{A,1} \equiv \exp\chi\,X \otimes Y = (W^{-1} \otimes W^{-1})\,F_{A,1/2}\,\Delta(W)\,, \tag{3.21}$$

where $W = \exp(\frac{\chi}{2} XY)$.[18]

Assuming X, Y real (antireal), i.e. $X^* = \pm X, Y^* = \pm Y$, the formal parameter χ has to be pure imaginary and all twist $F_{A,s}(\chi)$ are unitary. Consequently, each such twist

[18] We remind that a coboundary twist $F_W^{\rm cob}$ for a given Hopf algebra is constructed out of any invertible element W according to the following prescription

$$F_W^{\rm cob} = (W^{-1} \otimes W^{-1})\Delta(W)$$

and leads via twisting (3.15) to the isomorphic Hopf algebras (see e.g. [50]).

can be used to deform equivalently $\circledast$-Hopf algebras. However, there is an advantage of using (3.19). In this case the element u (see (3.16)) reduces to the unit and the antipodes map remains unchanged.

If $X^\circledast = \pm Y, Y^\circledast = \pm X$ then only (3.19) is unitary for χ real. Thus the inverse transformation $F_{A,1/2} = (W \otimes W)\, F_{A,1}\, \Delta(W^{-1})$ can be treated as unitarizing the non-unitary twist $F_{A,1}$ by the coboundary twist $(W \otimes W)\Delta(W^{-1})$ (cf. (3.21)). This property will be used later in section 5.4 for the quantized Abelian twist in quantization procedure of Lorentz algebra.

Alternatively, if we use the (non-standard) flipped conjugation on the tensor product (see [26])

$$(a \otimes b)^\circledast = b^\circledast \otimes a^\circledast \tag{3.22}$$

and in the formulae (2.11) one can get all the twist $F_{A,s}$ unitary as well for the imaginary parameter χ.

3.2 Two basic $\mathfrak{sl}(2;\mathbb{C})$ quantizations and their real versions

3.2.1 Quantization of Jordanian r-matrix r_J

The quantum twist F_J corresponding to the classical Jordanian[19] r-matrix r_J is well known since a long time [52]

$$F_J(\chi) = \exp\,(H \otimes \sigma), \qquad \sigma = \ln(1 + \chi E_+)\,. \tag{3.23}$$

The twisted coproducts and antipodes are easy to derive[20]

$$\begin{aligned}
\Delta_J(E_+) &= F\,(\chi)\Delta^{(0)}(E_+)\,F^{-1}(\chi) = E_+ \otimes e^{\sigma} + 1 \otimes E_+ \\
\Delta_J(H) &= H \otimes e^{-\sigma} + 1 \otimes H = H \otimes 1 + 1 \otimes H - \chi H \otimes E_+ e^{-\sigma} \\
\Delta_J(E_-) &= E_- \otimes e^{-\sigma} + 1 \otimes E_- + 2\chi H \otimes H e^{-\sigma} \\
&\quad - \chi^2 H(H-1) \otimes E_+ e^{-2\sigma}\,,
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
S_J(E_+) &= -E_+\, e^{-\sigma}, \qquad\qquad S_J(H) = -H\, e^{\sigma} \\
S_J(E_-) &= -E_-\, e^{\sigma} + 2\chi H^2 e^{\sigma} + \chi^2 H(H-1) E_+ e^{\sigma}\,.
\end{aligned} \tag{3.25}$$

The quantum R-matrix takes the form $(R_J = F_J^{21} F_J^{-1})$

$$R_J(\chi) = F_J^{21}(\chi) F_J^{-1}(\chi) = \exp\,(\sigma \otimes H)\exp\,(-H \otimes \sigma). \tag{3.26}$$

The only compatible reality condition for the $\mathfrak{sl}(2;\mathbb{C})$ Jordanian deformation is of non-compact type (see (3.8)) obtained if the parameter $\chi \in \imath\mathbb{R}$. In such case the Jordanian twist $F_J = \exp\,(H \otimes \ln(1 + \chi E_+))$ is unitary, provides deformed coproducts and antipodes satisfying automatically the conditions (2.11). Therefore, it provides (see (3.18)) the universal R-matrix $R_J = \exp\,(\ln(1 + \chi E_+) \otimes H)\exp\,(-H \otimes \ln(1 + \chi E_+))$.

[19] The name "Jordanian" is linked with Jordanian deformation of quantum plane $xy = yx + y^2$ introduced in [51]. Quantized enveloping algebra $U_J(\mathfrak{sl}(2))$ is a quantum symmetry of such plane.

[20] From the commutation relations (3.3) one finds $[f(E_+), H] = -E_+ f'(E_+)$, $[f(E_+), E_-] = 2H f'(E_+) - E_+ f''(E_+)$, where f is an analytic function of one variable. In particular $[\sigma, H] = -\chi E_+ e^{-\sigma}$, $[\sigma, E_-] = 2\chi H e^{-\sigma} + \chi^2 E_+ e^{-2\sigma}$.

3.2.2 Standard quantization of $\mathfrak{sl}(2;\mathbb{C})$

The standard (non-triangular, quasitriangular) quantum deformation is corresponding to the solution of CYBE given by (3.4). It is described by the q-analogue (in case of simple Lie algebra called also Drinfeld-Jimbo quantum deformation) with algebraic and coalgebraic sectors given by the following formulae (see e.g. [16]–[19])

$$q^{\mathtt{h}}\mathtt{e}_\pm = q^{\pm 1}\mathtt{e}_\pm\, q^{\mathtt{h}}\,, \qquad [\mathtt{e}_+,\,\mathtt{e}_-] = \frac{q^{2\mathtt{h}} - q^{-2\mathtt{h}}}{q - q^{-1}}\,, \tag{3.27}$$

$$\Delta_q(q^{\pm\mathtt{h}}) = q^{\pm\mathtt{h}} \otimes q^{\pm\mathtt{h}}\,, \qquad \Delta_q(\mathtt{e}_\pm) = \mathtt{e}_\pm \otimes q^{\mathtt{h}} + q^{-\mathtt{h}} \otimes \mathtt{e}_\pm\,, \tag{3.28}$$

$$S_q(q^{\pm\mathtt{h}}) = q^{\mp\mathtt{h}}\,, \qquad S_q(\mathtt{e}_\pm) = -q^{\pm 1}\mathtt{e}_\pm\,, \tag{3.29}$$

$$\epsilon_q(q^{\pm\mathtt{h}}) = 1\,, \qquad \epsilon_q(\mathtt{e}_\pm) = 0\,, \tag{3.30}$$

where we denote by $(q^{\mathtt{h}}, \mathtt{e}_\pm)$ the q-deformed or quantum CW basis.[21] The quantum universal R-matrix satisfying QYBE (2.7) as well as the conditions (2.5)–(2.6) is given by the formula:

$$R_q = \exp_{q^{-2}}\Big((q-q^{-1})\mathtt{e}_+\, q^{-\mathtt{h}} \otimes q^{\mathtt{h}}\mathtt{e}_-\Big) q^{2\mathtt{h}\otimes\mathtt{h}} = q^{2\mathtt{h}\otimes\mathtt{h}} \exp_{q^{-2}}\Big((q-q^{-1})\mathtt{e}_+\, q^{\mathtt{h}} \otimes q^{-\mathtt{h}}\mathtt{e}_-\Big)\,, \tag{3.31}$$

where we use the standard definition of q-exponential $\exp_{q^{-2}}$ (cf. appendix B)

$$\exp_q(x) := \sum_{n\geq 0} \frac{x^n}{(n)_q!}\,, \qquad (n)_q! := (1)_q(2)_q\cdots(n)_q, \qquad (n)_q = \frac{1-q^n}{1-q}\,. \tag{3.32}$$

The alternative second version of the universal R-matrix has the form:

$$R_q^{\tau -1} = \exp_{q^2}\Big((q^{-1}-q)\mathtt{e}_-\, q^{-\mathtt{h}} \otimes q^{\mathtt{h}}\mathtt{e}_+\Big) q^{-2\mathtt{h}\otimes\mathtt{h}} = q^{-2\mathtt{h}\otimes\mathtt{h}} \exp_{q^2}\Big((q^{-1}-q)\mathtt{e}_-\, q^{\mathtt{h}} \otimes q^{-\mathtt{h}}\mathtt{e}_+\Big) \tag{3.33}$$

and provides nontrivial element $Q_q = R_q R_q^\tau$. This quantum R-matrices in the limit $\gamma \mapsto 0$, $q \mapsto 1$ described by their linear term the non-skewsymmetric classical r-matrices in the Belavin-Drinfeld form (3.4).

We recall that the Jordanian Lie bialgebra has no effective deformation parameter and in rigorous mathematical sense only the presence of a (formal) parameter χ permits to construct the twist and the quantum R-matrix as a infinite power series, which are elements of $\mathrm{U}(\mathfrak{sl}(2;\mathbb{C})) \otimes \mathrm{U}(\mathfrak{sl}(2;\mathbb{C}))[[\chi]]$. In contrast, Lie bialgebras corresponding to standard deformations can be parametrized by numerical (complex or real) factor γ, describing effective deformation parameter. In fact, standard (i.e. nontriangular) semi-simple (coboundary) Lie bialgebras can be quantized in two different ways [16]–[19]; in particular

i) If it is given by the relations (3.27)–(3.30) the parameter γ is formal, with the element $q^{\mathtt{h}} = \exp \frac{1}{2}\gamma\mathtt{h} \in \mathrm{U}(\mathfrak{sl}(2;\mathbb{C}))[[\gamma]]$ given by a formal power series in γ. Such approach is more suitable for dealing with quantum R-matrices (see e.g. [18] section XVII.4 or [19] section 3.1.5). In the real case the formal parameter one should constrain by the reality condition ($\gamma^\dagger = \gamma$ or $\gamma^\dagger = -\gamma$);

[21] Non-standard, e.g. Jordanian, deformation can be also expressed by nonclassical quantum Lie algebra generators [53] obtained from twist (3.23) (see also [54]).

ii) one can hide formal series by introducing new generators $\mathtt{k}^{\pm 1} = q^{\pm \mathtt{h}}$ and rewrite the Hopf algebra redefining the relations (3.27)–(3.30) in terms of $(\mathtt{k}^{\pm 1}, \mathtt{e}_\pm)$ generators with the numerical (complex or real) deformation parameter q (so-called specialization [16])

$$\mathtt{k}\mathtt{e}_\pm\mathtt{k}^{-1} = q^{\pm 1}\mathtt{e}_\pm\,, \qquad \mathtt{k}\mathtt{k}^{-1} = \mathtt{k}^{-1}\mathtt{k} = 1\,, \quad [\mathtt{e}_+, \mathtt{e}_-] = \frac{\mathtt{k}^2 - \mathtt{k}^{-2}}{q - q^{-1}}\,, \tag{3.34}$$

$$\Delta_q(\mathtt{k}^{\pm 1}) = \mathtt{k}^{\pm 1} \otimes \mathtt{k}^{\pm 1}\,, \qquad \Delta_q(\mathtt{e}_\pm) = \mathtt{e}_\pm \otimes \mathtt{k} + \mathtt{k}^{-1} \otimes \mathtt{e}_\pm\,, \tag{3.35}$$

$$S_q(\mathtt{k}^{\pm 1}) = \mathtt{k}^{\mp 1}\,, \qquad S_q(\mathtt{e}_\pm) = -q^{\pm 1}\mathtt{e}_\pm\,, \tag{3.36}$$

$$\epsilon_q(\mathtt{k}^{\pm 1}) = 1\,, \qquad \epsilon_q(\mathtt{e}_\pm) = 0\,. \tag{3.37}$$

