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Recurrence relations for the $\mathcal{W}_3$ conformal blocks and $\mathcal{N} = 2$ SYM partition functions

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ABSTRACT: Recursion relations for the sphere 4-point and torus 1-point $\mathcal{W}_3$ conformal blocks, generalizing Alexei Zamolodchikov's famous relation for the Virasoro conformal blocks are proposed. One of these relations is valid for any 4-point conformal block with two arbitrary and two special primaries with charge parameters proportional to the highest weight of the fundamental irrep of $SU(3)$. The other relation is designed for the torus conformal block with a special (in above mentioned sense) primary field insertion. AGT relation maps the sphere conformal block and the torus block to the instanton partition functions of the $\mathcal{N} = 2$ $SU(3)$ SYM theory with 6 fundamental or an adjoint hypermultiplets respectively. AGT duality played a central role in establishing these recurrence relations, whose gauge theory counterparts are novel relations for the $SU(3)$ partition functions with $N_f = 6$ fundamental or an adjoint hypermultiplets. By decoupling some (or all) hypermultiplets, recurrence relations for the asymptotically free theories with $0 \leq N_f < 6$ are found.

KEYWORDS: Conformal and W Symmetry, Extended Supersymmetry, Gauge Symmetry, Supersymmetric Gauge Theory

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To the memory of Alexei Zamolodchikov

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1 Introduction

Conformal blocks play central role in any 2d CFT since they are holomorphic building constituents of the correlation functions of primary fields [1]. In the case when the theory possesses no extra holomorphic current besides the spin 2 energy-momentum tensor, the conformal block is fixed by the Virasoro symmetry solely. However a direct computation is practical up to first few levels of the intermediate state. Upon increasing the level such computation soon becomes intractable. Some three decades ago Alexei Zamolodchikov found a brilliant solution to this problem. Based on analysis of the poles and respective residues of the 4-point conformal block considered as a function of the intermediate conformal weight and thorough investigation of the semiclassical limit, a very efficient recursion formula has been discovered [2, 3]. Successful applications of this recurrence relation include Liouville theory [4], 4d $\mathcal{N} = 2$ SYM [5], topological strings [6, 7], partition function and Donaldson polynomials on $\mathbb{CP}_2$ [8] et al.

Analogous recurrence relations has been found much later also for torus 1-point Virasoro block [5] (see also [9]) and for $\mathcal{N} = 1$ Super-conformal blocks [10, 11].

The case when the theory admits higher spin $\mathcal{W}$ -algebra symmetry [12–14] is much more complicated. Holomorphic blocks of correlation functions of generic $\mathcal{W}$ -primary fields can not be found on the basis of the $\mathcal{W}$ -algebra Ward identities solely. Still, it is known

that if an n -point ($n \geq 4$) contains $n - 2$ partially degenerate primaries,¹ the $\mathcal{W}$ -algebra is restrictive enough to determine (in principle) such blocks. It appears that exactly at this situation an alternative way to obtain $\mathcal{W}$ -conformal blocks based on AGT relation [15–17] is available.

Note that though AGT relations provide combinatorial formulae for computing such conformal blocks, a recursion formulae like the one originally proposed by Zamolodchikov have an obvious advantage. Besides being very efficient for numerical calculations [4], such recursive formulae are very well suited for the investigation of analyticity properties and asymptotic behavior of the conformal blocks (or their AGT dual instanton partition functions [5]). Instead the individual terms of the instanton sum have many spurious poles that cancel out only after summing over all, rapidly growing number of terms of given order which leaves the final analytic structure more obscure.

In this paper recursion formulae are proposed for $\mathcal{N} = 2$ SU(3) gauge theory instanton partition function in Ω -background (Nekrasov’s partition function) with $0 \leq N_f \leq 6$ fundamental hypermultiplets as well as for the case with an adjoint hypermultiplet ($\mathcal{N} = 2^*$ theory). As a byproduct all instanton exact formula is conjectured for the partition function in an one-parameter family of vacua, which is a natural generalization of the special vacuum introduced in [18] and recently investigated in [19]. The IR-UV relation discovered in [19, 20] was very helpful in finding these results.

Using AGT relation the analogs of Zamolodchikov’s recurrence relations are proposed for the (special) $\mathcal{W}_3$ 4-point blocks on sphere and for the torus 1-point block. Though CFT point of view makes many of the features of the recurrence relations natural, unfortunately rigorous derivations are still lacking.

The organization of the paper is as follows.

In section 2. After a short review of instanton counting in the theory with 6 fundamentals, it is shown how investigation of the poles and residues of the partition function incorporated with the known UV - IR relation and the insight coming from the 2d CFT experience leads to the recurrence relation. Then, subsequently decoupling the hypermultiplets by sending their masses to infinity corresponding recurrence relations for smaller number of flavours are found. The simplest case of pure theory ($N_f = 0$) is presented in more details.

Then a similar analysis is carried out and as a result, corresponding recurrence relation is found for the SU(3), $\mathcal{N} = 2^*$ theory.

In section 3. Using AGT relation, the recurrence relations are constructed for the 4-point $\mathcal{W}_3$ sphere blocks with two arbitrary and two partially degenerate insertions and for the torus block with a partially degenerate insertion. In both cases exact formulae for the large $\mathcal{W}_3$ current zero mode limit are presented. It is argued that the location of the poles as well as the structure of the residues which were instrumental in finding the recurrence relations of section 2., are related to the degeneracy condition and the structure of OPE of $\mathcal{W}_3$ CFT.

