

Light-like scattering in quantum gravity

N.E.J. Bjerrum-Bohr,^a John F. Donoghue,^b Barry R. Holstein,^{b,c} Ludovic Planté^d and Pierre Vanhove^d

^a*Niels Bohr International Academy & Discovery Center, Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, Copenhagen Ø, DK-2100 Denmark*

^b*Department of Physics-LGRT, University of Massachusetts, Amherst, MA, 01003 U.S.A.*

^c*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA, 93016 U.S.A.*

^d*CEA, DSM, Institut de Physique Théorique, IPhT, CNRS MPPU, URA2306, Saclay, Gif-sur-Yvette, F-91191 France*

E-mail: bjbohr@nbi.dk, donoghue@physics.umass.edu,
holstein@physics.umass.edu, ludovic.plante@cea.fr,
pierre.vanhove@cea.fr

ABSTRACT: We consider scattering in quantum gravity and derive long-range classical and quantum contributions to the scattering of light-like bosons and fermions (spin-0, spin- $\frac{1}{2}$, spin-1) from an external massive scalar field, such as the Sun or a black hole. This is achieved by treating general relativity as an effective field theory and identifying the non-analytic pieces of the one-loop gravitational scattering amplitude. It is emphasized throughout the paper how modern amplitude techniques, involving spinor-helicity variables, unitarity, and squaring relations in gravity enable much simplified computations. We directly verify, as predicted by general relativity, that all classical effects in our computation are universal (in the context of matter type and statistics). Using an eikonal procedure we confirm the post-Newtonian general relativity correction for light-like bending around large stellar objects. We also comment on treating effects from quantum \hbar dependent terms using the same eikonal method.

KEYWORDS: Classical Theories of Gravity, Effective field theories, Scattering Amplitudes

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1 Introduction

The possible existence of a quantum field theoretical framework for general relativity valid at all energy scales is clearly a fundamental question, and since the original formulation of quantum field theory, a technique by which general relativity and quantum mechanics can be combined has been sought [1–7]. Though such a theory has yet to be found, today we can address profound practical and reliable (low energy) consequences of the (currently unknown) underlying quantum theory through the modern viewpoint of effective field theory (EFT) [8–23]. The EFT framework allows direct exploration of various quantitative phenomenological applications, see for example [24, 25]. In addition, the recent detection of gravitational waves GW150914 by the LIGO experiment is an important test of general relativity [26] and opens up exciting prospects for testing low-energy effective theories of gravity [27]. In the analysis below we will be seeking the classical and quantum long-range (power law falloff) corrections to the familiar $1/r$ Newtonian potential describing the gravitational interaction between two systems. The effective potential describing this interaction is defined as the Fourier transform of the gravitational scattering amplitude

and $1/r^n$, $n \geq 2$ behavior can only arise from non-analytic components associated with quantized graviton loop effects. Analytic pieces lead only to short-distance (delta function and its derivatives) behavior and can be dropped when only concerned with long range physics.

In this paper we focus on providing further details on the effective field theory computation of light-like scattering in quantum gravity. In particular we will extend our previous results for light-like scattering from bosons to fermions. Thus we can now address massless gravitational neutrino scattering in one-loop quantum gravity. The outline of our presentation will be as follows: we first discuss the framework for our computation; especially we will show how modern computational techniques, unitarity and spinor-helicity are important inputs for streamlining the computations. Next we will present details of the calculation, and finally discuss how to interpret our results.

2 General relativity as an effective field theory and one-loop amplitudes

Including gravitational interactions in particle physics models is a straightforward exercise employing ideas from effective field theory. The starting point is the gravitational effective field theory action

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{2}{\kappa^2} \mathcal{R} + S_{\text{model}} + S_{\text{EF}} \right], \quad (2.1)$$

where \mathcal{R} is the scalar curvature and $g_{\mu\nu}$ the metric. One can write the metric as $\eta_{\mu\nu} + \kappa h_{\mu\nu}$, with $\kappa^2 = 32\pi G_N/c^4$, where G_N is Newton's constant and $h_{\mu\nu}$ is the quantized gravitational field. Expanding all terms in $h_{\mu\nu}$, \mathcal{R} contains the propagator for the gravitational field as well as all the pure gravitational vertices. Interaction with matter is contained in the term S_{model} where the flat space Lagrangian for a given particle physics model is made generally covariant by replacing flat space derivatives with general covariant derivatives expanded in powers of $\kappa h_{\mu\nu}$ (see e.g. refs. [28, 29] for details regarding such expansions). Finally S_{EF} contains an infinite series of higher derivative operators (basically any operator allowed by general covariance) associated with new gravitational couplings, and ensures that, order by order in the energy expansion, *any* UV divergence due to loop effects [30] can be absorbed in the effective action. In this way the construction, albeit “effective”, is UV consistent up to the cut-off determined by the validity of the energy expansion, typically $\mathcal{O}(m_{\text{Planck}})$, where $m_{\text{Planck}} = \sqrt{\frac{\hbar c}{G}} \sim 10^{19}$ GeV is the Planck mass.

Having constructed the effective action, we have now, in principle, a straightforward path by which to derive transition amplitudes. The action term corresponding to the matter coupling S_{model} can take different forms depending on the specific theory we wish to study. We will, in this presentation, need only the minimal couplings of gravitons to scalars, photons and massless fermions.

The action for a massless scalar φ or massive scalar field Φ of mass M is

$$\mathcal{S}_{\text{scalar}} = \int d^4x \sqrt{-g} \left(-\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} \left((\partial_\mu \Phi)^2 - M^2 \Phi^2 \right) \right), \quad (2.2)$$

while the coupling to a massless spin- $\frac{1}{2}$ fermion is given by

$$\mathcal{S}_{\text{fermion}} = \frac{i}{2} \int d^4x \sqrt{-g} \bar{\chi} \not{D} \chi, \quad (2.3)$$

where $\not{D} = \gamma^\mu (\partial_\mu + \frac{i}{4} \omega_\mu^{ab} \gamma^{ab})$, ω_μ^{ab} is the spin connection, and γ^μ are 4×4 Dirac matrices with $\gamma^{ab} = \frac{1}{2} [\gamma^a, \gamma^b]$ the Lorentz generator. Finally, the coupling to an Abelian spin-1 massless field is given by

$$\mathcal{S}_{\text{QED}} = -\frac{1}{4} \int d^4x \sqrt{-g} (\nabla_\mu A_\nu - \nabla_\nu A_\mu)^2, \quad (2.4)$$

where $\nabla_\mu A^\nu := \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda$ and $\Gamma^\lambda_{\mu\nu} := \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$. A full list of propagators and vertices needed for the Feynman graphs computation can, e.g., be found in [17, 28, 29, 31].¹

Having an effective action, the traditional way to proceed is to simply work out all necessary Feynman rules to a particular loop order and then generate amplitudes perturbatively using the standard off-shell formalism. This, for instance, was the path taken in refs. [11–13]. However, at high orders such an approach is clearly not very practical, since it quickly leads to very unmanageable computations, in part because of vast off-shell vertices and the occurrence of tensor contractions everywhere. It is thus natural to take advantage of *any* available simplification and the on-shell techniques provided by the compact formalism of spinor-helicity and unitarity seems particularly ideal in this regard [32–34]. Also, importantly, recent progress in the computation of gluon and QCD amplitudes (see e.g. [35–38]) can be adapted to gravity [39, 40] using the Kawai-Lewellen-Tye (KLT) string theory relations [41, 42]. Using these methods, the only required input for effective field theory computations is that of compact on-shell tree amplitudes, since loop amplitudes can be written in terms of trees by the use of unitarity as the central consistency requirement.

To illustrate how this program is carried out in practice, we follow the procedure outlined in ref. [21] where we considered two-graviton-exchange between massive scalar sources. The only difference between that approach and the present one is that we here extend the analysis to two-graviton-exchange between a massless field X with spin S and a massive scalar field Φ with spin $S = 0$ and mass M . Given the necessary tree amplitudes, we compute the discontinuity across the two-particle t -channel cut via

$$\begin{aligned} \mathcal{M}_X^{(2)}(p_1, p_2, p_3, p_4) \Big|_{\text{disc}} &:= \frac{1}{2! i} \mu^{2\epsilon} \int d\text{LIPS}(\ell_1, -\ell_2) (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \\ &\times \sum_{\lambda_1, \lambda_2} \mathcal{M}_{X^2 G^2}^{(1)}(p_1, \ell_1, p_2 - \ell_2) \times \mathcal{M}_{\phi^2 G^2}^{(1)}(p_3, \ell_2, p_4, -\ell_1)^\dagger \end{aligned} \quad (2.5)$$

where $d\text{LIPS}(\ell_1, -\ell_2) = d^4\ell_1 d^4\ell_2 \delta^{(+)}(\ell_1^2) \delta^{(+)}(\ell_2^2) \delta(p_1 + p_2 + \ell_1 - \ell_2)$. We follow here the notation of ref. [20] and will everywhere employ $D = 4 - 2\epsilon$, defining $q := p_1 + p_2 = -p_3 - p_4 = \ell_2 - \ell_1$, and $p_3^2 := p_4^2 := M^2$. We use the mostly minus metric convention $(+, -, -, -)$. The Mandelstam variables are $t := q^2$, $s = (p_1 + p_4)^2$ and $u = (p_1 + p_3)^2$.

