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Full diffeomorphism and Lorentz invariance in 4D $\mathcal{N} = 1$ superfield description of 6D SUGRA

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Abstract: We complete the four-dimensional $\mathcal{N} = 1$ superfield description of six-dimensional supergravity. The missing ingredients in the previous works are the superfields that contain the sechsbein $e_4{}^{\nu}$, $e_5{}^{\nu}$, $e_{\mu}{}^{4}$, $e_{\mu}{}^{5}$ and the second gravitino. They are necessary to make the action invariant under the diffeomorphisms and the Lorentz transformations involving the extra dimensions. We find the corresponding superfield transformation laws, and show the invariance of the action under them. We also check that the resultant action reproduces the known superfield description of five-dimensional supergravity through the dimensional reduction.

Keywords: Field Theories in Higher Dimensions, Supergravity Models, Superspaces

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1 Introduction

When we consider higher-dimensional supersymmetric (SUSY) theories, it is useful to describe the action in terms of $\mathcal{N} = 1$ superfields [1]–[9] for various reasons.[1] It makes the expression of the action much more compact than the component field expression. In particular, the complicated spacetime index structures become much simpler. In higher than six dimensions (6D), however, the full superspace formulation is not known due to the extended SUSY structure. Even in such cases, the $\mathcal{N} = 1$ superfield expression is still possible because only partial SUSY structure is respected. Such an expression is useful to discuss a system in which the spacetime is compactified to four dimensions (4D) and the $\mathcal{N} = 1$ SUSY is preserved. We can derive the 4D effective action directly from the higher dimensional theory, keeping the $\mathcal{N} = 1$ superspace structure manifest. Especially, when the system contains lower dimensional branes or orbifold fixed points in the compactified space, the bulk-brane interactions are described in a transparent manner because all the sectors are expressed on the common $\mathcal{N} = 1$ superspace. Besides, the $\mathcal{N} = 1$ superfield formalism is familiar to many researchers, and is easy to handle.

For global SUSY theories, the $\mathcal{N} = 1$ superfield description of the action has been already provided in 5-10 dimensions [2]. We have to extend it to the supergravity (SUGRA) in order to discuss the moduli stabilization, the interactions to the moduli or the higher dimensional gravitational multiplet, and so on. However, such an extension is not straightforward. First, it is a nontrivial task to identify the component fields of the $\mathcal{N} = 1$ superfields. It usually happens that the non-gravitational fields form the superfields with the help of the gravitational fields, such as the vierbein and the gravitini. Of course, these superfields should reduce to the ones in ref. [2] if the gravitational fields are replaced with their background values in the flat spacetime. However, such an observation alone is not enough to identify the dependence of each component of the superfield on the gravitational fields. The complete identification can be achieved by requiring the invariance of the action under various symmetry transformations, such as the gauge transformations, the diffeomorphisms, the Lorentz transformations, etc. We should note that the diffeomorphisms and the Lorentz transformations have to be divided into the 4D parts and the extra-dimensional parts, and treated separately because we only respect the $\mathcal{N} = 1$ SUSY. The invariance under their 4D parts is obvious. In contrast, the invariances under the diffeomorphism in the extra dimensions and the Lorentz transformations that mix the 4D index with the extra-dimensional one are less trivial, but they are also expressed as the

[1] "$\mathcal{N} = 1$" denotes SUSY with four supercharges in this paper.

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$\mathcal{N}=1$ superfield transformations. Besides, we should also note that the $\mathcal{N}=1$ superconformal parameters[2] depend on the extra-dimensional coordinates, and that the desired superfield action involves the derivatives with respective to such coordinates. Therefore, we need to covariantize such derivatives. The corresponding connection superfields contain the "off-diagonal" components of the vierbein $e_\mu{}^{\underline{n}}$ and $e_{\underline{m}}{}^{\nu}$, where $\{\mu,\nu\}$ and $\{m,n\}$ denote the 4D and the extra-dimensional indices, respectively.

The simplest background for the extra-dimensional models is the five-dimensional (5D) spacetime. The $\mathcal{N}=1$ description of the 5D SUGRA action is provided in refs. [8, 9]. These works specify the dependence of the action on the "modulus" superfields that contains the extra-dimensional component of the fünfbein $e_4{}^{\underline{4}}$. This superfield description makes it possible to derive the 4D effective action for various setups systematically [15–18]. However, the superfield action in refs. [8, 9] does not contain the "off-diagonal" components of the fünfbein $e_\mu{}^{\underline{4}}$, $e_4{}^{\underline{\nu}}$ and their $\mathcal{N}=1$ SUSY partners. Thus, the action is not invariant under the diffeomorphism in the extra dimension and the Lorentz transformations that mix the 4D and the fifth dimensions. Those missing ingredients are incorporated at the linearized level in ref. [19], and play an important role in the calculation of the one-loop effective potential [20–22].

In this paper, we focus on 6D SUGRA [23–25]. The 6D spacetime is the next simplest setup for the extra-dimensional models, and the minimal setup where the shape modulus for the extra-dimensional space appears. 6D SUGRA generically contains the Weyl multiplet as the gravitational multiplet, and $n_{\rm H}$ hypermultiplets, $n_{\rm V}$ vector multiplets and $n_{\rm T}$ tensor multiplets as the mattter multiplets. From the anomaly cancellation condition, the numbers of the multiplets are constrained by $29n_{\rm T}+n_{\rm H}-n_{\rm V}=273$ [26–28]. In contrast to 5D SUGRA, the Weyl multiplet contains the anti-self-dual tensor field T^-_{MN} ($M,N=0,1,\cdots,5$), and a 6D tensor multiplet contains the self-dual tensor field B^+_{MN}. In general, the (anti-)self-dual condition is an obstacle to the Lagrangian formulation, similar to that of type IIB SUGRA. However, when $n_{\rm T}=1$, this difficulty can be solved because we can construct an unconstrained tensor field B_{MN} by combining T^-_{MN} with B^+_{MN} [25, 29]. When $n_{\rm T}\neq 1$, the (anti-)self-dual conditions remain, and thus the theory cannot be described by the Lagrangian. Hence, we focus on the case of $n_{\rm T}=1$ in this paper.

In our previous work [30], we found the $\mathcal{N}=1$ superfield description of the vector-tensor couplings in 6D global SUSY theories, which is derived from the invariant action [31] in the projective superspace [32–34].[3] Then, we extend this result to 6D SUGRA in ref. [38] by identifying the "moduli superfields" that contain the extra-dimensional components of the sechsbein $e_{\underline{m}}{}^{\underline{n}}$ ($m,n=4,5$), and inserting them into the result in ref. [30]. We have checked that the resultant action is invariant under the supergauge transformation, and reproduces the known 5D SUGRA action after the dimensional reduction. In this paper, we complete the $\mathcal{N}=1$ superfield description of 6D SUGRA by incorporating the missing ingredients, i.e., the "off-diagonal" components of the sechsbein $e_\mu{}^{\underline{n}}$ and $e_{\underline{m}}{}^{\nu}$ ($m,n=4,5$) and their $\mathcal{N}=1$ superpartners. The identification of the corresponding superfields and

[2] 4D $\mathcal{N}=1$ SUGRA can be described by the superconformal formulation [10–12], which is also expressed by the corresponding superspace formulation [13, 14].

[3] 6D projective superspace is also discussed in refs. [35–37].

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the dependence of the action on them are determined by the invariance under the full 6D diffeomorphisms. These newly incorporated superfields, which are the real superfields U^m and the spinor superfields Ψ^α_m $(m = 4, 5)$, are also necessary for the invariance under the Lorentz transformations that mix the 4D and the extra-dimensional indices. This work corresponds to the 6D extension of ref. [19]. We will treat the 4D $\mathcal{N} = 1$ SUGRA part at the linearized level for a technical reason. Due to this approximation, we can only determine the dependence of the action on Ψ^α_m at the linearized level. In contrast, we clarify the dependence on U^m at the full order[4] because it is determined only by the invariance under diffeomorphisms in the extra dimensions, independently of the 4D diffeomorphism.

The paper is organized as follows. We provide a brief review of our previous work [38] in the next section. In section 3, we require the invariance of the action under the diffeomorphisms in the extra dimensions, and introduce the connection superfields U^m $(m = 4, 5)$ that contain the "off-diagonal" components of the sechsbein. In section 4, we covariantize the derivatives with respective to the extra-dimensional coordinates by introducing another connection superfields Ψ^α_m $(m = 4, 5)$. In section 5, we address the Lorentz transformations that mix the 4D and the extra-dimensional indices, and show the invariance of the action under them. In section 6, we check that the resultant superfield action of 6D SUGRA reduces to the known 5D SUGRA action after the dimensional reduction. Section 7 is devoted to the summary. In appendix A, we collect the results of ref. [14] that discusses the 4D linearized SUGRA and the superfield description of the $\mathcal{N} = 1$ superconformal transformation. In appendices B and C, we show the diffeomorphisms and the Lorentz transformations in the component field expression, and provide the correspondence to the superfield description.

2 Review of our previous work

The 6D spacetime indices $M, N, \cdots = 0, 1, 2, \cdots, 5$ are divided into the 4D part $\mu, \nu, \cdots = 0, 1, 2, 3$ and the extra-dimensional part $m, n, \cdots = 4, 5$. The corresponding local Lorentz indices are denoted by the underbarred ones. We assume that the 4D part of the spacetime has the flat background geometry, and follow the notation of ref. [40] for the 2-component spinors.

2.1 $\mathcal{N} = 1$ decomposition of 6D supermultiplets

The 6D Weyl multiplet $\mathbb{E}$ consists of the sechsbein $e_M^{\ \underline{N}}$, the gravitino $\psi^i_{M\bar{\alpha}}$, the $\mathrm{SU}(2)_{\mathbf{U}}$ (auxiliary) gauge fields V^{ij}_M, and the other auxiliary fields, where $\bar{\alpha}$ is a 6D spinor index, and $i, j = 1, 2$ are the $\mathrm{SU}(2)_{\mathbf{U}}$-doublet indices. The gravitino has the 6D chirality $+$, and is the $\mathrm{SU}(2)_{\mathbf{U}}$-Majorana-Weyl fermion, which can be decomposed into the two 4D Dirac fermions.

$$\psi^1_M = \begin{pmatrix} \psi^+_{M\alpha} \\ \bar{\psi}^{-\dot{\alpha}}_M \end{pmatrix}, \qquad \psi^2_M = \begin{pmatrix} -\psi^-_{M\alpha} \\ \bar{\psi}^{+\dot{\alpha}}_M \end{pmatrix}, \tag{2.1}$$

where $\alpha, \dot{\alpha} = 1, 2$ are the 2-component spinor indices. If we choose ϵ^+_α and $\bar{\epsilon}^{+\dot{\alpha}}$ in the 6D SUSY transformation parameter $\epsilon^i_{\bar{\alpha}}$ as the 4D $\mathcal{N} = 1$ SUSY one we respect, the

[4] Some of the U^m-dependent terms are treated at the linearized level due to technical difficulties.

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fields $\left\{e_\mu{}^{\underline{\nu}}, \psi_\mu^+, \cdots\right\}$ form the 4D Weyl multiplet. We can construct the real superfield U^μ from them as (see appendix A)

$$U^\mu = (\theta\sigma^\nu\bar{\theta})\tilde{e}_\nu{}^\mu + i\bar{\theta}^2\left(\theta\sigma^\nu\bar{\sigma}^\mu\psi_\nu^+\right) - i\theta^2\left(\bar{\theta}\bar{\sigma}^\nu\sigma^\mu\bar{\psi}_\nu^+\right) + \cdots, \tag{2.2}$$

where $\tilde{e}_\nu{}^\mu$ is the fluctuation field around the background defined as (A.1), and

$$\sigma^\mu \equiv \langle e_{\underline{\nu}}{}^\mu\rangle\sigma^{\underline{\nu}}, \qquad \bar{\sigma}^\mu \equiv \langle e_{\underline{\nu}}{}^\mu\rangle\bar{\sigma}^{\underline{\nu}}. \tag{2.3}$$

Note that we need not discriminate the flat and the curved 4D indices for $\tilde{e}_\nu{}^\mu$ at the linearized order since the 4D part of the background spacetime is assumed to be flat ($\langle e_\nu{}^{\underline{\mu}}\rangle = \delta_\nu{}^\mu$). As explicitly shown in appendix A.3, once the matter action is given, we can always obtain its 4D gravitational couplings. Thus, we will omit the dependences on U^μ to simplify the expressions in the following.

In our previous work [38], we have found that the extra dimensional components of the sechsbein $e_m{}^{\underline{n}}$ and its superpartners form the chiral superfield $S_{\rm E}$ and the real general superfield $V_{\rm E}$ as

$$\begin{aligned} S_{\rm E} &= \sqrt{\frac{E_4}{E_5}} + \mathcal{O}(\theta), \\ V_{\rm E} &= e^{(2)} + \mathcal{O}(\theta), \end{aligned} \tag{2.4}$$

where $E_m \equiv e_m{}^{\underline{4}} + ie_m{}^{\underline{5}}$ and $e^{(2)} \equiv \det(e_m{}^{\underline{n}}) = e_4{}^{\underline{4}}e_5{}^{\underline{5}} - e_4{}^{\underline{5}}e_5{}^{\underline{4}}$. These correspond to the shape and the volume moduli, respectively.

The matter field content consists of hypermultiplets $\mathbb{H}^A$ ($A = 1, 2, \cdots, n_{\rm H}$), vector multiplets $\mathbb{V}^I$ ($I = 1, 2, \cdots, n_{\rm V}$),[5] and a tensor multiplet $\mathbb{T}$. They are decomposed into $\mathcal{N} = 1$ superfields as

$$\mathbb{H}^A = (H^{2A-1}, H^{2A}), \qquad \mathbb{V}^I = (V^I, \Sigma^I), \qquad \mathbb{T} = (\Upsilon_{{\rm T}\alpha}, V_{\rm T4}, V_{\rm T5}), \tag{2.5}$$

where $H^{2A-1}, H^{2A}, \Sigma^I, \Upsilon_{{\rm T}\alpha}$ are chiral superfields, and V^I, $V_{\rm T4}$ and $V_{\rm T5}$ are real superfields. Here, $\mathbb{H}^A$ contains the hyperscalars $(\phi_i^{2A-1}, \phi_i^{2A})$, which is subject to the reality condition: $\left(\phi_1^{2A-1}\right)^* = \phi_2^{2A}$, $\left(\phi_1^{2A}\right)^* = -\phi_2^{2A-1}$, $\mathbb{V}^I$ contains a 6D vector field A_M^I, and $\mathbb{T}$ contains a real scalar field σ and an anti-symmetric tensor field B_{MN}. The hypermultiplets $\mathbb{H}^A$ are divided into the compensator multiplets $A = 1, 2, \cdots, n_{\rm comp}$ and the physical ones $A = n_{\rm comp} + 1, \cdots, n_{\rm comp} + n_{\rm phys}$. The lowest bosonic components of the superfields are[6]

$$\begin{aligned} H^{\bar{A}} &= (E_4E_5)^{1/4}\,\phi_2^{\bar{A}} + \mathcal{O}(\theta), \\ V^I &= -(\theta\sigma^{\underline{\mu}}\bar{\theta})A_{\underline{\mu}}^I + \mathcal{O}(\theta^3), \\ \Sigma^I &= \frac{i}{2}\left(\frac{1}{S_{\rm E}|}A_4^I - S_{\rm E}|A_5^I\right) + \mathcal{O}(\theta), \end{aligned}$$

[5] The anomaly cancellation conditions constrain the numbers of the multiplets (see the introduction) and the gauge group [26–28]. In this paper, we do not consider such constraints, and assume that the gauge groups are Abelian, for simplicity.

[6] The factor $i/2$ was missing for the lowest component of Σ^I in ref. [38]. Besides, $V_{{\rm T}m} = -8X_m$ ($m = 4, 5$) and $\Upsilon_{{\rm T}\alpha} = 8\bar{D}^2Y_\alpha$ in the notation of ref. [38].

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$$\Upsilon_{\mathrm{T}\alpha} = -\theta_\alpha \left(2B_{\underline{45}} + i\sigma\right) - 2i\left(\sigma^{\underline{\mu\nu}}\theta\right)_\alpha B_{\underline{\mu\nu}} + \mathcal{O}(\theta^2),$$
$$V_{\mathrm{T}m} = -2(\theta\sigma^{\underline{\mu}}\bar{\theta})B_{\underline{\mu}m} + \mathcal{O}(\theta^3), \qquad (m = 4, 5) \tag{2.6}$$

where $\bar{A} = 2A - 1, 2A$, and $S_{\mathrm{E}}| = \sqrt{E_4/E_5}$ is the lowest component of S_E.

The supergauge transformations are given by

$$\delta_\Lambda V^I = \Lambda^I + \bar{\Lambda}^I, \qquad \delta_\Lambda \Sigma^I = \partial_{\mathrm{E}}\Lambda^I, \tag{2.7}$$

where the transformation parameters Λ^I are chiral superfields, and

$$\partial_{\mathrm{E}} \equiv \frac{1}{S_{\mathrm{E}}}\partial_4 - S_{\mathrm{E}}\partial_5. \tag{2.8}$$

The gauge-invariant field strength superfields are given by

$$\mathcal{W}^I_\alpha \equiv -\frac{1}{4}\bar{D}^2 D_\alpha V^I. \tag{2.9}$$

The SUSY extension of the tensor gauge transformation: $B_{MN} \to B_{MN} + \partial_M\lambda_N - \partial_N\lambda_M$ (λ_M: real parameter) is expressed as

$$\delta_{\mathrm{G}} V_{\mathrm{T}4} = -\partial_4 V_{\mathrm{G}} + \mathrm{Re}\left(S_{\mathrm{E}}\Sigma_{\mathrm{G}}\right), \qquad \delta_{\mathrm{G}} V_{\mathrm{T}5} = -\partial_5 V_{\mathrm{G}} + \mathrm{Re}\left(\frac{\Sigma_{\mathrm{G}}}{S_{\mathrm{E}}}\right),$$
$$\delta_{\mathrm{G}}\Upsilon_{\mathrm{T}\alpha} = -\frac{1}{4}\bar{D}^2 D_\alpha V_{\mathrm{G}}, \tag{2.10}$$

where the transformation parameters V_{G} and Σ_{G} are a real and a chiral superfields respectively, which form a 6D vector multiplet $\mathbb{V}_{\mathrm{G}}$.

$$V_{\mathrm{G}} = -2(\theta\sigma^{\underline{\mu}}\bar{\theta})\lambda_{\underline{\mu}} + \mathcal{O}(\theta^3),$$
$$\Sigma_{\mathrm{G}} = \frac{2\left|S_{\mathrm{E}}\right|^2}{\mathrm{Im}\, S^2_{\mathrm{E}}|}\left(\frac{1}{\bar{S}_{\mathrm{E}}|}\lambda_4 - \bar{S}_{\mathrm{E}}|\lambda_5\right) + \mathcal{O}(\theta). \tag{2.11}$$

The superfields other than $\mathbb{T}$ are neutral. The field strengths invariant under this transformation are

$$\mathcal{X}_{\mathrm{T}} \equiv \frac{1}{2}\mathrm{Im}\left(D^\alpha\Upsilon_{\mathrm{T}\alpha}\right),$$
$$\mathcal{Y}_{\mathrm{T}\alpha} \equiv \frac{1}{2S_{\mathrm{E}}}\mathcal{W}_{\mathrm{T}4\alpha} + \frac{S_{\mathrm{E}}}{2}\mathcal{W}_{\mathrm{T}5\alpha} + \frac{1}{2}S_{\mathrm{E}}\mathcal{O}_{\mathrm{E}}\Upsilon_{\mathrm{T}\alpha}, \tag{2.12}$$

where

$$\mathcal{W}_{\mathrm{T}m\alpha} \equiv -\frac{1}{4}\bar{D}^2 D_\alpha V_{\mathrm{T}m}, \qquad (m = 4, 5)$$
$$\mathcal{O}_{\mathrm{E}} \equiv \frac{1}{S^2_{\mathrm{E}}}\partial_4 + \partial_5. \tag{2.13}$$

Namely, $\mathcal{X}_{\mathrm{T}}$ and $\mathcal{Y}_{\mathrm{T}\alpha}$ are real linear and chiral superfields, respectively. The tensor multiplet $(\Upsilon_{\mathrm{T}\alpha}, V_{\mathrm{T}m})$ is subject to the constraints:

$$\frac{1}{S_{\mathrm{E}}}\mathcal{W}_{\mathrm{T}4\alpha} - S_{\mathrm{E}}\mathcal{W}_{\mathrm{T}5\alpha} + \partial_{\mathrm{E}}\Upsilon_{\mathrm{T}\alpha} = 0,$$
$$\bar{D}^2 D_\alpha\left(\mathcal{X}_{\mathrm{T}}V_{\mathrm{E}}\right) = -4\left\{\partial_E\mathcal{Y}_{\mathrm{T}\alpha} - (\mathcal{O}_{\mathrm{E}}S_{\mathrm{E}})\mathcal{Y}_{\mathrm{T}\alpha}\right\}. \tag{2.14}$$

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In the global SUSY limit, these constraints reduce to the superfield version of the self-dual condition:

$$\partial_{[M} B^{+}_{NL]} = \frac{1}{6}\epsilon_{MNLPQR}\partial^{P} B^{+QR}. \tag{2.15}$$

In fact, in the limit of $S_{\rm E} \to e^{-i\pi/4}$ and $V_{\rm E} \to 1$, (2.14) is reduced to

$$\begin{aligned} &\mathcal{W}_{\rm T4\alpha} + i\mathcal{W}_{\rm T5\alpha} + (\partial_4 + i\partial_5)\,\Upsilon_{\rm T\alpha} = 0, \\ &\bar{D}^2 D_\alpha \mathcal{X}_{\rm T} = -4e^{i\pi/4}\,(\partial_4 + i\partial_5)\,\mathcal{Y}_{\rm T\alpha}. \end{aligned} \tag{2.16}$$

The field strength superfield $\mathcal{Y}_{\rm T\alpha}$ becomes

$$\begin{aligned} \mathcal{Y}_{\rm T\alpha} &= \frac{e^{i\pi/4}}{2}\left\{\mathcal{W}_{\rm T4\alpha} - i\mathcal{W}_{\rm T5\alpha} + (\partial_4 - i\partial_5)\,\Upsilon_{\rm T\alpha}\right\} \\ &= e^{i\pi/4}\left(\mathcal{W}_{\rm T4\alpha} + \partial_4\Upsilon_{\rm T\alpha}\right) = e^{-i\pi/4}\left(\mathcal{W}_{\rm T5\alpha} + \partial_5\Upsilon_{\rm T\alpha}\right). \end{aligned} \tag{2.17}$$

In the second line, we have used the first constraint in (2.16). From these expressions, we obtain

$$D^\alpha \mathcal{Y}_{\rm T\alpha} = -2e^{-i\pi/4}\,(\partial_4 - i\partial_5)\,\mathcal{X}_{\rm T}. \tag{2.18}$$

This and the second constraint in (2.16) contain the self-dual condition (2.15). Thus, the antisymmetric tensor B_{MN} in $\Upsilon_{\rm T\alpha}$ and $V_{{\rm T}m}$ becomes the self-dual tensor B^{+}_{MN} in the global SUSY limit.

