Holographic correlation functions in Critical Gravity

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ABSTRACT: We compute the holographic stress tensor and the logarithmic energy-momentum tensor of Einstein-Weyl gravity at the critical point. This computation is carried out performing a holographic expansion in a bulk action supplemented by the Gauss-Bonnet term with a fixed coupling. The renormalization scheme defined by the addition of this topological term has the remarkable feature that all Einstein modes are identically cancelled both from the action and its variation. Thus, what remains comes from a nonvanishing Bach tensor, which accounts for non-Einstein modes associated to logarithmic terms which appear in the expansion of the metric. In particular, we compute the holographic 1-point functions for a generic boundary geometric source.

KEYWORDS: AdS-CFT Correspondence, Gauge-gravity correspondence, Models of Quantum Gravity

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1 Introduction

Critical Gravity belongs to a class of theories characterized by the presence of higher curvature terms in the action. Higher-derivative gravity theories were introduced as possible toy models that might provide insight in some aspects of quantum gravity. The failure of General Relativity (GR) to be perturbatively non-renormalizable, leads to a theory that is UV divergent, and consequently, it is not consistent at a quantum level [1].

One of the proposals to solve the problem of renormalizability was the addition of quadratic curvature terms on top of the Einstein-Hilbert action. Seminal papers on the topic show that these theories are renormalizable [2, 3]. The spectrum of the theory, for a flat spacetime, consists of massless spin-2, massive spin-2 and massive scalar excitations. However, as it was later pointed out, the massive graviton is a ghost mode (negative energy) in a generic higher-derivative theory [4].

In quest for consistent gravity theories, 3D massive gravity provided intuition and the appropriate tools to overcome the pathologies mentioned above [5, 6]. Some of the desirable features in these theories can be extended to higher dimensions. More specifically, the phenomenon of criticality, which represents the existence of a point in the parametric space of the coupling constants where the linearized EOM degenerate and the massive gravitons turns to massless, has been extended in 4D giving rise to Critical Gravity [7]. At this specific point the scalar excitations vanish whereas new modes with logarithmic behavior arise.

The logarithmic modes can be discarded imposing standard AdS boundary conditions. In this case, the theory is proved to be trivial as the energy of the massless excitations as well as the energy and the entropy of the Schwarzschild-AdS black hole vanish [7, 8]. This is a consequence of the on-shell equivalence between Einstein-AdS and Conformal Gravity when switching off the non-Einstein modes, as previously seen in refs. [9, 10].

If, on the contrary, one imposes a relaxed set of AdS boundary conditions, then the logarithmic modes can be included in the spectrum of the theory. Such asymptotic conditions have been discussed in refs. [11–13]. The behavior of the new modes is captured by terms with logarithmic dependence in the radial coordinate in the Fefferman-Graham (FG) expansion. The term at leading log order is the source of a logarithmic operator living on the boundary. The boundary field theory is a Logarithmic Conformal Field Theory (LCFT), instead of a regular CFT.

In general, the presence of a logarithmic source modifies the asymptotic structure and the spacetime fails to be asymptotically AdS in the standard sense. As a consequence, standard holographic description at its boundary breaks down. The alternative is treating the problem perturbatively, with a log contribution which is very small, such that the conformal structure at asymptotic infinity is preserved and the holographic dictionary is still valid.

LCFTs emerge in different fields in Physics, but they are mainly associated to critical behavior of disordered systems. Other setups where they are physically relevant include the description of polymers, percolation, turbulence, Quantum Hall plateau phase transition and in string theory, as well. LCFTs are characterized by the presence of logarithmic terms in the operator product expansion (OPE) [14] which, despite its logarithmic behavior, respect conformal invariance. Logarithmic operators, which in the gravity dual description are sourced by $b_{(0)ij}$, extend the notion of a primary operator for non-diagonalizable matrices. In particular, they arise as logarithmic partners coupled to zero norm primary states with degenerate scaling dimensions [15]. The Hamiltonian corresponding to these states is not Hermitian, and, therefore, they are associated to theories which are non-unitary.

In view of all above arguments, it is clear that Critical Gravity provides important intuition on properties of the AdS/LCFT correspondence. Some interesting properties of this gravity theory were made manifest in ref. [10], where it was shown that the only non-trivial contributions in Critical Gravity are coming from the non-Einstein sector of the theory, as the on-shell action is quadratic in the Bach tensor. In the present paper, we exploit this feature in order to gain a new insight into the properties of the theory. We identify the non-Einstein modes as the source of the divergences and propose a new set of counterterms which depend on the extrinsic curvature and its covariant derivative, in order to regulate the action. This formulation provides a shortcut in the derivation of holographic correlation functions, as it substantially simplifies the computations respect to similar approaches in the literature (e.g., ref. [16]).

2 Critical Gravity

Critical Gravity in 4D is defined by the action

$$I_{\text{critical}} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} \left[R - 2\Lambda + \frac{3}{2\Lambda} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) \right] \quad (2.1)$$

where $\Lambda = -3/\ell^2$ is the cosmological constant (in terms of the AdS radius ℓ). An equivalent form for the Critical Gravity action was provided in refs. [10, 17]

$$I_{\text{critical}} = I_{MM} - \frac{\ell^2}{64\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} W^{\alpha\beta\mu\nu} W_{\alpha\beta\mu\nu} \quad (2.2)$$

where I_{MM} stands for the Einstein-AdS action suitably regulated by the addition of the Gauss-Bonnet term

$$I_{MM} = \frac{\ell^2}{256\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} \delta_{[\mu_1\mu_2\mu_3\mu_4]}^{[\nu_1\nu_2\nu_3\nu_4]} \left(R_{\nu_1\nu_2}^{\mu_1\mu_2} + \frac{1}{\ell^2} \delta_{[\nu_1\nu_2]}^{[\mu_1\mu_2]} \right) \left(R_{\nu_3\nu_4}^{\mu_3\mu_4} + \frac{1}{\ell^2} \delta_{[\nu_3\nu_4]}^{[\mu_3\mu_4]} \right). \quad (2.3)$$

This particular form of the AdS gravity action is referred to as MacDowell-Mansouri (Stelle-West) form in the literature [18]. It was shown in ref. [19], that the addition of the Gauss-Bonnet term to the Einstein-Hilbert action with negative cosmological constant induces an extrinsic regularization scheme for AdS gravity. It was later shown in ref. [20], that the use of holographic techniques in asymptotically AdS spaces allows to expand the fields and to prove that the above action is the renormalized action that appears in the context of AdS/CFT correspondence [21, 22].

Henceforth, we shall adopt the name I_{MM} for the EH plus GB action, as we anticipate that this part of the Critical Gravity action, only by itself, will no longer be renormalized when log terms are present.

$$W_{\mu\nu}^{\alpha\beta} = R_{\mu\nu}^{\alpha\beta} - \frac{1}{2} \left(R_{\mu}^{\alpha} \delta_{\nu}^{\beta} - R_{\mu}^{\beta} \delta_{\nu}^{\alpha} - R_{\nu}^{\alpha} \delta_{\mu}^{\beta} + R_{\nu}^{\beta} \delta_{\mu}^{\alpha} \right) + \frac{R}{6} \delta_{[\mu\nu]}^{[\alpha\beta]}, \quad (2.4)$$

is the Weyl tensor of the spacetime. Here, we will refer to the Weyl² part within the action of Critical Gravity (2.2) as Conformal Gravity (CG), even though it comes with a specific coupling

$$I_{\text{CG}} = \frac{\ell^2}{256\pi G} \delta_{[\mu_1\mu_2\mu_3\mu_4]}^{[\nu_1\nu_2\nu_3\nu_4]} W_{\nu_1\nu_2}^{\mu_1\mu_2} W_{\nu_3\nu_4}^{\mu_3\mu_4}. \quad (2.5)$$

The coupling in front of the above action is such that the Einstein modes are exactly cancelled out from eq. (2.1). In other words, as it was shown in refs. [10, 17], the Critical Gravity action is identically zero for Einstein spacetimes.

As it is useful for the present derivation, we briefly review this result below.

