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General formulae for dipole Wilson line correlators with the Color Glass Condensate

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ABSTRACT: We present general formulae to compute Wilson line correlators with the Color Glass Condensate described by the McLerran-Venugopalan model. We explicitly construct a complete and non-orthogonal set of color-singlet bases and write matrix elements down, so that the exponential of the matrix leads to the Wilson line correlators. We further develop a systematic perturbative expansion of dipole Wilson line correlators in terms of $1/N_c$ where N_c is the color number. As a phenomenological application we calculate the flow harmonics $v_n\{m\}$ in the dipole model and discuss the N_c scaling.

KEYWORDS: Resummation, Perturbative QCD, Quark-Gluon Plasma

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1 Introduction

Quantum Chromodynamics (QCD) has a special property called the asymptotic freedom that implies that the strong coupling constant runs to a smaller value in parton reactions involving harder momentum scales. In this way the perturbative calculation of QCD is a reliable theoretical description in high-energy nuclear physics. In reality, however, a naïve perturbative expansion breaks down for diffractive type reactions in which exchanged momenta are not necessary hard as compared to the collision energy. Then, it is indispensable to make a resummation over large logarithmic enhancement factors which appear kinematically. Such a resummation leads to a picture of increasing parton or gluon density with increasing scattering energy, and, one would anticipate that the parton density would eventually enter a new regime where the effect of parton overlapping is significant. In such a regime of dense partons, QCD is still perturbative in a sense that the running coupling constant g is small, but is highly non-linear due to the gluon amplitude A^μ as large as $\sim 1/g$ [and thus $gA^\mu \sim \mathcal{O}(1)$]. This non-linear dynamics is a manifestation of the parton or gluon saturation (see refs. [1–5] for pioneering extensions including the non-linearity, see also refs. [6, 7] for recent reviews).

The virtue of the gluon saturation is that physical observables exhibit universal behavior in terms of scaling variables, from which the saturation momentum, Q_s , can be experimentally fixed [8]. The theoretical framework in the saturation regime to compute physical observables as functions of Q_s has been well established and known as the Color Glass Condensate (CGC) [9–11]. The CGC effective theory is elegantly formulated in the renormalization group language [12–14], in which the soft gluons are given by the classical Yang-Mills fields from the hard parton color source [15] whose transverse density is

characterized by Q_s^2 . The non-linear quantum evolution equation for the color source distribution is known as the JIMWLK equation named after the authors of the pioneering works [12–14, 16, 17]. Although solving the JIMWLK equation demands huge computational resources (see, for example, ref. [18] for the reformulation using the Langevin equation, and refs. [19, 20] for numerical simulations), a Gaussian approximated solution has been derived [21, 22]. The simplest CGC model in the Gaussian approximation is commonly called the McLerran-Venugopalan (MV) model, which is often used as an initial input for the JIMWLK evolution, or, this model itself could provide us with a good description of the qualitative features of high-energy QCD processes.

One of the most interested and testable quantities calculated in the CGC framework is the particle production [23–26] and correlation [27–31], including electromagnetic probes [32–36]. In the MV model various correlations among gluons and quarks have been discussed, and these predictions are to be compared to the experimental data of jets and hadron correlations. One of the most well known examples is the CGC picture to understand the ridge structure seen in the di-jet or di-hadron correlations in rapidity space [37, 38]. Thus, it would be a natural extension to apply the CGC-based calculations to account for the collective behavior in small size systems. As the collision energy grows up, small size systems such as the p - A (proton-nucleus) and even the p - p collisions may look similar to the A - A collision, yielding systematic patterns of the flow observables (see ref. [39] for an experimental overview). The final state interaction might be still responsible for the flow observables, but at the same time, one must also estimate the initial state effect to quantify which of the initial and the final state interactions is more important. Recently, within the dipole model for the particle production the systematics of the higher order flow observables, $v_n\{m\}$ (i.e. n -th harmonics of the m -particle flow) has been quantified in the glasma graph approximation [40, 41] and in the full MV model [42, 43].

Generally, in the CGC model calculations, the most time-consuming part in numerics is the numerical computation of the expectation value of the Wilson lines. In the presence of the CGC background fields corresponding to the saturated soft gluons typically in the p - A collision and the forward p - p collision, one needs to take account of multiple scatterings which amount to the Wilson line in the eikonal approximation. It has been repeatedly discussed how to compute the Wilson line correlators efficiently in separate contexts (see ref. [44] for a very recent work toward general formulae for the Wilson line correlators, for instance). Therefore, it would be quite useful to establish a prescription to compute the Wilson line correlators with some generality but in a fairly straightforward way. There are several preceding works along these lines from an early attempt in ref. [45] to a very recent reformulation in ref. [46] on top of an explicit calculation in ref. [44]. The purpose of the present work is to translate the powerful method of ref. [45] to a more specific physics problem in a more handy form. In particular, the setup in ref. [45] was too general and it was not clear how to utilize the formulae for phenomenological applications. In the present work, hence, we focus on the special type of correlation function in terms of the *dipole operators* in the color fundamental representation, which is the basic building block in the dipole model [47]. For n -point dipole correlations the problem is reduced to the exponentiation of $n! \times n!$ matrices. The calculation of the exponential of the matrix

is numerically feasible but it is sometimes insightful to perform the large- N_c expansion, especially to identify the N_c scaling of observables. We emphasize that our formulae take a quite convenient form for the large- N_c expansion and we will demonstrate the expansion explicitly up to the order where the first nonzero flow cumulants appear. We check the validity of the large- N_c formulae by comparing to the exact results known for $n = 2$ with $N_c = 3$. For $n \geq 4$ we need to be very careful of the correct large- N_c counting and we will argue the subtlety in the numerical analysis.

2 Master formulae

The Wilson line correlators that we calculate in this paper are expressed in analogy to quantum mechanics as follows [45]:

$$\left\langle \prod_{i=1}^n U(\mathbf{x}_{\perp i})_{\beta_i \alpha_i} U^*(\mathbf{y}_{\perp i})_{\bar{\beta}_i \bar{\alpha}_i} \right\rangle = \exp[-(H_0 + V)]_{\beta_1 \bar{\beta}_1 \dots \beta_n \bar{\beta}_n; \alpha_1 \bar{\alpha}_1 \dots \alpha_n \bar{\alpha}_n}, \quad (2.1)$$

where the *free Hamiltonian*, H_0 , is defined as

$$H_0 = Q_s^2 \frac{2N_c}{N_c^2 - 1} L(0, 0) \left[\sum_{i=1}^n (T_{F_i}^a - T_{F_i}^{*a}) \right]^2. \quad (2.2)$$

Here, we take the summation over a in group space (which is always implicitly assumed). In this paper we limit our considerations to the fundamental representation only and $T_{F_i}^a$'s represent the elements of $\mathfrak{su}(N_c)$ algebra in the fundamental representation, while the formula is valid for any representation. Here, $L(0, 0)$ is quadratically divergent, so that only color states that have zero eigenvalues of H_0 can make finite contributions to the Wilson line correlators [and thus, the detailed definition of $L(0, 0)$ is not important here; see ref. [45] for the explicit form of $L(0, 0)$]. We note that the definition of Q_s is slightly different from refs. [42, 43] and we will come to this point later when we apply our formulae for the flow observables. For the moment, eq. (2.2) is understood as a definition of Q_s in our convention. The color matrix in the *interaction part* is

$$V = -Q_s^2 \frac{2N_c}{N_c^2 - 1} \left\{ \sum_{i>j}^n \left[T_{F_i}^a T_{F_j}^a \Gamma(|\mathbf{x}_{\perp i} - \mathbf{x}_{\perp j}|) + T_{F_i}^{*a} T_{F_j}^{*a} \Gamma(|\mathbf{y}_{\perp i} - \mathbf{y}_{\perp j}|) \right] - \sum_{i,j=1}^n T_{F_i}^a T_{F_j}^{*a} \Gamma(|\mathbf{x}_{\perp i} - \mathbf{y}_{\perp j}|) \right\}. \quad (2.3)$$