The number $\gamma = 2\ln q \in \mathbb{C}$ parametrizes corresponding Lie bialgebra structures: complex $\mathsf{sl}_\gamma(2;\mathbb{C})$ or real $\mathsf{su}_\gamma(2)$, $\mathsf{su}_\gamma(1,1)$, or $\mathsf{sl}_\gamma(2;\mathbb{R})$. In such cases quantum R-matrices can be defined, in strict mathematical sense, only in some special cases when $q^{2p} = 1$ (see [18] section IX.6) or for (finite-dimensional) representations (see [19] section 3.4.1)

The h-adic quantization employing formal indeterminate parameters is a rigorous algebraic tool to treat both standard and nonstandard deformations and discuss universal R- matrices described by infinite power series regardless of the admissible values of the parameters. The topological h-adic extensions are considered in mathematical literature (see e.g. [15, 18, 19]) as an algebra $A[[\xi]]$ of formal (infinite) series with arbitrary powers of indeterminate parameter ξ and coefficients belonging to the algebra A. The algebra $A[[\xi]]$ (over the ring $\mathbb{C}[[\xi]]$) is completed in so-called h-adic topology. Such mathematical framework has been applied to the description of universal R-matrices and twist two-tensors F, which appears to belong to the ring $U_\xi(\mathfrak{g}) \otimes U_\xi(\mathfrak{g})[[\xi]]$.[22]

Three standard real forms (3.5)–(3.7) impose the following reality conditions on q-deformed generators $(q^{\pm\mathtt{h}}, \mathtt{e}_\pm)$ (e.g. [19] section 3.1.4):

$$(q^{\mathtt{h}})^\dagger = q^{\mathtt{h}}, \qquad \mathtt{e}_\pm^\dagger = \mathtt{e}_\mp, \qquad q \in \mathbb{R} \Leftrightarrow \gamma \in \mathbb{R} \quad \text{for } \mathsf{su}_\gamma(2), \tag{3.38}$$

$$(q^{\mathtt{h}})^\# = q^{\mathtt{h}}, \qquad \mathtt{e}_\pm^\# = -\mathtt{e}_\mp, \qquad q \in \mathbb{R} \Leftrightarrow \gamma \in \mathbb{R} \quad \text{for } \mathsf{su}_\gamma(1,1), \tag{3.39}$$

$$(q^{\mathtt{h}})^\star = q^{\mathtt{h}}, \qquad \mathtt{e}_\pm^\star = -\mathtt{e}_\pm, \qquad |q| = 1 \Leftrightarrow \gamma \in \imath\mathbb{R} \quad \text{for } \mathsf{sl}_\gamma(2), \tag{3.40}$$

which turn, in each case, the Hopf algebra (3.27)–(3.30) into the real Hopf algebra with the coproducts satisfying the reality conditions (2.11). Taking into consideration the restriction on the values of q we see that reality conditions (3.5)–(3.7) for $(H, E_\pm)$ have the same form as for $(\mathtt{h}, \mathtt{e}_\pm)$ (see (3.38)–(3.40)). The last two (non-compact) real forms coincide in the classical limit $\gamma \mapsto 0$. The classical limit $q \mapsto 1$ implies that the deformed and undeformed generators can be identified, i.e. $\mathtt{h} \mapsto H$, $\mathtt{e}_\pm \mapsto E_\pm$, $q^{\mathtt{h}} \mapsto 1$ and we are left with undeformed

[22] In applications in theoretical physics the indeterminate formal parameter ξ has been replaced by numerical parameter, but the algebraic operations on power series have only formal meaning. However, if the algebraic structure of $A[[\xi]]$ can be represented as new algebra with finite set of generators, the approach with numerical parameters can be used as mathematically rigorous (see (3.34)–(3.37)).

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Hopf algebra structure on the universal enveloping algebra $U(\mathfrak{sl}(2;\mathbb{R}))$.[23] For the first two real forms (3.38)–(3.39) the corresponding universal R-matrix (3.30) is real, for the last case (3.7) is antireal.

4 Lie bialgebras for $\mathfrak{o}(4;\mathbb{C})$ algebra and its real forms

In this section we describe Lie bialgebras of $D = 4$ complex rotations $\mathfrak{o}(4;\mathbb{C})$ and its real forms: Euclidian, Lorentz, Kleinian and quaternionic orthogonal Lie algebras in terms of chiral left $(H, E_\pm)$ and right $(\bar{H}, \bar{E}_\pm)$ CW bases:[24]

$$[H,\, E_\pm] = \pm E_\pm\,, \quad [E_+,\, E_-] = 2H\,, \quad [\bar{H},\, \bar{E}_\pm] = \pm \bar{E}_\pm\,, \quad [\bar{E}_+,\, \bar{E}_-] = 2\bar{H}\,. \tag{4.1}$$

Due to the fact that each $\mathfrak{sl}(2;\mathbb{C})$ sector has two bialgebra structures (single Jordanian and standard one-parameter family) one can easily to identify three (up to the flip) types of bialgebra structures on $\mathfrak{o}(4;\mathbb{C})$, namely the direct sums

$$\mathsf{o}_{\gamma,\bar{\gamma}}(4;\mathbb{C}) = \mathsf{sl}_\gamma(2;\mathbb{C}) \oplus \bar{\mathsf{sl}}_{\bar{\gamma}}(2;\mathbb{C})\,, \tag{4.2}$$

$$\mathsf{o}_{\gamma,\bar{J}}(4;\mathbb{C}) = \mathsf{sl}_\gamma(2;\mathbb{C}) \oplus \bar{\mathsf{sl}}_J(2;\mathbb{C})\,, \tag{4.3}$$

$$\mathsf{o}_{J,\bar{J}}(4;\mathbb{C}) = \mathsf{sl}_J(2;\mathbb{C}) \oplus \bar{\mathsf{sl}}_J(2;\mathbb{C}) \tag{4.4}$$

with the classical r-matrices obtained by summing up the pair of chiral and antichiral contributions, e.g. $r_{J,\bar{J}} = r_J + \bar{r}_J = H \wedge E_+ + \bar{H} \wedge \bar{E}_+$ in (4.4), etc.. The list (4.2)–(4.4) does not exhaust all possible bialgebra structures because it does not take into account the mixed terms belonging to $\mathfrak{sl}(2;\mathbb{C}) \wedge \bar{\mathfrak{sl}}(2;\mathbb{C})$, which can also contribute to the classical r-matrices. In [20, 22] using purely algebraic methods we classified all $\mathfrak{o}(4;\mathbb{C})$ bialgebras. We found five families of complex skew-symmetric r-matrices: three, each with three-parameters, one two-parameter and one with one parameter.[25] The list of $\mathfrak{o}(4;\mathbb{C})$ r-matrices looks as follows [20, 22]:

$$r_I(\chi) = \chi(E_+ + \bar{E}_+) \wedge (H + \bar{H})\,, \tag{4.5}$$

$$r_{II}(\chi, \bar{\chi}, \varsigma) = \chi\, E_+ \wedge H + \bar{\chi}\, \bar{E}_+ \wedge \bar{H} + \varsigma E_+ \wedge \bar{E}_+\,, \tag{4.6}$$

$$r_{III}(\gamma, \bar{\gamma}, \eta) = \gamma\, E_+ \wedge E_- + \bar{\gamma}\, \bar{E}_+ \wedge \bar{E}_- + \eta\, H \wedge \bar{H}\,, \tag{4.7}$$

$$r_{IV}(\gamma, \varsigma) = \gamma \left(E_+ \wedge E_- - \bar{E}_+ \wedge \bar{E}_- - 2H \wedge \bar{H}\right) + \varsigma E_+ \wedge \bar{E}_+\,, \tag{4.8}$$

$$r_V(\gamma, \bar{\chi}, \rho) = \gamma\, E_+ \wedge E_- + \bar{\chi}\, \bar{E}_+ \wedge \bar{H} + \rho H \wedge \bar{E}_+\,. \tag{4.9}$$

Here all parameters γ, $\bar{\gamma}$, η, χ, $\bar{\chi}$, ς, ρ are arbitrary complex numbers and they are independent in different r-matrices.

The first two r-matrices $r_I(\chi)$ and $r_{II}(\chi, \bar{\chi}, \varsigma)$, generate twists and they satisfy the homogeneous CYBE (2.2). Moreover the first r-matrix $r_I(\chi)$ is pure Jordanian type and the second r-matrix $r_{II}(\chi, \bar{\chi}, \varsigma)$ is the sum of two Jordanian ones with third one describing Abelian twist: $r_{II}(\chi, \bar{\chi}, \varsigma) = r_{II}(\chi, 0, 0) + r_{II}(0, \bar{\chi}, 0) + r_{II}(0, 0, \varsigma)$. The third r-matrix $r_{III}(\gamma, \bar{\gamma}, \eta)$ is the sum of two standard r-matrices and one Abelian: $r_{III}(\gamma, \bar{\gamma}, \eta) = r_{III}(\gamma, 0, 0) + r_{III}(0, \bar{\gamma}, 0) + r_{III}(0, 0, \eta)$. The fourth r-matrix $r_{IV}(\gamma, \varsigma)$ is is the sum of special choice of the third r-matrix and the Abelian r-matrix: $r_{IV}(\gamma, \varsigma) := r_{III}(\gamma, -\gamma, -2\gamma) +$

[23] For precise definition of the classical limit see, e.g. [18] section VI.2 or [19] section 3.1.3.

[24] For the relation with other, physically more meaningful, Cartesian basis see e.g. (3.9) and [20].

[25] The list (4.5)–(4.9) is numbered in different way in comparison with original result [22]; notation for the parameters is slightly changed as well.

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$\varsigma E_+ \wedge \bar{E}_+$. The last r-matrices $r_V(\gamma, \bar{\chi}, \rho)$ is the sum of standard, Jordanian and Abelian r-matrices: $r_V(\gamma, \bar{\chi}, \rho) = r_V(\gamma, 0, 0) + r_V(0, \bar{\chi}, 0) + r_V(0, 0, \rho)$. The formulae for (r_{II}, r_{III}, r_V) are obtained by supplementing (4.2)–(4.4) with particular additional Abelian contributions belonging to $\mathfrak{sl}(2;\mathbb{C}) \wedge \bar{\mathfrak{sl}}(2;\mathbb{C})$.

