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Aspects of superconformal multiplets in $D > 4$

Matthew Buican,^{a,b} Joseph Hayling^a and Constantinos Papageorgakis^a

^a*CRST and School of Physics and Astronomy, Queen Mary University of London,
Mile End Road, London, U.K.*

^b*Enrico Fermi Institute and Department of Physics, The University of Chicago,
S Ellis Ave, Chicago, U.S.A.*

E-mail: m.buican@qmul.ac.uk, j.a.hayling@qmul.ac.uk,
c.papageorgakis@qmul.ac.uk

ABSTRACT: We explicitly construct and list all unitary superconformal multiplets, along with their index contributions, in five and six dimensions. From this data, we uncover various unifying themes in the representation theory of five- and six-dimensional superconformal field theories. At the same time, we provide a detailed argument for the complete classification of unitary irreducible representations in five dimensions using a combination of physical and mathematical techniques.

KEYWORDS: Conformal and W Symmetry, Conformal Field Theory, Extended Supersymmetry

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1 Introduction and summary

By virtue of their high degree of symmetry, superconformal field theories (SCFTs) are somewhat simpler arenas in which to test and understand general ideas in quantum field theory (QFT) like duality [1–6] and emergent symmetry [7]. Moreover, since the end-points of supersymmetric (SUSY) renormalisation group (RG) flows are often SCFTs, they constrain the asymptotics of QFT and give rise to striking manifestations of the idea of universality [8, 9].

To construct an SCFT, we start with a superconformal algebra (SCA). In his pioneering work, Nahm showed that these algebras admit a simple classification [10]. This list of allowed SCAs gives rise to important constraints even away from criticality.¹ At a superconformal point, we find the basic building blocks of the theory — the multiplets of local operators² — by studying the unitary irreducible representations (UIRs) of the SCA.

¹For example, Nahm’s classification shows that six-dimensional $(1, 1)$ QFTs do not flow to SCFTs at short distances.

²We should also supplement these degrees of freedom with the non-local operators of the theory.

These UIRs are of two general types: short representations and long representations. Short UIRs have primaries that are annihilated by certain non-trivial combinations of the Poincaré supercharges while long representations do not. Moreover, short representations can contribute to the superconformal index [11–13], can realise non-trivial structures like chiral algebras [14, 15] and chiral rings that enjoy various non-renormalisation properties, can be used to study the structure of anomalies [16], and can describe the SUSY-preserving relevant and marginal deformations of SCFTs [17, 18]. Furthermore, by understanding how short representations recombine to form long representations one can hope, when sufficient symmetry is present, to bootstrap non-trivial correlation functions of local operators and perhaps even whole theories (see [14, 19] for important recent progress on this front).

In this paper, we perform the conceptually straightforward, but computationally non-trivial, task of giving the level-by-level construction of all UIRs for the five-dimensional $\mathcal{N} = 1$ and six-dimensional (1,0) and (2,0) SCAs (these are the only allowed SCAs in five and six dimensions [10]). We also calculate the most general superconformal index associated with these multiplets. Our approach throughout is based on the presentation and conventions of [13, 20].³

We expect the results assembled here to be useful for more detailed studies of the many still-mysterious SCFTs in five and six dimensions (see, e.g., the theories described in the classic works [25–29] and the more recent literature [30, 31]) as well as for more general explorations of the space of SCFTs in these dimensions.

Before delving into technical details, we should note that although the precise construction of the various multiplets depends on the spacetime dimension and amount of supersymmetry, we find various unifying themes in five and six dimensions. For example, we will see that multiplets containing conserved currents or obeying equations of motion cannot take part in recombination rules.⁴ In particular, we will show that:

- (i) Multiplets containing higher-spin currents can never recombine into long multiplets. This statement is compatible with the fact that theories in 5D and 6D are isolated as SUSY theories [17, 18], because it implies that there are no exactly marginal SUSY deformations of (almost) free theories.⁵ This situation is unlike the one in four dimensions, where such pairing up is required in the decoupling limit of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal gauge theories.
- (ii) Multiplets containing conserved spin-two currents can never recombine into long multiplets. This statement is also compatible with the isolated nature of 5D and 6D

³Note that a comprehensive classification of unitary irreducible representations (UIRs) for all SCAs was carried out in [13, 20–24] and further discussed in [18]. However, in this paper we add to these works by giving the level-by-level construction of the corresponding multiplets as well as the resulting superconformal index contributions. Part of this work was already done in [14] for the 6D (2,0) SCA (but we will provide the full set of multiplets and index contributions for this algebra).

⁴Unlike in 4D, where one can sometimes tune an exactly marginal parameter and short multiplets may recombine into long ones, 5D and 6D superconformal theories are necessarily isolated [17, 18]. Therefore, recombination should be understood purely at the level of superconformal representation theory, i.e. how one can write a long multiplet in terms of short multiplets.

⁵Here we use [32–34].

theories [17, 18], because it implies that there are no marginal SUSY couplings between general isolated SCFTs. In four dimensions, such recombination is required in coupling isolated interacting $\mathcal{N} = 2$ SCFTs (see, e.g., [6, 35, 36] for examples of such couplings between theories and [37] for a more general discussion).

- (iii) Flavour-symmetry currents⁶ are present in 5D and 6D (1,0) linear multiplets, which cannot recombine into long multiplets (for a discussion in the six-dimensional context, see [38]). This situation is analogous to the one in four-dimensional $\mathcal{N} = 2$ theories.
- (iv) Certain classes of multiplets cannot appear in free SCFTs.⁷ These include certain 6D (1,0) \mathcal{B} -type and \mathcal{C} -type multiplets in section 4 as well as some 6D (2,0) \mathcal{B} -type, \mathcal{C} -type and \mathcal{D} -type multiplets in section 5.

The methodology we use to extract our results is rather general and well established [13, 14, 18, 20, 39, 40]. Indeed, we use a simple Verma-module construction to obtain all irreducible representations of the full SCA from irreducible representations of its maximal compact subalgebra. The UIRs are labelled by highest weights corresponding to superconformal primaries, from which all descendants are recovered by the action of momentum operators and supercharges. Hence, each UIR is uniquely identified by a string of quantum numbers, which characterises the superconformal primary state. As we described above, there are both long and short multiplets. The short multiplets have null states, which can be consistently deleted (hence the moniker, “short”). A complete classification of short UIRs can be obtained by imposing the condition of unitarity. For special values of the quantum numbers characterising short UIRs, additional null states can occur. The precise enumeration and analysis of all such possibilities using unitarity is an intricate task.

Once all null states have been identified, the Racah-Speiser (RS) algorithm simplifies the multiplet construction and clarifies the origin of equations of motion and conservation equations, whenever these are present.⁸ The RS algorithm provides a prescription for the Clebsch-Gordan decomposition of states in representation space. Since representations of the maximal compact subalgebra are labelled by highest weights, these take values in the fundamental Weyl chamber and the corresponding Dynkin labels are positive. After the Clebsch-Gordan decomposition, a representation in the sum with negative Dynkin labels lies outside the fundamental Weyl chamber and can no longer label an irreducible representation. The RS prescription involves applying successive Weyl reflections, which bounce the weight vector off the boundaries of the Weyl chamber. Each time a Weyl reflection is performed, the multiplicity of the representation flips sign. Therefore, if a representation is labelled by negative Dynkin labels, it gets reflected back into the fundamental Weyl chamber up to a sign. If it is labelled by a weight which lies exactly on the boundary

⁶In this paper, we define these to be currents for symmetries that commute with the full SCA. Therefore, these currents do not sit in multiplets with higher-spin symmetries. Note that, as in four dimensions, there are spin-one currents that give rise to charges that commute with the R symmetry and also sit in higher-spin multiplets, but the corresponding charges are necessarily part of a larger algebraic structure including supercharges and higher-spin charges. For example, we will see such a current in (2.30). This operator gives rise to a charge that acts on bosons but not on fermions.

⁷Throughout this paper we only consider unitary theories.

⁸A concise summary of the Racah-Speiser algorithm can be found in appendix B of [39].

of the fundamental Weyl chamber, the state has zero multiplicity and should be removed from the sum. A natural interpretation for representations with negative multiplicities is in terms of constraints imposed on operators inside the multiplet [39].

Since we study the 5D $\mathcal{N} = 1$, 6D (1, 0), and 6D (2, 0) SCAs, our presentation is split into three corresponding sections, all of which are largely self-contained. The reader who is familiar with the classification of UIRs and only interested in looking up the results can proceed directly to the relevant tables. Each multiplet is labelled by the quantum numbers designating its superconformal primary and the shortening conditions the latter obeys. Some multiplets with special values for their quantum numbers admit a distinct physical interpretation; these are dealt with separately. For those interested in the approach employed to obtain our diagrams, we provide a detailed discussion for the case of the 5D $\mathcal{N} = 1$ SCA in section 2, which extends naturally to 6D in section 4 and section 5. Throughout this analysis, special emphasis is put on identifying operator constraints, whenever present. Each section also contains expressions for recombination rules and indices for the superconformal multiplets under study. A short collection of simple applications arising from the results of our analysis is presented in section 6. Finally, section 3 contains an argument for the complete classification of 5D UIRs, which has been missing from the literature (paying attention to some recent observations made in [41–43]).

We also include various appendices. Appendix A, B and C contain conventions and results which are necessary for our multiplet construction but would shift the focus away from our aim in the main part of the text; SCA conventions for five and six dimensions, the construction of supercharacters and the superconformal index, as well as the relationship between the RS algorithm and the identification of operator constraints. Appendix D collects the superconformal indices for all 5D and 6D multiplets for quick reference. As the 6D (2,0) refined indices are cumbersome, we only ever write down their Schur limit. However, we also provide a complementary Mathematica file with all the refined superconformal indices in five and six dimensions. Finally, appendix E contains the explicit 6D (2,0) spectra, which are too unwieldy to present in section 5.

Note added. While finalising our construction of multiplets in $D > 4$, we became aware of an upcoming publication ([44] cited in [18]). This upcoming work promises to be broader in scope than our own and have overlap with some of our constructions. Knowledge of the multiplet structure in five and six dimensional SCFTs, in [44], is essential background material for the results in [16, 18]; see also [38, 45–48]. We would like to thank C. Córdova, T. Dumitrescu, and K. Intriligator for relevant correspondence, as well as for pointing out an error in the classification of 5D multiplets in a draft version of this paper.

2 Multiplets and superconformal indices for 5D $\mathcal{N} = 1$

We begin by providing a systematic analysis of all short multiplets admitted by the 5D $\mathcal{N} = 1$ SCA, $F(4)$. This involves a derivation of the superconformal unitarity bounds. By doing so we reproduce the results of [13]. We then proceed to write the complete multiplet spectra and compute their indices. Our notation and conventions for the 5D SCA are provided in appendix A.

2.1 UIR building with auxiliary Verma modules

The superconformal primaries of the 5D SCA $F(4)$ are designated $|\Delta; l_1, l_2; k\rangle$, where Δ is the conformal dimension, $l_1 \geq l_2 > 0$ are Lorentz symmetry quantum numbers in the orthogonal basis and k is an R -symmetry label. Each primary is in one-to-one correspondence with a highest weight state of the maximal compact subalgebra $\mathfrak{so}(5) \oplus \mathfrak{so}(2) \oplus \mathfrak{su}(2)_R \subset F(4)$.⁹ There are eight Poincaré and eight superconformal supercharges, denoted by $\mathcal{Q}_{\mathbf{A}a}$ and $\mathcal{S}_{\mathbf{A}a}$ respectively — where $a = 1, \dots, 4$ is an $\mathfrak{so}(5)$ Lorentz spinor index and $\mathbf{A} = \mathbf{1}, \mathbf{2}$ an index of $\mathfrak{su}(2)_R$. One also has five momenta \mathcal{P}_μ and special conformal generators \mathcal{K}_μ , where $\mu = 1, \dots, 5$ is a Lorentz vector index. The superconformal primary is annihilated by all $\mathcal{S}_{\mathbf{A}a}$ and \mathcal{K}_μ . A basis for the representation space of $F(4)$ can be constructed by considering the following Verma module

$$\prod_{\mathbf{A}, a} (\mathcal{Q}_{\mathbf{A}a})^{n_{\mathbf{A}, a}} \prod_{\mu} \mathcal{P}_\mu^{n_\mu} |\Delta; l_1, l_2; k\rangle^{\text{hw}} \quad (2.1)$$

for some ordering of operators,¹⁰ where $n = \sum_{\mathbf{A}, a} n_{\mathbf{A}, a}$ and $\hat{n} = \sum_{\mu} n_\mu$ denote the “level” of a superconformal or conformal descendant respectively. In order to obtain UIRs, the requirement of unitarity needs to be imposed level-by-level on the Verma module. This leads to bounds on the conformal dimension Δ .

The highest $\mathfrak{su}(2)_R$ -weight level-one superconformal-descendant states can be expressed in a particularly suitable alternative basis as $\Lambda_{\mathbf{1}}^a |\Delta; l_1, l_2; k\rangle^{\text{hw}}$, where we define

$$\Lambda_{\mathbf{1}}^a := \sum_{b=1}^a \mathcal{Q}_{\mathbf{1}b} \lambda_b^a. \quad (2.2)$$

The λ_b^a are functions of the $\mathfrak{so}(5)$ quantum numbers and Lorentz lowering operators. The combinations (2.2) have the property that, when acting upon a conformal-primary highest-weight state of $\mathfrak{so}(5) \oplus \mathfrak{so}(2)$, they produce another conformal-primary highest-weight state. They can be uniquely determined by imposing the requirement that all Lorentz raising operators and R -symmetry raising operators annihilate $\Lambda_{\mathbf{A}}^a |\Delta; l_1, l_2; k\rangle^{\text{hw}}$ and are given in appendix A.4. It turns out that the most stringent unitarity bounds emerge by studying the norms of states constructed by acting with the $\Lambda_{\mathbf{1}}^a$ s on the superconformal primary. We provide a detailed argument in favour of this fact in section 3.

Ill-defined states. The definition of the Λ generators — as explicitly given in appendix A.4 — is such that for certain values of the quantum numbers the resulting state is not well defined. Consider e.g. the state

$$\Lambda_{\mathbf{1}}^2 |\Delta; l_1, l_2; k\rangle = \left(\mathcal{Q}_{\mathbf{11}} \lambda_{\mathbf{1}}^1 + \mathcal{Q}_{\mathbf{12}} \lambda_{\mathbf{1}}^2 \right) |\Delta; l_1, l_2; k\rangle = \left(\mathcal{Q}_{\mathbf{11}} - \mathcal{Q}_{\mathbf{12}} \mathcal{M}_2^- \frac{1}{2\mathcal{H}_2} \right) |\Delta; l_1, l_2; k\rangle, \quad (2.3)$$

where $\mathcal{H}_2 |\Delta; l_1, l_2; k\rangle = l_2 |\Delta; l_1, l_2; k\rangle$. The above is clearly ill-defined for $l_2 = 0$: although $\mathcal{M}_2^- |\Delta; l_1, 0; k\rangle = 0$ the factor of $1/l_2$ diverges and the norm of (2.3) is indeterminate.

⁹The quantum numbers labelling the primary are eigenvalues for the Cartans of the maximal compact subalgebra in a particular basis.

¹⁰Any other ordering can be obtained using the superconformal algebra.

However, there exist cases where products of ill-defined Λ s can lead to well-defined states, through various cancellations. Hence, one has to perform a delicate analysis of such possibilities through explicit calculation.

This phenomenon will be very important in the classification of unitarity bounds below, where one needs to evaluate the norms of all well-defined, distinct (i.e. not related through commutation relations) products of Λ s.¹¹

Unitarity bounds for $l_1 \geq l_2 > 0$. We can calculate the norms of the superconformal descendant states at level one to be

$$\begin{aligned}
 \|\Lambda_1^4 |\Delta; l_1, l_2; k\rangle^{\text{hw}}\|^2 &= (\Delta - 3k - l_1 - l_2 - 4) \frac{(2l_1 + 3)(l_1 + l_2 + 2)(2l_2 + 1)}{4(l_1 + 1)l_2(l_1 + l_2 + 1)}, \\
 \|\Lambda_1^3 |\Delta; l_1, l_2; k\rangle^{\text{hw}}\|^2 &= (\Delta - 3k - l_1 + l_2 - 3) \frac{(2l_1 + 3)(l_1 - l_2 + 1)}{2(l_1 + 1)(l_1 - l_2)}, \\
 \|\Lambda_1^2 |\Delta; l_1, l_2; k\rangle^{\text{hw}}\|^2 &= (\Delta - 3k + l_1 - l_2 - 1) \frac{(2l_2 + 1)}{2l_2}, \\
 \|\Lambda_1^1 |\Delta; l_1, l_2; k\rangle^{\text{hw}}\|^2 &= (\Delta - 3k + l_1 + l_2), \tag{2.4}
 \end{aligned}$$

where we have normalised $\left\| |\Delta; l_1, l_2; k\rangle^{\text{hw}} \right\|^2 = 1$. Observe that these norms are all of the form

$$\left\| \Lambda_1^a |\Delta; l_1, l_2; k\rangle^{\text{hw}} \right\|^2 = \left(\Delta - f^a(l_1, l_2, k) \right) g^a(l_1, l_2) =: B^a(l_1, l_2, k) g^a(l_1, l_2), \tag{2.5}$$

where $g^a(l_1, l_2)$ is a positive-definite rational function in the fundamental Weyl chamber, $l_1 \geq l_2 > 0$. Unitarity demands that the norms are positive semidefinite and this imposes a bound on the conformal dimension via the functions $B^a(l_1, l_2, k)$. The strongest bound on the conformal dimension is provided by $B^4(l_1, l_2, k) \geq 0$. When $B^4(l_1, l_2, k) > 0$ the UIR can be obtained using (2.1). The resulting multiplet is called “long” and labelled \mathcal{L} .

When $B^4(l_1, l_2, k) = 0$ the state is null. This means that the primary obeys the “shortening condition” $\Lambda_1^4 |\Delta; l_1, l_2; k\rangle^{\text{hw}} = 0$. All such states can be consistently removed from the superconformal representation. The resulting multiplet is “short” and labelled as type \mathcal{A} . Since it can be reached from a long multiplet by continuously dialling Δ it is called a “regular” short multiplet. At higher levels, $\prod_{a=1}^n \Lambda_1^a |\Delta; l_1, l_2; k\rangle^{\text{hw}}$ with $n > 1$, the norms involve products of $B^a(l_1, l_2, k)$ s and the strongest bound still comes from $B^4(l_1, l_2, k) \geq 0$. Therefore, there will be no change to the bounds obtained at level one.

Unitarity bounds for $l_1 > l_2 = 0$. We now turn to the special case with $l_1 > 0$, $l_2 = 0$, where the concept of ill-defined states becomes important. When $l_2 = 0$ the operator Λ_1^4 is not well defined and we have to omit the level-one state $\Lambda_1^4 |\Delta; l_1, 0; k\rangle^{\text{hw}}$ from our spectrum. Naively, the strongest bound then arises from the norm of the state

¹¹For example, in our upcoming discussion of unitarity bounds for $l_1 = l_2 = 0$, $\Lambda_1^3 |\Delta; 0, 0; k\rangle^{\text{hw}}$ and $\Lambda_1^2 |\Delta; 0, 0; k\rangle^{\text{hw}}$ are individually ill defined, while $\Lambda_1^2 \Lambda_1^3 |\Delta; 0, 0; k\rangle^{\text{hw}}$ is not. This can in turn lead to the wrong identification of shortening conditions, since $\|\Lambda_1^2 \Lambda_1^3 |\Delta; 0, 0; k\rangle^{\text{hw}}\|^2 = B^3(0, 0, k) B^1(0, 0, k)$, whereas $\|\Lambda_1^1 \Lambda_1^2 |\Delta; 0, 0; k\rangle^{\text{hw}}\|^2 = B^2(0, 0, k) B^1(0, 0, k)$, with the first one leading to more stringent unitarity bounds.

$\Lambda_1^3 |\Delta; l_1, 0; k\rangle^{\text{hw}}$. However, the level-two state $\Lambda_1^3 \Lambda_1^4 |\Delta; l_1, 0; k\rangle^{\text{hw}}$ is actually well defined, as can be explicitly checked. Its norm is proportional to

$$\left\| \Lambda_1^3 \Lambda_1^4 |\Delta; l_1, 0; k\rangle^{\text{hw}} \right\|^2 \propto (\Delta - 3k - l_1 - 4)(\Delta - 3k - l_1 - 3) = B^3(l_1, 0, k)B^4(l_1, 0, k) \tag{2.6}$$

and the corresponding set of restrictions come from $B^4(l_1, 0, k) \geq 0$ or $B^3(l_1, 0, k) = 0$.