In the above the common building block, $\Gamma(\mathbf{x}_{\perp})$, is defined in the CGC formalism by the following integral:

$$\Gamma(|\mathbf{x}_{\perp}|) := 2g^4 \int_{\Lambda}^{\infty} \frac{dk}{2\pi} \frac{1}{k^3} \left[1 - J_0(k|\mathbf{x}_{\perp}|) \right] \simeq -\frac{g^4}{4\pi} |\mathbf{x}_{\perp}|^2 \ln(|\mathbf{x}_{\perp}| \bar{\Lambda}), \quad (2.4)$$

where $J_\alpha(x)$ is the Bessel function of the first kind, Λ is an infrared (IR) cutoff of order of Λ_{QCD} and we introduced a shorthand notation as $\bar{\Lambda} := \frac{1}{2} \Lambda e^{\gamma-1}$. This approximated

expression has undesirable behavior in the IR region especially when $|\mathbf{x}_\perp|\bar{\Lambda} < 1$. We cure this IR problem, according to ref. [43], by introducing another regulator as

$$\Gamma(|\mathbf{x}_\perp|) \simeq \frac{g^4}{4\pi} |\mathbf{x}_\perp|^2 \ln \left(\frac{1}{|\mathbf{x}_\perp|\bar{\Lambda}} + e \right). \quad (2.5)$$

We will adopt this regularized approximation for $\Gamma(\mathbf{x}_\perp)$ in our numerical calculations later in section 5.

Now we shall sketch our strategy to proceed to concrete calculations of the Wilson line correlators using eq. (2.1). We will first identify all the color-singlet bases with which H_0 vanishes. For the n -th power product of U and U^* , there are $2n$ Wilson lines, and then the number of independent color-singlet bases should be $n!$, as we will explain in the next section. We can easily construct the color index structures by taking the permutations. After confirming that H_0 surely vanishes with such bases, next, we will consider the matrix elements of V . In general it is not an easy task to find analytical expressions for the eigenvalues of V . Nevertheless, it is known that the large- N_c approximation works at good quantitative accuracy, and we will systematically make an expansion in power of $1/N_c$ to find analytical expressions.

3 Color singlet bases

Because of the dipole-type structures of the Wilson lines it is easy to find all the combinations of the color singlet indices. One trivial singlet is immediately found as

$$\langle \{\alpha\}; \{\bar{\alpha}\} | s_0 \rangle := N_c^{-n/2} \delta_{\alpha_1 \bar{\alpha}_1} \cdots \delta_{\alpha_n \bar{\alpha}_n}. \quad (3.1)$$

Clearly all the permutations are possible, i.e. we introduce $|s_p\rangle$ as a permutation of $|s_0\rangle$. For this purpose let us introduce the symmetry group S_n with elements π_p that denotes a permutation as

$$\pi_p = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}. \quad (3.2)$$

Then,

$$\langle \{\alpha\}; \{\bar{\alpha}\} | s_p \rangle := \langle \{\pi_p \alpha\}; \{\bar{\alpha}\} | s_0 \rangle = N_c^{-n/2} \delta_{\alpha_{p_1} \bar{\alpha}_1} \cdots \delta_{\alpha_{p_n} \bar{\alpha}_n}. \quad (3.3)$$

We note that any π_p can be expressed as a product of cycles. For example, it is easy to confirm the following relation,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 3 & 6 & 2 & 4 \end{pmatrix} = (152) \cdot (46), \quad (3.4)$$

where one-cycles, i.e. (3) in the above case, are trivial and not explicitly written. Let us see several useful formulae for later calculation checks. Because of the well-known relation,

$$2T_F^a_{\beta_i \alpha_i} T_F^a_{\beta_j \alpha_j} = \delta_{\beta_i \alpha_j} \delta_{\beta_j \alpha_i} - \frac{1}{N_c} \delta_{\beta_i \alpha_i} \delta_{\beta_j \alpha_j}, \quad (3.5)$$

we can easily prove that

$$\langle \{\beta\}; \{\bar{\beta}\} | 2T_{F_i}^a T_{F_j}^a | s_p \rangle = \langle \{(ij)\pi_p \beta\}; \{\bar{\beta}\} | s_0 \rangle - \frac{1}{N_c} \langle \{\pi_p \beta\}; \{\bar{\beta}\} | s_0 \rangle, \quad (3.6)$$

which is schematically expressed as

$$2T_{F_i}^a T_{F_j}^a \cdot \pi_p = \left[(ij) - \frac{1}{N_c} \right] \cdot \pi_p. \quad (3.7)$$

For the complex conjugate, because T_F^a 's are Hermitean, the following should hold:

$$2T_{F_{\bar{\beta}_i \bar{\alpha}_i}}^{*a} T_{F_{\bar{\beta}_j \bar{\alpha}_j}}^{*a} = \delta_{\bar{\alpha}_j \bar{\beta}_i} \delta_{\bar{\alpha}_i \bar{\beta}_j} - \frac{1}{N_c} \delta_{\bar{\alpha}_i \bar{\beta}_i} \delta_{\bar{\alpha}_j \bar{\beta}_j}, \quad (3.8)$$

which implies

$$\langle \{\beta\}; \{\bar{\beta}\} | 2T_{F_i}^{*a} T_{F_j}^{*a} | s_p \rangle = \langle \{\pi_p \beta\}; \{(ij)\bar{\beta}\} | s_0 \rangle - \frac{1}{N_c} \langle \{\pi_p \beta\}; \{\bar{\beta}\} | s_0 \rangle. \quad (3.9)$$

Thanks to the index structure of $|s_0\rangle$, we readily see; $\langle \{\pi_p \beta\}; \{(ij)\bar{\beta}\} | s_0 \rangle = \langle \{\pi_p (ij)^{-1} \beta\}; \{\bar{\beta}\} | s_0 \rangle$ and trivially $(ij)^{-1} = (ij)$, so that we can again give a schematic representation as

$$2T_{F_i}^{*a} T_{F_j}^{*a} \cdot \pi_p = \pi_p \cdot \left[(ij) - \frac{1}{N_c} \right]. \quad (3.10)$$

Also, another useful formula is

$$2T_{F_{\beta_i \alpha_i}}^a T_{F_{\bar{\beta}_j \bar{\alpha}_j}}^{*a} = \delta_{\beta_i \bar{\beta}_j} \delta_{\alpha_i \bar{\alpha}_j} - \frac{1}{N_c} \delta_{\beta_i \alpha_i} \delta_{\bar{\beta}_j \bar{\alpha}_j}. \quad (3.11)$$

In this case, if $p_j = i$, the contract of $\delta_{\beta_i \bar{\beta}_j} \delta_{\alpha_i \bar{\alpha}_j} = \delta_{\beta_{p_j} \bar{\beta}_j} \delta_{\alpha_{p_j} \bar{\alpha}_j}$ in the above and $\delta_{\alpha_{p_j} \bar{\alpha}_j}$ in $|s_p\rangle$ makes $N_c \delta_{\beta_{p_j} \bar{\beta}_j}$. For $p_j \neq i$ the contract of $\delta_{\beta_i \bar{\beta}_j} \delta_{\alpha_i \bar{\alpha}_j}$ and $\delta_{\alpha_{p_j} \bar{\alpha}_j} \delta_{\alpha_i \bar{\alpha}_{p_i^{-1}}}$ together with $\delta_{\beta_{p_j} \alpha_{p_j}} \delta_{\bar{\beta}_{p_i^{-1}} \bar{\alpha}_{p_i^{-1}}}$ (which is always implicitly taken for indices not involved in T_F^a nor T_F^{*a}) gives $\delta_{\beta_i \bar{\beta}_j} \delta_{\beta_{p_j} \bar{\beta}_{p_i^{-1}}} = (ip_j) \delta_{\beta_{p_j} \bar{\beta}_j} \delta_{\beta_i \bar{\beta}_{p_i^{-1}}}$, where p_i^{-1} indicates an index that satisfies $p_{p_i^{-1}} = i$. Therefore, we establish the following schematic relations:

$$2T_{F_i}^a T_{F_j}^{*a} \pi_p = \begin{cases} \left(N_c - \frac{1}{N_c} \right) \cdot \pi_p & \text{for } p_j = i \\ \left[(ip_j) - \frac{1}{N_c} \right] \cdot \pi_p & \text{for } p_j \neq i \end{cases} \quad (3.12)$$

In what follows below, we will utilize the formulae (3.7), (3.10), and (3.12) to perform calculations with a compact notation.

