Derivation of functional equations for Feynman integrals from algebraic relations

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ABSTRACT: New methods for obtaining functional equations for Feynman integrals are presented. The application of these methods to finding functional equations for various one- and two-loop integrals is described in detail. It is shown that with the aid of the functional equations Feynman integrals in general kinematics can be expressed in terms of simpler integrals.

KEYWORDS: Renormalization Regularization and Renormalons, Scattering Amplitudes

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1 Introduction

Recently it has been discovered that Feynman integrals obey functional equations [1, 2]. In these papers, a systematic method for deriving the functional equations was proposed. Different examples of the functional equations were presented in refs. [1–3]. In these articles, only one-loop integrals were considered.

A particular relationship between N -point one-loop integrals with nonzero masses connecting integrals with different sets of kinematical arguments was derived in ref. [4]. A detailed consideration of this relationship for the one-loop propagator, vertex- and box-type integrals was presented in [5].

In the present paper, we propose essentially new systematic methods for deriving functional equations. These methods are based on algebraic relations between propagators and they are suitable for deriving functional equations for multiloop integrals. Also, these methods can be used to derive functional equations for integrals with some propagators raised to non-integer powers.

Our paper is organized as follows. Section 2 gives a short review of the method proposed in ref. [1].

In section 3, a method for finding algebraic relations between products of propagators is formulated. We describe in detail a derivation of explicit relations for products of two, three and four propagators. Also, the algebraic relation for products of an arbitrary number of propagators is given. These relations are used in section 4. to obtain functional equations for one- as well as two-loop integrals. In particular, the functional equation for the massless one-loop vertex type integral is presented. Also, the functional equation for the two-loop vertex type integral with arbitrary masses is given.

Section 5 proposes another method for obtaining functional equations. The method is based on finding algebraic relations for ‘deformed propagators’ and a further conversion of the integrals with ‘deformed propagators’ to usual Feynman integrals by imposing conditions on the deformation parameters. To perform such conversion, the α -parametric representation for both types of integrals is exploited. The method was used to derive a functional equation for the two-loop vacuum type integral with arbitrary masses. From this functional equation we obtained a new hypergeometric representation for the one-loop massless vertex integral.

In conclusion, we formulate our vision of the future applications and developments of the proposed methods.

2 Deriving functional equations from recurrence relations

The method for deriving functional equations proposed in ref. [1] is based on the use of a different kind recurrence relations. In particular, in refs. [1–3] the generalized recurrence relations [6] were utilized to obtain functional equations for one-loop Feynman integrals. In general, such recurrence relations connect a combination of some number of integrals $I_{1,n}, \dots, I_{k,n}$ corresponding to diagrams, say, with n lines and integrals corresponding to diagrams with fewer lines. Diagrams with fewer lines can be obtained by contracting some lines in integrals with n lines. The integrals corresponding to such diagrams depend on fewer kinematical variables and masses compared to integrals with n lines. Such recurrence relations can be written in the following form:

$$\sum_j Q_j(\{m_i\}, \{s_q\}, \nu_l, d) I_{j, n} = \sum_{k, r < n} R_{k, r}(\{m_i\}, \{s_m\}, \nu_l, d) I_{k, r}, \quad (2.1)$$

where Q_j and R_k are polynomials depending on masses m_i , scalar products s_r of external momenta, powers of propagators ν_l and parameter of the space time dimension d . In the left-hand side of eq. (2.1) we combined integrals with n lines and in the right-hand side integrals with fewer lines.

In accordance with the method of ref. [1], to obtain functional equation from eq. (2.1), one should eliminate terms on the left-hand side by defining some kinematical variables from the set of equations:

$$Q_j(\{m_i\}, \{s_q\}, \nu_l, d) = 0. \quad (2.2)$$

If there are nontrivial solutions of this system and for these solutions some $R_{k, r}(\{m_i\}, \{s_m\}, \nu_l, d)$ are different from zero, then the right-hand side of eq. (2.1) will represent the functional equation.

Figure 1. One-loop diagram with n external legs.

As an illustrative example, we consider one-loop integrals with n propagators

$$I_n^{(d)}(\{m_j^2\}, \{s_{kl}\}) = \int \frac{d^d q}{i\pi^{d/2}} \prod_{j=1}^n \frac{1}{P_j^{\nu_j}}, \quad (2.3)$$

where

$$P_j = (q - p_j)^2 - m_j^2 + i\eta. \quad (2.4)$$

Here as usual an imaginary term $i\eta$ with infinitesimal $\eta > 0$ implies a prescription on how to treat the poles of the propagators. The diagram corresponding to this integral is given in figure 1. Different types of recurrence relations for $I_n^{(d)}(\{m_j^2\}, \{s_{kl}\})$ were given in refs. [6, 7].

In particular, in refs. [6, 7] the following relation was derived:

$$\begin{aligned} & \left(d - \sum_{i=1}^n \nu_i + 1 \right) G_{n-1} I_n^{(d+2)}(\{m_j^2\}, \{s_{kl}\}) - 2\Delta_n I_n^{(d)}(\{m_j^2\}, \{s_{kl}\}) \\ &= \sum_{k=1}^n (\partial_k \Delta_n) \mathbf{k}^- I_n^{(d)}(\{m_j^2\}, \{s_{kl}\}), \end{aligned} \quad (2.5)$$

where the operators $\mathbf{k}^-$ shift the index of propagators by one unit $\nu_k \rightarrow \nu_k - 1$,

$$G_{n-1} = -2^n \begin{vmatrix} (p_1 - p_n)(p_1 - p_n) & (p_1 - p_n)(p_2 - p_n) & \dots & (p_1 - p_n)(p_{n-1} - p_n) \\ (p_1 - p_n)(p_2 - p_n) & (p_2 - p_n)(p_2 - p_n) & \dots & (p_2 - p_n)(p_{n-1} - p_n) \\ \vdots & \vdots & \ddots & \vdots \\ (p_1 - p_n)(p_{n-1} - p_n) & (p_2 - p_n)(p_{n-1} - p_n) & \dots & (p_{n-1} - p_n)(p_{n-1} - p_n) \end{vmatrix}, \quad (2.6)$$

$$\begin{aligned} \Delta_n &= \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix}, \\ Y_{ij} &= m_i^2 + m_j^2 - s_{ij}, \quad s_{ij} = (p_i - p_j)^2. \end{aligned} \quad (2.7)$$

Here p_i, p_j are external momenta going through lines i, j , respectively, and m_j is the mass attributed to the j -th line. The Gram determinant G_{n-1} and modified Cayley determinant Δ_n are polynomials depending on scalar products of external momenta and masses. In the case where all $\nu_j = 1$ the recurrence relation (2.5) for the first time was derived in refs. [8, 9].

It is assumed that these scalar products are made of d dimensional vectors and G_{n-1} and Δ_n are not subject to any restriction or condition specific to some integer values of d . Equation (2.5) is written in the form corresponding to eq. (2.1). To eliminate integrals with n lines in the left-hand side of eq. (2.5), the following conditions are to be held:

$$G_{n-1} = 0, \quad \Delta_n = 0. \quad (2.8)$$

Equation (2.5) is valid for arbitrary kinematical variables and masses. The solution of eqs. (2.8) can be easily done with respect to two kinematical variables or masses. Starting from $n = 3$, substitution of such solutions into eq. (2.5) gives nontrivial functional equations [1].

The method for obtaining the functional equations by eliminating complicated integrals from the recurrence relations is quite general. However, for multiloop integrals depending on several kinematical variables, derivation of equations like eq. (2.5) is computationally challenging. In the next sections, we will describe easier and more powerful methods that can be used for deriving the functional equations for multiloop integrals.

3 Algebraic relations between products of propagators

Setting $\nu_j = 1$ in eq. (2.5) and imposing conditions (2.8) lead to the following equation:

$$\sum_{k=1}^n (\partial_k \Delta_n) \mathbf{k}^- I_n^{(d)} = 0. \quad (3.1)$$

In eq. (3.1) the integrands of $\mathbf{k}^- I_n^{(d)}$ are products of $n - 1$ propagators depending on different external momenta, i.e. each term in this relation corresponds to the same function but with different arguments. In fact, the functional equations considered in refs. [1–3] are of the same form as eq. (3.1). The question naturally arises: does this relationship hold for integrals or it can be obtained as a consequence of a relationship between integrands?

By inspecting eq. (3.1), one can suggest the following form of the algebraic relation between the products of propagators of integrands:

$$\prod_{r=1}^n \frac{1}{P_r} = \frac{1}{P_{n+1}} \sum_{r=1}^n x_r \prod_{\substack{j=1 \\ j \neq r}}^n \frac{1}{P_j}, \quad (3.2)$$

where

$$P_j = (k_1 - p_j)^2 - m_j^2 + i\eta. \quad (3.3)$$

In what follows we will omit the $i\eta$ term assuming that all masses have such a correction.