We shall calculate as well in next section for all five quantizations generated by (4.5)–(4.9) the universal R-matrices. Using formula (2.9) one obtains in third, fourth and fifth cases the following Belavin-Drinfeld type of matrices which appear in the expansion (2.9):

$$\begin{aligned} \tilde{r}_{III}(\gamma,\bar{\gamma},\eta) &= \gamma\,(E_+\otimes E_- + H\otimes H) + \bar{\gamma}\left(\bar{E}_+\otimes\bar{E}_- + \bar{H}\otimes\bar{H}\right) + \eta\, H\wedge\bar{H}\,, \\ \tilde{r}_{IV}(\gamma,\varsigma) &= \gamma\left(E_+\otimes E_- + H\otimes H - \bar{E}_+\otimes\bar{E}_- - \bar{H}\otimes\bar{H} - 2H\wedge\bar{H}\right) + \varsigma E_+\wedge\bar{E}_+\,, \\ \tilde{r}_V(\gamma,\bar{\chi},\rho) &= \gamma\,(E_+\otimes E_- + H\otimes H) + \bar{\chi}\,\bar{E}_+\wedge\bar{H} + \rho H\wedge\bar{E}_+\,. \end{aligned} \tag{4.10}$$

There is unique compact real form $\mathfrak{o}(4;\mathbb{C})$ and three real non-compact forms of $\mathfrak{o}(4)$: *the Lorentz algebra* $\mathfrak{o}(3,1) := \mathfrak{o}(3,1;\mathbb{R}) \cong \mathfrak{sl}(2;\mathbb{C})^{\mathbb{R}}$, *the Kleinian algebra* $\mathfrak{o}(2,2) := \mathfrak{o}(2,2;\mathbb{R}) \cong \mathfrak{o}(2,1) \oplus \mathfrak{o}(2,1)$ and *the quaternionic Lie algebra* $\mathfrak{o}^\star(4) := \mathfrak{o}(2;\mathbb{H}) \cong \mathfrak{o}(2,1) \oplus \mathfrak{o}(3)$. These real forms can be expressed as the following six direct sums of $\mathfrak{sl}(2;\mathbb{C})$- real forms listed in (3.5)–(3.7)

$$H^\dagger = H, \qquad E_\pm^\dagger = E_\mp, \qquad \bar{H}^\dagger = \bar{H}, \qquad \bar{E}_\pm^\dagger = \bar{E}_\mp \qquad \text{for } \mathfrak{o}(4), \tag{4.11}$$

$$\left.\begin{aligned} H^\dagger &= H, & E_\pm^\dagger &= E_\mp, & \bar{H}^\# &= \bar{H}, & \bar{E}_\pm^\# &= -\bar{E}_\mp, \\ H^\dagger &= H, & E_\pm^\dagger &= E_\mp, & \bar{H}^\star &= -\bar{H}, & \bar{E}_\pm^\star &= -\bar{E}_\pm \end{aligned}\right\} \text{for } \mathfrak{o}^\star(4), \tag{4.12}$$

$$\left.\begin{aligned} H^\# &= H, & E_\pm{}^\# &= -E_\mp, & \bar{H}^\# &= \bar{H}, & \bar{E}_\pm^\# &= -\bar{E}_\mp, \\ H^\# &= H, & E_\pm{}^\# &= -E_\mp, & \bar{H}^\star &= -\bar{H}, & \bar{E}_\pm^\star &= -\bar{E}_\pm, \\ H^\star &= -H, & E_\pm{}^\star &= -E_\pm, & \bar{H}^\star &= -\bar{H}, & \bar{E}_\pm^\star &= -\bar{E}_\pm \end{aligned}\right\} \text{for } \mathfrak{o}(2,2), \tag{4.13}$$

$$H^\ddagger = -\bar{H}, \qquad E_\pm^\ddagger = -\bar{E}_\pm, \qquad \bar{H}^\ddagger = -H, \qquad \bar{E}_\pm^\ddagger = -E_\pm \qquad \text{for } \mathfrak{o}(3,1). \tag{4.14}$$

Only the last real form (4.14), characterizing the Lorentz $\mathfrak{o}(3,1)$-algebra, does not preserve the chiral decomposition.

By imposing all real involutions in the list of classical complex r-matrices (4.5)–(4.9) we get complete set of real bialgebra structures on the Lie algebra $\mathfrak{o}(4;\mathbb{C})$ [22]. The list of all non-isomorphic real bialgebras for $\mathfrak{o}(4;\mathbb{C})$, together with specified values for the corresponding parameters, is presented in the table below, where $\mathfrak{o}^\star(4)$, $\mathfrak{o}'^\star(4)$ denote two Lie algebras after imposing the isomorphic reality conditions (4.12). Similarly, $\mathfrak{o}''(2,2)$, $\mathfrak{o}'(2,2)$, $\mathfrak{o}''(2,2)$ denote three isomorphic Lie algebras obtained by applying three reality conditions (4.13).

In the following section we shall describe (non-isomorphic) Hopf-algebraic quantizations of five complex $\mathfrak{o}(4;\mathbb{C})$ r-matrices (4.5)–(4.9) representing non-isomorphic bialgebra structures. Out of these five complex quantizations after imposing seven reality conditions we obtain sixteen non-isomorphic real Hopf algebra structures: r_{III} provides seven real forms, r_V-three, and each of remaining three leads to two real quantizations.

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	$r_I(\chi)$	$r_{II}(\chi,\bar\chi,\varsigma)$	$r_{III}(\gamma,\bar\gamma,\eta)$	$r_{IV}(\gamma,\varsigma)$	$r_V(\gamma,\bar\chi,\rho)$
$\mathfrak{o}(4)$			$\gamma,\bar\gamma\in\mathbb{R}\ ;\ \eta\in\imath\mathbb{R}$		
$\mathfrak{o}^\star(4)$			$\gamma,\bar\gamma\in\mathbb{R}\ ;\ \eta\in\imath\mathbb{R}$		
$\mathfrak{o}'^\star(4)$			$\gamma,\eta\in\mathbb{R}\ ;\ \bar\gamma\in\imath\mathbb{R}$		$\gamma,\rho\in\mathbb{R}\ ;\ \bar\chi\in\imath\mathbb{R}$
$\mathfrak{o}(2,2)$			$\gamma,\bar\gamma\in\mathbb{R}\ ;\ \eta\in\imath\mathbb{R}$		
$\mathfrak{o}'(2,2)$			$\gamma,\eta\in\mathbb{R}\ ;\ \bar\gamma\in\imath\mathbb{R}$		$\gamma,\rho\in\mathbb{R}\ ;\ \bar\chi\in\imath\mathbb{R}$
$\mathfrak{o}''(2,2)$	$\chi\in\imath\mathbb{R}$	$\chi,\bar\chi,\varsigma\in\imath\mathbb{R}$	$\gamma,\bar\gamma,\eta\in\imath\mathbb{R}$	$\gamma,\varsigma\in\imath\mathbb{R}$	$\gamma,\bar\chi,\rho\in\imath\mathbb{R}$
$\mathfrak{o}(3,1)$	$\chi\in\imath\mathbb{R}$	$\chi=\bar\chi\in\imath\mathbb{R}\ ;\ \varsigma\in\mathbb{R}$	$\bar\gamma=-\gamma^*\in\mathbb{C}\ ;\ \eta\in\mathbb{R}$	$\gamma,\varsigma\in\mathbb{R}$	

Table 1. All real Lie bialgebras for $\mathfrak{o}(4;\mathbb{C})$, see [22].

5 Explicit quantizations of $\mathfrak{o}(4;\mathbb{C})$ and their real forms

5.1 Jordanian quantization of $\mathfrak{o}(4;\mathbb{C})$ (r-matrix r_I)

Following the previous considerations (subsection 2.2) the quantum twist F_1 corresponding to the classical Jordanian r-matrix (4.5) can be written as

$$F_1(\chi)=\exp\left((H+\bar H)\otimes\sigma\right),\qquad \sigma=\ln(1+\chi(E_++\bar E_+))\,, \tag{5.1}$$

Coproducts and antipodes are easy to derive (cf. (3.23)–(3.26))

$$\begin{aligned}\Delta_1(E_{k+})&=F_1(\chi)\Delta^{(0)}(E_k)F_1^{-1}(\chi)=\Delta_1(E_{k+})=E_{k+}\otimes e^\sigma+1\otimes E_{k+}\,,\\ \Delta_1(H_k)&=H_k\otimes 1+1\otimes H_k-\chi(H+\bar H)\otimes E_{k+}e^{-\sigma}\,,\\ \Delta_1(E_{k-})&=E_{k-}\otimes e^{-\sigma}+1\otimes E_{k-}+2\chi(H+\bar H)\otimes H_ke^{-\sigma}\\ &\quad-\chi^2(H+\bar H)(H+\bar H-1)\otimes E_{k+}e^{-2\sigma}\,,\end{aligned}\tag{5.2}$$

where $k\in\{0,1\}\equiv\mathbb{Z}_2$ and in order to reduce the number of formulae we denoted $H=H_0$, $E_\pm=E_{0\pm}$ and $\bar H=H_1,\bar E_\pm=E_{1\pm}$.[26]

Similarly, the formulae for the antipodes look as follows

$$\begin{aligned}&S_1(E_{k+})=-E_{k+}\,e^{-\sigma}\,,\qquad S_1(H_k)=-H_k-\chi(H+\bar H)E_{k+}\,,\\ &S_1(E_{k-})=-E_{k-}\,e^{\sigma}+2\chi(H+\bar H)H_ke^\sigma+\chi^2(H+\bar H)(H+\bar H-1)E_{k+}e^\sigma\,.\end{aligned}\tag{5.3}$$

The universal quantum R-matrix takes the form ($R=F^{21}F^{-1}$)

$$R_1(\chi)=\exp\left(\sigma\otimes(H+\bar H)\right)\exp\left(-(H+\bar H)\otimes\sigma\right). \tag{5.4}$$

This simple one-parameter deformation admits two real quantum group structures as indicated below in the table 2.

Since the twist is Jordanian, the reality conditions (2.11) are valid if the deformation parameter χ is imaginary. The Lorentzian case requiring as well imaginary χ has been already studied with more details in [29].

[26] The same convention will be further used through the paper.

$\mathfrak{o}''(2,2)$	$\chi \in \imath\mathbb{R}$	$H^\star = -H, E_\pm^\star = -E_\pm$	$\bar{H}^\star = -\bar{H}, \bar{E}_\pm^\star = -\bar{E}_\pm$
$\mathfrak{o}(3,1)$	$\chi \in \imath\mathbb{R}$	$H^\ddagger = -\bar{H}, E_\pm^\ddagger = -\bar{E}_\pm$	$\bar{H}^\ddagger = -H, \bar{E}_\pm^\ddagger = -E_\pm$

Table 2. Real quantizations of $r_I(\chi) = \chi(E_+ + \bar{E}_+) \wedge (H + \bar{H})$.

5.2 Left and right Jordanian quantizations intertwined by Abelian twist (r-matrix r_{II})

We see that for $\varsigma = 0$ the r-matrix (4.6) describes two complex Jordanian r-matrices, each one for chiral sectors $\mathfrak{sl}(2, \mathbb{C})$ and $\overline{\mathfrak{sl}(2; \mathbb{C})}$. They do commute with each other and can be quantized as the product of two Ogievetsky twists ($k = 1, 2$) ([52] see also (3.23))

$$F_{J,0}(\chi) = \exp\left(H \otimes \Sigma\right) \quad F_{J,1}(\bar{\chi}) = \exp\left(\bar{H} \otimes \bar{\Sigma}\right), \tag{5.5}$$

where $\Sigma = ln(1 + \chi E_+)$, $\bar{\Sigma} = ln(1 + \bar{\chi}\bar{E}_+)$. The next step is to consider the Abelian part of the classical r-matrix r_{II} belonging to $\mathfrak{sl}(2; \mathbb{C}) \wedge \bar{\mathfrak{sl}}(2; \mathbb{C})$ which intertwine two chiral coalgebra sectors becoming not independent. Because the generators $(H,\, E_\pm)$ and $(\bar{H},\, \bar{E}_\pm)$ do commute the twist function corresponding to (4.6) is given by the following formula:

$$F_2(\chi, \bar{\chi}, \varsigma) = F_A(\chi, \bar{\chi}, \varsigma) F_{J,1}(\bar{\chi}) F_{J,0}(\chi) = F_A(\chi, \bar{\chi}, \varsigma) F_{J,0}(\chi) F_{J,1}(\bar{\chi})\,, \tag{5.6}$$

where the Abelian twist F_A takes the form[27]

$$F_A(\chi, \bar{\chi}, \varsigma) = \exp\left(\frac{\varsigma}{\chi\bar{\chi}} \Sigma \wedge \bar{\Sigma}\right) \tag{5.7}$$

which follows from the property that elements Σ, $\bar{\Sigma}$ are primitive after performing Jordanian deformation. We would like to mention here that the form of twist function given above by the formula (5.6) was proposed firstly by Kulish and Mudrov [55]. If we use (3.15)–(3.16), and (5.6) we obtain the following formulae for the coproducts of $sl(2; \mathbb{C}) \oplus \bar{sl}(2; \mathbb{C})$ generators $(H_k,\, E_{k+},\, E_{k-})$, $k = 0, 1 \in \mathbb{Z}_2$