It would be an instructive check to see how $H_0 |s_p\rangle = 0$ is satisfied for all $|s_p\rangle$. For this purpose we first need to expand the second-order Casimir operator in H_0 as

$$\left[\sum_{i=1}^n (T_{F_i}^a - T_{F_i}^{*a}) \right]^2 = \sum_{i=1}^n (T_{F_i}^{a2} + T_{F_i}^{*a2}) + 2 \sum_{i>j}^n (T_{F_i}^a T_{F_j}^a + T_{F_i}^{*a} T_{F_j}^{*a}) - 2 \sum_{i,j=1}^n T_{F_i}^a T_{F_j}^{*a}. \quad (3.13)$$

Here, again, we note that the summation over a is always implicitly assumed. The first term is nothing but the Casimir operator, so that it is simply given by

$$\sum_{i=1}^n (T_{F_i}^{a^2} + T_{F_i}^{*a^2}) = n \left(N_c - \frac{1}{N_c} \right), \quad (3.14)$$

according to the $\text{su}(N_c)$ algebra. Using the formulae (3.7), (3.10), and (3.12), we can simplify the rest (if applied to $|s_p\rangle$) as

$$\begin{aligned} & \left[2 \sum_{i>j}^n (T_{F_i}^a T_{F_j}^a + T_{F_i}^{*a} T_{F_j}^{*a}) - 2 \sum_{i,j=1}^n T_{F_i}^a T_{F_j}^{*a} \right] \cdot \pi_p \\ &= \sum_{i>j}^n \left\{ \left[(i j) - \frac{1}{N_c} \right] \cdot \pi_p + \pi_p \cdot \left[(i j) - \frac{1}{N_c} \right] \right\} - \sum_{i,j=1}^n \left[N_c \delta_{ip_j} + (i p_j)(1 - \delta_{ip_j}) - \frac{1}{N_c} \right] \cdot \pi_p \\ &= -n \left(N_c - \frac{1}{N_c} \right) \cdot \pi_p + \sum_{i>j}^n [(i j) \cdot \pi_p + \pi_p \cdot (i j)] - \sum_{i,j=1}^n (i p_j)(1 - \delta_{ip_j}) \cdot \pi_p. \end{aligned} \quad (3.15)$$

The first term cancels out with eq. (3.14). Since i and j run from 1 to n , we can equivalently take a summation with respect to j instead of using p_j , leading to

$$- \sum_{i,j=1}^n (i p_j)(1 - \delta_{ip_j}) \cdot \pi_p = - \sum_{i \neq j}^n (i j) \cdot \pi_p. \quad (3.16)$$

Also, we see $\pi_p \cdot (i j) = (p_i p_j) \cdot \pi_p$, which allows us to replace p_i and p_j with i and j in the summations. Finally we arrive at

$$\sum_{i>j}^n [(i j) \cdot \pi_p + \pi_p \cdot (i j)] - \sum_{i,j=1}^n (i p_j)(1 - \delta_{ip_j}) \cdot \pi_p = 0. \quad (3.17)$$

This completes our confirmation of $H_0|s_p\rangle = 0$ for any $|s_p\rangle$.

For the practical calculation the most important is the evaluation of the matrix elements of V . In our formalism we can infer the matrix elements from

$$\begin{aligned} V \cdot \pi_p &= -Q_s^2 \frac{N_c}{N_c^2 - 1} \left(\sum_{i>j}^n \left\{ \left[(i j) - \frac{1}{N_c} \right] \Gamma(|\mathbf{x}_{\perp i} - \mathbf{x}_{\perp j}|) + \left[(p_i p_j) - \frac{1}{N_c} \right] \Gamma(|\mathbf{y}_{\perp i} - \mathbf{y}_{\perp j}|) \right\} \right. \\ &\quad \left. - \sum_{i,j=1}^n \left[N_c \delta_{ip_j} + (i p_j)(1 - \delta_{ip_j}) - \frac{1}{N_c} \right] \Gamma(|\mathbf{x}_{\perp i} - \mathbf{y}_{\perp j}|) \right) \cdot \pi_p, \end{aligned} \quad (3.18)$$

which can be easily verified with eqs. (3.7), (3.10), and (3.12). We can further simplify the above expression by introducing a notation for a proper combination of Γ 's, i.e.,

$$F(\mathbf{x}_{\perp}, \mathbf{x}'_{\perp}; \mathbf{y}_{\perp}, \mathbf{y}'_{\perp}) := \Gamma(|\mathbf{x}'_{\perp} - \mathbf{y}_{\perp}|) + \Gamma(|\mathbf{x}_{\perp} - \mathbf{y}'_{\perp}|) - \Gamma(|\mathbf{x}_{\perp} - \mathbf{x}'_{\perp}|) - \Gamma(|\mathbf{y}_{\perp} - \mathbf{y}'_{\perp}|). \quad (3.19)$$

We then define the explicit components of the matrix elements as

$$V|s_p\rangle = \sum_{p'} |s_{p'}\rangle V_{p',p}. \quad (3.20)$$

Here we must be careful of the fact that $|s_p\rangle$'s span a complete set of bases but they are not orthogonal. Using these notations and definitions we can summarize the non-zero components as follows:

$$V_{p,p} = Q_s^2 \left[\sum_{i=1}^n \Gamma(|\mathbf{x}_{\perp p_i} - \mathbf{y}_{\perp i}|) - \frac{1}{N_c^2 - 1} \sum_{i>j}^n F(\mathbf{x}_{\perp p_i}, \mathbf{x}_{\perp p_j}; \mathbf{y}_{\perp i}, \mathbf{y}_{\perp j}) \right], \quad (3.21)$$

$$V_{p(ij),p} = Q_s^2 \frac{N_c}{N_c^2 - 1} F(\mathbf{x}_{\perp p_i}, \mathbf{x}_{\perp p_j}; \mathbf{y}_{\perp i}, \mathbf{y}_{\perp j}), \quad (3.22)$$

and other matrix elements vanish. These formulae are our central results in the present paper. For the actual application of the formulae, we should compute the exponential of V as seen in eq. (2.1).