Now let us consider in detail a derivation of the algebraic relations for products of 2, 3 and 4 propagators.

3.1 Explicit relations for products of two, three and four propagators

At $n = 2$ relation (3.2) reads:

$$\frac{1}{P_1 P_2} = \frac{x_1}{P_2 P_3} + \frac{x_2}{P_1 P_3}, \quad (3.4)$$

where

$$P_1 = (k_1 - p_1)^2 - m_1^2, \quad P_2 = (k_1 - p_2)^2 - m_2^2, \quad P_3 = (k_1 - p_3)^2 - m_3^2. \quad (3.5)$$

Multiplying both sides of eq. (3.4) by $P_1 P_2 P_3$ one gets:

$$P_3 = x_1 P_1 + x_2 P_2. \quad (3.6)$$

In what follows it will be assumed that k_1 is an integration momentum and three vectors p_1, p_2, p_3 do not depend on it. Differentiating eq. (3.6) with respect to $k_{1\mu}$ yields:

$$k_{1\mu} - p_{3\mu} = x_1(k_{1\mu} - p_{1\mu}) + x_2(k_{1\mu} - p_{2\mu}). \quad (3.7)$$

This equation is equivalent to a system of two equations:

$$\begin{aligned} 1 &= x_1 + x_2, \\ p_3 &= x_1 p_1 + x_2 p_2. \end{aligned} \quad (3.8)$$

The last equation in (3.8) means that the Gram determinant for the set of three vectors p_1, p_2, p_3 is equal to zero. Taking into account eqs. (3.8) another equation follows from eq. (3.6):

$$x_1 x_2 s_{12} - m_1^2 x_1 - m_2^2 x_2 + m_3^2 = 0. \quad (3.9)$$

Equation (3.9) and the first equation in (3.8) define the system of equations for x_1, x_2 with the solution

$$x_1 = \lambda_2, \quad x_2 = 1 - \lambda_2, \quad (3.10)$$

where λ_2 is the root of the equation

$$A_2 \lambda_2^2 + B_2 \lambda_2 + C_2 = 0, \quad (3.11)$$

with

$$A_2 = s_{12}, \quad B_2 = m_1^2 - m_2^2 - s_{12}, \quad C_2 = m_2^2 - m_3^2. \quad (3.12)$$

This solution can be rewritten in an explicit form:

$$\begin{aligned} x_1 &= \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \pm \frac{\sqrt{\Lambda_2 + 4s_{12}m_3^2}}{2s_{12}}, \\ x_2 &= \frac{m_1^2 - m_2^2 + s_{12}}{2s_{12}} \mp \frac{\sqrt{\Lambda_2 + 4s_{12}m_3^2}}{2s_{12}}, \end{aligned} \quad (3.13)$$

where

$$\Lambda_2 = s_{12}^2 + m_1^4 + m_2^4 - 2s_{12}m_1^2 - 2s_{12}m_2^2 - 2m_1^2m_2^2. \quad (3.14)$$

Now let us find an algebraic relation for the products of three propagators. At $n = 3$ eq. (3.2) reads:

$$\frac{1}{P_1 P_2 P_3} = \frac{x_1}{P_4 P_2 P_3} + \frac{x_2}{P_1 P_4 P_3} + \frac{x_3}{P_1 P_2 P_4}, \quad (3.15)$$

where P_1, P_2, P_3 are defined in eq. (3.5) and

$$P_4 = (k_1 - p_4)^2 - m_4^2. \quad (3.16)$$

In complete analogy with the previous case we multiply eq. (3.15) by the product of $P_1 P_2 P_3 P_4$ and obtain

$$P_4 = x_1 P_1 + x_2 P_2 + x_3 P_3. \quad (3.17)$$

Assuming that p_{j,x_r} do not depend on k_1 , and differentiating eq. (3.17) with respect to $k_{1\mu}$ one gets the equation linear in k_1 . This equation is equivalent to the system of two equations:

$$x_1 + x_2 + x_3 = 1, \quad (3.18)$$

$$p_4 = x_1 p_1 + x_2 p_2 + x_3 p_3. \quad (3.19)$$

Taking into account eqs. (3.18), (3.19), we obtain from eq. (3.17):

$$x_2 x_3 s_{23} + x_1 x_3 s_{13} + x_1 x_2 s_{12} - m_1^2 x_1 - m_2^2 x_2 - m_3^2 x_3 + m_4^2 = 0. \quad (3.20)$$

Solving eqs. (3.18), (3.20) for x_1, x_2 yields

$$x_1 = 1 - \lambda_3 - x_3, \quad x_2 = \lambda_3, \quad (3.21)$$

where λ_3 is a solution of the equation

$$A_3 \lambda_3^2 + B_3 \lambda_3 + C_3 = 0. \quad (3.22)$$

Here

$$\begin{aligned} A_3 &= s_{12}, \\ B_3 &= x_3(s_{13} + s_{12} - s_{23}) - m_1^2 + m_2^2 - s_{12}, \\ C_3 &= x_{43}^2 s_{13} + (m_3^2 - m_1^2 - s_{13})x_3 + m_1^2 - m_4^2. \end{aligned} \quad (3.23)$$

Let us now turn to the derivation of an algebraic relation for the product of four propagators. At $n = 4$ eq. (3.2) reads:

$$\frac{1}{P_1 P_2 P_3 P_4} = \frac{x_1}{P_5 P_2 P_3 P_4} + \frac{x_2}{P_1 P_5 P_3 P_4} + \frac{x_3}{P_1 P_2 P_5 P_4} + \frac{x_4}{P_1 P_2 P_3 P_5}, \quad (3.24)$$

where P_1, P_2, P_3, P_4 are defined in eqs. (3.5), (3.16),

$$P_5 = (k_1 - p_5)^2 - m_5^2. \quad (3.25)$$

Multiplying eq. (3.24) by the product $P_1 P_2 P_3 P_4 P_5$ we obtain:

$$P_5 = x_1 P_1 + x_2 P_2 + x_3 P_3 + x_4 P_4. \quad (3.26)$$

As it was in two previous cases, we assume that p_{j,x_r} do not depend on k_1 . Differentiating eq. (3.26) with respect to $k_{1\mu}$, one gets an equation linear in k_1 from which two equations follow

$$x_1 + x_2 + x_3 + x_4 = 1, \quad (3.27)$$

$$p_5 = x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4. \quad (3.28)$$

Simplification of eq. (3.26) using eqs. (3.27), (3.28) yields

$$x_1x_2s_{12} + x_1x_3s_{13} + x_2x_3s_{23} + x_1x_4s_{14} + x_2x_4s_{24} + x_3x_4s_{34} - m_1^2x_1 - m_2^2x_2 - m_3^2x_3 - m_4^2x_4 + m_5^2 = 0. \quad (3.29)$$

Solving the system of equations (3.27), (3.29) for x_1, x_4 we obtain

$$x_1 = \lambda_4, \quad x_4 = 1 - x_2 - x_3 - \lambda_4, \quad (3.30)$$

where λ_4 is a solution of the equation

$$A_4\lambda_4^2 + B_4\lambda_4 + C_4 = 0, \quad (3.31)$$

with

$$\begin{aligned} A_4 &= s_{14}, \\ B_4 &= (s_{24} - s_{12} + s_{14})x_2 + (s_{34} - s_{13} + s_{14})x_3 + m_1^2 - m_4^2 - s_{14}, \\ C_4 &= s_{24}x_2^2 + (s_{34} - s_{23} + s_{24})x_2x_3 + (m_2^2 - m_4^2 - s_{24})x_2 + s_{34}x_3^2 \\ &\quad + (m_3^2 - m_4^2 - s_{34})x_3 + m_4^2 - m_5^2. \end{aligned} \quad (3.32)$$

Eqs. (3.4), (3.15) and (3.24) will be used in the next sections to derive functional equations for the propagator, vertex and box type of integrals.