2.1 Field equations

The corresponding field equations for Critical Gravity are given by

$$G_{\nu}^{\mu} + \frac{\ell^2}{4} B_{\nu}^{\mu} = 0, \quad (2.6)$$

where G_{ν}^{μ} is the Einstein tensor with negative cosmological constant

$$G_{\nu}^{\mu} = R_{\nu}^{\mu} - \frac{1}{2} R \delta_{\nu}^{\mu} - \frac{3}{\ell^2} \delta_{\nu}^{\mu} = -\frac{1}{4} \delta_{[\nu\gamma\delta]}^{[\mu\alpha\beta]} \left(R_{\alpha\beta}^{\gamma\delta} + \frac{1}{\ell^2} \delta_{[\alpha\beta]}^{[\gamma\delta]} \right). \quad (2.7)$$

The Bach tensor B_ν^μ is a four-derivative object that involves the covariant derivative of the Cotton tensor and a part which is quadratic in the curvature. It can be also written down in terms of the Weyl tensor as

$$B_\nu^\mu = -4 \left(\nabla^\alpha \nabla_\beta W_{\alpha\nu}^{\beta\mu} + \frac{1}{2} R_\beta^\alpha W_{\alpha\nu}^{\beta\mu} \right). \quad (2.8)$$

The above relation makes manifest its traceless property. The fact B_ν^μ is covariantly constant derives from Bianchi identity.

Taking the trace of (2.6), we notice that the Ricci scalar does not differ from the case of General Relativity ($R = -12/\ell^2$). Plugging in the general form of the Ricci scalar into the EOM, we obtain

$$R_{\mu\nu} = -\frac{3}{\ell^2} g_{\mu\nu} - \frac{\ell^2}{4} B_{\mu\nu}, \quad (2.9)$$

what governs not only the bulk dynamics, but also determines the asymptotic form of the boundary terms.

2.2 On-shell action

When eq. (2.9) is substituted in eq. (2.4), one obtains a generic decomposition of the Weyl tensor into an Einstein and a non-Einstein parts

$$W_{\mu\nu}^{\alpha\beta} = W_{(E)\mu\nu}^{\alpha\beta} + W_{(NE)\mu\nu}^{\alpha\beta}, \quad (2.10)$$

where

$$W_{(E)\mu\nu}^{\alpha\beta} = R_{\mu\nu}^{\alpha\beta} + \frac{1}{\ell^2} \delta_{[\mu\nu]}^{[\alpha\beta]}, \quad (2.11)$$

$$W_{(NE)\mu\nu}^{\alpha\beta} = \frac{\ell^2}{8} \left(B_\mu^\alpha \delta_\nu^\beta - B_\mu^\beta \delta_\nu^\alpha - B_\nu^\alpha \delta_\mu^\beta + B_\nu^\beta \delta_\mu^\alpha \right). \quad (2.12)$$

Here $W_{(E)\mu\nu}^{\alpha\beta}$ corresponds to the Weyl tensor for Einstein spacetimes $R_{\mu\nu} = -3/\ell^2 g_{\mu\nu}$. The departure from the Einstein condition provides additional contributions to the Weyl tensor. In Einstein-Weyl gravity, the deviation from Einstein spaces involves linear terms in the Bach tensor.

Applying the Weyl decomposition (2.10) in the Weyl² part of the action (2.2) leads to the expression

$$I_{\text{CG}} = \frac{\ell^2}{256\pi G} \delta_{[\mu_1\mu_2\mu_3\mu_4]}^{[\nu_1\nu_2\nu_3\nu_4]} \left(W_{(E)\nu_1\nu_2}^{\mu_1\mu_2} W_{(E)\nu_3\nu_4}^{\mu_3\mu_4} + 2W_{(E)\nu_1\nu_2}^{\mu_1\mu_2} W_{(NE)\nu_3\nu_4}^{\mu_3\mu_4} + W_{(NE)\nu_1\nu_2}^{\mu_1\mu_2} W_{(NE)\nu_3\nu_4}^{\mu_3\mu_4} \right). \quad (2.13)$$

The first term in the above expression carries a particular coupling constant, such that it exactly cancels the I_{MM} part in the Critical Gravity action (2.2). This is a direct consequence of the equivalence between Conformal and Einstein gravity, once Neumann boundary conditions are imposed in order to get rid of higher-derivative modes [9]. An explicit proof of this statement, carried out in ref. [10], recovers Einstein gravity by imposing $B_{\mu\nu} = 0$ in the decomposition of the $Weyl^2$ term (2.13).

As a consequence, the only nonvanishing part of Critical Gravity action is given in terms of the Bach tensor

$$I_{\text{critical}} = -\frac{\ell^4}{64\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} \delta_{[\mu\nu]}^{[\kappa\lambda]} \left(\frac{\ell^2}{8} B_{\kappa}^{\mu} + G_{\kappa}^{\mu} \right) B_{\lambda}^{\nu}. \quad (2.14)$$

Using the equation of motion (2.6),

$$\begin{aligned} I_{\text{critical}} &= \frac{\ell^6}{512\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} \delta_{[\mu\nu]}^{[\kappa\lambda]} B_{\kappa}^{\mu} B_{\lambda}^{\nu} \\ &= -\frac{\ell^6}{512\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} B_{\kappa}^{\mu} B_{\mu}^{\kappa}, \end{aligned} \quad (2.15)$$

one can notice that Critical Gravity action involves only the non-Einstein part of the Weyl tensor in the form of the square of the Bach tensor.

2.3 Surface terms

An arbitrary variation of the action (2.3) is given by

$$\delta I_{MM} = \frac{\ell^2}{64\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{-h} \delta_{[\mu_1\mu_2\mu_3\mu_4]}^{[\nu_1\nu_2\nu_3\nu_4]} n_{\nu_1} \delta\Gamma_{\kappa\nu_2}^{\mu_1} g^{\mu_2\kappa} W_{(E)\nu_3\nu_4}^{\mu_3\mu_4}. \quad (2.16)$$

Similarly, the surface terms coming from the variation of the Weyl² term are cast into the form

$$\delta I_{CG} = \frac{\ell^2}{64\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{-h} \delta_{[\mu_1\mu_2\mu_3\mu_4]}^{[\nu_1\nu_2\nu_3\nu_4]} \left[n_{\nu_1} \delta\Gamma_{\kappa\nu_2}^{\mu_1} g^{\mu_2\kappa} W_{\nu_3\nu_4}^{\mu_3\mu_4} + n^{\mu_1} \nabla_{\nu_1} W_{\nu_2\nu_3}^{\mu_2\mu_3} (g^{-1}\delta g)_{\nu_4}^{\mu_4} \right].$$

Combining these two contributions, we get the total surface term of Critical Gravity action

$$\begin{aligned} \delta I_{\text{critical}} = \delta I_{MM} - \delta I_{CG} &= \frac{\ell^2}{64\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{-h} \delta_{[\mu_1\mu_2\mu_3\mu_4]}^{[\nu_1\nu_2\nu_3\nu_4]} \left[n_{\nu_1} \delta\Gamma_{\kappa\nu_2}^{\mu_1} g^{\mu_2\kappa} \left(W_{(E)\nu_3\nu_4}^{\mu_3\mu_4} - W_{\nu_3\nu_4}^{\mu_3\mu_4} \right) \right. \\ &\quad \left. - n^{\mu_1} \nabla_{\nu_1} W_{\nu_2\nu_3}^{\mu_2\mu_3} (g^{-1}\delta g)_{\nu_4}^{\mu_4} \right]. \end{aligned} \quad (2.17)$$

Applying the Weyl decomposition (2.10) and the Bianchi identity in the previous expression, one gets

$$\begin{aligned} \delta I_{\text{critical}} &= -\frac{\ell^2}{64\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{-h} \delta_{[\mu_1\mu_2\mu_3\mu_4]}^{[\nu_1\nu_2\nu_3\nu_4]} \\ &\quad \times \left[n_{\nu_1} \delta\Gamma_{\kappa\nu_2}^{\mu_1} g^{\mu_2\kappa} W_{(NE)\nu_3\nu_4}^{\mu_3\mu_4} + n^{\mu_1} \nabla_{\nu_1} W_{(NE)\nu_2\nu_3}^{\mu_2\mu_3} (g^{-1}\delta g)_{\nu_4}^{\mu_4} \right]. \end{aligned} \quad (2.18)$$

In the last step, the Bianchi identity is applied as follows

$$\delta_{[\mu_1\mu_2\mu_3\mu_4]}^{[\nu_1\nu_2\nu_3\nu_4]} \nabla_{\nu_1} W_{(E)\nu_2\nu_3}^{\mu_2\mu_3} = 0.$$

In order to reveal that the variation of the Critical Gravity action is linear to the Bach tensor, we substitute the expression (2.12) in eq. (2.18), what leads to

$$\delta I_{\text{critical}} = -\frac{\ell^4}{128\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{-h} \delta_{[\mu_1\mu_2\mu_3]}^{[\nu_1\nu_2\nu_3]} \left[n_{\nu_1} \delta\Gamma_{\kappa\nu_2}^{\mu_1} g^{\mu_2\kappa} B_{\nu_3}^{\mu_3} + n^{\mu_1} \nabla_{\nu_1} B_{\nu_2}^{\mu_2} (g^{-1}\delta g)_{\nu_3}^{\mu_3} \right]. \quad (2.19)$$

A direct consequence of the above formula is the vanishing of the energy for Einstein spacetimes. This has been pointed out in, e.g., in refs. [7, 23, 24], based on a rather case-by-case analysis. A more general proof that Einstein spacetimes have zero energy can be made by using Noether-Wald charges [8].