¹This was studied in an unpublished Ph.D. Thesis, but the analysis contains some mistakes and in addition our interpretation differs significantly.

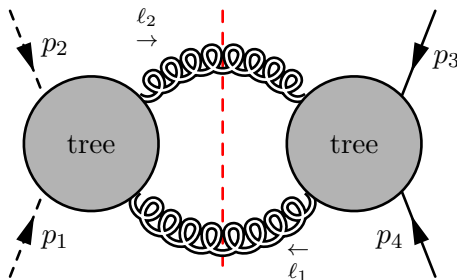


Figure 1. The one-loop scattering of one massless scalars (dotted line) and one massive scalars (solid line) mediated by a graviton (curly line). The discontinuity cut is represented by the red line.

Here $\mathcal{M}_{X^2G^2}^{(1)}$ is the gravitational Compton amplitude analyzed in [21] (see also appendix A) and the summation is over all possible helicity configurations λ_1 and λ_2 across the cut. The two-particle cut can be pictorially represented as in figure 1.

On the cut lines the gravitons are on-shell so that we have the constraint $\ell_1^2 = 0$, $\ell_2^2 = 0$. The discontinuity is given by the sum of four box integrals with the same numerator factor (see appendix A)

$$\mathcal{M}_{\varphi}^{(2)}(p_1, p_2, p_3, p_4) = -\frac{\kappa^4}{32t^2} \sum_{i=1}^2 \sum_{j=3}^4 \int \frac{d^D \ell \mu^{2\epsilon}}{(2\pi)^D} \frac{\mathcal{N}^S}{\ell_1^2 \ell_2^2 (p_i \cdot \ell_1) (p_j \cdot \ell_1)}. \quad (2.6)$$

With this construction one captures all the t -channel massless thresholds, which are the only terms (the non-analytic ones) of interest to us. Notice that the structure of the cut is very similar to that evaluated in [20]. The numerator \mathcal{N}^S receives contributions from the singlet graviton cut (i.e. where the helicities of the two cut gravitons are identical) as well as from the non-singlet graviton cut (i.e. where the helicities of the two cut gravitons are opposite).

- In the case of the massless external field, the properties of the gravitational Compton amplitudes (reviewed in appendix A.2) imply that

$$\mathcal{N}_{\text{singlet}}^S = 0, \quad \text{for } S = 0, \frac{1}{2}, 1. \quad (2.7)$$

- The non-singlet cut for the massless scalar $S = 0$ can be obtained by applying the equations (III.24) and (III.27) of [20] with $m_1 = 0$ and $m_2 = M$, yielding (we refer to the appendix A.1 for conventions and notations)²

$$\mathcal{N}_{\text{non-singlet}}^0 = \frac{1}{2} \left[(\text{tr}_-(\ell_1 p_1 \ell_2 p_3))^4 + (\text{tr}_-(\ell_2 p_1 \ell_1 p_3))^4 \right]. \quad (2.8)$$

- For the photon, we denote the non-singlet cut by $\mathcal{N}_{\text{non-singlet}}^{1h_1 h_2}$, where the polarization of the incoming photon is h_1 and the polarization of the outgoing photon is $-h_2$. The

²The computations performed in this work only involve parity even contributions, therefore the four-dimension Levi-Civita epsilon tensor $\varepsilon^{\mu\nu\rho\sigma}$ will never appear and there will be no issue in evaluating of the one-loop amplitude using dimensional regularisation.

only non-vanishing amplitudes are those preserving photon helicity

$$\mathcal{N}_{\text{non-singlet}}^{1+-} = \frac{(\text{tr}_-(\ell_2 p_1 \ell_1 p_3) \text{tr}_+(\ell_2 p_3 \ell_1 p_1 p_3 p_2))^2 + (\ell_1 \leftrightarrow \ell_2)}{\langle p_1 | p_3 | p_2 \rangle^2}, \quad (2.9)$$

and $(\mathcal{N}_{\text{non-singlet}}^{1+-})^* = \mathcal{N}_{\text{non-singlet}}^{1+-}$. Remarking that for the four-point amplitude the $\epsilon^{\mu\nu\rho\sigma}$ terms do not contribute, we conclude that

$$\langle p_1 | p_3 | p_2 \rangle^2 \mathcal{N}^{1+-} = \langle p_2 | p_3 | p_1 \rangle^2 \mathcal{N}^{1-+} = 2 \Re \left[(\text{tr}_-(\ell_2 p_1 \ell_1 p_3) \text{tr}_+(\ell_2 p_3 \ell_1 p_1 p_3 p_2))^2 \right]. \quad (2.10)$$

- For the massless fermion the non-singlet cut is non-vanishing as well only for the helicity conserving case, and we have

$$\mathcal{N}_{\text{non-singlet}}^{\frac{1}{2}+-} = \frac{(\text{tr}_-(\ell_1 p_1 \ell_2 p_3)^3 \text{tr}_+(p_1 p_3 p_2 \ell_1 p_3 \ell_2)) - (\ell_1 \leftrightarrow \ell_2)}{\langle p_2 | p_3 | p_1 \rangle}, \quad (2.11)$$

and $(\mathcal{N}_{\text{non-singlet}}^{\frac{1}{2}+-})^* = -\mathcal{N}_{\text{non-singlet}}^{\frac{1}{2}+-}$. The polarization of the external state appears only in the numerator, which takes the form

$$\langle p_2 | p_3 | p_1 \rangle \mathcal{N}_{\text{non-singlet}}^{\frac{1}{2}+-} = \langle p_1 | p_3 | p_2 \rangle \mathcal{N}_{\text{non-singlet}}^{\frac{1}{2}-+} = 2i \Im \left[\text{tr}_-(\ell_1 p_1 \ell_2 p_3)^3 \text{tr}_+(p_1 p_3 p_2 \ell_1 p_3 \ell_2) \right], \quad (2.12)$$

and, by multiplying by an appropriate factor, one can remove this polarization dependence. Therefore, for the photon, we can define the coefficients as coming from the expansion of

$$\mathcal{M}_\gamma^{(2)} := \langle p_1 | p_3 | p_2 \rangle^2 \mathcal{M}_\gamma^{+- (2)} = \langle p_2 | p_3 | p_1 \rangle^2 \mathcal{M}_\gamma^{-+ (2)}, \quad (2.13)$$

and similarly, for the fermion amplitude we define the coefficients from the polarization-stripped expression

$$\mathcal{M}_\chi^{(2)} := \langle p_1 | p_3 | p_2 \rangle \mathcal{M}_\chi^{+- (2)} = \langle p_2 | p_3 | p_1 \rangle \mathcal{M}_\chi^{-+ (2)}. \quad (2.14)$$

Performing the tensor integral reductions, the amplitude can be decomposed in terms of integral functions containing the two-massless-particle t -channel cut

$$\begin{aligned} \kappa^{-4} \mathcal{M}_\chi^{(2)}(p_1, p_2, p_3, p_4) \Big|_{t\text{-cut}} &= bo^S(t, s) I_4(t, s) + bo^S(t, u) I_4(t, u) \\ &\quad + t_{12}^S(t) I_3(t, 0) + t_{34}^S(t) I_3(t, M^2) + bu^S(t, 0) I_2(t, 0), \end{aligned} \quad (2.15)$$

where $I_4(t, s)$ and $I_4(t, u)$ are scalar box integrals given in eq. (B.1), $I_3(t)$ is the massless triangle integral of eq. (B.3), $I_3(t, M^2)$ the massive triangle integral of eq. (B.5), and $I_2(t)$ is the massless scalar bubble integral given in eq. (B.4). Explicit expressions for these integrals can be found in appendix B. The *full* integral reduction gives, in addition, massive bubbles, tadpoles, and as well as rational pieces that are (restricting to four dimensions) *not* contained in the massless t -cut. These terms are analytic in t and are not of interest to our analysis.

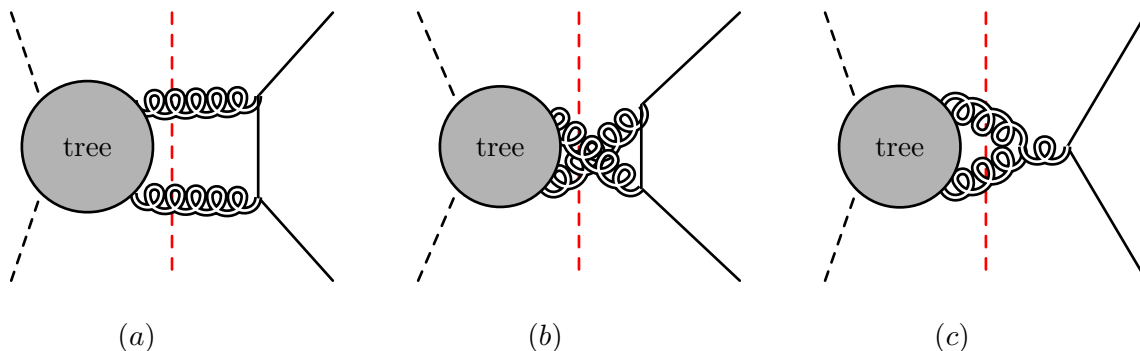


Figure 2. The monodromy BCJ relations in the cut (in red) that link the one-loop integral coefficients.