When $B^4(l_1, 0, k) = 0$ one recovers a regular short representation of type \mathcal{A} . Instead, one could also have $B^3(l_1, 0, k) = 0$; this gives rise to the null state $\Lambda_1^3 |\Delta; l_1, 0; k\rangle^{\text{hw}}$. Making that choice leads to an “isolated” short multiplet of type \mathcal{B} .¹²

Unitarity bounds for $l_1 = l_2 = 0$. The same logic extends to $l_1 = l_2 = 0$: at level one the only well-defined state is $\Lambda_1^1 |\Delta; 0, 0; k\rangle^{\text{hw}}$. However, there exist well-defined states at levels two and four, obtained by $\Lambda_1^2 \Lambda_1^3 |\Delta; 0, 0; k\rangle^{\text{hw}}$, $\Lambda_1^1 \Lambda_1^2 \Lambda_1^3 \Lambda_1^4 |\Delta; 0, 0; k\rangle^{\text{hw}}$. These give rise to the conditions $B^1(0, 0, k) = 0$, $B^3(0, 0, k) = 0$ or $B^4(0, 0, k) \geq 0$ and lead to the new set of isolated short multiplets \mathcal{D} . We summarise their properties and list all short multiplets for the 5D SCA in table 1.

Additional unitarity bounds. Finally, there are supplementary unitarity restrictions and associated null states originating from conformal descendants. These have been analysed in detail in [20, 40], the results of which we use. Saturating a conformal bound results in a “momentum-null” state, where the corresponding shortening condition is an operator constraint involving momentum analogues of the superconformal Λ s [40]. In that reference, a prescription is given for removing the associated states, $\mathcal{P}_\mu |\Delta; l_1, l_2; k\rangle^{\text{hw}}$, from the auxiliary Verma-module construction, again in analogy with the superconformal procedure.¹³ However, we will choose not to exclude any momenta from the basis of Verma-module generators (2.1). After using the RS algorithm this choice will allow us to explicitly recover highest weight states corresponding to the operator constraints from the general multiplet structure.

One can combine the conformal and superconformal bounds to predict that operator constraints will appear in the following short multiplets:

$$\mathcal{B}[d_1, 0; 0], \quad \mathcal{D}[0, 0; \{1, 2\}]. \tag{2.7}$$

The multiplet $\mathcal{D}[0, 0; 0]$ does not belong to this list as it is the vacuum.

Highest weight construction through the auxiliary Verma module. The Λ basis through which we obtained the unitarity bounds could in principle be used to construct the full multiplet. However, executing this procedure would require knowledge of the full Clebsh-Gordan decomposition for the resulting states. This is a very difficult task to carry out in practice. For that reason we will resort to constructing the highest weights of the superconformal representation using the auxiliary Verma module via the Racah-Speiser

¹²The name isolated is due to the fact that there are no states in the gap between $B^3(l_1, 0, k) = 0$ and $B^4(l_1, 0, k) \geq 0$.

¹³Note that this does not mean that all conformal descendants of a particular type should be removed from the set of local operators.

Multiplet	Shortening Condition	Conformal Dimension
$\mathcal{A}[l_1, l_2; k]$	$\Lambda_1^4 \Psi = 0$	$\Delta = 3k + l_1 + l_2 + 4$
$\mathcal{A}[l_1, 0; k]$	$\Lambda_1^3 \Lambda_1^4 \Psi = 0$	$\Delta = 3k + l_1 + 4$
$\mathcal{A}[0, 0; k]$	$\Lambda_1^1 \Lambda_1^2 \Lambda_1^3 \Lambda_1^4 \Psi = 0$	$\Delta = 3k + 4$
$\mathcal{B}[l_1, 0; k]$	$\Lambda_1^3 \Psi = 0$	$\Delta = 3k + l_1 + 3$
$\mathcal{B}[0, 0; k]$	$\Lambda_1^2 \Lambda_1^3 \Psi = 0$	$\Delta = 3k + 3$
$\mathcal{D}[0, 0; k]$	$\Lambda_1^1 \Psi = 0$	$\Delta = 3k$

Table 1. A list of all short multiplets for the 5D $\mathcal{N} = 1$ SCA, along with the conformal dimension of the superconformal primary and the corresponding shortening condition. The Λ_1^q in the shortening conditions are defined in (2.2) and (A.17). The first of these multiplets (\mathcal{A}) is a regular short representation, whereas the rest (\mathcal{B}, \mathcal{D}) are isolated short representations. Here Ψ denotes the superconformal primary state for each multiplet.

algorithm. This greatly simplifies the Glebsch-Gordan decomposition by implementing it at the level of highest weights.

Having motivated the use of the auxiliary Verma module basis, we will implement the conjectural recipe of [13, 39, 49] to generate the spectrum; see also appendix C of [14]. According to these references, in addition to removing the supercharge associated with the shortening condition, one is instructed to also remove any other supercharge combination that annihilates the auxiliary primary.

To make this point more transparent, let us consider the example of the $\mathcal{B}[l_1, 0; k]$ multiplet. The shortening condition dictates that we remove $\Lambda_3^1 \Psi \Rightarrow \mathcal{Q}_{13} \Psi_{\text{aux}}$, where note that Ψ and Ψ_{aux} have the same quantum numbers. Since $l_2 = 0$ we have that $\mathcal{M}_2^- \Psi_{\text{aux}} = 0$ and therefore

$$0 = \mathcal{M}_2^- \mathcal{Q}_{13} \Psi_{\text{aux}} = \mathcal{Q}_{14} \Psi_{\text{aux}}. \tag{2.8}$$

Hence we are required to also remove \mathcal{Q}_{14} from the auxiliary Verma-module basis. Note that (2.8) does *not* imply $\Lambda_4^1 \Psi = 0$ but is merely a prescription for obtaining the correct set of highest weights. One could similarly use any lowering operator of the maximal compact subalgebra. E.g. for $k = 0$ additional conditions can be generated by acting on the existing ones with R -symmetry lowering operators,¹⁴ resulting in the removal of more combinations of supercharges from the set of auxiliary Verma-module generators. We will mention explicitly the full set of such “absent supercharges” at the beginning of each case in our upcoming analysis.

We emphasise that this is an *auxiliary* Verma-module construction which leads to the same spectrum in terms of highest weights. If one is interested in the precise form of the operators, the much more involved Λ -basis should be used.

¹⁴See e.g. the discussion in appendix 6.2.1 of [14] in the context of the 6D (2,0) SCA.

2.2 The procedure

Based on the above ingredients, let us summarise our strategy for constructing the superconformal UIRs:

1. For a given multiplet type, begin with a superconformal primary and consider the highest-weight component of the corresponding irreducible representation of the Lorentz and R -symmetry algebras.
2. Implement the conjecture of [13, 14, 39, 49] to determine all combinations of supercharges which need to be removed from the auxiliary Verma-module basis (2.1).
3. Use the remaining auxiliary Verma-module generators to determine the highest weights for all descendant states. This may result in some of the quantum numbers labelling the highest weight state becoming negative.
4. Apply the Racah-Speiser algorithm to recover a spectrum with only positive quantum numbers.¹⁵ This could result in some states being projected out, while others acquiring a “negative multiplicity”. The latter can cancel out against other states with the same quantum numbers but positive multiplicity.¹⁶ Any remaining states with negative multiplicity can be interpreted as operator constraints. This conjectural identification follows [39] and is based on a large number of examples, but can be additionally supported using supercharacters; cf. appendix C.2.
5. In some special cases, the supercharges that have been removed from the auxiliary Verma-module basis anticommute into momentum generators, which should also be removed. This has the effect of projecting out states corresponding to operator constraints. The operator constraints can be restored using the discussion in appendix C.

The spectrum of a given superconformal multiplet can always be obtained following these steps and we believe the results (in all $D > 4$ SCAs that we have considered) to be correct: they satisfy the expected recombination rules, which have been checked using supercharacters. Of the multiplets that do not appear in recombination rules, for 5D $\mathcal{N} = 1$ and 6D $(1, 0)$ SCAs, we have also explicitly constructed the states using free fields. For 6D $(2, 0)$ the multiplets that do not appear have had the Schur limit of their superconformal indices matched with the results of [14]. Finally, an additional check on the computational implementation of this procedure is via supercharacters, which can be calculated in two ways: either by evaluating them over the states of a given short multiplet obtained using the RS algorithm or directly using the Weyl character formula; cf. appendix B. Both of these agree for all multiplets listed in our work.

¹⁵The details of the Racah-Speiser algorithm needed for this step can be found in appendix C.

¹⁶There are instances when this general procedure leads to ambiguities, i.e. there is more than one choice for performing the cancellations; see also [18]. However, these can be resolved uniquely by the requirement that all highest weight states should be reached by the successive action of allowed supercharges on the superconformal primary. For the examples that we investigate in this article, this phenomenon only appears in the $(2, 0)$ SCA $(\mathcal{B}[c_1, c_2, 0; 0, 0], \mathcal{C}[c_1, 0, 0; 0, 0])$, in which case the multipet spectra have also been compared to the construction by successive \mathcal{Q} -actions.

2.3 5D $\mathcal{N} = 1$ multiplet recombination rules

For the purposes of listing the recombination rules as well as for explicitly constructing the multiplets, we will find it more convenient to switch to the Dynkin basis for the various quantum numbers. That is, we will use

$$d_1 = l_1 - l_2, \quad d_2 = 2l_2, \quad K = 2k. \quad (2.9)$$

Short multiplets can recombine to form long multiplets when the conformal dimension for the latter approaches the unitarity bound, that is when $\Delta + \epsilon \rightarrow \frac{3}{2}K + d_1 + d_2 + 4$. It can then be checked, using the results that we will present in the following sections, that

$$\begin{aligned} \mathcal{L}[\Delta + \epsilon; d_1, d_2; K] &\xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[d_1, d_2; K] \oplus \mathcal{A}[d_1, d_2 - 1; K + 1], \\ \mathcal{L}[\Delta + \epsilon; d_1, 0; K] &\xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[d_1, 0; K] \oplus \mathcal{B}[d_1 - 1, 0; K + 2], \\ \mathcal{L}[\Delta + \epsilon; 0, 0; K] &\xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[0, 0; K] \oplus \mathcal{D}[0, 0; K + 4]. \end{aligned} \quad (2.10)$$

The following multiplets do not appear in a recombination rule:

$$\begin{aligned} &\mathcal{B}[d_1, 0; \{0, 1\}], \\ &\mathcal{D}[0, 0; \{0, 1, 2, 3\}]. \end{aligned} \quad (2.11)$$

2.4 The 5D $\mathcal{N} = 1$ superconformal index

We define the superconformal index with respect to the supercharge \mathcal{Q}_{14} , in accordance with [13, 50]. This is given by

$$\mathcal{I}(x, y) = \text{Tr}_{\mathcal{H}}(-1)^F e^{-\beta\delta} x^{\frac{2}{3}\Delta + \frac{1}{3}(d_1 + d_2)} y^{d_1}, \quad (2.12)$$

where making use of the spin-statistics theorem the fermion number is $F = d_2$ and the trace is over the Hilbert space of operators of the theory. The states that are counted by this index satisfy $\delta = 0$, where

$$\delta := \{\mathcal{Q}_{14}, \mathcal{S}_{21}\} = \Delta - \frac{3}{2}K - d_1 - d_2. \quad (2.13)$$

It is easy to see that as a result long multiplets can never contribute to the index. The charges d_1 and $\frac{2}{3}\Delta + \frac{1}{3}(d_1 + d_2)$ appearing in the exponents of (2.12) are eigenvalues for the generators commuting with $\mathcal{Q}_{14}, \mathcal{S}_{21}$ and consequently with δ . In practice, this index can be explicitly evaluated as a supercharacter for each of the multiplets constructed below. A detailed construction of characters for superconformal representations is reviewed in appendix B.

2.5 Long multiplets

We can now go ahead with the explicit construction of multiplets. Since long multiplets are not associated with any shortening conditions, we can proceed as per (2.1) acting with all supercharges and momenta on the superconformal primary to obtain the unitary superconformal representation.

We will choose to group the supercharges together as $Q = (\mathcal{Q}_{\mathbf{A}1}, \mathcal{Q}_{\mathbf{A}2})$ and $\tilde{Q} = (\mathcal{Q}_{\mathbf{A}3}, \mathcal{Q}_{\mathbf{A}4})$, purely for book-keeping purposes. The explicit quantum numbers of these supercharges are given by

$$\begin{aligned} \mathcal{Q}_{11} &\sim (1)_{(0,1)}, & \mathcal{Q}_{12} &\sim (1)_{(1,-1)}, & \mathcal{Q}_{13} &\sim (1)_{(-1,1)}, & \mathcal{Q}_{14} &\sim (1)_{(0,-1)}, \\ \mathcal{Q}_{21} &\sim (-1)_{(0,1)}, & \mathcal{Q}_{22} &\sim (-1)_{(1,-1)}, & \mathcal{Q}_{23} &\sim (-1)_{(-1,1)}, & \mathcal{Q}_{24} &\sim (-1)_{(0,-1)}. \end{aligned} \quad (2.14)$$

With this information in hand, it is straightforward to map out their action starting from a superconformal primary, labelled by $(K)_{(d_1, d_2)}$:

$$\begin{aligned} (K)_{(d_1, d_2)} &\xrightarrow{Q} (K+1)_{(d_1, d_2+1), (d_1+1, d_2-1)}, (K-1)_{(d_1, d_2+1), (d_1+1, d_2-1)}, \\ &\xrightarrow{Q^2} (K+2)_{(d_1+1, d_2)}, (K)_{(d_1, d_2+2), (d_1+1, d_2)^2, (d_1+2, d_2-2)}, (K-2)_{(d_1+1, d_2)}, \\ &\xrightarrow{Q^3} (K+1)_{(d_1+1, d_2+1), (d_1+2, d_2-1)}, (K-1)_{(d_1+1, d_2+1), (d_1+2, d_2-1)}, \\ &\xrightarrow{Q^4} (K)_{(d_1+2, d_2)}, \\ (K)_{(d_1, d_2)} &\xrightarrow{\tilde{Q}} (K+1)_{(d_1, d_2-1), (d_1-1, d_2+1)}, (K-1)_{(d_1, d_2-1), (d_1-1, d_2+1)}, \\ &\xrightarrow{\tilde{Q}^2} (K+2)_{(d_1-1, d_2)}, (K)_{(d_1, d_2-2), (d_1-1, d_2)^2, (d_1-2, d_2+2)}, (K-2)_{(d_1-1, d_2)}, \\ &\xrightarrow{\tilde{Q}^3} (K+1)_{(d_1-1, d_2-1), (d_1-2, d_2+1)}, (K-1)_{(d_1-1, d_2-1), (d_1-2, d_2+1)}, \\ &\xrightarrow{\tilde{Q}^4} (K)_{(d_1-2, d_2)}, \end{aligned} \quad (2.15)$$

where we have split the actions of Q and \tilde{Q} into two ‘‘chains’’. Since the Dynkin labels are generic, there is no need to implement the RS algorithm. By definition, these multiplets do not contribute to the superconformal index.

2.6 \mathcal{A} -type multiplets

Recall from table 1 that \mathcal{A} -type multiplets obey three types of shortening conditions depending on the quantum numbers of the superconformal primary. These result in the removal of the following combinations of supercharges from the basis of auxiliary Verma-module generators (2.1):¹⁷

$$\begin{aligned} \mathcal{A}[d_1, d_2; K] &: \mathcal{Q}_{14}, \\ \mathcal{A}[d_1, 0; K] &: \mathcal{Q}_{13} \mathcal{Q}_{14}, \\ \mathcal{A}[0, 0; K] &: \mathcal{Q}_{11} \mathcal{Q}_{12} \mathcal{Q}_{13} \mathcal{Q}_{14}. \end{aligned} \quad (2.16)$$

Let us consider the first case. On the one hand, acting with the allowed set of supercharges yields the same result as found in (2.15) for the Q -supercharge set. On the other,

¹⁷Once again, we are using a Dynkin basis for the quantum numbers.

for the \tilde{Q} -supercharge set we have

$$\begin{aligned}
 (K)_{(d_1, d_2)} &\xrightarrow{\tilde{Q}} (K+1)_{(d_1-1, d_2+1)}, (K-1)_{(d_1, d_2-1), (d_1-1, d_2+1)}, \\
 &\xrightarrow{\tilde{Q}^2} (K)_{(d_1-2, d_2+2), (d_1-1, d_2)}, (K-2)_{(d_1-1, d_2)}, \\
 &\xrightarrow{\tilde{Q}^3} (K-1)_{(d_1-2, d_2+1)}.
 \end{aligned}
 \tag{2.17}$$

The two remaining cases with $d_2 = 0$ and $d_1 = d_2 = 0$ can be obtained by implementing the recipe at the end of section 2.1, by e.g. first constructing all the states using the combinations of the Q -chain of eq. (2.15) and \tilde{Q} -chain of (2.17) and then setting $d_2 = 0$. Applying the RS algorithm will produce some negative-multiplicity states, all of which cancel with positive-multiplicity states with the same quantum numbers. The remaining states comprise the spectrum of the $\mathcal{A}[d_1, 0; K]$ multiplet and one can proceed analogously for $\mathcal{A}[0, 0; K]$.

For $K = 0$ and d_1, d_2 generic the primary still lies above the unitarity bound for all conformal descendants and there are no momentum-null states. Moreover, one also needs to remove \mathcal{Q}_{24} , $\mathcal{Q}_{23}\mathcal{Q}_{24}$ and $\mathcal{Q}_{21}\mathcal{Q}_{22}\mathcal{Q}_{23}\mathcal{Q}_{24}$ from the construction of the respective multiplets for $d_1, d_2 > 0$, $d_2 = 0$ and $d_1 = d_2 = 0$ using the auxiliary Verma module. The resulting spectrum is no different from setting $K = 0$ and running the RS algorithm for $\mathfrak{su}(2)_R$.

The index over the spectrum of all \mathcal{A} -type multiplets is given by

$$\mathcal{I}_{\mathcal{A}[d_1, d_2; K]}(x, y) = (-1)^{d_2+1} \frac{x^{d_1+d_2+K+4}}{(1-xy^{-1})(1-xy)} \chi_{d_1}(y),
 \tag{2.18}$$

by appropriately tuning d_1, d_2 and K , including $d_2 = 0$ and $d_1 = d_2 = 0$, where we have used the $\mathfrak{su}(2)$ character for the spin- $\frac{l}{2}$ representation

$$\chi_l(y) = \frac{y^{l+1} - y^{-l-1}}{y - y^{-1}}.
 \tag{2.19}$$

One readily sees that (2.18) is compatible with the recombination rule (2.10):

$$\lim_{\epsilon \rightarrow 0} \mathcal{I}_{\mathcal{L}[\Delta+\epsilon; d_1, d_2; K]}(x, y) = \mathcal{I}_{\mathcal{A}[d_1, d_2; K]}(x, y) + \mathcal{I}_{\mathcal{A}[d_1, d_2-1; K+1]}(x, y) = 0.
 \tag{2.20}$$

2.7 \mathcal{B} -type Multiplets

For \mathcal{B} -type multiplets, the supercharges that need to be removed from the auxiliary Verma-module basis (2.1) due to null states are

$$\begin{aligned}
 \mathcal{B}[d_1, 0; K] &: \mathcal{Q}_{13}, \\
 \mathcal{B}[0, 0; K] &: \mathcal{Q}_{12}\mathcal{Q}_{13}.
 \end{aligned}
 \tag{2.21}$$

For $\mathcal{B}[d_1, 0; K]$, following the argument in (2.8), one finds that \mathcal{Q}_{14} should also be removed from the basis of auxiliary Verma-module generators. Acting on the primary — while keeping the quantum numbers generic — one has the same action as (2.15) for Q , while the \tilde{Q} -chain is

$$\begin{aligned}
 (K)_{(d_1, d_2)} &\xrightarrow{\tilde{Q}} (K-1)_{(d_1, d_2-1), (d_1-1, d_2+1)}, \\
 &\xrightarrow{\tilde{Q}^2} (K-2)_{(d_1-1, d_2)}.
 \end{aligned}
 \tag{2.22}$$

We may then combine the action of these supercharges to construct the following grid; acting with Q is captured by *southwest* motion on the diagram, while \tilde{Q} by *southeast* motion. This module is well defined for all $d_1 \geq 1$ values:

$$\begin{array}{c}
 \Delta \\
 3 + d_1 + \frac{3K}{2} \\
 \frac{7}{2} + d_1 + \frac{3K}{2} \\
 4 + d_1 + \frac{3K}{2} \\
 \frac{9}{2} + d_1 + \frac{3K}{2} \\
 5 + d_1 + \frac{3K}{2} \\
 \frac{11}{2} + d_1 + \frac{3K}{2} \\
 6 + d_1 + \frac{3K}{2}
 \end{array}
 \begin{array}{c}
 \boxed{(K)_{(d_1,0)}} \\
 \begin{array}{cc}
 \boxed{(K \pm 1)_{(d_1,1)}} & \boxed{(K - 1)_{(d_1-1,1)}}
 \end{array} \\
 \begin{array}{ccc}
 \boxed{(K \pm 2)_{(d_1+1,0)}} & \boxed{(K)_{(d_1-1,2),(d_1,0)}} & \boxed{(K - 2)_{(d_1-1,0)}} \\
 \boxed{(K)_{(d_1,2),(d_1+1,0)}} & \boxed{(K - 2)_{(d_1-1,2),(d_1,0)}} &
 \end{array} \\
 \begin{array}{ccc}
 \boxed{(K \pm 1)_{(d_1+1,1)}} & \boxed{(K \pm 1)_{(d_1,1)}} & \boxed{(K - 1)_{(d_1-1,1)}} \\
 & \boxed{(K - 1)_{(d_1,1),(d_1-1,3)}} & \boxed{(K - 3)_{(d_1-1,1)}} \\
 & \boxed{(K - 3)_{(d_1,1)}} &
 \end{array} \\
 \begin{array}{ccc}
 \boxed{(K)_{(d_1+2,0)}} & \boxed{(K - 2)_{(d_1,2),(d_1+1,0)}} & \boxed{(K - 2)_{(d_1-1,2),(d_1,0)}} \\
 & \boxed{(K)_{(d_1,2),(d_1+1,0)}} & \boxed{(K - 4), (K)_{(d_1,0)}}
 \end{array} \\
 \begin{array}{cc}
 \boxed{(K - 1)_{(d_1+1)}} & \boxed{(K - 1), (K - 3)_{(d_1,1)}}
 \end{array} \\
 \boxed{(K - 2)_{(d_1+1,0)}}
 \end{array}
 \tag{2.23}$$

Let us next look at some special values of the R -symmetry quantum number. For $K = 1$ the states with values $(K - 4)$ and $(K - 3)$ are reflected to $-(-1)$ and $-(-0)$ respectively via the RS algorithm, where they subsequently cancel with other descendants with identical quantum numbers but non-negative multiplicity. The $(K - 2)$ states are simply deleted as they lie on the boundary of the Weyl chamber for $K = 1$. Likewise if $K = 2$, the $(K - 3)$ states are deleted and the $(K - 4)$ state is reflected to $-(-0)$ where it cancels against another state with non-negative multiplicity.