We can replace the Bach with the Einstein tensor using the EOM (2.6), such that

$$\delta I_{\text{critical}} = \frac{\ell^2}{32\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{-h} \delta_{[\mu_1\mu_2\mu_3]}^{[\nu_1\nu_2\nu_3]} \left[n_{\nu_1} \delta\Gamma_{\kappa\nu_2}^{\mu_1} g^{\kappa\mu_2} G_{\nu_3}^{\mu_3} + n^{\mu_1} \nabla_{\nu_1} G_{\nu_2}^{\mu_2} (g^{-1}\delta g)_{\nu_3}^{\mu_3} \right]. \quad (2.20)$$

Notice that some of the terms of the second part of eq. (2.20) will vanish due to the Bianchi identity, once the antisymmetric Kronecker delta is expanded. Equipped with the generic form of the variation of the action (2.20), a suitable intermediate step towards the derivation of the holographic correlation functions is to cast the corresponding surface terms in Gaussian coordinates,

$$ds^2 = N^2(\rho) d\rho^2 + h_{ij}(\rho, x) dx^i dx^j. \quad (2.21)$$

This far, Greek letters represent spacetime indices. In what follows, Latin will denote letters boundary indices. In this frame, the first part of eq. (2.20) becomes

$$\delta_{[\mu_1\mu_2\mu_3]}^{[\nu_1\nu_2\nu_3]} n_{\nu_1} \delta\Gamma_{\kappa\nu_2}^{\mu_1} g^{\kappa\mu_2} G_{\nu_3}^{\mu_3} = N \delta_{[k\ell]}^{[ij]} \left[\delta\Gamma_{mi}^{\rho} h^{mk} G_j^{\ell} - \delta\Gamma_{\rho i}^k g^{\rho\rho} G_j^{\ell} + \delta\Gamma_{mi}^k h^{m\ell} G_j^{\rho} \right]. \quad (2.22)$$

The first two terms of eq. (2.22) are of the form

$$N \delta_{[k\ell]}^{[ij]} \left[\delta \left(\frac{1}{N} K_{mi} \right) h^{km} G_j^{\ell} - \delta\Gamma_{\rho i}^k g^{\rho\rho} G_j^{\ell} \right] = \delta_{[k\ell]}^{[ij]} \left[K_i^m (h^{-1}\delta h)_m^k + 2\delta K_i^k \right] G_j^{\ell}. \quad (2.23)$$

Moreover, the last term of eq. (2.22) can be written as

$$\int_{\partial\mathcal{M}} d^3x \sqrt{-h} \delta_{[k\ell]}^{[ij]} N \delta\Gamma_{mi}^k h^{\ell m} G_j^{\rho} = - \int_{\partial\mathcal{M}} d^3x \sqrt{-h} N \delta_{[k\ell]}^{[ij]} (h^{-1}\delta h)_i^k D^{\ell} G_j^{\rho}, \quad (2.24)$$

where integration by parts was performed. Here, D_i is the covariant derivative defined in the boundary metric.

Summing up the contributions from eqs. (2.23) and (2.24), one shows that eq. (2.22) adopts the form

$$\delta_{[\mu_1\mu_2\mu_3]}^{[\nu_1\nu_2\nu_3]} n_{\nu_1} \delta\Gamma_{\kappa\nu_2}^{\mu_1} g^{\kappa\mu_2} G_{\nu_3}^{\mu_3} = \delta_{[k\ell]}^{[ij]} \left[\left(K_i^m (h^{-1}\delta h)_m^k + 2\delta K_i^k \right) G_j^{\ell} - N D^k G_i^{\rho} (h^{-1}\delta h)_j^{\ell} \right]. \quad (2.25)$$

In addition to this, the remaining contribution coming from eq. (2.20) adopts the form

$$\delta_{[\mu_1\mu_2\mu_3]}^{[\nu_1\nu_2\nu_3]} n^{\mu_1} \nabla_{\nu_1} G_{\nu_2}^{\mu_2} (g^{-1}\delta g)_{\nu_3}^{\mu_3} = \delta_{[k\ell]}^{[ij]} \frac{1}{N} \left(\nabla_{\rho} G_i^k - \nabla_i G_{\rho}^k \right) (h^{-1}\delta h)_j^{\ell}. \quad (2.26)$$

Hence, the variation of the Critical Gravity action in Gauss-normal coordinates becomes

$$\begin{aligned} \delta I_{\text{critical}} = & \frac{\ell^2}{32\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{-h} \delta_{[k\ell]}^{[ij]} \left[\left(2\delta K_i^k + K_i^m (h^{-1}\delta h)_m^k \right) G_j^\ell \right. \\ & \left. + \frac{1}{N} \left(\nabla_\rho G_i^k - \nabla_i G_\rho^k - N^2 D^k G_i^\rho \right) (h^{-1}\delta h)_j^\ell \right]. \end{aligned} \quad (2.27)$$

3 Holographic renormalization in Critical Gravity

In Critical Gravity, new modes appear as a consequence of the coalescence of massive spin-2 modes with the massless ones. These modes have logarithmic dependence in the radial coordinate and spoil the standard asymptotically AdS (AAdS) fall-off of the spacetime. Choosing suitable boundary conditions, the logarithmic modes can be discarded. Thus, one reproduces the standard AdS/CFT dictionary, sourced by Einstein modes at the boundary.

By keeping the logarithmic modes, one gains intuition on holographic duals to higher-derivative gravity theories at critical points. In particular, we focus on aspects of AdS/LCFT correspondence associated to the computation of the holographic stress tensors which are defined as the functional derivatives of the variation of the action with respect to the independent sources [25].

The new branch of solutions is consistent with a relaxed set of AdS boundary conditions [11, 26], which is expressed in the radial foliation (2.21) setting $N = \frac{\ell}{2\rho}$ and

$$h_{ij}(\rho, x) = \frac{1}{\rho} \tilde{g}_{ij}(\rho, x), \quad (3.1)$$

$$\tilde{g}_{ij}(\rho, x) = g_{(0)ij} + b_{(0)ij} \log \rho + \rho (g_{(2)ij} + b_{(2)ij} \log \rho) + \rho^{3/2} (g_{(3)ij} + b_{(3)ij} \log \rho) + \dots \quad (3.2)$$

3.1 Generic boundary geometry

In the treatment below, for simplicity, we choose unit AdS radius ($\ell = 1$). The inverse of the boundary metric reads

$$\begin{aligned} \tilde{g}^{ij}(\rho, x) = & g_{(0)}^{ij} - b_{(0)}^{ij} \log \rho + \rho \left[-g_{(2)}^{ij} - b_{(2)}^{ij} \log \rho + 2(b_{(0)}g_{(2)})^{ij} \log \rho + 2(b_{(0)}b_{(2)})^{ij} \log^2 \rho \right] \\ & + \rho^{3/2} \left[-g_{(3)}^{ij} - b_{(3)}^{ij} \log \rho + 2(b_{(0)}g_{(3)})^{ij} \log \rho + 2(b_{(0)}b_{(3)})^{ij} \log^2 \rho \right] + \dots \end{aligned} \quad (3.3)$$