The massless triangle coefficient $t_{12}(t)$ is related to the coefficients of the box $bo^S(t, s)$ and crossed-box $bo^S(t, u)$ by

$$\frac{bo^S(t, s)}{M^2 - s} + \frac{bo^S(t, u)}{M^2 - u} = t_{12}^S(t), \quad \text{for } S = 0, \frac{1}{2}, 1 \quad (2.16)$$

and this universal identity, which is a consequence of the monodromy BCJ relations [44] between the four point tree-level amplitude in the two-particle cut as depicted in figure 2, is a very useful check on computations. The consequence of the monodromy BCJ relations for one-loop integral coefficients have been studied in [45] and [46], while a string theory based systematic derivation of these relations was given in [47].

3 The one-loop integral coefficients

We now provide explicit expressions for the integral coefficients of the one-loop amplitudes in eq. (2.15) for the massless scalar $X = \varphi$, the photon $X = \gamma$, and the massless fermion $X = \chi$.

The box coefficients are given by

- for the scalar

$$bo^\varphi(t, s) = \frac{1}{4} (M^2 - s)^4, \quad (3.1)$$

and with u replacing s for the coefficient $bo^\varphi(t, u)$ of the cross box.

- for the photon

$$bo^\gamma(t, s) = \frac{(M^2 - s)^2}{8} (2M^8 - 8M^6s + 2M^4s(t + 6s) - 4M^2s^2(t + 2s) + s^2(t^2 + 2ts + 2s^2)), \quad (3.2)$$

and with u replacing s for the coefficient $bo^\gamma(t, u)$ of the cross box.

- for the fermion

$$bo^\chi(t, s) = \frac{(M^2 - s)^3}{8} (2M^4 - 4M^2s + s(t + 2s)), \quad (3.3)$$

and with u replacing s for the coefficient $bo^\chi(t, u)$ of the cross box.

The massless triangle coefficients are given by

- for the scalar

$$t_{12}^{\varphi}(t, s) = \frac{1}{4}(M^2 - s)^3 + \frac{1}{4}(M^2 - u)^3, \quad (3.4)$$

- for the photon

$$t_{12}^{\gamma}(t, s) = \frac{t}{8} \left(6M^8 - 2M^6(5t + 12s) + 2M^4(5t^2 + 16ts + 18s^2) - M^2(5t^3 + 20t^2s + 34ts^2 + 24s^3) + (t^2 + 2ts + 2s^2)(t^2 + 3ts + 3s^2) \right), \quad (3.5)$$

- for the fermion

$$t_{12}^{\chi}(t, s) = \frac{t}{8} (s^3 + 3M^4(s - u) - 2s^2u + 2su^2 - u^3 - 2M^2(s^2 - u^2)). \quad (3.6)$$

The massive triangle coefficients are given by

- for the scalar

$$t_{34}^{\varphi}(t, s) = \frac{1}{4(t - 4M^2)^2} \left(-12M^{10} + 6M^8t - 12M^6(s^2 - 3su + u^2) + M^4t(23s^2 - 44su + 23u^2) - 3M^2t^2(3s^2 - 4su + 3u^2) + t^3(s^2 - su + u^2) \right), \quad (3.7)$$

- for the photon

$$t_{34}^{\gamma}(t, s) = \frac{1}{8(t - 4M^2)^2} \left(-4M^{12}(s + u) + 4M^{10}(s - u)^2 - 16M^8(s - u)^2(s + u) - 12M^6su(s - u)^2 + 6M^4(s + u)(s^4 + s^3u - 2s^2u^2 + su^3 + u^4) - M^2(s - u)^2(s + u)^2(s^2 - 4su + u^2) - (s + u)^3(s^2 + u^2)(s^2 - su + u^2) \right), \quad (3.8)$$

- for the fermion

$$t_{34}^{\chi}(t, s) = \frac{2M^2 - t - 2s}{8(t - 4M^2)^2} \left(60M^{10} - 2M^8(61t + 60s) - t^3(t^2 + 3ts + 3s^2) + 6M^2t^2(2t^2 + 6ts + 5s^2) \right). \quad (3.9)$$

The massless bubble coefficients are

- for the scalar

$$bu^{\varphi}(t) = \frac{1}{120(t - 4M^2)^2} \left(-56M^8 + 72M^6t + M^4(23s^2 + 10su + 23u^2) - M^2t(13s^2 + 218su + 13u^2) + t^2(s^2 + 41su + u^2) \right), \quad (3.10)$$

- for the photon

$$bu^\gamma(t) = \frac{su - M^4}{240(t - 4M^2)^2} \left(1288M^8 - 8M^6(373t + 322s) + M^4(2083t^2 + 3416ts + 1288s^2) - 2M^2t(300t^2 + 695ts + 532s^2) + t^2(60t^2 + 163ts + 163s^2) \right), \quad (3.11)$$

- for the fermion

$$bu^\chi(t) = \frac{124M^4 - 272M^2t + 49t^2}{240(t - 4M^2)^2} \left(2M^6 + 2M^2s(2t + 3s) - M^4(t + 6s) - s(t^2 + 3ts + 2s^2) \right). \quad (3.12)$$

4 The low energy limit of the cut constructible one-loop amplitude

In the previous section we have provided the full non-analytic contributions to the one-loop amplitudes of gravitational scattering of scalars, photons and fermions from a large scalar mass. However, for many applications and specifically for the calculation done in this paper, only the leading low energy limit is needed, and we present it here. In the low energy limit, the energy $E = \hbar\omega$ of the massless particle is much smaller than the mass of the massive scalar $\omega \ll M$ and the momentum transfer $t \sim -\mathbf{q}^2 \ll M^2$ is tiny as well.

In this limit the one-graviton-exchange amplitude of a massless particle X from the massive scalar Φ is given by (cf. appendix A.3)

$$\mathcal{M}_X^{(1)} \simeq \frac{\mathcal{N}^X}{\hbar} \kappa^2 \frac{(2M\omega)^2}{4t}, \quad (4.1)$$

where $\mathcal{N}^\varphi = 1$ for the massless scalar, while for the photon $\mathcal{N}^\gamma = (2M\omega)^2 / (2\langle p_1 | p_3 | p_2 \rangle^2)$ for the $(+-)$ photon helicity contribution and its complex conjugate for the $(-+)$ photon helicity contribution, and $\mathcal{N}^\chi = M\omega / \langle p_1 | p_3 | p_2 \rangle$ for the $(+-)$ fermion helicity contribution and its complex conjugate for the $(-+)$ configuration. That the photon amplitude vanishes for the polarization configurations $(++)$ and $(--)$ is a direct consequence of the properties of the tree-amplitudes in eq. (A.10). Note that $|\mathcal{N}^\gamma|^2 \rightarrow 1$ and $|\mathcal{N}^\chi|^2 \rightarrow 1$ in the low-energy limit and therefore this pre-factor does not affect the cross-section.

The corresponding low energy one-loop amplitudes have the form

$$i\mathcal{M}_X^{(2)} \simeq \frac{\mathcal{N}^X}{\hbar} \left[\frac{\kappa^4}{4} \left(4(M\omega)^4 (I_4(t, u) + I_4(t, s)) + 3(M\omega)^2 t I_3(t) - 15(M^2\omega)^2 I_3(t, M) + bu^X (M\omega)^2 I_2(t) \right) \right], \quad (4.2)$$

where the coefficients of the bubble contributions are

$$bu^\varphi = \frac{3}{40}, \quad bu^\gamma = -\frac{161}{120}, \quad bu^\chi = -\frac{31}{30}. \quad (4.3)$$

Since in this limit $t \sim -\mathbf{q}^2$, $s \simeq (M + E)^2$, and $u \simeq (M - E)^2 + \mathbf{q}^2$, with $E = \hbar\omega$, the finite component of the integral functions I_n are found to be (cf. appendix B for more details)

$$i(I_4(t, s) + I_4(t, u)) \simeq \frac{1}{2tME} (4i\pi) \log\left(\frac{-t}{M^2}\right) \quad (4.4)$$

$$iI_3(t) \simeq -\frac{1}{2t} \log^2\left(\frac{-t}{\mu^2}\right) \quad (4.5)$$

$$iI_3(t, M^2) \simeq -\frac{1}{32\pi^2 M^2} \left(\log\left(\frac{-t}{M^2}\right) + \frac{\pi^2 M}{\sqrt{-t}} \right) \quad (4.6)$$

$$iI_2(t) \simeq -\frac{1}{16\pi^2} \left(2 - \log\left(\frac{-t}{\mu^2}\right) \right). \quad (4.7)$$