In the SUGRA case, the second constraint in (2.14) can be solved as follows. Using the first constraint in (2.14), $\mathcal{Y}_{\rm T\alpha}$ can be expressed as

$$\begin{aligned} \mathcal{Y}_{\rm T\alpha} &= \frac{1}{S_{\rm E}}\left(\mathcal{W}_{\rm T4\alpha} + \partial_4\Upsilon_{\rm T\alpha}\right) \\ &= S_{\rm E}\left(\mathcal{W}_{\rm T5\alpha} + \partial_5\Upsilon_{\rm T\alpha}\right). \end{aligned} \tag{2.19}$$

Thus the second constraint in (2.14) is rewritten as

$$\begin{aligned} \bar{D}^2 D_\alpha\left(\mathcal{X}_{\rm T} V_{\rm E}\right) &= -4\partial_4\left(\frac{\mathcal{Y}_{\rm T\alpha}}{S_{\rm E}}\right) + 4\partial_5\left(S_{\rm E}\mathcal{Y}_{\rm T\alpha}\right) \\ &= -4\partial_4\left(\mathcal{W}_{\rm T5\alpha} + \partial_5\Upsilon_{\rm T\alpha}\right) + 4\partial_5\left(\mathcal{W}_{\rm T4\alpha} + \partial_4\Upsilon_{\rm T\alpha}\right) \\ &= \bar{D}^2 D_\alpha\left(\partial_4 V_{\rm T5} - \partial_5 V_{\rm T4}\right), \end{aligned} \tag{2.20}$$

which can be solved as

$$\mathcal{X}_{\rm T} V_{\rm E} = \partial_4 V_{\rm T5} - \partial_5 V_{\rm T4} + \Sigma_{\rm T} + \bar{\Sigma}_{\rm T} \equiv \mathcal{V}_{\rm T}, \tag{2.21}$$

where $\Sigma_{\rm T}$ is a chiral superfield. The lowest component of $\Sigma_{\rm T}$ is identified as

$$\Sigma_{\rm T}| = \frac{1}{2}e^{(2)}\sigma - iB_{45}. \tag{2.22}$$

eq. (2.21) indicates that the "volume modulus" superfield $V_{\rm E}$ is expressed by $\Upsilon_{\rm T\alpha}$, $V_{{\rm T}m}$ and $\Sigma_{\rm T}$, and is not an independent degree of freedom.

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2.2 Invariant action

The $\mathcal{N}=1$ superfield description of (the U^μ-independent part of) the 6D SUGRA action provided in ref. [38] is

$$
\begin{aligned}
S &= \int d^6x\ (\mathcal{L}_{\rm H} + \mathcal{L}_{\rm VT})\,, \\
\mathcal{L}_{\rm H} &= -\int d^4\theta\ 2V_{\rm E}^{1/2} R_{\rm E}^{1/2} \left(H_{\rm odd}^\dagger \tilde{d} e^V H_{\rm odd} + H_{\rm even}^\dagger \tilde{d} e^{-V} H_{\rm even} \right) \\
&\quad + \left[\int d^2\theta\ \left\{ H_{\rm odd}^t \tilde{d} \left(\partial_{\rm E} - \Sigma\right) H_{\rm even} - H_{\rm even}^t \tilde{d} \left(\partial_{\rm E} + \Sigma\right) H_{\rm odd} \right\} + {\rm h.c.} \right], \\
\mathcal{L}_{\rm VT} &= \int d^4\theta\ f_{IJ} \left[\left\{ -2\Sigma^I D^\alpha V^J \mathcal{Y}_{{\rm T}\alpha} + \frac{1}{2} \left(\partial_{\rm E} V^I D^\alpha V^J - \partial_{\rm E} D^\alpha V^I V^J \right) \mathcal{Y}_{{\rm T}\alpha} + {\rm h.c.} \right\} \right. \\
&\qquad + \mathcal{X}_{\rm T} V_{\rm E} \left(D^\alpha V^I \mathcal{W}_\alpha^J + \frac{1}{2} V^I D^\alpha \mathcal{W}_\alpha^J + {\rm h.c.} \right) \\
&\qquad + \frac{\mathcal{X}_{\rm T}}{R_{\rm E}} \left\{ 4 \left(\bar{\partial}_{\rm E} V^I - \bar{\Sigma}^I \right) \left(\partial_{\rm E} V^J - \Sigma^J \right) - 2 \bar{\partial}_{\rm E} V^I \partial_{\rm E} V^J \right. \\
&\qquad \left. \left. + \frac{2S_{\rm E}}{\bar{S}_{\rm E}} \Sigma^I \Sigma^J + \frac{2\bar{S}_{\rm E}}{S_{\rm E}} \bar{\Sigma}^I \bar{\Sigma}^J \right\} \right],
\end{aligned}
\tag{2.23}
$$

where $H_{\rm odd} = (H^1, H^3, H^5, \cdots)^t$, $H_{\rm even} = (H^2, H^4, H^6, \cdots)^t$, $\tilde{d} = {\rm diag}(\mathbf{1}_{n_{\rm comp}}, -\mathbf{1}_{n_{\rm phys}})$ is the metric of the hyperscalar space that discriminates the compensator multiplets from the physical ones,[7] $f_{IJ} = f_{JI}$ are real constants, and[8]

$$
R_{\rm E} \equiv {\rm Im}\, \frac{\bar{S}_{\rm E}}{S_{\rm E}}, \qquad V \equiv t_I V^I, \qquad \Sigma \equiv t_I \Sigma^I. \tag{2.24}
$$

The matrices t_I are the generators for the Abelian gauge groups.

The above action is invariant under the gauge transformation:

$$
\begin{aligned}
H_{\rm odd} &\to e^{-\Lambda} H_{\rm odd}, & H_{\rm even} &\to e^{\Lambda} H_{\rm even}, & (\Lambda \equiv t_I \Lambda^I) \\
V^I &\to V^I + \Lambda^I + \bar{\Lambda}^I, & \Sigma^I &\to \Sigma^I + \partial_{\rm E} \Lambda^I,
\end{aligned}
\tag{2.25}
$$

where Λ^I are chiral superfields, and the other superfields are neutral. We should also note that (2.23) becomes the 5D SUGRA action in refs. [8, 9] with the norm function: $\mathcal{N}(X) = f_{IJ} X^I X^J X^T$ (the index T denotes the 5D vector multiplet originated from the 6D tensor multiplet) after the dimensional reduction.

We list the Weyl weights of the $\mathcal{N}=1$ superfields in table 1.

3 Diffeomorphism invariance in extra dimensions

Now we modify (2.23) by introducing the "off-diagonal" components of the 6D Weyl multiplet. For this purpose, we require the action to be invariant under the diffeomorphism in

[7] In contrast to 4D SUGRA, an arbitrary number of the compensators is possible in 5D and 6D SUGRAs. When $n_{\rm comp} > 1$, the superconformal gauge-fixing conditions cannot eliminate all the degrees of freedom of the compensators. So some auxiliary multiplets are necessary to eliminate them. (See ref. [39], for example.) The number $n_{\rm comp}$ determines the geometry of the space spanned by the physical hyperscalars.

[8] $R_{\rm E}$ is denoted as $U_{\rm E}^2$ in ref. [38].

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$\mathbb{E}$					$\mathbb{H}^A$	$\mathbb{V}^I$		$\mathbb{T}$			field strength			
U^μ	U^m	Ψ^α_m	$S_{\rm E}$	$V_{\rm E}$	$H^{\bar{A}}$	V^I	Σ^I	$\Upsilon_{{\rm T}\alpha}$	$V_{{\rm T}m}$	$\Sigma_{\rm T}$	$\mathcal{W}^I_\alpha$	$\mathcal{X}_{\rm T}$	$\mathcal{Y}_{{\rm T}\alpha}$	$\mathcal{V}_{\rm T}$
0	0	$-3/2$	0	-2	$3/2$	0	0	$3/2$	0	0	$3/2$	2	$3/2$	0

Table 1. The Weyl weights of the $\mathcal{N}=1$ superfields. The 4D gravitational superfield U^μ is explained in appendix A, and the "off-diagonal" gravitational superfields U^m and Ψ^α_m are introduced in section 3 and section 4, respectively.

the extra dimensions, i.e., $\delta_\xi x^m = \xi^m$. The component field transformations are collected in appendix B. It should be noted that we now have to discriminate the flat and the curved 4D indices even for the flat 4D background.

3.1 Hyper sector

3.1.1 Chiral superspace

First, we focus on the chiral superspace in the hypersector.

In the $\mathcal{N}=1$ chiral superspace, the transformation parameters ξ^m are promoted to the chiral superfields as

$$\Xi^m(x,\theta) = \xi^m(x) + ia^m(x) + \mathcal{O}(\theta), \tag{3.1}$$

where a^m are real functions. From (B.3), (B.11) and (B.13), the chiral superfields $S_{\rm E}$, $H_{\rm odd}$, $H_{\rm even}$ and Σ^I transform as

$$\begin{aligned}
\delta_\Xi S_{\rm E} &= \Xi^m\partial_m S_{\rm E} + \frac{1}{2}\left(\partial_4\Xi^4 - \partial_5\Xi^5 + \frac{1}{S_{\rm E}^2}\partial_4\Xi^5 - S_{\rm E}^2\partial_5\Xi^4\right) S_{\rm E},\\
\delta_\Xi H &= \Xi^m\partial_m H + \frac{1}{4}\left(\partial_m\Xi^m + \frac{1}{S_{\rm E}^2}\partial_4\Xi^5 + S_{\rm E}^2\partial_5\Xi^4\right) H,\\
\delta_\Xi \Sigma^I &= \Xi^m\partial_m \Sigma^I + \frac{1}{2}\left(\partial_m\Xi^m - \frac{1}{S_{\rm E}^2}\partial_4\Xi^5 - S_{\rm E}^2\partial_5\Xi^4\right) \Sigma^I,
\end{aligned} \tag{3.2}$$

where $H = H_{\rm odd}, H_{\rm even}$. Because the first terms in the right-hand sides correspond to the shift of the coordinates x^m, they have the universal structure for all the chiral superfields. In fact, noticing that

$$\begin{aligned}
\delta_\Xi(\partial_{\rm E} H) &= -(\delta_\Xi S_{\rm E})\mathcal{O}_{\rm E} H + \partial_{\rm E}\left(\delta_\Xi H\right)\\
&= \Xi^m\partial_m\left(\partial_{\rm E} H\right) + \frac{1}{4}\left(3\partial_m\Xi^m - \frac{1}{S_{\rm E}^2}\partial_4\Xi^5 - S_{\rm E}^2\partial_5\Xi^4\right)\partial_{\rm E} H\\
&\quad + \frac{1}{4}\partial_{\rm E}\left(\partial_m\Xi^m + \frac{1}{S_{\rm E}^2}\partial_4\Xi^5 + S_{\rm E}^2\partial_5\Xi^4\right) H,
\end{aligned} \tag{3.3}$$

we can show that the chiral superspace part of the action (2.23), i.e., the second line of $\mathcal{L}_{\rm H}$, is invariant under (3.2) up to total derivatives.

$$\delta_\Xi L^{(1)}_{\rm H} = \partial_m\left(\Xi^m L^{(1)}_{\rm H}\right), \tag{3.4}$$

where

$$L^{(1)}_{\rm H} \equiv H^t_{\rm odd}\tilde{d}\left(\partial_{\rm E} - \Sigma\right) H_{\rm even} - H^t_{\rm even}\tilde{d}\left(\partial_{\rm E} + \Sigma\right) H_{\rm odd}. \tag{3.5}$$

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3.1.2 Full superspace

Next we consider the invariance in the full superspace. There, terms originating from the shift of x^m in the δ_Ξ-transformation should have a common form for all superfields. However, those for the chiral and the anti-chiral superfields have different forms. In order to accommodate them, we introduce the real superfields U^m $(m = 4, 5)$, and introduce the operator $\mathcal{P}_U$ that shifts x^m by iU^m.

$$\mathcal{P}_U : x^m \to x^m + iU^m(x, \theta, \bar{\theta}) \tag{3.6}$$

Then, for a chiral superfield Φ (i.e., $\delta_\Xi \Phi = \Xi^m \partial_m \Phi + \cdots$),

$$\hat{\Phi}(x, \theta, \bar{\theta}) \equiv \mathcal{P}_U \Phi(x, \theta, \bar{\theta}) = \Phi(x^\mu, x^m + iU^m(x, \theta, \bar{\theta}), \theta, \bar{\theta}) \tag{3.7}$$

transforms as[9]

$$\begin{aligned} \delta_\Xi \hat{\Phi} &= \widehat{\delta_\Xi \Phi} + i(\delta_\Xi U^m)\widehat{\partial_m \Phi} \\ &= \hat{\Xi}^m \widehat{\partial_m \Phi} + i(\delta_\Xi U^m)\widehat{\partial_m \Phi} + \cdots \\ &= (\mathrm{Re}\, \hat{\Xi}^m)\partial_m \hat{\Phi} + \cdots , \end{aligned} \tag{3.8}$$

if we assume that

$$\delta_\Xi U^m = -\mathrm{Im}\, \hat{\Xi}^m + (\mathrm{Re}\, \hat{\Xi}^n)\partial_n U^m. \tag{3.9}$$

Since U^m transform nonlinearly, these correspond to the gauge fields for the δ_Ξ-transformation. The components of U^m are identified as

$$\begin{aligned} U^m = &(\theta\sigma^{\underline{\mu}}\bar{\theta}) e_{\underline{\mu}}{}^m - \bar{\theta}^2 (\theta\sigma^{\underline{\mu}} \bar{\psi}^-_{\underline{\mu}}) \left(e_{\underline{4}}{}^m + i e_{\underline{5}}{}^m \right) \\ &+ \theta^2 \left(\bar{\theta} \bar{\sigma}^{\underline{\mu}} \psi^-_{\underline{\mu}} \right) \left(e_{\underline{4}}{}^m - i e_{\underline{5}}{}^m \right) + \cdots . \end{aligned} \tag{3.10}$$

Then, (3.9) is consistent with the component transformation (B.8).[10]

For an anti-chiral superfield $\bar{\Phi}$,

$$\hat{\bar{\Phi}}(x, \theta, \bar{\theta}) \equiv \bar{\mathcal{P}}_U \bar{\Phi}(x, \theta, \bar{\theta}) = \bar{\Phi}(x^\mu, x^m - iU^m(x, \theta, \bar{\theta}), \theta, \bar{\theta}) \tag{3.11}$$

transforms as

$$\delta_\Xi \hat{\bar{\Phi}} = (\mathrm{Re}\, \hat{\Xi}^m)\partial_m \hat{\bar{\Phi}} + \cdots , \tag{3.12}$$

which has the same form as (3.8).

With the $\mathcal{P}_U$ operation, (3.2) becomes

$$\begin{aligned} \delta_\Xi \hat{S}_{\mathrm{E}} &= (\mathrm{Re}\, \hat{\Xi}^m)\partial_m \hat{S}_{\mathrm{E}} + \frac{1}{2} \left(\widehat{\partial_4 \Xi^4} - \widehat{\partial_5 \Xi^5} + \frac{1}{\hat{S}^2_{\mathrm{E}}} \widehat{\partial_4 \Xi^5} - \hat{S}^2_{\mathrm{E}} \widehat{\partial_5 \Xi^4} \right) \hat{S}_{\mathrm{E}}, \\ \delta_\Xi \hat{H} &= (\mathrm{Re}\, \hat{\Xi}^m)\partial_m \hat{H} + \frac{1}{4} \left(\widehat{\partial_m \Xi^m} + \frac{1}{\hat{S}^2_{\mathrm{E}}} \widehat{\partial_4 \Xi^5} + \hat{S}^2_{\mathrm{E}} \widehat{\partial_5 \Xi^4} \right) \hat{H}, \\ \delta_\Xi \hat{\Sigma}^I &= (\mathrm{Re}\, \hat{\Xi}^m)\partial_m \hat{\Sigma}^I + \frac{1}{2} \left(\widehat{\partial_m \Xi^m} - \frac{1}{\hat{S}^2_{\mathrm{E}}} \widehat{\partial_4 \Xi^5} - \hat{S}^2_{\mathrm{E}} \widehat{\partial_5 \Xi^4} \right) \hat{\Sigma}^I. \end{aligned} \tag{3.13}$$

[9] Note that $\partial_m \hat{\Phi} = \widehat{\partial_m \Phi} + i \partial_m U^n \widehat{\partial_n \Phi}$.

[10] As we will explain in section 3.1.3, the $\theta\bar{\theta}$-component of $\mathrm{Im}\, \hat{\Xi}^m$ is $(\theta\sigma^{\underline{\mu}}\bar{\theta})\partial_{\underline{\mu}} \xi^m$.

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From (2.6) and (B.12), the δ_Ξ-transformation of the vector superfield V^I is found to be

$$\delta_\Xi V^I = (\mathrm{Re}\,\hat{\Xi}^m)\partial_m V^I. \tag{3.14}$$

Therefore, the combination

$$L_{\mathrm{H}}^{(2)} \equiv \hat{H}_{\mathrm{odd}}^\dagger \tilde{de}^V \hat{H}_{\mathrm{odd}} + \hat{H}_{\mathrm{even}}^\dagger \tilde{de}^{-V} \hat{H}_{\mathrm{even}} \tag{3.15}$$

in the first line of $\mathcal{L}_{\mathrm{H}}$ transforms as

$$\delta_\xi L_{\mathrm{H}}^{(2)} = (\mathrm{Re}\,\hat{\Xi}^m)\partial_m L_{\mathrm{H}}^{(2)} + \frac{1}{2}\mathrm{Re}\left(\widehat{\partial_m \Xi^m} + \frac{1}{\hat{S}_{\mathrm{E}}^2}\widehat{\partial_4 \Xi^5} + \hat{S}_{\mathrm{E}}^2 \widehat{\partial_5 \Xi^4}\right) L_{\mathrm{H}}^{(2)}. \tag{3.16}$$

As for the factor in front of $L_{\mathrm{H}}^{(2)}$ in $\mathcal{L}_{\mathrm{H}}$, we should note that the combination $V_{\mathrm{E}}R_{\mathrm{E}}$ transforms as

$$\delta_\Xi\left(V_{\mathrm{E}}R_{\mathrm{E}}\right) = (\mathrm{Re}\,\hat{\Xi}^m)\partial_m\left(V_{\mathrm{E}}R_{\mathrm{E}}\right) + \mathrm{Re}\left(\widehat{\partial_m \Xi^m} - \frac{1}{\hat{S}_{\mathrm{E}}^2}\widehat{\partial_4 \Xi^5} - \hat{S}_{\mathrm{E}}^2 \widehat{\partial_5 \Xi^4}\right)\left(V_{\mathrm{E}}R_{\mathrm{E}}\right), \tag{3.17}$$

which is consistent with (B.10). This transformation law is derived from (3.52) and (3.57) explained later.