By definition, the extrinsic curvature is given by $K_{ij} = -\frac{1}{2N} \partial_\rho h_{ij}$ what, in the above frame is expressed as

$$\begin{aligned} K_j^i = & \delta_j^i - b_{(0)j}^i + \rho \left[-b_{(2)j}^i - g_{(2)j}^i - b_{(2)j}^i \log \rho + (b_{(0)}g_{(2)})_j^i + 2(b_{(0)}b_{(2)})_j^i \log \rho + (b_{(0)}g_{(2)})_j^i \log \rho \right. \\ & \left. + (b_{(0)}b_{(2)})_j^i \log^2 \rho \right] + \rho^{3/2} \left[-b_{(3)j}^i - \frac{3}{2}g_{(3)j}^i - \frac{3}{2}b_{(3)j}^i \log \rho + (b_{(0)}g_{(3)})_j^i + 2(b_{(0)}b_{(3)})_j^i \log \rho \right. \\ & \left. + \frac{3}{2}(b_{(0)}g_{(3)})_j^i \log \rho + \frac{3}{2}(b_{(0)}b_{(3)})_j^i \log^2 \rho \right] + \dots \end{aligned} \quad (3.4)$$

As it can be easily seen from the asymptotic expansion of the extrinsic curvature (3.4), the log term prevents δK_j^i from vanishing at leading (finite) order. Thus, on top of the

boundary metric $g_{(0)ij}$, which is the source of the boundary stress-energy tensor, a new independent source arises, $b_{(0)ij}$. Furthermore, the presence of the new source modifies the asymptotic expansion of the curvature as follows,

$$\begin{aligned} R_{j\rho}^{i\rho} &= -\delta_j^i + 2b_{(0)j}^i + \mathcal{O}(\rho) \\ R_{jk}^{i\rho} &= 2\rho \left(D_k b_{(0)j}^i - D_j b_{(0)k}^i \right) + \mathcal{O}(\rho^2) \\ R_{kl}^{ij} &= -\delta_{[kl]}^{[ij]} + b_{(0)k}^i \delta_l^j - b_{(0)l}^i \delta_k^j - b_{(0)k}^j \delta_l^i + b_{(0)l}^j \delta_k^i + \mathcal{O}(\rho) . \end{aligned}$$

Thus, the spacetime is no longer AAdS and as a result the dual CFT description is not valid anymore. Nevertheless, considering a non-vanishing but sufficiently small $b_{(0)ij}$, one avoids spoiling the asymptotic conformal structure of the AAdS spacetime. For this reason, in the present section, we proceed perturbatively in $b_{(0)ij}$. The dual theory living on the boundary is now a Logarithmic Conformal Field Theory (LCFT), instead of a CFT. As a result, a new operator arises, which is identified as the logarithmic stress energy tensor t_{ij} , i.e., the response to $b_{(0)ij}$. In the dual description, the logarithmic stress tensor is an irrelevant operator.

The eq. (2.27), evaluated in the FG expansion for relaxed AdS boundary conditions (3.2), provides the holographic one-point functions of the boundary field theory. These are expressed as the coefficients of $\delta g_{(0)ij}$ and $\delta b_{(0)ij}$, which are the sources of the dual stress tensors. This derivation requires the asymptotic resolution of the EOM, by substituting the expression (3.2) in (2.6). As a result, one obtains algebraic equations relating the coefficients in FG expansion of the metric.

3.2 Vanishing log source ($b_{(0)ij} = 0$)

As a warmup computation, here we derive the energy-momentum tensor for a given boundary background metric $g_{(0)ij}$, while setting the leading logarithmic mode $b_{(0)ij}$ as zero.

Thus, the trace of the equation of motion ($R = -12$) gives rise to

$$Tr g_{(3)} = Tr b_{(3)} = 0 = Tr b_{(2)} , \quad (3.5)$$

$$4Tr g_{(2)} + R(g_{(0)}) = 0 . \quad (3.6)$$

For the (ρi) component of the EOM one finds

$$\nabla_j b_{(3)i}^j = \nabla_j g_{(3)i}^j = \nabla_j b_{(2)i}^j = 0 , \quad (3.7)$$

$$\nabla_i Tr g_{(2)} - \nabla_j g_{(2)i}^j = 0 . \quad (3.8)$$

Finally, for the (ij) part of the EOM

$$g_{(2)j}^i - Tr g_{(2)} \delta_j^i + R_j^i(g_{(0)}) - \frac{1}{2} R(g_{(0)}) \delta_j^i = 0 , \quad (3.9)$$

$$b_{(2)j}^i = 0 . \quad (3.10)$$

Thus, the vanishing of $b_{(0)}$ leads to a vanishing $b_{(2)}$ in (3.2), but there is yet a remaining logarithmic contribution coming from the subdominant $b_{(3)}$ term. Due to the absence of logarithmic source, the corresponding energy-momentum tensor t_{ij} is zero.

In this case the extrinsic curvature is expanded asymptotically as

$$K_j^i = \delta_j^i - \rho g_{(2)j}^i + \rho^{3/2} \left(b_{(3)j}^i - \frac{3}{2} g_{(3)j}^i - \frac{3}{2} b_{(3)j}^i \log \rho \right) + \dots \quad (3.11)$$

Calculating the contributions appearing in the action (2.27), we obtain

$$\delta_{[k\ell]}^{[ij]} K_i^m G_j^\ell = 3\rho^{3/2} b_{(3)k}^m \quad (3.12)$$

$$\frac{1}{N} \delta_{[k\ell]}^{[ij]} \left(\nabla_\rho G_i^k - \nabla_i G_\rho^k \right) = 6\rho^{3/2} b_{(3)\ell}^j \quad (3.13)$$

$$N \delta_{[k\ell]}^{[ij]} D^k G_i^\rho = 0 \quad (3.14)$$

Moreover, the absence of $b_{(0)ij}$ turns the variation of the extrinsic curvature into a term of order $\mathcal{O}(\rho)$. Considering the corresponding asymptotic expansion of the fields

$$(h^{-1} \delta h)_j^i = (\tilde{g}^{-1} \delta \tilde{g})_j^i = \left(g_{(0)}^{-1} \delta g_{(0)} \right)_j^i + \mathcal{O}(\rho) \quad (3.15)$$

$$\sqrt{-h} = \frac{\sqrt{-\tilde{g}}}{\rho^{3/2}} = \frac{\sqrt{-g_{(0)}}}{\rho^{3/2}} + \mathcal{O}(\rho^{-1/2}), \quad (3.16)$$

the variation of K_j^i is subdominant with respect to the variation of the metric, as in standard AAdS spacetimes. Therefore, this terms does not contribute at the conformal boundary. Thus, eq. (2.27) adopts the form

$$\delta I_{\text{critical}} = \frac{9}{32\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{-g_{(0)}} b_{(3)j}^i \left(g_{(0)}^{-1} \delta g_{(0)} \right)_i^j. \quad (3.17)$$

No infrared divergences appear and the variational principle is well defined for the Dirichlet boundary condition, $\delta g_{(0)ij} = 0$. Hence, it turns out that no counterterms are needed on top of the Critical Gravity action (2.1).

The holographic stress tensor is obtained as the functional variation of the regular part of the surface term respect to the metric source [25]

$$\langle T_{ij} \rangle = - \frac{2}{\sqrt{-g_{(0)}}} \frac{\delta I}{\delta g_{(0)}^{ij}}. \quad (3.18)$$

Reading off from the formula (3.17), one gets

$$\langle T_{ij} \rangle = \frac{9}{16\pi G} b_{(3)ij}, \quad (3.19)$$

for the holographic one-point function dual to the boundary metric $g_{(0)ij}$. This formula recovers the result in ref. [16] without assuming any particular form on the boundary geometry.

It is clear that, for Einstein spaces, the above stress tensor is zero.

4 Linearized analysis ($b_{(0)ij} \neq 0$)

The calculation of the holographic correlation functions, when $b_{(0)ij}$ is switched on, turns considerably cumbersome. In order to simplify the discussion, one tackles the problem perturbatively around AdS_4 [16, 27, 28]. Linearizing the EOM, one gets the on-shell action up to quadratic order in the perturbation. This analysis is sufficient for the derivation of the two-point functions. In this section, taking advantage of the alternative form of the Critical Gravity action (2.15), we evaluate the linearized AdS_4 metric and introduce proper counterterms in order to cancel the emerging infinities.