The total gravitational scattering amplitude

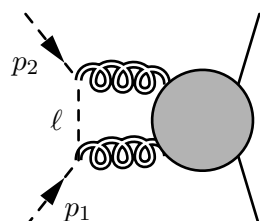
$$i\mathcal{M}_X = \frac{i}{\hbar} \mathcal{M}_X^{(1)} + i\mathcal{M}_X^{(2)}, \quad (4.8)$$

then has the low-energy expansion

$$\begin{aligned} i\mathcal{M}_X^{(2)} \simeq & \frac{\mathcal{N}^X}{\hbar} (M\omega)^2 \\ & \times \left[-\frac{\kappa^2}{\mathbf{q}^2} + \kappa^4 \frac{15}{512} \frac{M}{|\mathbf{q}|} + \hbar\kappa^4 \frac{15}{512\pi^2} \log\left(\frac{\mathbf{q}^2}{M^2}\right) - \hbar\kappa^4 \frac{bu^X}{(8\pi)^2} \log\left(\frac{\mathbf{q}^2}{\mu^2}\right) \right. \\ & \left. + \hbar\kappa^4 \frac{3}{128\pi^2} \log^2\left(\frac{\mathbf{q}^2}{\mu^2}\right) - \kappa^4 \frac{M\omega}{8\pi} \frac{i}{\mathbf{q}^2} \log\left(\frac{\mathbf{q}^2}{M^2}\right) \right], \end{aligned} \quad (4.9)$$

where μ^2 is a mass scale parameter used in dimensional regularization. This result confirms ref. [22] and extends it to the case of fermionic scattering. Note that eq. (4.9) contains both classical (independent of \hbar) and quantum mechanical ($\propto \hbar$) loop contributions. It was shown in [48] why classical post-Newtonian corrections appear in one-loop gravitational amplitudes. While most field theories identify classical and quantum effects by separating tree from loop topologies, this is not a fundamental distinction and has in fact more to do with linearity vs. non-linearity of the field equations; it is then natural that a quantum field theory constructed from the non-linear Einstein-Hilbert action receives classical contributions from loop topologies. At one-loop order, in the above computation, this contribution is of the type $\sim \kappa^4/|\mathbf{q}|$ and it is a very pleasing check of our computation that we observe universality in both particle type and statistics for this contribution, as expected from general relativity.

The one-loop amplitude has infrared divergences arising from the propagation of the graviton between the massless external legs $p_1^2 = p_2^2 = 0$,



$$\propto \mathcal{M}_X^{(1)} \times \int_0 \frac{d^{4-2\epsilon} \ell \mu^{2\epsilon}}{\ell^2 2\ell \cdot p_1 2\ell \cdot p_2} \sim \frac{(t \mu^{-2})^{-\epsilon}}{t \epsilon^2} \mathcal{M}_X^{(1)}. \quad (4.10)$$

The resulting infrared divergence is contained in the scalar boxes and the massless triangle in the decomposition (4.2). The infrared divergences of gravitational theories have been studied in [49–53]. At one-loop the amplitude is the sum, $\mathcal{M}_X^{(2)} = \mathcal{S} \mathcal{M}^{(1)} + \mathcal{H}$, consisting of an infrared divergent part from the soft region in (4.10), $\mathcal{S} = (t\mu^{-2})^{-\epsilon}/(t\epsilon^2)$ times $\mathcal{M}^{(1)}$ (the tree-level one-graviton amplitude is evaluated in appendix A.3) and a finite hard part \mathcal{H} . When forming cross-sections, we know how to resolve them. Soft graviton bremsstrahlung radiation also contributes to the measured cross-section if the radiated graviton is below the resolution of the detector. Including the cross-section for bremsstrahlung with a finite detector resolution ΔE has the effect of converting the scale μ into the detector resolution, potentially along with some finite constants depending on the specifics of the detector and the cross-section definition. This has been checked explicitly for massless gravitons in the process of graviton-graviton scattering [53].

In our case, we are about to use this amplitude in the process of light bending. Again, very soft gravitational bremsstrahlung cannot be differentiated from the non-radiative light bending, and should be included in the measurement. As with the cross-section, this should eliminate the IR divergences and replace the scale μ the logarithm by a factor depending on the resolution of the measurement. We have not done an explicit calculation of this process. If the quantum correction were close to being observable and if detectors capable of resolving graviton bremsstrahlung existed, one would be motivated to perform a careful analysis. However, a couple of options present themselves. If the light was a monochromatic beam, the detector resolution could be a resolution in energy of the photon. This could be either an energy independent resolution factor, or one which is proportional to some fraction of the original energy. These two cases would then have different infrared factors in the bending angle, indicating that there is not a unique detector-independent factor to include in a bending angle formula. Alternatively, the angular resolution of the detector could be used to define the acceptance factor for graviton bremsstrahlung. In the absence of a full calculation, we simply replace the scale μ in the logarithm by an infrared scale which we will call $1/b_0$ below.

5 Bending of light

5.1 Bending formula from general relativity

Perhaps the most famous verification of Einstein’s general theory of relativity is its prediction for the bending angle of light passing the rim of the Sun, since its 1919 measurement during a total solar eclipse led to worldwide publicity and acceptance of Einstein’s theory. The standard derivation of this result in general relativity follows from considering a spherically symmetric metric parameterized as

$$ds^2 = A(r)dt^2 - B(r)^2 dr^2 - r^2 d\Omega^2 . \tag{5.1}$$

In the case of the Schwarzschild metric we have

$$A(r) = \frac{1}{B(r)} = 1 - \frac{2G_N M}{r} , \tag{5.2}$$

so that geometrically the deflection angle is given the standard formula

$$\theta = 2 \int_0^1 \frac{du}{\sqrt{1 - u^2 - \frac{2G_N M}{R}(1 - u^3)}} - \pi, \quad (5.3)$$

where we have defined $u = \frac{R}{r}$. Here R is the distance of closest approach in Schwarzschild coordinates. The integration in eq. (5.3) can be performed exactly in terms of elliptic functions, but since near the solar rim $2G_N M/R \simeq 10^{-3} \ll 1$, we can instead use a perturbative solution

$$\begin{aligned} \theta &= 2 \int_0^1 du \left[\frac{1}{\sqrt{(1-u)(1+u)}} + \frac{G_N M}{R} \frac{1+u+u^2}{\sqrt{(1-u)(1+u)^3}} \right. \\ &\quad \left. + \frac{3}{2} \frac{G_N^2 M^2}{R^2} \frac{(1+u+u^2)^2}{\sqrt{(1-u)(1+u)^5}} + \dots \right] - \pi \\ &= \frac{4G_N M}{R} + \frac{4G_N^2 M^2}{R^2} \left(\frac{15\pi}{16} - 1 \right) + \dots \end{aligned} \quad (5.4)$$

However, instead of using the coordinate-dependent quantity R , the bending angle should be written in terms of the impact parameter b , defined as

$$b = \sqrt{B(R)}R = \frac{R}{\sqrt{1 - \frac{2G_N M}{R}}} = R + G_N M + \dots \quad (5.5)$$

It is important to note that b is a coordinate-*independent* quantity whereas R depends on the coordinate system (see [54] for a nice discussion of the coordinate dependence on the expression of the deflection angle). We arrive then at the universal (matter-independent) formula for the bending angle

$$\frac{b}{2G_N M} = \frac{2}{\theta} + \frac{15\pi}{32} + \mathcal{O}(\theta), \quad (5.6)$$

or

$$\theta = \frac{4G_N M}{b} + \frac{15\pi G_N^2 M^2}{4b^2} + \mathcal{O}\left(\frac{1}{b^3}\right). \quad (5.7)$$

This is the standard derivation and arises from considering light as particles (photons) traversing a classical trajectory. Below we show how we can reproduce this expansion from the (low energy) limit of the one-loop scattering amplitude, which can be thought of as a quantum mechanical (wavelike) derivation.

5.2 Leading Newtonian correction

We will first reproduce the leading term by evaluating the (classical) elastic differential cross-section using only the first Newtonian (tree-level) contribution. Writing the cross-section out as a perturbative expansion we have

$$\mathcal{M} = \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \dots \quad (5.8)$$

and thus

$$|\mathcal{M}|^2 = |\mathcal{M}^{(1)}|^2 + 2 \Re(\mathcal{M}^{(1)}(\mathcal{M}^{(2)})^*) + \dots \quad (5.9)$$

Since we are interested in the low energy limit $E \ll m$, we can make the approximations $t = (p_1 + p_2)^2 = (p_3 + p_4)^2 \simeq -\mathbf{q}^2 = -4E^2 (\sin \theta/2)^2$, with $q = p_1 + p_2$, $s = (p_1 + p_4)^2 \simeq (M + E)^2 \simeq M^2 + 2ME$, and $u = (p_1 + p_3)^2 \simeq (M - E)^2 + \mathbf{q}^2 \simeq M^2 - 2ME + \mathbf{q}^2$. We also employ a small angle scattering approximation so that $|\mathbf{q}^2| \ll E^2$. Recalling that $\kappa^2 = 32\pi G_N/c^4$, the tree-level contribution to the cross-section is given by

$$\left. \frac{d\sigma}{dt} \right|_{\varphi^2}^{\text{tree}} = \frac{\kappa^4 (u - M^2)^2}{16^3 \pi t^2} \simeq \pi \left(\frac{4G_N M E}{t} \right)^2. \quad (5.10)$$