Following this reasoning, it quickly becomes obvious that only through setting $K = 0$ does one end up with negative multiplicities being present after cancellations, which we can observe from the last two levels of the module. In that case, the $(K - 1)$ highest weights are deleted to leave the states $-[11/2 + d_1; d_1, 1; 1]$ and $-[6 + d_1; d_1 + 1, 0; 0]$. We will study this $\mathcal{B}[d_1, 0; 0]$ multiplet separately below.

The second type of \mathcal{B} -multiplet to consider is $\mathcal{B}[0, 0; K]$, for which one is instructed to remove from the auxiliary Verma-module basis (2.1) the supercharge combinations $\mathcal{Q}_{12}\mathcal{Q}_{13}$, $\mathcal{Q}_{12}\mathcal{Q}_{14}$ and $\mathcal{Q}_{13}\mathcal{Q}_{14}$. The superconformal representation constructed using the remaining

supercharges is

$$\begin{array}{c}
 \Delta \\
 3 + \frac{3K}{2} \\
 \frac{7}{2} + \frac{3K}{2} \\
 4 + \frac{3K}{2} \\
 \frac{9}{2} + \frac{3K}{2} \\
 5 + \frac{3K}{2} \\
 \frac{11}{2} + \frac{3K}{2} \\
 6 + \frac{3K}{2}
 \end{array}
 \begin{array}{c}
 \boxed{(K)_{(0,0)}} \\
 \boxed{(K \pm 1)_{(0,1)}} \\
 \begin{array}{l}
 \boxed{(K)_{(0,2),(1,0)}} \\
 \boxed{(K \pm 2)_{(1,0)}}
 \end{array}
 \quad \boxed{(K-2), (K)_{(0,0)}} \\
 \begin{array}{l}
 \boxed{(K \pm 1)_{(1,1)}} \\
 \boxed{2(K-1), (K+1)_{(0,1)}} \\
 \boxed{(K-3)_{(0,1)}}
 \end{array} \\
 \begin{array}{l}
 \boxed{(K)_{(2,0)}} \\
 \begin{array}{l}
 \boxed{(K-2)_{(1,0),(0,2)}} \\
 \boxed{(K)_{(1,0),(0,2)}}
 \end{array}
 \quad \boxed{(K-4), (K-2)_{(0,0)}} \\
 \quad \quad \quad \boxed{(K)_{(0,0)}}
 \end{array} \\
 \begin{array}{l}
 \boxed{(K-1)_{(1,1)}} \\
 \boxed{(K-1), (K-3)_{(0,1)}}
 \end{array} \\
 \boxed{(K-2)_{(1,0)}}
 \end{array}
 \tag{2.24}$$

This UIR can also be obtained from $\mathcal{B}[d_1, 0; K]$ by setting $d_1 = 0$. The same arguments as in the $\mathcal{B}[d_1, 0; K]$ case can be applied to study the behaviour of the multiplet for concrete K values. Again, we find that non-cancelling negative-multiplicity states only appear at $K = 0$.

The superconformal index for all values of d_1 and $K > 0$ is calculated to be

$$\mathcal{I}_{\mathcal{B}[d_1, 0; K]}(x, y) = \frac{x^{d_1+K+3}}{(1-xy^{-1})(1-xy)} \chi_{d_1+1}(y). \tag{2.25}$$

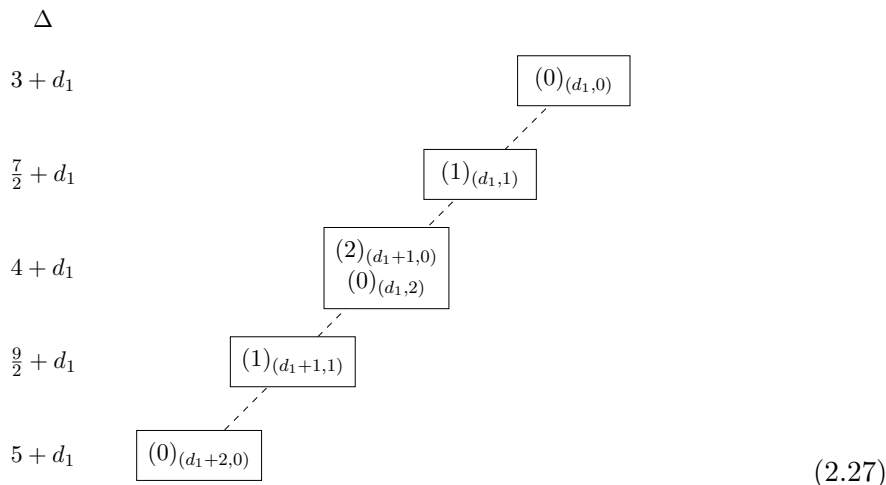
One observes that this satisfies the recombination rules (2.10):

$$\lim_{\epsilon \rightarrow 0} \mathcal{I}_{\mathcal{L}[\Delta+\epsilon; d_1, 0; K]}(x, y) = \mathcal{I}_{\mathcal{A}[d_1, 0; K]}(x, y) + \mathcal{I}_{\mathcal{B}[d_1-1, 0; K+2]}(x, y) = 0. \tag{2.26}$$

2.7.1 Higher-spin-current multiplets: $\mathcal{B}[d_1, 0; 0]$

Let us now address the special case with $K = 0$. For $d_1 \neq 0$ this family of \mathcal{B} -type multiplets contains higher-spin currents and has a primary corresponding to the symmetric traceless representation of $\mathfrak{so}(5)$. One finds that $\mathcal{Q}_{\mathbf{A}_3}$ and $\mathcal{Q}_{\mathbf{A}_4}$ need to be removed from the auxiliary Verma-module basis (2.1). Therefore the entire representation is built by just acting on

the superconformal primary with the set of Q s from (2.14) and is

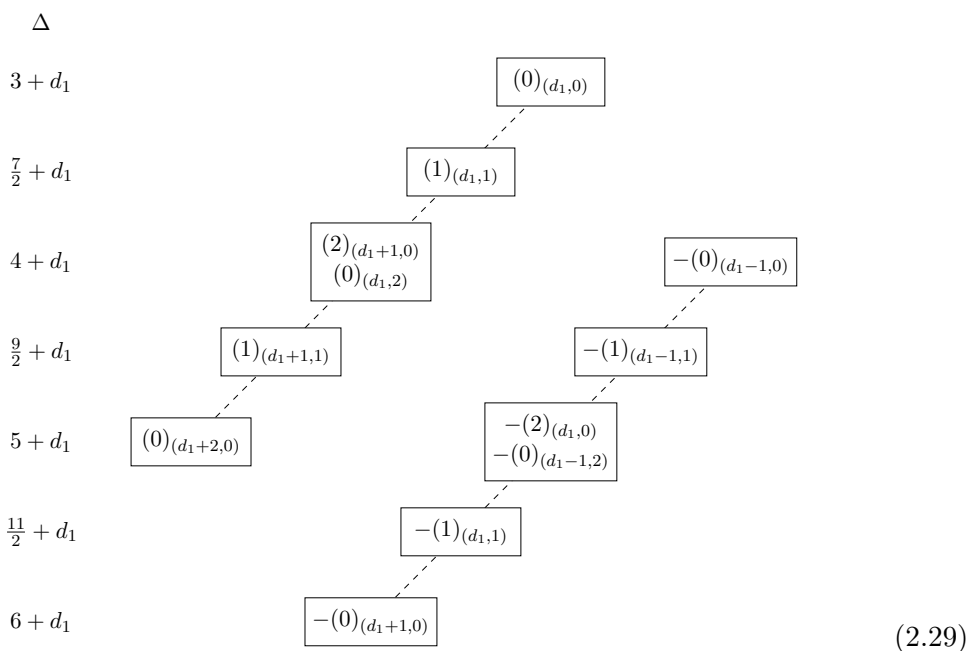


This result seems to contradict (2.7), which predicted the presence of operator constraints. However, note that the absent Q s anticommute into \mathcal{P}_5 , which has therefore been implicitly removed from the auxiliary Verma-module generators. This has the effect of projecting out states corresponding to operator constraints; we will henceforth refer to the remaining states as “reduced states”.

The operator constraints can actually be restored — cf. appendix C — by utilising the following relationship between characters

$$\hat{\chi}[\Delta; d_1, d_2; K] = \chi[\Delta; d_1, d_2; K] - \chi[\Delta + 1; d_1 - 1, d_2; K]. \tag{2.28}$$

Here the $\hat{\chi}$ and χ correspond to taking a character of the superconformal representation without/with the \mathcal{P}_5 respectively. We can appropriately account for the negative contributions to the r.h.s. via negative-multiplicity states. The full multiplet can then be expressed as:



An example of the above $[3 + d_1; d_1, 0; 0]$ primary is the operator

$$\mathcal{O}_{\mu_1 \dots \mu_{d_1}} = \epsilon_{\mathbf{AB}} \phi^{\mathbf{A}} \overset{\leftrightarrow}{\partial}_{\mu_1} \dots \overset{\leftrightarrow}{\partial}_{\mu_{d_1}} \phi^{\mathbf{B}}, \tag{2.30}$$

where $\phi^{\mathbf{A}}$ is a free hypermultiplet scalar. This object satisfies the generalised conservation equation

$$\partial^\mu \mathcal{O}_{\mu\mu_2 \dots \mu_{d_1-1}} = 0, \tag{2.31}$$

which is identified with the state $-[4 + d_1; d_1 - 1, 0; 0]$.

It is interesting to point out that for $d_1 = 1$ the superconformal primary is an R -neutral $\Delta = 4$ conserved current, which is not itself a higher-spin current. The higher-spin currents are instead found as its descendants. This logic also shows that the commutator of the conserved charge, \mathcal{T} , associated with the primary has a non-vanishing commutator with the supercharges

$$[\mathcal{Q}_{11}, \mathcal{T}] = \mathcal{T}'_{11}, \tag{2.32}$$

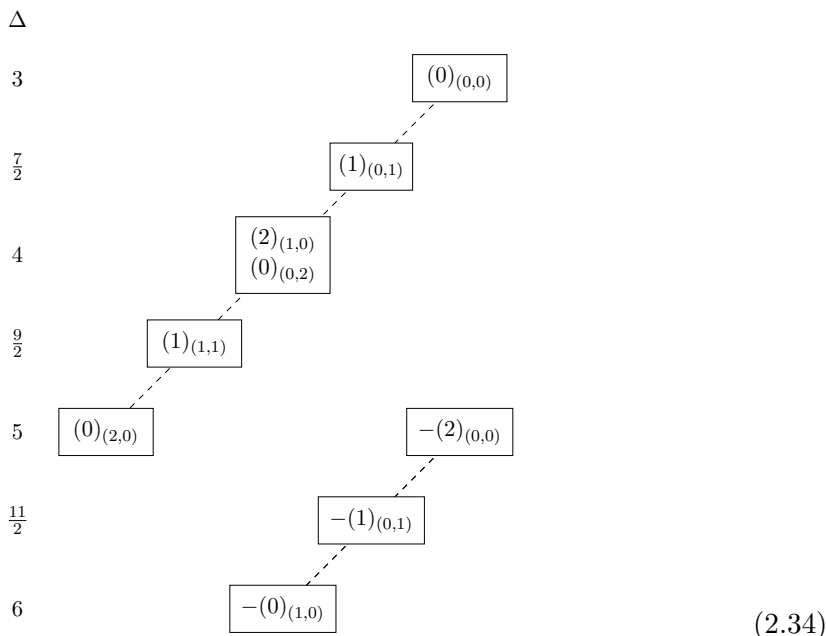
where \mathcal{T}'_{11} is the charge associated with the level-one descendant current. This commutator explains why we do not refer to \mathcal{T} as a flavour symmetry.

The index is of course insensitive to the above discussion; it gives the same answer when evaluated either on (2.27) or (2.29) and that is

$$\mathcal{I}_{\mathcal{B}[d_1, 0; 0]}(x, y) = \frac{x^{d_1+3}}{(1 - xy^{-1})(1 - xy)} \chi_{d_1+1}(y). \tag{2.33}$$

2.7.2 The stress-tensor multiplet: $\mathcal{B}[0, 0; 0]$

Let us finally turn to the case where one also sets $d_1 = 0$. The supercharges removed from the auxiliary Verma-module basis (2.1) should be those for $\mathcal{B}[0, 0; K]$, but since $K = 0$ one can also act on its shortening conditions with the R -symmetry lowering operator. This results in needing to remove $\mathcal{Q}_{1a} \mathcal{Q}_{1b}$, $\mathcal{Q}_{2a} \mathcal{Q}_{1b}$ and $\mathcal{Q}_{2a} \mathcal{Q}_{2b}$ with $b > a > 1$. Consequently the module is:



We recognise that this multiplet contains the R -symmetry current, the supersymmetry current and the stress tensor, as well as their corresponding equations of motion. In particular, we identify:

$$\begin{aligned}
 [4; 1, 0; 2] : J_\mu^{(\mathbf{AB})}, & & -[5; 0, 0; 2] : \partial^\mu J_\mu^{(\mathbf{AB})} = 0, \\
 [9/2; 1, 1; 1] : S_{\mu a}^{\mathbf{A}}, & & -[11/2; 0, 1; 1] : \partial^\mu S_{\mu a}^{\mathbf{A}} = 0, \\
 [5; 2, 0; 0] : \Theta_{\mu\nu}, & & -[6; 1, 0; 0] : \partial^\mu \Theta_{\mu\nu} = 0.
 \end{aligned}
 \tag{2.35}$$

This multiplet also contains three states that do not obey conservation equations.

The index over this multiplet counts just two components of the R -symmetry current, J_1^{11} and J_4^{11} . It is given by

$$\mathcal{I}_{\mathcal{B}[0,0;0]}(x, y) = \frac{x^3}{(1 - xy^{-1})(1 - xy)} \chi_1(y).
 \tag{2.36}$$

2.8 \mathcal{D} -type multiplets

For the $\mathcal{D}[0, 0; K]$ multiplet unitarity requires that the conformal dimension of the primary is $\Delta = \frac{3K}{2}$. The null state associated with this condition instructs us to remove \mathcal{Q}_{11} from the basis of auxiliary Verma-module generators, but due to having $d_1 = d_2 = 0$ one actually needs to remove the larger set of supercharges \mathcal{Q}_{1a} , $a = 1, \dots, 4$. In fact, in this case $\Lambda_1^1 \Psi = \mathcal{Q}_{11} \Psi = 0$ and one has the shortening condition

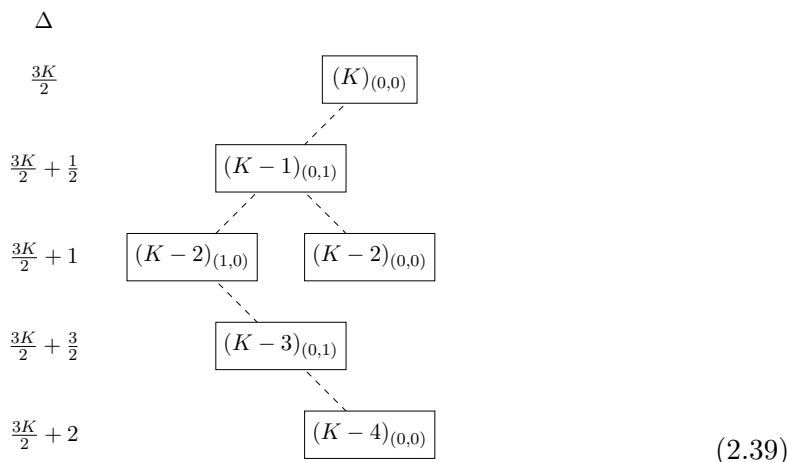
$$\mathcal{Q}_{1a} \Psi = 0,
 \tag{2.37}$$

which renders the multiplet $\frac{1}{2}$ BPS.

The action of the remaining supercharges on a primary state where the quantum numbers are kept generic is then

$$\begin{aligned}
 (K)_{(d_1, d_2)} &\xrightarrow{Q} (K-1)_{(d_1, d_2+1), (d_1+1, d_2-1)}, \\
 &\xrightarrow{Q^2} (K-2)_{(d_1+1, d_2)}, \\
 (K)_{(d_1, d_2)} &\xrightarrow{\tilde{Q}} (K-1)_{(d_1, d_2-1), (d_1-1, d_2+1)}, \\
 &\xrightarrow{\tilde{Q}^2} (K-2)_{(d_1-1, d_2)}.
 \end{aligned}
 \tag{2.38}$$

After setting $d_1 = d_2 = 0$ and employing RS the full multiplet is



For low enough values of K there can be additional reflections. If $K = 1$ or $K = 2$ these will correspond to operator constraints, to be discussed below. If $K = 3$ then the only problematic state will be the level four $[0, 0; K - 4]$, which becomes $[0, 0; -1]$; this is on the boundary of the Weyl chamber, and hence will also be deleted.