Following the analysis of the generic case in Gauss-normal coordinates (2.21), where the boundary metric is expressed as a deviation of the Minkowski background,

$$h_{ij}(\rho, x) = \frac{1}{\rho} \tilde{g}_{ij} = \frac{1}{\rho} (\eta_{ij} + c_{ij}) . \quad (4.1)$$

In this gauge, the holographic correlation functions are planar. The perturbation c_{ij} admits the FG expansion

$$c_{ij} = h_{(0)ij} + b_{(0)ij} \log \rho + \rho (g_{(2)ij} + b_{(2)ij} \log \rho) + \rho^{3/2} (g_{(3)ij} + b_{(3)ij} \log \rho) + \dots , \quad (4.2)$$

which is consistent within the logarithmic branch of the theory. The substitution of the FG expansion (4.2) in eq. (2.6), provides relations between the FG coefficients, which hold for the linearized version of the theory.

From the Ricci scalar $R = -12$, one is able to obtain

$$Tr b_{(0)} = Tr b_{(3)} = Tr g_{(3)} = 0 \quad (4.3)$$

$$4Tr b_{(2)} + \partial_i \partial_j b_{(0)}^{ij} = 0 \quad (4.4)$$

$$4Tr g_{(2)} + \partial_i \partial_j h_{(0)}^{ij} - \partial^m \partial_m Tr h_{(0)} = 0 . \quad (4.5)$$

The $(\rho\rho)$ component of the EOM (2.6) give

$$4Tr b_{(2)} - \partial_i \partial_j b_{(0)}^{ij} = 0 , \quad (4.6)$$

while the tracelessness of the Bach tensor leads to

$$2Tr b_{(2)} - \frac{7}{2} \partial_i \partial_j b_{(0)}^{ij} = 0 . \quad (4.7)$$

Combining these expressions with eq. (4.4), one concludes that

$$Tr b_{(2)} = \partial_i \partial_j b_{(0)}^{ij} = 0 . \quad (4.8)$$

The (ρi) terms give

$$\partial_j b_{(0)i}^j = \partial_j b_{(2)i}^j = \partial_j b_{(3)i}^j = 0 , \quad (4.9)$$

$$4\partial_j g_{(2)i}^j + \partial_i \partial_m \partial_k h_{(0)}^{mk} - \partial_i \partial^m \partial_m Tr h_{(0)} = 0 . \quad (4.10)$$

Finally, for the (ij) part of the EOM

$$\partial^m \partial_m b_{(0)j}^i = 2b_{(2)j}^i \quad (4.11)$$

$$(D^2 h_{(0)})_j^i = 2g_{(2)j}^i + 2\delta_j^i \text{Tr} g_{(2)} - 8b_{(2)j}^i \quad (4.12)$$

where

$$(D^2 g_{(n)})_{ij} = \partial_i \partial_j (\text{Tr} g_{(n)}) + \partial^m \partial_m g_{(n)ij} - \left(\partial_i \partial_k g_{(n)j}^k + \partial_j \partial_k g_{(n)i}^k \right) \quad (4.13)$$

is the general form of the D^2 operator, introduced in ref. [16], because the covariant derivatives are with respect to the background Minkowski metric η_{ij} .

Evaluating the AdS₄ metric (4.1), (4.2), the different parts of eq. (2.27) can be expanded in the following form. Initially the determinant and the variation of the metric give

$$\sqrt{-h} = \frac{\sqrt{-\tilde{g}}}{\rho^{3/2}} = \frac{1}{\rho^{3/2}} \left(1 + \frac{1}{2} \text{Tr} c \right), \quad (4.14)$$

$$(h^{-1} \delta h)_j^i = (\tilde{g}^{-1} \delta \tilde{g})_j^i = (\eta^{im} - c^{im}) \delta c_{mj}, \quad (4.15)$$

while the coefficient of δK reads

$$\delta_{[k\ell]}^{[ij]} G_j^\ell = -3b_{(0)k}^i - 3\rho b_{(2)k}^i + 3\rho^{3/2} b_{(3)k}^i. \quad (4.16)$$

The remaining terms are the coefficients of the variation of the metric and adopt the form

$$\begin{aligned} \delta_{[k\ell]}^{[ij]} K_i^m G_j^\ell = & -3b_{(0)k}^m + 3\rho \left[2(b_{(0)} b_{(2)})_k^m - b_{(2)k}^m + (b_{(0)} g_{(2)})_k^m + (b_{(0)} b_{(2)})_k^m \log \rho \right] \\ & + 3\rho^{3/2} \left[b_{(3)k}^m + \frac{3}{2} (b_{(0)} g_{(3)})_k^m + \frac{3}{2} (b_{(0)} b_{(3)})_k^m \log \rho \right], \end{aligned}$$

and

$$\begin{aligned} \delta_{[k\ell]}^{[ij]} \frac{1}{N} \left(\nabla_\rho G_i^k - \nabla_i G_\rho^k \right) = & 3b_{(0)\ell}^j + 3\rho \left[-2(b_{(0)} b_{(2)})_\ell^j - b_{(2)\ell}^j - (b_{(0)} g_{(2)})_\ell^j + \delta_\ell^j (2\text{Tr} b_{(0)} b_{(2)} + \right. \\ & + \text{Tr} b_{(0)} g_{(2)} + \text{Tr} b_{(0)} b_{(2)} \log \rho) - (b_{(0)} b_{(2)})_\ell^j \log \rho \left. \right] + 3\rho^{3/2} \left[2b_{(3)\ell}^j + \right. \\ & + \frac{3}{2} \delta_\ell^j \text{Tr} b_{(0)} g_{(3)} - \frac{3}{2} (b_{(0)} g_{(3)})_\ell^j - \frac{3}{2} (b_{(0)} b_{(3)})_\ell^j \log \rho + \frac{3}{2} \text{Tr} b_{(0)} b_{(3)} \delta_\ell^j \log \rho \left. \right]. \end{aligned}$$

The third term in the second line of (2.27) vanishes in the linearized case. Here, the terms $b_{(0)} b_{(2)}$, $b_{(0)} g_{(2)}$, $b_{(0)} b_{(3)}$ and $b_{(0)} g_{(3)}$ are of order $\mathcal{O}(c^2)$. Demanding an action up to quadratic order in c_{ij} , eq. (2.27) adopts the form

$$\delta I_{\text{critical}} = \frac{1}{32\pi G} \int_{\partial \mathcal{M}} d^3 x \left(6\rho^{-3/2} b_{(0)ij} \delta b_{(0)}^{ij} - 6b_{(3)ij} \delta b_{(0)}^{ij} + 9b_{(3)ij} \log \rho \delta b_{(0)}^{ij} + 9b_{(3)ij} \delta h_{(0)}^{ij} \right). \quad (4.17)$$

Despite the fact that the $b_{(2)ij}$ contribution is divergent, the field equations (4.11) show that it is a total derivative, so that it can be dropped. From the above derivation, it is

evident that the variation of the action is not finite, due to the presence of a logarithmic term. This actually corresponds to a divergent logarithmic stress-energy tensor, as it the conjugate of the source $b_{(0)ij}$. In turn, the holographic response to the Einstein source $h_{(0)ij}$ is finite. Following standard holographic renormalization and the formulation of refs. [27, 28], we track these divergences at the level of the action.

Using the EOM (2.6), the eq. (2.15) can be written as

$$I_{\text{critical}} = -\frac{1}{32\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} G_{\nu}^{\mu} G_{\mu}^{\nu}. \quad (4.18)$$

After some algebraic manipulation and taking into account the linearized EOM, the square of the linearized Einstein tensor reads

$$G_{\nu}^{\mu} G_{\mu}^{\nu} = 9Trb_{(0)}^2 + 18\rho Trb_{(0)}b_{(2)} - 18\rho^{3/2}Trb_{(0)}b_{(3)}. \quad (4.19)$$

Putting a cutoff scale at radius $\rho = \varepsilon$, the action (4.18) can be cast in the form,

$$\begin{aligned} I_{\text{critical}} &= -\frac{1}{64\pi G} \int d^3x \int_{\rho=\varepsilon} d\rho \frac{\sqrt{-\tilde{g}}}{\rho^{3/2+1}} G_{\nu}^{\mu} G_{\mu}^{\nu} \\ &= -\frac{9}{64\pi G} \int d^3x \int_{\rho=\varepsilon} d\rho \frac{\sqrt{-\tilde{g}}}{\rho^{3/2+1}} \left(Trb_{(0)}^2 + 2\rho Trb_{(0)}b_{(2)} - 2\rho^{3/2}Trb_{(0)}b_{(3)} \right) \\ &= \frac{9}{32\pi G} \int_{\partial\mathcal{M}} d^3x \left(Trb_{(0)}b_{(3)} \log \varepsilon + \frac{1}{3}\varepsilon^{-3/2}Trb_{(0)}^2 + 2\varepsilon^{-1/2}Trb_{(0)}b_{(2)} \right). \end{aligned} \quad (4.20)$$