Consider the Jacobian for $\mathcal{P}_U$, which is calculated as

$$\begin{aligned} J_{\mathcal{P}} &\equiv \mathrm{sdet}\left(\frac{\partial(x^\mu, x^m + iU^m(x,\theta), \theta^\alpha, \bar{\theta}_{\dot{\alpha}})}{\partial(x^\nu, x^n, \theta^\beta, \bar{\theta}_{\dot{\beta}})}\right) \\ &= 1 + i\partial_m U^m - \partial_4 U^4 \partial_5 U^5 + \partial_4 U^5 \partial_5 U^4, \end{aligned} \tag{3.18}$$

which satisfies

$$\int d^6x d^4\theta\; J_{\mathcal{P}}\hat{\Phi} = \int d^6x d^4\theta\; \Phi = 0, \tag{3.19}$$

for a chiral superfield Φ. After some calculations, we can show that $J_{\mathcal{P}}$ transforms as

$$\delta_\Xi J_{\mathcal{P}} = \partial_m\left\{(\mathrm{Re}\,\hat{\Xi}^m)J_{\mathcal{P}}\right\} - \widehat{\partial_m \Xi^m} J_{\mathcal{P}}. \tag{3.20}$$

Then, we obtain

$$\delta_\Xi\left|J_{\mathcal{P}}\right| = \mathrm{Re}\,\hat{\Xi}^m \partial_m\left|J_{\mathcal{P}}\right| + \mathrm{Re}\left(\partial_m \hat{\Xi}^m - \widehat{\partial_m \Xi^m}\right)\left|J_{\mathcal{P}}\right|. \tag{3.21}$$

Combining these transformation laws, we find

$$\delta_\Xi\left(\left|J_{\mathcal{P}}\right| V_{\mathrm{E}}^{1/2} R_{\mathrm{E}}^{1/2} L_{\mathrm{H}}^{(2)}\right) = \partial_m\left(\mathrm{Re}\,\hat{\Xi}^m \left|J_{\mathcal{P}}\right| V_{\mathrm{E}}^{1/2} R_{\mathrm{E}}^{1/2} L_{\mathrm{H}}^{(2)}\right). \tag{3.22}$$

3.1.3 Comment on $\mathcal{P}_U$

Here, we give a comment on the operator $\mathcal{P}_U$. Let us consider a chiral superfield Φ whose components are given by

$$\Phi = \phi + \theta\psi + \theta^2 F + i(\theta\sigma^{\underline{\mu}}\bar{\theta})\partial_\mu\phi - \frac{i}{2}\theta^2\partial_\mu\psi\sigma^{\underline{\mu}}\bar{\theta} + \frac{1}{4}\theta^2\bar{\theta}^2\Box_4\phi, \tag{3.23}$$

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where $\Box_4 \equiv \partial_\mu \partial^\mu$. After the $\mathcal{P}_U$ operation, this becomes

$$\begin{aligned}\hat{\Phi}(x,\theta) &= \Phi(x,\theta) + iU^m \partial_m \Phi + \mathcal{O}(U^2) \\ &= \phi + \theta\psi + \theta^2 F + i(\theta\sigma^{\underline{\mu}}\bar{\theta})\left(\partial_\mu \phi + e_{\underline{\mu}}{}^m \partial_m \phi\right) \\ &\quad -\frac{i}{2}\theta^2\left(\partial_\mu \psi + e_{\underline{\mu}}{}^m \partial_m \psi\right)\sigma^{\underline{\mu}}\bar{\theta} + \cdots .\end{aligned} \tag{3.24}$$

Namely, the operator $\mathcal{P}_U$ replaces the derivative ∂_μ appearing in the components with

$$\begin{aligned}\partial_{\underline{\mu}} &= e_{\underline{\mu}}{}^N \partial_N = e_{\underline{\mu}}{}^\nu \partial_\nu + e_{\underline{\mu}}{}^m \partial_m \\ &= \partial_\mu + e_{\underline{\mu}}{}^m \partial_m .\end{aligned} \tag{3.25}$$

We have dropped the fluctuation of $e_\mu{}^{\underline{\nu}}$ around the background $\delta_\mu{}^{\underline{\nu}}$, and terms beyond linear in the "off-diagonal" components of the sechsbein. Recall that the index of $\sigma^{\underline{\mu}}$ is the flat one. So the 4D indices contracted with it should also be the flat ones. In higher-dimensional SUGRA, this means that terms involving the "off-diagonal" components of the vielbein must be incorporated, which are missing in the original superfield Φ. The operator $\mathcal{P}_U$ provides such missing terms.

For later convenience, we "covariantize" the spinor derivatives D_α and $\bar{D}_{\dot{\alpha}}$ as

$$D_\alpha^{\mathcal{P}} \equiv \bar{\mathcal{P}}_U D_\alpha \bar{\mathcal{P}}_U^{-1}, \qquad \bar{D}_{\dot{\alpha}}^{\mathcal{P}} \equiv \mathcal{P}_U \bar{D}_{\dot{\alpha}} \mathcal{P}_U^{-1}. \tag{3.26}$$

Then, we can also see the same effect of $\mathcal{P}_U$ in the $\mathcal{N} = 1$ SUSY algebra.

$$\begin{aligned}\left\{D_\alpha^{\mathcal{P}}, \bar{D}_{\dot{\alpha}}^{\mathcal{P}}\right\} &= \left\{D_\alpha + iD_\alpha U^m \partial_m + \mathcal{O}(U^2), \bar{D}_{\dot{\alpha}} - i\bar{D}_{\dot{\alpha}} U^n \partial_n + \mathcal{O}(U^2)\right\} \\ &= \left\{D_\alpha, \bar{D}_{\dot{\alpha}}\right\} - i\left[D_\alpha, \bar{D}_{\dot{\alpha}}\right] U^m \partial_m + \mathcal{O}(U^2) \\ &= -2i\sigma^{\underline{\mu}}_{\alpha\dot{\alpha}} \partial_\mu - i\left(2\sigma^{\underline{\mu}}_{\alpha\dot{\alpha}} e_{\underline{\mu}}{}^m \partial_m + \cdots\right) + \mathcal{O}(U^2) \\ &= -2i\sigma^{\underline{\mu}}_{\alpha\dot{\alpha}} \partial_{\underline{\mu}} + \cdots ,\end{aligned} \tag{3.27}$$

where $\mathcal{O}(U^2)$ denotes terms beyond linear in U^m.

3.2 Vector-tensor sector

3.2.1 Field strength superfields

From (3.14), we can show that[11]

$$\begin{aligned}\delta_\Xi\left(\mathcal{P}_U^{-1} V^I\right) &= \Xi^m \partial_m \left(\mathcal{P}_U^{-1} V^I\right), \\ \delta_\Xi\left(\bar{\mathcal{P}}_U^{-1} V^I\right) &= \bar{\Xi}^m \partial_m \left(\bar{\mathcal{P}}_U^{-1} V^I\right).\end{aligned} \tag{3.29}$$

[11] Notice that $\mathcal{P}_U^{-1}$ is different from $\bar{\mathcal{P}}_U$ because

$$\begin{aligned}\mathcal{P}_U^{-1} x^m &= x^m - iU^m(\mathcal{P}_U^{-1} x, \theta) \\ &= x^m - iU^m(x,\theta) - U^n(x,\theta)\partial_n U^m(x,\theta) + \cdots .\end{aligned} \tag{3.28}$$

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Hence, if we modify the field strength superfield $\mathcal{W}_\alpha^I$ in (2.9) as

$$\hat{\mathcal{W}}_\alpha^I \equiv -\frac{1}{4}(\bar{D}^{\mathcal{P}})^2 D_\alpha^{\mathcal{P}} V^I, \tag{3.30}$$

it transforms as

$$\delta_\Xi \hat{\mathcal{W}}_\alpha^I = (\mathrm{Re}\,\hat{\Xi}^m)\partial_m \hat{\mathcal{W}}_\alpha^I, \tag{3.31}$$

which is consistent with the component transformation. However, this is not gauge-invariant under

$$\delta_\Lambda V^I = \hat{\Lambda}^I + \bar{\hat{\Lambda}}^I \tag{3.32}$$

because

$$\begin{aligned}\delta_\Lambda \hat{\mathcal{W}}_\alpha^I &= -\frac{1}{4}\left(\bar{D}^{\mathcal{P}}\right)^2 D_\alpha^{\mathcal{P}} \hat{\Lambda}^I \\ &= \mathcal{P}_U\left(-\frac{i}{2}\bar{D}^2 D_\alpha U^m \partial_m \Lambda\right) + \mathcal{O}(U^2).\end{aligned} \tag{3.33}$$

This stems from the fact that $\mathcal{W}_\alpha^I$ should include the field strength $F_{\underline{\mu}\underline{\nu}}$, and

$$\begin{aligned}F_{\underline{\mu}\underline{\nu}} &= e_{\underline{\mu}}{}^L e_{\underline{\nu}}{}^P \partial_L A_P - (\mu \leftrightarrow \nu) \\ &= e_{\underline{\mu}}{}^L \left(\partial_L A_{\underline{\nu}} - \partial_L e_{\underline{\nu}}{}^P A_P\right) - (\mu \leftrightarrow \nu) \\ &= \left(\partial_{\underline{\mu}} A_{\underline{\nu}} - \partial_{\underline{\nu}} A_{\underline{\mu}}\right) - \left(\partial_{\underline{\mu}} e_{\underline{\nu}}{}^n - \partial_{\underline{\nu}} e_{\underline{\mu}}{}^n\right) A_n + \cdots,\end{aligned} \tag{3.34}$$

where the ellipsis denotes terms beyond the linear order in the "off-diagonal" components $\{e_m{}^{\underline{\nu}}, e_{\underline{\mu}}{}^n\}$, or terms involving the fluctuation of $e_\mu{}^{\underline{\nu}}$. The superfield defined in (3.30) only contains the first term in (3.34). Thus, we have to modify (3.30) by adding terms that depend on U^m and Σ^I, in order to cancel the variation (3.33). The identification of the additional terms is left for the subsequent paper, in which the gauge group is extended to non-Abelian, but such correction terms should be determined so that the transformation law (3.31) is maintained.

Next we consider the tensor multiplet. The δ_Ξ-transformations of $\Upsilon_{\mathrm{T}\alpha}$, $V_{\mathrm{T}m}$ and Σ_{T} are found from (B.1) and (B.14) as

$$\begin{aligned}\delta_\Xi \Upsilon_{\mathrm{T}\alpha} &= \Xi^m \partial_m \Upsilon_{\mathrm{T}\alpha}, \\ \delta_\Xi V_{\mathrm{T}m} &= \mathrm{Re}\,\hat{\Xi}^n \partial_n V_{\mathrm{T}m} + (\mathrm{Re}\,\partial_m \hat{\Xi}^n) V_{\mathrm{T}n}, \\ \delta_\Xi \Sigma_{\mathrm{T}} &= \partial_m \left(\Xi^m \Sigma_{\mathrm{T}}\right).\end{aligned} \tag{3.35}$$

The definition of the field strength $\mathcal{X}_{\mathrm{T}}$ in (2.12) is modified as

$$\mathcal{X}_{\mathrm{T}} \equiv \frac{1}{2}\mathrm{Im}\,\left(D^{\mathcal{P}\alpha}\Upsilon_{\mathrm{T}\alpha}\right). \tag{3.36}$$

Then, it transforms as

$$\delta_\Xi \mathcal{X}_{\mathrm{T}} = \mathrm{Re}\,\hat{\Xi}^m \partial_m \mathcal{X}_{\mathrm{T}}. \tag{3.37}$$

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The second term in $\delta_\Xi V_{\mathrm{T}m}$ exists because $V_{\mathrm{T}m}$ has an external index m. Thus we extend the operator $\mathcal{P}_U$ as follows. For a chiral superfield Φ_m, we define the operator $\mathcal{Q}_U$ as[12]

$$\mathcal{Q}_U \Phi_m = \hat{\Phi}_m + i\partial_m U^n \hat{\Phi}_n. \tag{3.38}$$

Since Φ_m has an external index m, its δ_Ξ-transformation has a form of

$$\delta_\Xi \Phi_m = \Xi^n \partial_n \Phi_m + \partial_m \Xi^n \Phi_n + \cdots, \tag{3.39}$$

Then we can show that

$$\delta_\Xi (\mathcal{Q}_U \Phi_m) = \mathrm{Re}\, \hat{\Xi}^n \partial_n (\mathcal{Q}_U \Phi_m) + (\mathrm{Re}\, \partial_m \hat{\Xi}^n) \mathcal{Q}_U \Phi_n + \cdots. \tag{3.40}$$

Note that this has the same form as $\delta_\Xi V_{\mathrm{T}m}$ in (3.35). Hence, it follows that[13]

$$\begin{aligned} \delta_\Xi (\mathcal{Q}_U^{-1} V_{\mathrm{T}m}) &= \Xi^n \partial_n (\mathcal{Q}_U^{-1} V_{\mathrm{T}m}) + \partial_m \Xi^n (\mathcal{Q}_U^{-1} V_{\mathrm{T}n}), \\ \delta_\Xi (\bar{\mathcal{Q}}_U^{-1} V_{\mathrm{T}m}) &= \bar{\Xi}^n \partial_n (\bar{\mathcal{Q}}_U^{-1} V_{\mathrm{T}m}) + \partial_m \bar{\Xi}^n (\bar{\mathcal{Q}}_U^{-1} V_{\mathrm{T}n}). \end{aligned} \tag{3.42}$$

Making use of these properties, $\mathcal{W}_{\mathrm{T}m\alpha}$ in (2.13) should be modified as

$$\begin{aligned} \mathcal{W}_{\mathrm{T}m\alpha} &= -\frac{1}{4} \bar{D}^2 \mathcal{Q}_U^{-1} \bar{\mathcal{Q}}_U D_\alpha \bar{\mathcal{Q}}_U^{-1} V_{\mathrm{T}m} \\ &= \mathcal{Q}_U^{-1} \left\{ -\frac{1}{4} (\bar{D}^{\mathcal{Q}})^2 D_\alpha^{\mathcal{Q}} V_{\mathrm{T}m} \right\}, \end{aligned} \tag{3.43}$$

where

$$D_\alpha^{\mathcal{Q}} \equiv \bar{\mathcal{Q}}_U D_\alpha \bar{\mathcal{Q}}_U^{-1}, \qquad \bar{D}_{\dot{\alpha}}^{\mathcal{Q}} \equiv \mathcal{Q}_U \bar{D}_{\dot{\alpha}} \mathcal{Q}_U^{-1}. \tag{3.44}$$

Then, it transforms as

$$\delta_\Xi \mathcal{W}_{\mathrm{T}m\alpha} = \Xi^n \partial_n \mathcal{W}_{\mathrm{T}m\alpha} + \partial_m \Xi^n \mathcal{W}_{\mathrm{T}n\alpha}, \tag{3.45}$$

which leads to

$$\begin{aligned} \delta_\Xi \left(\frac{\mathcal{W}_{\mathrm{T}4\alpha}}{S_{\mathrm{E}}} + S_{\mathrm{E}} \mathcal{W}_{\mathrm{T}5\alpha} \right) = \Xi^m \partial_m \left(\frac{\mathcal{W}_{\mathrm{T}4\alpha}}{S_{\mathrm{E}}} + S_{\mathrm{E}} \mathcal{W}_{\mathrm{T}5\alpha} \right) \\ + \frac{1}{2} \left(\partial_m \Xi^m + \frac{1}{S_{\mathrm{E}}^2} \partial_4 \Xi^5 + S_{\mathrm{E}}^2 \partial_5 \Xi^4 \right) \left(\frac{\mathcal{W}_{\mathrm{T}4\alpha}}{S_{\mathrm{E}}} + S_{\mathrm{E}} \mathcal{W}_{\mathrm{T}5\alpha} \right) \\ - \left(\frac{1}{S_{\mathrm{E}}^2} \partial_4 \Xi^5 - S_{\mathrm{E}}^2 \partial_5 \Xi^4 \right) \left(\frac{\mathcal{W}_{\mathrm{T}4\alpha}}{S_{\mathrm{E}}} - S_{\mathrm{E}} \mathcal{W}_{\mathrm{T}5\alpha} \right). \end{aligned} \tag{3.46}$$

[12] The operators $\mathcal{P}_U$ and $\mathcal{Q}_U$ are understood as $e^{i\mathcal{L}_U}$, where $\mathcal{L}_U$ is the Lie derivative along U^m.

[13] Specifically, $\mathcal{Q}_U^{-1} V_{\mathrm{T}m}$ is

$$\begin{aligned} \mathcal{Q}_U^{-1} V_{\mathrm{T}m}(x) &= V_{\mathrm{T}m}(\mathcal{P}_U^{-1} x) - i(\mathcal{P}_U^{-1} \partial_m U^n) \left\{ \mathcal{Q}_U^{-1} V_{\mathrm{T}n} \right\}(x) \\ &= V_{\mathrm{T}m}(\mathcal{P}_U^{-1} x) - i(\mathcal{P}_U^{-1} \partial_m U^n) V_{\mathrm{T}n}(\mathcal{P}_U^{-1} x) \\ &\quad -(\mathcal{P}_U^{-1} \partial_m U^n)(\mathcal{P}_U^{-1} \partial_n U^l) V_{\mathrm{T}l}(\mathcal{P}_U^{-1} x) + \cdots. \end{aligned} \tag{3.41}$$

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From (3.35), we can show that

$$\begin{aligned}\delta_\Xi\left(S_{\rm E}\mathcal{O}_{\rm E}\Upsilon_{{\rm T}\alpha}\right) &= S_{\rm E}\mathcal{O}_{\rm E}\left(\delta_\Xi\Upsilon_{{\rm T}\alpha}\right)-\frac{\delta_\Xi S_{\rm E}}{S_{\rm E}}\partial_{\rm E}\Upsilon_{{\rm T}\alpha}\\ &= \Xi^m\partial_m\left(S_{\rm E}\mathcal{O}_{\rm E}\Upsilon_{{\rm T}\alpha}\right)+\frac{1}{2}\left(\partial_m\Xi^m+\frac{1}{S_{\rm E}^2}\partial_4\Xi^5+S_{\rm E}^2\partial_5\Xi^4\right)S_{\rm E}\mathcal{O}_{\rm E}\Upsilon_{{\rm T}\alpha}\\ &\quad-\left(\frac{1}{S_{\rm E}^2}\partial_4\Xi^5-S_{\rm E}^2\partial_5\Xi^4\right)\partial_{\rm E}\Upsilon_{{\rm T}\alpha}.\end{aligned}\tag{3.47}$$

Summing (3.46) and (3.47), we obtain the δ_Ξ-transformation of $\mathcal{Y}_{{\rm T}\alpha}$ defined in (2.12) as

$$\delta_\Xi\mathcal{Y}_{{\rm T}\alpha}=\Xi^m\partial_m\mathcal{Y}_{{\rm T}\alpha}+\frac{1}{2}\left(\partial_m\Xi^m+\frac{1}{S_{\rm E}^2}\partial_4\Xi^5+S_{\rm E}^2\partial_5\Xi^4\right)\mathcal{Y}_{{\rm T}\alpha}.\tag{3.48}$$

We have used the constraint (2.14).