Making the (classical) assumption that we can determine the impact parameter $\rho(\theta)$ from the cross-section $d\sigma = \pi d\rho^2$, we have

$$\rho^2 = \int_{4E^2(\sin \theta/2)^2}^{4E^2} \frac{d\sigma}{dt} \frac{dt}{\pi}, \quad (5.11)$$

and in the small angle approximation $\theta \ll 1$

$$\rho^2 \simeq (4G_N M E)^2 \int_{4E^2(\theta/2)^2}^{4E^2} \frac{dt}{t^2} \simeq \left(\frac{4G_N M}{\theta} \right)^2, \quad (5.12)$$

which gives the relation between the bending angle and the impact parameter

$$\theta \simeq \frac{4G_N M}{\rho}. \quad (5.13)$$

It is clear, given eq. (4.9), that this result for the leading contribution to the bending angle is universal for both particle type and statistics and agrees with general relativity. Of course, the validity of this semi-classical derivation of the leading contribution (Coulomb-type potential scattering) to the bending angle is *not* guaranteed when considered within a full quantum mechanical framework. It is, however, still true, up to a phase, due to the fact that even quantum mechanically angular momentum remains conserved for a Coulombic potential, so that the classical cross-section formula at leading order is valid even in the full quantum regime.

5.3 Bending via the eikonal approximation

An appropriate quantum mechanical treatment of the light bending problem makes use of the eikonal formulation, which describes the scattering in terms of an impact parameter representation. In impact parameter space, the scattering amplitude exponentiates into an eikonal phase and evaluation using the stationary phase method yields the classical result for the bending angle, together with quantum effects.

There are two important aspects to the eikonal approximation. One is kinematic. When the impact parameter is large, the bending angle is small. The small-angle approximation means that the momentum transfer is small — $t \sim -\mathbf{q}^2 \ll s$. This condition is easy to implement in our amplitude. The second approximation is diagrammatic. The leading

eikonal approximation involves the iteration of one graviton exchange in all permutations. The first correction to this approximation involves more complicated diagrams, such as loop processes, in addition to the permutations of graviton exchange. The leading eikonal phase is of order G_N , and the first correction to the phase will be of order G_N^2 . We will impose eikonal kinematics and proceed to the first diagrammatic correction.

Our guide in this approach is the discussion of the next-to-leading eikonal amplitude by Akhoury et al. in [55].³ This method can be readily generalized to include the quantum terms within the same diagrams as well as the purely quantum diagrams (the bubble diagrams) at the same order. We determine the eikonal phase by matching the amplitudes at one-loop order. In high energy small-angle scattering the dominant four-momentum transfer is in the transverse spatial direction. For photons traveling in the z direction we have $p_3 = p_1 + q$ so that, squaring this equation, we obtain $0 = 2E(q_0 - q_z) + \mathbf{q}^2$. A similar calculation for the heavy scalar yields $0 = -2Mq_0 + \mathbf{q}^2$, which tells us that both $q_{\pm} = q_0 \pm q_z$ are suppressed compared to the transverse components $\mathbf{q}^2 \sim -\mathbf{q}_{\perp}^2$ by at least a factor of $2E$. This condition on the overall momentum transfer gets reflected in the same condition on the exchanged gravitons, so that the dominant momentum transfer inside loops is also transverse. In the effective theory of high energy scattering, the Soft Collinear Effective Theory (SCET), these are called Glauber modes [56] and carry momentum scaling $(k_+, k_-, k_{\perp}) \sim \sqrt{s}(\lambda^2, \lambda^2, \lambda)$ where $\lambda \sim \sqrt{-t/s}$.

The one-graviton amplitude in this limit is

$$\mathcal{M}_1^{(1)}(\mathbf{q}) = \kappa^2 M^2 E^2 \frac{1}{\mathbf{q}^2}, \tag{5.14}$$

and, after some manipulations, the multiple exchanges of this amplitude can be arranged into a form which exponentiates

$$\mathcal{M}_{\text{sum}}^{(1)}(\mathbf{q}) = 32\pi M E \sum_n \frac{1}{n!} (i2G_N M E)^n \prod_{i=1}^n \int \frac{d^2 k_i}{(2\pi)^2} \frac{1}{\mathbf{k}_i^2} \delta^2\left(\sum k_i - \mathbf{q}\right). \tag{5.15}$$

In order to bring this amplitude into impact parameter space, one defines the Fourier transform, with impact parameter \mathbf{b} being transverse to the initial motion.

$$i\mathcal{M}(\mathbf{b}) = \int \frac{d^2 \mathbf{q}}{(2\pi)^2} e^{-i\mathbf{q}\cdot\mathbf{b}} \mathcal{M}_{\text{sum}}^{(1)}(\mathbf{q}). \tag{5.16}$$

Writing the sum as an exponential, the result (with the prefactor relevant for gravity) for the scattering of a massless particle from a massive one is given by

$$i\mathcal{M}(\mathbf{b}) = 2(s - M^2) \left(e^{i\chi_1(\mathbf{b})} - 1 \right). \tag{5.17}$$

Here $\chi_1(\mathbf{b})$ is the Fourier transform of the one graviton exchange, with some kinematics factored out

$$\chi_1(\mathbf{b}) = \frac{1}{2M2E} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} e^{-i\mathbf{q}\cdot\mathbf{b}} \mathcal{M}_1(\mathbf{q}), \tag{5.18}$$

³We note that, at the time our manuscript is being written, the preprint version of their work contains a clear error in the summary of their amplitude. We have confirmed this with the authors and a corrected version of this work will appear soon. Once this mistake is rectified, their result reproduces the correct next-to-leading classical bending angle.

and can be evaluated using dimensional regularization,

$$\int \frac{d^2 \mathbf{q}}{(2\pi)^2} e^{-i\mathbf{q}\cdot\mathbf{b}} |\mathbf{q}|^\alpha = \int_0^\infty \frac{dq}{2\pi} q^{\alpha+d-1} J_0(qb) = \frac{1}{4\pi(2b)^{\alpha+d}} \frac{\Gamma\left(\frac{\alpha+d}{2}\right)}{\Gamma\left(\frac{2-\alpha-d}{2}\right)}, \quad (5.19)$$

and taking the limit $d \rightarrow 2$ in the final expression. Therefore (using $\kappa^2 = 32\pi G_N$)

$$\begin{aligned} \chi_1(\mathbf{b}) &= \frac{\kappa^2 M E}{4} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} e^{-i\mathbf{q}\cdot\mathbf{b}} \frac{1}{\mathbf{q}^2} \\ &\simeq 4G_N M E \left[\frac{1}{d-2} - \log(b/2) - \gamma_E \right], \end{aligned} \quad (5.20)$$

with E being the energy of the massless particle. Only the $\log b$ term will be important in the following treatment.

At order G_N^2 , the matrix element picks up corrections which we can describe by

$$\mathcal{M}(\mathbf{q}) = \mathcal{M}_1^{(1)}(\mathbf{q}) + \mathcal{M}^{(2)}(\mathbf{q}), \quad (5.21)$$

where $\mathcal{M}^{(2)}(\mathbf{q})$ is our calculated amplitude evaluated in this kinematic limit. Including the dressing of $\mathcal{M}^{(2)}(\mathbf{q})$ by permutations of one graviton exchange, it was shown in [55] that there is again an exponentiation of the simple exchange

$$\mathcal{M}_{\text{sum}}^{(2)}(\mathbf{q}) = (32\pi)^2 M E \sum_n \frac{(i2G_N M E)^{n-2}}{(n-2)!} \int d^2 k_j \mathcal{M}^{(2)}(k_j) \prod_{i \neq j}^n \int \frac{d^2 k_i}{(2\pi)^2} \frac{1}{\mathbf{k}_i^2} \delta^2\left(\sum k_i - \mathbf{q}\right). \quad (5.22)$$

In impact parameter space then we can write (5.21)

$$i\mathcal{M}(\mathbf{b}) = 2(s - M^2) [(1 + i\chi_2)e^{i\chi_1} - 1] \simeq 2(s - M^2) [e^{i(\chi_1 + \chi_2)} - 1], \quad (5.23)$$

with the second expression being valid to this order in G_N . The second order phase χ_2 is given by

$$\chi_2(\mathbf{b}) = \frac{1}{2M^2 E} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} e^{-i\mathbf{q}\cdot\mathbf{b}} \mathcal{M}^{(2)}(\mathbf{q}). \quad (5.24)$$

For the classical correction we need the integral

$$\int \frac{d^2 \mathbf{q}}{(2\pi)^2} e^{-i\mathbf{q}\cdot\mathbf{b}} \frac{1}{|\mathbf{q}|} = \frac{1}{2\pi b}, \quad (5.25)$$

while for the quantum terms we require

$$\int \frac{d^2 q}{(2\pi)^2} e^{-i\mathbf{q}\cdot\mathbf{b}} \log \mathbf{q}^2 = -\frac{1}{\pi b^2}, \quad (5.26)$$

$$\int \frac{d^2 q}{(2\pi)^2} e^{-i\mathbf{q}\cdot\mathbf{b}} \log^2 \mathbf{q}^2 = \frac{4}{\pi b^2} \log \frac{2}{b}. \quad (5.27)$$