4.1 Counterterms

All terms tend to infinity at the conformal boundary ($\varepsilon = 0$). These divergences generate the infinities previously seen at the variation of the action (4.17). In order to render the action finite, proper counterterms have to be added. In the first place we invert the series as follows

$$b_{(0)ij} = \rho \partial_{\rho} c_{ij} - \rho \left(b_{(2)ij} + g_{(2)ij} + b_{(2)ij} \log \rho \right) - \rho^{3/2} \left(g_{(3)ij} + b_{(3)ij} \log \rho \right). \quad (4.21)$$

The combination

$$\begin{aligned} \frac{1}{3}\rho^{1/2}\partial_{\rho}c_{ij}\partial_{\rho}c^{ij} &= \frac{2}{3}Trb_{(0)}b_{(3)} + Trb_{(0)}g_{(3)} + Trb_{(0)}b_{(3)} \log \rho + \frac{1}{3}\rho^{-3/2}Trb_{(0)}^2 \\ &\quad + \frac{2}{3}\rho^{-1/2} \left(Trb_{(0)}b_{(2)} + Trb_{(0)}g_{(2)} + Trb_{(0)}b_{(2)} \log \rho \right), \end{aligned} \quad (4.22)$$

cancels the leading order logarithmic divergence of the action but introduces new infinities plus a finite contribution. Taking into account the eqs. (4.11), (4.12), the following term can also be written as

$$\begin{aligned} Trb_{(0)}g_{(2)} &= b_{(0)}^{ij}g_{(2)ij} = \frac{1}{2}b_{(0)}^{ij} \left(D^2h_{(0)} \right)_{ij} - Trb_{(0)}Trg_{(2)} + 4b_{(0)}^{ij}b_{(2)ij} \\ &= \frac{1}{2}b_{(0)}^{ij} \left(D^2h_{(0)} \right)_{ij} + 4Trb_{(0)}b_{(2)} \\ &= \frac{1}{2}h_{(0)}^{ij}\partial^m\partial_m b_{(0)ij} + 4Trb_{(0)}b_{(2)} \\ &= h_{(0)}^{ij}b_{(2)ij} + 4Trb_{(0)}b_{(2)} \\ &= Trh_{(0)}b_{(2)} + 4Trb_{(0)}b_{(2)}, \end{aligned} \quad (4.23)$$

where integration by parts was performed passing from the second to the third line. Moreover, inverting the series, one produces the expressions

$$c^{ij} \partial^m \partial_m \partial_\rho c_{ij} = 2\rho^{-1} (Tr h_{(0)} b_{(2)} + Tr b_{(0)} b_{(2)} \log \rho) + \mathcal{O}(\rho^0) , \quad (4.24)$$

$$\partial_\rho c^{ij} \partial^m \partial_m \partial_\rho c_{ij} = \rho^{-2} b_{(0)}^{ij} \partial^m \partial_m b_{(0)ij} + \mathcal{O}(\rho^{-1}) = 2\rho^{-2} Tr b_{(0)} b_{(2)} + \mathcal{O}(\rho^{-1}) . \quad (4.25)$$

There is a linear combination of the terms in eqs. (4.22)–(4.25), that cancels the divergences up to finite terms. More specifically, this can be written as

$$\begin{aligned} \frac{1}{3} \rho^{1/2} (\partial_\rho c_{ij} \partial_\rho c^{ij} - c^{ij} \partial^m \partial_m \partial_\rho c_{ij} - 2\rho \partial_\rho c^{ij} \partial^m \partial_m \partial_\rho c_{ij}) = \\ = Tr b_{(0)} b_{(3)} \log \rho + 2\rho^{-1/2} Tr b_{(0)} b_{(2)} + \frac{1}{3} \rho^{-3/2} Tr b_{(0)}^2 + \frac{2}{3} Tr b_{(0)} b_{(3)} + Tr b_{(0)} g_{(3)} . \end{aligned}$$

Hence, the counterterm action obtains the form

$$I_{ct} = -\frac{3}{32\pi G} \int_{\partial\mathcal{M}} d^3x \rho^{1/2} (\partial_\rho c_{ij} \partial_\rho c^{ij} - c^{ij} \partial^m \partial_m \partial_\rho c_{ij} - 2\rho \partial_\rho c^{ij} \partial^m \partial_m \partial_\rho c_{ij}) . \quad (4.26)$$

This expression can be covariantized after performing the proper rescaling of the metric and its perturbation. The respective metric field can be written as $h_{ij} = (\eta_{ij} + c_{ij}) / \rho$. The extrinsic curvature obtains the form $K_{ij} = \frac{1}{\rho} \eta_j^i - \kappa_{ij}$ where $\kappa_{ij} = \partial_\rho c_{ij}$. Hence, the fully covariant form of the counterterms can be cast in the following form

$$I_{ct} = \frac{3}{32\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{-h} \left(2K - K_{ij} K^{ij} - 3 + \frac{1}{N} K^{ij} D^m D_m K_{ij} - \frac{1}{2N} D^m D_m K \right) . \quad (4.27)$$

Our renormalized AdS action relies on the addition of extrinsic counterterms on top of a bulk topological invariant. This, in principle, provides a different starting point from the one proposed in ref. [16]. The difference stems from the use, in the latter reference, of a Dirichlet boundary conditions for the metric h_{ij} .

The variation of the counterterm gives

$$\begin{aligned} \delta I_{ct} &= -\frac{3}{32\pi G} \int_{\partial\mathcal{M}} d^3x \rho^{1/2} [-\partial^m \partial_m \partial_\rho c_{ij} \delta c^{ij} + (2\partial_\rho c_{ij} - \partial_m \partial^m c_{ij} - 4\rho \partial^m \partial_m \partial_\rho c_{ij}) \delta (\partial_\rho c^{ij})] \\ &= -\frac{3}{16\pi G} \int_{\partial\mathcal{M}} d^3x \rho^{1/2} \partial_\rho c_{ij} \delta (\partial_\rho c^{ij}) , \end{aligned} \quad (4.28)$$

where the rest of the terms have been dropped as total derivatives.

Thus, evaluating eq. (4.28) and adding on top of eq. (4.17), the variation of the total action $I_{\text{tot}} = I_{\text{critical}} + I_{ct}$ reads

$$\begin{aligned} \delta I_{\text{total}} &= \frac{1}{32\pi G} \int_{\partial\mathcal{M}} d^3x \left(6\rho^{-3/2} b_{(0)ij} \delta b_{(0)}^{ij} - 6b_{(3)ij} \delta b_{(0)}^{ij} + 9b_{(3)ij} \log \rho \delta b_{(0)}^{ij} + 9b_{(3)ij} \delta h_{(0)}^{ij} \right. \\ &\quad \left. - 6\rho^{-3/2} b_{(0)ij} \delta b_{(0)}^{ij} - 6b_{(3)ij} \delta b_{(0)}^{ij} - 9b_{(3)ij} \log \rho \delta b_{(0)}^{ij} - 9g_{(3)ij} \delta b_{(0)}^{ij} \right) \\ &= \frac{1}{32\pi G} \int_{\partial\mathcal{M}} d^3x \left(-12b_{(3)ij} \delta b_{(0)}^{ij} - 9g_{(3)ij} \delta b_{(0)}^{ij} + 9b_{(3)ij} \delta h_{(0)}^{ij} \right) . \end{aligned} \quad (4.29)$$

4.2 Holographic correlation functions

The functional derivatives of the sources are finite giving rise to holographic energy-momentum tensors. Hence, the one-point functions around a flat background are given by

$$\langle T_{ij} \rangle = 2 \frac{\delta I_{\text{total}}}{\delta h_{(0)}^{ij}} = \frac{9}{16\pi G} b_{(3)ij}, \quad (4.30)$$

what is the holographic dual to the Einstein source $h_{(0)ij}$, and

$$\langle t_{ij} \rangle = 2 \frac{\delta I_{\text{total}}}{\delta b_{(0)}^{ij}} = -\frac{3}{16\pi G} (4b_{(3)ij} + 3g_{(3)ij}), \quad (4.31)$$

is the dual to the logarithmic source $b_{(0)ij}$.