3.2 Algebraic relationship for products of any number of propagators

The relations between the products of five and more propagators can be easily derived in the same way as it was done for the products of two-, three- and four-propagators. From eq. (3.2) one can derive a system of equations and find its solution for arbitrary n . Multiplying both sides of eq. (3.2) by the product $\prod_{j=1}^{n+1} P_j$ yields

$$P_{n+1} = \sum_{r=1}^n x_r P_r, \quad (3.33)$$

or

$$k_1^2 - 2k_1p_{n+1} + p_{n+1}^2 = \sum_{r=1}^n x_r (k_1^2 - 2k_1p_r + p_r^2 - m_r^2). \quad (3.34)$$

Again we assume that k_1 will be our integration momentum and p_j, x_r do not depend on it. Differentiating eq. (3.34) with respect to $k_{1\mu}$ one gets a linear equation in k_1 from which two equations follow

$$1 = \sum_{r=1}^n x_r, \quad (3.35)$$

$$p_{n+1} = \sum_{j=1}^n x_j p_j. \quad (3.36)$$

Figure 2. n -propagator functional equation.

Substituting eqs. (3.35), (3.36) into eq. (3.34) yields the following equation:

$$m_{n+1}^2 - \sum_{k=1}^n x_k m_k^2 + \sum_{j=1}^n \sum_{l=1}^{j-1} x_j x_l s_{lj} = 0. \quad (3.37)$$

Solving eq. (3.35) for one of the x_j and substituting this solution into eq. (3.37) gives a quadratic equation for the remaining x_i . This quadratic equation can be solved with respect to one of the parameters x_j . Thus, the solution of the system of equations (3.35), (3.37) will depend on $n - 2$ remaining arbitrary parameters x_i and one arbitrary mass m_n .

It is interesting to note that for any n , the functional equations for integrals with all masses being equal to zero coincide with the functional equations for integrals with all equal masses and different from zero. In the case of equal masses, two mass dependent terms in eq. (3.37) cancel each other due to eq. (3.35). In both cases systems of equations for x_i are the same and therefore arguments of integrals are the same.

Eq. (3.4) is analogous to the equation for splitting propagators presented in ref. [10]. Eq. (3.15) is a generalization of eq. (3.4). Indeed, setting $x_3 = 0$, canceling common factor P_3 on both sides of eq. (3.15) yields a relation similar to (3.4). In turn, eq. (3.24) is a generalization of eq. (3.15).

3.3 Prototypes of functional equations

Multiplying algebraic relations (3.4), (3.15), (3.24) by products of any number of propagators raised to arbitrary powers ν_j

$$\prod_{j=n_0}^N \frac{1}{[(k_1 - p_j) - m_j^2]^{\nu_j}} \quad (3.38)$$

and integrating with respect to k_1 we get a functional equation for one-loop integrals. Eqs. (3.4), (3.15), (3.24) can also be used to derive functional equations for integrals with any number of loops. Multiplying algebraic relations for propagators by the function corresponding to Feynman integral depending on momentum k_1 and any number of external momenta and then integrating with respect to k_1 will produce functional equations. Just for demonstrational purposes in figure 2 we present graphically a functional equation based on n propagator relation. The blob on this picture corresponds to either a product of the propagators raised to arbitrary powers or an integral with any number of loops and external legs. Also, the blob can be a product of the propagators multiplied by an integral. One of the external momenta of this multiloop integral should be k_1 .

4 Some examples of functional equations

In this section several particular examples of the functional equations resulting from algebraic relations for the products of propagators will be considered.

4.1 Functional equation for the one-loop propagator type integral

First, we consider the simplest case, namely, a functional equation for the integral $I_2^{(d)}$:

$$I_2^{(d)}(m_j^2, m_k^2; s_{jk}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{[(k_1 - p_j)^2 - m_j^2][(k_1 - p_k)^2 - m_k^2]}. \quad (4.1)$$

Integrating both sides of eq. (3.4) with respect to k_1 , we get:

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = x_1 I_2^{(d)}(m_2^2, m_3^2, s_{23}) + x_2 I_2^{(d)}(m_1^2, m_3^2, s_{13}). \quad (4.2)$$

The arguments s_{13} , s_{23} of the integrals on the right-hand side depend on x_1 , x_2

$$\begin{aligned} s_{13} &= (p_1 - p_3)^2 = (x_1 - 1)^2 p_1^2 + 2x_2(x_1 - 1)p_1 p_2 + x_2^2 p_2^2, \\ s_{23} &= (p_2 - p_3)^2 = p_1^2 x_1^2 + 2(x_2 - 1)x_1 p_1 p_2 + (x_2 - 1)^2 p_2^2. \end{aligned} \quad (4.3)$$

Substituting solution for x_j from eq. (3.10) into eq. (4.3) yields:

$$\begin{aligned} s_{13} &= \frac{\Lambda_2 + 2s_{12}(m_1^2 + m_3^2)}{2s_{12}} \pm \frac{m_2^2 - m_1^2 - s_{12}}{2s_{12}} \sqrt{\Lambda_2 + 4s_{12}m_3^2}, \\ s_{23} &= \frac{\Lambda_2 + 2s_{12}(m_2^2 + m_3^2)}{2s_{12}} \pm \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \sqrt{\Lambda_2 + 4s_{12}m_3^2}. \end{aligned} \quad (4.4)$$

In these equations m_3^2 is an arbitrary parameter and it can be taken at will. The functional equation (4.2) is in agreement with the result presented in refs. [1, 2].

4.2 Functional equations for the one-loop vertex type integral

Functional equations for the vertex type integral

$$\begin{aligned} &I_3^{(d)}(m_1^2, m_2^2, m_3^2, s_{23}, s_{13}, s_{12}) \\ &= \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{[(k_1 - p_1)^2 - m_1^2][(k_1 - p_2)^2 - m_2^2][(k_1 - p_3)^2 - m_3^2]}, \end{aligned} \quad (4.5)$$

can be obtained from eq. (3.4) as well as from eq. (3.15). Multiplying both sides of eq. (3.4) by the factor $1/P_4$ where

$$P_4 = (k_1 - p_4)^2 - m_4^2, \quad (4.6)$$

and integrating over k_1 leads to the equation

$$\int \frac{d^d k_1}{P_1 P_2 P_4} = x_1 \int \frac{d^d k_1}{P_2 P_3 P_4} + x_2 \int \frac{d^d k_1}{P_1 P_3 P_4}. \quad (4.7)$$

In terms of integrals $I_3^{(d)}$ this equation reads

$$\begin{aligned} &I_3^{(d)}(m_1^2, m_2^2, m_4^2; s_{24}, s_{14}, s_{12}) \\ &= x_1 I_3^{(d)}(m_2^2, m_3^2, m_4^2; s_{34}, s_{24}, s_{23}) + x_2 I_3^{(d)}(m_1^2, m_3^2, m_4^2; s_{34}, s_{14}, s_{13}), \end{aligned} \quad (4.8)$$

where s_{12} , s_{14} , s_{24} are independent scalar invariants, s_{13}, s_{23} are given in eq. (4.4) and

$$s_{34} = \frac{1}{2}(s_{14} + s_{24} - m_1^2 - m_2^2 + 2m_3^2) - \frac{(m_1^2 - m_2^2 - s_{14} + s_{24})(m_2^2 - m_1^2 \pm \sqrt{\Lambda_2 + 4m_3^2})}{2s_{12}}. \quad (4.9)$$

Two more functional equations can be obtained from eq. (4.7) by symmetric permutations $(m_1, p_1) \leftrightarrow (m_4, p_4)$ and $(m_2, p_2) \leftrightarrow (m_4, p_4)$.

Another functional equation for the vertex type integral can be obtained by integrating both sides of eq. (3.15) with respect to k_1 :

$$I_3^{(d)}(m_1^2, m_2^2, m_3^2, s_{23}, s_{13}, s_{12}) = (1 - \lambda_3 - x_3)I_3^{(d)}(m_4^2, m_2^2, m_3^2, s_{23}, s_{34}, s_{24}) + \lambda_3 I_3^{(d)}(m_1^2, m_4^2, m_3^2, s_{34}, s_{13}, s_{14}) + x_3 I_3^{(d)}(m_1^2, m_2^2, m_4^2, s_{24}, s_{14}, s_{12}), \quad (4.10)$$

where s_{12} , s_{13} , s_{23} are arbitrary scalar invariants and

$$\begin{aligned} s_{14} &= (m_1^2 - m_2^2 + s_{12})\lambda_3 + (m_1^2 - m_3^2 + s_{13})x_3 + m_4^2 - m_1^2, \\ s_{24} &= (m_1^2 - s_{12} - m_2^2)\lambda_3 + (m_1^2 - m_3^2 - s_{12} + s_{23})x_3 + m_4^2 - m_1^2 + s_{12}, \\ s_{34} &= (m_1^2 - s_{13} + s_{23} - m_2^2)\lambda_3 + (m_1^2 - m_3^2 - s_{13})x_3 + m_4^2 - m_1^2 + s_{13}. \end{aligned} \quad (4.11)$$