From (3.35), we also obtain

$$\begin{aligned}\delta_\Xi\hat{\Sigma}_{\rm T} &= {\rm Re}\,\hat{\Xi}^m\partial_m\hat{\Sigma}_{\rm T}+\widehat{\partial_m\Xi^m}\hat{\Sigma}_{\rm T},\\ \delta_\Xi\left(\partial_4V_{\rm T5}-\partial_5V_{\rm T4}\right) &= \partial_4\left(\delta_\Xi V_{\rm T5}\right)-\partial_5\left(\delta_\Xi V_{\rm T4}\right)\\ &= \partial_m\left\{{\rm Re}\,\hat{\Xi}^m\left(\partial_4V_{\rm T5}-\partial_5V_{\rm T4}\right)\right\},\\ \delta_\Xi\left(J_{\mathcal{P}}\hat{\Sigma}_{\rm T}\right) &= \left(\delta_\Xi J_{\mathcal{P}}\right)\hat{\Sigma}_{\rm T}+J_{\mathcal{P}}\delta_\Xi\hat{\Sigma}_{\rm T}\\ &= \partial_m\left\{({\rm Re}\,\hat{\Xi}^m)J_{\mathcal{P}}\hat{\Sigma}_{\rm T}\right\}.\end{aligned}\tag{3.49}$$

Therefore, if we modify the definition of $\mathcal{V}_{\rm T}$ in (2.21) as

$$\mathcal{V}_{\rm T}\equiv\partial_4V_{\rm T5}-\partial_5V_{\rm T4}+J_{\mathcal{P}}\hat{\Sigma}_{\rm T}+\bar{J}_{\mathcal{P}}\bar{\hat{\Sigma}}_{\rm T},\tag{3.50}$$

we find that

$$\delta_\Xi\left(\mathcal{V}_{\rm T}\right)=\partial_m\left({\rm Re}\,\hat{\Xi}^m\mathcal{V}_{\rm T}\right).\tag{3.51}$$

Recall that $V_{\rm E}=\mathcal{V}_{\rm T}/\mathcal{X}_{\rm T}$ from (2.21). Thus, from (3.37) and (3.51), we obtain

$$\begin{aligned}\delta_\Xi V_{\rm E} &= \delta_\Xi\left(\frac{\mathcal{V}_{\rm T}}{\mathcal{X}_{\rm T}}\right)=\partial_m\left({\rm Re}\,\hat{\Xi}^m\frac{\mathcal{V}_{\rm T}}{\mathcal{X}_{\rm T}}\right)\\ &= {\rm Re}\,\hat{\Xi}^m\partial_m\left(\frac{\mathcal{V}_{\rm T}}{\mathcal{X}_{\rm T}}\right)+\left({\rm Re}\,\widehat{\partial_m\Xi^m}-\partial_mU^n{\rm Im}\,\widehat{\partial_n\Xi^m}\right)\frac{\mathcal{V}_{\rm T}}{\mathcal{X}_{\rm T}},\end{aligned}\tag{3.52}$$

which is consistent with (B.9). However, this and (3.13) are not consistent with (3.17). Hence, we modify the definition of $R_{\rm E}$ given in (2.24) in such a way that $V_{\rm E}R_{\rm E}$ transforms as (3.17). We modify $R_{\rm E}$ as

$$R_{\rm E}\equiv\frac{1}{2}{\rm Im}\left(J_S^{(2)}\frac{\bar{\hat{S}}_{\rm E}}{\hat{S}_{\rm E}}-J_S^{(1)}\frac{\hat{S}_{\rm E}}{\bar{\hat{S}}_{\rm E}}\right),\tag{3.53}$$

where

$$\begin{aligned}J_S^{(1)} &\equiv 1+i\left(\partial_4U^4-\partial_5U^5\right)-2i\bar{\hat{S}}_{\rm E}^2\partial_5U^4+\mathcal{O}(U^2),\\ J_S^{(2)} &\equiv 1-i\left(\partial_4U^4-\partial_5U^5\right)-\frac{2i}{\bar{\hat{S}}_{\rm E}^2}\partial_4U^5+\mathcal{O}(U^2).\end{aligned}\tag{3.54}$$

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The higher order terms $\mathcal{O}(U^2)$ are determined so that $J_S^{(1)}$ and $J_S^{(2)}$ transform as

$$\begin{aligned}
\delta_\Xi J_S^{(1)} &= \mathrm{Re}\,\hat{\Xi}^m\partial_m J_S^{(1)} - i\left\{\mathrm{Im}\left(\widehat{\partial_4\Xi^4} - \widehat{\partial_5\Xi^5}\right) - 2\bar{\hat{S}}_{\mathrm{E}}^2\mathrm{Im}\,\widehat{\partial_5\Xi^4}\right\} J_S^{(1)} \\
&\quad - \left\{\partial_m U^n \mathrm{Im}\,\widehat{\partial_n\Xi^m} - 2i|\hat{S}_{\mathrm{E}}|^2\left(\frac{R_{\mathrm{E}}}{J_S^{(1)}} - \mathrm{Im}\,\frac{\bar{\hat{S}}_{\mathrm{E}}}{\hat{S}_{\mathrm{E}}}\right)\widehat{\partial_5\Xi^4}\right\} J_S^{(1)}, \\
\delta_\Xi J_S^{(2)} &= \mathrm{Re}\,\hat{\Xi}^m\partial_m J_S^{(2)} + i\left\{\mathrm{Im}\left(\widehat{\partial_4\Xi^4} - \widehat{\partial_5\Xi^5}\right) + \frac{2}{\bar{\hat{S}}_{\mathrm{E}}^2}\mathrm{Im}\,\widehat{\partial_4\Xi^5}\right\} J_S^{(2)} \\
&\quad - \left\{\partial_m U^n \mathrm{Im}\,\widehat{\partial_n\Xi^m} + \frac{2i}{|\hat{S}_{\mathrm{E}}|^2}\left(\frac{R_{\mathrm{E}}}{J_S^{(2)}} - \mathrm{Im}\,\frac{\bar{\hat{S}}_{\mathrm{E}}}{\hat{S}_{\mathrm{E}}}\right)\widehat{\partial_4\Xi^5}\right\} J_S^{(2)}.
\end{aligned} \tag{3.55}$$

These lead to

$$\begin{aligned}
\delta_\Xi\left(J_S^{(1)}\frac{\hat{S}_{\mathrm{E}}}{\bar{\hat{S}}_{\mathrm{E}}}\right) &= \mathrm{Re}\,\hat{\Xi}^m\partial_m\left(J_S^{(1)}\frac{\hat{S}_{\mathrm{E}}}{\bar{\hat{S}}_{\mathrm{E}}}\right) + 2i\hat{S}_{\mathrm{E}}^2\widehat{\partial_5\Xi^4}R_{\mathrm{E}} \\
&\quad + \left\{-\mathrm{Re}\left(i\partial_m U^n\widehat{\partial_n\Xi^m}\right) + i\mathrm{Im}\left(\frac{1}{\hat{S}_{\mathrm{E}}^2}\widehat{\partial_4\Xi^5} + \hat{S}_{\mathrm{E}}^2\widehat{\partial_5\Xi^4}\right)\right\} J_S^{(1)}\frac{\hat{S}_{\mathrm{E}}}{\bar{\hat{S}}_{\mathrm{E}}}, \\
\delta_\Xi\left(J_S^{(2)}\frac{\bar{\hat{S}}_{\mathrm{E}}}{\hat{S}_{\mathrm{E}}}\right) &= \mathrm{Re}\,\hat{\Xi}^m\partial_m\left(J_S^{(2)}\frac{\bar{\hat{S}}_{\mathrm{E}}}{\hat{S}_{\mathrm{E}}}\right) - \frac{2i}{\hat{S}_{\mathrm{E}}^2}\widehat{\partial_4\Xi^5}R_{\mathrm{E}} \\
&\quad + \left\{-\mathrm{Re}\left(i\partial_m U^n\widehat{\partial_n\Xi^m}\right) + i\mathrm{Im}\left(\frac{1}{\hat{S}_{\mathrm{E}}^2}\widehat{\partial_4\Xi^5} + \hat{S}_{\mathrm{E}}^2\widehat{\partial_5\Xi^4}\right)\right\} J_S^{(2)}\frac{\bar{\hat{S}}_{\mathrm{E}}}{\hat{S}_{\mathrm{E}}}.
\end{aligned} \tag{3.56}$$

As a result, R_{E} transforms as

$$\delta_\Xi R_{\mathrm{E}} = \mathrm{Re}\,\hat{\Xi}^m\partial_m R_{\mathrm{E}} - \mathrm{Re}\left(\frac{1}{\hat{S}_{\mathrm{E}}^2}\widehat{\partial_4\Xi^5} + \hat{S}_{\mathrm{E}}^2\widehat{\partial_5\Xi^4} + i\partial_m U^n\widehat{\partial_n\Xi^m}\right)R_{\mathrm{E}}. \tag{3.57}$$

From (3.52) and (3.57), we certainly obtain the transformation law (3.17).

3.2.2 Invariance of action

Let us first consider the δ_Ξ-invariance of the first line of $\mathcal{L}_{\mathrm{VT}}$ in (2.23). If we define

$$\partial_{\mathrm{E}}^{\mathcal{P}} \equiv \mathcal{P}_U\partial_{\mathrm{E}}\mathcal{P}_U^{-1}, \tag{3.58}$$

we find that

$$\delta_\Xi\left(\partial_{\mathrm{E}}^{\mathcal{P}}V^I\right) = (\mathrm{Re}\,\hat{\Xi}^m)\partial_m\left(\partial_{\mathrm{E}}^{\mathcal{P}}V^I\right) + \frac{1}{2}\left(\widehat{\partial_m\Xi^m} - \frac{1}{\hat{S}_{\mathrm{E}}^2}\widehat{\partial_4\Xi^5} - \hat{S}_{\mathrm{E}}^2\widehat{\partial_5\Xi^4}\right)\partial_{\mathrm{E}}^{\mathcal{P}}V^I. \tag{3.59}$$

This is the same transformation law as that of $\hat{\Sigma}^I$. Similarly, $\partial_{\mathrm{E}}^{\mathcal{P}}D_\alpha^{\mathcal{P}}V^I$ also has the same transformation law. Combining these properties with (3.48), we can show that

$$\delta_\Xi\left(L_{\mathrm{V}}^{(1)\alpha}\hat{\mathcal{Y}}_{\mathrm{T}\alpha}\right) = (\mathrm{Re}\,\hat{\Xi}^m)\partial_m\left(L_{\mathrm{V}}^{(1)\alpha}\hat{\mathcal{Y}}_{\mathrm{T}\alpha}\right) + \widehat{\partial_m\Xi^m}\left(L_{\mathrm{V}}^{(1)\alpha}\hat{\mathcal{Y}}_{\mathrm{T}\alpha}\right), \tag{3.60}$$

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where

$$L_{\rm V}^{(1)\alpha} \equiv f_{IJ}\left\{-2\hat{\Sigma}^I D^{\mathcal{P}\alpha}V^J + \frac{1}{2}\left(\partial_{\rm E}^{\mathcal{P}}V^I D^{\mathcal{P}\alpha}V^J - \partial_{\rm E}^{\mathcal{P}}D^{\mathcal{P}\alpha}V^I V^J\right)\right\}. \tag{3.61}$$

Recalling (3.20), we find that

$$\delta_\Xi\left(J_{\mathcal{P}}L_{\rm V}^{(1)\alpha}\hat{\mathcal{Y}}_{{\rm T}\alpha}\right) = \partial_m\left\{(\mathrm{Re}\,\hat{\Xi}^m)J_{\mathcal{P}}L_{\rm V}^{(1)\alpha}\hat{\mathcal{Y}}_{{\rm T}\alpha}\right\}. \tag{3.62}$$

Next, consider the second line of $\mathcal{L}_{\rm VT}$. Since the combination

$$L_{\rm V}^{(2)} \equiv f_{IJ}\left(D^{\mathcal{P}\alpha}V^I\hat{\mathcal{W}}_\alpha^J + \frac{1}{2}V^I D^{\mathcal{P}\alpha}\hat{\mathcal{W}}_\alpha^J + \text{h.c.}\right) \tag{3.63}$$

transforms as

$$\delta_\Xi L_{\rm V}^{(2)} = (\mathrm{Re}\,\hat{\Xi}^m)\partial_m L_{\rm V}^{(2)}, \tag{3.64}$$

we find that

$$\begin{aligned}\delta_\Xi\left(\mathcal{V}_{\rm T}L_{\rm V}^{(2)}\right) &= \partial_m\left(\mathrm{Re}\,\hat{\Xi}^m\mathcal{V}_{\rm T}\right)L_{\rm V}^{(2)} + \mathcal{V}_{\rm T}\cdot(\mathrm{Re}\,\hat{\Xi}^m)\partial_m L_{\rm V}^{(2)}\\ &= \partial_m\left(\mathrm{Re}\,\hat{\Xi}^m\mathcal{V}_{\rm T}L_{\rm V}^{(2)}\right).\end{aligned} \tag{3.65}$$

As for the third line of $\mathcal{L}_{\rm VT}$, the combination

$$L_{\rm V}^{(3)} \equiv f_{IJ}\left\{4\left(\partial_{\rm E}^{\mathcal{P}}V^I - \hat{\Sigma}^I\right)^\dagger\left(\partial_{\rm E}^{\mathcal{P}}V^J - \hat{\Sigma}^J\right) - 2\left(\partial_{\rm E}^{\mathcal{P}}V^I\right)^\dagger\partial_{\rm E}^{\mathcal{P}}V^J\right\} \tag{3.66}$$

transforms as

$$\delta_\Xi L_{\rm V}^{(3)} = (\mathrm{Re}\,\hat{\Xi}^m)\partial_m L_{\rm V}^{(3)} + \mathrm{Re}\left(\widehat{\partial_m\Xi^m} - \frac{1}{\hat{S}_{\rm E}^2}\widehat{\partial_4\Xi^5} - \hat{S}_{\rm E}^2\widehat{\partial_5\Xi^4}\right)L_{\rm V}^{(3)}. \tag{3.67}$$

From (3.37) and (3.57), we obtain

$$\delta_\Xi\left(\frac{\mathcal{X}_{\rm T}}{R_{\rm E}}\right) = \mathrm{Re}\,\hat{\Xi}^m\partial_m\left(\frac{\mathcal{X}_{\rm T}}{R_{\rm E}}\right) + \mathrm{Re}\left(\frac{1}{\hat{S}_{\rm E}^2}\widehat{\partial_4\Xi^5} + \hat{S}_{\rm E}^2\widehat{\partial_5\Xi^4} + i\partial_m U^n\widehat{\partial_n\Xi^m}\right)\frac{\mathcal{X}_{\rm T}}{R_{\rm E}}. \tag{3.68}$$

Therefore, we find that

$$\delta_\Xi\left(\frac{\mathcal{X}_{\rm T}}{R_{\rm E}}L_{\rm V}^{(3)}\right) = \partial_m\left(\mathrm{Re}\,\hat{\Xi}^m\frac{\mathcal{X}_{\rm T}}{R_{\rm E}}L_{\rm V}^{(3)}\right). \tag{3.69}$$

Finally, consider the last line of $\mathcal{L}_{\rm VT}$. Combining (3.13), (3.20), (3.56) and (3.68), we can see that

$$\delta_\Xi\left(J_{\mathcal{P}}f_{IJ}\frac{\mathcal{X}_{\rm T}}{R_{\rm E}}J_S^{(1)}\frac{\hat{S}_{\rm E}}{\bar{\hat{S}}_{\rm E}}\hat{\Sigma}^I\hat{\Sigma}^J\right) = \partial_m\left(\mathrm{Re}\,\hat{\Xi}^m J_{\mathcal{P}}f_{IJ}\frac{\mathcal{X}_{\rm T}}{R_{\rm E}}J_S^{(1)}\frac{\hat{S}_{\rm E}}{\bar{\hat{S}}_{\rm E}}\hat{\Sigma}^I\hat{\Sigma}^J\right). \tag{3.70}$$

We have used the property (3.19), which also ensures that

$$J_{\mathcal{P}}f_{IJ}\frac{\mathcal{X}_{\rm T}}{R_{\rm E}}J_S^{(1)}\frac{\hat{S}_{\rm E}}{\bar{\hat{S}}_{\rm E}}\hat{\Sigma}^I\hat{\Sigma}^J = J_{\mathcal{P}}f_{IJ}\frac{\mathcal{X}_{\rm T}}{R_{\rm E}}J_S^{(2)}\frac{\bar{\hat{S}}_{\rm E}}{\hat{S}_{\rm E}}\hat{\Sigma}^I\hat{\Sigma}^J. \tag{3.71}$$

Using the results obtained in this section, we can modify the action in (2.23) so that it is δ_Ξ-invariant up to total derivatives. We will provide the modified Lagrangian in section 5.3.

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4 Covariantization of ∂_{E}

So far, we have concentrated on the δ_Ξ-transformation, i.e., a diffeomorphism in the extra dimensions. In this section, we argue the consistency with its 4D counterpart, i.e., the 4D $\mathcal{N}=1$ superconformal transformation. Notice that ∂_m does not preserve the proper transformation laws for the $\mathcal{N}=1$ superconformal transformation collected in appendix A.2. Thus we need to introduce the connection superfields Ψ_m^α that transform as $\delta_L \Psi_m^\alpha = -\partial_m L^\alpha$ (L^α is the $\mathcal{N}=1$ superconformal transformation parameter), and covariantize ∂_m.

4.1 Chiral superspace

On a chiral superfield, we define the covariant derivative ∇_m as

$$\nabla_m \equiv \partial_m - \left(\frac{1}{4}\bar{D}^2 \Psi_m^\alpha D_\alpha - i\sigma^{\underline{\mu}}_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} \Psi_m^\alpha \partial_\mu + \frac{w}{12}\bar{D}^2 D^\alpha \Psi_{m\alpha} \right), \tag{4.1}$$

where w is the Weyl weight. Then, $\nabla_m H$ ($H = H_{\mathrm{odd}}, H_{\mathrm{even}}$) transforms as

$$\delta_L (\nabla_m H) = \left(-\frac{1}{4}\bar{D}^2 L^\alpha D_\alpha + i\sigma^{\underline{\mu}}_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} L^\alpha \partial_\mu - \frac{1}{8}\bar{D}^2 D^\alpha L_\alpha \right) \nabla_m H, \tag{4.2}$$

at the leading order in Ψ^α.[14] This is the same law as $\delta_L H$. (See (A.6).) Hence, (3.5) is modified as

$$L_{\mathrm{H}}^{(1)} = H_{\mathrm{odd}}^t \tilde{d} \left(\nabla_{\mathrm{E}} - \Sigma \right) H_{\mathrm{even}} - H_{\mathrm{even}}^t \tilde{d} \left(\nabla_{\mathrm{E}} + \Sigma \right) H_{\mathrm{odd}}, \tag{4.3}$$

where

$$\nabla_{\mathrm{E}} \equiv \frac{1}{S_{\mathrm{E}}} \nabla_4 - S_{\mathrm{E}} \nabla_5. \tag{4.4}$$

This is invariant under the δ_L-transformation up to total derivatives.

Next we consider the δ_Ξ-transformation. This should commute with the δ_L-transformation in order for the chiral property of the $\mathcal{N}=1$ chiral superfields to be preserved. From this requirement, the δ_Ξ-transformation of Ψ_m^α is found to be

$$\delta_\Xi \Psi_m^\alpha = \Xi^n \left(\partial_n \Psi_m^\alpha - \partial_m \Psi_n^\alpha \right). \tag{4.5}$$

In fact, we can see that

$$\delta_L \delta_\Xi \Psi_m^\alpha = \delta_\Xi \delta_L \Psi_m^\alpha = 0. \tag{4.6}$$

The transformation law (4.5) is consistent with the component field transformation (B.6) under the constraint $\partial_m \xi^{\underline{\mu}} = 0$ if we identify the $\bar{\theta}$-component of $\Psi_{m\alpha}$ as

$$\Psi_{m\alpha} = \frac{i}{2} \left(\sigma^{\underline{\mu}} \bar{\theta} \right)_\alpha e_{m\underline{\mu}} + \cdots. \tag{4.7}$$

Then, $\nabla_m H$ transforms as

$$\begin{aligned} \delta_\Xi (\nabla_m H) &= \nabla_m (\delta_\Xi H) - \frac{1}{4}\bar{D}^2 (\delta_\Xi \Psi_m^\alpha D_\alpha H) - \frac{1}{8} \left(\bar{D}^2 D^\alpha \delta_\Xi \Psi_{m\alpha} \right) H \\ &= \Xi^n \nabla_n (\nabla_m H) + \nabla_m \Xi^n \nabla_n H + \nabla_m (X_\Xi H), \end{aligned} \tag{4.8}$$

[14] In this paper, we consider the superconformal transformations at the linearized order in Ψ_m^α.

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where

$$X_\Xi \equiv \frac{1}{4}\left(\nabla_m \Xi^m + \frac{1}{S_{\rm E}^2}\nabla_4 \Xi^5 + S_{\rm E}^2 \nabla_5 \Xi^4\right). \tag{4.9}$$

We have used that

$$\begin{aligned}\nabla_m\left(\nabla_n H\right) =\ & \nabla_n\left(\nabla_m H\right) - \frac{1}{4}\bar{D}^2\left\{\left(\partial_m \Psi_n^\alpha - \partial_n \Psi_m^\alpha\right) D_\alpha H\right\} \\ & -\frac{1}{8}\left\{\bar{D}^2 D^\alpha\left(\partial_m \Psi_{n\alpha} - \partial_n \Psi_{m\alpha}\right)\right\} H + \mathcal{O}(\Psi^2).\end{aligned} \tag{4.10}$$

As a result, the δ_Ξ-transformation of (4.3) becomes total derivatives.

$$\begin{aligned}\delta_\Xi L_{\rm H}^{(1)} =\ & \nabla_m\left(\Xi^m L_{\rm H}^{(1)}\right) \\ =\ & \partial_m\left(\Xi^m L_{\rm H}^{(1)}\right) - \frac{1}{4}\bar{D}^2 D^\alpha\left(\Psi_{m\alpha}\Xi^m L_{\rm H}^{(1)}\right).\end{aligned} \tag{4.11}$$

Note that $L_{\rm H}^{(1)}$ has the Weyl weight 3.