Following the AdS/CFT dictionary, the variation of these correlators with respect to the sources provide the two-point correlation functions. As a result, they read

$$\langle T_{ij}(x) T_{kl}(x') \rangle = -2i \frac{\delta}{\delta h_{(0)}^{kl}(x')} \langle T_{ij}(x) \rangle = -\frac{9i}{8\pi G} \frac{\delta b_{(3)ij}(x)}{\delta h_{(0)}^{kl}(x')} = 0 \quad (4.32)$$

$$\begin{aligned} \langle T_{ij}(x) t_{kl}(x') \rangle &= -2i \frac{\delta}{\delta b_{(0)}^{kl}(x')} \langle T_{ij}(x) \rangle = -2i \frac{\delta}{\delta h_{(0)}^{kl}(x')} \langle t_{ij}(x) \rangle \\ &= -\frac{9i}{8\pi G} \frac{\delta b_{(3)ij}(x)}{\delta b_{(0)}^{kl}(x')} = \frac{9i}{8\pi G} \frac{\delta g_{(3)ij}(x)}{\delta h_{(0)}^{kl}(x')} \end{aligned} \quad (4.33)$$

$$\langle t_{ij}(x) t_{kl}(x') \rangle = -2i \frac{\delta}{\delta b_{(0)}^{kl}(x')} \langle t_{ij}(x) \rangle = \frac{3i}{8\pi G} \left(4 \frac{\delta b_{(3)ij}(x)}{\delta b_{(0)}^{kl}(x')} + 3 \frac{\delta g_{(3)ij}(x)}{\delta b_{(0)}^{kl}(x')} \right) \quad (4.34)$$

The i factor in the two-point functions comes from the generating functional when written in Lorentzian signature. More precisely, in the context of AdS/CFT correspondence a relation between the generating functional and the on-shell action of the type $W_L \sim iI_L$. This choice yields the formulas displayed above [28].

One can notice that the norm of the stress-energy tensor is zero, as expected in a LCFT. This is an immediate consequence of the fact that no Einstein mode can source a logarithmic mode. The latter assertion was remarked in ref. [16], where the mode analysis allows to calculate the aforementioned functional derivatives.

Actually, the EOM (4.3)–(4.9) show that b_3 is a transverse and traceless mode while g_3 it is just traceless. The latter is a consequence of the presence of non-Einstein modes in the theory, whereas in Einstein Gravity $g_{(3)ij}$ is both transverse and traceless and determines the holographic stress-energy tensor. This property leads to the York decompositions of the $g_{(3)}$ mode, which reads:

$$g_{(3)ij} = \nabla_i V_j^{(3)} + \nabla_j V_i^{(3)} + g_{(3)ij}^{TT} + \left(\nabla_i \nabla_j - \frac{1}{3} \eta_{ij} \nabla^2 \right) S^{(3)}. \quad (4.35)$$

Each one of the terms contribute independently to different pieces of the one-point function t_{ij} which now consists of: i) a transverse vector V_i , ii) a transverse traceless part t_{ij}^{TT} , which is the logarithmic conjugate of T_{ij} , iii) and a scalar S .

Consequently, from eq. (4.31) we get the following three operators:

$$\langle t_{ij}^{TT} \rangle = -\frac{3}{16\pi G} \left(4b_{(3)ij} + 3g_{(3)ij}^{TT} \right) \quad (4.36)$$

$$\langle V_i \rangle = -\frac{9}{16\pi G} V_i^{(3)} \quad (4.37)$$

$$\langle S \rangle = -\frac{9}{16\pi G} S^{(3)}. \quad (4.38)$$

These values are justified considering that the vector and the scalar operator contributions come explicitly from $g_{(3)ij}$, while the logarithmic stress-energy tensor t_{ij}^{TT} is sourced by both parts.

Considering that $b_{(3)ij}$ is transverse and traceless leads to only one non-vanishing mixed correlator in eq. (4.33), the one between the two stress-energy tensors. Given the mode dependence on the sources in [16], we obtain that

$$\langle T_{ij}(x) t_{kl}^{TT}(0) \rangle = -\frac{1}{2\pi^3} \frac{3}{2G} \hat{\Delta}_{ij,kl} \frac{1}{|x^2|}, \quad (4.39)$$

where

$$\hat{\Delta}_{ij,kl} = \frac{1}{2} \left(\hat{\Theta}_{ik} \hat{\Theta}_{jl} + \hat{\Theta}_{il} \hat{\Theta}_{jk} - \hat{\Theta}_{ij} \hat{\Theta}_{kl} \right) \quad (4.40)$$

$$\hat{\Theta}_{ij} = \partial_i \partial_j - \eta_{ij} \square. \quad (4.41)$$

Finally, from (4.34) we get three different correlators, each one corresponding to the vector, scalar and transverse traceless operators. The former ones obtain contributions only from the $g_{(3)ij}$ functional derivatives. Hence:

$$\begin{aligned} \langle V_i(x) V_j(0) \rangle &= \frac{9i}{8\pi G} \left(\frac{\delta g_{(3)ij}(x)}{\delta b_{(0)}^{kl}(0)} \right)_V = \frac{1}{2\pi^3} \frac{9i}{8\pi G} \int d^3p e^{ipx} \left(\frac{\delta g_{(3)ij}(p)}{\delta b_{(0)}^{kl}(0)} \right)_V \\ &= -\frac{1}{2\pi^3} \frac{1}{4G} \hat{\Theta}_{ij} \frac{1}{|x^2|} \end{aligned} \quad (4.42)$$

and

$$\begin{aligned} \langle S(x) S(0) \rangle &= \frac{9i}{8\pi G} \left(\frac{\delta g_{(3)ij}(x)}{\delta b_{(0)}^{kl}(0)} \right)_S = \frac{1}{2\pi^3} \frac{9i}{8\pi G} \int d^3p e^{ipx} \left(\frac{\delta g_{(3)ij}(p)}{\delta b_{(0)}^{kl}(0)} \right)_S \\ &= \frac{1}{2\pi^3} \frac{3}{8G} \frac{1}{|x^2|}. \end{aligned} \quad (4.43)$$

The latter two-point function is the one corresponding to the logarithmic stress-energy tensors. It receives contributions from the transverse traceless part of both $b_{(3)i}$ and $g_{(3)}$. One may rewrite eq. (4.34) as

$$\langle t_{ij}^{TT}(x) t_{kl}^{TT}(x') \rangle = -\frac{4}{3} \langle T_{ij}(x) t_{kl}^{TT}(x') \rangle + \frac{9i}{8\pi G} \left(\frac{\delta g_{(3)ij}(x)}{\delta b_{(0)}^{kl}(x')} \right)_{TT}. \quad (4.44)$$

In this case we obtain that

$$\begin{aligned}\langle t_{ij}^{TT}(x) t_{kl}^{TT}(0) \rangle &= -\frac{4}{3} \langle T_{ij}(x) t_{kl}^{TT}(0) \rangle + \frac{1}{2\pi^3} \frac{9i}{8\pi G} \int d^3p e^{ipx} \left(\frac{\delta g_{(3)ij}}{\delta b_{(0)}^{kl}}(p) \right)_{TT} \\ &= -\frac{1}{2\pi^3} \frac{3}{2G} \hat{\Delta}_{ij,kl} \frac{\log|x^2| + C + 4\gamma - 4/3}{|x^2|},\end{aligned}\quad (4.45)$$

where C is a numerical constant. In general, the logarithmic stress tensor is defined up to the addition of a multiple of $\langle T_{ij} \rangle$. Therefore, taking advantage of this freedom, we redefine t_{ij}^{TT} as $t_{ij}^{TT} \rightarrow -(C/4 + \gamma - 1/3) T_{ij}$, canceling all the numerical constants appearing in the numerator. Hence, we obtain that

$$\langle t_{ij}^{TT}(x) t_{kl}^{TT}(0) \rangle = -\frac{1}{2\pi^3} \frac{3}{2G} \hat{\Delta}_{ij,kl} \frac{\log|x^2|}{|x^2|}. \quad (4.46)$$

5 Conclusions

In the present paper, we have computed holographic correlation functions in Einstein-Weyl gravity at the critical point. We have applied holographic techniques to an equivalent form of the Critical Gravity action, given by eq. (2.2), where the curvature-squared part are expressed as the difference between the Weyl² and the GB terms. The GB term, with its coupling fixed by the above argument, provides partial renormalization of the variation of the action, such that the divergent pieces can be attributed to the non-Einstein part in the curvature (Bach tensor).