There is an essential difference between the functional equation eq. (4.8) obtained from eq. (3.4) and the functional equation (4.10) derived from eq. (3.15). For example, at $m_1^2 = m_2^2 = m_3^2 = m_4^2 = 0$, eq. (4.8) becomes trivial while from eq. (4.10) for the integral

$$I_3^{(d)}(s_{23}, s_{13}, s_{12}) \equiv I_3^{(d)}(0, 0, 0, s_{23}, s_{13}, s_{12}) \quad (4.12)$$

we obtain a nontrivial functional equation

$$\begin{aligned} I_3^{(d)}(s_{23}, s_{13}, s_{12}) &= (1 - \lambda_3 - x_3)I_3^{(d)}(s_{12}(1 - \lambda_3 - x_3) + s_{23}x_3, s_{23}, \lambda_3(s_{23} - s_{13}) + s_{13}(1 - x_3)) \\ &\quad + \lambda_3 I_3^{(d)}(s_{13}x_3 + \lambda_3 s_{12}, \lambda_3(s_{23} - s_{13}) + s_{13}(1 - x_3), s_{13}) \\ &\quad + x_3 I_3^{(d)}(s_{12}, s_{12}(1 - \lambda_3 - x_3) + s_{23}x_3, s_{13}x_3 + \lambda_3 s_{12}), \end{aligned} \quad (4.13)$$

where λ_3 is a root of the quadratic equation

$$s_{12}\lambda_3^2 - (s_{12} - s_{12}x_3 - s_{13}x_3 + s_{23}x_3)\lambda_3 + x_3^2 s_{13} - x_3 s_{13} = 0. \quad (4.14)$$

If one argument of $I_3^{(d)}(s_{23}, s_{13}, s_{12})$ is zero, then by applying functional equation (4.13) such an integral can be expressed in terms of integrals $I_3^{(d)}$ with two arguments equal to zero. For example, at $s_{23} = 0$ and $x_3 = s_{12}/(s_{12} - s_{13})$ the relation (4.13) becomes

$$I_3^{(d)}(0, s_{13}, s_{12}) = \frac{s_{12}}{s_{12} - s_{13}} I_3^{(d)}(s_{12}, 0, 0) - \frac{s_{13}}{s_{12} - s_{13}} I_3^{(d)}(0, 0, s_{13}). \quad (4.15)$$

This is a typical example how functional equations can be used to simplify evaluation of an integral by reducing it to a combination of integrals with fewer arguments.

At $m_1^2 = m_2^2 = m_3^2 = m_4^2 = m^2$, similar to the previous case, eq. (4.8) degenerate while from eq. (4.10) for the integral

$$I_3^{(d)}(m^2, m^2, m^2, s_{23}, s_{13}, s_{12}) \equiv I_3^{(d)}(m^2; s_{23}, s_{13}, s_{12}) \quad (4.16)$$

we obtain a nontrivial functional equation

$$\begin{aligned} & I_3^{(d)}(m^2; s_{23}, s_{13}, s_{12}) \\ &= (1 - \lambda_3 - x_3) I_3^{(d)}(m^2; s_{12}(1 - \lambda_3 - x_3) + s_{23}x_3, s_{23}, \lambda_3(s_{23} - s_{13}) + s_{13}(1 - x_3)) \\ &+ \lambda_3 I_3^{(d)}(m^2; s_{13}x_3 + \lambda_3 s_{12}, \lambda_3(s_{23} - s_{13}) + s_{13}(1 - x_3), s_{13}) \\ &+ x_3 I_3^{(d)}(m^2; s_{12}, s_{12}(1 - \lambda_3 - x_3) + s_{23}x_3, s_{13}x_3 + \lambda_3 s_{12}), \end{aligned} \quad (4.17)$$

where λ_3 is a root of the quadratic equation

$$s_{12}\lambda_3^2 - (s_{12} - s_{12}x_3 - s_{13}x_3 + s_{23}x_3)\lambda_3 + x_3^2 s_{13} - x_3 s_{13} = 0. \quad (4.18)$$

Eqs. (4.17), (4.18) are identical to eqs. (4.13), (4.14) respectively and therefore the functional equation for the integral with massless propagators and the functional equation for the integral with all masses equal are the same. Setting $s_{23} = 0$ and $x_3 = s_{12}/(s_{12} - s_{13})$ in eq. (4.17) leads to the relation similar to (4.15):

$$I_3^{(d)}(m^2; 0, s_{13}, s_{12}) = \frac{s_{12}}{s_{12} - s_{13}} I_3^{(d)}(m^2; s_{12}, 0, 0) - \frac{s_{13}}{s_{12} - s_{13}} I_3^{(d)}(m^2; 0, 0, s_{13}). \quad (4.19)$$

This is not surprising because the coefficients of the eq. (4.17) are mass independent and in the integrand m^2 and $i\eta$ appear in the covariant combination $m^2 - i\eta$. For this reason, the similarity of functional equations for massless integrals and integrals with all masses equal take place for integrals with more external legs and more loops.

4.3 Functional equations for one-loop box type integrals

Functional equations for the box type integrals can be obtained by multiplying relation (3.4) by two propagators, or by multiplying relation (3.15) by one propagator and then integrating over momentum k_1 . Yet another relation can be obtained just by integrating eq. (3.24) over momentum k_1 :

$$\begin{aligned} & I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2; s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) \\ &= \lambda_4 I_4^{(d)}(m_5^2, m_2^2, m_3^2, m_4^2; s_{25}, s_{23}, s_{34}, s_{45}, s_{24}, s_{35}) \\ &+ x_2 I_4^{(d)}(m_1^2, m_5^2, m_3^2, m_4^2; s_{15}, s_{35}, s_{34}, s_{14}, s_{45}, s_{13}) \\ &+ x_3 I_4^{(d)}(m_1^2, m_2^2, m_5^2, m_4^2; s_{12}, s_{25}, s_{45}, s_{14}, s_{24}, s_{15}) \\ &+ (1 - x_2 - x_3 - \lambda_4) I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_5^2; s_{12}, s_{23}, s_{35}, s_{15}, s_{25}, s_{13}). \end{aligned} \quad (4.20)$$

Figure 3. Diagram corresponding to $R(m_1^2, m_2^2, m_3^2, m_4^2; q_1^2, q_2^2, q_3^2)$.

Here λ_4 is defined in eq. (3.31) and m_5, x_2, x_3 are arbitrary parameters and

$$\begin{aligned}
 s_{15} &= \lambda_4(m_4^2 - s_{14} - m_1^2) + x_3(m_4^2 - m_3^2 - s_{14} + s_{13}) + x_2(m_4^2 - m_2^2 - s_{14} + s_{12}) \\
 &\quad + s_{14} + m_5^2 - m_4^2, \\
 s_{25} &= \lambda_4(m_4^2 - m_1^2 - s_{24} + s_{12}) + x_3(m_4^2 - m_3^2 - s_{24} + s_{23}) + x_2(m_4^2 - m_2^2 - s_{24}) \\
 &\quad + s_{24} + m_5^2 - m_4^2, \\
 s_{35} &= \lambda_4(m_4^2 - m_1^2 - s_{34} + s_{13}) + x_3(m_4^2 - m_3^2 - s_{34}) + x_2(m_4^2 - m_2^2 - s_{34} + s_{23}) \\
 &\quad + s_{34} + m_5^2 - m_4^2, \\
 s_{45} &= \lambda_4(s_{14} + m_4^2 - m_1^2) + x_3(s_{34} + m_4^2 - m_3^2) + x_2(s_{24} + m_4^2 - m_2^2) \\
 &\quad + m_5^2 - m_4^2.
 \end{aligned} \tag{4.21}$$

In this functional equation the arbitrary parameters can be chosen from the requirement of simplicity of evaluation of integrals on the right-hand side of eq. (4.20) or from some other requirements. For example, one can choose these parameters by transforming arguments to a certain kinematical region needed for the analytic continuation of the original integral.