¹In this paper the term partially degenerate refers to the primary fields which admit a single null-vector on level 1.

Figure 1. On the left: the quiver diagram for the conformal $SU(3)$ gauge theory with 6 fundamental hypermultiplets. On the right: the dual $\mathcal{W}_3$ conformal block.

2 Instanton partition function in Ω background

2.1 $SU(3)$ theory with $N_f = 6$ fundamental hypermultiplets

Graphically this theory can be depicted as a quiver diagram on the left side of figure 1. The parameters $a_{0,i}$, $a_{2,i}$ are related to the hypermultiplet masses while a_i (i runs over 1, 2, 3) are the expectation values of the vector multiplet. The instanton part of the partition function is given as a sum over triple of Young diagrams $\vec{Y} = (Y_1, Y_2, Y_3)$ (see [21–23])

$$Z = \sum_{\vec{Y}} Z_{\vec{Y}} x^{|\vec{Y}|}, \quad (2.1)$$

where x is the exponentiated coupling (the instanton counting parameter), $|\vec{Y}|$ is the total number of boxes of Young diagrams. The coefficients $Z_{\vec{Y}}$ can be represented as

$$Z_{\vec{Y}} = \prod_{i,j=1}^3 \frac{Z_{bf}(\emptyset, a_{i,0}|Y_j, a_j) Z_{bf}(Y_i, a_i|\emptyset, a_{2,j})}{Z_{bf}(Y_i, a_i|Y_j, a_j)} \quad (2.2)$$

where

$$\begin{aligned} Z_{bf}(\lambda, a|\mu, b) &= \prod_{s \in \lambda} (a - b - L_\mu(s)\epsilon_1 + (1 + A_\lambda(s))\epsilon_2) \\ &\times \prod_{s \in \mu} (a - b + (1 + L_\lambda(s))\epsilon_1 - A_\mu(s)\epsilon_2). \end{aligned} \quad (2.3)$$

Here $A_\lambda(s)$ ($L_\lambda(s)$) is the distance in vertical (horizontal) direction from the upper (right) border of the box s to the outer boundary of the diagram λ as demonstrated in figure 2. As usual ϵ_1 and ϵ_2 denote the parameters of the Ω background. Without loss of generality one may assume that $a_1 + a_2 + a_3 = 0$. Then this parameters can be reexpressed in terms of the independent differences $a_{12} \equiv a_1 - a_2$ and $a_{23} \equiv a_2 - a_3$

$$(a_1, a_2, a_3) = \left(\frac{2a_{12} + a_{23}}{3}, -\frac{a_{12} - a_{23}}{3}, -\frac{a_{12} + 2a_{23}}{3} \right). \quad (2.4)$$

The masses of 6 fundamental hypermultiplets can be identified as

$$\begin{aligned} m_i &= -a_{0,i} & \text{for } i &= 1, 2, 3, \\ m_i &= \epsilon_1 + \epsilon_2 - a_{0,i-3} & \text{for } i &= 4, 5, 6. \end{aligned} \quad (2.5)$$

Figure 2. Arm and leg length with respect to the Young diagram with column lengths $\{4, 3, 3, 1, 1, 1\}$. The thick solid line outlines its outer border. $A(s_1) = -2$, $L(s_1) = -2$, $A(s_2) = 2$, $L(s_2) = 3$, $A(s_3) = -3$, $L(s_3) = -4$.

The advantage of the definition above is that the partition function is symmetric with respect to permutations of $N_f = 6$ masses $m_1, \dots, m_6$. For later convenience let us introduce also notations (elementary symmetric functions of masses)

$$T_n = \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N_f} m_{i_1} \cdots m_{i_n}. \quad (2.6)$$

Let us fix an instanton number k and perform partial summation in (2.1) over all diagrams with total number of boxes equal to k . Many spurious poles present in individual terms cancel and one gets a rational expression whose denominator is

$$(\epsilon_1 \epsilon_2)^k \prod (a_{12}^2 - \epsilon_{r,s}^2) (a_{23}^2 - \epsilon_{r,s}^2) ((a_{12} + a_{23})^2 - \epsilon_{r,s}^2), \quad (2.7)$$

where the product is over the positive integers $r \geq 1$, $s \geq 1$ such that $rs \leq k$ and

$$\epsilon_{r,s} = r\epsilon_1 + s\epsilon_2. \quad (2.8)$$

It is not difficult to check this statement explicitly for small k . Under AGT map this is equivalent to the well known fact that the 2d CFT blocks as a function of the parameters of the intermediate state acquire poles exactly at the degeneration points. Anticipating this relation let us introduce parameters

$$\begin{aligned} u &= a_{12}^2 + a_{12}a_{23} + a_{23}^2, \\ v &= (a_{12} - a_{23})(2a_{12} + a_{23})(a_{12} + 2a_{23}). \end{aligned} \quad (2.9)$$

We'll see in section 3.1 that u is closely related to the dimension and v to the W zero mode eigenvalue of the intermediate state. For what follows it will be crucial to note that the factors of (2.7) in terms of newly introduced parameters can be rewritten as

$$-27 (a_{12}^2 - \epsilon_{r,s}^2) (a_{23}^2 - \epsilon_{r,s}^2) ((a_{12} + a_{23})^2 - \epsilon_{r,s}^2) = v^2 - v_{r,s}(u)^2, \quad (2.10)$$

where

$$v_{r,s}(u) = (3\epsilon_{r,s}^2 - u) \sqrt{4u - 3\epsilon_{r,s}^2}. \quad (2.11)$$

Using (2.9) also in the numerator we can expel the parameters a_{12} , a_{23} in favor of v and u . Moreover for fixed u one gets a polynomial dependence on v . Thus, to recover the partition function one needs

- the residues at $v = v_{r,s}(u)$;
- the asymptotic behaviour of the partition function for a fixed value of u and large v .