To understand how the formulae work, let us consider the simplest example of $n = 2$. The matrix elements of 2×2 matrix V read

$$V = \begin{pmatrix} V_{0,0} & V_{0,(21)} \\ V_{(21),0} & V_{(21),(21)} \end{pmatrix} = Q_s^2 \frac{2N_c}{N_c^2 - 1} \begin{pmatrix} \frac{N_c^2 - 1}{N_c} \gamma - \frac{1}{N_c} (\alpha - \beta) & \gamma - \beta \\ \alpha - \beta & \frac{N_c^2 - 1}{N_c} \alpha - \frac{1}{N_c} (\gamma - \beta) \end{pmatrix}, \quad (3.23)$$

where,

$$\begin{aligned} 2\alpha &:= \Gamma(|\mathbf{x}_{\perp 1} - \mathbf{y}_{\perp 2}|) + \Gamma(|\mathbf{y}_{\perp 1} - \mathbf{x}_{\perp 2}|), \\ 2\beta &:= \Gamma(|\mathbf{x}_{\perp 1} - \mathbf{x}_{\perp 2}|) + \Gamma(|\mathbf{y}_{\perp 1} - \mathbf{y}_{\perp 2}|), \\ 2\gamma &:= \Gamma(|\mathbf{x}_{\perp 1} - \mathbf{y}_{\perp 1}|) + \Gamma(|\mathbf{x}_{\perp 2} - \mathbf{x}_{\perp 2}|). \end{aligned} \quad (3.24)$$

It is a straightforward calculation to obtain two eigenvalues as $\lambda_{\pm} = \frac{1}{2}(\text{tr}V \pm \varphi)$ with

$$\varphi := \sqrt{(\text{tr}V)^2 - 4 \det V} = Q_s^2 \frac{2N_c^2}{N_c^2 - 1} \sqrt{(\alpha - \gamma)^2 + \frac{4}{N_c^2} (\beta - \alpha)(\beta - \gamma)}. \quad (3.25)$$

Now we can express the exponential of V in a simple form as

$$e^{-V} = e^{-\frac{1}{2} \text{tr}V} \begin{pmatrix} \cosh \frac{1}{2} \varphi - \frac{\sinh \frac{1}{2} \varphi}{\varphi} (V_{0,0} - V_{(21),(21)}) & -2 \frac{\sinh \frac{1}{2} \varphi}{\varphi} V_{0,(21)} \\ -2 \frac{\sinh \frac{1}{2} \varphi}{\varphi} V_{(21),0} & \cosh \frac{1}{2} \varphi + \frac{\sinh \frac{1}{2} \varphi}{\varphi} (V_{0,0} - V_{(21),(21)}) \end{pmatrix}. \quad (3.26)$$

Although the calculation machinery is rather simple, a larger n would cause a huge computational cost. Hence, we will seek for an algorithmic expansion to approximate e^{-V} without complicated matrix algebra.

4 Dipole Wilson line correlators and the large- N_c expansion

For the application for the particle production problem in the relativistic heavy-ion collision [42, 43], we are specifically interested in the correlation functions of the dipole operators. The definition of the dipole operator is

$$D(\mathbf{x}_{\perp}, \mathbf{y}_{\perp}) := \frac{1}{N_c} \text{tr}[U(\mathbf{x}_{\perp}) U^{\dagger}(\mathbf{y}_{\perp})] = \frac{1}{N_c} \delta_{\beta\bar{\beta}} \delta_{\alpha\bar{\alpha}} U(\mathbf{x}_{\perp})_{\beta\alpha} U^*(\mathbf{y}_{\perp})_{\bar{\beta}\bar{\alpha}}. \quad (4.1)$$

From this form it is obvious that the n dipole expectation value is given by an matrix element of e^{-V} evaluated with $|s_0\rangle$, i.e.,

$$\left\langle \prod_{i=1}^n D(\mathbf{x}_{\perp i}, \mathbf{y}_{\perp i}) \right\rangle = \langle s_0 | e^{-V} | s_0 \rangle. \quad (4.2)$$

We may be able to do a direct computation, but we can develop a more sophisticated method assuming that N_c is large enough. In view of eqs. (3.21) and (3.22), the off-diagonal components are suppressed by $1/N_c$, so that we can avoid exponentiating V but make a systematic expansion in terms of $V_{p(ij),p}$.

For notational brevity we shall denote the diagonal and the off-diagonal parts of V as $V^{(0)}$ and $V^{(1)}$, respectively. Then, the starting point for the systematic perturbative expansion is the *interaction picture* as in quantum mechanics expressed as

$$e^{-V} = e^{-V^{(0)}} \mathcal{T}_\tau \exp \left[- \int_0^1 d\tau V^{(1)}(\tau) \right], \quad (4.3)$$

where $\mathcal{T}_\tau$ stands for the time-ordered product in terms of τ and the time dependent $V^{(1)}(\tau)$ in the interaction picture is defined as

$$V^{(1)}(\tau) := e^{\tau V^{(0)}} V^{(1)} e^{-\tau V^{(0)}}. \quad (4.4)$$

Thus, up to the second order in $V^{(1)}$ for example, the perturbative expansion reads

$$e^{-V} \simeq e^{-V^{(0)}} \left[1 - \int_0^1 d\tau V^{(1)}(\tau) + \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 V^{(1)}(\tau_1) V^{(1)}(\tau_2) \right]. \quad (4.5)$$

Because $V^{(0)}$ is a diagonal matrix, its matrix elements, $V_{p,p}$, are the eigenvalues of V . Then, let us introduce an eigenvector $|p\rangle$ with an eigenvalue E_p for $V^{(0)}$. That is,

$$V^{(0)}|p\rangle = E_p|p\rangle = (E_p^{(0)} + E_p^{(2)})|p\rangle, \quad (4.6)$$

where we decomposed the eigenvalue according to the $1/N_c$ order as

$$E_p^{(0)} := Q_s^2 \sum_{i=1}^n \Gamma(\mathbf{x}_{\perp p_i} - \mathbf{y}_{\perp i}), \quad (4.7)$$

$$E_p^{(2)} := -Q_s^2 \frac{1}{N_c^2 - 1} \sum_{i>j}^n F(\mathbf{x}_{\perp p_i}, \mathbf{x}_{\perp p_j}; \mathbf{y}_{\perp i}, \mathbf{y}_{\perp j}) = -\frac{1}{N_c} \sum_{i>j}^n V_{p(ij),p}^{(1)}. \quad (4.8)$$

It is then easy to re-express the first perturbative correction as

$$\begin{aligned} -e^{-V^{(0)}} \int_0^1 d\tau V^{(1)}(\tau) |p\rangle &= - \sum_q |q\rangle e^{-E_q} \int_0^1 d\tau V_{qp}^{(1)} e^{\tau(E_q - E_p)} \\ &= \sum_q |q\rangle \frac{e^{-E_q} - e^{-E_p}}{E_q - E_p} V_{qp}^{(1)}. \end{aligned} \quad (4.9)$$

Here, $V_{qp}^{(1)}$ is defined as $V^{(1)}|p\rangle = \sum_q |q\rangle V_{qp}^{(1)}$, where we note that this $V^{(1)}$ is an original matrix, not the one in the interaction picture.

In the same way, we can proceed to the second perturbative correction as

$$\begin{aligned}
 & e^{-V^{(0)}} \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 V^{(1)}(\tau_1) V^{(1)}(\tau_2) |p\rangle \\
 &= \sum_{q,r} |q\rangle e^{-E_q} \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 e^{\tau_1(E_q - E_r) + \tau_2(E_r - E_p)} V_{qr}^{(1)} V_{rp}^{(1)} \\
 &= \sum_{q,r} |q\rangle \left(\frac{e^{-E_q} - e^{-E_r}}{E_q - E_r} - \frac{e^{-E_q} - e^{-E_p}}{E_q - E_p} \right) \frac{V_{qr}^{(1)} V_{rp}^{(1)}}{E_r - E_p}. \quad (4.10)
 \end{aligned}$$

At this point, we can see a general algorithm to go to arbitrary high orders. The next order, for example, is generated automatically via one more iteration as

$$\begin{aligned}
 & \sum_{q,s,r} |q\rangle \left[\left(\frac{e^{-E_q} - e^{-E_s}}{E_q - E_s} - \frac{e^{-E_q} - e^{-E_r}}{E_q - E_r} \right) \frac{1}{E_s - E_r} \right. \\
 & \quad \left. - \left(\frac{e^{-E_q} - e^{-E_s}}{E_q - E_s} - \frac{e^{-E_q} - e^{-E_p}}{E_q - E_p} \right) \frac{1}{E_s - E_p} \right] \frac{V_{qs}^{(1)} V_{sr}^{(1)} V_{rp}^{(1)}}{E_r - E_p}. \quad (4.11)
 \end{aligned}$$