4.4 Functional equation for two-loop vertex type integral

The method described in the previous section can be applied to multiloop integrals. Consider, for example, an integral corresponding to the diagram given in figure 3. If we multiply eq. (3.4) by the one-loop integral depending on k_1

$$\int \frac{d^d k_2}{[k_2^2 - m_4^2][(k_1 - k_2)^2 - m_5^2]} \tag{4.22}$$

and integrate with respect to momentum k_1 then we obtain the functional equation

$$\begin{aligned}
 &R(m_1^2, m_2^2, m_3^2, m_4^2; q_1^2, q_2^2, q_3^2) \\
 &= \alpha R(0, m_2^2, m_3^2, m_4^2; Q^2, q_2^2, (m_2^2 - m_1^2 + q_3^2)\alpha - m_2^2) \\
 &\quad + (1 - \alpha) R(m_1^2, 0, m_3^2, m_4^2; q_1^2, Q^2, (m_2^2 - m_1^2 - q_3^2)\alpha + q_3^2 - m_2^2),
 \end{aligned} \tag{4.23}$$

where

$$R(m_1^2, m_2^2, m_3^2, m_4^2; q_1^2, q_2^2, q_3^2) = \int \int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{[(k_2 + q_1)^2 - m_1^2][(k_2 - q_2)^2 - m_2^2][k_1^2 - m_3^2][(k_1 - k_2)^2 - m_4^2]}, \quad (4.24)$$

$$Q^2 = (q_1^2 - q_2^2 - m_1^2 + m_2^2)\alpha + q_2^2 - m_2^2, \quad (4.25)$$

$$\alpha = \frac{q_3^2 - m_1^2 + m_2^2 \pm \sqrt{\Delta}}{2q_3^2}, \quad (4.26)$$

$$\Delta = q_3^4 + m_1^4 + m_2^4 - 2q_3^2 m_1^2 - 2q_3^2 m_2^2 - 2m_1^2 m_2^2. \quad (4.27)$$

Integrals of this type arise, for example, in calculations of two-loop radiative corrections in the electroweak theory. Instead of the integral $R(m_1^2, m_2^2, m_3^2, m_4^2; q_1^2, q_2^2, q_3^2)$, one can consider a derivative of R with respect to m_3^2 which is UV finite:

$$\begin{aligned} \frac{\partial R}{\partial m_3^2} &\equiv R_3(m_1^2, m_2^2, m_3^2, m_4^2; q_1^2, q_2^2, q_3^2) \\ &= \int \int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{[(k_2 + q_1)^2 - m_1^2][(k_2 - q_2)^2 - m_2^2][k_1^2 - m_3^2]^2[(k_1 - k_2)^2 - m_4^2]}. \end{aligned} \quad (4.28)$$

Integral R_3 satisfies the following functional equations:

$$\begin{aligned} R_3(m_1^2, m_2^2, m_3^2, m_4^2; q_1^2, q_2^2, q_3^2) \\ = \alpha R_3(0, m_2^2, m_3^2, m_4^2; Q^2, q_2^2, (m_2^2 - m_1^2 + q_3^2)\alpha - m_2^2) \\ + (1 - \alpha) R_3(m_1^2, 0, m_3^2, m_4^2; q_1^2, Q^2, (m_2^2 - m_1^2 - q_3^2)\alpha + q_3^2 - m_2^2). \end{aligned} \quad (4.29)$$

This relation can be used for computing the basis integral arising in calculation of the two-loop radiative correction to the ortho -positronium lifetime. In particular, one of these basis integrals corresponds to kinematics $m_1^2 = m_2^2 = m_3^2 = m_4^2 = m^2$, $q_1^2 = q_2^2 = m^2$, $q_3^2 = 4m^2$. In this case relation (4.29) reads

$$R_3(m^2, m^2, m^2, m^2; m^2, m^2, 4m^2) = R_3(0, m^2, m^2, m^2; 0, m^2, m^2). \quad (4.30)$$

In fact the integral on the right-hand side is a propagator type integral with one massless line. Applying the recurrence relations given in ref. [11], this integral can be reduced to a simpler integral:

$$\begin{aligned} R_3(0, m^2, m^2, m^2; 0, m^2, m^2) \\ = \frac{1}{(i\pi^{d/2})^2} \int \int \frac{d^d k_1 d^d k_2}{k_1^2(k_2^2 - m^2)[(k_1 - k_2)^2 - m^2]^2[(k_1 + q_1)^2 - m^2]} \\ = \frac{2}{3(d-3)} J_{111}^{(d-2)}(m^2), \end{aligned} \quad (4.31)$$

where

$$J_{111}^{(d)}(q^2) = \frac{1}{(i\pi^{d/2})^2} \int \int \frac{d^d k_1 d^d k_2}{(k_1^2 - m^2)[(k_1 - k_2)^2 - m^2][(k_2 - q)^2 - m^2]}. \quad (4.32)$$

Figure 4. Two-loop vacuum type diagram.

At $q^2 = m^2$, the result for $J_{111}^{(d)}(q^2)$ is known [12]:

$$\begin{aligned} & \frac{m^{6-2d}}{8} (d-4)^2 (d-2) (d-3) J_{111}^{(d)}(m^2) \\ &= {}_3F_2 \left[\begin{matrix} 1, 3-d, \frac{1}{2} \\ \frac{5-d}{2}, \frac{d}{2} \end{matrix}; 1 \right] + (d-3) {}_3F_2 \left[\begin{matrix} 1, \frac{4-d}{2}, \frac{d-1}{2} \\ \frac{3}{2}, d-1 \end{matrix}; 1 \right], \end{aligned} \quad (4.33)$$

and it can be used for the ε expansion of R and R_3 . As was already mentioned, at $q_1^2 = q_2^2 = m^2$, $q_3^2 = 4m^2$ integrals on the right-hand side of eq. (4.23) correspond to the propagator type integrals. Analytic result for R reads

$$\begin{aligned} & (d-3)R(m^2, m^2, m^2, m^2, m^2, m^2, 4m^2) \\ &= \frac{1}{8m^4(d-4)} \left[2m^2(3d-8)(d-3)J_{111}^{(d)} - 3(d-2)^2 \left(T_1^{(d)} \right)^2 \right], \end{aligned} \quad (4.34)$$

where

$$T_1^{(d)} = -\Gamma \left(1 - \frac{d}{2} \right) (m^2)^{(d-2)/2}. \quad (4.35)$$

We checked that the first few terms in the $\varepsilon = (4-d)/2$ expansion of R and R_3 are in agreement with results of [13]. The main profit from the functional equations for R and R_3 comes from the fact that vertex integrals were expressed in terms of simpler, propagator type integrals.

5 Deriving functional equation by deforming propagators

The method described in the previous section does not work for deriving functional equations for all kinds of Feynman integrals. For example, we did not find a functional equation for the two-loop vacuum type integral given in figure 4. In this section we shall describe another method that extends the class of integrals for which we can obtain functional equations. The method is based on transformation of functional equations for some auxiliary integrals depending on arbitrary parameters into functional equations for integrals of interest. Such functional equations will be derived from algebraic relations for ‘deformed propagators’ which will be defined in the next section. These auxiliary integrals will be transformed into α parametric representation. In general characteristic polynomials of these integrals in α parametric representation differ from those for the investigated integral. The functional equation for the integral of interest can be obtained in case when it is possible to map characteristic polynomials of auxiliary integrals with ‘deformed propagators’ to characteristic polynomials of this integral. Such a mapping will be performed by rescaling α parameters and appropriate choice of arbitrary ‘deforming parameters’.

5.1 Algebraic relations for products of deformed propagators

In the previous section, in order to derive the functional equation, we added to our consideration a propagator with combination of external momenta taken with arbitrary scalar coefficients. Now we consider a generalization of this method.

To find functional equation for L -loop Feynman integral depending on E -external momenta, we start from the relation of the form

$$\prod_{r=1}^n \frac{1}{D_r} = \frac{1}{D_{n+1}} \sum_{r=1}^n x_r \prod_{\substack{j=1 \\ j \neq r}}^n \frac{1}{D_j}, \quad (5.1)$$

where D_j is defined as

$$D_j = Q_j^2 - m_j^2 + i\eta, \quad (5.2)$$

with

$$Q_j = \sum_{l=1}^L a_{jl}^{(j)} k_l + \sum_{l=1}^{E+1} b_{jl}^{(j)} p_l, \quad (5.3)$$

and $a_{jl}^{(j)}, b_{jl}^{(j)}$ for the time being are arbitrary scalar parameters. Some of these parameters as well as x_r will be fixed from eq. (5.1). Another part of these parameters will be fixed from the requirement that the product of propagators in eq. (5.1) should correspond to the integrand of the integral with the considered topology. We would like to remark that instead of the deformation of the propagators proposed in eqs. (5.2), (5.3), one can use other deformations. For example, all terms in denominators of propagators can be taken with arbitrary scalar coefficients:

$$Q_j^2 = \sum_{l,r=1}^L a_{rl}^{(j)} (k_r \cdot k_l) + \sum_{l=1}^L \sum_{r=1}^{E+1} b_{rl}^{(j)} (p_r \cdot k_l) + \sum_{r,l=1}^{E+1} c_{rl}^{(j)} (p_r \cdot p_l). \quad (5.4)$$

To establish algebraic relation (5.1), we put all terms over a common denominator and then equate coefficients in front of scalar products depending on integration momenta. Solving the obtained system of equations gives some restrictions on the scalar parameters.