The index for this multiplet is

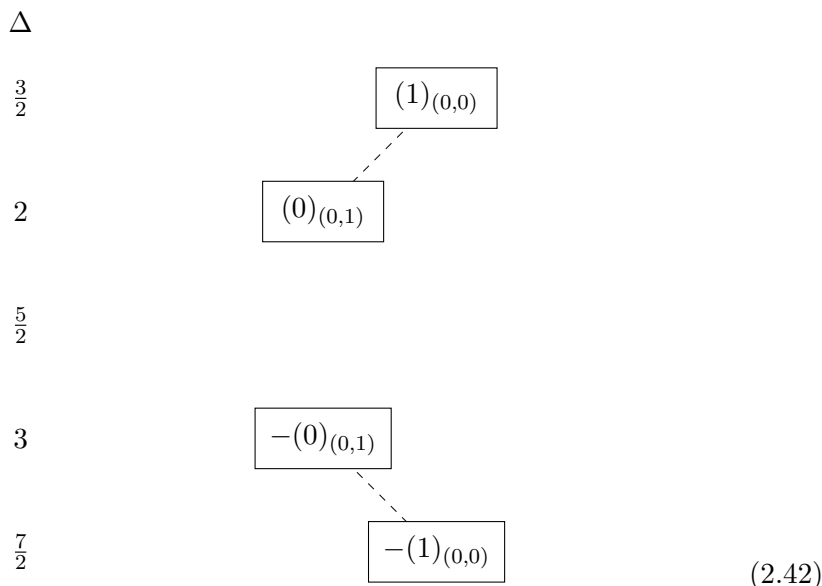
$$\mathcal{I}_{\mathcal{D}[0,0;K]}(x, y) = \frac{x^K}{(1 - xy^{-1})(1 - xy)}. \tag{2.40}$$

This satisfies a recombination rule from (2.10):

$$\lim_{\epsilon \rightarrow 0} \mathcal{I}_{\mathcal{L}[\Delta+\epsilon;0,0;K]}(x, y) = \mathcal{I}_{\mathcal{A}[0,0;K]}(x, y) + \mathcal{I}_{\mathcal{D}[0,0;K+4]}(x, y) = 0. \tag{2.41}$$

2.8.1 The hypermultiplet: $\mathcal{D}[0, 0; 1]$

When $K = 1$ no additional shortening conditions arise, therefore we may proceed directly using (2.39). One recovers:



We can recognise these highest weight states as

$$\begin{aligned}
 [3/2; 0, 0; 1] &: \phi^{\mathbf{A}}, \\
 [2; 0, 1; 0] &: \lambda_a, \\
 -[3; 0, 1; 0] &: \not\partial^{ab} \bar{\lambda}_b = 0, \\
 -[7/2; 0, 0; 1] &: \partial^2 \phi^{\mathbf{A}} = 0.
 \end{aligned} \tag{2.43}$$

The index for this multiplet counts the first operator in the primary, $\phi^{\mathbf{1}}$ along with the \mathcal{P}_1 and \mathcal{P}_2 conformal descendants. Therefore we have

$$\mathcal{I}_{\mathcal{D}[0,0;1]}(x, y) = \frac{x}{(1 - xy^{-1})(1 - xy)}. \tag{2.44}$$

This is the single-letter index obtained in [50, 51].

2.8.2 The flavour-current multiplet: $\mathcal{D}[0, 0; 2]$

One of the \mathcal{D} -type multiplets predicted to contain operator constraints is $\mathcal{D}[0, 0; 2]$. As above, we may jump straight into the multiplet structure by setting $K = 2$ in the $\mathcal{D}[0, 0; K]$ module. This produces

$$\begin{array}{c}
 \Delta \\
 3 \qquad \qquad \qquad \boxed{(2)_{(0,0)}} \\
 \qquad \qquad \qquad \text{---} \\
 \frac{7}{2} \qquad \qquad \boxed{(1)_{(0,1)}} \\
 \qquad \qquad \text{---} \qquad \text{---} \\
 4 \qquad \boxed{(0)_{(1,0)}} \qquad \boxed{(0)_{(0,0)}} \\
 \qquad \qquad \text{---} \qquad \text{---} \\
 \frac{9}{2} \\
 5 \qquad \qquad \qquad \boxed{-(0)_{(0,0)}}
 \end{array} \tag{2.45}$$

We recognise these fields to be a scalar $\mu^{(\mathbf{AB})}$ in the $\mathbf{3}$ of $\mathfrak{su}(2)_R$, a symplectic Majorana fermion $\psi_a^{\mathbf{A}}$ in the $\mathbf{2}$ of $\mathfrak{su}(2)_R$, a vector J_μ and an R-neutral scalar M . Furthermore the negative-multiplicity state is the equation of motion for the vector current $\partial^\mu J_\mu = 0$. This multiplet is also known as the linear multiplet and appeared in the UV symmetry-enhancement discussion of [52].

The corresponding index is:

$$\mathcal{I}_{\mathcal{D}[0,0;2]}(x, y) = \frac{x^2}{(1 - xy^{-1})(1 - xy)}. \tag{2.46}$$

This concludes our listing of superconformal multiplets for the 5D SCA.

3 On the complete classification of UIRs for the 5D SCA

We will next give an argument that the conditions imposed in section 2.1 on the dimension of the superconformal primary — giving rise to the $\mathcal{A}[d_1, d_2; K]$, $\mathcal{B}[d_1, 0; K]$ and $\mathcal{D}[0, 0; K]$ short-multiplet types — are necessary and sufficient for unitarity. To the best of our knowledge, there is no such argument in the literature for 5D; for proofs in 4D and 6D see [21–24].

Our argument will proceed in three steps. We first use the results of [43] to establish when the Verma module can admit null states (i.e., when the Verma module is reducible). We then show that only the 5D multiplet types $\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{L}$ can be unitary (determining necessity). Finally, we argue that all multiplets of the above type are indeed unitary (determining sufficiency).

3.1 Reducibility conditions for $F(4)$

We begin by determining the necessary and sufficient conditions for when a representation of the 5D SCA can contain null states (i.e., when the $F(4)$ parabolic Verma module is reducible). This question has recently been revisited in [42, 43], the results of which we use. Towards that end, let us first establish some notation following, e.g., [53, 54].

The 5D SCA, $F(4)$, is a Lie superalgebra for which we choose the following simple-root decomposition

$$\Pi = \left\{ \alpha_1 = \delta, \alpha_2 = \frac{1}{2}(-\delta + \epsilon_1 - \epsilon_2 - \epsilon_3), \alpha_3 = \epsilon_3, \alpha_4 = \epsilon_2 - \epsilon_3 \right\}, \quad (3.1)$$

with corresponding Dynkin diagram and Cartan matrix:

$$[a_{ij}] = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

The $\alpha_1, \alpha_3, \alpha_4 \in \Pi_{\bar{0}}$ are even simple roots, while $\alpha_2 \in \Pi_{\bar{1}}$ is an odd simple root. These have been expressed in an orthogonal basis in terms of ϵ_i , for $i = 1, 2, 3$, and δ such that $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\delta, \delta) = -3$ and $(\epsilon_i, \delta) = 0$.

The simple roots can be used to obtain the even and odd positive roots of $F(4)$

$$\Phi_{\bar{0}}^+ = \{\delta, \epsilon_i \pm \epsilon_j \text{ (for } i < j), \epsilon_i\}, \quad \Phi_{\bar{1}}^+ = \left\{ \frac{1}{2}(\pm\delta + \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \right\}, \quad (3.2)$$

where the signs are not correlated. One also typically defines the set $\bar{\Phi}_{\bar{0}}$ and the set of isotropic roots $\bar{\Phi}_{\bar{1}}$ through

$$\bar{\Phi}_{\bar{0}} := \{\alpha \in \Phi_{\bar{0}} \mid \alpha/2 \notin \Phi_{\bar{1}}\} \subseteq \Delta_{\bar{0}}, \quad \bar{\Phi}_{\bar{1}} := \{\alpha \in \Phi_{\bar{1}} \mid 2\alpha \notin \Phi_{\bar{0}}\} \subseteq \Delta_{\bar{1}}, \quad (3.3)$$

although it is easy to see that in our case $\bar{\Phi}_{\bar{0}} = \Phi_{\bar{0}}$ and $\bar{\Phi}_{\bar{1}} = \Phi_{\bar{1}}$.

The difference between the half-sum of the positive even roots $\rho_{\bar{0}}$ and half-sum of the positive odd roots $\rho_{\bar{1}}$ denotes the Weyl vector, which can be evaluated to be

$$\rho := \rho_{\bar{0}} - \rho_{\bar{1}} = \frac{1}{2}(\delta + \epsilon_1 + 3\epsilon_2 + \epsilon_3). \quad (3.4)$$

A highest weight representation of $F(4)$, λ , may be expanded in terms of the fundamental weights of the bosonic subalgebra $\mathfrak{g}_{\bar{0}} = \mathfrak{su}(2)_R \oplus \mathfrak{so}(5, 2)$ as

$$\lambda = \sum_a \omega_a H_a, \quad (3.5)$$

where the ω_a for $a = 1$ and $a = 2, 3, 4$ are the fundamental weights associated with the simple roots of $\mathfrak{su}(2)_R$ and $\mathfrak{so}(5, 2)$ respectively and the H_a are the Cartans. For each of these bosonic subalgebras, the fundamental weights are related to the simple roots via the inverse of the Cartan matrix. For $\mathfrak{su}(2)_R$ and $\mathfrak{so}(5, 2)$ this allows us to express

$$\omega_1 = \frac{1}{2}\delta \quad \text{and} \quad \omega_2 = \epsilon_1, \quad \omega_3 = \epsilon_1 + \epsilon_2, \quad \omega_4 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3). \quad (3.6)$$

Since in our discussion thus far we have been labelling representations in terms of their $\mathfrak{l} = \mathfrak{su}(2)_R \oplus \mathfrak{so}(5) \oplus \mathfrak{so}(2) \subset \mathfrak{g}$ Dynkin labels, we will also need the $\mathfrak{so}(5)$ fundamental weights

$$\hat{\omega}_1 = \epsilon_2 \quad \hat{\omega}_2 = \frac{1}{2}(\epsilon_2 + \epsilon_3), \quad (3.7)$$

which we use to write

$$\lambda = \frac{1}{2}\delta H_1 + \omega_2 \left(H_2 + H_3 + \frac{1}{2}H_4 \right) + \hat{\omega}_1 H_3 + \hat{\omega}_2 H_4. \quad (3.8)$$

Now, by definition we have that

$$H_1 = K, \quad H_3 = d_1, \quad H_4 = d_2, \quad (3.9)$$

since $H_{1,3,4}$ multiply the fundamental weights of $\mathfrak{su}(2)_R$ and $\mathfrak{so}(5)$ respectively. We can naturally assign the remaining label to the conformal dimension $H_2 + H_3 + \frac{1}{2}H_4 = -\Delta$, where the minus sign is there to account for the signature of $\mathfrak{so}(5,2)$. This means that

$$H_2 = -\Delta - d_1 - \frac{1}{2}d_2. \quad (3.10)$$

Expressing the highest weight using the above in the orthogonal basis

$$\lambda = \delta \frac{K}{2} - \epsilon_1 \Delta + \epsilon_2 \left(d_1 + \frac{d_2}{2} \right) + \epsilon_3 \frac{d_2}{2}. \quad (3.11)$$

Finally, the following two sets have to be defined before proceeding, where we have adapted the definitions of [43] to the case of $F(4)$:

$$\Psi_{\lambda, \text{iso}} := \left\{ \alpha \in \Phi_{\mathfrak{l}}^+ \mid (\lambda + \rho, \alpha) = 0 \right\} \quad (3.12)$$

and

$$\Psi_{\lambda, \text{non-iso}} := \left\{ \alpha \in \Phi_{n, \bar{0}}^+ \mid n_\alpha := \frac{2(\lambda + \rho, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}_{>0} \right\}, \quad (3.13)$$

where $\Phi_{n, \bar{0}} := \Phi_n \cap \Phi_{\bar{0}}$, with $\Phi_n = \Phi^+ \setminus \Phi_{\mathfrak{l}}$. Hence we have

$$\begin{aligned} \Phi_{\mathfrak{l}} &= \{ \pm\delta, \pm\epsilon_2 \pm \epsilon_3, \pm\epsilon_2, \pm\epsilon_3 \}, \\ \Phi_{n, \bar{0}}^+ &= \{ \epsilon_1 \pm \epsilon_2, \epsilon_1 \pm \epsilon_3, \epsilon_1 \}, \end{aligned} \quad (3.14)$$

where the signs are not correlated.

With this information at hand, we can now apply the criteria of [43] regarding the necessary and sufficient conditions for irreducibility:

- If an $F(4)$ Verma module is irreducible, then $\Psi_{\lambda, \text{iso}} = \emptyset$ and for all $\alpha \in \Psi_{\lambda, \text{non-iso}}$ there exists $\beta \in \Phi_{\bar{0}}$ such that $(\lambda + \rho, \beta) = 0$. ([43], Proposition 4)
- An $F(4)$ Verma module is irreducible if $\Psi_{\lambda, \text{iso}} = \emptyset$, and, for all $\alpha \in \Psi_{\lambda, \text{non-iso}}$, there exists $\beta \in \Phi_{\bar{0}}$ such that $(\lambda + \rho, \beta) = 0$ and $s_\alpha(\beta) := \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Phi_{\mathfrak{l}}$. ([43], Proposition 5)

From the first bullet point above we immediately determine that the module is reducible when $\Psi_{\lambda, \text{iso}} \neq \emptyset$. This phenomenon occurs when eight conditions are satisfied, one for each of the positive odd roots,

$$\begin{aligned}
 \Delta = \frac{3K}{2} - d_1 - d_2 &:= f^1, & \Delta = -\frac{3K}{2} - d_1 - d_2 - 3 &:= \tilde{f}^1, \\
 \Delta = \frac{3K}{2} - d_1 + 1 &:= f^2, & \Delta = -\frac{3K}{2} - d_1 - 2 &:= \tilde{f}^2, \\
 \Delta = \frac{3K}{2} + d_1 + 3 &:= f^3, & \Delta = -\frac{3K}{2} + d_1 &:= \tilde{f}^3, \\
 \Delta = \frac{3K}{2} + d_1 + d_2 + 4 &:= f^4, & \Delta = -\frac{3K}{2} + d_1 + d_2 + 1 &:= \tilde{f}^4.
 \end{aligned} \tag{3.15}$$

These equations correspond exactly to the conditions for which the eight level-one norms become null, eq. (A.18).

Note that, according to the second bullet point above, there can exist additional reducible representations only if $\Psi_{\lambda, \text{non-iso}} \neq \emptyset$. This latter condition is equivalent to the following equations with $n_\alpha \in \mathbb{Z}_{>0}$:

$$\begin{aligned}
 n_{\epsilon_1} &= 1 - 2\Delta, \\
 n_{\epsilon_1 + \epsilon_2} &= d_1 + \frac{d_2}{2} + 2 - \Delta, \\
 n_{\epsilon_1 - \epsilon_2} &= -d_1 - \frac{d_2}{2} - 1 - \Delta, \\
 n_{\epsilon_1 + \epsilon_3} &= \frac{d_2}{2} + 1 - \Delta, \\
 n_{\epsilon_1 - \epsilon_3} &= -\frac{d_2}{2} - \Delta.
 \end{aligned} \tag{3.16}$$

However, the conformal dimensions of these modules are below the unitarity bounds associated with conformal descendants:

$$\begin{aligned}
 \text{Scalar} : \Delta &\geq \frac{3}{2}, \\
 \text{Operator with spin } \frac{d_2}{2} : \Delta &\geq 2 + \frac{d_2}{2}, \\
 \text{Composite operator with spin } d_1 + \frac{d_2}{2} : \Delta &\geq 3 + d_1 + \frac{d_2}{2}.
 \end{aligned} \tag{3.17}$$

Hence, these additional modules are not unitary and will not be useful for the discussion of UIRs.

We conclude that $F(4)$ admits reducible modules precisely when the eight level-one norms become null, as predicted by [20].

3.2 An argument for necessity

For the necessity part of the argument we will begin from the various unitarity bounds for the \mathcal{A}, \mathcal{B} and \mathcal{D} -type multiplets. As shown in section 2.1, these can be expressed as

$$\Delta = f^a(d_1, d_2, K) \quad \text{with} \quad a = 1, 3, 4, \tag{3.18}$$

where the $f^a(d_1, d_2, K)$ are given in (3.15) and we observe that $f_4 > f_3 > f_2 > f_1$. We will next show that there can be no UIRs of the 5D SCA except for the ones satisfying (3.18) and $\Delta > f^4$, i.e. that there is at least one negative-norm state in the intervals (f^3, f^4) , (f^1, f^3) and $(-\infty, f^1)$.

Towards that end, consider the norms of the following well-defined states:

$$\begin{aligned}\mathbb{F}_4 &= \|\Lambda_1^4 \Lambda_1^3 \Lambda_1^2 \Lambda_1^1 |\Delta; d_1, d_2; K\rangle^{\text{hw}}\|^2 = (\Delta - f^4)(\Delta - f^3)(\Delta - f^2)(\Delta - f^1), \\ \mathbb{F}_3 &= \|\Lambda_1^3 \Lambda_1^1 |\Delta; d_1, d_2; K\rangle^{\text{hw}}\|^2 = (\Delta - f^3)(\Delta - f^1), \\ \mathbb{F}_2 &= \|\Lambda_1^2 \Lambda_1^3 |\Delta; 0, 0; K\rangle^{\text{hw}}\|^2 = (\Delta - f^3)(\Delta - f^1), \\ \mathbb{F}_1 &= \|\Lambda_1^1 |\Delta; d_1, d_2; K\rangle^{\text{hw}}\|^2 = (\Delta - f^1).\end{aligned}\tag{3.19}$$

1. **$\mathbf{d}_1, \mathbf{d}_2 \neq \mathbf{0}$** : \mathbb{F}_a is positive for $\Delta > f^4$, this describes a long multiplet $\mathcal{L}[\Delta; d_1, d_2; K]$. When $\Delta = f^4$ this describes an $\mathcal{A}[d_1, d_2; K]$ multiplet. If $f^3 < \Delta < f^4$ then \mathbb{F}_4 is negative. If $f^1 < \Delta < f^3$ then \mathbb{F}_3 is negative. The condition $0 < \Delta < f^1$ would result in the positivity of \mathbb{F}_4 and \mathbb{F}_3 , but leads to \mathbb{F}_1 being negative. $\Delta < 0$ is forbidden by unitarity since it would result in a negative conformal dimension for the superconformal primary. If $\Delta = f^a$ for $a = 1, 2, 3$ then, despite the fact that $\mathbb{F}_4 = 0$, the level-one norm of the state $\Lambda_1^4 \Psi$ will be negative, and is therefore not allowed.
2. **$\mathbf{d}_1 \neq \mathbf{0}, \mathbf{d}_2 = \mathbf{0}$** : the same logic can be applied to this case when taking $d_2 = 0$. However, alongside $\Delta = f^4$, it is possible to have $\Delta = f^3$, describing $\mathcal{B}[d_1, 0; K]$. This is because the level-one state $\Lambda_1^4 |\Delta; d_1, 0; K\rangle^{\text{hw}}$ does not exist as it is ill-defined for $d_2 = 0$, hence there would be no negative-norm state associated with it.
3. **$\mathbf{d}_1 = \mathbf{d}_2 = \mathbf{0}$** : there is now one additional way to saturate a unitarity bound. The states $\Lambda_1^{a=2,3,4} |\Delta; 0, 0; K\rangle^{\text{hw}}$ are all ill-defined for these d_i values. Therefore one can achieve $\mathbb{F}_2 = 0$ with $\Delta = f^1$, which describes a $\mathcal{D}[0, 0; K]$ multiplet, in addition to $\Delta = f^3$, which describes the $\mathcal{B}[0, 0; K]$ multiplet.

We conclude that the necessary conditions for unitarity are $\Delta \geq f^4$ for $d_1, d_2 \neq 0$; $\Delta \geq f^4$ or $\Delta = f^3$ for $d_1 \neq 0, d_2 = 0$; $\Delta \geq f^4$ or $\Delta = f^{a=1,3}$ for $d_1 = d_2 = 0$, in line with the classification of section 2.1.¹⁸

3.3 An argument for sufficiency

For this part one needs to show that all states contained in the $\mathcal{A}, \mathcal{B}, \mathcal{D}$ short multiplets and the long multiplets \mathcal{L} are indeed unitary.

1. **$\mathcal{D}[\mathbf{0}, \mathbf{0}; \mathbf{K}]$** : all superconformal-primary norms for this multiplet can be explicitly calculated using the results of appendix A and determined to be positive.¹⁹ As an additional check, one recognises that in a free theory one can construct the $\mathcal{D}[0, 0; K]$ multiplets from K free hypermultiplets, $\mathcal{D}[0, 0; 1]$, and therefore the former are necessarily unitary.

¹⁸In a previous version of this paper, an additional necessary condition for unitarity had been incorrectly identified for $d_1 = d_2 = 0$ at $\Delta = f^2$. We thank the authors of [44] for bringing this to our attention.

¹⁹The relevant Mathematica notebooks can be obtained from the authors upon request.

2. $\mathcal{B}[\mathbf{d}_1, \mathbf{0}; \mathbf{K}]$: these multiplets can admit a realisation in terms of free hypermultiplets. Similarly to (2.30), for a free theory we may write a superconformal primary with quantum numbers $[3 + d_1 + \frac{3K}{2}; d_1, 0; K]$ as

$$\mathcal{O}_{\mu_1 \dots \mu_{d_1}}^{(\mathbf{A}_1 \dots \mathbf{A}_K)} = \epsilon_{\mathbf{BC}} \phi^{\mathbf{B}} \overset{\leftrightarrow}{\partial}_{\mu_1} \dots \overset{\leftrightarrow}{\partial}_{\mu_{d_1}} \phi^{\mathbf{C}} \phi^{(\mathbf{A}_1} \dots \phi^{\mathbf{A}_K)}. \quad (3.20)$$

Therefore, this multiplet must necessarily be unitary.