2.1.1 The residues

It follows from the remarkable identity (2.10) that the residues at $v = \pm v_{r,s}$ is related to the residue with respect to the variable a_{12} at $a_{12} = \epsilon_{r,s}$ in a simple way:²

$$Res|_{v=\pm v_{r,s}} = \frac{27\epsilon_{r,s}}{\epsilon_{r,s} + 2a_{23}} Res|_{a_{12}=\epsilon_{r,s}}. \quad (2.12)$$

To restore the u -dependence in right hand side of (2.12) due to (2.10) one should substitute

$$a_{23} = \frac{-\epsilon_{r,s} \pm \sqrt{4u - 3\epsilon_{r,s}^2}}{2}. \quad (2.13)$$

A careful examination shows that the residue of $k = rs$ instanton term at $a_{12} = \epsilon_{r,s}$ receives a nonzero contribution only from the triple $(Y_1, \emptyset, \emptyset)$, where Y_1 is a rectangular diagram of size $r \times s$. Using eqs. (2.1), (2.2), (2.3) it is straightforward to evaluate this contribution. The result has a nice factorized form

$$Res|_{a_{12}=\epsilon_{r,s}} Z_{r,s} = - \prod_{i=1-r}^r \prod_{j=1-s}' \epsilon_{i,j}^{-1} \times \prod_{i=1}^r \prod_{j=1}^s \frac{\prod_{f=1}^6 (m_f + \frac{1}{3}a_{23} + \frac{2}{3}\epsilon_{r,s} - \epsilon_{i,j})}{(a_{23} + \epsilon_{r-i,s-j})(a_{23} + \epsilon_{i,j})}, \quad (2.14)$$

where the prime over the product means that the term with $i = j = 0$ should be omitted.³

2.1.2 Large v limit

Now let us consider the limit $v \rightarrow \infty$ for fixed u . This is equivalent to choosing

$$a_{23} = \frac{\sqrt{4u - 3a_{12}^2} - a_{12}^2}{2} \quad (2.15)$$

and taking large a_{12} limit. Here are the first few terms of this expansion

$$a_{23} = e^{-\frac{i\pi}{3}} a_{12} - \frac{i u}{\sqrt{3} a_{12}} - \frac{i u^2}{3 \sqrt{3} a_{12}^3} - \frac{2 i u^3}{9 \sqrt{3} a_{12}^5} - \frac{5 i u^4}{27 \sqrt{3} a_{12}^7} - \frac{14 i u^5}{81 \sqrt{3} a_{12}^9} + \dots \quad (2.16)$$

²This is a choice of branch of the inverse map $(v, u) \rightarrow (a_{12}, a_{23})$. We could consider the poles at $a_{23} = \epsilon_{r,s}$ or $a_{12} + a_{23} = \epsilon_{r,s}$ instead.

³In generic $SU(n)$ case with no hypers a nice formula has been found earlier [24] for the *multiple* residues at the values of parameters $a_{1,n}, a_{2,n} \dots a_{n-1,n}$ specialized like in (?). Unfortunately, these residues alone are not sufficient to derive a recurrence relation for the partition function.

I performed instanton calculation in this limit up to the order x^5 . The result up to the order x^4 reads:

$$\begin{aligned} \epsilon_1 \epsilon_2 \log Z \sim & x \left(\frac{m_1 \epsilon}{3} - \frac{m_2}{3} - \frac{2\epsilon^2}{9} - \frac{4u}{27} \right) + x^2 \left(\frac{5m_1 \epsilon}{27} - \frac{m_1^2}{54} - \frac{7m_2}{54} - \frac{10\epsilon^2}{81} - \frac{14u}{243} \right) \\ & + x^3 \left(\frac{283m_1 \epsilon}{2187} - \frac{40m_1^2}{2187} - \frac{163m_2}{2187} - \frac{566\epsilon^2}{6561} - \frac{1948u}{59049} \right) \\ & + x^4 \left(\frac{655m_1 \epsilon}{6561} - \frac{433m_1^2}{26244} - \frac{1321m_2}{26244} - \frac{1310\epsilon^2}{19683} - \frac{3931u}{177147} \right) + \dots, \end{aligned} \quad (2.17)$$

where (and further on) for shortness I use the notation $\epsilon = \epsilon_1 + \epsilon_2$. Notice that at $u = 0$ the choice of VEV (2.15), (2.16) coincides with the special vacuum investigated in [18, 19]. In [19, 20] an exact relation between the UV coupling and effective IR coupling has been established. It was shown that a central role is played by the congruence subgroup $\Gamma_1(3)$ of the duality group $\text{SL}(2, \mathbb{Z})$ [25, 26] and that the relation

$$x = -27 \left(\frac{\eta(q^3)}{\eta(q)} \right)^{12} \quad (2.18)$$

between $x = \exp 2\pi i \tau_{uv}$ and $q = \exp 2\pi i \tau_{ir}$, where $\eta(q)$ is Dedekind's eta function