In general, the integrals obtained by integrating products of ‘deformed propagators’ will not correspond to usual Feynman integrals. Further restrictions on parameters should be imposed in order to obtain relations between the integrals corresponding to Feynman integrals coming from a realistic quantum field theory models.

5.2 Functional equation for two-loop vacuum type integral with arbitrary masses

As an example, let us consider a derivation of the functional equation for the two-loop vacuum type integral given in figure 4:

$$J_0^{(d)}(m_1^2, m_2^2, m_3^2) = \iint \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{(k_1^2 - m_1^2)((k_1 - k_2)^2 - m_2^2)(k_2^2 - m_3^2)}. \quad (5.5)$$

An analytic expression for this integral was presented in ref. [14]. Instead of this integral we will first consider an auxiliary integral with the integrand made from the ‘deformed propagators’ defined in eqs. (5.2), (5.3):

$$\tilde{J}_0^{(d)}(m_1^2, m_2^2, m_3^2) = \iint \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{D_1 D_2 D_3}, \quad (5.6)$$

where

$$\begin{aligned} D_1 &= (a_1 k_1 + a_2 k_2)^2 - m_1^2 + i\eta, & D_2 &= (b_1 k_1 + b_2 k_2)^2 - m_2^2 + i\eta, \\ D_3 &= (h_1 k_1 + h_2 k_2)^2 - m_3^2 + i\eta. \end{aligned} \quad (5.7)$$

For the product of three deformed propagators one can try to find an algebraic relation of the form

$$\frac{1}{D_1 D_2 D_3} = \frac{x_1}{D_4 D_2 D_3} + \frac{x_2}{D_1 D_4 D_3} + \frac{x_3}{D_1 D_2 D_4}, \quad (5.8)$$

where D_1, D_2, D_3 are defined in eq. (5.7) and

$$D_4 = (r_1 k_1 + r_2 k_2)^2 - m_4^2 + i\eta. \quad (5.9)$$

Here m_k are arbitrary masses, x_k, a_j, b_i, h_s, r_l are undetermined parameters and k_1, k_2 will be integration momenta.

Putting in eq. (5.8) all over a common denominator and equating to zero coefficients in front of different products of k_1^2, k_2^2 and $k_1 k_2$ leads to the system of equations:

$$\begin{aligned} r_1^2 - x_1 a_1^2 - x_2 b_1^2 - x_3 h_1^2 &= 0, & r_1 r_2 - x_1 a_1 a_2 - x_2 b_1 b_2 - x_3 h_1 h_2 &= 0, \\ r_2^2 - x_1 a_2^2 - x_2 b_2^2 - x_3 h_2^2 &= 0, & m_4^2 - x_1 m_1^2 - x_2 m_2^2 - x_3 m_3^2 &= 0. \end{aligned} \quad (5.10)$$

Solving this system for r_1, x_1, x_2, x_3 we have:

$$\begin{aligned} r_1 &= r_2 \lambda, \\ Ax_1 &= r_2^2 (h_1 h_2 m_2^2 - b_2 b_1 m_3^2) + b_2 h_2 (b_1 h_2 - b_2 h_1) m_4^2 - r_2^2 (h_2^2 m_2^2 - m_3^2 b_2^2) \lambda, \\ Ax_2 &= -r_2^2 (m_1^2 h_2 h_1 - a_2 a_1 m_3^2) - a_2 h_2 (a_1 h_2 - a_2 h_1) m_4^2 + r_2^2 (h_2^2 m_1^2 - m_3^2 a_2^2) \lambda, \\ Ax_3 &= r_2^2 (m_1^2 b_2 b_1 - a_2 a_1 m_2^2) + a_2 b_2 (a_1 b_2 - a_2 b_1) m_4^2 - r_2^2 (b_2^2 m_1^2 - m_2^2 a_2^2) \lambda, \end{aligned} \quad (5.11)$$

where λ is a root of the quadratic equation

$$A\lambda^2 + B\lambda + C = 0, \quad (5.12)$$

and

$$\begin{aligned} A &= b_2 h_2 (b_1 h_2 - h_1 b_2) m_1^2 + a_2 h_2 (h_1 a_2 - a_1 h_2) m_2^2 + a_2 b_2 (a_1 b_2 - b_1 a_2) m_3^2, \\ B &= (h_1 b_2 - b_1 h_2) (b_1 h_2 + h_1 b_2) m_1^2 + (a_1 h_2 - h_1 a_2) (a_1 h_2 + h_1 a_2) m_2^2 \\ &\quad + (b_1 a_2 - a_1 b_2) (a_1 b_2 + b_1 a_2) m_3^2, \\ C &= b_1 h_1 (b_1 h_2 - h_1 b_2) m_1^2 + a_1 h_1 (a_2 h_1 - a_1 h_2) m_2^2 + a_1 b_1 (a_1 b_2 - b_1 a_2) m_3^2 \\ &\quad + (b_2 h_1 - b_1 h_2) (a_1 h_2 - a_2 h_1) (a_1 b_2 - a_2 b_1) m_4^2. \end{aligned} \quad (5.13)$$

In order to obtain functional equation for the integral $J_0^{(d)}$, we integrate first both sides of the eq. (5.8) with respect to k_1, k_2 and then convert these integrals into the α -parametric representation. Transforming all propagators into a parametric form

$$\frac{1}{(k^2 - m^2 + i\eta)^\nu} = \frac{i^{-\nu}}{\Gamma(\nu)} \int_0^\infty d\alpha \alpha^{\nu-1} \exp[i\alpha(k^2 - m^2 + i\eta)], \quad (5.14)$$

and using the d -dimensional Gaussian integration formula

$$\int d^d k \exp[i(a k^2 + 2(pk))] = i \left(\frac{\pi}{ia} \right)^{\frac{d}{2}} \exp\left[-\frac{ip^2}{a}\right], \quad (5.15)$$

we can easily evaluate the integrals over loop momenta. The final result is:

$$\tilde{J}_0^{(d)}(m_1^2, m_2^2, m_3^2) = i^{1-d} \int_0^\infty \int_0^\infty \int_0^\infty \frac{d\alpha_1 d\alpha_2 d\alpha_3}{[\tilde{D}(\alpha)]^{\frac{d}{2}}} \exp\left[-i \sum_{l=1}^3 \alpha_l (m_l^2 - i\eta)\right], \quad (5.16)$$

where the polynomial

$$\tilde{D}(\alpha) = (a_1 b_2 - a_2 b_1)^2 \alpha_1 \alpha_2 + (a_1 h_2 - a_2 h_1)^2 \alpha_1 \alpha_3 + (b_1 h_2 - b_2 h_1)^2 \alpha_2 \alpha_3, \quad (5.17)$$

differs from the appropriate D polynomial of the two-loop vacuum integral defined in eq. (5.5)

$$D(\alpha) = \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3. \quad (5.18)$$

Changing in eq. (5.16) integration variables

$$\alpha_j \rightarrow \alpha_j \theta_j^2 \quad (5.19)$$

with

$$\theta_1 = b_1 h_2 - b_2 h_1, \quad \theta_2 = a_1 h_2 - a_2 h_1, \quad \theta_3 = a_1 b_2 - a_2 b_1, \quad (5.20)$$

leads to the relation:

$$\tilde{D} = (b_1 h_2 - b_2 h_1)^2 (a_1 h_2 - a_2 h_1)^2 (a_1 b_2 - a_2 b_1)^2 (\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3). \quad (5.21)$$

Therefore integral $\tilde{J}_0^{(d)}$ with deformed propagators is proportional to $J_0^{(d)}$ with modified arguments:

$$\begin{aligned} \int \int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{D_1 D_2 D_3} &= i^{1-d} [\theta_1^2 \theta_2^2 \theta_3^2]^{\frac{2-d}{2}} \int_0^\infty \int_0^\infty \int_0^\infty \frac{d\alpha_1 d\alpha_2 d\alpha_3}{D^{\frac{d}{2}}} \exp[-i\mathcal{M}] \\ &= [\theta_1^2 \theta_2^2 \theta_3^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_1^2 m_1^2, \theta_2^2 m_2^2, \theta_3^2 m_3^2) \end{aligned} \quad (5.22)$$

where

$$\mathcal{M} = \alpha_1 \theta_1^2 m_1^2 + \alpha_2 \theta_2^2 m_2^2 + \alpha_3 \theta_3^2 m_3^2. \quad (5.23)$$