Now, we are ready to compute $\langle s_0 | e^{-V} | s_0 \rangle$ up to the N_c^{-2} order. Noting that $V^{(1)}$ has a matrix element between p and $p(ij)$, we can write down

$$\begin{aligned}
 \langle s_0 | e^{-V} | s_0 \rangle &\simeq \langle s_0 | s_0 \rangle e^{-E_0} + \sum_{i>j}^n \langle s_0 | (ij) s_0 \rangle \frac{e^{-E_{(ij)}} - e^{-E_0}}{E_{(ij)} - E_0} V_{(ij),0}^{(1)} \\
 &\quad + \sum_{i>j}^n \langle s_0 | s_0 \rangle \left(\frac{e^{-E_0} - e^{-E_{(ij)}}}{E_0 - E_{(ij)}} + e^{-E_0} \right) \frac{V_{0,(ij)}^{(1)} V_{(ij),0}^{(1)}}{E_{(ij)} - E_0}. \quad (4.12)
 \end{aligned}$$

As we already mentioned, $|s_p\rangle$'s are not orthogonal for different p 's, and a simple calculation leads to the normalization as $\langle s_0 | s_p \rangle = N_c^{-n+n_p}$ where n_p denotes the number of cycles of π_p . Because the second and the third terms are already suppressed by N_c^{-2} , we can replace $E_{(ij)}$ with $E_{(ij)}^{(0)}$ in the above expansion. Then, we notice that $E_{(ij)}^{(0)}$ is always accompanied by $E_0^{(0)}$, which motivates us to introduce a new notation as

$$\Delta E_{(ij)}^{(0)} := E_{(ij)}^{(0)} - E_0^{(0)} = Q_s^2 [\Gamma(|\mathbf{x}_{\perp j} - \mathbf{y}_{\perp i}|) + \Gamma(|\mathbf{x}_{\perp i} - \mathbf{y}_{\perp j}|) - \Gamma(|\mathbf{x}_{\perp i} - \mathbf{y}_{\perp i}|) - \Gamma(|\mathbf{x}_{\perp j} - \mathbf{y}_{\perp j}|)]. \quad (4.13)$$

Now, by expanding E_0 and using the above relations, we can reach the result from the large- N_c expansion up to the second order as

$$\left\langle \prod_{i=1}^n D(\mathbf{x}_{\perp i}, \mathbf{y}_{\perp i}) \right\rangle = e^{-E_0^{(0)}} \left\{ 1 + \sum_{i>j}^n \left(1 - \frac{1 - e^{-\Delta E_{(ij)}^{(0)}}}{\Delta E_{(ij)}^{(0)}} \right) \left(\frac{V_{(ij),0}^{(1)}}{N_c} + \frac{V_{0,(ij)}^{(1)} V_{(ij),0}^{(1)}}{\Delta E_{(ij)}^{(0)}} \right) \right\}. \quad (4.14)$$

This is our final expression expanded up to the N_c^{-2} order in the MV model.

Figure 1. Validity check for the large- N_c approximation (4.14) as compared to the analytically exact result (3.26) for the $n = 2$ dipole correlation with $Q_s = 5$ at $\mathbf{x}_{\perp 1} = (0, 0)$, $\mathbf{y}_{\perp 1} = (0.6, 0.6)$, $\mathbf{x}_{\perp 2} = (x_2, 0)$, $\mathbf{y}_{\perp 2} = (0.3, 0.3)$ (in the unit of $\bar{\Lambda}$).

5 Comparison to the exact answer

In this section let us make a comparison between numerical results from our expansion (4.14) and the exact answer. In particular for the $n = 2$ case as we discussed around eq. (3.26), the full analytical expression for the dipole Wilson line correlation is known for general N_c , which provides us with a useful benchmark to quantify the validity of the large- N_c approximation in eq. (4.14). Because our present purpose is to check our formulae (4.14), concrete values of model parameters are not relevant. The coupling g always appears as a combination of $g^4 Q_s^2$, so we can take $g = 1$ without loss of generality and change Q_s . We measure all variables in the unit of $\bar{\Lambda}$ here. This means, for example, $Q_s = 5$ in this section is actually $Q_s = 5\bar{\Lambda}$, etc.

Figure 1 shows the validity check between eqs. (3.26) and (4.14) for the $n = 2$ dipole correlator with $N_c = 3$. The agreement generally depends on $\mathbf{x}_{\perp i}$ and $\mathbf{y}_{\perp i}$, but our formulae (4.14) work quite well for almost all $\mathbf{x}_{\perp i}$ and $\mathbf{y}_{\perp i}$ as long as Q_s is not too large (when Q_s is too large, the outputs are too small, and the errors become relatively larger). Here, in figure 1, we chose $Q_s = 5$ and $\mathbf{x}_{\perp 1} = (0, 0)$, $\mathbf{y}_{\perp 1} = (0.6, 0.6)$, $\mathbf{x}_{\perp 2} = (x_2, 0)$, $\mathbf{y}_{\perp 2} = (0.3, 0.3)$, which is intentionally chosen to make the difference as visible as possible for the small x_2 region, and so, the overall agreement is better than shown in figure 1.

We next see the Q_s dependence of the $n = 2$ dipole correlator together with the Abelian approximation, as depicted in figure 2. We introduce the Abelian approximation as employed in refs. [42, 43] so that the $n = 1$ expectation value can reproduce the exact result. For example of the $n = 2$ case, the Abelian approximation reads

$$e^{-Q_s^2[\Gamma(|\mathbf{x}_{\perp 1} - \mathbf{y}_{\perp 1}|) - \Gamma(|\mathbf{x}_{\perp 1} - \mathbf{x}_{\perp 2}|) + \Gamma(|\mathbf{x}_{\perp 1} - \mathbf{y}_{\perp 2}|) + \Gamma(|\mathbf{x}_{\perp 2} - \mathbf{y}_{\perp 1}|) - \Gamma(|\mathbf{y}_{\perp 1} - \mathbf{y}_{\perp 2}|) + \Gamma(|\mathbf{x}_{\perp 2} - \mathbf{y}_{\perp 2}|)]}, \quad (5.1)$$

which would agree with the exact answer in the limit of $\mathbf{x}_{\perp 1} = \mathbf{y}_{\perp 1}$ or $\mathbf{x}_{\perp 2} = \mathbf{y}_{\perp 2}$ (in which the correlator reduces to the $n = 1$ one), but deviates from the exact answer for general $\mathbf{x}_{\perp i}$ and $\mathbf{y}_{\perp i}$. In this sense, the expression like eq. (5.1) is to be regarded as an Abelianized

Figure 2. Validity check for the large- N_c approximation (4.14) and the Abelian approximation as compared to the analytically exact result (3.26) for the $n = 2$ dipole correlation. The positions are chosen as $\mathbf{x}_{\perp 1} = (0, 0)$, $\mathbf{y}_{\perp 1} = (0.6, 0.6)$, $\mathbf{x}_{\perp 2} = (0.1, 0)$, $\mathbf{y}_{\perp 2} = (0.3, 0.3)$.

extrapolation from the $n = 1$ answer. Figure 2 clearly shows that a small disagreement magnified in figure 1 is actually a negligibly small difference in the whole profile over various Q_s . The Abelian approximation captures a qualitative dependence with increasing Q_s , while the quantitative values should be considered as only order estimates.