4.2 Full superspace

In the full superspace, ∇_m in (4.1) is modified as

$$\begin{aligned}\tilde{\nabla}_m \equiv\ & \partial_m - \left(\frac{1}{4}\bar{D}^2 \Psi_m^\alpha D_\alpha + \frac{1}{2}\bar{D}^{\dot\alpha}\Psi_m^\alpha \bar{D}_{\dot\alpha} D_\alpha + \frac{w+n}{24}\bar{D}^2 \mathcal{R}_U D^\alpha \mathcal{R}_U^{-1}\Psi_{m\alpha}\right) \\ & -\mathcal{R}_U\left(\frac{1}{4}D^2\bar{\Psi}_{m\dot\alpha}\bar{D}^{\dot\alpha} + \frac{1}{2}D^\alpha \bar{\Psi}_m^{\dot\alpha} D_\alpha \bar{D}_{\dot\alpha} + \frac{w-n}{24}D^2 \mathcal{R}_U^{-1}\bar{D}_{\dot\alpha}\mathcal{R}_U \bar{\Psi}_m^{\dot\alpha}\right),\end{aligned} \tag{4.12}$$

where n is the chiral weight (i.e., the $\mathrm{U}(1)_A$ charge), and the operator $\mathcal{R}_U$ is defined by

$$\mathcal{R}_U X_m = X_m - 2iU^n\left(\partial_n X_m - \partial_m X_n\right) + \mathcal{O}(U^2). \tag{4.13}$$

Then, from the relation:

$$\begin{aligned}\tilde{\nabla}_m\tilde{\nabla}_n =\ & \tilde{\nabla}_n\tilde{\nabla}_m - \left\{\frac{1}{4}\bar{D}^2\left(\partial_m\Psi_n^\alpha - \partial_n\Psi_m^\alpha\right)D_\alpha + \frac{1}{2}\bar{D}^{\dot\alpha}\left(\partial_m\Psi_n^\alpha - \partial_n\Psi_m^\alpha\right)\bar{D}_{\dot\alpha}D_\alpha\right. \\ & \left.+\frac{w+n}{24}\bar{D}^2 D^\alpha\left(\partial_m\Psi_{n\alpha} - \partial_n\Psi_{m\alpha}\right) + \text{h.c.}\right\} + \mathcal{O}(U^m),\end{aligned} \tag{4.14}$$

and the transformation law:

$$\delta_\Xi\left(\mathcal{P}_U^{-1}V^I\right) = \Xi^n\tilde{\nabla}_n\left(\mathcal{P}_U^{-1}V^I\right), \tag{4.15}$$

we find that

$$\delta_\Xi\left\{\tilde{\nabla}_m\left(\mathcal{P}_U^{-1}V^I\right)\right\} = \Xi^n\tilde{\nabla}_n\left\{\tilde{\nabla}_m\left(\mathcal{P}_U^{-1}V^I\right)\right\} + \tilde{\nabla}_m\Xi^n\tilde{\nabla}_n\left(\mathcal{P}_U^{-1}V^I\right) + \mathcal{O}(U^m), \tag{4.16}$$

which leads to

$$\begin{aligned}\delta_\Xi\left\{\tilde{\nabla}_{\rm E}\left(\mathcal{P}_U^{-1}V\right)\right\} =\ & \Xi^n\tilde{\nabla}_n\left\{\tilde{\nabla}_{\rm E}\left(\mathcal{P}_U^{-1}V^I\right)\right\} \\ & +\frac{1}{2}\left(\tilde{\nabla}_n\Xi^n - \frac{1}{S_{\rm E}^2}\tilde{\nabla}_4\Xi^5 - S_{\rm E}^2\tilde{\nabla}_5\Xi^4\right)\tilde{\nabla}_{\rm E}\left(\mathcal{P}_U^{-1}V^I\right) + \mathcal{O}(U^m),\end{aligned} \tag{4.17}$$

where

$$\tilde{\nabla}_{\rm E} \equiv \frac{1}{S_{\rm E}}\tilde{\nabla}_4 - S_{\rm E}\tilde{\nabla}_5. \tag{4.18}$$

Therefore, $L_{\rm V}^{(1)\alpha}$ in (3.61) and $L_{\rm V}^{(3)}$ in (3.66) are modified as

$$\begin{aligned} L_{\rm V}^{(1)\alpha} &= f_{IJ}\left\{-2\hat{\Sigma}^I D^{\mathcal{P}\alpha}V^J + \frac{1}{2}\left(\nabla_{\rm E}^{\mathcal{P}}V^I D^{\mathcal{P}\alpha}V^J - \nabla_{\rm E}^{\mathcal{P}}D^{\mathcal{P}\alpha}V^I V^J\right)\right\}, \\ L_{\rm V}^{(3)} &= f_{IJ}\left\{4\left(\nabla_{\rm E}^{\mathcal{P}}V^I - \hat{\Sigma}^I\right)^\dagger\left(\nabla_{\rm E}^{\mathcal{P}}V^J - \hat{\Sigma}^J\right) - 2\left(\nabla_{\rm E}^{\mathcal{P}}V^I\right)^\dagger\nabla_{\rm E}^{\mathcal{P}}V^J\right\}, \end{aligned} \tag{4.19}$$

where

$$\nabla_{\rm E}^{\mathcal{P}} \equiv \mathcal{P}_U\tilde{\nabla}_{\rm E}\mathcal{P}_U^{-1}. \tag{4.20}$$

Besides, the δ_Ξ-transformations (3.13), (3.14) and (3.35) are modified as

$$\begin{aligned} \delta_\Xi\hat{S}_{\rm E} &= ({\rm Re}\,\hat{\Xi}^m)\nabla_m^{\mathcal{P}}\hat{S}_{\rm E} + \frac{1}{2}\left(\widehat{\nabla_4\Xi^4} - \widehat{\nabla_5\Xi^5} + \frac{1}{\hat{S}_{\rm E}^2}\widehat{\nabla_4\Xi^5} - \hat{S}_{\rm E}^2\widehat{\nabla_5\Xi^4}\right)\hat{S}_{\rm E}, \\ \delta_\Xi\hat{H} &= ({\rm Re}\,\hat{\Xi}^m)\nabla_m^{\mathcal{P}}\hat{H} + \frac{1}{4}\left(\widehat{\nabla_m\Xi^m} + \frac{1}{\hat{S}_{\rm E}^2}\widehat{\nabla_4\Xi^5} + \hat{S}_{\rm E}^2\widehat{\nabla_5\Xi^4}\right)\hat{H}, \\ \delta_\Xi\hat{\Sigma}^I &= ({\rm Re}\,\hat{\Xi}^m)\nabla_m^{\mathcal{P}}\hat{\Sigma}^I + \frac{1}{2}\left(\widehat{\nabla_m\Xi^m} - \frac{1}{\hat{S}_{\rm E}^2}\widehat{\nabla_4\Xi^5} - \hat{S}_{\rm E}^2\widehat{\nabla_5\Xi^4}\right)\hat{\Sigma}^I, \\ \delta_\Xi V^I &= ({\rm Re}\,\hat{\Xi}^m){\rm Re}\left(\nabla_m^{\mathcal{P}}V^I\right), \\ \delta_\Xi\Upsilon_{{\rm T}\alpha} &= \Xi^m\nabla_m\Upsilon_{{\rm T}\alpha}, \\ \delta_\Xi V_{{\rm T}m} &= ({\rm Re}\,\hat{\Xi}^n){\rm Re}\left(\nabla_n^{\mathcal{P}}V_{{\rm T}m}\right) + ({\rm Re}\,\nabla_m^{\mathcal{P}}\hat{\Xi}^n)V_{{\rm T}n}, \\ \delta_\Xi\Sigma_{\rm T} &= \nabla_m\left(\Xi^m\Sigma_{\rm T}\right), \end{aligned} \tag{4.21}$$

where

$$\nabla_m^{\mathcal{P}} \equiv \mathcal{P}_U\tilde{\nabla}_m\mathcal{P}_U^{-1}, \qquad \widehat{\nabla_m\Xi^n} \equiv \mathcal{P}_U\left(\nabla_m\Xi^n\right). \tag{4.22}$$

5 Rotations that mix 4D and extra dimensions

Here we consider the Lorentz transformations that mix 4D and the extra dimensions. In order to simplify the discussion, we treat the "off-diagonal" superfields U^m and Ψ_m^α at the linearized level in this section. Then, the corresponding superfield transformation laws are given by

$$\begin{aligned} &\delta_N U^\mu = 0, \qquad \delta_N U^4 = {\rm Re}\left(\frac{N}{S_{\rm E}}\right), \qquad \delta_N U^5 = -{\rm Re}\left(NS_{\rm E}\right), \\ &\delta_N\tilde{V}_{\rm E} = 2\tilde{V}_{\rm E}^{1/2}{\rm Im}\,\partial_{\rm E}\left(N\tilde{V}_{\rm E}^{1/2}\right), \qquad \delta_N S_{\rm E} = 0, \\ &\delta_N\Psi_4^\alpha = -\frac{iV_{\rm E}}{2}D^\alpha{\rm Im}\left(NS_{\rm E}\right), \qquad \delta_N\Psi_5^\alpha = -\frac{iV_E}{2}D^\alpha{\rm Im}\left(\frac{N}{S_{\rm E}}\right), \\ &\delta_N H_{\rm odd} = -\frac{i}{4}\bar{D}^2\left(N\tilde{V}_{\rm E}^{1/2}e^{-V}\bar{H}_{\rm even}\right), \qquad \delta_N H_{\rm even} = \frac{i}{4}\bar{D}^2\left(N\tilde{V}_{\rm E}^{1/2}e^{V}\bar{H}_{\rm odd}\right), \\ &\delta_N V^I = {\rm Im}\left\{N\left(\partial_{\rm E}V^I - 2\Sigma^I\right)\right\}, \qquad \delta_N\Sigma^I = -\frac{i}{8}\bar{D}^2\left(\tilde{V}_{\rm E}D^\alpha\bar{N}D_\alpha V^I\right), \end{aligned} \tag{5.1}$$

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where $\tilde{V}_{\rm E} \equiv V_{\rm E} R_{\rm E}$, and the transformation parameter N is a complex general superfield whose $\theta\bar{\theta}$-component is

$$N = (\theta\sigma^{\underline{\mu}}\bar{\theta})\left(\lambda_{\underline{\mu}\underline{4}} - i\lambda_{\underline{\mu}\underline{5}}\right)\frac{\sqrt{E_4 E_5}}{ie^{(2)}} + \cdots . \tag{5.2}$$

5.1 Invariance in hyper sector

The invariance of the action under the δ_N-transformation is less manifest than the δ_L- and the δ_Ξ-transformations because the cancellation between the $\int d^4\theta$- and the $\int d^2\theta$-integrals occurs in the δ_N-transformation. Here, we show the invariance in the hyper sector to illustrate such cancellation.

From (5.1), the hatted superfields transform as

$$\begin{aligned}
\delta_N \hat{H}_{\rm odd} &= \frac{i}{2}\left(N\partial_{\rm E} + \bar{N}\bar{\partial}_{\rm E}\right) H_{\rm odd} - \frac{i}{4}\bar{D}^2\left(N\tilde{V}_{\rm E}^{1/2} e^{-V}\bar{H}_{\rm even}\right), \\
\delta_N \hat{H}_{\rm even} &= \frac{i}{2}\left(N\partial_{\rm E} + \bar{N}\bar{\partial}_{\rm E}\right) H_{\rm even} + \frac{i}{4}\bar{D}^2\left(N\tilde{V}_{\rm E}^{1/2} e^{V}\bar{H}_{\rm odd}\right).
\end{aligned} \tag{5.3}$$

After some straightforward calculations, we can see that $L_{\rm H}^{(1)}$ in (4.3) and $L_{\rm H}^{(2)}$ in (3.15) transform as

$$\begin{aligned}
\delta_N L_{\rm H}^{(1)} = -\frac{i}{4}\bar{D}^2\Big[&2N\tilde{V}_{\rm E}^{1/2}\left\{H_{\rm odd}^\dagger \tilde{d}\left(\partial_{\rm E} + \Sigma\right) H_{\rm odd} + H_{\rm even}^\dagger \tilde{d}\left(\partial_{\rm E} - \Sigma\right) H_{\rm even}\right\} \\
&+N\tilde{V}_{\rm E}^{1/2}(\mathcal{O}_{\rm E} S_{\rm E})\left(H_{\rm odd}^\dagger \tilde{d} H_{\rm odd} + H_{\rm even}^\dagger \tilde{d} H_{\rm even}\right) \\
&+\frac{1}{2}\tilde{V}_{\rm E} D^\alpha \bar{N}\left(H_{\rm odd}^t \tilde{d} D_\alpha H_{\rm even} - H_{\rm even}^t \tilde{d} D_\alpha H_{\rm odd} - 2D_\alpha V H_{\rm odd}^t \tilde{d} H_{\rm even}\right)\Big],
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
\delta_N\left(-2\tilde{V}_{\rm E}^{1/2} L_{\rm H}^{(2)}\right) = {\rm Im}\Big[&4N\tilde{V}_{\rm E}^{1/2}\left\{H_{\rm odd}^\dagger \tilde{d} e^V\left(\partial_{\rm E} + \Sigma\right) H_{\rm odd} + H_{\rm even}^\dagger \tilde{d} e^{-V}\left(\partial_{\rm E} - \Sigma\right) H_{\rm even}\right\} \\
&-2N\tilde{V}_{\rm E}^{1/2}(\mathcal{O}_{\rm E} S_{\rm E})\left(H_{\rm odd}^\dagger \tilde{d} e^V H_{\rm odd} + H_{\rm even}^\dagger \tilde{d} e^{-V} H_{\rm even}\right) \\
&+\tilde{V}_{\rm E}^{1/2}\Big\{H_{\rm odd}^t \tilde{d} e^V D^2\left(\bar{N}\tilde{V}_{\rm E}^{1/2} e^{-V} H_{\rm even}\right) \\
&\qquad -H_{\rm even}^t \tilde{d} e^{-V} D^2\left(\bar{N}\tilde{V}_{\rm E}^{1/2} e^{V} H_{\rm odd}\right)\Big\}\Big],
\end{aligned} \tag{5.5}$$

up to total derivatives. We have dropped the U^m- and the Ψ_m^α-dependent terms in the right-hand-sides. The last line in $\delta_N L_{\rm H}^{(2)}$ can be rewritten as

$$\begin{aligned}
A &\equiv \tilde{V}_{\rm E}^{1/2}\left\{H_{\rm odd}^t \tilde{d} e^V D^2\left(\bar{N}\tilde{V}_{\rm E}^{1/2} e^{-V} H_{\rm even}\right) - H_{\rm even}^t \tilde{d} e^{-V} D^2\left(\bar{N}\tilde{V}_{\rm E}^{1/2} e^{V} H_{\rm odd}\right)\right\} \\
&= 2D^\alpha \bar{N}\left(\tilde{H}_{\rm o}^t \tilde{d} D_\alpha \tilde{H}_{\rm e} - \tilde{H}_{\rm e}^t \tilde{d} D_\alpha \tilde{H}_{\rm o}\right) + \bar{N}\left(\tilde{H}_{\rm o}^t \tilde{d} D^2 \tilde{H}_{\rm e} - \tilde{H}_{\rm e}^t \tilde{d} D^2 \tilde{H}_{\rm o}\right),
\end{aligned} \tag{5.6}$$

where

$$\tilde{H}_{\rm o} \equiv \tilde{V}_{\rm E}^{1/2} e^V H_{\rm odd}, \qquad \tilde{H}_{\rm e} \equiv \tilde{V}_{\rm E}^{1/2} e^{-V} H_{\rm even}. \tag{5.7}$$

This can also be rewritten as

$$A = \bar{N}\left(\tilde{H}_{\rm e}^t \tilde{d} D^2 \tilde{H}_{\rm o} - \tilde{H}_{\rm o}^t \tilde{d} D^2 \tilde{H}_{\rm e}\right), \tag{5.8}$$

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up to total derivatives. Therefore, we obtain

$$\begin{aligned} A &= D^\alpha \bar{N} \left(\tilde{H}_{\rm o} \tilde{d} D_\alpha \tilde{H}_{\rm e} - \tilde{H}_{\rm e}^t \tilde{d} D_\alpha \tilde{H}_{\rm o} \right) \\ &= \tilde{V}_{\rm E} D^\alpha \bar{N} \left(H_{\rm odd}^t \tilde{d} D_\alpha H_{\rm even} - H_{\rm even}^t \tilde{d} D_\alpha H_{\rm odd} - 2 D_\alpha V H_{\rm odd}^t \tilde{d} H_{\rm even} \right). \end{aligned} \tag{5.9}$$

We should also note that

$$\delta_N |J_{\mathcal{P}}| = \mathcal{O}(U^m), \tag{5.10}$$

since $|J_{\mathcal{P}}| = 1 + \mathcal{O}((U^m)^2)$.

Making use of these, we can show that

$$\delta_N \mathcal{L}_{\rm H} = \delta_N \left\{ -2 \int d^4\theta \ |J_{\mathcal{P}}| \, \tilde{V}_{\rm E}^{\ 1/2} L_{\rm H}^{(2)} + \left(\int d^2\theta \ L_{\rm H}^{(1)} + \text{h.c.} \right) \right\} = 0, \tag{5.11}$$

up to total derivatives. We have used the relation $\int d^2\bar{\theta} = -\frac{1}{4}\bar{D}^2$ in the $d^2\theta$-integration.

5.2 Kinetic terms for U^m and Ψ_m^α

Now we consider the kinetic terms for the gravitational superfields, which originate from the 6D Weyl multiplet. Among $\{U^\mu, U^m, \Psi_m^\alpha, V_{\rm E}, S_{\rm E}\}$, only $V_{\rm E}$ and $S_{\rm E}$ have nonvanishing background values. Here, we treat the superfields $\{U^\mu, U^m, \Psi_m^\alpha\}$ and the fluctuation parts of $V_{\rm E}$ and $S_{\rm E}$ at the linearized order, and neglect terms beyond quadratic in them. As shown in appendix A, the kinetic term for U^μ, $\mathcal{L}_{\rm E}^{N=1}$, is given by (A.12). There is an additional term that involves the "off-diagonal" component superfields U^m and Ψ_m^α.

We define the covariant derivatives of U^μ as

$$\tilde{\nabla}_m U^\mu \equiv \partial_m U^\mu - \frac{1}{2} \sigma_{\alpha\dot{\alpha}}^\mu \left(\bar{D}^{\dot{\alpha}} \Psi_m^\alpha - D^\alpha \bar{\Psi}_m^{\dot{\alpha}} \right), \tag{5.12}$$

where $\sigma_{\alpha\dot{\alpha}}^\mu = \langle e_{\underline{\nu}}^{\ \mu} \rangle \sigma_{\alpha\dot{\alpha}}^{\underline{\nu}}$. This has the Weyl weight 0, and is invariant under the δ_L-transformation. In order to construct the δ_N-invariant term, we redefine the above covariant derivatives as

$$\begin{aligned} \nabla_4 U^\mu &\equiv \tilde{\nabla}_4 U^\mu + V_{\rm E} \left\{ \left(\text{Im}\, S_{\rm E}^2 \right) \partial^\mu U^4 - \frac{i \text{Re}\, S_{\rm E}^2}{\bar{S}_{\rm E}^2} \partial^\mu U^5 \right\}, \\ \nabla_5 U^\mu &\equiv \tilde{\nabla}_5 U^\mu + V_{\rm E} \left\{ \frac{\text{Im}\, S_{\rm E}^2}{|S_{\rm E}|^4} \partial^\mu U^5 + \frac{i \text{Re}\, S_{\rm E}^2}{S_{\rm E}^2} \partial^\mu U^4 \right\}, \end{aligned} \tag{5.13}$$

where $\partial^\mu \equiv \langle e_{\underline{\rho}}^{\ \mu} \rangle \langle e_{\underline{\tau}}^{\ \nu} \rangle \eta^{\underline{\rho\tau}} \partial_\nu$. Then, the combination:

$$\begin{aligned} \mathcal{C}_{\rm E}^\mu &\equiv \frac{1}{S_{\rm E}} \nabla_4 U^\mu - S_{\rm E} \nabla_5 U^\mu \\ &= \frac{1}{S_{\rm E}} \tilde{\nabla}_4 U^\mu - S_{\rm E} \tilde{\nabla}_5 U^\mu - i V_{\rm E} \left(S_{\rm E} \partial^\mu U^4 + \frac{\partial^\mu U^5}{S_{\rm E}} \right) \end{aligned} \tag{5.14}$$

is δ_L- and δ_N-invariant at the linearized order.

$$\delta_L \mathcal{C}_{\rm E}^\mu = \mathcal{O}(U^m), \qquad \delta_N \mathcal{C}_{\rm E}^\mu = \mathcal{O}(U^m, D^\alpha V_{\rm E}, D^\alpha S_{\rm E}). \tag{5.15}$$

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Using this combination, we can construct the following δ_L- and δ_N-invariant Lagrangian term.