With the aid of eq. (5.22), the relation (5.8) integrated with respect to k_1, k_2 can be written as a combination of integrals $J_0^{(d)}$ with different arguments:

$$\begin{aligned}
 & [\theta_1^2 \theta_2^2 \theta_3^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_1^2 m_1^2, \theta_2^2 m_2^2, \theta_3^2 m_3^2) \\
 &= x_1 [\theta_1^2 \theta_6^2 \theta_4^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_1^2 m_4^2, \theta_6^2 m_2^2, \theta_4^2 m_3^2) \\
 &+ x_2 [\theta_2^2 \theta_5^2 \theta_6^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_6^2 m_1^2, \theta_2^2 m_4^2, \theta_5^2 m_3^2) \\
 &+ x_3 [\theta_3^2 \theta_4^2 \theta_5^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_4^2 m_1^2, \theta_5^2 m_2^2, \theta_3^2 m_4^2),
 \end{aligned} \tag{5.24}$$

where $\theta_1, \theta_2, \theta_3$ are defined in (5.20) and

$$\theta_4 = r_1 b_2 - r_2 b_1, \quad \theta_5 = r_1 a_2 - r_2 a_1, \quad \theta_6 = r_1 h_2 - r_2 h_1. \tag{5.25}$$

Now we consider relation (5.24) as an equation for integrals in momentum representation. By rescaling integration variables k_1, k_2 in the integral on the left-hand side

$$k_1 = (\theta_1 \theta_2 \theta_3)^{\frac{1}{2}} \tilde{k}_1, \quad k_2 = (\theta_1 \theta_2 \theta_3)^{\frac{1}{2}} \tilde{k}_2, \tag{5.26}$$

and performing analogous changes for the integrals on the right-hand side, we obtain the relation

$$\begin{aligned}
 & \frac{1}{\theta_1 \theta_2 \theta_3} J_0^{(d)} \left(\frac{\theta_1}{\theta_2 \theta_3} m_1^2, \frac{\theta_2}{\theta_1 \theta_3} m_2^2, \frac{\theta_3}{\theta_1 \theta_2} m_3^2 \right) \\
 &= \frac{x_1}{\theta_1 \theta_4 \theta_6} J_0^{(d)} \left(\frac{\theta_1}{\theta_4 \theta_6} m_4^2, \frac{\theta_6}{\theta_1 \theta_4} m_2^2, \frac{\theta_4}{\theta_1 \theta_6} m_3^2 \right) \\
 &+ \frac{x_2}{\theta_2 \theta_5 \theta_6} J_0^{(d)} \left(\frac{\theta_6}{\theta_2 \theta_5} m_1^2, \frac{\theta_2}{\theta_6 \theta_5} m_4^2, \frac{\theta_5}{\theta_2 \theta_6} m_3^2 \right) \\
 &+ \frac{x_3}{\theta_3 \theta_4 \theta_5} J_0^{(d)} \left(\frac{\theta_4}{\theta_3 \theta_5} m_1^2, \frac{\theta_5}{\theta_3 \theta_4} m_2^2, \frac{\theta_3}{\theta_5 \theta_4} m_4^2 \right).
 \end{aligned} \tag{5.27}$$

In terms of redefined masses M_1, M_2, M_3 related to original masses m_1, m_2, m_3 as

$$m_1^2 = \frac{\theta_2 \theta_3}{\theta_1} M_1^2, \quad m_2^2 = \frac{\theta_1 \theta_3}{\theta_2} M_2^2, \quad m_3^2 = \frac{\theta_1 \theta_2}{\theta_3} M_3^2, \tag{5.28}$$

equation (5.27) reads

$$\begin{aligned}
 J_0^{(d)}(M_1^2, M_2^2, M_3^2) &= \frac{\theta_2 \theta_3}{\theta_4 \theta_6} \tilde{x}_1 J_0^{(d)} \left(\frac{\theta_1}{\theta_4 \theta_6} m_4^2, \frac{\theta_3 \tilde{\theta}_6}{\theta_2 \theta_4} M_2^2, \frac{\theta_2 \tilde{\theta}_4}{\theta_3 \theta_6} M_3^2 \right) \\
 &+ \frac{\theta_1 \theta_3}{\theta_5 \theta_6} \tilde{x}_2 J_0^{(d)} \left(\frac{\theta_3 \tilde{\theta}_6}{\theta_1 \theta_5} M_1^2, \frac{\theta_2}{\theta_5 \theta_6} m_4^2, \frac{\theta_1 \tilde{\theta}_5}{\theta_3 \theta_6} M_3^2 \right) \\
 &+ \frac{\theta_1 \theta_2}{\theta_4 \theta_5} \tilde{x}_3 J_0^{(d)} \left(\frac{\theta_2 \tilde{\theta}_4}{\theta_1 \theta_5} M_1^2, \frac{\theta_1 \tilde{\theta}_5}{\theta_2 \theta_4} M_2^2, \frac{\theta_3}{\theta_4 \theta_5} m_4^2 \right),
 \end{aligned} \tag{5.29}$$

where $\tilde{x}_i, \tilde{\theta}_j$ are defined from eqs. (5.11), (5.13) with redefined masses. After simplifications eq. (5.29) takes a simpler form:

$$\begin{aligned}
 J_0^{(d)}(M_1^2, M_2^2, M_3^2) &= J_0^{(d)}(\rho_1 + \rho_2 + M_1^2 - M_2^2 - M_3^2, \rho_1, \rho_2) \\
 &\quad + J_0^{(d)}(\rho_3, M_3^2 - \rho_2, \rho_3 - \rho_2 - M_1^2 + M_2^2) \\
 &\quad - J_0^{(d)}(\rho_3 - M_1^2, \rho_1 - M_2^2, \rho_1 + \rho_3 - M_3^2),
 \end{aligned} \tag{5.30}$$

where

$$\begin{aligned}
 \rho_1 &= \frac{(M_3^2 + M_2^2 - M_1^2)\theta_1\theta_2\theta_3 - 2b_2h_2\theta_2\theta_3m_4^2 - \sqrt{\delta_2}}{2\theta_1\theta_2\theta_3M_2^2 - 2b_2^2\theta_2^2m_4^2} M_2^2 \\
 \rho_2 &= \frac{(M_3^2 + M_2^2 - M_1^2)\theta_1\theta_2\theta_3 - 2b_2h_2\theta_2\theta_3m_4^2 + \sqrt{\delta_2}}{2\theta_1\theta_2\theta_3M_3^2 - 2h_2^2\theta_3^2m_4^2} M_3^2 \\
 \rho_3 &= \frac{(M_1^2 - M_2^2 + M_3^2)\theta_1\theta_2\theta_3 + 2a_2h_2\theta_1\theta_3m_4^2 + \sqrt{\delta_2}}{2\theta_1\theta_2\theta_3M_1^2 - 2a_2^2\theta_1^2m_4^2} M_1^2,
 \end{aligned} \tag{5.31}$$

with

$$\begin{aligned}
 \delta_2 &= \theta_1^2\theta_2^2\theta_3^2\Delta_2 + 4A\theta_1\theta_2\theta_3m_4^2, \\
 \Delta_2 &= M_1^4 + M_2^4 + M_3^4 - 2M_1^2M_2^2 - 2M_1^2M_3^2 - 2M_2^2M_3^2.
 \end{aligned} \tag{5.32}$$

The coefficients A , B and C can be expressed in terms of M_i , θ_j :

$$\begin{aligned}
 A &= b_2h_2\theta_2\theta_3M_1^2 - a_2h_2\theta_1\theta_3M_2^2 + a_2b_2\theta_1\theta_2M_3^2, \\
 B &= -(b_1h_2 + b_2h_1)\theta_2\theta_3M_1^2 + (a_1h_2 + a_2h_1)\theta_1\theta_3M_2^2 - (a_1b_2 + a_2b_1)\theta_1\theta_2M_3^2, \\
 C &= b_1h_1\theta_2\theta_3M_1^2 - \theta_1\theta_3a_1h_1M_2^2 + a_1b_1\theta_1\theta_2M_3^2 - \theta_1\theta_2\theta_3m_4^2.
 \end{aligned} \tag{5.33}$$

One can easily observe that due to relations

$$\begin{aligned}
 \rho_1\rho_2 &= M_2^2M_3^3, \\
 \rho_1\rho_3 &= M_1^2\rho_1 + M_2^2\rho_3, \\
 \rho_2\rho_3 &= M_3^2\rho_3 - M_1^2M_3^2,
 \end{aligned} \tag{5.34}$$

parameters ρ_2 and ρ_3 can be expressed in terms of ρ_1 and therefore all the auxiliary parameters introduced in the derivation of the functional equation will be absorbed only in one parameter- ρ_1 .