6 Flow harmonics and higher order contributions

We follow the calculations of the flow observables, $v_n\{m\}$, in the dilute-dense system according to refs. [42, 43], i.e. we focus on only the CGC effects on $v_n\{m\}$ apart from other fluctuations as implemented in the Monte-Carlo Glauber model. We here emphasize that our present goal is not giving a full theoretical prediction for the observables but quantifying the parton level contributions captured by the CGC model. The flows are characteristic angular distributions defined from the m -particle inclusive spectra, which are in the dipole model given by

$$\begin{aligned} & \frac{d^m N}{d^2 \mathbf{p}_{\perp 1} \cdots d^2 \mathbf{p}_{\perp m}} \\ &= \frac{1}{(4\pi^3 B)^m} \prod_{i=1}^m \int d^2 \mathbf{x}_{\perp i} d^2 \mathbf{y}_{\perp i} e^{-\frac{\mathbf{x}_{\perp i}^2 + \mathbf{y}_{\perp i}^2}{2B} + i(\mathbf{x}_{\perp i} - \mathbf{y}_{\perp i}) \cdot \mathbf{p}_{\perp i}} \left\langle \prod_{j=1}^m D(\mathbf{x}_{\perp j}, \mathbf{y}_{\perp j}) \right\rangle, \end{aligned} \quad (6.1)$$

where B is a dipole model parameter. The dipole description is valid in the large N_c limit in which gluons are represented by dipoles. In principle, the dipole distribution should be fixed by the solution of the quantum evolution equation, but in the dipole model the distribution is simply approximated in a Gaussian form. Typically, the Gaussian size B should be of order of the dilute-system (or proton) size $\sim 1 \text{ fm} \sim (0.2 \text{ GeV})^{-1}$, and we take $\sqrt{B} = 2 \text{ GeV}^{-1}$.

We note that, if (multiple) gluon radiations occur, the above formula would be modified. Such processes (after the collision) are parametrically suppressed by higher powers of

the strong coupling constant, but may be collinearly enhanced, leading to a parton shower. In the present analysis we implicitly assume that the initial parton correlations are not diminished by the hadronization process. The gluon splitting processes are also possible before the collision, but they should be identified as the parton evolution in the projectile wave-function, which is here approximated by the dipole model.

The general analysis for the flow properties is presented in ref. [48] and the n -th moment of the m -particle correlation is introduced as

$$\kappa_n\{m\} := \prod_{i=1}^m \int \frac{d^2 \mathbf{p}_{\perp i}}{(2\pi)^2} e^{in(-1)^{i+1}\phi_i} \frac{d^m N}{d^2 \mathbf{p}_{\perp 1} \cdots d^2 \mathbf{p}_{\perp m}}, \quad (6.2)$$

where ϕ_i represents the azimuthal angle, i.e. $\mathbf{p}_{\perp i} = |\mathbf{p}_{\perp i}|(\cos \phi_i, \sin \phi_i)$. Then, in the dipole model, we can perform the momentum integrations to find the following expression,

$$\kappa_n\{m\} = \frac{1}{(4\pi^3 B)^m} \prod_{i=1}^m \int d^2 \mathbf{x}_{\perp i} d^2 \mathbf{y}_{\perp i} e^{-\frac{\mathbf{x}_{\perp i}^2 + \mathbf{y}_{\perp i}^2}{2B}} K_n^{((-1)^{i+1})}(\mathbf{x}_{\perp i} - \mathbf{y}_{\perp i}) \left\langle \prod_{j=1}^m D(\mathbf{x}_{\perp j}, \mathbf{y}_{\perp j}) \right\rangle. \quad (6.3)$$

Here, using the regularized generalized hypergeometric function, we defined,

$$\begin{aligned} K_n^{(\pm)}(\mathbf{x}_{\perp}) &:= \int^{p_{\max}} \frac{d^2 \mathbf{p}_{\perp}}{(2\pi)^2} e^{\pm in\phi + i\mathbf{x}_{\perp} \cdot \mathbf{p}_{\perp}} \\ &= \frac{i^n p_{\max}^2 e^{\pm in\theta}}{2\pi(n+2)n!} \left(\frac{|\mathbf{x}_{\perp}| p_{\max}}{2} \right)^n {}_1F_2 \left(1 + \frac{n}{2}; \left\{ 1 + n, 2 + \frac{n}{2} \right\}; -\frac{1}{4} |\mathbf{x}_{\perp}|^2 p_{\max}^2 \right), \end{aligned} \quad (6.4)$$

where θ is the azimuthal angle of $\mathbf{x}_{\perp}$. In particular, we will frequently use the $n = 0$ function for the normalization, which is given by

$$K_0^{(\pm)}(\mathbf{x}_{\perp}) = \tilde{\delta}_{p_{\max}}^{(2)}(\mathbf{x}_{\perp}) := \frac{p_{\max}}{2\pi |\mathbf{x}_{\perp}|} J_1(|\mathbf{x}_{\perp}| p_{\max}). \quad (6.5)$$

It is important to notice that $\pm$ is irrelevant for $n = 0$ and there is no angular dependence any more in $K_0^{(\pm)}(\mathbf{x}_{\perp})$. Also, we must point out that $\tilde{\delta}_{p_{\max}}^{(2)}(\mathbf{x}_{\perp})$ should approach $\delta^{(2)}(\mathbf{x}_{\perp})$ in the $p_{\max} \rightarrow \infty$ limit.

Let us first consider the case with $m = 2$ using our large- N_c expansion. The N_c^0 order term in eq. (4.14) does not contribute to $\kappa_n\{2\}$ due to the phase factor in $K_n^{(\pm)}(\mathbf{x}_{\perp i} - \mathbf{y}_{\perp i})$. As a result, we can write $\kappa_n\{2\}$ using the N_c^{-2} order results as

$$\kappa_n\{2\} = D_n^{(+-)}, \quad (6.6)$$

where

$$\begin{aligned} D_n^{(+-)} &:= \frac{1}{(4\pi^3 B)^2} \int d^2 \mathbf{x}_{\perp 1} d^2 \mathbf{y}_{\perp 1} d^2 \mathbf{x}_{\perp 2} d^2 \mathbf{y}_{\perp 2} e^{-\frac{\mathbf{x}_{\perp 1}^2 + \mathbf{y}_{\perp 1}^2 + \mathbf{x}_{\perp 2}^2 + \mathbf{y}_{\perp 2}^2}{2B}} \\ &\quad \times e^{-Q_s^2 \Gamma(\mathbf{x}_{\perp 1} - \mathbf{y}_{\perp 1})} K_n^{(+)}(\mathbf{x}_{\perp 1} - \mathbf{y}_{\perp 1}) e^{-Q_s^2 \Gamma(\mathbf{x}_{\perp 2} - \mathbf{y}_{\perp 2})} K_n^{(-)}(\mathbf{x}_{\perp 2} - \mathbf{y}_{\perp 2}) \\ &\quad \times \left(1 - \frac{1 - e^{-\Delta E_{(21)}^{(0)}}}{\Delta E_{(21)}^{(0)}} \right) \left(\frac{V_{(21),0}^{(1)}}{N_c} + \frac{V_{0,(21)}^{(1)} V_{(21),0}^{(1)}}{\Delta E_{(21)}^{(0)}} \right). \end{aligned} \quad (6.7)$$

Figure 3. 2-particle flow harmonics using the analytical (unapproximated) expression (3.26) and our formulae (4.14). The saturation momentum Q_s is the one defined in eq. (6.11) as in ref. [43].

One could think of $D_n^{(++)}$ and $D_n^{(--)}$ in a similar manner but they are also vanishing because of the phase factors in $K_n^{(\pm)}$. The last part of the integrand is a function of modulus of various combinations of $\mathbf{x}_{\perp 1}$, $\mathbf{x}_{\perp 2}$, $\mathbf{y}_{\perp 1}$, $\mathbf{y}_{\perp 2}$. Here, it is crucially important to understand that any term in the integrand which is factorized into a function of $|\mathbf{x}_{\perp i} - \mathbf{y}_{\perp i}|$ alone would vanish due to the phase factors in $K_n^{(\pm)}(\mathbf{x}_{\perp i} - \mathbf{y}_{\perp i})$ in the factorized integrations.