$$\begin{aligned}
\Delta_2(E_{k+}) &= F_2(\chi, \bar{\chi}, \varsigma) \Delta^{(0)}(E_k) F_2^{-1}(\chi, \bar{\chi}, \varsigma) = E_{k+} \otimes e^{\Sigma_k} + 1 \otimes E_{k+}\,, \\
\Delta_2(H_k) &= H_k \otimes e^{-\Sigma_k} + 1 \otimes H_k + \\
&\quad (-1)^k \frac{\varsigma}{\chi_{k+1}} \left(\Sigma_{k+1} \otimes E_{k+} e^{-\Sigma_k} - E_{k+} e^{-\Sigma_k} \otimes \Sigma_{k+1} e^{-\Sigma_k}\right), \\
\Delta_2(E_{k-}) &= E_{k-} \otimes e^{-\Sigma_k} + 1 \otimes E_{k-} + 2\chi_k H_k \otimes H_k e^{-\Sigma_k} + \chi_k H_k (H_k - 1) \otimes \Lambda_k + \\
&\quad + (-1)^k \frac{2\varsigma}{\chi_{k+1}} \left(H_k e^{-\Sigma_k} \otimes \Sigma_{k+1} e^{-\Sigma_k} - H_k \Sigma_{k+1} \otimes \Lambda_k - \Sigma_{k+1} \otimes H_k e^{-\Sigma_k}\right) \\
&\quad + (-1)^k \frac{2\varsigma}{\chi_{k+1}} \left(\Lambda_k e^{\Sigma_k} \otimes H_k \Sigma_{k+1} e^{-\Sigma_k} + H_k \Lambda_k e^{\Sigma_k} \otimes \Sigma_{k+1} \Lambda_k\right) \\
&\quad + (-1)^k \frac{\varsigma}{\chi_{k+1}} \left(\left(1 - e^{-2\Sigma_k}\right) \otimes \Sigma_{k+1} \Lambda_k + \Sigma_{k+1} \otimes \Lambda_k - \Lambda_k \otimes \Sigma_{k+1} e^{-\Sigma_k}\right) \\
&\quad + \frac{1}{\chi_k} \left(\frac{\varsigma}{\chi_{k+1}}\right)^2 \left(\Lambda_k^2 e^{2\Sigma_k} \otimes \Sigma_{k+1}^2 \Lambda_k + \Lambda_k \otimes \Sigma_{k+1}^2 e^{-\Sigma_k} + \Sigma_{k+1}^2 \otimes \Lambda_k\right) \\
&\quad - \frac{2}{\chi_k} \left(\frac{\varsigma}{\chi_{k+1}}\right)^2 \Lambda_k \Sigma_{k+1} e^{\Sigma_k} \otimes \Sigma_{k+1} \Lambda_k\,,
\end{aligned} \tag{5.8}$$

[27] The normalization $\frac{\varsigma}{\chi\bar{\chi}}$ in the deformation parameter is necessary in order to recover correct formula in the limit $\chi, \bar{\chi} \mapsto 0$.

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$\mathfrak{o}''(2,2)$	$\chi, \bar{\chi}, \varsigma \in \imath\mathbb{R}$	$H^\star = -H, E_\pm^\star = -E_\pm$	$\bar{H}^\star = -\bar{H}, \bar{E}_\pm^\star = -\bar{E}_\pm$
$\mathfrak{o}(3,1)$	$\chi = \bar{\chi}^* \in \imath\mathbb{R}$; $\varsigma \in \mathbb{R}$	$H^\ddagger = -\bar{H}, E_\pm^\ddagger = -\bar{E}_\pm$	$\bar{H}^\ddagger = -H, \bar{E}_\pm^\ddagger = -E_\pm$

Table 3. Real quantizations of $r_{II}(\chi, \bar{\chi}, \varsigma) = \chi\, E_+ \wedge H + \bar{\chi}\, \bar{E}_+ \wedge \bar{H} + \varsigma E_+ \wedge \bar{E}_+$.

where $\Sigma_0 = \Sigma$, $\Sigma_1 = \bar{\Sigma}$ and Σ_{k+1} is denoted with index mod 2, i.e. Σ_{k+1} is equal to Σ_0 for $k = 1$; further $\Lambda_k = e^{-2\Sigma_k} - e^{-\Sigma_k} = -\chi_k E_{k+} e^{-2\Sigma_k}$.[28]

Using further the relations (3.15)–(3.16) one obtains the following formulae for the antipodes

$$\begin{aligned} &S_2(E_{k+}) = -E_{k+}\, e^{-\Sigma_k}\,, \qquad\qquad S_2(H_k) = -H_k e^{\Sigma_k}\,, \\ &S_2(E_{k-}) = -E_{k-}\, e^{\Sigma_k} + \chi_k H_k^2 e^{\Sigma_k}(e^{\Sigma_k}+1) - \chi_k^2 H_k E_{k+} e^{\Sigma_k}\,. \end{aligned} \tag{5.9}$$

We notice that the Abelian twist (5.7) does not contribute to the antipodes (5.9).

The quantum universal R-matrix $R_2 \equiv R_2(\chi, \bar{\chi}, \varsigma)$ takes the form

$$R_2 = \exp\left(\frac{-\varsigma}{\chi\bar{\chi}}\Sigma\wedge\bar{\Sigma}\right)\exp(\Sigma\otimes H)\exp(-H\otimes\Sigma)\exp(\bar{\Sigma}\otimes\bar{H})\exp(-\bar{H}\otimes\bar{\Sigma})\exp\left(\frac{-\varsigma}{\chi\bar{\chi}}\Sigma\wedge\bar{\Sigma}\right). \tag{5.10}$$

The formulae (5.8)–(5.10) present the general three-parameter deformation which can be studied in various two-parameter limits. For example, if $\chi \mapsto 0$ one should take into account that $\lim_{\chi\mapsto 0}\chi^{-1}\Sigma = E_+$, $\lim_{\chi\mapsto 0}\Lambda = 0$ and $\lim_{\chi\mapsto 0}\chi^{-1}\Lambda = -E_+$. In this limit the left chiral sector will be deformed only by Abelian twist. The case $\varsigma = 0$ provides obviously the product of two independent Jordanian deformations.

In real cases the independence of parameters may be not valid. Only for the real $\mathfrak{o}(2,2)$ deformation all three parameters are imaginary and independent. In the Lorentzian case[29] two Jordanian parameters $(\chi, \bar{\chi})$ are replace by one as follows from the condition $\chi = (\bar{\chi})^*$ in the table 3 above. All the twists present in the formula (5.6) are unitary (if the corresponding parameters are as indicated in the table 3) and the reality conditions (2.11) are satisfied.

5.3 Twisted pair of two $\mathfrak{sl}(2;\mathbb{C})$ q-analogues (r-matrix r_{III})

From the structure of the classical r-matrix $r_{III}(\gamma.\bar{\gamma}, 0) = r_{3'}$ (see (4.7)) for $\eta = 0$ follows that the quantum deformation $U_{r_3'}(\mathfrak{o}(3;\mathbb{C}))$ is the combination of two independent q-analogues (standard deformations) of $\mathrm{U}(\mathfrak{sl}(2;\mathbb{C}))$, with parameters $q \equiv q_0 = \exp\frac{1}{2}\gamma$ and $\bar{q} \equiv q_1 = \exp\frac{1}{2}\bar{\gamma}$. Moreover one has the splitting $U_{r_3'}(\mathfrak{o}(3;\mathbb{C})) \equiv U_{(q,\bar{q})}(\mathfrak{o}(4;\mathbb{C})) \cong U_q(\mathfrak{sl}(2;\mathbb{C})) \otimes U_{\bar{q}}(\mathfrak{sl}(2;\mathbb{C}))$.

Our starting point for further considerations is a pair of standard q-analogues (Drinfeld-Jimbo) deformations in chiral and antichiral sectors with two independent deformation parameters $\gamma, \bar{\gamma}$. They are described by nonlinear (quantum) generators $q_k^{\pm \mathsf{h}_k}$, $\mathsf{e}_{k\pm}$ $(k = 0, 1)$

[28] One has $[\Sigma, H] = -\chi E_+ e^{-\Sigma} = \Lambda e^{\Sigma}$, $[\Sigma, E_-] = 2\chi H e^{-\Sigma} - \chi\Lambda$.
[29] Studied first time in [28].

which satisfy the following defining relations (cf. (3.27))

$$q_k^{\mathtt{h}_k}\,\mathtt{e}_{k\pm} = q_k^{\pm 1}\mathtt{e}_{k\pm}\,q_k^{\mathtt{h}_k}\,, \qquad [\mathtt{e}_{k+},\,\mathtt{e}_{k-}] = \frac{q_k^{2\mathtt{h}_k} - q_k^{-2\mathtt{h}_k}}{q_k - q_k^{-1}}\,. \tag{5.11}$$

The co-products $\Delta_{3'}$ and antipodes $S_{3'}$ for this deformation are given by the formulas (cf. (3.28)–(3.29)):

$$\Delta_{3'}(q_k^{\pm\mathtt{h}_k}) = q_k^{\pm\mathtt{h}_k}\otimes q_k^{\pm\mathtt{h}_k}\,, \qquad \Delta_{3'}(\mathtt{e}_{k\pm}) = \mathtt{e}_{k\pm}\otimes q_k^{\mathtt{h}_k} + q_k^{-\mathtt{h}_k}\otimes \mathtt{e}_{k\pm}\,, \tag{5.12}$$

$$S_{3'}(q_k^{\pm\mathtt{h}_k}) = q_k^{\mp\mathtt{h}_k}\,, \qquad S_{3'}(\mathtt{e}_{k\pm}) = -q_k^{\pm 1}\mathtt{e}_{k\pm}\,. \tag{5.13}$$

The universal R-matrices $R_{3'k}$ for each chiral sector are well-known and using deformed CW generators (5.11) take the form ($q_k = \exp\frac{1}{2}\gamma_k$):

$$R_{3'k}(\gamma_k) = \exp_{q_k^{-2}}\Big((q_k - q_k^{-1})\mathtt{e}_{k+}\,q_k^{-\mathtt{h}_k}\otimes q_k^{\mathtt{h}_k}\mathtt{e}_{k-}\Big)q_k^{2\mathtt{h}_k\otimes\mathtt{h}_k}\,. \tag{5.14}$$

Following the discussion of nontriangular case in section 2.1, there exists alternative universal R-matrix in the form

$$(R^{\tau}_{3'k})^{-1} = q_k^{-2\mathtt{h}_k\otimes\mathtt{h}_k}\ \exp_{q_k^{2}}\Big((q_k^{-1} - q_k)E_{k-}\,q_k^{\mathtt{h}_k}\otimes q_k^{-\mathtt{h}_k}\,\mathtt{e}_{k+}\Big)\,. \tag{5.15}$$

Therefore, taking into account (5.14), (5.15) we see that the universal R-matrix $R_{3'}$ connecting the coproducts $\Delta_{3'}$ with the flipped one $\Delta^{21}_{3'}$ can be written in four equivalent forms, e.g.

$$R_{3'}(\gamma,\bar\gamma) = R_{3'0}(\gamma)R_{3'1}(\bar\gamma) = R_{3'1}(\bar\gamma)R_{3'0}(\gamma)\,. \tag{5.16}$$

Expanding (5.16) up to first order in deformation parameters $(\gamma,\bar\gamma)$ one gets

$$R_{3'}(\gamma,\bar\gamma) = 1 + r_{3'BD} + O(\gamma^2,\gamma\bar\gamma,\bar\gamma^2)\,, \tag{5.17}$$

where $r_{3'BD}$ is in $\mathfrak{sl}(2;\mathbb{C})\oplus\mathfrak{sl}(2;\mathbb{C})$ Belavin-Drinfeld form[30]

$$r_{3'BD} = \gamma\big(E_+\otimes E_- + H\otimes H\big) + \bar\gamma\big(\bar E_+\otimes \bar E_- + \bar H\otimes \bar H\big)\,. \tag{5.18}$$

This r-matrix is not skew-symmetric and satisfies the condition

$$r^{12}_{BD} + r^{21}_{BD} = \omega\,, \tag{5.19}$$

where ω is the quadratic split Casimir of $\mathfrak{o}(4;\mathbb{C})$

$$\begin{aligned}\omega = \gamma\big(&E_+\otimes E_- + E_-\otimes E_+ + 2H\otimes H\big)\\ &+\bar\gamma\big(\bar E_+\otimes \bar E_- + \bar E_-\otimes \bar E_+ + 2\bar H\otimes \bar H\big)\,.\end{aligned} \tag{5.20}$$

We recall that the Belavin-Drinfeld r-matrix r_{BD} satisfies CYBE and the r-matrix r'_3 is its skew-symmetric part.