We would like to notice that eq. (5.30) is valid for integrals but not for their integrands. This is due to the fact that the factor in front of integral that comes from the scaling of α parameters in the parametric integral is not fully compensated by the scaling momenta given in eq. (5.26).

At $m_4 = 0$ the dependence on all parameters a_i, b_j, h_k in eqs. (5.29), (5.30) drops out and the integral $J_0^{(d)}(M_1^2, M_2^2, M_3^2)$ reduces to a combination of simpler integrals:

$$\begin{aligned} J_0^{(d)}(M_1^2, M_2^2, M_3^2) &= J_0^{(d)}\left(0, \frac{-M_1^2 + M_2^2 + M_3^2 + \sqrt{\Delta_2}}{2}, \frac{-M_1^2 + M_2^2 + M_3^2 - \sqrt{\Delta_2}}{2}\right) \\ &+ J_0^{(d)}\left(\frac{M_1^2 - M_2^2 + M_3^2 + \sqrt{\Delta_2}}{2}, 0, \frac{M_1^2 - M_2^2 + M_3^2 - \sqrt{\Delta_2}}{2}\right) \\ &- J_0^{(d)}\left(\frac{-M_1^2 - M_2^2 + M_3^2 + \sqrt{\Delta_2}}{2}, \frac{-M_1^2 - M_2^2 + M_3^2 - \sqrt{\Delta_2}}{2}, 0\right). \end{aligned} \quad (5.35)$$

The analytic expression for the integral $J_0^{(d)}$ with one mass equal to zero is known [14]. Under assumption that $|m_3^2 - m_2^2| \leq |m_3^2|$ it reads

$$J_0^{(d)}(0, m_2^2, m_3^2) = \frac{\pi^2(2-d)(m_2^2 m_3^2)^{\frac{d-2}{2}}}{2(d-3)m_3^2 \left(\Gamma\left(\frac{d}{2}\right) \sin \frac{\pi d}{2}\right)^2} {}_2F_1\left[1, 2 - \frac{d}{2}; 4 - d; 1 - \frac{m_2^2}{m_3^2}\right]. \quad (5.36)$$

From functional equation (5.35) as a by-product one can get a new hypergeometric representation for the one-loop massless vertex type integral. In ref. [15] an interesting relation between the dimensionally regularized one-loop vertex type integral $I_3^{(d)}(m_1^2, m_2^2, m_3^2)$ and the two-dimensional integral $J_0^{(6-d)}(m_1^2, m_2^2, m_3^2)$ was discovered

$$I_3^{(d)}(s_{23}, s_{13}, s_{12}) = \frac{\Gamma\left(3 - \frac{d}{2}\right)}{\Gamma(d-3)} (-s_{23}s_{13}s_{12})^{\frac{d-4}{2}} J_0^{(6-d)}(s_{23}, s_{13}, s_{12}). \quad (5.37)$$

The functional equation (5.35) with $J_0^{(d)}(0, m_2^2, m_3^2)$ defined in eq. (5.36) provides us with a new hypergeometric representation for the integral $I_3^{(d)}$ with massless propagators. Formula for the one-loop massless vertex integral in terms of other Gauss' hypergeometric functions is given in ref. [16].

6 Conclusions

Finally, we summarize what we have accomplished in this paper.

First of all, we formulated new methods for deriving functional equations for Feynman integrals. These methods are quite simple and do not use any kind of integration by parts techniques.

Second, it has been shown that integrals with many kinematic arguments can be reduced to a combination of simpler integrals with fewer arguments. In our future publications we are going to demonstrate that in some cases applying functional equations one can reduce, the so-called, master integrals to a combination of simpler integrals from, what we would like to call, a ‘universal’ basis of integrals.

The method based on algebraic relations for ‘deformed propagators’ can be used not only for the vacuum type integrals, but also for the integrals depending on external momenta. In this paper we consider rather particular cases of the functional equations. The

systematic investigation and classification of the proposed functional equations requires application of the methods of algebraic geometry and group theory.

At the moment, it is not quite clear whether functional equations derivable from recurrence relations can be reproduced by the methods of algebraic relations between products of propagators described in section 3 and section 5.

A detailed consideration of our functional equations and their application to the one-loop integrals with four, five and six external legs as well as to some two- and three-loop Feynman integrals will be presented in future publications.

Finally, we would like to make a short comparison of our functional equations and the relationship presented in ref. [4]. The relationship given in ref. [4] corresponds to a particular choice of arbitrary parameters appearing in our functional equations. It has no free parameters. In general, our functional equations depend on several arbitrary parameters. This arbitrariness allows one to analytically continue the integrals to the required regions of kinematical variables.

In our approach, by choosing the arbitrary parameters, one can get also relationships between the integrals depending only on the arguments that are free of square roots of kinematical determinants. This was explicitly demonstrated in ref. [2].

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References

- [1] O.V. Tarasov, *New relationships between Feynman integrals*, *Phys. Lett. B* **670** (2008) 67 [[arXiv:0809.3028](https://arxiv.org/abs/0809.3028)] [[INSPIRE](#)].
- [2] O.V. Tarasov, *Functional equations for Feynman integrals*, *Phys. Part. Nucl. Lett.* **8** (2011) 419 [[INSPIRE](#)].
- [3] B.A. Kniehl and O.V. Tarasov, *Functional equations for one-loop master integrals for heavy-quark production and Bhabha scattering*, *Nucl. Phys. B* **820** (2009) 178 [[arXiv:0904.3729](https://arxiv.org/abs/0904.3729)] [[INSPIRE](#)].
- [4] A.I. Davydychev and R. Delbourgo, *A Geometrical angle on Feynman integrals*, *J. Math. Phys.* **39** (1998) 4299 [[hep-th/9709216](https://arxiv.org/abs/hep-th/9709216)] [[INSPIRE](#)].
- [5] A.I. Davydychev, *Geometrical splitting and reduction of Feynman diagrams*, *J. Phys. Conf. Ser.* **762** (2016) 012068 [[arXiv:1605.04828](https://arxiv.org/abs/1605.04828)] [[INSPIRE](#)].
- [6] O.V. Tarasov, *Connection between Feynman integrals having different values of the space-time dimension*, *Phys. Rev. D* **54** (1996) 6479 [[hep-th/9606018](https://arxiv.org/abs/hep-th/9606018)] [[INSPIRE](#)].

- [7] J. Fleischer, F. Jegerlehner and O.V. Tarasov, *Algebraic reduction of one loop Feynman graph amplitudes*, *Nucl. Phys. B* **566** (2000) 423 [[hep-ph/9907327](#)] [[INSPIRE](#)].
- [8] Z. Bern, L.J. Dixon and D.A. Kosower, *Dimensionally regulated one loop integrals*, *Phys. Lett. B* **302** (1993) 299 [*Erratum ibid.* **B 318** (1993) 649] [[hep-ph/9212308](#)] [[INSPIRE](#)].
- [9] Z. Bern, L.J. Dixon and D.A. Kosower, *Dimensionally regulated pentagon integrals*, *Nucl. Phys. B* **412** (1994) 751 [[hep-ph/9306240](#)] [[INSPIRE](#)].
- [10] G. 't Hooft and M.J.G. Veltman, *Scalar One Loop Integrals*, *Nucl. Phys. B* **153** (1979) 365 [[INSPIRE](#)].
- [11] O.V. Tarasov, *Generalized recurrence relations for two loop propagator integrals with arbitrary masses*, *Nucl. Phys. B* **502** (1997) 455 [[hep-ph/9703319](#)] [[INSPIRE](#)].
- [12] D.J. Broadhurst, J. Fleischer and O.V. Tarasov, *Two loop two point functions with masses: Asymptotic expansions and Taylor series, in any dimension*, *Z. Phys. C* **60** (1993) 287 [[hep-ph/9304303](#)] [[INSPIRE](#)].
- [13] B.A. Kniehl, A.V. Kotikov and O.L. Veretin, *Irrational constants in positronium decays*, [arXiv:0811.0306](#) [[INSPIRE](#)].
- [14] A.I. Davydychev and J.B. Tausk, *Two loop selfenergy diagrams with different masses and the momentum expansion*, *Nucl. Phys. B* **397** (1993) 123 [[INSPIRE](#)].
- [15] A.I. Davydychev and J.B. Tausk, *A magic connection between massive and massless diagrams*, *Phys. Rev. D* **53** (1996) 7381 [[hep-ph/9504431](#)] [[INSPIRE](#)].
- [16] A.I. Davydychev, *Explicit results for all orders of the ϵ -expansion of certain massive and massless diagrams*, *Phys. Rev. D* **61** (2000) 087701 [[hep-ph/9910224](#)] [[INSPIRE](#)].