3. $\mathcal{A}[\mathbf{d}_1, \mathbf{d}_2; \mathbf{K}]$: from the set of the $\mathcal{A}[d_1, d_2; K]$ multiplets, only the $\mathcal{A}[d_1, 1; K]$ can admit a free-field realisation. For those cases and in a free theory, the primary can be expressed as

$$\mathcal{O}_{a, \mu_1 \dots \mu_{d_1}}^{(\mathbf{A}_1 \dots \mathbf{A}_K)} = \epsilon_{\mathbf{BC}} \phi^{\mathbf{B}} \overset{\leftrightarrow}{\partial}_{\mu_1} \dots \overset{\leftrightarrow}{\partial}_{\mu_{d_1}} \phi^{\mathbf{C}} \lambda_a \phi^{(\mathbf{A}_1} \dots \phi^{\mathbf{A}_K)}. \quad (3.21)$$

However, we can argue for the unitarity of other choices of quantum numbers using the recombination rules

$$\mathcal{L}[\Delta + \epsilon; d_1, d_2; K] \xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[d_1, d_2; K] \oplus \mathcal{A}[d_1, d_2 - 1; K + 1], \quad (3.22)$$

for $\Delta = 4 + d_1 + d_2 + \frac{3K}{2}$ as follows.

Let us formally denote the subsets of long-multiplet states related to the $\mathcal{A}[d_1, d_2; K]$ and $\mathcal{A}[d_1, d_2 - 1; K + 1]$ ones above as $\Psi_{\mathcal{L}_{A_1}, \mathcal{L}_{A_2}} \xrightarrow{\epsilon \rightarrow 0} \Psi_{A_1, A_2}$; that is, the states in $\Psi_{\mathcal{L}_{A_1}}$ can be completed into a long multiplet by including the states $\Psi_{\mathcal{L}_{A_2}}$. We may therefore write

$$\Psi_{\mathcal{L}_{A_2}} = \Lambda_{\mathbf{1}}^4 \Psi_{\mathcal{L}_{A_1}}, \quad (3.23)$$

for all states in $\Psi_{\mathcal{L}_{A_2}}$. Their norms can be explicitly calculated and are related by

$$\|\Psi_{\mathcal{L}_{A_2}}\|^2 = \epsilon \frac{(2d_1 + d_2 + 3)(d_1 + 2)(d_2 + 1)}{2(d_1 + d_2/2 + 1)d_2(d_1 + d_2 + 1)} \|\Psi_{\mathcal{L}_{A_1}}\|^2, \quad (3.24)$$

where note that the coefficient on the r.h.s. is always positive.

Since the multiplet $\mathcal{A}[d_1, 1; K]$ can appear in the recombination rule (3.22) as

$$\mathcal{L}[\Delta + \epsilon; d_1, 1; K] \xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[d_1, 1; K] \oplus \mathcal{A}[d_1, 0; K + 1], \quad (3.25)$$

we can use the above argument to conclude that in the $\epsilon \rightarrow 0$ limit $\mathcal{A}[d_1, 0; K + 1]$ is also unitary. Moreover, one can also write

$$\mathcal{L}[\Delta + \epsilon; d_1, 2; K - 1] \xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[d_1, 2; K - 1] \oplus \mathcal{A}[d_1, 1; K], \quad (3.26)$$

and use the same logic to conclude that the multiplet $\mathcal{A}[d_1, 2; K]$ is unitary. This method can be used recursively to demonstrate that all $\mathcal{A}[d_1, d_2; K]$ are unitary with the exception of $\mathcal{A}[0, 0; 0]$. However, the unitarity of $\mathcal{A}[0, 0; 0]$ can be argued for by appealing to the index.

To that end, recall the index of a unitary SCFT is a sum over contributions from individual local operators of the theory transforming in UIRs. Now, the index for a

5D SCFT with gauge group $SU(2)$ and global symmetry group $E_2 = SU(2) \times U(1)$ was evaluated in [50]; see also [55]. Moreover, the arguments in [50] suggest that this result can be understood as an index coming from one of Seiberg's unitary 5D SCFTs [26] that appear in the low-energy limit of string theory. Assuming this identification is correct (or, least restrictively, that the index computed in [50] corresponds to an index of some unitary theory with E_2 global symmetry), then it is straightforward to show that $\mathcal{A}[0, 0; 0]$ is also a unitary representation.

To understand this last statement, note that the authors of [50] found the index in question admits the following expansion

$$\begin{aligned} \mathcal{I}(x, y) = & 1 + [1 + \chi_2(q)]x^2 + \chi_1(y)[2 + \chi_2(q)]x^3 \\ & + \left(\chi_2(y)[2 + \chi_2(q)] + 1 - \chi_4(f) \right)x^4 + \mathcal{O}(x^5), \end{aligned} \quad (3.27)$$

where q is a fugacity associated with the instanton contributions and

$$\chi_4(f) = (e^{\frac{i\rho}{2}} + e^{-\frac{i\rho}{2}})\chi_1(q), \quad (3.28)$$

for some $U(1)$ flavour charge ρ . It is then possible to rewrite this expression in terms of contributions from local operators as follows

$$\begin{aligned} \mathcal{I}(x, y) = & 1 + [1 + \chi_2(q)]\mathcal{I}_{\mathcal{D}[0,0;2]}(x, y) + \mathcal{I}_{\mathcal{B}[0,0;0]}(x, y) + \chi_3(q)\mathcal{I}_{\mathcal{D}[0,0;4]} \\ & + \chi_4(f)\mathcal{I}_{\mathcal{A}[0,0;0]}(x, y) + \mathcal{O}(x^5). \end{aligned} \quad (3.29)$$

Note that the $-\chi_4(f)x^4$ term in (3.27) can only be accounted for by using the index for $\mathcal{A}[0, 0; 0]$ since all other indices at $\mathcal{O}(x^4)$ either have positive sign or include y -dependent factors. We therefore conclude that $\mathcal{A}[0, 0; 0]$ is also unitary.

4. $\mathcal{L}[\mathbf{f}_4 + \epsilon; \mathbf{d}_1, \mathbf{d}_2; \mathbf{K}]$: for the long multiplets approaching the unitarity bound $\Delta = 4 + d_1 + d_2 + \frac{3K}{2}$, one can conclude that, since the $\mathcal{L}[f_4 + \epsilon; d_1, d_2 \neq 0; K]$ can be written as a direct sum of two unitary multiplets as in (3.22), it too must be unitary. The remaining two recombination rules of (2.10)

$$\begin{aligned} \mathcal{L}[\Delta + \epsilon; d_1, 0; K] & \xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[d_1, 0; K] \oplus \mathcal{B}[d_1 - 1, 0; K + 2], \\ \mathcal{L}[\Delta + \epsilon; 0, 0; K] & \xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[0, 0; K] \oplus \mathcal{D}[0, 0; K + 4], \end{aligned} \quad (3.30)$$

can be similarly used to determine that all $\mathcal{L}[\Delta + \epsilon; d_1, d_2; K]$ are unitary.

5. $\mathcal{L}[\Delta; \mathbf{d}_1, \mathbf{d}_2; \mathbf{K}]$: in the neighbourhood $\Delta = 4 + d_1 + d_2 + \frac{3K}{2} + \epsilon = f^4 + \epsilon$, every state has positive norm. Since the norm of a state is a smooth function of Δ , in order for it to become negative as we move arbitrarily far from f^4 , it must first become null. By definition this scenario would lead to a reducible Verma module. However, in section 3.1, we determined all the values for which such a situation can arise. It is easy to see that f^4 is the largest of these and therefore the norm can never be negative.

We therefore conclude that the multiplets $\mathcal{L}[\Delta; d_1, d_2; K]$, $\mathcal{A}[d_1, d_2; K]$, $\mathcal{B}[d_1, 0; K]$ and $\mathcal{D}[0, 0; K]$ are unitary hence concluding our argument.

4 Multiplets and superconformal indices for 6D (1,0)

We now switch to six dimensions. In this section we provide a systematic analysis of all superconformal multiplets admitted by the 6D $\mathcal{N} = (1, 0)$ algebra, $\mathfrak{osp}(8^*|2)$.²⁰ Since the method for building UIRs is completely analogous to section 2.1 we will be brief and merely sketch the derivation of the unitarity bounds obtained in detail by [13]. We then proceed to write out the complete set of spectra for superconformal multiplets along with their corresponding superconformal indices.

4.1 UIR building with auxiliary Verma modules

The superconformal primaries of the algebra $\mathfrak{osp}(8^*|2)$ are designated $|\Delta; c_1, c_2, c_3; K\rangle$ and labelled by the conformal dimension Δ , the Lorentz quantum numbers for $\mathfrak{su}(4)$ in the Dynkin basis c_i and an R -symmetry label K . Each primary is in one-to-one correspondence with a highest weight labeling irreducible representations of the maximal compact subalgebra $\mathfrak{so}(6) \oplus \mathfrak{so}(2) \oplus \mathfrak{su}(2)_R \subset \mathfrak{osp}(8^*|2)$. There are eight Poincaré and superconformal supercharges, denoted by $\mathcal{Q}_{\mathbf{A}a}$ and $\mathcal{S}_{\mathbf{A}\dot{a}}$ — where $a, \dot{a} = 1, \dots, 4$ are $\mathfrak{su}(4)$ (anti)fundamental indices and $\mathbf{A} = 1, 2$ an index of $\mathfrak{su}(2)_R$. One also has six momenta \mathcal{P}_μ and special conformal generators \mathcal{K}_μ , where $\mu = 1, \dots, 6$ is a Lorentz vector index. The superconformal primary is annihilated by all $\mathcal{S}_{\mathbf{A}\dot{a}}$ and \mathcal{K}_μ . A basis for the representation space of $\mathfrak{osp}(8^*|2)$ can be constructed by considering the following Verma module

$$\prod_{\mathbf{A}, a} (\mathcal{Q}_{\mathbf{A}a})^{n_{\mathbf{A}, a}} \prod_{\mu} \mathcal{P}_\mu^{n_\mu} |\Delta; c_1, c_2, c_3; K\rangle^{\text{hw}}, \quad (4.1)$$

where $n = \sum_{\mathbf{A}, a} n_{\mathbf{A}, a}$ and $\hat{n} = \sum_{\mu} n_\mu$ denote the level of a superconformal or conformal descendant respectively. UIRs can be obtained from the above after also imposing the requirement of unitarity level-by-level.

We will now review the conditions imposed by unitarity, starting with the superconformal descendants [13]. For descendants of level $n > 0$ it suffices to calculate the norms of states in the highest weight of $\mathfrak{su}(2)_R$, since these provide the most stringent set of unitarity bounds [13, 20, 24].²¹ Hence the unitarity bounds stem from the study of superconformal descendants arising from the action of supercharges of the form $\mathcal{Q}_{\mathbf{1}a}$. Before proceeding, it is convenient to define the basis $A^a |\Delta; c_1, c_2, c_3; K\rangle^{\text{hw}}$, where

$$A^a = \sum_{b=1}^a \mathcal{Q}_{\mathbf{1}b} \Upsilon_b^a. \quad (4.2)$$

The Υ_b^a are functions of $\mathfrak{u}(4)$ Cartans and Lorentz lowering operators, the explicit form of which can be found in [13]. They map highest weights to highest weights.

²⁰Some of the multiplets up to spin 2 discussed in this section have been previously constructed using the dual gravity description in [56].

²¹In the language of [43], the proof for the 6D SCAs in [24] only deals with the conditions associated with $\Psi_{\lambda, \text{iso}} \neq \emptyset$. We have investigated the possible reducible modules coming from $\Psi_{\lambda, \text{non-iso}} \neq \emptyset$ and, similar to our findings in section 3, all predicted cases are below the unitarity bounds associated with conformal descendants.

Multiplet	Shortening Condition	Conformal Dimension
$\mathcal{A}[c_1, c_2, c_3; K]$	$A^4\Psi = 0$	$\Delta = 2K + \frac{c_1}{2} + c_2 + \frac{3c_3}{2} + 6$
$\mathcal{A}[c_1, c_2, 0; K]$	$A^3A^4\Psi = 0$	$\Delta = 2K + \frac{c_1}{2} + c_2 + 6$
$\mathcal{A}[c_1, 0, 0; K]$	$A^2A^3A^4\Psi = 0$	$\Delta = 2K + \frac{c_1}{2} + 6$
$\mathcal{A}[0, 0, 0; K]$	$A^1A^2A^3A^4\Psi = 0$	$\Delta = 2K + 6$
$\mathcal{B}[c_1, c_2, 0; K]$	$A^3\Psi = 0$	$\Delta = 2K + \frac{c_1}{2} + c_2 + 4$
$\mathcal{B}[c_1, 0, 0; K]$	$A^2A^3\Psi = 0$	$\Delta = 2K + \frac{c_1}{2} + 4$
$\mathcal{B}[0, 0, 0; K]$	$A^1A^2A^3\Psi = 0$	$\Delta = 2K + 4$
$\mathcal{C}[c_1, 0, 0; K]$	$A^2\Psi = 0$	$\Delta = 2K + \frac{c_1}{2} + 2$
$\mathcal{C}[0, 0, 0; K]$	$A^1A^2\Psi = 0$	$\Delta = 2K + 2$
$\mathcal{D}[0, 0, 0; K]$	$A^1\Psi = 0$	$\Delta = 2K$

Table 2. A list of all short multiplets for the 6D (1,0) SCA, along with the conformal dimension of the superconformal primary and the corresponding shortening condition. The A^a in the shortening conditions are defined in (4.2) and [13]. The first of these multiplets (\mathcal{A}) is a regular short representation, whereas the rest ($\mathcal{B}, \mathcal{C}, \mathcal{D}$) are isolated short representations. Here Ψ denotes the superconformal primary state for each multiplet.

One then conducts a level-by-level analysis of the norms for the superconformal descendants, while also keeping track of whether the states $A^a |\Delta; c_1, c_2, c_3; K\rangle^{\text{hw}}$ are well defined when setting $c_i = 0$. This gives rise to a set of regular and isolated short multiplets, obeying certain shortening conditions [13]. The precise form of the A^a is unimportant; they have the same quantum numbers as the supercharge \mathcal{Q}_{1a} . The shortening conditions can then be translated into absent generators in the auxiliary Verma module (4.1) and we are instructed to remove the corresponding combinations of supercharges in a straightforward way. These results are summarised in table 2.

Additional absent supercharges can be obtained when $K = 0$ by acting on the existing null states with Lorentz and R -symmetry lowering operators. As a result additional supercharges need to be removed from (4.1). These occurrences will be dealt with on a case-by-case basis.

Finally, there can be supplementary unitarity restrictions and associated null states arising from considering conformal descendants. These have been studied in detail in [20, 40]. In analogy with the 5D approach, we will choose not to exclude any momenta from the basis of auxiliary Verma-module generators (4.1); with the help of the RS algorithm this will account for states corresponding to equations of motion and conservation equations.

We can combine the information from the conformal and superconformal unitarity bounds to predict when a multiplet will contain operator constraints. These are found to be

$$\mathcal{B}[c_1, c_2, 0; 0], \quad \mathcal{C}[c_1, 0, 0; \{0, 1\}], \quad \mathcal{D}[0, 0, 0; \{1, 2\}] \tag{4.3}$$

and will be treated separately in the following sections. The multiplet $\mathcal{D}[0, 0, 0; 0]$ does not belong to this list as it is the vacuum, which is annihilated by all supercharges and momenta.

4.2 6D (1,0) recombination rules

Short multiplets can recombine into a long multiplet \mathcal{L} . This occurs when the conformal dimension of \mathcal{L} approaches the unitarity bound, that is when $\Delta + \epsilon \rightarrow 2K + \frac{1}{2}(c_1 + 2c_2 + 3c_3) + 6$. We find that

$$\begin{aligned}
 \mathcal{L}[\Delta + \epsilon; c_1, c_2, c_3; K] &\xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[c_1, c_2, c_3; K] \oplus \mathcal{A}[c_1, c_2, c_3 - 1; K + 1], \\
 \mathcal{L}[\Delta + \epsilon; c_1, c_2, 0; K] &\xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[c_1, c_2, 0; K] \oplus \mathcal{B}[c_1, c_2 - 1, 0; K + 2], \\
 \mathcal{L}[\Delta + \epsilon; c_1, 0, 0; K] &\xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[c_1, 0, 0; K] \oplus \mathcal{C}[c_1 - 1, 0, 0; K + 3], \\
 \mathcal{L}[\Delta + \epsilon; 0, 0, 0; K] &\xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[0, 0, 0; K] \oplus \mathcal{D}[0, 0, 0; K + 4].
 \end{aligned} \tag{4.4}$$

These identities can be explicitly checked, e.g. by using supercharacters for the multiplets that we discuss below.

A small number of short multiplets do not appear in any recombination rule. These are

$$\begin{aligned}
 &\mathcal{B}[c_1, c_2, 0; \{0, 1\}], \\
 &\mathcal{C}[c_1, 0, 0; \{0, 1, 2\}], \\
 &\mathcal{D}[0, 0, 0; \{0, 1, 2, 3\}].
 \end{aligned} \tag{4.5}$$

4.3 The 6D (1,0) superconformal index

We define the 6D (1,0) superconformal index with respect to the supercharge \mathcal{Q}_{14} , in accordance with [13]. This is given by

$$\mathcal{I}(p, q, s) = \text{Tr}_{\mathcal{H}}(-1)^F e^{-\beta\delta} q^{\Delta - \frac{1}{2}K} p^{c_2} s^{c_1}, \tag{4.6}$$

where the fermion number is $F = c_1 + c_3$. The states that are counted satisfy $\delta = 0$, where

$$\delta := \{\mathcal{Q}_{14}, \mathcal{S}_{24}\} = \Delta - 2K - \frac{1}{2}(c_1 + 2c_2 + 3c_3). \tag{4.7}$$

The Cartan combinations $\Delta - \frac{1}{2}K$, c_2 and c_1 are generators of the subalgebra that commutes with $\mathcal{Q}_{14}, \mathcal{S}_{24}$ and δ . This index can be evaluated as a 6D supercharacter, as discussed in appendix B.

4.4 Long multiplets

Long multiplets are generated by the action of all supercharges. We will choose to group them into $Q = (\mathcal{Q}_{A1}, \mathcal{Q}_{A2})$ and $\tilde{Q} = (\mathcal{Q}_{A3}, \mathcal{Q}_{A4})$, with their individual quantum numbers given by

$$\begin{aligned}
 \mathcal{Q}_{11} &\sim (1)_{(1,0,0)}, & \mathcal{Q}_{12} &\sim (1)_{(-1,1,0)}, & \mathcal{Q}_{13} &\sim (1)_{(0,-1,1)}, & \mathcal{Q}_{14} &\sim (1)_{(0,0,-1)}, \\
 \mathcal{Q}_{21} &\sim (-1)_{(1,0,0)}, & \mathcal{Q}_{22} &\sim (-1)_{(-1,1,0)}, & \mathcal{Q}_{23} &\sim (-1)_{(0,-1,1)}, & \mathcal{Q}_{24} &\sim (-1)_{(0,0,-1)}.
 \end{aligned} \tag{4.8}$$

The action of these supercharges on a superconformal primary $(K)_{(c_1, c_2, c_3)}$ is given by

$$\begin{aligned}
 (K)_{(c_1, c_2, c_3)} &\xrightarrow{Q} (K \pm 1)_{(c_1+1, c_2, c_3), (c_1-1, c_2+1, c_3)}, \\
 &\xrightarrow{Q^2} (K \pm 2)_{(c_1, c_2+1, c_3)}, (K)_{(c_1+2, c_2, c_3), (c_1, c_2+1, c_3)^2, (c_1-2, c_2+2, c_3)}, \\
 &\xrightarrow{Q^3} (K \pm 1)_{(c_1+1, c_2+1, c_3), (c_1-1, c_2+2, c_3)}, \\
 &\xrightarrow{Q^4} (K)_{(c_1, c_2+2, c_3)}, \\
 (K)_{(c_1, c_2, c_3)} &\xrightarrow{\tilde{Q}} (K \pm 1)_{(c_1, c_2-1, c_3+1), (c_1, c_2, c_3-1)}, \\
 &\xrightarrow{\tilde{Q}^2} (K \pm 2)_{(c_1, c_2-1, c_3)}, (K)_{(c_1, c_2-2, c_3+2), (c_1, c_2-1, c_3)^2, (c_1, c_2, c_3-2)}, \\
 &\xrightarrow{\tilde{Q}^3} (K \pm 1)_{(c_1, c_2-1, c_3-1), (c_1, c_2-2, c_3+1)}, \\
 &\xrightarrow{\tilde{Q}^4} (K)_{(c_1, c_2-2, c_3)}. \tag{4.9}
 \end{aligned}$$

4.5 \mathcal{A} -type multiplets

Recall from table 2 that the \mathcal{A} -multiplets obey four kinds of shortening conditions. These result in the removal of the following combinations of supercharges from the basis of Verma module generators (4.1):

$$\begin{aligned}
 \mathcal{A}[c_1, c_2, c_3; K] &: \mathcal{Q}_{14}, \\
 \mathcal{A}[c_1, c_2, 0; K] &: \mathcal{Q}_{13} \mathcal{Q}_{14}, \\
 \mathcal{A}[c_1, 0, 0; K] &: \mathcal{Q}_{12} \mathcal{Q}_{13} \mathcal{Q}_{14}, \\
 \mathcal{A}[0, 0, 0; K] &: \mathcal{Q}_{11} \mathcal{Q}_{12} \mathcal{Q}_{13} \mathcal{Q}_{14}. \tag{4.10}
 \end{aligned}$$

With regards to the spectrum, let us first consider the $\mathcal{A}[c_1, c_2, c_3; K]$ multiplet. The action of the Q -set of supercharges is the same as for the long multiplet (4.9), however since we are also instructed to remove \mathcal{Q}_{14} the \tilde{Q} -chain becomes

$$\begin{aligned}
 (K)_{(c_1, c_2, c_3)} &\xrightarrow{\tilde{Q}} (K \pm 1)_{(c_1, c_2-1, c_3+1)}, (K-1)_{(c_1, c_2, c_3-1)}, \\
 &\xrightarrow{\tilde{Q}^2} (K)_{(c_1, c_2-2, c_3+2), (c_1, c_2-1, c_3)}, (K-2)_{(c_1, c_2-1, c_3)}, \\
 &\xrightarrow{\tilde{Q}^3} (K-1)_{(c_1, c_2-2, c_3+1)}. \tag{4.11}
 \end{aligned}$$

The remaining four cases can be obtained straightforwardly in a similar way. The resulting multiplet spectra turn out to be the same as starting with (4.11), substituting for specific c_i values and running the RS algorithm.