[30]In (5.18) and in other formulas describing classical r-matrices, the generators $E_\pm$, $\bar E_\pm$ are not deformed.

Now we consider deformation of the quantum algebra $U_{(q_0,q_1)}(\mathfrak{o}(4;\mathbb{C})) \cong U_{q_0}(\mathfrak{sl}(2;\mathbb{C}))\otimes U_{q_1}(\mathfrak{sl}(2;\mathbb{C}))$ generated by the r-matrix $r_3'' = \eta H \otimes \bar{H}$, (see (4.7)). Since the generators $\mathtt{h}$ and $\bar{\mathtt{h}}$ have the primitive coproduct

$$\Delta_{3'}(\mathtt{h}_k) = \mathtt{h}_k \otimes 1 + 1 \otimes \mathtt{h}_k \quad (k = 0, 1)\,, \tag{5.21}$$

the Abelian two-tensor ($\tilde{q} = \exp \frac{1}{4}\eta$)

$$F_{3''}(\eta) := \tilde{q}^{\mathtt{h}\wedge\bar{\mathtt{h}}} \tag{5.22}$$

satisfies the 2-cocycle condition (3.17). Thus the complete three-parameter deformation generated by the r-matrix r_3 is the twist deformation of $U_{(\gamma,\bar{\gamma})}(\mathfrak{o}(4;\mathbb{C}))$; the resulting coproduct Δ_3 is given as follows

$$\Delta_3(a) = F_{3''}\Delta_{3'}(a)F_{3''}^{-1} \quad (\forall a \in U_{r_3'}(\mathfrak{o}(4;\mathbb{C}))\,, \tag{5.23}$$

and the antipode S_3 is not changed ($S_3 = S_{3'}$). Applying the twist (5.22) to the formulas (5.12) we obtain

$$\Delta_3(q_k^{\pm\mathtt{h}_k}) = q_k^{\pm\mathtt{h}_k} \otimes q_k^{\pm\mathtt{h}_k}, \tag{5.24}$$

$$\Delta_3(\mathtt{e}_{k\pm}) = \mathtt{e}_{k\pm} \otimes q_k^{\mathtt{h}_k}\tilde{q}^{\pm(-)^k\mathtt{h}_{k+1}} + q_k^{-\mathtt{h}_k}\tilde{q}^{\mp(-)^k\mathtt{h}_{k+1}} \otimes \mathtt{e}_{k\pm}\,. \tag{5.25}$$

The universal R-matrix, $R_3(\gamma,\bar{\gamma},\eta)$, generated from the complete r-matrix r_3, has the form

$$R_3(\gamma,\bar{\gamma},\eta) = R_{30}(\gamma,\eta)R_{31}(\bar{\gamma},\eta)\tilde{q}^{2\bar{\mathtt{h}}\wedge\mathtt{h}} \;=\; R_{31}(\bar{\gamma},\eta)R_{30}(\gamma,\eta)\tilde{q}^{2\bar{\mathtt{h}}\wedge\mathtt{h}}\,, \tag{5.26}$$

where $R_3(\gamma,\bar{\gamma},\eta) = \tilde{q}^{\bar{\mathtt{h}}\wedge\mathtt{h}}R_{3'}(\gamma,\bar{\gamma})\tilde{q}^{\bar{\mathtt{h}}\wedge\mathtt{h}}$ and

$$R_{3k}(\gamma_k,\eta) = \exp_{q_k^{-2}}\Big((q_k - q_k^{-1})\mathtt{e}_{k+}q_k^{-\mathtt{h}_k}\tilde{q}^{(-)^{k+1}\mathtt{h}_{k+1}} \otimes q_k^{\mathtt{h}_k}\tilde{q}^{(-)^{k+1}\mathtt{h}_{k+1}}\mathtt{e}_{k-}\Big)\, q_k^{2\mathtt{h}_k\otimes\mathtt{h}_k}\,.$$

In the linear limit we obtain (cf. (5.16), (5.17))

$$R_3 \sim 1 + r_3\,. \tag{5.27}$$

This deformation admits seven real forms which employ all four conjugations (cf. (4.11)–(4.14)). The list of real forms with corresponding restricted values of the deformation parameters $\gamma,\bar{\gamma},\eta$ is presented in the table 4, with real bialgebras denoted in the first column (cf. table 1). The letters in the last column (R=real, A=antireal, H=hybrid) indicate the properties of the R-matrix under respective conjugation: $R^\star = R^\tau$ for real, $R^\star = R^{-1}$ for antireal cases. In the hybrid case the R-matrix decomposes into a product of three factors, with first real, second antireal and third is given by twist which satisfies both reality conditions.

It should be mentioned that only the classical r-matrix r_{III} provides the quantum deformations of real $\mathfrak{o}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ algebra (see first line in table 4). Particular case, with $\eta = 0$, was derived as describing quantum symmetries of $D = 3$ LQG [7].

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$\mathfrak{o}(4)$	$\gamma, \bar\gamma \in \mathbb{R}$; $\eta \in \imath\mathbb{R}$	$(q^{\mathtt{h}})^\dagger = q^{\mathtt{h}}, \mathtt{e}_\pm^\dagger = \mathtt{e}_\mp$	$(\bar q^{\bar{\mathtt{h}}})^\dagger = \bar q^{\bar{\mathtt{h}}}, \bar{\mathtt{e}}_\pm^\dagger = \bar{\mathtt{e}}_\mp$	R
$\mathfrak{o}^\star(4)$	$\gamma, \bar\gamma \in \mathbb{R}$; $\eta \in \imath\mathbb{R}$	$(q^{\mathtt{h}})^\dagger = q^{\mathtt{h}}, \mathtt{e}_\pm^\dagger = \mathtt{e}_\mp$	$(\bar q^{\bar{\mathtt{h}}})^\# = q^{\bar{\mathtt{h}}}, \bar{\mathtt{e}}_\pm^\# = -\bar{\mathtt{e}}_\mp$	R
$\mathfrak{o}'^\star(4)$	$\gamma, \eta \in \mathbb{R}$; $\bar\gamma \in \imath\mathbb{R}$	$(q^{\mathtt{h}})^\dagger = q^{\mathtt{h}}, \mathtt{e}_\pm^\dagger = \mathtt{e}_\mp$	$(\bar q^{\bar{\mathtt{h}}})^\star = \bar q^{\bar{\mathtt{h}}}, \bar{\mathtt{e}}_\pm^\star = -\bar{\mathtt{e}}_\pm$	H
$\mathfrak{o}(2,2)$	$\gamma, \bar\gamma \in \mathbb{R}$; $\eta \in \imath\mathbb{R}$	$(q^{\mathtt{h}})^\# = q^{\mathtt{h}}, \mathtt{e}_\pm^\# = -\mathtt{e}_\mp$	$(\bar q^{\bar{\mathtt{h}}})^\# = \bar q^{\bar{\mathtt{h}}}, \bar{\mathtt{e}}_\pm^\# = -\bar{\mathtt{e}}_\mp$	R
$\mathfrak{o}'(2,2)$	$\gamma, \eta \in \mathbb{R}$; $\bar\gamma \in \imath\mathbb{R}$	$(q^{\mathtt{h}})^\# = q^{\mathtt{h}}, \mathtt{e}_\pm^\# = -\mathtt{e}_\mp$	$(\bar q^{\bar{\mathtt{h}}})^\star = \bar q^{\bar{\mathtt{h}}}, \bar{\mathtt{e}}_\pm^\star = -\bar{\mathtt{e}}_\pm$	H
$\mathfrak{o}''(2,2)$	$\gamma, \bar\gamma, \eta \in \imath\mathbb{R}$	$(q^{\mathtt{h}})^\star = q^{\mathtt{h}}, \mathtt{e}_\pm^\star = -\mathtt{e}_\pm$	$(\bar q^{\bar{\mathtt{h}}})^\star = \bar q^{\bar{\mathtt{h}}}, \bar{\mathtt{e}}_\pm^\star = -\bar{\mathtt{e}}_\pm$	A
$\mathfrak{o}(3,1)$	$\bar\gamma = -\gamma^* \in \mathbb{C}$; $\eta \in \mathbb{R}$	$(q^{\mathtt{h}})^\ddagger = \bar q^{\bar{\mathtt{h}}}, \mathtt{e}_\pm^\ddagger = -\bar{\mathtt{e}}_\pm$	$(\bar q^{\bar{\mathtt{h}}})^\ddagger = q^{\mathtt{h}}, \bar{\mathtt{e}}_\pm^\ddagger = -\mathtt{e}_\pm$	A

Table 4. Real quantizations of $r_{III}(\gamma, \bar\gamma, \eta) = \gamma\, E_+ \wedge E_- + \bar\gamma\, \bar E_+ \wedge \bar E_- + \eta\, H \wedge \bar H$.

5.4 Twisting of $\mathfrak{o}(4;\mathbb{C})$ Belavin-Drinfeld triple (r-matrix r_{IV})

Next, we describe quantum deformation corresponding to the classical r-matrix r_{IV} (4.8). Since the r-matrix $r_{IV}(\gamma, 0) := r_{4'}$ is a particular case of $r_{III}(\gamma, \bar\gamma, \eta)$, namely $r_{IV'}(\gamma) = r_{III}(\gamma, -\gamma, -2\gamma)$, $\gamma \in \mathbb{C}$, the quantum deformation corresponding to the r-matrix $r_{4'}$ is obtained from the formulae in section 5.3 by setting $\bar q = \tilde q = q^{-1}$. The quantum deformation corresponding to r_{IV} is generated by the elements $q^{\pm \mathtt{h}_k}$, $\mathtt{e}_{k\pm}$ (k=0,1) with the following defining relations (cf. (5.11))

$$q^{\mathtt{h}_k}\, \mathtt{e}_{k\pm} = q^{\pm 1} \mathtt{e}_{k\pm}\, q^{\mathtt{h}_k}\,, \qquad [\mathtt{e}_{k+}, \mathtt{e}_{k-}] = \frac{q^{2\mathtt{h}_k} - q^{-2\mathtt{h}_k}}{q - q^{-1}} \tag{5.28}$$

constituting the algebra $U_q(\mathfrak{sl}(2;\mathbb{C})) \otimes U_{q^{-1}}(\mathfrak{sl}(2;\mathbb{C}))$. The co-products $\Delta_{4'}$ and antipodes $S_{4'}$ generated by $r_{IV'}$ are given by the formulas (cf. (5.24)–(5.25)):

$$\begin{aligned} \Delta_{4'}(q^{\pm \mathtt{h}_k}) &= q^{\pm \mathtt{h}_k} \otimes q^{\pm \mathtt{h}_k}\,, \\ \Delta_{4'}(\mathtt{e}_{k\pm}) &= \mathtt{e}_{k\pm} \otimes q^{(-)^k(\mathtt{h}_k \pm \mathtt{h}_{k+1})} + q^{(-)^{k+1}(\mathtt{h}_k \pm \mathtt{h}_{k+1})} \otimes \mathtt{e}_{k\pm}\,, \\ S_{4'}(q^{\pm \mathtt{h}_k}) &= q^{\mp \mathtt{h}_k}, \qquad S_{4'}(\mathtt{e}_{k\pm}) = -q^{\pm(-)^k} \mathtt{e}_{k\pm}\,. \end{aligned} \tag{5.29}$$

The full deformation of the quantum algebra (5.28)–(5.29) is obtained after performing the twist quantization generated by the remaining part of the r-matrix r_{IV}, namely $r_{IV''} = \varsigma E_+ \wedge \bar E_+$, described by the following quantum Abelian twist factor [56]:

$$F_{4''}(\gamma, \varsigma) := \exp_{q^2}\left(\varsigma \mathtt{e}_+ q^{\mathtt{h}+\bar{\mathtt{h}}} \otimes q^{\mathtt{h}+\bar{\mathtt{h}}} \bar{\mathtt{e}}_+\right). \tag{5.30}$$

It can be shown that the two-tensor (5.30) satisfies the 2-cocycle equation (3.17).