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad (2.19)$$

is valid. It should not come as a surprise also that the unique degree 1 modular form of $\Gamma_1(3)$

$$f_1(q) = \left(\left(\frac{\eta^3(q)}{\eta(q^3)} \right)^3 + 27 \left(\frac{\eta^3(q^3)}{\eta(q)} \right)^3 \right)^{1/3} \quad (2.20)$$

and its “ingredients” have a role to play. Indeed the expression

$$\epsilon_1 \epsilon_2 \log \left(\left(-\frac{x}{27q} \right)^{\frac{u}{3\epsilon_1 \epsilon_2}} \left(\frac{\eta(q^3)}{\eta^3(q)} \right)^{\frac{3T_2 - T_1^2}{\epsilon_1 \epsilon_2}} f_1(q)^{\frac{T_1^2 - 3T_1 \epsilon + 2\epsilon^2}{2\epsilon_1 \epsilon_2}} \right) \quad (2.21)$$

nicely matches the expansion (2.17) up to quite high orders in q and there is little doubt that the argument of logarithm in (2.21) indeed gives the large v limit of the partition function exactly.

2.1.3 The recurrence relation

Using AGT relation it is not difficult to establish that the residue of the partition function at $v = \pm v_{r,s}(u)$ is proportional to the partition function with expectation values specified as

$$v \rightarrow \pm v_{r,-s}(u - 3\epsilon_1 \epsilon_2 r s); \quad u \rightarrow u - 3\epsilon_1 \epsilon_2 r s. \quad (2.22)$$

On CFT side these are exactly the values corresponding to the null vector built from the given degenerate intermediate state related to the choice $v = \pm v_{r,s}(u)$. Let us represent the partition function as

$$Z(v, u, q) = \left(-\frac{x}{27q}\right)^{\frac{u}{3\epsilon_1\epsilon_2}} \left(\frac{\eta(q^3)}{\eta^3(q)}\right)^{\frac{3T_2-T_1^2}{\epsilon_1\epsilon_2}} f_1(q)^{\frac{T_1^2-3T_1\epsilon+2\epsilon^2}{2\epsilon_1\epsilon_2}} H(v, u|q). \quad (2.23)$$

Note that

$$H(v, u|q) = 1 + O(v^{-1}). \quad (2.24)$$

Incorporating information about residues establish above we finally arrive at the recurrent relation

$$H(v, u|q) = 1 + \sum_{r,s=1}^{\infty} \sum_{\sigma=\pm} \frac{(-27q)^{rs} R_{r,s}^{(\sigma)}(u)}{v - \sigma v_{r,s}(u)} H(\sigma v_{r,-s}(u - 3\epsilon_1\epsilon_2 rs), u - 3\epsilon_1\epsilon_2 rs|q), \quad (2.25)$$

where due to eqs. (2.12), (2.14)

$$R_{r,s}^{(\pm)} = \frac{27\epsilon_{r,s}(u - \epsilon_{r,s}^2)}{\mp \sqrt{4u - 3\epsilon_{r,s}^2}} \prod_{i=1-r}^r \prod_{j=1-s}' \epsilon_{i,j}^{-1} \\ \times \prod_{i=1}^r \prod_{j=1}^s \frac{\prod_{l=1}^{N_f} \left(m_l - \frac{1}{2} \epsilon_{2i-r, 2j-s} \pm \frac{1}{6} \sqrt{4u - 3\epsilon_{r,s}^2}\right)}{u - \epsilon_{r,s}^2 + \epsilon_{i,j} \epsilon_{r-i, s-j}}. \quad (2.26)$$

Using the recurrence relation I have computed the partition function up to the order x^8 and compared it with the result of the direct instanton calculation. The agreement was perfect.

2.2 $N_f < 6$ cases

It is straightforward to decouple some of 6 hypermultiplets sending their masses to infinity.

Let us choose $m_{N_f+1} = \dots = m_6 = \Lambda$, renormalize the coupling constant as $x \rightarrow -\frac{x}{\Lambda^{6-N_f}}$ and take the large Λ limit.⁴ The net effect is that instead of the recursion relation (2.25) one obtains

$$H(v, u|x) = 1 + \sum_{r,s=1}^{\infty} \sum_{\sigma=\pm} \frac{(-x)^{rs} R_{r,s}^{(\sigma)}(u)}{v - \sigma v_{r,s}(u)} H(\sigma v_{r,-s}(u - 3\epsilon_1\epsilon_2 rs), u - 3\epsilon_1\epsilon_2 rs|x), \quad (2.27)$$

where for the residues the same formula (2.26) with appropriate number of hypermultiplets N_f is valid. The relation between Z and H becomes much simpler. Using eq. (2.17) we immediately see that for $N_f = 5$ the appropriate relation is

$$Z_{N_f=5} = \exp\left(\frac{x(18(T_1 - \epsilon) - x)}{54\epsilon_1\epsilon_2}\right) H(v, u|x), \quad (2.28)$$

⁴The minus sign is due to a subtle difference between fundamental and anti-fundamental hypermultiplets. With this sign included we get N_f anti-fundamentals in conventions of [15].

and, for $N_f = 4$:

$$Z_{N_f=4} = \exp\left(\frac{x}{3\epsilon_1\epsilon_2}\right) H(v, u|x). \quad (2.29)$$

Finally in the cases $N_f = 0, 1, 2, 3$ the functions Z and H simply coincide.