Because there is no finite contribution of disconnected parts in the two particle correlation, we can immediately compute the two particle flow harmonics, $v_n\{2\}$, from

$$v_n\{2\} = \sqrt{\frac{\kappa_n\{2\}}{\kappa_0\{2\}}}, \quad (6.8)$$

where the denominator is obtained with eq. (6.5), i.e. if we keep using the expanded expression up to the N_c^{-2} order for later convenience, we have

$$\kappa_0\{2\} = D_0^2 + D_0^{(+-)}. \quad (6.9)$$

Here, we defined

$$D_0 := \frac{1}{4\pi^3 B} \int d^2\mathbf{x}_{\perp} d^2\mathbf{y}_{\perp} e^{-\frac{\mathbf{x}_{\perp}^2 + \mathbf{y}_{\perp}^2}{2B}} e^{-Q_s^2 \Gamma(\mathbf{x}_{\perp} - \mathbf{y}_{\perp})} \tilde{\delta}_{p_{\max}}^{(2)}(\mathbf{x}_{\perp} - \mathbf{y}_{\perp}). \quad (6.10)$$

Of course, up to this order, keeping $D_0^{(+-)}$ in the denominator is in principle irrelevant since it gives a higher order correction which we should neglect.

We summarize our numerical results in figure 3. We have performed the 8 dimensional numerical integration with respect to $\{\mathbf{x}_{\perp i}, \mathbf{y}_{\perp i}\}$ using the Monte-Carlo method by taking 10^8 sampling points. To draw figure 3 we chose $\bar{\Lambda} = 0.241$ GeV and $p_{\max} = 2$ GeV in accord with ref. [43]. We also note that, only in this section, we change the definition of the saturation momentum from our original Q_s in eqs. (2.2) and (2.3) to new $\bar{Q}_s$ defined by

$$\bar{Q}_s^2 = \frac{1}{2\Gamma(|\sqrt{2}/Q_s|)} \quad (6.11)$$

according to refs. [42, 43]. Since there is no confusion, in this section, we will omit bar and simply denote Q_s to mean $\bar{Q}_s$. Then, we can make a direct comparison of our outputs to figure 1 of ref. [42]. The dashed curved in figure 3 must precisely reproduce figure 1 of ref. [42]. At a glance of our numerical calculations we see quantitatively good agreement. The most interesting question is how useful our large- N_c formulae (4.14) can be for the $N_c = 3$ case. The comparison between the dashed curves (full analytical results) and the solid curves (large- N_c approximations) in figure 3 concludes that the errors are of only a few (at most ~ 5) % level except for the $n = 3$ case.

Next, it is intriguing to see what happens for $m = 4$. In this case, $\kappa_n\{4\}$ involves $K_n^{(+)}(\mathbf{x}_{\perp 1} - \mathbf{y}_{\perp 1})$, $K_n^{(-)}(\mathbf{x}_{\perp 2} - \mathbf{y}_{\perp 2})$, $K_n^{(+)}(\mathbf{x}_{\perp 3} - \mathbf{y}_{\perp 3})$, and $K_n^{(-)}(\mathbf{x}_{\perp 4} - \mathbf{y}_{\perp 4})$. It is then easy to understand that our formula of the N_c^{-2} order in eq. (4.14) is insufficient to get nonvanishing contributions. Terms of the formula (4.14) are functions of, say, $\mathbf{x}_{\perp 1}$, $\mathbf{x}_{\perp 2}$, $\mathbf{y}_{\perp 1}$, $\mathbf{y}_{\perp 2}$ for $i = 2$ and $j = 1$, and then the angle integrations of $\mathbf{x}_{\perp 3} - \mathbf{y}_{\perp 3}$ and $\mathbf{x}_{\perp 4} - \mathbf{y}_{\perp 4}$ become zero. Therefore, one permutation is not enough, but two permutations are necessary to shuffle all $\mathbf{x}_{\perp 1}$, $\mathbf{x}_{\perp 2}$, $\mathbf{x}_{\perp 3}$, and $\mathbf{x}_{\perp 4}$; we must go to the next N_c^{-4} order to compute a first nonzero term in $\kappa_n\{4\}$.

It is a straightforward but tedious calculation to pick all the N_c^{-4} order terms up from the expansion in eq. (4.11). We can slightly simplify the problem by discarding terms which do not contribute to $\kappa_n\{4\}$. The N_c^{-4} order terms generally contain the product of the interaction matrix elements like

$$\sim V_{0,(pq)}^{(1)} V_{(pq),(ij)(kl)}^{(1)} V_{(ij)(kl),(ij)}^{(1)} V_{(ij),0}^{(1)}, \quad (6.12)$$

but we already saw that, if an unexchanged pair exists, the angle integration is vanishing. For example, if the above matrix elements are $\sim V_{0,(21)}^{(1)} V_{(21),0}^{(1)} V_{0,(21)}^{(1)} V_{(21),0}^{(1)}$, which itself is nonzero, the angle integrations of $\mathbf{x}_{\perp 3} - \mathbf{y}_{\perp 3}$ and $\mathbf{x}_{\perp 4} - \mathbf{y}_{\perp 4}$ are zero. Thus, among all possible combinations of the matrix elements, there are finite contributions only from

$$\begin{aligned} & V_{0,(43)}^{(1)} V_{(43),(0)}^{(1)} V_{(0),(21)}^{(1)} V_{(21),0}^{(1)}, \quad V_{0,(42)}^{(1)} V_{(42),(0)}^{(1)} V_{(0),(31)}^{(1)} V_{(31),0}^{(1)}, \\ & V_{0,(32)}^{(1)} V_{(32),(0)}^{(1)} V_{(0),(41)}^{(1)} V_{(41),0}^{(1)}. \end{aligned} \quad (6.13)$$

Here, we used relations such as $V_{(ij)(kl),(ij)}^{(1)} = V_{(kl),0}^{(1)}$, $V_{(kl),(ij)(kl)}^{(1)} = V_{0,(ij)}^{(1)}$, etc for $(ij) \neq (kl)$. After long calculations we arrive at a final form which turned out to be factorized as

$$\begin{aligned} \kappa_n\{4\} = & \frac{1}{(4\pi^3 B)^4} \prod_{i=1}^4 \int d^2 \mathbf{x}_{\perp i} d^2 \mathbf{y}_{\perp i} e^{-\frac{\mathbf{x}_{\perp i}^2 + \mathbf{y}_{\perp i}^2}{2B}} e^{-Q_s^2 \Gamma(\mathbf{x}_{\perp i} - \mathbf{y}_{\perp i})} K_n^{((-1)^{i+1})}(\mathbf{x}_{\perp i} - \mathbf{y}_{\perp i}) \\ & \times \frac{1}{2!} \sum_{(ij) \neq (kl)} \left(1 - \frac{1 - e^{-\Delta E_{(ij)}^{(0)}}}{\Delta E_{(ij)}^{(0)}} \right) \left(\frac{V_{(ij),0}^{(1)}}{N_c} + \frac{V_{0,(ij)}^{(1)} V_{(ij),0}^{(1)}}{\Delta E_{(ij)}^{(0)}} \right) \\ & \times \left(1 - \frac{1 - e^{-\Delta E_{(kl)}^{(0)}}}{\Delta E_{(kl)}^{(0)}} \right) \left(\frac{V_{(kl),0}^{(1)}}{N_c} + \frac{V_{0,(kl)}^{(1)} V_{(kl),0}^{(1)}}{\Delta E_{(kl)}^{(0)}} \right), \end{aligned} \quad (6.14)$$

which is the full expression of the N_c^{-4} order without any truncation like the glasma graph approximation. This result looks quite reasonable, but we emphasize that the complete