We find then

$$\chi_2(\mathbf{b}) = G_N^2 M^2 E \frac{15\pi}{4b} + \frac{G_N^2 M^2 E}{2\pi b^2} \left(8bu^\eta - 15 + 48 \log \frac{2b_0}{b} \right). \quad (5.28)$$

The light bending analysis is now straightforward and involves determining the stationary phase of the exponent, which can be argued to dominate the momentum space integration, via

$$\frac{\partial}{\partial b} (-\mathbf{q} \cdot \mathbf{b} + \chi_1(b) + \chi_2(b) + \dots) = \frac{\partial}{\partial b} (qb + \chi_1(b) + \chi_2(b) + \dots) = 0. \quad (5.29)$$

Using $q = 2E \sin(\theta/2)$ this condition reads

$$2 \sin \frac{\theta}{2} \simeq \theta = -\frac{1}{E} \frac{\partial}{\partial b} (\chi_1(b) + \chi_2(b)), \quad (5.30)$$

and yields

$$\theta \simeq \frac{4G_N M}{b} + \frac{15}{4} \frac{G_N^2 M^2 \pi}{b^2} + \left(8bu^S + 9 - 48 \log \frac{b}{2b_0} \right) \frac{\hbar G_N^2 M}{\pi b^3} + \dots \quad (5.31)$$

Here $1/b_0$ in the logarithm is the infrared cutoff which removes the IR singularities of the amplitude. We see that the eikonal approximation leads to the expected classical general relativity contributions, in agreement with the next-to-leading correction of [55] and [57], as well as producing the leading quantum correction. Treating the quantum effect using the eikonal procedure, we recover the results of [22] derived with a semiclassical potential method.

The quantum effect has the power-law dependence in impact parameter as the classical post-post-Newtonian contribution. This second post-Newtonian contribution of order $G_N^3 M^3 / b^3$ arises as a classical piece from two-loop amplitudes with momentum dependence q^0 / \hbar . These two contributions lead to very distinct analytic structure to the S -matrix and are easily separated. The classical corrections is much larger than the quantum effect by the ratio the square of the Schwarzschild radius to the Planck length $G_N^2 M^2 \gg \hbar G_N$.

5.4 Bending via geometrical optics

There is an equivalence between the eikonal method described above and the semiclassical potential method which we used in [22]. In order to elucidate this, it is useful to consider the bending in terms of a wave picture of light propagation. Since the wavelength of the light is much smaller than size of the massive scalar object (Sun or black hole) around which the bending occurs, the analysis can be done using the methods of geometrical (ray) optics. This formalism is developed in many places, e.g. [58], and leads to the equation

$$\frac{d}{ds} n \frac{d\mathbf{r}}{ds} = \nabla n, \quad (5.32)$$

where n is the index of refraction and $\mathbf{r}(s)$ is the trajectory as a function of the path length s . For light we can write $ds \simeq c dt$ so that eq. (5.32) becomes

$$\frac{1}{c^2} \frac{d^2 \mathbf{r}}{dt^2} = \frac{1}{n} \nabla n. \quad (5.33)$$

In our case, at leading order, the index of refraction is determined from the general relativity/optical-mechanical analogy [59] which, for a line element

$$ds^2 = A(r) dt^2 - B(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (5.34)$$

yields

$$n(r) = \sqrt{\frac{B(r)}{A(r)}}. \quad (5.35)$$

For the Schwarzschild metric we have then

$$n(r) = \frac{1}{1 - \frac{2G_{\text{N}}M}{r}} \simeq 1 + \frac{2G_{\text{N}}M}{r} = 1 - \frac{1}{E_m} V_0(r), \quad (5.36)$$

where

$$V_0(r) = -\frac{2G_{\text{N}}ME_m}{r}, \quad (5.37)$$

is the leading order potential energy for a photon of energy E_m interacting with a massive scalar of mass M . Following [22], the generalization to the full interaction is then achieved by replacing the lowest order potential $V_0(r)$ by the full interaction potential $V_{\text{int}}(r)$ generated from the low energy approximation with $t \simeq -\mathbf{q}^2$ of the total gravitational scattering amplitude⁴ in eq. (4.9) with the second Born (rescattering) term excised

$$\begin{aligned} V_{\text{int}}(r) &= \frac{\hbar}{4M\omega} \int \left(\mathcal{M}_X(\mathbf{q}) + \kappa^4 \frac{M\omega}{8\pi\mathbf{q}^2} \log \frac{\mathbf{q}^2}{M^2} \right) e^{i\mathbf{q}\cdot\mathbf{r}} \frac{d^3\mathbf{q}}{(2\pi)^3} \\ &= -\frac{2G_{\text{N}}ME}{r} + \frac{15}{4} \frac{(G_{\text{N}}M)^2 E}{r^2} + \frac{8bu^S - 15 + 48 \log(r/r_0)}{4\pi} \frac{\hbar G_{\text{N}}^2 ME}{r^3}. \end{aligned} \quad (5.39)$$

(Note that this interaction potential is *not* derivable from an quantum corrected effective background metric. We will comment more about this fact in the section 6.) In the absence of a potential, we can imagine a photon incident in the \hat{e}_y direction with impact parameter b on a massive scalar target located at the origin. The trajectory is then characterized by

$$\mathbf{r}(t) = b\hat{e}_x + ct\hat{e}_y, \quad -\infty < t < \infty. \quad (5.40)$$

If we now impose a potential, there will exist a small deviation from this straight line trajectory with

$$\Delta \frac{1}{c^2} \frac{d\mathbf{r}}{dt} = - \int_{-\infty}^{+\infty} dt V'_{\text{int}}(r) \hat{\mathbf{r}}. \quad (5.41)$$

We have then

$$\frac{2}{c} \sin \frac{1}{2}\theta \simeq \frac{1}{E_m} \int_{-\infty}^{+\infty} dt V'_{\text{int}}(\sqrt{b^2 + c^2 t^2}) \frac{b}{\sqrt{b^2 + c^2 t^2}}, \quad (5.42)$$

where θ is the scattering angle. Changing variables to $t = bu/c$, we find

$$\frac{2}{c} \sin \frac{1}{2}\theta \simeq \frac{1}{c} \theta = -\frac{b}{E_m} \int_{-\infty}^{+\infty} V'_{\text{int}}(b\sqrt{1+u^2}) \frac{du}{\sqrt{1+u^2}}, \quad (5.43)$$

yielding

$$\theta = \frac{b}{E_m} \int_{-\infty}^{+\infty} V'_{\text{int}}(b\sqrt{1+u^2}) \frac{du}{\sqrt{1+u^2}}. \quad (5.44)$$

⁴We made use of the following Fourier transformations

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{\mathbf{q}^2} = \frac{1}{4\pi r}, \quad \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{|\mathbf{q}|} = \frac{1}{2\pi^2 r^2}, \quad \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \log(\mathbf{q}^2) = -\frac{1}{2\pi r^3}. \quad (5.38)$$

Substituting the interaction potential eq. (5.40) and performing the requisite integration, we arrive at

$$\theta = \frac{4G_N M}{b} + \frac{15\pi G_N^2 M^2}{4b^2} + \left(8bu^S + 9 - 48 \log \frac{b}{2b_0} \right) \frac{\hbar G_N^2 M}{\pi b^3} + \dots \quad (5.45)$$

The first two (classical) pieces of eq. (5.45) agree with the standard post-Newtonian analysis given above, but they are accompanied by a small quantum mechanical correction term found in our eikonal analysis. It may seem surprising that small angle scattering theory, involving the two-dimensional eikonal Fourier transform yields a result identical to that found from the semiclassical potential result, which is given in terms of a three-dimensional Fourier transform of the transition amplitude. However, this equality is made clear from the mathematical identity

$$\int_0^\infty dq q^2 J_1(qb) F(\mathbf{q}^2) = \frac{b}{\pi} \int_{-\infty}^\infty \frac{du}{\sqrt{1+u^2}} \int_0^\infty dq q^3 j_1(qb\sqrt{1+u^2}) F(\mathbf{q}^2) \quad (5.46)$$

which is valid for any sufficiently smooth function $F(\mathbf{q}^2)$ [60].⁵

6 Discussion

We have in this presentation derived one-loop scattering results for all types of massless matter interacting gravitationally with a massive scalar source. While we have found universality and agreement with general relativity for the classical physics component of the result, i.e., the so called post-Newtonian corrections, field theory has also produced a new type of contribution of quantum origin, which has no precedence in classical general relativity. This pattern of new contributions will persist to all loop orders, and thus produce terms having varying powers of \hbar that all are unique signatures of quantum effects in the theory of gravity. We will here comment on the role of such terms.