2.2.1 Pure SU(3) theory

This is the simplest case. It is easy to realize that the partition function is even with respect to the parameter v , so that the expansion (2.25) can be organized according to the poles in the variable v^2 :

$$Z(v^2, u|x) = 1 + \sum_{r,s=1}^{\infty} \frac{(-x)^{rs} R_{r,s}(u)}{v^2 - v_{r,s}^2(u)} Z(v_{r,-s}^2(u - 3\epsilon_1\epsilon_2 rs), u - 3\epsilon_1\epsilon_2 rs|x), \quad (2.30)$$

where

$$R_{r,s} = 54\epsilon_{r,s} (u - \epsilon_{r,s}^2) (u - 3\epsilon_{r,s}^2) \prod_{i=1-r}^r \prod_{j=1-s}^{s'} \epsilon_{i,j}^{-1} \prod_{i=1}^r \prod_{j=1}^s (u - \epsilon_{r,s}^2 + \epsilon_{i,j}\epsilon_{r-i,s-j})^{-1}. \quad (2.31)$$

2.3 $N = 2^*$ theory

The analysis of the SU(3) theory with an adjoint hypermultiplet can be carried out in a similar manner. The coefficients $Z_{\vec{Y}}$ of the instanton partition function (2.1) in this case is given by

$$Z_{\vec{Y}} = \prod_{i,j=1}^3 \frac{Z_{bf}(Y_i, a_i - m|Y_j, a_j)}{Z_{bf}(Y_i, a_i|Y_j, a_j)}, \quad (2.32)$$

where m is the mass of the adjoint hypermultiplet. The structure of poles is the same as in the previous cases. Due to symmetry under permutation $a_{12} \leftrightarrow a_{23}$ the partition function, as in the case of pure theory, is a function of v^2 . The residue of the $k = rs$ instanton charge sector of the partition function at $v^2 = v_{r,s}^2$ and fixed u is related to the residue in variable a_{12} at $a_{12} = \epsilon_{r,s}$ (with a_{23} fixed)

$$Res|_{v^2=v_{r,s}^2} = -54\epsilon_{r,s}(a_{23}^2 - \epsilon_{r,s}^2)(2\epsilon_{r,s}a_{23} + a_{23}^2) Res|_{a_{12}=\epsilon_{r,s}}. \quad (2.33)$$

As in the case of fundamental hypermultiplets the residue of $k = rs$ instanton term at $a_{12} = \epsilon_{r,s}$ receives a nonzero contribution only from the triple of Young diagrams $(Y_1, \emptyset, \emptyset)$ with Y_1 being a rectangular diagram of size $r \times s$. A direct calculation, using eqs. (2.3), (2.32) shows that

$$\begin{aligned} Res|_{a_{12}=\epsilon_{r,s}} Z_{r,s} &= \prod_{i=1-r}^r \prod_{j=1-s}^{s'} \frac{\epsilon_{i,j} - m}{\epsilon_{i,j}} \\ &\times \prod_{i=1}^r \prod_{j=1}^s \frac{(a_{23} + \epsilon_{r-i,s-j} + m)(a_{23} + \epsilon_{i,j} - m)}{(a_{23} + \epsilon_{r-i,s-j})(a_{23} + \epsilon_{i,j})}. \end{aligned} \quad (2.34)$$

Investigation of the large v^2 behavior in this case is simpler compared to the theory with 6 fundamentals. Computations in first few instanton orders shows that (in this section a more conventional notation q instead of x for the instanton counting parameter is restored)

$$\epsilon_1 \epsilon_2 \log Z_{\mathcal{N}=2^*} = -3(m - \epsilon_1)(m - \epsilon_2) \log \left(q^{-\frac{1}{24}} \eta(q) \right) + O(v^{-2}). \quad (2.35)$$

This is a suggestive result. Recall that in the case of $SU(2)$ gauge group one gets the same answer with the only difference that the overall factor 3 is replaced by 2 [5].

Further steps are straightforward. Introducing the function H via

$$Z_{\mathcal{N}=2^*} = \left(q^{-\frac{1}{24}} \eta(q) \right)^{-\frac{3(m-\epsilon_1)(m-\epsilon_2)}{\epsilon_1 \epsilon_2}} H(v^2, u, q) \quad (2.36)$$

we get the recurrence relation

$$H(v^2, u|q) = 1 + \sum_{r,s=1}^{\infty} \frac{q^{rs} R_{r,s}(u)}{v^2 - v_{r,s}^2(u)} H(v_{r,-s}^2(u - 3\epsilon_1 \epsilon_2 r s), u - 3\epsilon_1 \epsilon_2 r s | q), \quad (2.37)$$

where

$$\begin{aligned} R_{r,s} = & -54 m \epsilon_{r,s} (u - \epsilon_{r,s}^2) (u - 3\epsilon_{r,s}^2) \prod_{i=1-r}^r \prod_{j=1-s}^s \frac{\epsilon_{i,j} - m}{\epsilon_{i,j}} \\ & \times \prod_{i=1}^r \prod_{j=1}^s \frac{u - \epsilon_{r,s}^2 + (\epsilon_{i,j} - m)(\epsilon_{r-i,s-j} + m)}{u - \epsilon_{r,s}^2 + \epsilon_{i,j} \epsilon_{r-i,s-j}}. \end{aligned} \quad (2.38)$$

This recurrence relation has been checked by instanton calculation up to the order q^{10} .