For $K = 0$ one should also remove the additional supercharge combinations obtained by acting with the R -symmetry lowering operators on the supercharges mentioned in (4.10) from the basis of auxiliary Verma-module generators. Once again, the resulting spectra are equivalent to starting with the $K \neq 0$ multiplets, explicitly setting $K = 0$ and running the RS algorithm for $\mathfrak{su}(2)_R$.

The superconformal index for all values of c_i and K is given by

$$\mathcal{I}_{\mathcal{A}[c_1, c_2, c_3; K]}(p, q, s) = \frac{(-1)^{c_1+c_3} q^{6+\frac{3K}{2}+\frac{1}{2}(c_1+2c_2+3c_3)}}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(p-qs)} \quad (4.12)$$

$$\times \left\{ p^{-c_2+1} s^{c_2+4} (p^{-c_1} - ps^{c_1+1}) + p^{c_2+4} s^{c_1+4} - p^{-c_1+2} s^{2-c_2} + p^{c_1+4} s^{-c_1+1} (s^{-c_2} - sp^{c_2+1}) \right\}.$$

This index satisfies the following recombination rules

$$\lim_{\epsilon \rightarrow 0} \mathcal{I}_{\mathcal{L}[\Delta+\epsilon; c_1, c_2, c_3; K]}(p, q, s) = \mathcal{I}_{\mathcal{A}[c_1, c_2, c_3; K]}(p, q, s) + \mathcal{I}_{\mathcal{A}[c_1, c_2, c_3-1; K+1]}(p, q, s) = 0. \quad (4.13)$$

4.6 \mathcal{B} -type Multiplets

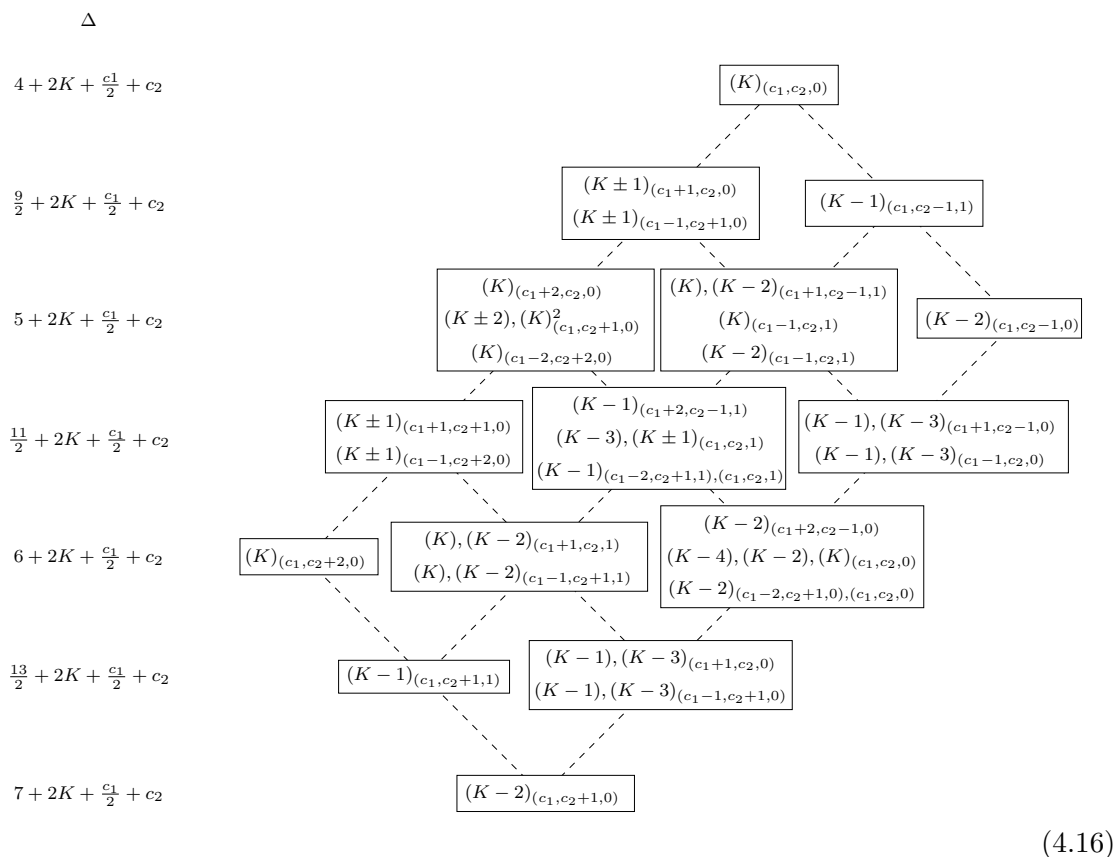
For the \mathcal{B} -type multiplets, the supercharges that need to be removed from the basis (4.1) are

$$\begin{aligned} \mathcal{B}[c_1, c_2, 0; K] &: \mathcal{Q}_{13}, \\ \mathcal{B}[c_1, 0, 0; K] &: \mathcal{Q}_{12} \mathcal{Q}_{13}, \\ \mathcal{B}[0, 0, 0; K] &: \mathcal{Q}_{11} \mathcal{Q}_{12} \mathcal{Q}_{13}. \end{aligned} \quad (4.14)$$

For the first type of multiplet, $\mathcal{B}[c_1, c_2, 0; K]$, one should also remove \mathcal{Q}_{14} . Acting on a primary with generic quantum numbers, $(K)_{(c_1, c_2, c_3)}$, one has the same action as (4.9) for Q . The \tilde{Q} -chain is modified to

$$\begin{aligned} (K)_{(c_1, c_2, c_3)} &\xrightarrow{\tilde{Q}} (K-1)_{(c_1, c_2-1, c_3+1), (c_1, c_2, c_3-1)}, \\ &\xrightarrow{\tilde{Q}^2} (K-2)_{(c_1, c_2-1, c_3)}. \end{aligned} \quad (4.15)$$

We can represent this multiplet on a grid, where acting with Q s corresponds to *southwest* motion on the diagram, while acting with \tilde{Q} to *southeast* motion



The next multiplet type is $\mathcal{B}[c_1, 0, 0; K]$, where one is instructed to remove $\mathcal{Q}_{12}\mathcal{Q}_{13}$ due to the shortening condition. Since $c_2 = c_3 = 0$, one can deduce that $\mathcal{Q}_{12}\mathcal{Q}_{14}$ and $\mathcal{Q}_{13}\mathcal{Q}_{14}$ should also be removed. The multiplet spectrum is given by

$$\begin{array}{l}
 \Delta \\
 4 + 2K + \frac{c_1}{2} \\
 \frac{9}{2} + 2K + \frac{c_1}{2} \\
 5 + 2K + \frac{c_1}{2} \\
 \frac{11}{2} + 2K + \frac{c_1}{2} \\
 6 + 2K + \frac{c_1}{2} \\
 \frac{13}{2} + 2K + \frac{c_1}{2} \\
 7 + 2K + \frac{c_1}{2}
 \end{array}
 \begin{array}{c}
 \boxed{(K)_{(c_1,0,0)}} \\
 \boxed{\begin{array}{l} (K \pm 1)_{(c_1+1,0,0)} \\ (K \pm 1)_{(c_1-1,1,0)} \end{array}} \\
 \boxed{\begin{array}{l} (K)_{(c_1+2,0,0)} \\ (K \pm 2), (K)^2_{(c_1,1,0)} \\ (K)_{(c_1-2,2,0)} \end{array}} \quad \boxed{\begin{array}{l} (K)_{(c_1-1,0,1)} \\ (K-2)_{(c_1-1,0,1)} \end{array}} \\
 \boxed{\begin{array}{l} (K \pm 1)_{(c_1+1,1,0)} \\ (K \pm 1)_{(c_1-1,2,0)} \end{array}} \quad \boxed{\begin{array}{l} (K-3), (K \pm 1)_{(c_1,0,1)} \\ (K-1)_{(c_1-2,1,1),(c_1,0,1)} \end{array}} \quad \boxed{(K-1), (K-3)_{(c_1-1,0,0)}} \\
 \boxed{(K)_{(c_1,2,0)}} \quad \boxed{\begin{array}{l} (K), (K-2)_{(c_1+1,0,1)} \\ (K), (K-2)_{(c_1-1,1,1)} \end{array}} \quad \boxed{\begin{array}{l} (K-4), (K-2), (K)_{(c_1,0,0)} \\ (K-2)_{(c_1-2,1,0),(c_1,0,0)} \end{array}} \\
 \boxed{(K-1)_{(c_1,1,1)}} \quad \boxed{\begin{array}{l} (K-1), (K-3)_{(c_1+1,0,0)} \\ (K-1), (K-3)_{(c_1-1,1,0)} \end{array}} \\
 \boxed{(K-2)_{(c_1,1,0)}}
 \end{array}
 \tag{4.17}$$

The last \mathcal{B} -type multiplet to consider is $\mathcal{B}[0, 0, 0; K]$. Its shortening condition is $A^1 A^2 A^3 \Psi = 0$ implying that $\mathcal{Q}_{11} \mathcal{Q}_{12} \mathcal{Q}_{13}$ should be removed from the auxiliary Verma-module basis. One further deduces that $\mathcal{Q}_{1a} \mathcal{Q}_{1b} \mathcal{Q}_{1c}$ should also be removed from the set of generators since $c_i = 0$. The multiplet spectrum is determined to be

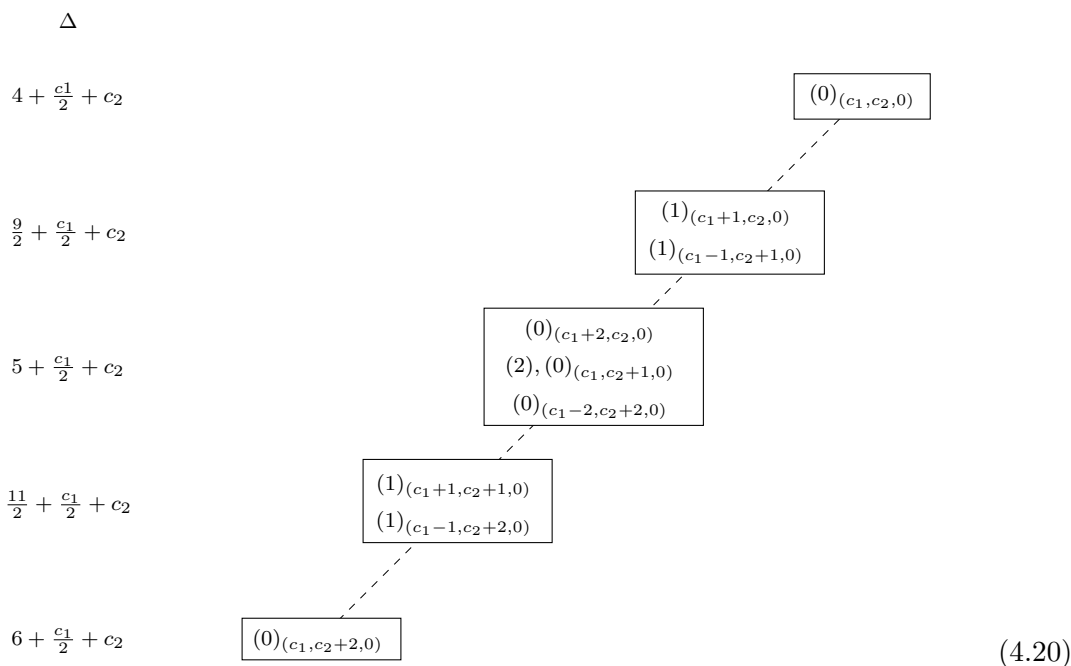
$$\begin{array}{c}
 \Delta \\
 4 + 2K \\
 \frac{9}{2} + 2K \\
 5 + 2K \\
 \frac{11}{2} + 2K \\
 6 + 2K \\
 \frac{13}{2} + 2K \\
 7 + 2K
 \end{array}
 \begin{array}{c}
 (K)_{(0,0,0)} \\
 (K \pm 1)_{(1,0,0)} \\
 (K)_{(2,0,0)} \\
 (K \pm 2), (K)_{(0,1,0)} \\
 (K \pm 1)_{(1,1,0)} \\
 (K - 3), (K \pm 1)_{(0,0,1)} \\
 (K)_{(0,2,0)} \\
 (K - 2), (K)_{(1,0,1)} \\
 (K - 4), (K - 2), (K)_{(0,0,0)} \\
 (K - 1)_{(0,1,1)} \\
 (K - 3), (K - 1)_{(1,0,0)} \\
 (K - 2)_{(0,1,0)}
 \end{array}
 \tag{4.18}$$

The index over \mathcal{B} -type multiplets for all values of c_i and $K \neq 0$ is given by

$$\begin{aligned}
 \mathcal{I}_{\mathcal{B}[c_1, c_2, 0; K]}(p, q, s) = & (-1)^{c_1+1} q^{4+\frac{3K}{2}+\frac{c_1}{2}+c_2} \left\{ \frac{p^{-c_2} s^{c_2+5} (p^{-c_1} - p s^{c_1+1}) - p^{-c_1+2} s^{1-c_2}}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(p-qs)} \right. \\
 & \left. + \frac{p^{c_2+5} s^{c_1+4} + p^{c_1+4} s^{-c_1} (s^{-c_2} - s^2 p^{c_2+2})}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(p-qs)} \right\}. \tag{4.19}
 \end{aligned}$$

4.6.1 Higher-spin-current multiplets: $\mathcal{B}[c_1, c_2, 0; 0]$

In eq. (4.3) we claimed that the special subset of \mathcal{B} -type multiplets with $K = 0$ should contain operator constraints. Recall that for $\mathcal{B}[c_1, c_2, 0; K]$ one should remove \mathcal{Q}_{14} and \mathcal{Q}_{13} from the basis of auxiliary Verma-module generators. Since $K = 0$ we also remove two more supercharges, \mathcal{Q}_{24} and \mathcal{Q}_{23} . That is, we should completely remove the set of \tilde{Q} supercharges of (4.8) and the spectrum is generated solely by acting with the set of all Q s:



This result seems to contradict (4.3), which predicted the presence of operator constraints. However, note that the absent Q s anticommute into \mathcal{P}_6 , which has therefore been implicitly removed from the auxiliary Verma-module generators. This has the effect of projecting out states corresponding to operator constraints and hence the spectrum only contains reduced states. The operator constraints can be restored using the dictionary developed in appendix C. This can be done using the character expression

$$\hat{\chi}[\Delta; c_1, c_2, c_2; K] = \chi[\Delta; c_1, c_2, c_3; K] - \chi[\Delta + 1; c_1, c_2 - 1, c_3; K], \quad (4.21)$$

where the hat indicates a character of the reduced (i.e. \mathcal{P}_6 -removed) Verma module. The

result is

$$\begin{array}{l}
 \Delta \\
 4 + \frac{c_1}{2} + c_2 \\
 \frac{9}{2} + \frac{c_1}{2} + c_2 \\
 5 + \frac{c_1}{2} + c_2 \\
 \frac{11}{2} + \frac{c_1}{2} + c_2 \\
 6 + \frac{c_1}{2} + c_2 \\
 \frac{13}{2} + \frac{c_1}{2} + c_2 \\
 7 + \frac{c_1}{2} + c_2
 \end{array}
 \begin{array}{c}
 \boxed{(0)_{(c_1, c_2, 0)}} \\
 \begin{array}{c} \boxed{(1)_{(c_1+1, c_2, 0)}} \\ \boxed{(1)_{(c_1-1, c_2+1, 0)}} \end{array} \\
 \begin{array}{c} \boxed{(0)_{(c_1+2, c_2, 0)}} \\ \boxed{(2), (0)_{(c_1, c_2+1, 0)}} \\ \boxed{(0)_{(c_1-2, c_2+2, 0)}} \end{array} \\
 \begin{array}{c} \boxed{(1)_{(c_1+1, c_2+1, 0)}} \\ \boxed{(1)_{(c_1-1, c_2+2, 0)}} \end{array} \\
 \begin{array}{c} \boxed{(0)_{(c_1, c_2+2, 0)}} \\ \begin{array}{c} \boxed{-(0)_{(c_1+2, c_2-1, 0)}} \\ \boxed{-(2), -(0)_{(c_1, c_2, 0)}} \\ \boxed{-(0)_{(c_1-2, c_2+1, 0)}} \end{array} \end{array} \\
 \begin{array}{c} \boxed{-(1)_{(c_1+1, c_2, 0)}} \\ \boxed{-(1)_{(c_1-1, c_2+1, 0)}} \end{array} \\
 \boxed{-(0)_{(c_1, c_2+1, 0)}}
 \end{array}
 \begin{array}{c}
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 \\
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 \\
 \end{array}
 \begin{array}{c}
 \\
 \\
 \begin{array}{c} \boxed{-(0)_{(c_1, c_2-1, 0)}} \\ \\ \end{array} \\
 \\
 \\
 \\
 \\
 \end{array}
 \tag{4.22}$$

An example of a superconformal primary for a $\mathcal{B}[0, c_2, 0; 0]$ multiplet with $c_2 > 0$ is

$$\mathcal{O}_{\mu_1 \dots \mu_{c_2}} = \epsilon_{\mathbf{AB}} \phi^{\mathbf{A}} \overset{\leftrightarrow}{\partial}_{\mu_1} \dots \overset{\leftrightarrow}{\partial}_{\mu_{c_2}} \phi^{\mathbf{B}}, \tag{4.23}$$

where $\phi^{\mathbf{A}}$ is a free hypermultiplet scalar. It is interesting to point out that for $c_1 = 1, c_2 = 0$ or $c_1 = 0, c_2 = 1$ the superconformal primary is not higher spin. The higher-spin currents are instead found as their descendants. Moreover, the superconformal primary in the multiplet of $\mathcal{B}[0, 1, 0; 0]$ is a $\Delta = 5$, R-neutral conserved current. However, as explained around (2.32) in the case of 5D, this is not a flavour current.