While many field theories have limits wherein quantum effects can be dealt with and motivated in a semi-classical/semi-quantum context (even in QCD!) it is particularly hard finding such limits in general relativity, given its geometric nature and local description. Concepts like a free falling elevator and motion along geodesics have no known simple quantum mechanical equivalent. An interesting observation, however, is that such quantum terms, except for the bubble coefficients, are universal. The bubble non-universality could be interpreted as a violation of some classical descriptions of the equivalence principle, in that massless particles do not follow null geodesics, and different types of massless particles follow different trajectories. However this is not a fundamental violation of the equivalence principle in the larger sense, as the action which defines the theory is compatible with the equivalence principle. However, in the scattering process tidal effects offer another possible interpretation of the result, since we have the quantum loops of massless particles involving long-range propagation in a non-homogenous gravitational field. Construction of a gedanken experiment featuring a homogeneous gravitational field could thus be an interesting exercise.

⁵We have used integral 6.654.2 with $\nu = \frac{3}{2}$ and $a = -i$.

One might consider the possibility that the quantum correction to the gravitational interaction between two massive particles could have a geometrical interpretation in terms of an effective particle evolving in a quantum-corrected metric. However, this seems not to be feasible since the effective one-particle reducible potential that would result from propagation in a quantum corrected metric would be gauge-dependent. This is already the case for the interaction potential between two massive particles. It appears that a fully quantum mechanical description, such as we have presented, is required.

We conclude that the best way to deal with this situation is to simply compute a cross-section for scattering and use this to compare observational data to theory. The effects are seen to be too small to be observed experimentally, yet they can yield interesting theoretical insights, such as the evidence that massless particles no longer follow null geodesics, and that the cross-section is not universal as it depends on the type of massless particle.

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A Gravitational photon and scalar tree amplitudes

A.1 Helicity formalism conventions

This appendix contains a brief account of the conventions and the notation in the paper. We follow the notations and conventions of ref. [32].

We employ the mostly minus metric signature $\eta^{\mu\nu} = \text{diag}(+, -, -, -)$ and use Dirac matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, i.e.,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}; \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.1})$$

We have $(\sigma^\mu) = (1, \sigma^i)$ and $(\bar{\sigma}^\mu) = (-1, \bar{\sigma}^i)$ where σ^i are the standard Pauli matrices. We use some places the notation $\gamma^\mu p_\mu = \not{p}$. The Dirac matrices satisfy the Clifford algebra (we refer to [43, appendix A.2] for details) $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and γ_5 is the chirality operator, satisfying $\text{tr}(\gamma_5 \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4}) = 4i \epsilon^{\mu\nu\rho\sigma}$.

We have the following conventions for traces. They are defined by $\text{tr}_\pm(a_1 \cdots a_r) := \text{tr}(\frac{1 \pm \gamma_5}{2} \gamma^{\mu_1} \cdots \gamma^{\mu_r}) a_{1\mu_1} \cdots a_{r\mu_r}$. We note in particular that $\text{tr}_\pm(abcd) = 2(a \cdot b c \cdot d - a \cdot c b \cdot d + a \cdot d b \cdot c) \pm 2i \epsilon^{\mu\nu\rho\sigma} a_\mu b_\nu c_\rho d_\sigma$.

The Levi-Civita epsilon tensor $\epsilon^{\mu\nu\rho\sigma}$ takes the value 1 if $\{\mu, \nu, \rho, \sigma\}$ is an even permutation of $\{0, 1, 2, 3\}$, -1 if $\{\mu, \nu, \rho, \sigma\}$ is an odd permutation of $\{0, 1, 2, 3\}$, and 0 otherwise.

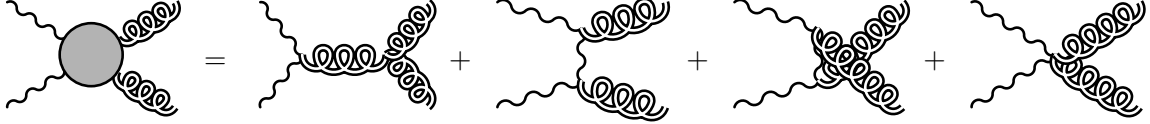


Figure 3. Gravitational Compton scattering given by the tree-level scattering a massless particle (wavy line) on graviton (curly line) with only gravitational interactions.

For a light-like momentum p the positive energy solution to the Dirac equation is $\not{p}u_h(p) = 0$ both for positive and negative helicities, i.e., $h = +1$ and $h = -1$. This solution satisfy the usual chirality condition $(1 \mp \gamma_5)/2 u_{\pm}(p) = 0$ and $(1 \pm \gamma_5)/2 \bar{u}_{\pm}(p) = 0$.

We will also make use of the following conventions

$$|k\rangle \equiv u_+(k); \quad |k] \equiv u_-(k) \quad (\text{A.2})$$

$$\langle k| \equiv \bar{u}_-(k); \quad [k| \equiv \bar{u}_+(k). \quad (\text{A.3})$$

and spinor products will be defined according to

$$\langle p q \rangle \equiv \bar{u}_-(p)u_+(q); \quad [p q] \equiv \bar{u}_+(p)u_-(q), \quad (\text{A.4})$$

where $(p + q)^2 = 2p \cdot q = \langle p q \rangle [p q]$.

This yields the following completeness relation

$$\sum_{h=\pm 1} u_h(k)\bar{u}_h(k) = \not{k} = |k\rangle[k| + |k][k| \quad (\text{A.5})$$

and we arrive at

$$\not{\epsilon}_{\mu}^{+}(k, p_{\text{ref}}) = \frac{[k|\gamma_{\mu}|p_{\text{ref}}\rangle}{\sqrt{2}\langle p_{\text{ref}}k\rangle}; \quad \not{\epsilon}_{\mu}^{-}(k, p_{\text{ref}}) = -\frac{\langle k|\gamma_{\mu}|p_{\text{ref}}\rangle}{\sqrt{2}[p_{\text{ref}}k]}, \quad (\text{A.6})$$

for the polarisation tensor for the photon of light-like momentum k where p_{ref} is an arbitrary light-like reference momentum.

For p and q light-like momenta and k a four-momentum vector we have

$$\langle p|k|q\rangle = \langle p|\gamma^{\mu}|q\rangle k_{\mu}; \quad [p|kq] = (\langle p|k|q\rangle)^*. \quad (\text{A.7})$$

A.2 The gravitational Compton amplitudes

In this section we review the gravitational Compton scattering at tree-level represented in figure 3 and discussed in detail in [15, 17, 20] and in [21]. We are interested in the gravitational Compton scattering of a graviton g from a massless target of spin 0 (scalar φ), spin $\frac{1}{2}$ (fermion χ), and spin 1 (photon γ). The only interactions that we consider are gravitational interactions.

A remarkable property of the gravitational Compton scattering from a target X of spin S and mass M (which can be vanishing) is its factorization onto a product of Abelian QED Compton amplitudes [20]

$$i\mathcal{M}_{X^2G^2}(p_1, k_1, p_2, k_2) = \frac{\kappa^2}{8e^4} \frac{p_1 \cdot k_1 p_1 \cdot k_2}{k_1 \cdot k_2} A_S^{\text{Compton}}(p_1, k_2, p_2, k_1) A_0^{\text{Compton}}(p_1, k_2, p_2, k_1), \quad (\text{A.8})$$

where A_S^{Compton} is the Compton amplitude associated with scattering a photon from a target of spin S , and A_0^{Compton} is the Compton amplitude obtained by scattering a photon on scalar target.

We express gravitational Compton amplitudes in the helicity formalism, using the notation for the polarization dependence of the external states

$$\mathcal{M}_{\gamma^2 G^2}^{(h_1 h_2 | \lambda_1 \lambda_2)} := \mathcal{M}_{\gamma}^{\text{G-Compton}}(p_1^{h_1}, k_1^{\lambda_1}, p_2^{h_2}, k_2^{\lambda_2}). \quad (\text{A.9})$$

The amplitudes $\mathcal{M}_{\gamma^2 G^2}^{(++|++)}$, $\mathcal{M}_{\gamma^2 G^2}^{(++|+-)}$, $\mathcal{M}_{\gamma^2 G^2}^{(++|-+)}$, $\mathcal{M}_{\gamma^2 G^2}^{(+ -|++)}$, $\mathcal{M}_{\gamma^2 G^2}^{(-+|++)}$ and their complex conjugate vanish since the four-gluon tree-level amplitude $A_1^{\text{Compton}}(p_1^{h_1}, k_1^{\lambda_1}, p_2^{h_2}, k_2^{\lambda_2})$ is zero for these configurations of polarizations as they are not MHV amplitudes. In addition, the gravitational amplitude $\mathcal{M}_{\gamma^2 G^2}^{(++|--)}$ (and its complex conjugate) vanishes because the scalar amplitudes $A_0^{\text{Compton}}(p_1, k_1^-, p_2, k_2^-)$ (and its complex conjugate) vanishes for massless scalars. Thus, the only non-vanishing gravitational Compton amplitudes for photons are, see for example [32–34].

$$\mathcal{M}_{\gamma^2 G^2}^{(+ -|+-)} = \frac{\kappa^2 [p_1 k_1]^2 \langle p_2 k_2 \rangle^2 \langle k_2 | p_1 | k_1 \rangle^2}{8 (p_1 \cdot p_2)(p_1 \cdot k_1)(p_1 \cdot k_2)}, \quad (\text{A.10})$$

$$\mathcal{M}_{\gamma^2 G^2}^{(-+|+-)} = \frac{\kappa^2 [p_2 k_1]^2 \langle p_1 k_2 \rangle^2 \langle k_2 | p_2 | k_1 \rangle^2}{8 (p_1 \cdot p_2)(p_1 \cdot k_1)(p_1 \cdot k_2)}, \quad (\text{A.11})$$

and their complex conjugates.