3 Recurrence relation for $\mathcal{W}_3$ conformal blocks

In this section using AGT relations [15–17] the recurrence relations for $\mathcal{N} = 2$ SYM partition functions will be translated into recurrence relations for certain $\mathcal{W}_3$ -algebra four-point conformal blocks on sphere (AGT counterpart of $N_f = 6$ theory) and one-point torus blocks (AGT dual of $\mathcal{N} = 2^*$). This recurrence relations generalize Alexei Zamolodchikov's famous relation established for the four point Virasoro conformal blocks [2, 3]. The recurrent relation for Virasoro 1-point torus block was proposed in [5] (see also [9]). It should be emphasised nevertheless, that the $\mathcal{W}_3$ blocks considered here are not quite general, two of four primary fields of the sphere block as well as that of the 1-point torus block are specific. The charge vectors defining their dimensions and $\mathcal{W}_3$ zero-mode eigenvalues are taken to be multiples of the highest weight of the fundamental (or anti-fundamental) representation of $SU(3)$. Unfortunately effective methods to understand generic $\mathcal{W}$ -blocks (to my knowledge) are still lacking.

3.1 Preliminaries on A_2 Toda CFT

These are 2d CFT theories which, besides the spin 2 holomorphic energy momentum current $\mathcal{W}^{(2)}(z) \equiv T(z)$ are endowed with additional higher spin $s = 3$ current $\mathcal{W}^{(3)}$ [12, 13, 27]. The Virasoro central charge is conventionally parameterised as

$$c = 2 + 24Q^2,$$

where the “background charge” Q is given by

$$Q = b + \frac{1}{b},$$

and b is the dimensionless coupling constant of Toda theory. In what follows it would be convenient to represent roots, weights and Cartan elements of the Lie algebra A_2 as 3-component vectors satisfying the condition that the sum of the components is zero. It is assumed also that the scalar product is the usual Kronecker one. Obviously this is equivalent to a more conventional representation of these quantities as diagonal traceless 3×3 matrices with pairing given by trace. In this representation the Weyl vector is given by

$$\boldsymbol{\rho} = (1, 0, -1). \quad (3.1)$$

For further reference let us quote here explicit expressions for the highest weight $\boldsymbol{\omega}_1$ of the first fundamental representation and for its complete set of weights $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$

$$\begin{aligned} \boldsymbol{\omega}_1 &= \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right), \\ (\mathbf{h}_l)_i &= \delta_{l,i} - 1/3. \end{aligned} \quad (3.2)$$

The primary fields $V_{\boldsymbol{\alpha}}$ (in this paper we concentrate only on the left moving holomorphic parts) are parameterized by vectors $\boldsymbol{\alpha}$ with vanishing center of mass. Their conformal weights are given by

$$h_{\boldsymbol{\alpha}} = \frac{(\boldsymbol{\alpha}, 2Q\boldsymbol{\rho} - \boldsymbol{\alpha})}{2}. \quad (3.3)$$

Sometimes it is convenient to parameterize primary fields (or states) in terms of the Toda momentum vector $\mathbf{p} = Q\boldsymbol{\rho} - \boldsymbol{\alpha}$ instead of $\boldsymbol{\alpha}$. In what follows a special role is played by the fields $V_{\lambda\boldsymbol{\omega}_1}$ with dimensions

$$h_{\lambda\boldsymbol{\omega}_1} = \lambda \left(Q - \frac{\lambda}{3} \right). \quad (3.4)$$

For generic λ these fields admit a single null vector at the first level.

Besides the dimension, the fields are characterized also by the zero mode eigenvalue of the $\mathcal{W}_3$ current

$$w = -\frac{i}{27} \sqrt{\frac{48}{22+5c}} v, \quad (3.5)$$

where v is defined in terms of the momentum vector $\mathbf{p}$ as

$$v = 27p_1p_2p_3 = (p_{12} - p_{23})(p_{12} + 2p_{23})(2p_{12} + p_{23}) \quad (3.6)$$

and $p_{12} = p_1 - p_2$, $p_{23} = p_2 - p_3$. It is convenient to introduce also the parameter

$$u = p_{12}^2 + p_{23}^2 + p_{12}p_{23} \quad (3.7)$$

so that the conformal dimension (3.4) can be rewritten as

$$h = Q^2 - \frac{u}{3}. \quad (3.8)$$

The pair v, u characterizes primary fields more faithfully, than the charge vector, since they are invariant under the Weyl group action.

3.1.1 Sphere 4-point block

The object of our interest in this section will be the conformal block

$$\langle V_{\alpha_4}(\infty) V_{\lambda_3 \omega_1}(1) V_{\lambda_2 \omega_1}(x) V_{\alpha_1}(0) \rangle_{\mathbf{p}} \sim x^{h_{\alpha} - h_1 - h_2} G(v, u|x), \quad (3.9)$$

where $\langle \dots \rangle_{\mathbf{p}}$ denotes the holomorphic part of the correlation function with a specified intermediate state of momentum $\mathbf{p} = Q\boldsymbol{\rho} - \boldsymbol{\alpha}$. It is assumed that the function $G(v, u|x)$ is normalized so that $G(v, u|x) = 1 + O(x)$ (we explicitly display only dependence on the parameters v, u , which specify the intermediate state). Due to AGT relation, the function $G(v, u|x)$ is directly connected to the instanton partition function of SU(3) gauge theory with $N_f = 6$ hypermultiplets discussed earlier (see figure 1). Here is the map between parameters of the CFT and Gauge Theory (GT) sides:

$$b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}; \quad u_{\text{CFT}} = \frac{u_{\text{GT}}}{\epsilon_1 \epsilon_2}; \quad v_{\text{CFT}} = \frac{v_{\text{GT}}}{(\epsilon_1 \epsilon_2)^{3/2}}; \quad (3.10)$$

$$\lambda^{(2)} = \frac{3\epsilon - m_4 - m_5 - m_6}{\sqrt{\epsilon_1 \epsilon_2}}; \quad \lambda^{(3)} = \frac{m_1 + m_2 + m_3}{\sqrt{\epsilon_1 \epsilon_2}}; \quad (3.11)$$

$$\begin{aligned} \mathbf{p}^{(1)} &= Q\boldsymbol{\rho} - \boldsymbol{\alpha}^{(1)} \\ &= \left(\frac{-2m_4 + m_5 + m_6}{\sqrt{\epsilon_1 \epsilon_2}}, \frac{-2m_5 + m_4 + m_6}{\sqrt{\epsilon_1 \epsilon_2}}, \frac{-2m_6 + m_4 + m_5}{\sqrt{\epsilon_1 \epsilon_2}} \right); \end{aligned} \quad (3.12)$$

$$\begin{aligned} \mathbf{p}^{(4)} &= Q\boldsymbol{\rho} - \boldsymbol{\alpha}^{(4)} \\ &= \left(\frac{-2m_1 + m_2 + m_3}{\sqrt{\epsilon_1 \epsilon_2}}, \frac{-2m_2 + m_1 + m_3}{\sqrt{\epsilon_1 \epsilon_2}}, \frac{-2m_3 + m_1 + m_2}{\sqrt{\epsilon_1 \epsilon_2}} \right). \end{aligned} \quad (3.13)$$

Under this identification of parameters the relation between the gauge theory (with $N_f = 6$ fundamentals) partition function and the CFT conformal block is very simple:

$$Z = (1 - x)^{\lambda^{(3)}(Q - \frac{1}{3}\lambda^{(2)})} G. \quad (3.14)$$

Now it is quite easy to rephrase the recurrence relation for the partition function in terms of CFT language. Define a function $H(v, u|q)$ through

$$G(v, u|x) = \left(-\frac{x}{27q} \right)^{\frac{u}{3}} \left(\frac{\eta(q^3)}{\eta^3(q)} \right)^{3(h_1 + h_4) - 6Q^2} f_1(q)^{\frac{-3(h_2 + h_3) + 2Q^2}{2}} H(v, u|q), \quad (3.15)$$

where q and x are related as in (2.18). Then, due to (2.23), (2.25), (3.14) and (3.15) for $H(v, u|q)$ we get essentially the same recurrence relation (2.25)

$$H(v, u|q) = 1 + \sum_{r,s=1}^{\infty} \sum_{\sigma=\pm} \frac{(-27q)^{rs} R_{r,s}^{(\sigma)}(u)}{v - \sigma v_{r,s}(u)} H(\sigma v_{r,-s}(u - 3rs), u - 3rs|q), \quad (3.16)$$

where similar to (2.11)

$$v_{r,s}(u) = (3Q_{r,s}^2 - u) \sqrt{4u - 3Q_{r,s}^2} \quad (3.17)$$

with (cf. (2.8))

$$Q_{r,s} = br + \frac{s}{b} \quad (3.18)$$

and the residues are given by

$$R_{r,s}^{(\pm)} = \frac{27Q_{r,s}(u - Q_{r,s}^2)}{\mp \sqrt{4u - Q_{r,s}^2}} \prod_{i=1-r}^r \prod_{j=1-s}^{s'} Q_{i,j}^{-1} \\ \times \prod_{i=1}^r \prod_{j=1}^s \frac{\prod_{l=1}^6 \left(\mu_l - \frac{1}{2} Q_{2i-r, 2j-s} \pm \frac{1}{6} \sqrt{4u - 3Q_{r,s}^2} \right)}{u - Q_{r,s}^2 + Q_{i,j} Q_{r-i, s-j}}, \quad (3.19)$$

where CFT counterparts of gauge theory masses $\mu_l = m_l / \sqrt{\epsilon_1 \epsilon_2}$ are related to the parameters of the inserted fields via (3.11)–(3.13).

It follows from the analog of the Kac determinant for $\mathcal{W}_3$ -algebra [28], that the conformal block truncated up to the order x^k should have simple poles in the variable v (for u fixed) located at $v = \pm v_{r,s}(u)$ with $r \geq 1$, $s \geq 1$ and $rs \leq k$. The relation

$$v^2 - v_{r,s}^2(u) = 0 \quad (3.20)$$

among parameters v , u is the condition of existence of a null vector at the level rs . This null vector originates a $\mathcal{W}_3$ -algebra representation with parameters

$$u \rightarrow u - 3rs; \quad v \rightarrow \pm v_{r,-s}(u - 3rs). \quad (3.21)$$

Though we arrived to the recurrence relation starting from the gauge theory side, in fact many features of this relation are transparent from the CFT side and it is reasonable to expect that a rigorous proof may be found generalizing arguments of Alexei Zamolodchikov from Virasoro to the $\mathcal{W}$ -algebra case. Indeed (3.16) states that the residues at the poles $v = \pm v_{r,s}(u)$ (3.17), are proportional to the conformal block with internal channel parameters (3.21) corresponding to the null vector at the level rs .