The corresponding superconformal index for any $c_1, c_2 \geq 0$ reads

$$\begin{aligned}
 \mathcal{I}_{\mathcal{B}[c_1, c_2, 0; 0]}(p, q, s) = & (-1)^{c_1+1} q^{4+\frac{1}{2}(c_1+2c_2)} \left\{ \frac{p^{-c_2} s^{c_2+5} (p^{-c_1} - p s^{c_1+1}) - p^{-c_1+2} s^{1-c_2}}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(p-qs)} \right. \\
 & \left. + \frac{p^{c_2+5} s^{c_1+4} + p^{c_1+4} s^{-c_1} (s^{-c_2} - s^2 p^{c_2+2})}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(p-qs)} \right\}. \tag{4.24}
 \end{aligned}$$

4.6.2 The stress-tensor multiplet: $\mathcal{B}[0, 0, 0; 0]$

The last \mathcal{B} -type multiplet of note is the stress tensor. Ordinarily, the shortening conditions require that we remove $\mathcal{Q}_{1a}\mathcal{Q}_{1b}\mathcal{Q}_{1c}$ from the basis of generators, but since $K = 0$ we also need to remove contributions obtained by acting repeatedly with R -symmetry lowering operators. The resulting multiplet is therefore

$$\begin{array}{c}
 \Delta \\
 4 \qquad \qquad \qquad \boxed{(0)_{(0,0,0)}} \\
 \frac{9}{2} \qquad \qquad \qquad \boxed{(1)_{(1,0,0)}} \\
 5 \qquad \qquad \qquad \boxed{\begin{array}{l} (0)_{(2,0,0)} \\ (2)_{(0,1,0)} \end{array}} \\
 \frac{11}{2} \qquad \qquad \boxed{(1)_{(1,1,0)}} \\
 6 \qquad \qquad \boxed{(0)_{(0,2,0)}} \qquad \qquad \boxed{-(2)_{(0,0,0)}} \\
 \frac{13}{2} \qquad \qquad \qquad \boxed{-(1)_{(1,0,0)}} \\
 7 \qquad \qquad \qquad \boxed{-(0)_{(0,1,0)}}
 \end{array} \tag{4.25}$$

We recognise these states as being associated with the fields

$$\begin{array}{ll}
 [5; 0, 1, 0; 2] : J_{\mu}^{(\mathbf{AB})}, & -[6; 0, 0, 0; 2] : \partial^{\mu} J_{\mu}^{(\mathbf{AB})} = 0, \\
 [11/2; 1, 1, 0; 1] : S_{\mu a}^{\mathbf{A}}, & -[13/2; 1, 0, 0; 1] : \partial^{\mu} S_{\mu a}^{\mathbf{A}} = 0, \\
 [6; 0, 2, 0; 0] : \Theta_{\mu\nu}, & -[7; 0, 1, 0; 0] : \partial^{\mu} \Theta_{\mu\nu} = 0,
 \end{array} \tag{4.26}$$

namely the 6D R -symmetry current, supersymmetry current and stress tensor. We also have three states that do not obey equations of motion. These are

$$\begin{array}{ll}
 [4; 0, 0, 0; 0] : \Sigma, \\
 [9/2; 1, 0, 0; 1] : \zeta_a^{\mathbf{A}}, \\
 [5; 2, 0, 0; 0] : Z_{(ab)}^+,
 \end{array} \tag{4.27}$$

where $+$ denotes the selfdual part of the operator.

The index over this stress-tensor multiplet is

$$\mathcal{I}_{\mathcal{B}[0,0,0;0]}(p, q, s) = q^4 \frac{s^{-1} + p + p^{-1}s}{(1-pq)(1-qs^{-1})(1-p^{-1}qs)}, \quad (4.28)$$

counting three components of the R -symmetry current plus conformal descendants.

4.7 \mathcal{C} -type multiplets

The two distinct \mathcal{C} -type multiplets are $\mathcal{C}[c_1, 0, 0; K]$ and $\mathcal{C}[0, 0, 0; K]$, with the requirement that we remove \mathcal{Q}_{1a} for $a \neq 1$ and $\mathcal{Q}_{1a}\mathcal{Q}_{1b}$ respectively from the basis of auxiliary Verma-module generators. On the one hand, since the generators \mathcal{Q}_{13} and \mathcal{Q}_{14} are absent, the set of available \tilde{Q} s is the same as for the $\mathcal{B}[c_1, c_2, 0; K]$ case, that is $\mathcal{Q}_{23}, \mathcal{Q}_{24}$. On the other, only \mathcal{Q}_{12} is removed from the set of Q s. Therefore the two resulting chains of supercharge actions on a generic state $(K)_{(c_1, c_2, c_3)}$ are

$$\begin{aligned} (K)_{(c_1, c_2, c_3)} &\xrightarrow{Q} (K \pm 1)_{(c_1+1, c_2, c_3)}, (K-1)_{(c_1-1, c_2+1, c_3)}, \\ &\xrightarrow{Q^2} (K-2)_{(c_1, c_2+1, c_3)}, (K)_{(c_1+2, c_2, c_3), (c_1, c_2+1, c_3)}, \\ &\xrightarrow{Q^3} (K-1)_{(c_1+1, c_2+1, c_3)}, \\ (K)_{(c_1, c_2, c_3)} &\xrightarrow{\tilde{Q}} (K-1)_{(c_1, c_2-1, c_3+1), (c_1, c_2, c_3-1)}, \\ &\xrightarrow{\tilde{Q}^2} (K-2)_{(c_1, c_3+1, c_3)}. \end{aligned} \quad (4.29)$$

After substituting in the relevant c_i values for the primary and implementing the RS algorithm we obtain for $\mathcal{C}[c_1, 0, 0; K]$

$$\begin{array}{c} \Delta \\ 2 + 2K + \frac{c_1}{2} \\ \frac{5}{2} + 2K + \frac{c_1}{2} \\ 3 + 2K + \frac{c_1}{2} \\ \frac{7}{2} + 2K + \frac{c_1}{2} \\ 4 + 2K + \frac{c_1}{2} \\ \frac{9}{2} + 2K + \frac{c_1}{2} \end{array} \quad \begin{array}{c} \boxed{(K)_{(c_1, 0, 0)}} \\ \text{---} \\ \boxed{\begin{array}{l} (K \pm 1)_{(c_1+1, 0, 0)} \\ (K-1)_{(c_1-1, 1, 0)} \end{array}} \\ \text{---} \\ \begin{array}{cc} \boxed{\begin{array}{l} (K-2), (K)_{(c_1, 1, 0)} \\ (K)_{(c_1+2, 0, 0)} \end{array}} & \boxed{(K-2)_{(c_1-1, 0, 1)}} \end{array} \\ \text{---} \\ \begin{array}{ccc} \boxed{(K-1)_{(c_1+1, 1, 0)}} & \boxed{(K-1), (K-3)_{(c_1, 0, 1)}} & \boxed{(K-3)_{(c_1-1, 0, 0)}} \end{array} \\ \text{---} \\ \begin{array}{cc} \boxed{(K-2)_{(c_1+1, 0, 1)}} & \boxed{(K-2), (K-4)_{(c_1, 0, 0)}} \end{array} \\ \text{---} \\ \boxed{(K-3)_{(c_1+1, 0, 0)}} \end{array} \quad (4.30)$$

The corresponding superconformal index is

$$\mathcal{I}_{\mathcal{C}[c_1,0,0;K]}(p, q, s) = (-1)^{c_1} q^{2+\frac{3K}{2}+\frac{c_1}{2}} \frac{p^{c_1+5} s^{-c_1} (ps-1) + p^2 s^{c_1+5} (s-p^2) + p^{-c_1} s^2 (p-s^2)}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(qs-p)}, \quad (4.31)$$

which can be used in conjunction with $\mathcal{A}[c_1, 0, 0; K]$ to verify the recombination rules

$$\lim_{\epsilon \rightarrow 0} \mathcal{I}_{\mathcal{L}[\Delta+\epsilon; c_1, 0, 0; K]}(p, q, s) = \mathcal{I}_{\mathcal{A}[c_1, 0, 0; K]}(p, q, s) + \mathcal{I}_{\mathcal{C}[c_1-1, 0, 0; K+3]}(p, q, s) = 0. \quad (4.32)$$

Turning our attention to $\mathcal{C}[0, 0, 0; K]$, we recall that the shortening conditions require the combinations $\mathcal{Q}_{1a} \mathcal{Q}_{1b}$ to be absent from the basis of auxiliary Verma-module generators. The resulting spectrum is alternatively obtained by setting $c_1 = 0$ in $\mathcal{C}[c_1, 0, 0; K]$ and running the RS algorithm. This is a simple task: the lowest value of c_1 that appears in $\mathcal{C}[c_1, 0, 0; K]$ is $c_1 - 1$ and only these states are deleted when setting $c_1 = 0$ — they will be on the boundary of the Weyl chamber. There is no need to perform any of the more elaborate Weyl reflections on any other state.

Therefore the spectrum for $\mathcal{C}[0, 0, 0; K]$ is given by

Δ	
$2 + 2K$	$(K)_{(0,0,0)}$
$\frac{5}{2} + 2K$	$(K \pm 1)_{(1,0,0)}$
$3 + 2K$	$(K-2), (K)_{(0,1,0)}$ $(K)_{(2,0,0)}$
$\frac{7}{2} + 2K$	$(K-1)_{(1,1,0)}$ $(K-1), (K-3)_{(0,0,1)}$
$4 + 2K$	$(K-2)_{(1,0,1)}$ $(K-2), (K-4)_{(0,0,0)}$
$\frac{9}{2} + 2K$	$(K-3)_{(1,0,0)}$

(4.33)

The associated index for $K > 1$ evaluates to

$$\mathcal{I}_{\mathcal{C}[0,0,0;K]}(p, q, s) = -q^{2+\frac{3K}{2}} \frac{s^{-1}p + s + p^{-1}}{(1-pq)(1-qs^{-1})(1-p^{-1}qs)}. \quad (4.34)$$

The cases with $K = 0, 1$ are predicted to contain operator constraints from (4.3) and will be dealt with separately below.

4.7.1 $\mathcal{C}[c_1, 0, 0; 0]$

For this class of multiplets, $K = 0$ and we obtain additional shortening conditions. These can be translated into the requirement that $\mathcal{Q}_{\mathbf{A}a}$ for $a \neq 1$ be removed from the basis of auxiliary Verma-module generators. The latter then in turn imply that $\mathcal{P}_{23} \sim \{\mathcal{Q}_{12}, \mathcal{Q}_{23}\}$, $\mathcal{P}_{24} \sim \{\mathcal{Q}_{12}, \mathcal{Q}_{24}\}$ and $\mathcal{P}_{34} \sim \{\mathcal{Q}_{13}, \mathcal{Q}_{24}\}$ have also been removed from the auxiliary Verma-module basis. In vector notation, these momentum operators correspond respectively to \mathcal{P}_3 , \mathcal{P}_5 and \mathcal{P}_6 . The implication of this fact is that the multiplet construction will only reproduce the reduced states, with the operator constraints having been projected out. Thus the $\mathcal{C}[c_1, 0, 0; 0]$ multiplet is very simply:

$$\begin{array}{c}
 \Delta \\
 2 + \frac{c_1}{2} \quad \boxed{(0)_{(c_1, 0, 0)}} \\
 \frac{5}{2} + \frac{c_1}{2} \quad \boxed{(1)_{(c_1+1, 0, 0)}} \\
 3 + \frac{c_1}{2} \quad \boxed{(0)_{(c_1+2, 0, 0)}}
 \end{array} \tag{4.35}$$

The operator constraints can be restored by means of appendix C. Implementing this would result in introducing — for each of the three states in (4.35) — the following combinations

$$(K)_{(c_1, 0, 0)}^\Delta : \quad -(K)_{(c_1-1, 0, 1)}^{\Delta+1} + (K)_{(c_1-2, 1, 0)}^{\Delta+2} - (K)_{(c_1-2, 0, 0)}^{\Delta+3}, \tag{4.36}$$

for a total of nine additional states corresponding to equations of motion. As a side comment, note that this spectrum is not the one we would have obtained had we just set $K = 0$ in (4.30) and implemented the RS algorithm. As such, the index that one obtains is different to just setting $K = 0$ in (4.31), and is given by

$$\begin{aligned}
 \mathcal{I}_{\mathcal{C}[c_1, 0, 0; 0]} = & (-1)^{c_1} q^{2+\frac{c_1}{2}} \left\{ \frac{p^{c_1+6} s^{1-c_1} - p^{c_1+5} s^{-c_1} + s^2 p^{-c_1} (p - s^2) - p^4 s^{c_1+5} + p^2 s^{c_1+6}}{(pq - 1)(p^2 - s)(ps - 1)(p - s^2)(q - s)(qs - p)} \right. \\
 & \left. + \frac{p^{c_1+5} s^{-c_1} (ps - 1) + p^2 s^{c_1+5} (s - p^2) + p^{-c_1} s^2 (p - s^2)}{(pq - 1)(p^2 - s)(ps - 1)(p - s^2)(q - s)(qs - p)} \right\}. \tag{4.37}
 \end{aligned}$$

4.7.2 The free-tensor multiplet: $\mathcal{C}[0, 0, 0; 0]$

When $c_1 = 0$ we recover the free-tensor multiplet. In this case the lowering operator of $\mathfrak{su}(2)_R$ acts on the null-state condition $\mathcal{Q}_{11}\mathcal{Q}_{12}\Psi_{\text{aux}} = 0$ to create additional shortening conditions. This translates into the requirement that $\mathcal{Q}_{1a}\mathcal{Q}_{1b}$, $\mathcal{Q}_{2a}\mathcal{Q}_{1b}$ and $\mathcal{Q}_{2a}\mathcal{Q}_{2b}$ for $a \neq b$ should be removed from the basis of auxiliary Verma-module generators. The

multiplet can be constructed using the remaining supercharges and is given by

$$\begin{array}{c}
 \Delta \\
 2 \\
 \frac{5}{2} \\
 3
 \end{array}
 \begin{array}{c}
 \boxed{(0)_{(0,0,0)}} \\
 \text{---} \\
 \boxed{(1)_{(1,0,0)}} \\
 \text{---} \\
 \boxed{(0)_{(2,0,0)}}
 \end{array}
 \tag{4.38}$$

We identify the states with the following fields

$$\begin{aligned}
 [2; 0, 0; 0; 0] &: \varphi, \\
 [5/2; 1, 0, 0; 1] &: \lambda_a^{\mathbf{A}}, \\
 [3; 2, 0, 0; 0] &: H_{[\mu\nu\rho]}^+.
 \end{aligned}
 \tag{4.39}$$

Similar to the previous subsection, (4.38) only contains reduced states and no equations of motion. When the latter are restored by means of appendix C we recover the expected

$$\begin{aligned}
 \partial^2 \varphi = 0 &: - [4; 0, 0, 0; 0], \\
 \not{\partial}^{\dot{a}b} \tilde{\lambda}_b^{\mathbf{A}} = 0 &: - [7/2; 0, 0, 1; 1], \\
 \partial_{[\sigma} H_{\mu\nu\rho]}^+ = 0 &: - [4; 1, 0, 1; 0] + [5; 0, 1, 0; 0] - [6; 0, 0, 0; 0].
 \end{aligned}
 \tag{4.40}$$

The index for this configuration evaluates to

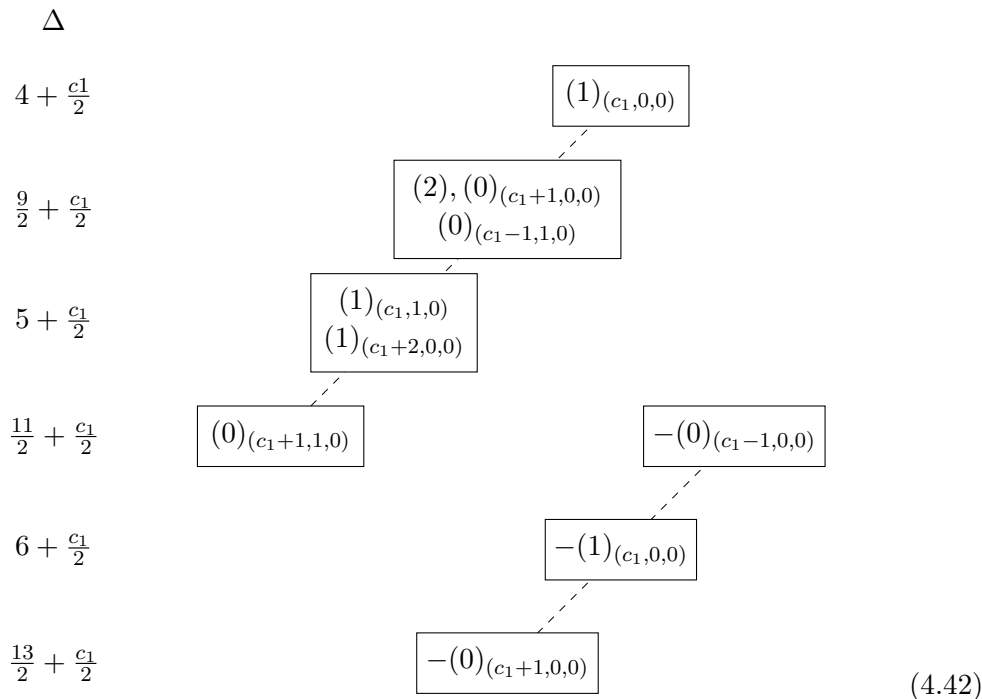
$$\mathcal{I}_{C[0,0,0;0]}(p, q, s) = -q^2 \frac{s^{-1}p + s + p^{-1} - q}{(1 - pq)(1 - qs^{-1})(1 - p^{-1}qs)},
 \tag{4.41}$$

counting three components of the fermion $\lambda_a^{\mathbf{A}}$ alongside its equation of motion.

4.7.3 Higher-spin-current multiplets: $\mathcal{C}[c_1, 0, 0; 1]$

These multiplets are simple to construct since, contrary to the above case, we need not act with the R -symmetry lowering operator. As such, we can take the spectrum of (4.30) and set $K = 1$ without incurring new shortening conditions. The resulting $\mathcal{C}[c_1, 0, 0; 1]$

multiplet is



which includes conservation equations. It is worth pointing out that the multiplet with $c_1 = 1$ contains a $\Delta = 5$, R-neutral conserved current as a level one descendant. However, we see that there is also a spin-2 current with $\Delta = 6$ at level three.

The corresponding index evaluates to

$$\mathcal{I}_{\mathcal{C}[c_1,0,0;1]}(p, q, s) = q^{\frac{7}{2} + \frac{c_1}{2}} \frac{p^{c_1+5} s^{-c_1} (ps-1) + p^2 s^{c_1+5} (s-p^2) + p^{-c_1} s^2 (p-s^2)}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(qs-p)}. \quad (4.43)$$

Note that this index is valid for $c_1 = 0$. This is because setting $K = 1$ does not introduce any additional shortening conditions. However, when $c_1 = 0$ this multiplet does not contain currents with spin $j \geq 2$, and will be discussed below.

4.7.4 “Extra”-supercurrent multiplet: $\mathcal{C}[0, 0, 0; 1]$

These multiplets can be constructed by substituting $K = 1$ into the spectrum of $\mathcal{C}[0, 0, 0; K]$. The result is

Δ		
4		$(1)_{(0,0,0)}$
		┆
$\frac{9}{2}$		$(2), (0)_{(1,0,0)}$
		┆
5		$(1)_{(0,1,0)}$ $(1)_{(2,0,0)}$
		┆
$\frac{11}{2}$		$(0)_{(1,1,0)}$
		┆
6		$-(1)_{(0,0,0)}$
		┆
$\frac{13}{2}$		$-(0)_{(1,0,0)}$

(4.44)

This multiplet contains a conserved $K = 1$ spin-1 current and a conserved R-neutral spin- $\frac{3}{2}$ current. These generate additional global and supersymmetry transformations, which — as we will discuss in section 6.2 — are necessary for the description of the (2,0) stress-tensor multiplet in the language of the (1,0) SCA.

Its index is then given by

$$\mathcal{I}_{\mathcal{C}[0,0,0;1]}(p, q, s) = -q^{\frac{7}{2}} \frac{s^{-1}p + s + p^{-1}}{(1 - pq)(1 - qs^{-1})(1 - p^{-1}qs)}. \quad (4.45)$$

4.8 \mathcal{D} -type multiplets

We finally turn to the $\mathcal{D}[0, 0, 0; K]$ multiplets. These are the simplest of the 6D (1, 0) SCA for generic values of K . Since the shortening condition is $A^1\Psi = \mathcal{Q}_{11}\Psi = 0$, acting with all Lorentz lowering operators leads to

$$\mathcal{Q}_{1a}\Psi = 0 \quad (4.46)$$

and hence the multiplet is $\frac{1}{2}$ -BPS.

Starting with a generic superconformal primary state $(K)_{(c_1, c_2, c_3)}$, the two chains obtained from the action of Q s and \tilde{Q} s are

$$\begin{aligned}
 (K)_{(c_1, c_2, c_3)} &\xrightarrow{Q} (K-1)_{(c_1+1, c_2, c_3), (c_1-1, c_2+1, c_3)}, \\
 &\xrightarrow{Q^2} (K-2)_{(c_1, c_2+1, c_3)}, \\
 (K)_{(c_1, c_2, c_3)} &\xrightarrow{\tilde{Q}} (K-1)_{(c_1, c_2-1, c_3+1), (c_1, c_2, c_3-1)}, \\
 &\xrightarrow{\tilde{Q}^2} (K-2)_{(c_1, c_3+1, c_3)}.
 \end{aligned}
 \tag{4.47}$$

One then needs to set $c_i = 0$ and run the RS algorithm. Upon doing so, the result is

$$\begin{array}{r}
 \Delta \\
 2K \\
 2K + \frac{1}{2} \\
 2K + 1 \\
 2K + \frac{3}{2} \\
 2K + 2
 \end{array}
 \begin{array}{c}
 \phantom{\boxed{(K)_{(0,0,0)}}} \\
 \phantom{\boxed{(K-1)_{(1,0,0)}}} \\
 \phantom{\boxed{(K-2)_{(0,1,0)}}} \\
 \phantom{\boxed{(K-3)_{(0,0,1)}}} \\
 \phantom{\boxed{(K-4)_{(0,0,0)}}} \\
 \phantom{\boxed{(K-4)_{(0,0,0)}}}
 \end{array}
 \tag{4.48}$$

We notice that for $K \leq 2$ the above will lead to negative-multiplicity representations via the RS algorithm and hence operator constraints, to be discussed in the next section.