cancellation of $(ij)(kl)$ intermediate states with an energy denominator, $E_{(ij)} + E_{(kl)}$, is far from trivial. The sum with respect to (ij) and (kl) should run over all the permutations of the combinations as listed in eq. (6.13). Among all the combinations of indices, $[(ij) = (31), (kl) = (42)]$ and $[(ij) = (42), (kl) = (31)]$ are irrelevant because $D_n^{(++)} = D_n^{(--) = 0}$ (for $n > 0$) as we already pointed out. Therefore, the remaining four combinations of $[(ij) = (21), (kl) = (43)]$, $[(ij) = (43), (kl) = (21)]$, $[(ij) = (41), (kl) = (32)]$, and $[(ij) = (32), (kl) = (41)]$ lead to

$$\kappa_n\{4\} = 2[D_n^{(+-)}]^2. \quad (6.15)$$

Then, with extra terms corresponding to $D_0^{(++)}$ and $D_0^{(--)}$ which are nonzero, the normalization is written as

$$\kappa_0\{4\} = D_0^4 + 6D_0^2 D_0^{(+-)}, \quad (6.16)$$

up to the N_c^{-2} order in the same way as in eq. (6.9). Now, the cumulant is then given by

$$c_n\{4\} = \frac{\kappa_n\{4\}}{\kappa_0\{4\}} - 2\left(\frac{\kappa_n\{2\}}{\kappa_0\{2\}}\right)^2. \quad (6.17)$$

The above quantity itself is zero in the strict order counting for $c_n\{4\}$ up to N_c^{-4} . Thus, our conclusion is, even in the full MV model beyond the glasma graph approximation, no connected cumulant remains for $c_n\{4\}$ at the N_c^{-4} order.

In this way we can understand that the first connected contribution to cumulants appears from the N_c^{-2m+2} order; for $m = 4$, thus, we need to go to the N_c^{-6} order and then a completely nested combination of four indices like $(43), (32), (21)$ is possible. Therefore, the flow harmonics from the fully nested permutations must scale as

$$v_n\{m\} \sim N_c^{-2+2/m}. \quad (6.18)$$

This is our conclusion on the N_c scaling in the full MV model. We make a remark that this conclusion is robust even if the MV model is extended to include non-Gaussian (cubic, quartic, etc) terms. For example, one insertion of quartic vertex would decrease one $1/N_c$ factor to make a completely nested diagram, but one extra $g^2 \sim N_c$ factor from the quartic coupling would compensate for it to give the same scaling after all.

Although our conclusion of $c_n\{4\} = 0$ at the N_c^{-4} order is solid, there may be a subtle point in the large- N_c scaling due to finiteness of $p_{\max}$. If one just uses our formulae up to the N_c^{-4} order to estimate $\kappa_n\{2\}$, $\kappa_0\{2\}$, $\kappa_n\{4\}$, and $\kappa_0\{4\}$, one would naïvely find that the cancellation is incomplete at the N_c^{-6} order and a finite remainder is given by

$$\delta c_n\{4\} = \frac{2[D_n^{(+-)}]^2}{D_0^4 + 6D_0^2 D_0^{(+-)}} - 2\left(\frac{D_n^{(+-)}}{D_0^2 + D_0^{(+-)}}\right)^2 \simeq -\frac{8[D_n^{(+-)}]^2 D_0^{(+-)}}{D_0^6} \simeq -\frac{8D_0^{(+-)}}{D_0^2} (v_n\{2\})^4. \quad (6.19)$$

This correction is beyond the N_c^{-4} order and should be identified as a part of the N_c^{-6} terms. One can easily check that this N_c^{-6} order correction is vanishing for $p_{\max} \rightarrow \infty$, in which $D_0^{(+-)} \rightarrow 0$. However, for a finite $p_{\max}$ especially comparable to Q_s , the correction could be sizable and the flow harmonics is modified even from the disconnected piece. To

Figure 4. $\delta v_n\{4\}$ estimated from eq. (6.20) using $v_n\{2\}$ obtained in the large- N_c limit. Our choice of the parameter is $p_{\max} = 2 \text{ GeV}$.

see this effect quantitatively, let us compute the flow harmonics corresponding to $\delta c\{4\}$, which leads to

$$\delta v_n\{4\} = (-\delta c_n\{4\})^{1/4} = 2^{3/4} \left(\frac{D_0^{(+-)}}{D_0^2} \right)^{1/4} v_n\{2\}. \quad (6.20)$$

We make a plot of $\delta v_n\{4\}$ as a function of Q_s in figure 4. One must be very careful of the physical interpretation of this correction by $\delta v_n\{4\}$. Even though this is non-negligible as seen in figure 4, the physical origin lies in not the connected correlator but in the normalization. Moreover, this normalization effect makes the N_c counting skewed to become

$$\delta v_n\{m\} \sim N_c^{-1-2/m} \quad (m \geq 4), \quad (6.21)$$

which starts differing from anticipated eq. (6.18) for $m > 4$. Thus, to distinguish the connected contribution from the normalization effect, one can test the N_c scaling properties as in eqs. (6.18) and (6.21) and also check the $p_{\max}$ dependence since $\delta v_n\{m\}$ from the normalization effect is very sensitive to $p_{\max}/Q_s$ as perceived from figure 4.

7 Conclusions

We have established general formulae and machineries to compute $2n$ -point Wilson line (or n dipole) correlators with the color group representation $(N_c \otimes \bar{N}_c)^n$ in the McLerran-Venugopalan model. The color structure accommodates a huge representation but the nonzero contribution to the Wilson line correlators reduces to the $n! \times n!$ matrix, whose bases correspond to the color singlets. In particular, we have derived the explicit expression of the matrix elements [see eqs. (3.21) and (3.22)] in the color singlet bases constructed by permutations. The formulae are quite powerful not only for the direct numerical evaluation of the matrix but also for the analytical large- N_c expansion. We have developed the systematic large- N_c expansion in a way analogous to time-dependent perturbation theory

in quantum mechanics. We have then shown the explicit expression up to N_c^{-2} order for the dipole correlators as given in eq. (4.14). As a check of the validity, we have compared results from the exact answers and those in the large- N_c expansion for the two-particle flow harmonics, $v_n\{2\}$ ($n = 2, 3, 4, 5$), which shows quantitatively good agreement. We have continued our large- N_c expansion to higher orders to discuss the flow harmonics with more particles. Then, we have discovered the N_c scaling as $v_n\{m\} \sim N_c^{-2+2/m}$ even beyond the glasma graph approximation but in the full MV model. We have also pointed out that a slightly different N_c scaling could emerge from the normalization effect at finite cutoff of the transverse momenta of integrated particles.

Although we focused on only the dipole correlators in the present paper, our general formulae also provide us with useful approaches to evaluate Wilson line correlators in channels relevant for the particle production rate in the p - A collision generally. In fact, not only fundamental but also adjoint Wilson line correlators appearing in the multi-gluon production can be derived from our results in eq. (2.1) through the relation $[U_{\text{adj}}]_{ba}(\mathbf{x}_\perp) = 2\text{tr}[U(\mathbf{x}_\perp)T_F^a U^\dagger(\mathbf{x}_\perp)T_F^b] = T_{F\alpha\bar{\alpha}}^a T_{F\bar{\beta}\beta}^b U_{\beta\alpha}(\mathbf{x}_\perp)U_{\bar{\beta}\bar{\alpha}}^*(\mathbf{x}_\perp)$.

As we emphasized, our scheme of the large- N_c expansion takes a nice form which is easily implemented in numerical algorithms to go to arbitrarily higher orders. Such higher order numerical evaluations are left as an intriguing future problem. It would be also an important question to think about generalizations beyond the MV model. Further systematic considerations on the Wilson line correlators should deserve more investigations in the future.

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