The factor $R_{r,s}^{(\pm)}$ (3.19) also has many expected features. Its denominator vanishes exactly when the parameter u is specified so that a second independent degenerate state arises. The factors in the numerator reflect the structure of OPE with degenerate field (see [14]) exactly as it was in the case of Virasoro block considered by Alexei Zamolodchikov. It seems more subtle to justify presence of the u independent factors $Q_{i,j}^{-1}$.

Our result predicts the following large v behavior of the $\mathcal{W}_3$ block

$$G(v, u|x) \sim \left(-\frac{x}{27q} \right)^{\frac{u}{3}} \left(\frac{\eta(q^3)}{\eta^3(q)} \right)^{3(h_1+h_4)-6Q^2} f_1(q)^{\frac{-3(h_2+h_3)+2Q^2}{2}} + O(v^{-1}). \quad (3.22)$$

A good starting point to prove this relation might be the deformed Seiberg-Witten curve DSFT [29–31] or, equivalently, the quasiclassical null vector decoupling equation for $\mathcal{W}$ -blocks derived in [32].

3.1.2 Torus 1-point block

Since the torus 1-point block (below α is the charge parameter of the intermediate states)

$$\mathcal{F}_\alpha^\lambda(q) = q^{\frac{c}{24} - h_\alpha} \text{tr}_\alpha \left(q^{L_0 - \frac{c}{24}} V_{\lambda\omega_1}(1) \right) \quad (3.23)$$

is related to the partition function of the gauge theory with adjoint hypermultiplet via [33]

$$Z_{\mathcal{N}=2^*} = \left(q^{-\frac{1}{24}} \eta(q) \right)^{-\lambda(Q - \frac{\lambda}{3}) - 1} \mathcal{F}_\alpha^\lambda(q). \quad (3.24)$$

The parameter λ is related to the adjoint hypermultiplet mass m :

$$\lambda = \frac{3m}{\sqrt{\epsilon_1 \epsilon_2}} \quad (3.25)$$

and as earlier the intermediate momentum parameter $\mathbf{p} = Q\boldsymbol{\rho} - \alpha$ is related to the VEV of the vector multiplet $\mathbf{a}$ as

$$p_i = \frac{a_i}{\sqrt{\epsilon_1 \epsilon_2}}; \quad i = 1, 2, 3. \quad (3.26)$$

Thus, comparing with (2.36), (2.37), (2.38), we see that the function $H(v^2, u, q)$ defined by the equality

$$\mathcal{F}_\alpha^\lambda(q) = \left(q^{-\frac{1}{24}} \eta(q) \right)^{-2} H(v^2, u, q), \quad (3.27)$$

(v and u in terms of the momentum p were defined in (3.6), (3.7)) satisfies the recurrence relation

$$H(v^2, u|q) = 1 + \sum_{r,s=1}^{\infty} \frac{q^{rs} R_{r,s}(u)}{v^2 - v_{r,s}^2(u)} H(v_{r,-s}^2(u - 3rs), u - 3rs|q), \quad (3.28)$$

where

$$\begin{aligned} R_{r,s} = & -18 \lambda Q_{r,s} (u - Q_{r,s}^2) (u - 3Q_{r,s}^2) \prod_{i=1-r}^r \prod_{j=1-s}' \frac{Q_{i,j} - \frac{\lambda}{3}}{Q_{i,j}} \\ & \times \prod_{i=1}^r \prod_{j=1}^s \frac{u - Q_{r,s}^2 + (Q_{i,j} - \frac{\lambda}{3})(Q_{r-i,s-j} + \frac{\lambda}{3})}{u - Q_{r,s}^2 + Q_{i,j} Q_{r-i,s-j}}. \end{aligned} \quad (3.29)$$

4 Summary and discussion

To summarize let me quote the main results of this paper:

- the recurrence relation (see (2.21), (2.25), (2.26)) for the instanton partition function of $\mathcal{N} = 2$ SU(3) gauge theory with 6 fundamental hypermultiplets. This recurrence relation suggests an exact in all instanton orders formula for the partition function and prepotential for the theory in a generalized version of the special vacuum considered in [18, 19];

- recurrence relations for smaller number of hypermultiplets (see section 2.2) and for pure $N_f = 0$ theory (section 2.2.1);
- recurrence relations for the theory with an adjoint hypermultiplet, commonly referred as $\mathcal{N} = 2^*$ theory (see section 2.3);
- the analogs of Zamolodchikov's recurrence relations are constructed for 4-point sphere $\mathcal{W}_3$ -blocks with two arbitrary and two partially degenerate insertions (see (3.15), (3.16), (3.19)) and for the torus $\mathcal{W}_3$ -block with a partially degenerate insertion (see (3.27), (3.28), (3.29)). For both cases recursion formulae provide explicit expressions for the large $\mathcal{W}_3$ zero mode limit.

Though many details of the recurrence relations are transparent either from the 4d gauge theory or from the 2d CFT points of view, still full derivation is lacking. I hope to come back to these questions in a future publication.

Of course, generalization to the case of generic $SU(n)/\mathcal{W}_n$ cases would be an interesting development.

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