$$\mathcal{L}_{\mathcal{C}} = \int d^4\theta\, a\bar{\mathcal{C}}^{\mu}_{\rm E}\mathcal{C}_{{\rm E}\mu}, \tag{5.16}$$

where a is a real constant. The invariance of the action under the δ_Ξ-transformation determines a. Restoring the U^μ-dependence (see appendix A.3), the 6D Lagrangian should have the form of

$$\mathcal{L} = \mathcal{L}^{N=1}_{\rm E} + \mathcal{L}_{\mathcal{C}} + \int d^4\theta\, \left(1 + \frac{1}{12}\bar{\sigma}^{\dot{\alpha}\alpha}_{\mu}\left[D_\alpha, \bar{D}_{\dot{\alpha}}\right]U^\mu\right)\Omega + \left(\int d^2\theta\, W + \text{h.c.}\right), \tag{5.17}$$

where Ω and W are real and holomorphic functions respectively, whose explicit forms will be given in section 5.3. Recall that $\delta_\Xi\Omega = \partial_m\left(\text{Re}\,\hat{\Xi}^m\Omega\right)$ from the results in section 3. Then, we have

$$\begin{aligned}
\delta_\Xi\mathcal{L} &= \delta_\Xi\mathcal{L}_{\mathcal{C}} + \int d^4\theta\, \left(\frac{1}{12}\bar{\sigma}^{\dot{\alpha}\alpha}_{\underline{\mu}}\left[D_\alpha, \bar{D}_{\dot{\alpha}}\right]U^\mu\right)\partial_m\left(\text{Re}\,\Xi^m\Omega\right) + \cdots \\
&= \delta_\Xi\mathcal{L}_{\mathcal{C}} - \int d^4\theta\, \frac{\Omega}{12}\partial_m U^\mu \bar{\sigma}^{\dot{\alpha}\alpha}_{\underline{\mu}}\left[D_\alpha, \bar{D}_{\dot{\alpha}}\right]\text{Re}\,\Xi^m + \cdots \\
&= \delta_\Xi\mathcal{L}_{\mathcal{C}} - \int d^4\theta\, \frac{\Omega}{3}\partial_m U^\mu \text{Im}\,\partial_\mu\Xi^m + \cdots,
\end{aligned} \tag{5.18}$$

where we have dropped total derivatives, and also dropped the fluctuation part of Ω.[15] Here, since

$$\delta_\Xi\mathcal{C}^{\mu}_{\rm E} = iV_{\rm E}\left(S_{\rm E}\text{Im}\,\partial^\mu\Xi^4 + \frac{1}{S_{\rm E}}\text{Im}\,\partial^\mu\Xi^5\right) + \cdots, \tag{5.19}$$

where the ellipses are of $\mathcal{O}(U^\mu, U^m, \Psi_m, D^\alpha V_{\rm E}, D^\alpha S_{\rm E})$, we can see that

$$\begin{aligned}
\delta_\Xi\mathcal{L}_{\mathcal{C}} &= \int d^4\theta\, a\left\{\bar{\partial}_{\rm E}U^\mu \cdot iV_{\rm E}\left(S_{\rm E}\text{Im}\,\partial_\mu\Xi^4 + \frac{1}{S_{\rm E}}\text{Im}\,\partial_\mu\Xi^5\right)\right\} + \text{h.c.} + \cdots \\
&= -\int d^4\theta\, 2aV_{\rm E}\text{Im}\left(\frac{S_{\rm E}}{\bar{S}_{\rm E}}\partial_4 U^\mu \text{Im}\,\partial_\mu\Xi^4 - \frac{\bar{S}_{\rm E}}{S_{\rm E}}\partial_5 U^\mu \text{Im}\,\partial_\mu\Xi^5\right) + \cdots \\
&= \int d^4\theta\, 2aV_{\rm E}R_{\rm E}\partial_m U^\mu \text{Im}\,\partial_\mu\Xi^m + \cdots.
\end{aligned} \tag{5.20}$$

Therefore, from the δ_Ξ-invariance of the action, we find

$$a = \left\langle \frac{\Omega}{6V_{\rm E}R_{\rm E}} \right\rangle. \tag{5.21}$$

5.3 6D SUGRA Lagrangian

Here we summarize our results. The 6D SUGRA Lagrangian is expressed as

$$\mathcal{L} = \int d^4\theta\, L_{\rm E} + \int d^4\theta\, \left(1 + \frac{1}{12}\bar{\sigma}^{\dot{\alpha}\alpha}_{\mu}\left[D_\alpha, \bar{D}_{\dot{\alpha}}\right]U^\mu\right)\Omega_{\rm HVT} + \left(\int d^2\theta\, L^{(1)}_{\rm H} + \text{h.c.}\right), \tag{5.22}$$

[15] The superconformal gauge-fixing condition $\Omega|_{\theta=0} = -3M^4_{\rm 6D}$ must be imposed in order to obtain the Poincaré SUGRA. ($M_{\rm 6D}$ is the 6D Planck mass.)

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where

$$
\begin{aligned}
L_{\rm E} &\equiv \frac{\langle \Omega_{\rm HVT}\rangle}{3}\left\{\frac{1}{8}U^\mu D^\alpha \bar{D}^2 D_\alpha U_\mu + \frac{1}{48}\left(\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\left[D_\alpha, \bar{D}_{\dot{\alpha}}\right]U^\mu\right)^2 - (\partial_\mu U^\mu)^2 + \frac{\bar{\mathcal{C}}_{\rm E}^\mu \mathcal{C}_{{\rm E}\mu}}{2\langle V_{\rm E} R_{\rm E}\rangle}\right\},\\
\mathcal{C}_{\rm E}^\mu &\equiv \partial_{\rm E} U^\mu - \frac{1}{2}\sigma^\mu_{\alpha\dot{\alpha}}\left\{\frac{1}{S_{\rm E}}\left(\bar{D}^{\dot{\alpha}}\Psi_4^\alpha - D^\alpha \bar{\Psi}_4^{\dot{\alpha}}\right) - S_{\rm E}\left(\bar{D}^{\dot{\alpha}}\Psi_5^\alpha - D^\alpha\bar{\Psi}_5^{\dot{\alpha}}\right)\right\}\\
&\quad -iV_{\rm E}\partial^\mu\left(S_{\rm E}U^4 + \frac{U^5}{S_{\rm E}}\right),\\
\Omega_{\rm HVT} &\equiv -2\left|J_{\mathcal{P}}\right| V_{\rm E}^{1/2} R_{\rm E}^{1/2} L_{\rm H}^{(2)} + \left(J_{\mathcal{P}} L_{\rm V}^{(1)\alpha}\hat{\mathcal{Y}}_{{\rm T}\alpha} + {\rm h.c.}\right) + \mathcal{V}_{\rm T} L_{\rm V}^{(2)}\\
&\quad + \frac{\mathcal{X}_{\rm T}}{R_{\rm E}} L_{\rm V}^{(3)} + \left(J_{\mathcal{P}}\frac{\mathcal{X}_{\rm T}}{R_{\rm E}} L_{\rm V}^{(4)} + {\rm h.c.}\right),
\end{aligned}
\tag{5.23}
$$

and

$$
\begin{aligned}
L_{\rm H}^{(1)} &\equiv H_{\rm odd}^t \tilde{d}\left(\nabla_{\rm E} - \Sigma\right) H_{\rm even} - H_{\rm even}^t \tilde{d}\left(\nabla_{\rm E} + \Sigma\right) H_{\rm odd},\\
L_{\rm H}^{(2)} &\equiv \hat{H}_{\rm odd}^\dagger \tilde{d} e^V \hat{H}_{\rm odd} + \hat{H}_{\rm even}^\dagger \tilde{d} e^{-V}\hat{H}_{\rm even},\\
L_{\rm V}^{(1)\alpha} &\equiv f_{IJ}\left\{-2\hat{\Sigma}^I D^{\mathcal{P}\alpha} V^J + \frac{1}{2}\left(\nabla_{\rm E}^{\mathcal{P}} V^I D^{\mathcal{P}\alpha} V^J - \nabla_{\rm E}^{\mathcal{P}} D^{\mathcal{P}\alpha} V^I V^J\right)\right\},\\
L_{\rm V}^{(2)} &\equiv f_{IJ}\left(D^{\mathcal{P}\alpha} V^I \hat{\mathcal{W}}_\alpha^J + \frac{1}{2} V^I D^{\mathcal{P}\alpha}\hat{\mathcal{W}}_\alpha^J + {\rm h.c.}\right),\\
L_{\rm V}^{(3)} &\equiv f_{IJ}\left\{4\left(\nabla_{\rm E}^{\mathcal{P}} V^I - \hat{\Sigma}^I\right)^\dagger\left(\nabla_{\rm E}^{\mathcal{P}} V^J - \hat{\Sigma}^J\right) - 2\left(\nabla_{\rm E}^{\mathcal{P}} V^I\right)^\dagger \nabla_{\rm E}^{\mathcal{P}} V^J\right\},\\
L_{\rm V}^{(4)} &\equiv f_{IJ} J_S^{(1)} \frac{2\hat{S}_{\rm E}}{\bar{\hat{S}}_{\rm E}}\hat{\Sigma}^I\hat{\Sigma}^J.
\end{aligned}
\tag{5.24}
$$

The covariant derivatives $\nabla_{\rm E}$, $\nabla_{\rm E}^{\mathcal{P}}$ and $\nabla_m^{\mathcal{P}}$ are defined in section 4, and the field strengths are given by

$$
\begin{aligned}
\hat{\mathcal{W}}_\alpha^I &\equiv -\frac{1}{4}(\bar{D}^{\mathcal{P}})^2 D_\alpha^{\mathcal{P}} V^I + \mathcal{O}(U^\mu, U^m\Sigma),\\
\mathcal{X}_{\rm T} &\equiv \frac{1}{2}{\rm Im}\left(D^{\mathcal{P}\alpha}\Upsilon_{{\rm T}\alpha}\right),\\
\mathcal{Y}_{{\rm T}\alpha} &\equiv \frac{1}{2S_{\rm E}}\mathcal{W}_{{\rm T}4\alpha} + \frac{S_{\rm E}}{2}\mathcal{W}_{{\rm T}5\alpha} + \frac{1}{2}\left(\frac{1}{S_{\rm E}}\nabla_4 + S_{\rm E}\nabla_5\right)\Upsilon_{{\rm T}\alpha},\\
\mathcal{V}_{\rm T} &\equiv {\rm Re}\left(\nabla_4^{\mathcal{P}} V_{\rm T5} - \nabla_5^{\mathcal{P}} V_{\rm T4}\right) + J_{\mathcal{P}}\hat{\Sigma}_{\rm T} + \bar{J}_{\mathcal{P}}\bar{\hat{\Sigma}}_{\rm T},
\end{aligned}
\tag{5.25}
$$

and $J_{\mathcal{P}}$, $R_{\rm E}$ and $J_S^{(1)}$ are defined in (3.18), (3.53) and (3.54), respectively.

We have revived the U^μ-dependence. Thus, for a chiral superfield Φ, $\hat{\Phi}$ should be understood as

$$
\hat{\Phi}(x^M, \theta, \bar{\theta}) = \Phi(x^M + iU^M(\theta,\bar{\theta}), \theta, \bar{\theta}).
\tag{5.26}
$$

The U^μ-dependence of $\hat{\mathcal{W}}_\alpha^I$ is given by (A.10).

The real superfield $V_{\rm E}$ is expressed as

$$
V_{\rm E} = \frac{\mathcal{V}_{\rm T}}{\mathcal{X}_{\rm T}},
\tag{5.27}
$$

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and the chiral superfield $\Upsilon_{\mathrm{T}\alpha}$ is subject to the constraint:

$$\frac{1}{S_{\mathrm{E}}}\mathcal{W}_{\mathrm{T}4\alpha} - S_{\mathrm{E}}\mathcal{W}_{\mathrm{T}5\alpha} + \nabla_{\mathrm{E}}\Upsilon_{\mathrm{T}\alpha} = 0. \tag{5.28}$$

Note that this contains Ψ_m^α $(m = 4, 5)$. This constraint indicates that either Ψ_4^α or Ψ_5^α is a dependent superfield, i.e., it can be expressed in terms of the other superfields.

6 Dimensional reduction to 5D

We consider the situation that the two extra dimensions are compactified on a torus, i.e., $x^m \in [0, L_m]$. We take the coordinates so that $L_m = \mathcal{O}(1)$. Since the "off-diagonal" components of the sechsbein do not have nonvanishing background values, the line element along the extra dimensions is expressed as

$$ds^2 = \langle e_m{}^4 e_n{}^4 + e_m{}^5 e_n{}^5 \rangle dx^m dx^n = |\langle E_m \rangle dx^m|^2 . \tag{6.1}$$

Hence, the ratio of the sizes of the two extra dimensions is parameterized by the background value of S_{E} because

$$|\langle S_{\mathrm{E}} \rangle|^2 = \frac{|\langle E_4 \rangle|}{|\langle E_5 \rangle|}. \tag{6.2}$$

Therefore, the limit that the sixth (fifth) dimension shrinks to zero corresponds to the limit $|S_{\mathrm{E}}| \to \infty$ ($|S_{\mathrm{E}}| \to 0$). Since the extra dimensions are compactified, there are mass gaps between the zero-modes and the KK excited modes. For the latter, ∂_m gives $\mathcal{O}(1)$ factors because we have taken L_m as $\mathcal{O}(1)$. When $|S_{\mathrm{E}}| \to \infty$ ($|S_{\mathrm{E}}| \to 0$), terms involving ∂_5 (∂_4) in ∇_{E} grow infinitely large and drop out of the path integral. So we can neglect such terms because only the contributions from the zero-modes survive. In such a case, we should drop the covariant derivative ∇_5 (∇_4) in order to maintain the 4D diffeomorphism invariance. As a result, we can replace ∇_E with $\frac{1}{S_{\mathrm{E}}}\nabla_4$ ($-S_{\mathrm{E}}\nabla_5$) in this limit.

Let us consider the limit $|S_{\mathrm{E}}| \to \infty$ as an example.[16] In this case, we can neglect the x^5-dependence of the superfields, and the only extra-dimensional coordinate is $y \equiv x^4$. Thus $\mathcal{P}_U$ is understood as the operator that shifts y as $y \to y + iU^4$.

6.1 Hyper sector

First, we consider the hyper sector. The covariant derivative ∇_{E} becomes

$$\begin{aligned} \nabla_{\mathrm{E}} &\to \frac{1}{S_{\mathrm{E}}}\nabla_y^{(5\mathrm{D})}, \\ \nabla_y^{(5\mathrm{D})} &\equiv \partial_y - \frac{1}{4}\bar{D}^2\left(\Psi_y^\alpha D_\alpha\right) - \frac{w}{12}\bar{D}^2 D^\alpha \Psi_{y\alpha}, \end{aligned} \tag{6.3}$$

where $\Psi_y^\alpha \equiv \Psi_4^\alpha$. Thus, $L_{\mathrm{H}}^{(1)}$ in (5.24) becomes

$$L_{\mathrm{H}}^{(1)} \to H_{\mathrm{odd}}^{(5\mathrm{D})t}\tilde{d}\left(\nabla_y^{(5\mathrm{D})} - \Sigma^{(5\mathrm{D})}\right)H_{\mathrm{even}}^{(5\mathrm{D})} - H_{\mathrm{even}}^{(5\mathrm{D})t}\tilde{d}\left(\nabla_y^{(5\mathrm{D})} + \Sigma^{(5\mathrm{D})}\right)H_{\mathrm{odd}}^{(5\mathrm{D})}, \tag{6.4}$$

[16] The procedure in the limit $|S_{\mathrm{E}}| \to 0$ is similar if we use the relation (3.71).

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where

$$H_{\rm odd}^{(5{\rm D})} \equiv S_{\rm E}^{-1/2} H_{\rm odd}, \qquad H_{\rm even}^{(5{\rm D})} \equiv S_{\rm E}^{-1/2} H_{\rm even}, \qquad \Sigma^{(5{\rm D})I} \equiv S_{\rm E} \Sigma^I. \tag{6.5}$$

As for the full superspace part, we obtain

$$|J_{\mathcal{P}}|\, V_{\rm E}^{1/2} R_{\rm E}^{1/2} L_{\rm H}^{(2)} \to |J_y|\, V_{\rm E}^{(5{\rm D})1/2} \left(\hat{H}_{\rm odd}^{(5{\rm D})\dagger} \tilde{d} e^V \hat{H}_{\rm odd}^{(5{\rm D})} + \hat{H}_{\rm even}^{(5{\rm D})\dagger} \tilde{d} e^{-V} \hat{H}_{\rm even}^{(5{\rm D})} \right), \tag{6.6}$$

where

$$J_y \equiv 1 + i\partial_y U^4, \qquad V_{\rm E}^{(5{\rm D})} \equiv V_{\rm E} R_{\rm E} |\hat{S}_{\rm E}|^2. \tag{6.7}$$

The integrands (6.4) and (6.6) agree with those in ref. [19] at the linearized order in U^4.

6.2 Vector-tensor sector

Next consider the vector-tensor sector. Noting that

$$\begin{aligned} \partial_4^{\mathcal{P}} V^I &= \mathcal{P}_U \partial_4 \mathcal{P}_U^{-1} V^I \\ &= \partial_4 V^I - i \partial_4 U^m \partial_m V^I + (-i)^2 \partial_4 U^m \partial_m U^n \partial_n V^I + \cdots \\ &\to \sum_{n=0}^{\infty} \left(-i \partial_4 U^4 \right)^n \partial_4 V^I = \frac{1}{1 + i \partial_4 U^4} \partial_4 V^I = \frac{\partial_4 V^I}{J_y}, \end{aligned} \tag{6.8}$$

the covariant derivative $\nabla_{\rm E}^{\mathcal{P}}$ becomes

$$\nabla_{\rm E}^{\mathcal{P}} \to \frac{1}{J_y \hat{S}_{\rm E}} \nabla_y^{(5{\rm D})\mathcal{P}} + \mathcal{O}(\Psi_y U^4), \tag{6.9}$$

where

$$\nabla_y^{(5{\rm D})\mathcal{P}} \equiv \partial_y - \left(\frac{1}{4} \bar{D}^2 \Psi_y^\alpha D_\alpha + \frac{1}{2} \bar{D}^{\dot{\alpha}} \Psi_y^\alpha \bar{D}_{\dot{\alpha}} D_\alpha + \frac{w+n}{24} \bar{D}^2 D^\alpha \Psi_{y\alpha} + {\rm h.c.} \right) + \mathcal{O}(\Psi_y U^4). \tag{6.10}$$

Therefore, we obtain

$$L_{\rm V}^{(1)\alpha} \to \frac{f_{IJ}}{\hat{S}_{\rm E}} \left\{ -2 \hat{\Sigma}^{(5{\rm D})I} D^{\mathcal{P}\alpha} V^J + \frac{1}{2 J_y} \left(\nabla_y^{(5{\rm D})\mathcal{P}} V^I D^{\mathcal{P}\alpha} V^J - \nabla_y^{(5{\rm D})\mathcal{P}} D^{\mathcal{P}\alpha} V^I V^J \right) \right\}. \tag{6.11}$$

The field strengths $\mathcal{Y}_{{\rm T}\alpha}$ and $\mathcal{V}_{\rm T}$ become

$$\begin{aligned} \mathcal{Y}_{{\rm T}\alpha} &= S_{\rm E} \mathcal{W}_{{\rm T}5\alpha} + S_{\rm E} \nabla_5 \Upsilon_{{\rm T}\alpha} \\ &\to S_{\rm E} \mathcal{W}_\alpha^T, \\ \mathcal{V}_{\rm T} &\to \mathcal{V}^T \equiv \nabla_y^{(5{\rm D})\mathcal{P}} V^T - \left(J_y \hat{\Sigma}^{(5{\rm D})T} + {\rm h.c.} \right) + \mathcal{O}(\Psi_y U^4, (U^4)^2), \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} \mathcal{W}_\alpha^T &\equiv \mathcal{W}_{\rm T5} = -\frac{1}{4} \bar{D}^2 D_\alpha V^T, \\ V^T &\equiv V_{\rm T5}, \qquad \Sigma^{(5{\rm D})T} \equiv -\Sigma_{\rm T}. \end{aligned} \tag{6.13}$$

Thus, we obtain

$$\frac{\mathcal{X}_{\rm T}}{R_{\rm E}} = \frac{\mathcal{V}_{\rm T}}{V_{\rm E} R_{\rm E}} \to \frac{\mathcal{V}^T |\hat{S}_{\rm E}|^2}{V_{\rm E}^{(5{\rm D})}}, \tag{6.14}$$

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and

$$\begin{aligned}
&\frac{\mathcal{X}_{\rm T}}{R_{\rm E}}L_{\rm V}^{(3)}+\left(J_{\mathcal{P}}\frac{\mathcal{X}_{\rm T}}{R_{\rm E}}L_{\rm V}^{(4)}+\text{h.c.}\right)\\
&\quad\to\frac{\mathcal{V}^T|\hat{S}_{\rm E}|^2}{V_{\rm E}^{(5{\rm D})}}f_{IJ}\left\{\frac{4}{|\hat{S}_{\rm E}|^2}\left(\frac{1}{J_y}\nabla_y^{(5{\rm D})\mathcal{P}}V^I-\hat{\Sigma}^{(5{\rm D})I}\right)^\dagger\left(\frac{1}{J_y}\nabla_y^{(5{\rm D})\mathcal{P}}V^J-\hat{\Sigma}^{(5{\rm D})J}\right)\right.\\
&\qquad\qquad\left.+\left(\frac{2J_y}{\bar{J}_y|\hat{S}_{\rm E}|^2}\hat{\Sigma}^{(5{\rm D})I}\hat{\Sigma}^{(5{\rm D})J}+\text{h.c.}\right)\right\}\\
&\quad=\frac{2\mathcal{V}^Tf_{IJ}}{V_{\rm E}^{(5{\rm D})}}\mathcal{V}^I\mathcal{V}^J,
\end{aligned}\tag{6.15}$$

up to $\mathcal{O}(\Psi_4U^4,(U^4)^2)$, where

$$\mathcal{V}^I\equiv\nabla_y^{(5{\rm D})\mathcal{P}}V^I-\left(J_y\hat{\Sigma}^{(5{\rm D})I}+\text{h.c.}\right).\tag{6.16}$$

We have used the limit of $J_S^{(1)}\to1/\bar{J}_y+\mathcal{O}((U^4)^2)$.