For scalar target Compton scattering, the helicity amplitudes derived in [20] are given by

$$A_0^{\text{Compton}}(p_1, p_2, k_2^+, k_1^+) = -\frac{M^2 [k_1 k_2]^2}{k_1 \cdot k_2 2k_1 \cdot p_1}, \quad A_0^{\text{Compton}}(p_1, p_2, k_2^-, k_1^+) = \frac{\langle k_2 | p_1 | k_1 \rangle^2}{k_1 \cdot k_2 2k_1 \cdot p_1}, \quad (\text{A.12})$$

with the complex conjugated expressions $A_0^{\text{Compton}}(p_1, p_2, k_2^-, k_1^-) = (A_0^{\text{Compton}}(p_1, p_2, k_2^+, k_1^+))^*$ and $A_0^{\text{Compton}}(p_1, p_2, k_2^+, k_1^-) = (A_0^{\text{Compton}}(p_1, p_2, k_2^-, k_1^+))^*$.

The gravitational Compton amplitude in eq. (A.8) then reads in the helicity formalism

$$\begin{aligned} \mathcal{M}_{\Phi^2 G^2}(p_1, k_1^+, p_2, k_2^+) &= \frac{\kappa^2}{16} \frac{1}{(k_1 \cdot k_2)} \frac{M^4 [k_1 k_2]^4}{(k_1 \cdot p_1)(k_1 \cdot p_2)}, \\ \mathcal{M}_{\Phi^2 G^2}(p_3, k_1^-, p_4, k_2^+) &= \frac{\kappa^2}{16} \frac{1}{(k_1 \cdot k_2)} \frac{\langle k_1 | p_3 | k_2 \rangle^2 \langle k_1 | p_4 | k_2 \rangle^2}{(k_1 \cdot p_3)(k_1 \cdot p_4)}. \end{aligned} \quad (\text{A.13})$$

Note that for the same reason as in the photon case, the $\mathcal{M}_{\Phi^2 G^2}(p_1, k_1^+, p_2, k_2^+)$ amplitude vanishes for a massless target, and in the same way we find the Compton amplitude for massless fermions to be

$$\mathcal{M}_{\chi^2 G^2}(p_1^+, p_2^-, k_1^+, k_2^-) = \frac{\kappa^2}{16} \frac{1}{(k_1 \cdot k_2)} \frac{\langle k_2 | p_1 | k_1 \rangle^3 [k_1 p_2] \langle p_1 k_2 \rangle}{(k_1 \cdot p_1)(k_1 \cdot p_2)}. \quad (\text{A.14})$$

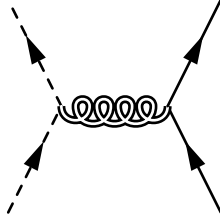


Figure 4. One graviton exchange between a massless fields and a massive scalar.

A.3 The one-graviton tree-level amplitudes

We give the full tree-level one-graviton exchange amplitude of figure 4 between the massive scalar Φ of mass M and the massless scalar φ , the photon γ and the massless fermion χ . The massless particles have momenta p_1 and p_2 , the massive scalar has momenta p_3 and p_4 , with $p_1 + p_2 + p_3 + p_4 = 0$. The kinematic invariants are $t = (p_1 + p_2)^2$, $s = (p_1 + p_4)^2$ and $u = (p_1 + p_3)^2$.

- The tree-level gravitational interaction between a massless scalar φ and a massive scalar Φ is given by

$$\mathcal{M}_{\varphi^2\Phi^2}^{(1)} = -\frac{\kappa^2}{4} \frac{(s - M^2)(u - M^2)}{t}. \quad (\text{A.15})$$

- The tree-level gravitational interaction between a photon γ and a massive scalar Φ is given by

$$\mathcal{M}_{\gamma^2\Phi^2}^{(1)(++)} = 0; \quad \mathcal{M}_{\gamma^2\Phi^2}^{(1)(+-)} = \kappa^2 \frac{\langle p_2|p_3|p_1\rangle^2}{4(p_1 + p_2)^2} = \kappa^2 \frac{(M^4 - su)^2}{4t \langle p_1|p_3|p_2\rangle^2}, \quad (\text{A.16})$$

where the superscript denotes the helicity of the external photons, and we used that $\langle p_2|p_3|p_1\rangle^2 \langle p_1|p_3|p_2\rangle^2 = (\text{tr}_-(p_2 p_3 p_1 p_3))^2 = (M^4 - su)^2$ with equivalent expressions for their complex conjugate helicity configurations. The vanishing of the amplitude when the helicity of the configuration of the incoming and outgoing photons are the same is expected from general relativity since two parallel beams of light do not interact gravitationally [61].

- The tree-level gravitational interaction between a massless spin- $\frac{1}{2}$ field χ and a massive scalar Φ is given by

$$\mathcal{M}_{\chi^2\Phi^2}^{(1)(+-)} = \kappa^2 \frac{(s - u)}{8t} \langle p_2|p_3|p_1\rangle = \kappa^2 \frac{(s - u)(M^2 - su)}{8t \langle p_1|p_3|p_2\rangle} \quad (\text{A.17})$$

with an equivalent expression for the complex conjugate helicity configuration.

B Integrals

In this appendix we use the same convention as in the main text $q = p_1 + p_2 = -p_3 - p_4$, $t = (p_1 + p_2)^2$, $s = (p_1 + p_4)^2$ and $u = (p_1 + p_3)^2$.

The infrared divergent integral. The boxes are defined and evaluated [62, 63] as

$$I_4(t, s) = \frac{1}{i\pi^{2-\epsilon} r_\Gamma} \int \frac{d^{4-2\epsilon} \ell \mu^{2\epsilon}}{\ell^2(\ell+q)^2(\ell+p_1)^2((\ell-p_4)^2-M^2)}, \quad (\text{B.1})$$

with the following ϵ expansion

$$I_4(t, s) = -\frac{1}{t(M^2-s)} \left(\frac{\mu^2}{M^2}\right)^\epsilon \left[\frac{2}{\epsilon^2} - \frac{1}{\epsilon} \left(2 \log \frac{M^2-s}{M^2} + \log \frac{-t}{M^2} \right) + 2 \log \frac{M^2-s}{M^2} \log \frac{-t}{M^2} - \frac{\pi^2}{2} + O(\epsilon) \right], \quad (\text{B.2})$$

where $r_\Gamma = (\Gamma(1-\epsilon))^2 \Gamma(1+\epsilon) / \Gamma(1-2\epsilon)$. The box scalar integral $I_4(t, u)$ is obtained by replacing s by u in the previous expressions.

The massless triangle integral is defined by

$$I_3(t) = \frac{1}{i\pi^{2-\epsilon} r_\Gamma} \int \frac{d^{4-2\epsilon} \ell \mu^{2\epsilon}}{\ell^2(\ell-p_1)^2(\ell+p_2)^2} = -\frac{1}{\epsilon^2 t} \left(\frac{-t}{\mu^2}\right)^{-1-\epsilon} = -\frac{1}{t\epsilon^2} - \frac{\log(-t/\mu^2)}{t\epsilon} - \frac{(\log(-t/\mu^2))^2}{2t} + O(\epsilon). \quad (\text{B.3})$$

The ultraviolet divergent integral. The massless bubble integral is defined as

$$I_2(t) = \frac{1}{i\pi^{2-\epsilon} r_\Gamma} \int \frac{d^{4-2\epsilon} \ell \mu^{2\epsilon}}{\ell^2(\ell+q)^2} = \frac{1}{\epsilon} + 2 - \log(-t/\mu^2) + O(\epsilon). \quad (\text{B.4})$$

The finite integral. The massive triangle is given by

$$I_3(t, M^2) = \frac{1}{i\pi^{2-\epsilon} r_\Gamma} \int \frac{d^{4-2\epsilon} \ell \mu^{2\epsilon}}{(\ell+p_4)^2(\ell-p_3)^2(\ell^2-M^2)} = \frac{1}{t\beta} \left[4\zeta(2) + 2\text{Li}_2\left(\frac{\beta-1}{\beta+1}\right) + \frac{1}{2} \log^2\left(\frac{\beta-1}{\beta+1}\right) \right], \quad (\text{B.5})$$

where $\text{Li}_2(x) = \sum_{n \geq 1} \frac{x^n}{n^2}$ and $\beta^2 = 1 - \frac{4M^2}{t}$. The non-relativistic limit leads to

$$I_3(t, M^2) \simeq -\frac{1}{32\pi^2 M^2} \left(\log\left(\frac{-t}{M^2}\right) + \frac{\pi^2 M}{\sqrt{-t}} \right). \quad (\text{B.6})$$

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