In turn, for Einstein modes, both the action and its variation are not only finite, but identically zero [10, 17]. The vanishing of the holographic stress tensor for Einstein spacetimes [8], together with a zero mass and entropy for Einstein black holes, indicates that Critical Gravity turns somehow trivial within that sector.

Additional counterterms, which depend on the extrinsic curvature and its covariant derivatives, are needed when the logarithmic source is switched on at the boundary. The departure from the Einstein condition by including log terms in the metric modifies the asymptotic form of the Riemann tensor at leading order.

The addition of these terms makes the action principle not suitable for imposing a Dirichlet boundary condition in h_{ij} . However, variation of the action is finite and written down in terms of variations of $h_{(0)ij}$ and $b_{(0)ij}$. In other words, the counterterms in (4.27) provide a well-posed variational principle by fixing the holographic sources on the conformal boundary. In this sense, our boundary conditions are compatible with the holographic description of AdS gravity with relaxed asymptotic behavior.

In Einstein gravity with standard AdS boundary conditions, the fall-off of the curvature tensor determines the coupling of the GB term in eq. (2.3). The locally equivalent boundary term to the GB invariant is the 2nd Chern form, which is a given polynomial of the extrinsic and intrinsic curvatures. Intrinsic counterterms presented in refs. [21, 22] are worked out as a truncation of the series coming from taking a FG expansion on the extrinsic curvature [20].

A similar comparison, this time, between the counterterms in eq. (4.27) and the ones presented in ref. [16] might be worked out adding and subtracting the corresponding

generalized Gibbons-Hawking term for Critical Gravity as a higher-derivative theory

$$I_{GGH} = \frac{1}{2\kappa^2} \int_{\partial\mathcal{M}} d^3x F^{ij} (K\gamma_{ij} - K_{ij}) . \quad (5.1)$$

On the other hand, and because $b_{(0)}$ is neither a parameter of the theory nor a covariant field in the Lagrangian, there is no direct way to fine tune the GB coupling to incorporate the information on the modified asymptotic curvature. Therefore, it remains as an open problem how to mimic the effect of Topological Regularization in presence of a log boundary source.

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References

- [1] M.H. Goroff and A. Sagnotti, *The Ultraviolet Behavior of Einstein Gravity*, *Nucl. Phys. B* **266** (1986) 709 [[INSPIRE](#)].
- [2] K.S. Stelle, *Renormalization of Higher Derivative Quantum Gravity*, *Phys. Rev. D* **16** (1977) 953 [[INSPIRE](#)].
- [3] S.L. Adler, *Einstein Gravity as a Symmetry Breaking Effect in Quantum Field Theory*, *Rev. Mod. Phys.* **54** (1982) 729 [Erratum *ibid.* **55** (1983) 837] [[INSPIRE](#)].
- [4] K.S. Stelle, *Classical Gravity with Higher Derivatives*, *Gen. Rel. Grav.* **9** (1978) 353 [[INSPIRE](#)].
- [5] E.A. Bergshoeff, O. Hohm and P.K. Townsend, *Massive Gravity in Three Dimensions*, *Phys. Rev. Lett.* **102** (2009) 201301 [[arXiv:0901.1766](#)] [[INSPIRE](#)].
- [6] S. Deser and B. Tekin, *Massive, topologically massive, models*, *Class. Quant. Grav.* **19** (2002) L97 [[hep-th/0203273](#)] [[INSPIRE](#)].
- [7] H. Lü and C.N. Pope, *Critical Gravity in Four Dimensions*, *Phys. Rev. Lett.* **106** (2011) 181302 [[arXiv:1101.1971](#)] [[INSPIRE](#)].
- [8] G. Anastasiou, R. Olea and D. Rivera-Betancour, *Noether-Wald energy in Critical Gravity*, [arXiv:1707.00341](#) [[INSPIRE](#)].
- [9] J. Maldacena, *Einstein Gravity from Conformal Gravity*, [arXiv:1105.5632](#) [[INSPIRE](#)].
- [10] G. Anastasiou and R. Olea, *From conformal to Einstein Gravity*, *Phys. Rev. D* **94** (2016) 086008 [[arXiv:1608.07826](#)] [[INSPIRE](#)].

- [11] M. Alishahiha and R. Fareghbal, *D-Dimensional Log Gravity*, *Phys. Rev. D* **83** (2011) 084052 [[arXiv:1101.5891](#)] [[INSPIRE](#)].
- [12] E.A. Bergshoeff, O. Hohm, J. Rosseel and P.K. Townsend, *Modes of Log Gravity*, *Phys. Rev. D* **83** (2011) 104038 [[arXiv:1102.4091](#)] [[INSPIRE](#)].
- [13] I. Gullu, M. Gurses, T.C. Sisman and B. Tekin, *AdS Waves as Exact Solutions to Quadratic Gravity*, *Phys. Rev. D* **83** (2011) 084015 [[arXiv:1102.1921](#)] [[INSPIRE](#)].
- [14] V. Gurarie, *Logarithmic operators in conformal field theory*, *Nucl. Phys. B* **410** (1993) 535 [[hep-th/9303160](#)] [[INSPIRE](#)].
- [15] J.S. Caux, I.I. Kogan and A.M. Tsvelik, *Logarithmic operators and hidden continuous symmetry in critical disordered models*, *Nucl. Phys. B* **466** (1996) 444 [[hep-th/9511134](#)] [[INSPIRE](#)].
- [16] N. Johansson, A. Naseh and T. Zojer, *Holographic two-point functions for 4d log-gravity*, *JHEP* **09** (2012) 114 [[arXiv:1205.5804](#)] [[INSPIRE](#)].
- [17] O. Mišković, R. Olea and M. Tsoukalas, *Renormalized AdS action and Critical Gravity*, *JHEP* **08** (2014) 108 [[arXiv:1404.5993](#)] [[INSPIRE](#)].
- [18] S.W. MacDowell and F. Mansouri, *Unified Geometric Theory of Gravity and Supergravity*, *Phys. Rev. Lett.* **38** (1977) 739 [Erratum *ibid.* **38** (1977) 1376] [[INSPIRE](#)].
- [19] R. Olea, *Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes*, *JHEP* **06** (2005) 023 [[hep-th/0504233](#)] [[INSPIRE](#)].
- [20] O. Mišković and R. Olea, *Topological regularization and self-duality in four-dimensional anti-de Sitter gravity*, *Phys. Rev. D* **79** (2009) 124020 [[arXiv:0902.2082](#)] [[INSPIRE](#)].
- [21] V. Balasubramanian and P. Kraus, *A Stress tensor for Anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [[hep-th/9902121](#)] [[INSPIRE](#)].
- [22] R. Emparan, C.V. Johnson and R.C. Myers, *Surface terms as counterterms in the AdS/CFT correspondence*, *Phys. Rev. D* **60** (1999) 104001 [[hep-th/9903238](#)] [[INSPIRE](#)].
- [23] H. Lü, Y. Pang and C.N. Pope, *Conformal Gravity and Extensions of Critical Gravity*, *Phys. Rev. D* **84** (2011) 064001 [[arXiv:1106.4657](#)] [[INSPIRE](#)].
- [24] M. Porrati and M.M. Roberts, *Ghosts of Critical Gravity*, *Phys. Rev. D* **84** (2011) 024013 [[arXiv:1104.0674](#)] [[INSPIRE](#)].
- [25] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [26] D. Grumiller and N. Johansson, *Consistent boundary conditions for cosmological topologically massive gravity at the chiral point*, *Int. J. Mod. Phys. D* **17** (2009) 2367 [[arXiv:0808.2575](#)] [[INSPIRE](#)].
- [27] M. Alishahiha and A. Naseh, *Holographic renormalization of new massive gravity*, *Phys. Rev. D* **82** (2010) 104043 [[arXiv:1005.1544](#)] [[INSPIRE](#)].
- [28] K. Skenderis, M. Taylor and B.C. van Rees, *Topologically Massive Gravity and the AdS/CFT Correspondence*, *JHEP* **09** (2009) 045 [[arXiv:0906.4926](#)] [[INSPIRE](#)].