$\mathfrak{o}''(2,2)$	$\gamma,\varsigma\in\imath\mathbb{R}$	$(q^{\mathtt{h}})^\star=q^{\mathtt{h}},\mathtt{e}_\pm^\star=-\mathtt{e}_\pm$	$(\bar q^{\bar{\mathtt{h}}})^\star=\bar q^{\bar{\mathtt{h}}},\bar{\mathtt{e}}_\pm^\star=-\bar{\mathtt{e}}_\pm$	A
$\mathfrak{o}(3,1)$	$\gamma\in\mathbb{R}\ ,\ \varsigma=0$	$(q^{\mathtt{h}})^\ddagger=q^{-\bar{\mathtt{h}}},\mathtt{e}_\pm^\ddagger=-\bar{\mathtt{e}}_\pm$	$(q^{\bar{\mathtt{h}}})^\ddagger=q^{-\mathtt{h}},\bar{\mathtt{e}}_\pm^\ddagger=-\mathtt{e}_\pm$	A

Table 5. Real quantizations of $r_{IV}(\gamma,\varsigma)=\gamma\left(E_+\wedge E_- - \bar E_+\wedge\bar E_- - 2H\wedge\bar H\right)+\varsigma E_+\wedge\bar E_+$.

Explicit form of the co-products $\Delta_4(\cdot)=F_{4''}\Delta_{4'}(\cdot)F_{4''}^{-1}$ in the complex Cartan-Weyl bases of $U_{r'_4}(\mathfrak{o}(4;\mathbb{C}))$ can be calculated using q-analogue of Hadamard formula (appendix B)

$$\begin{aligned}
\Delta_4(q^{\pm(\mathtt{h}-\bar{\mathtt{h}})}) &= q^{\pm(\mathtt{h}-\bar{\mathtt{h}})}\otimes q^{\pm(\mathtt{h}-\bar{\mathtt{h}})}\,,\\
\Delta_4(q^{\,\mathtt{h}+\bar{\mathtt{h}}}) &= \mathbb{X}^{-1}\,q^{\mathtt{h}+\bar{\mathtt{h}}}\otimes q^{\mathtt{h}+\bar{\mathtt{h}}}\,,\\
\Delta_4(q^{\,-\mathtt{h}-\bar{\mathtt{h}}}) &= q^{-\mathtt{h}-\bar{\mathtt{h}}}\otimes q^{-\mathtt{h}-\bar{\mathtt{h}}}\,\mathbb{X}\,,\\
\Delta_4(\mathtt{e}_+) &= \mathtt{e}_+\otimes q^{\mathtt{h}+\bar{\mathtt{h}}}+q^{-\mathtt{h}-\bar{\mathtt{h}}}\otimes\mathtt{e}_+\,\mathbb{X}\,,\\
\Delta_4(\bar{\mathtt{e}}_+) &= \bar{\mathtt{e}}_+\otimes q^{-\mathtt{h}-\bar{\mathtt{h}}}\mathbb{X}+q^{\mathtt{h}+\bar{\mathtt{h}}}\otimes\bar{\mathtt{e}}_+\,,\\
\Delta_4(\mathtt{e}_-) &= \mathtt{e}_-\otimes q^{\mathtt{h}-\bar{\mathtt{h}}}+q^{\bar{\mathtt{h}}-\mathtt{h}}\otimes\mathtt{e}_- -\\
&\quad -\frac{\varsigma}{q-q^{-1}}\left(q^{-4\mathtt{h}}\otimes 1-\mathbb{X}^{-1}\right)\left(q^{3\mathtt{h}+\bar{\mathtt{h}}}\otimes\bar{\mathtt{e}}_+q^{2\mathtt{h}}\right),\\
\Delta_4(\bar{\mathtt{e}}_-) &= \bar{\mathtt{e}}_-\otimes q^{\mathtt{h}-\bar{\mathtt{h}}}+q^{\bar{\mathtt{h}}-\mathtt{h}}\otimes\bar{\mathtt{e}}_- -\\
&\quad -\frac{\varsigma}{q-q^{-1}}\left(1\otimes q^{-4\bar{\mathtt{h}}}-\mathbb{X}^{-1}\right)\left(\mathtt{e}_+q^{2\bar{\mathtt{h}}}\otimes q^{\mathtt{h}+3\bar{\mathtt{h}}}\right),
\end{aligned}\tag{5.31}$$

where

$$\mathbb{X}:=1\otimes 1\,+\,\varsigma(q^2-1)\mathtt{e}_+q^{\mathtt{h}+\bar{\mathtt{h}}}\otimes q^{\mathtt{h}+\bar{\mathtt{h}}}\bar{\mathtt{e}}_+\,.\tag{5.32}$$

Explicit formulas for antipodes $S_4(\cdot)=uS_{4'}(\cdot)u^{-1}$ where

$$u^{-1}=m\circ(S_{4'}\otimes\mathrm{id})\exp_{q^2}\left(\varsigma\,\mathtt{e}_+q^{\mathtt{h}+\bar{\mathtt{h}}}\otimes q^{\mathtt{h}+\bar{\mathtt{h}}}\bar{\mathtt{e}}_+\right)=\exp_{q^2}\left(\varsigma\,\mathtt{e}_+\bar{\mathtt{e}}_+\right),\tag{5.33}$$

are given (as results from q-Hadamard formula) below

$$\begin{aligned}
S_4(q^{\pm(\mathtt{h}-\bar{\mathtt{h}})}) &= q^{\mp(\mathtt{h}-\bar{\mathtt{h}})}\,, & S_4(\mathtt{e}_{k+}) &= -q^{(-)^k}\mathtt{e}_{k+}\,,\\
S_4(\,q^{\mathtt{h}+\bar{\mathtt{h}}}) &= q^{-\mathtt{h}-\bar{\mathtt{h}}}X^{-1}\,, & S_4(q^{\,-\mathtt{h}-\bar{\mathtt{h}}}) &= X\,q^{\mathtt{h}+\bar{\mathtt{h}}}\,,\\
S_4(\mathtt{e}_{k-}) &= -q^{(-1)^{k+1}}\mathtt{e}_{k-}+\frac{(-)^k\varsigma}{q^{2(-)^k}-1}\,\mathtt{e}_{(k+1)+}\left(q^{2\mathtt{h}_k}-q^{-2\mathtt{h}_k}X^{-1}\right),
\end{aligned}\tag{5.34}$$

where

$$X:=1+\varsigma(q^2-1)\mathtt{e}_+\bar{\mathtt{e}}_+\,.\tag{5.35}$$

Therefore a total universal R-matrix for this case is the following product (now $\bar\gamma=-\gamma\Leftrightarrow\bar q=q^{-1}$)

$$R_4(\gamma,\varsigma)=F_{4''}^{\tau}(\gamma,\varsigma)R_{3'0}(\gamma)R_{3'1}(-\gamma)F_{4''}^{-1}(\gamma,\varsigma)\,.\tag{5.36}$$

Two real qunatizations are described in the table 5.

It should be noted that the value $\varsigma=0$ in the Lorentzian case is due to the property that twist (5.30), in contrast to the $\mathfrak{o}''(2,2)$ case where $|q|=1$, is not unitary for real q. In order to have formulae (5.31)–(5.34) compatible with the Lorentzian conjugation (4.14) it is helpful to introduce flipped conjugation (3.22) on the tensor product of quantized algebras (see [29]).

Alternatively, one can keep the standard (non-flipped) conjugation and seek for the unitarizing coboundary twist - the quantum analogue of (3.21).[31] Examples of quantum coboundary twists can be found e.g. in [58, 59]. The realization of this task is postponed to our future work.

Yet another method relying on quantum deformation of the real involution ($\star$-involution) has been studied in [17, 60, 61]. Assuming q real, the quantum twist (5.30) in the real Hopf algebra $(U_q(\mathfrak{sl}(2;\mathbb{C})) \otimes U_{q^{-1}}(\mathfrak{sl}(2;\mathbb{C}), \Delta_{4'}, S_{4'}, \ddagger)$ satisfies the condition (see [17], Prop. 2.3.7, p. 59)

$$(S_{4'} \otimes S_{4'})(F_{4''}^{\ddagger\otimes\ddagger}) = F_{4''}^{\tau}$$

for ς real. This permits to introduce new quantum conjugation which is quantum-deformed by the similarity transformation

$$()^{\ddagger'} = u\,()^{\ddagger}\,u^{-1}$$

where u^{-1} is given by the formula (5.33) (in our case $S^{-1}(u) = u$). Explicit calculations with the help of q-Hadamard formula leads to the following results

$$\begin{aligned} (q^{\pm(\mathtt{h}-\bar{\mathtt{h}})})^{\ddagger'} &= (q^{\pm(\mathtt{h}-\bar{\mathtt{h}})})^{\ddagger} = q^{\pm(\mathtt{h}-\bar{\mathtt{h}})}\,, \qquad (\mathtt{e}_{k+})^{\ddagger'} = (\mathtt{e}_{k+})^{\ddagger} = -\mathtt{e}_{(k+1)+}\,, \\ (\,q^{\mathtt{h}+\bar{\mathtt{h}}})^{\ddagger'} &= q^{-\mathtt{h}-\bar{\mathtt{h}}}X^{-1}\,, \qquad (q^{\,-\mathtt{h}-\bar{\mathtt{h}}})^{\ddagger'} = X\,q^{\mathtt{h}+\bar{\mathtt{h}}}\,, \\ (\mathtt{e}_{k-})^{\ddagger'} &= -\mathtt{e}_{(k+1)-} + \frac{\varsigma}{q-q^{-1}}\,\mathtt{e}_{k+}\big(q^{2\mathtt{h}_{k+1}} - q^{-2\mathtt{h}_{k+1}}X^{-1}\big)\,, \end{aligned} \tag{5.37}$$

where X is given by (5.35). In this way Belavin-Drinfeld type quantum deformation of the Lorentz algebra is described by the real Hopf algebra $(U_q(\mathfrak{sl}(2;\mathbb{C}))\otimes U_{q^{-1}}(\mathfrak{sl}(2;\mathbb{C})[[\varsigma]], \Delta_4, S_4, \ddagger')$.

5.5 Left q-analogue and right Jordanian deformation intertwined by Abelian twist (r-matrix r_V)

In this case we start with the left sector as q-deformed with $q = \exp\frac{1}{2}\gamma$

$$q^{\mathtt{h}}\mathtt{e}_{\pm} = q^{\pm 1}\mathtt{e}_{\pm}\,q^{\mathtt{h}}\,, \qquad [\mathtt{e}_+,\,\mathtt{e}_-] = \frac{q^{2\mathtt{h}} - q^{-2\mathtt{h}}}{q-q^{-1}}\,. \tag{5.38}$$

The right sector is deformed by Jordanian twist F_J expressed in undeformed CW basis (cf. section 3.2.1)

$$[\bar{H},\,\bar{E}_{\pm}] = \bar{E}_{\pm}\,, \quad [\bar{E}_+,\,\bar{E}_-] = 2\bar{H}\,. \tag{5.39}$$

Further we perform the subsequent quantization by using the quantized Abelian twist

$$F_{5''}(\bar{\chi},\rho) = \tilde{q}^{\mathtt{h}\wedge\bar{\Sigma}}, \qquad \tilde{q} = \exp\frac{\rho}{4\bar{\chi}}\,.$$

[31] This method has been e.g. used in [57] in order to unitarize superextension of the Jordanian deformation.