As a result, the Lagrangian in the vector-tensor sector becomes

$$\begin{aligned}
\mathcal{L}_{\rm VT}\;\equiv\;&\int d^4\theta\left\{\left(J_{\mathcal{P}}L_{\rm V}^{(1)\alpha}\hat{\mathcal{Y}}_{{\rm T}\alpha}+\text{h.c.}\right)+\mathcal{V}_{\rm T}L_{\rm V}^{(2)}+\frac{\mathcal{X}_{\rm T}}{R_{\rm E}}L_{\rm V}^{(3)}+\left(J_{\mathcal{P}}\frac{\mathcal{X}_{\rm T}}{R_{\rm E}}L_{\rm V}^{(4)}+\text{h.c.}\right)\right\}\\
\to\;&\int d^4\theta\;\left[f_{IJ}\left\{-2J_y\hat{\Sigma}^{(5{\rm D})I}D^{\mathcal{P}\alpha}V^J\right.\right.\\
&\qquad\left.+\frac{1}{2J_y}\left(\nabla_y^{(5{\rm D})\mathcal{P}}V^ID^{\mathcal{P}\alpha}V^J-\nabla_y^{(5{\rm D})\mathcal{P}}D^{\mathcal{P}\alpha}V^IV^J\right)\right\}\hat{\mathcal{W}}_\alpha^T\\
&\qquad+f_{IJ}\mathcal{V}^T\left(D^{\mathcal{P}\alpha}V^I\hat{\mathcal{W}}_\alpha^J+\frac{1}{2}V^ID^{\mathcal{P}\alpha}\hat{\mathcal{W}}_\alpha^J+\text{h.c.}\right)+\left.\frac{2f_{IJ}}{V_{\rm E}^{(5{\rm D})}}\mathcal{V}^T\mathcal{V}^I\mathcal{V}^J\right]\\
=\;&\left(-\int d^2\theta\;C_{\bar{I}\bar{J}\bar{K}}\Sigma^{(5{\rm D})\bar{I}}\mathcal{W}^{\bar{J}}\mathcal{W}^{\bar{K}}+\text{h.c.}\right)\\
&+\int d^4\theta\;\left\{\frac{C_{\bar{I}\bar{J}\bar{K}}}{3J_y}\left(\partial_yV^{\bar{I}}D^\alpha V^{\bar{J}}-\partial_yD^\alpha V^{\bar{I}}V^{\bar{J}}\right)\hat{\mathcal{W}}_\alpha^{\bar{K}}+\text{h.c.}\right\}\\
&+\int d^4\theta\;\frac{2C_{\bar{I}\bar{J}\bar{K}}}{3V_{\rm E}^{(5{\rm D})}}\mathcal{V}^{\bar{I}}\mathcal{V}^{\bar{J}}\mathcal{V}^{\bar{K}},
\end{aligned}\tag{6.17}$$

up to $\mathcal{O}(\Psi_4U^4,(U^4)^2)$, where the indices $\bar{I},\bar{J},\bar{K}$ run over $T,1,2,\cdots$, and the completely symmetric constant tensor $C_{\bar{I}\bar{J}\bar{K}}$ is defined as $C_{IJT}=f_{IJ}$ and the other components are zero. This agrees with the 5D result in ref. [19] at the linearized order in Ψ_y^α and U^4. At the last step in (6.17), we have used the relation

$$\begin{aligned}
&\frac{f_{IJ}}{J_y}\left\{\left(\nabla_y^{(5{\rm D})\mathcal{P}}V^ID^{\mathcal{P}\alpha}V^J-\nabla_y^{(5{\rm D})\mathcal{P}}D^{\mathcal{P}\alpha}V^IV^J\right)\hat{\mathcal{W}}_{{\rm T}5\alpha}\right.\\
&\qquad\left.+\left(\nabla_y^{(5{\rm D})\mathcal{P}}V^TD^{\mathcal{P}\alpha}V^I-\nabla_y^{(5{\rm D})\mathcal{P}}D^{\mathcal{P}\alpha}V^TV^I\right)\hat{\mathcal{W}}_\alpha^J\right\}+\text{h.c.}\\
&\qquad=\frac{2f_{IJ}}{J_y}\left(\nabla_y^{(5{\rm D})\mathcal{P}}V^ID^{\mathcal{P}\alpha}V^T-\nabla_y^{(5{\rm D})\mathcal{P}}D^{\mathcal{P}\alpha}V^IV^T\right)\hat{\mathcal{W}}_\alpha^J+\text{h.c.},
\end{aligned}\tag{6.18}$$

which can be shown in the same way as appendix D in ref. [38].

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6.3 Gravitational sector

Finally, we consider the gravitational sector. Since $\mathcal{C}^{\mu}_{\rm E}$ in (5.14) becomes

$$\mathcal{C}^{\mu}_{\rm E} \to \frac{1}{S_{\rm E}}\nabla_4 U^{\mu} = \frac{1}{S_{\rm E}}\left\{\partial_4 U^{\mu} - \frac{1}{2}\sigma^{\mu}_{\alpha\dot{\alpha}}\left(\bar{D}^{\dot{\alpha}}\Psi_4^{\alpha} - D^{\alpha}\bar{\Psi}_4^{\dot{\alpha}}\right) + V_E(\mathrm{Im}\, S^2_{\rm E})\partial^{\mu}U^4\right\}, \tag{6.19}$$

we find that

$$\frac{\bar{\mathcal{C}}^{\mu}_{\rm E}\mathcal{C}_{{\rm E}\mu}}{2\langle V_{\rm E}R_{\rm E}\rangle} \to \frac{\mathcal{C}^{({\rm 5D})\mu}\mathcal{C}^{({\rm 5D})}_{\mu}}{2\langle V^{({\rm 5D})}_{\rm E}\rangle}, \tag{6.20}$$

where

$$\mathcal{C}^{({\rm 5D})\mu} \equiv \partial_y U^{\mu} - \frac{1}{2}\sigma^{\mu}_{\alpha\dot{\alpha}}\left(\bar{D}^{\dot{\alpha}}\Psi_y^{\alpha} - D^{\alpha}\bar{\Psi}_y^{\dot{\alpha}}\right) - V^{({\rm 5D})}_{\rm E}\partial^{\mu}U^4. \tag{6.21}$$

This agrees with the kinetic term for U^4 and Ψ^{α}_y in ref. [19].

Finally, we give a comment on the independence of $V^{({\rm 5D})}_{\rm E}$ defined in (6.7). Notice that $S_{\rm E}$ disappears in the 5D action, and $\Upsilon_{{\rm T}\alpha}$ appears only through $\mathcal{X}_{\rm T}$ in $V_{\rm E}$ after the dimensional reduction. (The $\Upsilon_{\rm T}$-dependence of $\mathcal{Y}_{{\rm T}\alpha}$ disappears as shown in (6.12).) Thus, although $V_{\rm E}$ in the 6D SUGRA action is not an independent degree of freedom (see (5.27)), $V^{({\rm 5D})}_{\rm E}$ is independent in the 5D SUGRA action. Namely, the degrees of freedom of $S_{\rm E}$ and $\Upsilon_{{\rm T}\alpha}$ are converted into that of $V^{({\rm 5D})}_{\rm E}$.

7 Summary

In this paper, we have completed the $\mathcal{N} = 1$ superfield description of 6D SUGRA. Specifically, we have clarified the dependence of the action on the $\mathcal{N} = 1$ superfields that contain the "off-diagonal" components of the sechsbein $e_m^{\ \underline{\nu}}$, $e_{\mu}^{\ \underline{n}}$, which were missing in our previous work [38]. These superfields are necessary for the invariance of the action under the full 6D diffeomorphisms and the Lorentz transformations in the $\mathcal{N} = 1$ superfield description. The corresponding superfields U^m and Ψ^{α}_m play roles of the gauge fields for those transformations. Although they do not have zero-modes in many extra-dimensional models, they can give significant effects on 4D effective theory when they are integrated out, as in the case of 5D SUGRA [17, 18].

Our results are collected in section 5.3. The superfields U^m and Ψ^{α}_m appear in the action in a nontrivial manner, but the resultant action is consistent with the 6D diffeomorphisms, 6D Lorentz transformations and the transformation laws of the component fields. Besides, it reduces to the known 5D SUGRA action in ref. [19]. These properties ensure the reliability of our result.

In this paper, Ψ^{α}_m are treated at the linearized level. This is because we have adopted the linearized 4D SUGRA formulation [14, 41, 42] to describe the 4D part of the 6D Weyl multiplet. In order to treat Ψ^{α}_m at full order, we need to use the complete conformal superspace formulation [13], which is technically more complicated.

Our 6D SUGRA description is useful to construct or analyze various setups for the braneworld models that contain lower-dimensional branes or the orbifold fixed points. Besides, it is also powerful for the systematic derivation of 4D effective action that keeps the $\mathcal{N} = 1$ superspace structure.

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We have focused on the case of the Abelian gauge group, for simplicity. In order to extend our result to the non-Abelian case, we need to include an additional term, which is the SUGRA counterpart of (3.9) in ref. [1] or (2.23) in ref. [2], to ensure the gauge invariance.

We will discuss these issues in a subsequent paper.

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A $\mathcal{N} = 1$ SUGRA couplings

In this section, we summarize the result of ref. [14], and show how to obtain the couplings to the $\mathcal{N} = 1$ SUGRA multiplet. This corresponds to the modification of the 4D linearized SUGRA [41, 42] to make the relation to the superconformal formulation in refs. [10]–[12] clearer. Before the gauge fixing of the extraneous symmetry, the action has the $\mathcal{N} = 1$ superconformal symmetry that consists of the invariance under the translation $\boldsymbol{P}$, SUSY $\boldsymbol{Q}$, the local Lorentz transformation $\boldsymbol{M}$, the dilatation $\boldsymbol{D}$, the automorphism $\mathrm{U}(1)_A$, the conformal boost $\boldsymbol{K}$, and the conformal SUSY $\boldsymbol{S}$. In ref. [14], we expressed this formulation in the language of the superfields at the linearized order in (the fluctuation part of) the gravitational fields. In this appendix, we neglect terms beyond this order, and the background spacetime is assumed to be 4D Minkowski spacetime.

A.1 Definition of superfields

The independent fields in the Weyl multiplet are the vierbein $e_\mu{}^{\underline{\nu}}$, the gravitino $\psi_{\mu\alpha}$, the $\mathrm{U}(1)_A$-gauge field A_μ, and the $\boldsymbol{D}$-gauge field b_μ. Among them, b_μ does not play any essential role, and can be set to zero, which corresponds to the $\boldsymbol{K}$-gauge fixing.

The vierbein $e_\mu{}^{\underline{\nu}}$ is divided into the background $\langle e_\mu{}^{\underline{\nu}} \rangle$ and the fluctuation $\tilde{e}_\mu{}^\nu$ as

$$e_\mu{}^{\underline{\nu}} = \langle e_\mu{}^{\underline{\rho}} \rangle \left(\delta_\rho{}^\nu + \tilde{e}_\rho{}^\nu \right), \tag{A.1}$$

where $\langle e_\mu{}^{\underline{\nu}} \rangle = \delta_\mu{}^\nu$ by our assumption.[17] Then we can form the following real superfield.

$$\begin{aligned} U^\mu \;=\; & (\theta\sigma^{\underline{\rho}}\bar{\theta}) \langle e_{\underline{\rho}}{}^\nu \rangle \tilde{e}_\nu{}^\mu + i\bar{\theta}^2 \langle e_{\underline{\rho}}{}^\mu \rangle \left(\theta\sigma^{\underline{\nu}}\bar{\sigma}^{\underline{\rho}}\psi_{\underline{\nu}} \right) - i\theta^2 \langle e_{\underline{\rho}}{}^\mu \rangle \left(\bar{\theta}\bar{\sigma}^{\underline{\nu}}\sigma^{\underline{\rho}}\bar{\psi}_{\underline{\nu}} \right) \\ & + \frac{1}{4}\theta^2\bar{\theta}^2 \left(3A^\mu - \epsilon^{\mu\nu\rho\tau}\partial_\nu \tilde{e}_{\rho\tau} \right). \end{aligned} \tag{A.2}$$

We have included $\langle e_\mu{}^\nu \rangle$ in the above expression in order to make the counting of the Weyl weight clear. This superfield has the Weyl weight 0.

We construct a chiral superfield from a (superconformal) chiral multiplet $[\phi, \chi_\alpha, F]$ as

$$\begin{aligned} \Phi &= \left(1 + \frac{w}{3}\mathcal{E} \right) \left(\phi + \theta\chi + \theta^2 F \right), \\ \mathcal{E} &\equiv \tilde{e}_\mu{}^\mu - 2i\theta\sigma^{\underline{\mu}}\bar{\psi}_{\underline{\mu}}, \end{aligned} \tag{A.3}$$

where w denotes the Weyl weight (i.e., the $\boldsymbol{D}$ charge) of this multiplet.

[17] We need not discriminate the curved indices μ from the flat one $\underline{\mu}$ for $\tilde{e}_\rho{}^\nu$ whose Weyl weight is 0.

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We also construct a real (unconstrained) superfield from a real general multiplet $[C, \zeta_\alpha, \mathcal{H}, B_\mu, \lambda_\alpha, D]$[18] as

$$V = \left\{1 + \frac{w}{6}\left(\mathcal{E} + \bar{\mathcal{E}}\right)\right\}\left\{C + i\theta\zeta - i\bar{\theta}\bar{\zeta} - \theta^2\mathcal{H} - \bar{\theta}^2\bar{\mathcal{H}} - (\theta\sigma^{\underline{\mu}}\bar{\theta})B'_{\underline{\mu}} + i\theta^2(\bar{\theta}\bar{\lambda}') - i\bar{\theta}^2(\theta\lambda') + \frac{1}{2}\theta^2\bar{\theta}^2 D'\right\}, \tag{A.4}$$

where

$$\begin{aligned} B'_\mu &\equiv B_\mu - \zeta\psi_\mu - \bar{\zeta}\bar{\psi}_\mu - \frac{w}{2}CA_\mu, \\ \lambda'_\alpha &\equiv \lambda_\alpha - \frac{i}{2}\left(\sigma^\mu\partial_\mu\bar{\zeta}\right)_\alpha - \left(\sigma^\mu\bar{\sigma}^\nu\psi_\mu\right)_\alpha B_\nu - \frac{w}{4}\left(\sigma^\mu\bar{\zeta}\right)_\alpha A_\mu, \\ D' &\equiv D - \frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu C + \cdots, \end{aligned} \tag{A.5}$$

and $\sigma^\mu_{\alpha\dot{\alpha}} \equiv \langle e_{\underline{\nu}}{}^\mu\rangle\sigma^{\underline{\nu}}_{\alpha\dot{\alpha}}$.

A.2 Superconformal transformation

With the above definitions of the superfields, the (linearized) superconformal transformations are expressed as[19]

$$\begin{aligned} \delta_L U^\mu &= -\frac{1}{2}\sigma^\mu_{\alpha\dot{\alpha}}\left(\bar{D}^{\dot{\alpha}}L^\alpha - D^\alpha\bar{L}^{\dot{\alpha}}\right), \\ \delta_L\Phi &= \left(-\frac{1}{4}\bar{D}^2L^\alpha D_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}}L^\alpha\partial_\mu - \frac{w}{12}\bar{D}^2D^\alpha L_\alpha\right)\Phi \\ &= -\frac{1}{4}\bar{D}^2\left(L^\alpha D_\alpha\Phi + \frac{w}{3}D^\alpha L_\alpha\Phi\right), \\ \delta_L V &= \left(-\frac{1}{4}\bar{D}^2L^\alpha D_\alpha + \frac{i}{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}}L^\alpha\partial_\mu - \frac{w}{24}\bar{D}^2D^\alpha L_\alpha + \text{h.c.}\right)V, \end{aligned} \tag{A.6}$$

where the transformation parameter L^α is an unconstrained complex spinor superfield. The components of L^α denoted as

$$\begin{aligned} \xi^\mu &\equiv -\text{Re}\left(i\sigma^\mu_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}}L^\alpha\right)\big|_{\theta=0}, & \epsilon_\alpha &\equiv -\frac{1}{4}\bar{D}^2L_\alpha\Big|_{\theta=0}, \\ \lambda_{\mu\nu} &\equiv -\frac{1}{2}\text{Re}\left\{\left(\sigma_{\mu\nu}\right)_\alpha{}^\beta D_\alpha\bar{D}^2L^\beta\right\}\Big|_{\theta=0}, & \varphi_D &\equiv \text{Re}\left(\frac{1}{4}D^\alpha\bar{D}^2L_\alpha\right)\Big|_{\theta=0}, \\ \vartheta_A &\equiv \text{Im}\left(-\frac{1}{6}D^\alpha\bar{D}^2L_\alpha\right)\Big|_{\theta=0}, & \eta_\alpha &\equiv -\frac{1}{32}D^2\bar{D}^2L_\alpha\Big|_{\theta=0}, \end{aligned} \tag{A.7}$$

represent the transformation parameters for $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{M}$, $\boldsymbol{D}$, $\text{U}(1)_A$ and $\boldsymbol{S}$, respectively. As we can see from (A.6), U^μ transforms nonlinearly, and thus it corresponds to the gauge (super)field for the δ_L-transformation. We should also note that this superfield transformation preserves the chirality condition: $\bar{D}_{\dot{\alpha}}\Phi = 0$.

[18] A complex scalar $\mathcal{H}$ is $\frac{1}{2}(H + iK)$ in the notation of ref. [12].

[19] We take the metric convention and the definitions of the spinor derivatives of ref. [40], which are different from those in ref. [14].

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A.3 Invariant action

For a given global SUSY Lagrangian:

$$\mathcal{L}_{\rm matter} = \int d^4\theta\, \Omega(\Phi, V) + \left[\int d^2\theta\, \left\{W(\Phi) - \frac{1}{4} f(\Phi)\mathcal{W}^\alpha \mathcal{W}_\alpha\right\} + \text{h.c.}\right], \tag{A.8}$$

where Ω is a real function, W and f are holomorphic functions, and $\mathcal{W}_\alpha \equiv -\frac{1}{4}\bar{D}^2 D_\alpha V$, we can make it invariant under the δ_L-transformation by inserting U^μ in the following way.

$$\mathcal{L} = \int d^4\theta\, \left(1 + \frac{1}{3}E_1\right)\Omega\left(\Phi_U, V\right) + \left[\int d^2\theta\, \left\{W(\Phi) - \frac{1}{4} f(\Phi)\mathcal{W}_U^\alpha \mathcal{W}_{U\alpha}\right\} + \text{h.c.}\right], \tag{A.9}$$

where

$$\begin{aligned} E_1 &\equiv \frac{1}{4}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\left[D_\alpha, \bar{D}_{\dot{\alpha}}\right] U^\mu, \qquad \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \equiv \langle e_\mu{}^{\underline{\nu}}\rangle \bar{\sigma}_{\underline{\nu}}^{\dot{\alpha}\alpha}, \\ \Phi_U &\equiv \left(1 + iU^\mu\partial_\mu\right)\Phi, \\ \mathcal{W}_{U\alpha} &\equiv -\frac{1}{4}\bar{D}^2\left(D_\alpha V + \frac{1}{4}D_\alpha U^\mu \bar{\sigma}_\mu^{\dot{\beta}\beta}\left[D_\beta, \bar{D}_{\dot{\beta}}\right] V - iU^\mu\partial_\mu D_\alpha V\right). \end{aligned} \tag{A.10}$$

Here, the operation of $(1 + iU^\mu\partial_\mu)$ on Φ is understood as the embedding of the chiral multiplet into a general multiplet. The modified field strength superfield $\mathcal{W}_{U\alpha}$ is invariant under the gauge transformation:

$$V \to V + \left(1 + iU^\mu\partial_\mu\right)\Lambda + \left(1 - iU^\mu\partial_\mu\right)\bar{\Lambda}, \tag{A.11}$$

where Λ is a chiral superfield.

The kinetic term for U^μ is given by[20]

$$\mathcal{L}_{\rm E}^{N=1} = \int d^4\theta\, \frac{\langle\Omega\rangle}{3}\left\{\frac{1}{8}U^\mu D^\alpha \bar{D}^2 D_\alpha U_\mu + \frac{1}{3}E_1^2 - \left(\partial_\mu U^\mu\right)^2\right\}, \tag{A.12}$$

where the Weyl weight of $U_\mu = \langle e_\mu{}^{\underline{\rho}}\rangle\langle e_\nu{}^{\underline{\tau}}\rangle\eta_{\underline{\rho\tau}}U^\nu$ is -2.

Using the above insertion of U^μ, the $\mathcal{N} = 1$ (linearized) SUGRA Lagrangian is obtained by choosing

$$\begin{aligned} \Omega &= -3\left|\Phi_U^{\rm com}\right|^2 e^{-K(\Phi_U, V)/3}, \\ W &= \left(\Phi^{\rm comp}\right)^3 W_{\rm SUGRA}(\Phi), \end{aligned} \tag{A.13}$$

where $\Phi^{\rm comp}$ is the compensator chiral superfield, Φ is the physical chiral superfield, the real function $K(\Phi_U, V)$ is the Kähler potential, and the holomorphic function $W_{\rm SUGRA}(\Phi)$ is the superpotential.