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$\mathfrak{o}'^{\star}(4)$	$\gamma,\rho\in\mathbb{R}\ ;\ \bar{\chi}\in\imath\mathbb{R}$	$(q^{\mathtt{h}})^{\dagger}=q^{\mathtt{h}},\mathtt{e}_{\pm}^{\dagger}=\mathtt{e}_{\mp}$	$\bar{H}^{\star}=-\bar{H},\bar{E}_{\pm}^{\star}=-\bar{E}_{\pm}$	R
$\mathfrak{o}'(2,2)$	$\gamma,\rho\in\mathbb{R}\ ;\ \bar{\chi}\in\imath\mathbb{R}$	$(q^{\mathtt{h}})^{\#}=q^{\mathtt{h}},\mathtt{e}_{\pm}^{\#}=-\mathtt{e}_{\mp}$	$\bar{H}^{\star}=-\bar{H},\bar{E}_{\pm}^{\star}=-\bar{E}_{\pm}$	R
$\mathfrak{o}''(2,2)$	$\gamma,\rho,\bar{\chi}\in\imath\mathbb{R}$	$(q^{\mathtt{h}})^{\star}=q^{\mathtt{h}},\mathtt{e}_{\pm}^{\star}=-\mathtt{e}_{\pm}$	$\bar{H}^{\star}=-\bar{H},\bar{E}_{\pm}^{\star}=-\bar{E}_{\pm}$	A

Table 6. Real quantizations of $r_V(\gamma,\rho,\bar{\chi})=\gamma\,E_+\wedge E_-+\bar{\chi}\,\bar{E}_+\wedge\bar{H}+\rho H\wedge\bar{E}_+$.

The explicit coproduct formulae are the following

$$\Delta_5(q^{\pm\mathtt{h}})=q^{\pm\mathtt{h}}\otimes q^{\pm\mathtt{h}}, \tag{5.40}$$

$$\Delta_5(\mathtt{e}_{\pm})=\mathtt{e}_{\pm}\otimes q^{\mathtt{h}}\tilde{q}^{\pm\bar{\Sigma}}+q^{-\mathtt{h}}\tilde{q}^{\mp\bar{\Sigma}}\otimes\mathtt{e}_{\pm}\,, \tag{5.41}$$

$$\Delta_5(\bar{E}_+)=\bar{E}_+\otimes e^{\bar{\Sigma}}+1\otimes\bar{E}_+\,, \tag{5.42}$$

$$\begin{aligned}
\Delta_5(\bar{H})&=\bar{H}\otimes e^{-\bar{\Sigma}}+1\otimes\bar{H}\;-\frac{\rho}{4}\left(\mathtt{h}\otimes\bar{E}_+e^{-\bar{\Sigma}}-\;\bar{E}_+e^{-\bar{\Sigma}}\otimes\mathtt{h}e^{-\bar{\Sigma}}\right),\\
\Delta_5(\bar{E}_-)&=\bar{E}_-\otimes e^{-\bar{\Sigma}}+1\otimes\bar{E}_-+2\bar{\chi}\bar{H}\otimes\bar{H}e^{-\bar{\Sigma}}+\bar{\chi}\bar{H}(\bar{H}-1)\otimes\bar{\Lambda}+\\
&\quad-\frac{\rho}{2}\left(\bar{H}e^{-\bar{\Sigma}}\otimes\mathtt{h}e^{-\bar{\Sigma}}-\bar{H}\mathtt{h}\otimes\bar{\Lambda}-\mathtt{h}\otimes\bar{H}e^{-\bar{\Sigma}}\right)\\
&\quad-\frac{\rho}{2}\left(\bar{\Lambda}e^{\bar{\Sigma}}\otimes\bar{H}\mathtt{h}e^{-\bar{\Sigma}}\;+\bar{H}\bar{\Lambda}e^{\bar{\Sigma}}\otimes\mathtt{h}\bar{\Lambda}\right)\\
&\quad-\frac{\rho}{4}\left(\left(1-e^{-2\bar{\Sigma}}\right)\otimes\mathtt{h}\bar{\Lambda}\;+\mathtt{h}\otimes\bar{\Lambda}\;-\bar{\Lambda}\otimes\mathtt{h}e^{-\bar{\Sigma}}\right)\\
&\quad+\frac{1}{\bar{\chi}}\left(\frac{\rho}{4}\right)^2\left(\bar{\Lambda}^2e^{2\bar{\Sigma}}\otimes\mathtt{h}^2\bar{\Lambda}+\bar{\Lambda}\otimes\mathtt{h}^2e^{-\bar{\Sigma}}+\mathtt{h}^2\otimes\bar{\Lambda}\right)\\
&\quad-\frac{2}{\bar{\chi}}\left(\frac{\rho}{4}\right)^2\;\bar{\Lambda}\mathtt{h}e^{\bar{\Sigma}}\otimes\mathtt{h}\bar{\Lambda}\,.
\end{aligned} \tag{5.43}$$

The antipodes do not depend on the Abelian twist and look as follows:

$$\begin{aligned}
S_5(q^{\pm\mathtt{h}})&=q^{\mp\mathtt{h}}\,, &\qquad S_5(\mathtt{e}_{\pm})&=-q^{\pm1}\mathtt{e}_{\pm}\,,\\
S_5(\bar{E}_+)&=-\bar{E}_+\,e^{-\bar{\Sigma}}, & S_5(\bar{H})&=-\bar{H}e^{\bar{\Sigma}}\,,\\
S_5(\bar{E}_-)&=-\bar{E}_-\,e^{\bar{\Sigma}}+\bar{\chi}\bar{H}^2e^{\bar{\Sigma}}(e^{\bar{\Sigma}}+1)-\bar{\chi}^2\bar{H}\bar{E}_+e^{\bar{\Sigma}}\,.
\end{aligned} \tag{5.44}$$

Quantum universal R-matrix generated from r_V takes the following form

$$R_5(\gamma,\bar{\chi},\rho)=\tilde{q}^{\bar{\Sigma}\wedge\mathtt{h}}R_{3'0}(\gamma)F^{\tau}_{J1}(\bar{\chi})F^{-1}_{J1}(\bar{\chi})\tilde{q}^{\bar{\Sigma}\wedge\mathtt{h}}\,. \tag{5.45}$$

Three real quantizations we describe in the table 6 above.

6 Concluding remarks and outlook

In this paper we presented the complete set of Hopf-algebraic quantum deformations generated by classical r-matrices for $\mathfrak{o}(4;\mathbb{C})$ and its real forms given in [20, 22]. We provide the explicit formulae describing algebraic and coalgebraic sectors as well as the universal R-matrices which describe the tensoring of quantum modules (representations of quantum-deformed Hopf algebras). In such a way the universal R-matrices lead to the braided structure of quantum-covariant tensor products of modules [17, 62] what has been useful in quantum-covariant NC field theory [63–65]. For quantum twist deformations of enveloping Lie algebras $U(\mathfrak{g})$ ($\mathfrak{g} = \mathfrak{o}(4;\mathbb{C}), \mathfrak{o}(4-k,k)$ ($k = 0,1,2$), and $\mathfrak{o}^*(4) = \mathfrak{o}(2;\mathbb{H})$); for $\mathfrak{o}(4;\mathbb{C})$ (see section 5.1, 5.2) the algebra of quantum modules, describing e.g. NC quantum fields, can be represented by the functions of commutative classical fields with twist-dependent nonlocal star product multiplication rule [66, 67].

The $D = 4$ Minkowski space $\mathbb{R}^{3,1}$, with signature $(+,+,+,-)$, and its Lorentz rotations $\mathfrak{o}(3,1)$ play basic role in relativistic physics. In order to describe the Lie algebra generating the full relativistic group of motion, one adds four generators $P_\mu \in \mathbb{T}^{3,1}$ of translations, i.e. one extends Lorentz algebra $\mathfrak{o}(3,1)$ to $D = 4$ Poincaré algebra $\mathfrak{o}(3,1) \ltimes \mathbb{T}^{3,1}$. It is known that only two out of four quantum deformations of $\mathfrak{o}(3,1)$ can be extended to quantum deformations of $D = 4$ Poincaré algebra (see [29, 68–70]). The studies providing the complete list of possible quantum deformations of inhomogeneous $D = 4$ Euclidean $\mathfrak{o}(4) \ltimes \mathbb{T}^4$ algebra and of inhomogeneous $D = 4$ Kleinian $\mathfrak{o}(2,2) \ltimes \mathbb{T}^{2,2}$ algebra still has not been given.[32] We add that inhomogeneous extension of quaternionic real form $\mathfrak{o}^*(4) \equiv \mathfrak{o}(2;\mathbb{H})$ of $\mathfrak{o}(4;\mathbb{C})$ contains four complex or two quaternionic translations and the application of such group of motion to the description of physical symmetries is, as far as we know, not presented.

The real forms of considered quantum groups describe the quantum symmetries of $D = 3$ compact Euclidean (S^3), de Sitter (dS_3) or anti-de-Sitter (AdS_3) spacetimes, with finite nonvanishing constant curvature and curved $D = 3$ Euclidean, dS_3 or AdS_3 translations. In $D = 4$ rotation algebras $\mathfrak{o}(3,1)$ ($\mathfrak{o}(2,2)$), the dS (AdS) radius $\mathcal{R}$ is introduced by suitable rescaling of the generators in the coset $\frac{\mathfrak{o}(4-k,k)}{\mathfrak{o}(3-k,k)}$ ($k = 1,2$), with $\Lambda = \mathcal{R}^{-1}$ which can be treated as a deformation parameter. The quantum deformations of $\mathfrak{o}(4-k,k)$ ($k = 1,2$) have been extensively studied as describing the NC geometry of $2+1$-dimensional QG with cosmological constant $\Lambda \neq 0$ [73–76]. The classical action of $D = 3$ gravity can be introduced geometrically as the gauge theory described by $D = 3$ Chern-Simons (CS) model. Following Fock-Rosly construction [77, 78], in such framework we describe gravitational degrees of freedom as parameterizing the Poisson-Lie group manifold. If we search for quantum deformations of Fock-Rosly construction, it appears that only classical r-matrices obtained from Drinfeld double (DD) structures [10, 11] are allowed [79–81]. The DD structures and corresponding classical r-matrices for $\mathfrak{o}(3,1)$ and $\mathfrak{o}(2,2)$ algebras were recently constructed and classified [74–76]. One can check that such quantum deformations, which are well adjusted to the description of quantum-deformed $D = 3$ gravity, are generated by the subclass of classical r-matrices, listed in [20, 22] and quantized in this paper.

The next step in our program is to construct the complete list of classical r-matrices for the $D = 4$ complex inhomogeneous Euclidean algebra $\mathcal{E}(4;\mathbb{C}) := \mathfrak{io}(4;\mathbb{C}) := \mathfrak{o}(4;\mathbb{C}) \ltimes \mathbf{T}(4;\mathbb{C})$ (orthogonal rotations together with translations) and for its real forms, in particular $\mathfrak{o}(4-k,k) \ltimes \mathbf{T}(4-k,k;\mathbb{R})$ ($k = 0,1,2$). Until present time the most complete results were obtained for $\mathfrak{o}(3,1) \ltimes \mathbf{T}(3,1)$ by Zakrzewski [69], who provided almost complete list of 21 different, not related by Poincaré automorphism real $D = 4$ Poincaré r-matrices (see also [70, 71]). It should be noticed that the complete classifications of r-matrices for both inhomogenous $D = 3$ Poincaré and $D = 3$ Euclidean algebras have been given by Stachura [82].

[32] For partial results in $D = 4$ Euclidean case see e.g. [72].

Recently in [83, 84] the present authors complexified Zakrzewski results and then imposed $D = 4$ Euclidean reality constraints. It appeared that 8 out of 21 complexified Zakrzewski r-matrices are consistent with the Euclidean conjugation in $\mathfrak{o}(4;\mathbb{C})$ (see (4.11)). It can be shown, however, that the complexified Zakrzewski r-matrices do not describe all r-matrices for $\mathcal{E}(4;\mathbb{C})$.[33] Using the constructive method analogous to the one proposed in this paper we plan to describe the complete classification of classical r-matrices for $D = 4$ complex inhomogeneous Lie algebra $\mathcal{E}(4;\mathbb{C})$ and for its all real forms.

We add that in [83, 84] we considered also the $N = 1$ superextension of Poincaré and Euclidean classical r-matrices. Recently we derived in analogous way a new class of $N = 2$ Poincaré and Euclidean supersymmetric r-matrices (see [72]). We hope that our constructive method of providing the complete list of classical r-matrices for the complex $\mathcal{E}(4;\mathbb{C})$ case can be applied as well to N-extended Euclidean superalgebras $\mathcal{E}(4|N;\mathbb{C})$ for $N = 1, 2, 4$ and further classify and quantize the supersymmetric r-matrices for the corresponding all real forms.

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A All $\mathfrak{o}(3)$ and $\mathfrak{o}(2,1)$ Lie bialgebras

Classification of r-matrices is the same task as classification of coboundary Lie bialgebras up to isomorphisms (an isomorphism which preserves the structure constants is called an automorphism). In geometric terms they can be seen as orbits of an action of the Lie algebra automorphism group in the space of skew-symmetric solution of mCYBE (see also [85, 86]). For simple algebras all bialgebra structures are coboundary due to the Whitehead lemma.

Let us consider, for completeness as well as for pedagogical reason, geometric classification scheme for classical r-matrices of simple 3-dimensional real rotational Lie algebras (for purely algebraic approach see [21]). Up to an isomorphism there are only two non-isomorphic real simple Lie algebras: compact $\mathfrak{o}(3)$ and non-compact $\mathfrak{o}(2,1)$, both are real form of $\mathfrak{o}(3;\mathbb{C})$.

Consider firstly the compact $\mathfrak{o}(3)$ case with the canonical vectorial basis ($I_k^\dagger = -I_k$ cf. (3.10)–(3.11))

$$[I_1, I_2] = I_3, \quad [I_1, I_3] = -I_2, \quad [I_2, I_3] = I_1\,. \tag{A.1}$$

We notice that any element $r(a,b,c) = aI_2 \wedge I_3 + bI_3 \wedge I_1 + cI_1 \wedge I_2 \in \mathfrak{o}(3) \wedge \mathfrak{o}(3)$ is a classical r-matrix since it satisfies

$$[[r(a,b,c), r(a,b,c)]] = (a^2 + b^2 + c^2)\Omega\,, \tag{A.2}$$

where $\Omega = I_1 \wedge I_2 \wedge I_3 \in \mathfrak{o}(3) \wedge \mathfrak{o}(3) \wedge \mathfrak{o}(3)$ is a unique up to the constant invariant element.

[33] Such conclusion follows e.g. from the observation that the list of the real r-matrices for $\mathfrak{o}(2,2) \ltimes \mathbf{T}(2,2)$ is longer then the Zakrzewski list (see [69]) for $D = 4$ Poincaré algebra.

The non-isomorphic Lie bialgebra structures for $\mathfrak{o}(3)$ case can be identify with orbits of the automorphism group in the space of free parameters $(a, b, c) \in \mathbb{R}^3$ with the Euclidean metric. The group of automorphisms contain SO(3) subgroup. Moreover, bivector and vector representations are equivalent in dimension 3. Due to this property we look only for SO(3)- orbits in $\mathbb{R}^3$. These are the 2-dimensional spheres represented by a radius $\xi > 0$ or by the vector $\xi(1, 0, 0)$. Thus as a result of final classification one gets the following family of non-trivial $\mathfrak{o}(3)$ r-matrices (the trivial $r = 0$ r-matrix corresponds to singular one-point orbit at $(0, 0, 0)$)

$$r_\xi = \xi I_1 \wedge I_2 \,. \tag{A.3}$$

Notice that the values of the real parameter $\xi > 0$ are effective and lead to nonequivalent Lie bialgebra structures.

Similar analysis applied to the non-compact real form $\mathfrak{o}(2, 1)$ provides qualitatively different results. Arbitrary $\mathfrak{o}(2, 1)$ r-matrix satisfies the following YB equation (a,b,c real)

$$[[r(a, b, c), r(a, b, c)]] = (a^2 - b^2 + c^2) J_1 \wedge J_2 \wedge J_3 \,, \tag{A.4}$$

where $r(a, b, c) = aJ_2 \wedge J_3 + bJ_3 \wedge J_1 + cJ_1 \wedge J_2 \in \mathfrak{o}(2, 1) \wedge \mathfrak{o}(2, 1)$ is written in the canonical $\mathfrak{o}(2, 1)$ basis. We choose noncompact vectorial generators J_1, J_2, J_3 following our choice of $\mathfrak{o}(2, 1)$ reality conditions $J_k^\star = -J_k$

$$[J_1, J_2] = J_3, \quad [J_1, J_3] = J_2, \quad [J_2, J_3] = J_1 \,. \tag{A.5}$$

The automorphisms group SO(2, 1) of the Lie algebra $\mathfrak{o}(2, 1)$ acts in three-dimensional Minkowski space $\mathbb{R}^{2,1}$. There are three types of non-trivial orbits in $\mathbb{R}^{2,1} = \{(a, b, c) : a, b, c \in \mathbb{R}\}$, characterizing three independent $\mathfrak{o}(2, 1)$ r-matrices:

1. single light-cone orbit represented by the light-like vector $\xi(1, 1, 0)$ which provides the solution of homogeneous classical YB equation (CYBE);

2. one-parameter family of space-like orbits represented by space-like vectors $\xi(1, 0, 0)$, $\xi \neq 0$ (solution of modified CYBE);

3. one-parameter families of time-like orbits represented by time-like vectors $\xi(0, 1, 0)$, $\xi \neq 0$ (solution of modified CYBE).

Three canonical $\mathfrak{o}(2, 1)$ r-matrices corresponding to three types of orbits take the form

$$r_\xi^0 = \xi(J_1 \wedge J_3 + J_1 \wedge J_2); \quad r_\xi^- = \xi J_3 \wedge J_2; \quad r_\xi^+ = \xi J_1 \wedge J_3 \,, \tag{A.6}$$

where in the first case one gets the same deformation for any value of the parameter $\xi \neq 0$, while in the remaining two cases different values of ξ lead to different Lie bialgebras. In this setting we get one reality condition and three different types of r-matrices representing nonequivalent bialgebra structures. However, for the quantization purposes we have used different perspective keeping one standard form of the classical r-matrix $r_{st} = E_+ \wedge E_-$ and changing reality conditions appropriately (see subsection 3.1 for details).

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B q-exponent and q-Hadamard formula

Our aim here is to introduce some formulas (mainly concerning a q-deformed Hadamard lemma), which were main tools for calculations presented in section 5.4.

Let A and B be two arbitrary elements of some quantum algebra and let $\exp_q(A)$ be a formal q-exponential

$$\exp_q(A) := \sum_{n\geq 0} \frac{A^n}{(n)_q!}\,, \qquad (n)_q! := (1)_q(2)_q\cdots(n)_q, \quad (n)_q = \frac{1-q^n}{1-q} \tag{B.1}$$

of the element A. As the q-exponential $\exp_{q^{-1}}(-A)$ is inverse to $\exp_q(A)$, i.e. $\big(\exp_q(A)\big)^{-1} = \exp_{q^{-1}}(-A)$ thus the q-analogue of Hadamard formula can be obtained as follows (see [25])

$$\begin{aligned} \exp_q(A)\,B\big(\exp_q(A)\big)^{-1} &= \exp_q(A)\,B\exp_{q^{-1}}(-A) \;\equiv\; \big(\mathrm{Ad}\exp_q(A)\big)(B) = \\ &= \left(\sum_{n\geq 0}\frac{1}{(n)_q!}(\mathrm{ad}_q\,A)^n\right)(B) = \big(\exp_q(\mathrm{ad}_q\,A)\big)(B)\,, \end{aligned} \tag{B.2}$$

where the q-adjoint action is defined by means of q-brackets ($[C,\,D]_q \equiv CD - qDC$):

$$\begin{aligned} &(\mathrm{ad}_q\,A)^0(B) \equiv B\,, \quad (\mathrm{ad}_q(A))^1(B) \equiv [A,\,B]\,, \quad (\mathrm{ad}_q(A))^2(B) \equiv [A,\,[A,\,B]]_q\,, \\ &(\mathrm{ad}_q(A))^3(B) \equiv [A,\,[A,\,[A,\,B]]_q]_{q^2}\,, \ldots, (\mathrm{ad}_q(A))^{n+1}(B) = [A,\,(\mathrm{ad}_q(A))^n(B)]_{q^n}\,. \end{aligned} \tag{B.3}$$

Considering the specific case ($q' \neq q$ in general)

$$[A,\,B]_{q'} = 0\,, \tag{B.4}$$

one gets

$$\begin{aligned} (\mathrm{ad}_q(A))^{n+1}(B) &= (1-q'^{-1}q^n)A\big(\mathrm{ad}_q(A)\big)^n(B) = \prod_{k=0}^{n}(1-q'^{-1}q^k)A^nB \\ &= (q'^{-1};q)_n\,A^nB \end{aligned} \tag{B.5}$$

using the standard notation $(a;q)_n$ from the theory of basic hypergeometric series (see e.g. [19, 87]). Substituting (B.5) in (B.2) we obtain

$$\begin{aligned} \exp_q(A)\,B\big(\exp_q(A)\big)^{-1} &= \left(\sum_{n=0}^{\infty}\frac{(q'^{-1};q)_n}{(q;q)_n}\,(1-q)^nA^n\right)B = \\ &= {}_1\phi_0(q'^{-1};-;q;(1-q)A)\,B = \frac{\big(q'^{-1}(1-q)A;q\big)_\infty}{((1-q)A;q)_\infty}\,B\,, \end{aligned} \tag{B.6}$$

as the result of the q-binomial theorem (see [19, 87]). In the particular case $q' = q^n$, $n = 0, 1, 2, \ldots$, the formula (B.2) reads

$$\begin{aligned} \exp_q(A)\,B(\exp_q(A))^{-1} &= \big(q^{-n}(1-q)A;q\big)_n\,B \\ &= q^{-n(n+1)/2}\big((q-1)A\big)^n\,\big(q/(1-q)A;q\big)_n\,B\,. \end{aligned} \tag{B.7}$$

In the case $q' = q^{-n}$, $n = 1, 2, \ldots$, for (B.2) we have

$$\exp_q(A)\, B(\exp_q(A))^{-1} = \big((1-q)A; q\big)_n^{-1} B\,. \tag{B.8}$$

Adopting to the situation in section 5.4 one has to substitute $A \to \mathbb{A} = \varsigma \mathtt{e}_+ q^{\mathtt{h}+\bar{\mathtt{h}}} \otimes q^{\mathtt{h}+\bar{\mathtt{h}}} \bar{\mathtt{e}}_+$ or $A = \varsigma \mathtt{e}_+ \bar{\mathtt{e}}_+$ and $n = 1, 2$ (cf. (5.32) or (5.34)).

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