[20]The $\boldsymbol{D}$-gauge-fixing condition that leads to the canonically normalized Einstein-Hilbert term is given by $\Omega|_{\theta=0} = -3$ in the unit of the Planck mass.

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B Diffeomorphism of component fields

Under the diffeomorphism, the coordinates and the fields transform as

$$\begin{aligned}
\delta_\xi x^M &= \xi^M \\
\delta_\xi e_M^{\ \underline{N}} &= \xi^L \partial_L e_M^{\ \underline{N}} + \partial_M \xi^L e_L^{\ \underline{N}}, \\
\delta_\xi \phi_i^{\bar{A}} &= \xi^M \partial_M \phi_i^{\bar{A}}, \\
\delta_\xi A_M^I &= \xi^N \partial_N A_M^I + \partial_M \xi^N A_N^I, \\
\delta_\xi \sigma &= \xi^M \partial_M \sigma, \\
\delta_\xi B_{MN} &= \xi^L \partial_L B_{MN} + \partial_M \xi^L B_{LN} + \partial_N \xi^L B_{ML}, \qquad \cdots,
\end{aligned} \tag{B.1}$$

where the transformation parameters $\xi^M(x)$ are real functions. The 6D diffeomorphism δ_ξ can be divided into the 4D part $\delta_\xi^{(1)}$ with ξ^μ, and the extra-dimensional part $\delta_\xi^{(2)}$ with ξ^m. In this section, we focus on the $\delta_\xi^{(2)}$-transformations of the component fields of the $\mathcal{N}=1$ superfields.

B.1 Weyl multiplet

From the second equation in (B.1), $E_m \equiv e_m^{\ \underline{4}} + i e_m^{\ \underline{5}}$ transforms as

$$\delta_\xi^{(2)} E_m = \partial_m \xi^n E_n + \xi^n \partial_n E_m, \tag{B.2}$$

which leads to

$$\begin{aligned}
\delta_\xi^{(2)} S_E| &= \xi^m \partial_m S_E| + \frac{1}{2}\left(\partial_4 \xi^4 - \partial_5 \xi^5 + \frac{1}{S_E^2|}\partial_4 \xi^5 - S_E^2| \partial_5 \xi^4\right) S_E|, \\
\delta_\xi^{(2)} \left(E_4 E_5\right) &= \xi^m \partial_m \left(E_4 E_5\right) + \left(\partial_m \xi^m + \frac{1}{S_{\mathrm{E}}^2|}\partial_4 \xi^5 + S_{\mathrm{E}}^2| \partial_5 \xi^4\right)\left(E_4 E_5\right),
\end{aligned} \tag{B.3}$$

where $S_E| \equiv \sqrt{E_4/E_5}$.

Here we impose the constraint:[21]

$$\partial_m \xi^{\underline{\mu}} = \partial_m \left(\xi^N e_N^{\ \underline{\mu}}\right) = 0. \tag{B.4}$$

Then the "off-diagonal" components $e_m^{\ \underline{\mu}}$ transform as

$$\begin{aligned}
\delta_\xi e_m^{\ \underline{\mu}} &= \xi^N \partial_N e_m^{\ \underline{\mu}} + \partial_m \xi^N e_N^{\ \underline{\mu}} \\
&= \xi^N \partial_N e_m^{\ \underline{\mu}} - \xi^N \partial_m e_N^{\ \underline{\mu}}.
\end{aligned} \tag{B.5}$$

Namely, its $\delta_\xi^{(2)}$-transformation is

$$\delta_\xi^{(2)} e_m^{\ \underline{\mu}} = \xi^n \left(\partial_n e_m^{\ \underline{\mu}} - \partial_m e_n^{\ \underline{\mu}}\right). \tag{B.6}$$

[21] This constraint preserves the values of $e_m^{\ \underline{\mu}}$ under the 4D diffeomorphism, but we do not take a gauge in which they are fixed to zero.

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Since

$$\begin{aligned}\delta_\xi e_{\underline{M}}{}^{N} &= -e_{\underline{M}}{}^{L}\left(\delta_\xi e_{L}{}^{P}\right)e_{\underline{P}}{}^{N}\\ &= \xi^P\partial_P e_{\underline{M}}{}^{N} - e_{\underline{M}}{}^{P}\partial_P\xi^N = \xi^P\partial_P e_{\underline{M}}{}^{N} - \partial_{\underline{M}}\xi^N,\end{aligned} \tag{B.7}$$

we obtain

$$\delta_\xi^{(2)} e_{\underline{\mu}}{}^{m} = -\partial_{\underline{\mu}}\xi^m + \xi^n\partial_n e_{\underline{\mu}}{}^{m}. \tag{B.8}$$

Besides, $e^{(2)} = e_4{}^{4}e_5{}^{5} - e_4{}^{5}e_5{}^{4}$ transforms as

$$\delta_\xi^{(2)} e^{(2)} = \partial_m\left(\xi^m e^{(2)}\right). \tag{B.9}$$

Hence, it follows that

$$\begin{aligned}\delta_\xi^{(2)}\tilde{V}_{\rm E}| &= \delta_\xi^{(2)}\left(\frac{e^{(2)2}}{|E_4E_5|}\right)\\ &= \frac{2e^{(2)}\delta_\xi^{(2)}e^{(2)}}{|E_4E_5|} - \frac{e^{(2)2}}{|E_4E_5|^3}\mathrm{Re}\left\{\bar{E}_4\bar{E}_5\delta_\xi^{(2)}\left(E_4E_5\right)\right\}\\ &= \xi^m\partial_m\tilde{V}_{\rm E}| + \mathrm{Re}\left(\partial_m\xi^m - \frac{1}{S_{\rm E}^2|}\partial_4\xi^5 - S_{\rm E}^2|\partial_5\xi^4\right)\tilde{V}_{\rm E}|,\end{aligned} \tag{B.10}$$

where $\tilde{V}_{\rm E} \equiv V_{\rm E}R_{\rm E}$.

B.2 Hypermultiplet

Combining the third equation in (B.1) with the second equation in (B.3), we obtain the transformation of $H^{\bar{A}}| \equiv (E_4E_5)^{1/4}\phi_2^{\bar{A}}$ as

$$\delta_\xi^{(2)}H^{\bar{A}}| = \xi^m\partial_m H^{\bar{A}}| + \frac{1}{4}\left(\partial_m\xi^m + \frac{1}{S_{\rm E}^2|}\partial_4\xi^5 + S_{\rm E}^2|\partial_5\xi^4\right)H^{\bar{A}}|. \tag{B.11}$$

B.3 Vector multiplet

Combining the fourth equation in (B.1) with (B.7), we can show that

$$\delta_\xi^{(2)}A_{\underline{\mu}}^I = \delta_\xi^{(2)}\left(e_{\underline{\mu}}{}^{N}A_N^I\right) = \xi^n\partial_n A_{\underline{\mu}}^I. \tag{B.12}$$

As for the extra-dimensional components, we see that

$$\begin{aligned}\delta_\xi^{(2)}\Sigma^I| &= \frac{i}{2}\left(-\frac{1}{S_{\rm E}^2|}A_4^I - A_5^I\right)\delta_\xi^{(2)}S_{\rm E}| + \frac{i}{2}\left(\frac{1}{S_{\rm E}|}\delta_\xi^{(2)}A_4^I - S_{\rm E}|\delta_\xi^{(2)}A_5^I\right)\\ &= \xi^m\partial_m\Sigma^I| + \frac{1}{2}\left(\partial_m\xi^m - \frac{1}{S_{\rm E}^2|}\partial_4\xi^5 - S_{\rm E}^2|\partial_5\xi^4\right)\Sigma^I|,\end{aligned} \tag{B.13}$$

where $\Sigma^I| = \frac{i}{2}\left(S_{\rm E}^{-1}|A_4^I - S_{\rm E}|A_5^I\right)$.

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B.4 Tensor multiplet

From the last equation in (B.1) and (B.7), we have

$$\begin{aligned}
\delta_\xi^{(2)} B_{\underline{\mu\nu}} &= \xi^n \partial_n B_{\underline{\mu\nu}}, \\
\delta_\xi^{(2)} B_{\underline{\mu}m} &= \xi^n \partial_n B_{\underline{\mu}m} + \partial_m \xi^n B_{\underline{\mu}n}, \\
\delta_\xi^{(2)} B_{45} &= \xi^n \partial_n B_{45} + \partial_4 \xi^4 B_{45} + \partial_5 \xi^5 B_{45} = \partial_n \left(\xi^n B_{45}\right), \\
\delta_\xi^{(2)} B_{\underline{45}} &= \xi^n \partial_n B_{\underline{45}}.
\end{aligned} \tag{B.14}$$

C Lorentz transformations of component fields

In this section we see the Lorentz transformations of the component fields of the superfields.

C.1 Weyl multiplet

The sechsbein $e_M^{\ \underline{N}}$ transforms as

$$\delta_\lambda e_M^{\ \underline{N}} = \lambda^{\underline{N}}_{\ \underline{L}} e_M^{\ \underline{L}}, \tag{C.1}$$

where the transformation parameters $\lambda^{\underline{N}}_{\ \underline{L}}$ are real, and $\lambda_{\underline{NL}} = -\lambda_{\underline{LN}}$.[22] In the following, we focus on the transformations by $\lambda^{\underline{\mu}}_{\ \underline{n}}$, which mix 4D and the extra dimensions.

First, note that

$$\begin{aligned}
\delta_\lambda E_m &= \delta_\lambda \left(e_m^{\ \underline{4}} + i e_m^{\ \underline{5}}\right) = \left(\lambda^{\underline{4}}_{\ \underline{\mu}} + i\lambda^{\underline{5}}_{\ \underline{\mu}}\right) e_m^{\ \underline{\mu}} \\
&= -\left(\lambda_{\underline{\mu 4}} + i\lambda_{\underline{\mu 5}}\right) e_m^{\ \underline{\mu}}, \\
\delta_\lambda e^{(2)} &= \delta_\lambda \mathrm{Im}\left(\bar{E}_4 E_5\right) \\
&= -\mathrm{Im}\left\{\left(\lambda_{\underline{\mu 4}} - i\lambda_{\underline{\mu 5}}\right)\left(E_5 e_4^{\ \underline{\mu}} - E_4 e_5^{\ \underline{\mu}}\right)\right\} \\
&= -e^{(2)} \mathrm{Re}\left\{\left(\lambda_{\underline{\mu 4}} - i\lambda_{\underline{\mu 5}}\right) \frac{\sqrt{E_4 E_5}}{i e^{(2)}} \times \left(\sqrt{\frac{E_5}{E_4}} e_4^{\ \underline{\mu}} - \sqrt{\frac{E_4}{E_5}} e_5^{\ \underline{\mu}}\right)\right\}.
\end{aligned} \tag{C.2}$$

Since these are proportional to $e_m^{\ \underline{\mu}}$, we can see that

$$\delta_\lambda \sqrt{\frac{E_4}{E_5}} = \mathcal{O}(e_m^{\ \underline{\mu}}), \qquad \delta_\lambda \left(\frac{e^{(2)2}}{|E_4 E_5|}\right) = \mathcal{O}(e_m^{\ \underline{\mu}}). \tag{C.3}$$

These are consistent with the first and the fourth transformations in (5.1) if we choose the lowest component of N as zero, $N| = 0$.

[22] The flat indices $\underline{M}, \underline{N}, \cdots$ are raised and lowered by $\eta^{\underline{MN}}$ and $\eta_{\underline{MN}}$, respectively.

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In the following, we neglect the "off-diagonal" components $e_{\underline{m}}{}^{\underline{\nu}}$ and $e_{\underline{\mu}}{}^{\underline{n}}$ in the right-hand sides. Then we can see that

$$\begin{aligned}
\delta_\lambda e_{\underline{\mu}}{}^{\underline{4}} &= \lambda_{\underline{\mu}}{}^{\underline{n}} e_{\underline{n}}{}^{\underline{4}} = \frac{1}{e^{(2)}}\left(\lambda_{\underline{\mu}}{}^{\underline{4}} e_{\underline{5}}{}^{\underline{5}} - \lambda_{\underline{\mu}}{}^{\underline{5}} e_{\underline{5}}{}^{\underline{4}}\right) \\
&= \mathrm{Re}\left\{\frac{1}{e^{(2)}}\left(-\lambda_{\underline{\mu}}{}^{\underline{5}} - i\lambda_{\underline{\mu}}{}^{\underline{4}}\right)\left(e_{\underline{5}}{}^{\underline{4}} + ie_{\underline{5}}{}^{\underline{5}}\right)\right\} \\
&= \mathrm{Re}\left\{\left(\lambda_{\underline{\mu}\underline{4}} - i\lambda_{\underline{\mu}\underline{5}}\right)\frac{\sqrt{E_4 E_5}}{ie^{(2)}} \times \sqrt{\frac{E_5}{E_4}}\right\}, \\
\delta_\lambda e_{\underline{\mu}}{}^{\underline{5}} &= \lambda_{\underline{\mu}}{}^{\underline{n}} e_{\underline{n}}{}^{\underline{5}} = \frac{1}{e^{(2)}}\left(-\lambda_{\underline{\mu}}{}^{\underline{4}} e_{\underline{4}}{}^{\underline{5}} + \lambda_{\underline{\mu}}{}^{\underline{5}} e_{\underline{4}}{}^{\underline{4}}\right) \\
&= \mathrm{Re}\left\{\frac{1}{e^{(2)}}\left(\lambda_{\underline{\mu}}{}^{\underline{5}} + i\lambda_{\underline{\mu}}{}^{\underline{4}}\right)\left(e_{\underline{4}}{}^{\underline{4}} + ie_{\underline{4}}{}^{\underline{5}}\right)\right\} \\
&= -\mathrm{Re}\left\{\left(\lambda_{\underline{\mu}\underline{4}} - i\lambda_{\underline{\mu}\underline{5}}\right)\frac{\sqrt{E_4 E_5}}{ie^{(2)}} \times \sqrt{\frac{E_4}{E_5}}\right\},
\end{aligned} \tag{C.4}$$

which are consistent with the second and the third transformations in (5.1). Besides, since

$$\begin{aligned}
\delta_\lambda\left(\frac{i}{2}e_{m\underline{\mu}}\right) &= \frac{i}{2}\left(\lambda_{\underline{\mu}}{}^{\underline{4}} e_{m\underline{4}} + \lambda_{\underline{\mu}}{}^{\underline{5}} e_{m\underline{5}}\right) \\
&= \frac{i}{2}\mathrm{Im}\left\{\left(\lambda_{\underline{\mu}\underline{5}} + i\lambda_{\underline{\mu}\underline{4}}\right)\left(e_m{}^{\underline{4}} + ie_m{}^{\underline{5}}\right)\right\} \\
&= -\frac{ie^{(2)}}{2}\mathrm{Im}\left\{\left(\lambda_{\underline{\mu}\underline{4}} - i\lambda_{\underline{\mu}\underline{5}}\right)\frac{E_m}{ie^{(2)}}\right\},
\end{aligned} \tag{C.5}$$

we obtain

$$\begin{aligned}
\delta_\lambda\left(\frac{i}{2}e_{4\underline{\mu}}\right) &= -\frac{ie^{(2)}}{2}\mathrm{Im}\left\{\left(\lambda_{\underline{\mu}\underline{4}} - i\lambda_{\underline{\mu}\underline{5}}\right)\frac{\sqrt{E_4 E_5}}{ie^{(2)}} \times \sqrt{\frac{E_4}{E_5}}\right\}, \\
\delta_\lambda\left(\frac{i}{2}e_{5\underline{\mu}}\right) &= -\frac{ie^{(2)}}{2}\mathrm{Im}\left\{\left(\lambda_{\underline{\mu}\underline{4}} - i\lambda_{\underline{\mu}\underline{5}}\right)\frac{\sqrt{E_4 E_5}}{ie^{(2)}} \times \sqrt{\frac{E_5}{E_4}}\right\},
\end{aligned} \tag{C.6}$$

which are consistent with the transformations in the third line of (5.1).

C.2 Hypermultiplet

Since

$$\delta_\lambda\left\{(E_4 E_5)^{1/4}\,\phi_2^{\bar{A}}\right\} = \frac{\phi_2^{\bar{A}}}{4(E_4 E_5)^{3/4}}\left(E_5\delta_\lambda E_4 + E_4\delta_\lambda E_5\right) = \mathcal{O}(e_m{}^{\underline{\mu}}), \tag{C.7}$$

the transformations in the fourth line of (5.1) are consistent with the component transformations. (Recall that we have chosen the lowest component of N as zero.)

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C.3 Vector multiplet

We can also see that the last two transformations in (5.1) are consistent with the δ_λ-transformations of the component fields because

$$\begin{aligned}
\delta_\lambda A^I_{\underline{\mu}} &= \lambda_{\underline{\mu}}{}^{\underline{n}} A^I_{\underline{n}} = \lambda_{\underline{\mu}}{}^{\underline{4}} \left(e_{\underline{4}}{}^4 A^I_4 + e_{\underline{4}}{}^5 A^I_5\right) + \lambda_{\underline{\mu}}{}^{\underline{5}} \left(e_{\underline{5}}{}^4 A^I_4 + e_{\underline{5}}{}^5 A^I_5\right) \\
&= \lambda_{\underline{\mu}\underline{4}} \frac{1}{e^{(2)}} \left(e_5{}^{\underline{5}} A^I_4 - e_4{}^{\underline{5}} A^I_5\right) + \lambda_{\underline{\mu}\underline{5}} \frac{1}{e^{(2)}} \left(-e_5{}^{\underline{4}} A^I_4 + e_4{}^{\underline{4}} A^I_5\right) \\
&= \frac{1}{e^{(2)}} \left\{\lambda_{\underline{\mu}\underline{4}} \mathrm{Im}\left(E_5 A^I_4 - E_4 A^I_5\right) - \lambda_{\underline{\mu}\underline{5}} \mathrm{Re}\left(E_5 A^I_4 - E_4 A^I_5\right)\right\} \\
&= \mathrm{Re}\left\{\frac{1}{ie^{(2)}} \left(\lambda_{\underline{\mu}\underline{4}} - i\lambda_{\underline{\mu}\underline{5}}\right)\left(E_5 A^I_4 - E_4 A^I_5\right)\right\} \\
&= 2\mathrm{Im}\left\{\left(\lambda_{\underline{\mu}\underline{4}} - i\lambda_{\underline{\mu}\underline{5}}\right) \frac{\sqrt{E_4 E_5}}{ie^{(2)}} \times \frac{i}{2}\left(\sqrt{\frac{E_5}{E_4}} A^I_4 - \sqrt{\frac{E_4}{E_5}} A^I_5\right)\right\},
\end{aligned} \tag{C.8}$$

$$\begin{aligned}
\delta_\lambda \left\{\frac{i}{2}\left(\sqrt{\frac{E_5}{E_4}} A^I_4 - \sqrt{\frac{E_4}{E_5}} A^I_5\right)\right\} &= \delta_N \left\{-\frac{e^{(2)}}{2\sqrt{E_4 E_5}} \left(A^I_{\underline{4}} + i A^I_{\underline{5}}\right)\right\} \\
&= -\frac{e^{(2)}}{2\sqrt{E_4 E_5}} \left(\lambda_{\underline{4}}{}^{\underline{\mu}} + i\lambda_{\underline{5}}{}^{\underline{\mu}}\right) A^I_{\underline{\mu}} \\
&= \frac{1}{2} \frac{e^{(2)}}{\sqrt{E_4 E_5}} \left(\lambda_{\underline{\mu}\underline{4}} + i\lambda_{\underline{\mu}\underline{5}}\right) A^{I\underline{\mu}} \\
&= -\frac{i}{2} \frac{e^{(2)2}}{|E_4 E_5|} \left\{-\left(\lambda_{\underline{\mu}\underline{4}} + i\lambda_{\underline{\mu}\underline{5}}\right) \frac{\sqrt{\bar{E}_4 \bar{E}_5}}{ie^{(2)}}\right\} A^{I\underline{\mu}} \\
&= -\frac{i}{2} \times \frac{e^{(2)2}}{|E_4 E_5|} \times \left\{\left(\lambda_{\underline{\mu}\underline{4}} - i\lambda_{\underline{\mu}\underline{5}}\right) \frac{\sqrt{E_4 E_5}}{ie^{(2)}}\right\}^* \times A^{I\underline{\mu}},
\end{aligned} \tag{C.9}$$

and

$$-\frac{i}{8}\bar{D}^2 \left(\tilde{V}_{\mathrm{E}} D^\alpha \bar{N} D_\alpha V^I\right)\Big| = -\frac{i}{2} \times \tilde{V}_{\mathrm{E}}| \times \Lambda_{\underline{\mu}} \times A^{I\underline{\mu}}. \tag{C.10}$$

where $\Lambda_{\underline{\mu}}$ denotes the $\theta\bar{\theta}$-component of N, i.e.,

$$\Lambda_{\underline{\mu}} \equiv \left(\lambda_{\underline{\mu}\underline{4}} - i\lambda_{\underline{\mu}\underline{5}}\right) \frac{\sqrt{E_4 E_5}}{ie^{(2)}}. \tag{C.11}$$

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