The superconformal index for $K > 2$ is given by

$$\mathcal{I}_{\mathcal{D}[0,0,0;K]}(p, q, s) = \frac{q^{\frac{3K}{2}}}{(1-pq)(1-qs^{-1})(1-p^{-1}qs)}.
 \tag{4.49}$$

4.8.1 The hypermultiplet: $\mathcal{D}[0, 0, 0; 1]$

Setting $K = 1$ in the $\mathcal{D}[0, 0, 0; K]$ multiplet will not incur any nontrivial changes to the spectrum, as it does not lead to additional shortening conditions. As a result, we may

These states can be identified with the operators $\mu^{(\mathbf{AB})}$, $\psi_a^{\mathbf{A}}$ and the $\mathfrak{su}(2)_R$ -singlet conserved current J_μ , with $\partial^\mu J_\mu = 0$. This multiplet is also known as a linear multiplet and appears in [52, 57, 58].

The index for this multiplet is given by

$$\mathcal{I}_{\mathcal{D}[0,0,0;2]}(p, q, s) = \frac{q^3}{(1-pq)(1-qs^{-1})(1-p^{-1}qs)}. \tag{4.53}$$

This concludes our discussion of the superconformal multiplets for the 6D (1,0) SCA.

5 Multiplets and superconformal indices for 6D (2,0)

We lastly turn to the construction of superconformal multiplets for the 6D (2,0) SCA, $\mathfrak{osp}(8^*|4)$. Since we now have sixteen Poincaré supercharges, the UIRs will be much larger compared to the ones obtained in section 2 and section 4 and representing them diagrammatically would not be particularly instructive. Similarly, the full expressions for the most general (“refined”) superconformal indices are unwieldy. Instead we will choose to detail the multiplet types, their shortening conditions and the “Schur” limit of their indices, although we do include the refined versions of the index in the accompanying Mathematica notebook.²²

5.1 UIR building with auxiliary Verma modules

The superconformal primaries of the algebra $\mathfrak{osp}(8^*|4)$ are designated $|\Delta; c_1, c_2, c_3; d_1, d_2\rangle$ and labelled by the conformal dimension Δ , the Lorentz quantum numbers for $\mathfrak{su}(4)$ in the Dynkin basis c_i and the R -symmetry quantum numbers in the Dynkin basis d_i . Each primary is in one-to-one correspondence with a highest weight labeling irreducible representations of the maximal compact subalgebra $\mathfrak{so}(6) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(5)_R \subset \mathfrak{osp}(8^*|4)$. There are sixteen Poincaré and superconformal supercharges, denoted by $\mathcal{Q}_{\mathbf{A}a}$ and $\mathcal{S}_{\mathbf{A}\hat{a}}$, where $\hat{a}, a = 1, \dots, 4$ are (anti)fundamental indices of $\mathfrak{su}(4)$ and $\mathbf{A} = 1, \dots, 4$ a spinor index of $\mathfrak{so}(5)_R$. One also has six momenta \mathcal{P}_μ and special conformal generators \mathcal{K}_μ , where μ is a vector index of the Lorentz group, $\mu = 1, \dots, 6$. The superconformal primary is annihilated by all $\mathcal{S}_{\mathbf{A}\hat{a}}$ and \mathcal{K}_μ . A basis for the representation space of $\mathfrak{osp}(8^*|4)$ can be constructed by considering the Verma module

$$\prod_{\mathbf{A},a} (\mathcal{Q}_{\mathbf{A}a})^{n_{\mathbf{A},a}} \prod_{\mu} \mathcal{P}_\mu^{n_\mu} |\Delta; c_1, c_2, c_3; d_1, d_2\rangle^{\text{hw}}, \tag{5.1}$$

where $n = \sum_{\mathbf{A},a} n_{\mathbf{A},a}$ and $\hat{n} = \sum_{\mu} n_\mu$ denote the level of a superconformal or conformal descendant respectively. In order to obtain UIRs, the requirement of unitarity needs to be imposed level-by-level on the Verma module. This leads to bounds on the conformal dimension Δ .

²²A discussion of the null states for the 6D (2,0) SCA, along with the calculation of Schur indices for the various short multiplets, can be found in appendix C of [14]. Here we additionally construct the full multiplets with an emphasis on the equations of motion and conservation equations. We also provide the refined indices for all multiplets.

Multiplet	Shortening Condition	Conformal Dimension
$\mathcal{A}[c_1, c_2, c_3; d_1, d_2]$	$A^4\Psi = 0$	$\Delta = 2d_1 + 2d_2 + \frac{c_1}{2} + c_2 + \frac{3c_3}{2} + 6$
$\mathcal{A}[c_1, c_2, 0; d_1, d_2]$	$A^3A^4\Psi = 0$	$\Delta = 2d_1 + 2d_2 + \frac{c_1}{2} + c_2 + 6$
$\mathcal{A}[c_1, 0, 0; d_1, d_2]$	$A^2A^3A^4\Psi = 0$	$\Delta = 2d_1 + 2d_2 + \frac{c_1}{2} + 6$
$\mathcal{A}[0, 0, 0; d_1, d_2]$	$A^1A^2A^3A^4\Psi = 0$	$\Delta = 2d_1 + 2d_2 + 6$
$\mathcal{B}[c_1, c_2, 0; d_1, d_2]$	$A^3\Psi = 0$	$\Delta = 2d_1 + 2d_2 + \frac{c_1}{2} + c_2 + 4$
$\mathcal{B}[c_1, 0, 0; d_1, d_2]$	$A^2A^3\Psi = 0$	$\Delta = 2d_1 + 2d_2 + \frac{c_1}{2} + 4$
$\mathcal{B}[0, 0, 0; d_1, d_2]$	$A^1A^2A^3\Psi = 0$	$\Delta = 2d_1 + 2d_2 + 4$
$\mathcal{C}[c_1, 0, 0; d_1, d_2]$	$A^2\Psi = 0$	$\Delta = 2d_1 + 2d_2 + \frac{c_1}{2} + 2$
$\mathcal{C}[0, 0, 0; d_1, d_2]$	$A^1A^2\Psi = 0$	$\Delta = 2d_1 + 2d_2 + 2$
$\mathcal{D}[0, 0, 0; d_1, d_2]$	$A^1\Psi = 0$	$\Delta = 2d_1 + 2d_2$

Table 3. A list of all short multiplets for the 6D (2,0) SCA, along with the shortening condition and conformal dimension of the superconformal primary. The A^a in the shortening conditions are defined in (4.2) and [13]. The first of these multiplets (\mathcal{A}) is a regular short representation, whereas the rest ($\mathcal{B}, \mathcal{C}, \mathcal{D}$) are isolated short representations. Here Ψ denotes the superconformal primary state for each multiplet.

Starting with the superconformal descendants, the conditions imposed by unitarity can be deduced as follows. In principle, one needs to calculate the norms of superconformal descendants for $n > 0$. However, since it is sufficient to perform this analysis in the highest weight of the R -symmetry group [13, 20, 24], the results of section 4.1 can be easily imported to the (2,0) case and we may still use the basis $A^{a_i} \dots A^{a_j} |\Delta; c_1, c_2, c_3; d_1, d_2\rangle^{\text{hw}}$.²³ We need only convert the Cartan for the highest weight of $\mathfrak{su}(2)_R$ to $\mathfrak{so}(5)_R$, which is done by simply replacing $K \rightarrow d_1 + d_2$. This gives rise to a group of similar short multiplets, which we collect in table 3. We will provide more details regarding the generators that are absent from the auxiliary Verma module when discussing individual multiplets. Furthermore, it will be important to clarify which additional absent generators can occur from tuning the R -symmetry quantum numbers, d_1 and d_2 . These turn out to be far more intricate than for (1,0).

The null states arising from conformal descendants are identical to section 4.1. One can combine the information from the conformal and superconformal unitarity bounds to predict when a multiplet will contain operator constraints [20, 40]. These special cases are

$$\mathcal{B}[c_1, c_2, 0; 0, 0], \quad \mathcal{C}[c_1, 0, 0; d_1, d_2] \quad \text{for } d_1 + d_2 \leq 1, \quad \mathcal{D}[0, 0, 0; d_1, d_2] \quad \text{for } d_1 + d_2 \leq 2. \tag{5.2}$$

The multiplet $\mathcal{D}[0, 0, 0; 0, 0]$ does not belong to this list as it is the vacuum, which is annihilated by all supercharges and momenta.

²³The analysis in [13, 20, 24] was performed for generic R -symmetry $\mathfrak{sp}(\mathcal{N})$. The construction is largely focussed around the Lorentz Cartans and raising operators.

5.2 6D (2,0) recombination rules

Short multiplets can recombine into a long multiplet \mathcal{L} . This occurs when the conformal dimension of \mathcal{L} approaches the unitarity bound, that is when $\Delta + \epsilon \rightarrow 2d_1 + 2d_2 + \frac{1}{2}(c_1 + 2c_2 + 3c_3) + 6$. As in [14], we find that

$$\begin{aligned} \mathcal{L}[\Delta + \epsilon; c_1, c_2, c_3; d_1, d_2] &\xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[c_1, c_2, c_3; d_1, d_2] \oplus \mathcal{A}[c_1, c_2, c_3 - 1; d_1, d_2 + 1], \\ \mathcal{L}[\Delta + \epsilon; c_1, c_2, 0; d_1, d_2] &\xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[c_1, c_2, 0; d_1, d_2] \oplus \mathcal{B}[c_1, c_2 - 1, 0; d_1, d_2 + 2], \\ \mathcal{L}[\Delta + \epsilon; c_1, 0, 0; d_1, d_2] &\xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[c_1, 0, 0; d_1, d_2] \oplus \mathcal{C}[c_1 - 1, 0, 0; d_1, d_2 + 3], \\ \mathcal{L}[\Delta + \epsilon; 0, 0, 0; d_1, d_2] &\xrightarrow{\epsilon \rightarrow 0} \mathcal{A}[0, 0, 0; d_1, d_2] \oplus \mathcal{D}[0, 0, 0; d_1, d_2 + 4]. \end{aligned} \quad (5.3)$$

A small number of short multiplets do not appear in a recombination rule. These are

$$\begin{aligned} &\mathcal{B}[c_1, c_2, 0; d_1, \{0, 1\}], \\ &\mathcal{C}[c_1, 0, 0; d_1, \{0, 1, 2\}], \\ &\mathcal{D}[0, 0, 0; d_1, \{0, 1, 2, 3\}]. \end{aligned} \quad (5.4)$$

5.3 The 6D (2,0) superconformal index

We define the 6D (2,0) superconformal index with respect to the supercharge \mathcal{Q}_{24} . This is given by

$$\mathcal{I}(p, q, s, t) = \text{Tr}_{\mathcal{H}}(-1)^F e^{-\beta \delta} q^{\Delta - d_1 - \frac{1}{2}d_2} p^{c_2 + c_3 - d_2} t^{-d_1 - d_2} s^{c_1 + c_2}, \quad (5.5)$$

where the fermion number is $F = c_1 + c_3$. The states that are counted satisfy $\delta = 0$, with

$$\delta := \{\mathcal{Q}_{24}, \mathcal{S}_{34}\} = \Delta - 2d_1 - \frac{1}{2}(c_1 + 2c_2 + 3c_3). \quad (5.6)$$

The exponents appearing in (5.5) are the eigenvalues for the generators commuting with \mathcal{Q}_{24} , \mathcal{S}_{34} and δ . This index can be evaluated as a 6D supercharacter; this is discussed in our appendix B and appendix C of [14].

5.3.1 The 6D Schur limit

The authors of [14] define the 6D Schur limit by taking $t \rightarrow 1$ in (5.5). Under the trace the resulting index can be rewritten as

$$\mathcal{I}^{\text{Schur}}(p, q, s) = \text{Tr}_{\mathcal{H}}(-1)^F e^{-\beta \delta} q^{\Delta - d_1 - \frac{1}{2}d_2} p^{\delta'} s^{c_1 + c_2}, \quad (5.7)$$

where

$$\delta' := \{\mathcal{Q}_{12}, \mathcal{S}_{42}\} = \Delta - 2d_1 - 2d_2 - \frac{1}{2}(c_1 - 2c_2 - c_3). \quad (5.8)$$

It can be seen that all exponents in (5.7) commute with the supercharge \mathcal{Q}_{12} and the 6D Schur index consequently counts operators annihilated by two supercharges. As a result the above index is independent of both $e^{-\beta}$ and p and the trace can be taken over operators satisfying $\delta = 0 = \delta'$:

$$\mathcal{I}^{\text{Schur}}(q, s) = \text{Tr}_{\mathcal{H}_{\delta=0=\delta'}}(-1)^F q^{\Delta - d_1 - \frac{1}{2}d_2} s^{c_1 + c_2}. \quad (5.9)$$

The operators contributing to the index in this limit satisfy

$$\begin{aligned}\Delta &= 2d_1 + d_2 + \frac{1}{2}(c_1 + c_3), \\ d_2 &= c_2 + c_3.\end{aligned}\tag{5.10}$$

5.4 Long multiplets

Long multiplets are constructed by acting with all supercharges on a superconformal primary $(d_1, d_2)_{(c_1, c_2, c_3)}$. This leads to lengthy expressions which, although not presented here, are available from the authors upon request. For book-keeping purposes we will group the supercharges into $Q = (Q_{2a}, Q_{3a})$ and $\tilde{Q} = (Q_{1a}, Q_{4a})$ for the remaining of the 6D (2,0) discussion.

5.5 \mathcal{A} -type multiplets

Recall from table 3 that the \mathcal{A} -type multiplets obey four kinds of shortening conditions. These result in the removal of the following supercharges from the basis of Verma module generators in (5.1)

$$\begin{aligned}\mathcal{A}[c_1, c_2, c_3; d_1, d_2] &: \mathcal{Q}_{14}, \\ \mathcal{A}[c_1, c_2, 0; d_1, d_2] &: \mathcal{Q}_{13}\mathcal{Q}_{14}, \\ \mathcal{A}[c_1, 0, 0; d_1, d_2] &: \mathcal{Q}_{12}\mathcal{Q}_{13}\mathcal{Q}_{14}, \\ \mathcal{A}[0, 0, 0; d_1, d_2] &: \mathcal{Q}_{11}\mathcal{Q}_{12}\mathcal{Q}_{13}\mathcal{Q}_{14}.\end{aligned}\tag{5.11}$$

The \mathcal{A} -type multiplets cannot contain operator constraints for any d_1 or d_2 .

One can arrive at new shortening conditions — and as a result a reduced number of \mathcal{Q} -generators — for $d_1 \neq 0$, $d_2 = 0$ and $d_1 = d_2 = 0$. Consider for instance the case $\mathcal{A}[c_1, c_2, c_3; d_1, 0]$. From table 3 the null state reads $A^4\Psi = 0$. In the auxiliary Verma-module basis, this corresponds to $\mathcal{Q}_{14}\Psi_{\text{aux}} = 0$. However, since $d_2 = 0$ one also finds that $\mathcal{R}_2^- \mathcal{Q}_{14}\Psi_{\text{aux}} = 0$. This corresponds to having to additionally remove \mathcal{Q}_{24} from our auxiliary Verma-module basis. Following on from this, when $d_1 = 0$ the operator \mathcal{R}_1^- also annihilates the superconformal primary and two new conditions are obtained: $\mathcal{R}_1^- \mathcal{R}_2^- \mathcal{Q}_{14}\Psi_{\text{aux}} = 0$ and $\mathcal{R}_2^- \mathcal{R}_1^- \mathcal{R}_2^- \mathcal{Q}_{14}\Psi_{\text{aux}} = 0$. These correspond to also removing \mathcal{Q}_{34} and \mathcal{Q}_{44} respectively from the basis of Verma-module generators.

All \mathcal{A} -type multiplets have zero contribution to the Schur limit of the 6D (2,0) index. We provide their spectra in appendix E.1.

5.6 \mathcal{B} -type multiplets

For the \mathcal{B} -type multiplets, the supercharges that need to be removed from the auxiliary Verma-module basis due to null states (5.1) are

$$\begin{aligned}\mathcal{B}[c_1, c_2, 0; d_1, d_2] &: \mathcal{Q}_{13}, \\ \mathcal{B}[c_1, 0, 0; d_1, d_2] &: \mathcal{Q}_{12}\mathcal{Q}_{13}, \\ \mathcal{B}[0, 0, 0; d_1, d_2] &: \mathcal{Q}_{11}\mathcal{Q}_{12}\mathcal{Q}_{13}.\end{aligned}\tag{5.12}$$

For the first type of multiplet, $\mathcal{B}[c_1, c_2, 0; d_1, d_2]$, one should also remove \mathcal{Q}_{14} . A similar argument can be made regarding $\mathcal{B}[c_1, 0, 0; d_1, d_2]$ to reach the conclusion that $\mathcal{Q}_{12}\mathcal{Q}_{13}$, $\mathcal{Q}_{12}\mathcal{Q}_{14}$ and $\mathcal{Q}_{13}\mathcal{Q}_{14}$ should be removed from the auxiliary Verma-module basis. For $\mathcal{B}[0, 0, 0; d_1, d_2]$ these additional supercharges are $\mathcal{Q}_{1a}\mathcal{Q}_{1b}\mathcal{Q}_{1c}$ with $a \neq b \neq c$, exactly as in section 4.6.

There are three distinct sub-cases that need to be considered when dialling d_1, d_2 . These are:

1. $\mathcal{B}[c_1, c_2, 0; d_1, 1]$: when $d_2 = 1$ we find that $(\mathcal{R}_2^-)^2 \Psi_{\text{aux}} = 0$. This leads to the combination $\mathcal{Q}_{23}\mathcal{Q}_{24}$ being additionally removed from the basis of auxiliary Verma-module generators [14].
2. $\mathcal{B}[c_1, c_2, 0; d_1, 0]$: having $d_2 = 0$ implies that $\mathcal{R}_2^- \Psi_{\text{aux}} = 0$. This means that $\mathcal{R}_2^- \mathcal{Q}_{13} \Psi_{\text{aux}} = 0 = \mathcal{R}_2^- \mathcal{Q}_{14} \Psi_{\text{aux}}$ and we consequently we remove \mathcal{Q}_{23} and \mathcal{Q}_{24} from the basis of auxiliary Verma-module generators.
3. $\mathcal{B}[c_1, c_2, 0; 0, 0]$: when $d_1 = d_2 = 0$ all R -symmetry lowering operators annihilate the primary. Hence we can apply the above logic while including the lowering operator \mathcal{R}_1^- . One finds that the supercharges $\mathcal{Q}_{33}, \mathcal{Q}_{43}, \mathcal{Q}_{34}, \mathcal{Q}_{44}$ should also be removed from the basis.

According to (5.2) the multiplets $\mathcal{B}[c_1, c_2, 0; 0, 0]$ should contain operator constraints. However, the removal of \mathcal{Q}_{A3} and \mathcal{Q}_{A4} from the basis of generators implicitly also removes

$$\mathcal{P}_{34} = \{\mathcal{Q}_{13}, \mathcal{Q}_{44}\} = \{\mathcal{Q}_{23}, \mathcal{Q}_{34}\} \propto \mathcal{P}_6. \tag{5.13}$$

This corresponds to projecting out all states associated with a conservation equation from the UIR and the resulting module will not include negative-multiplicity representations. The operator constraints can be restored using the dictionary of appendix C.

The spectra of the Q - and \tilde{Q} -actions for \mathcal{B} -type multiplets can be found in appendix E.2. Their contribution to the Schur limit of the 6D (2,0) superconformal index is vanishing, with the exception of $\mathcal{B}[c_1, c_2, 0; d_1, 0]$, for which

$$\mathcal{I}_{\mathcal{B}[c_1, c_2, 0; d_1, 0]}^{\text{Schur}}(q, s) = (-1)^{c_1} \frac{q^{4+d_1+\frac{c_1}{2}+c_2}}{1-q} \chi_{c_1+1}(s). \tag{5.14}$$

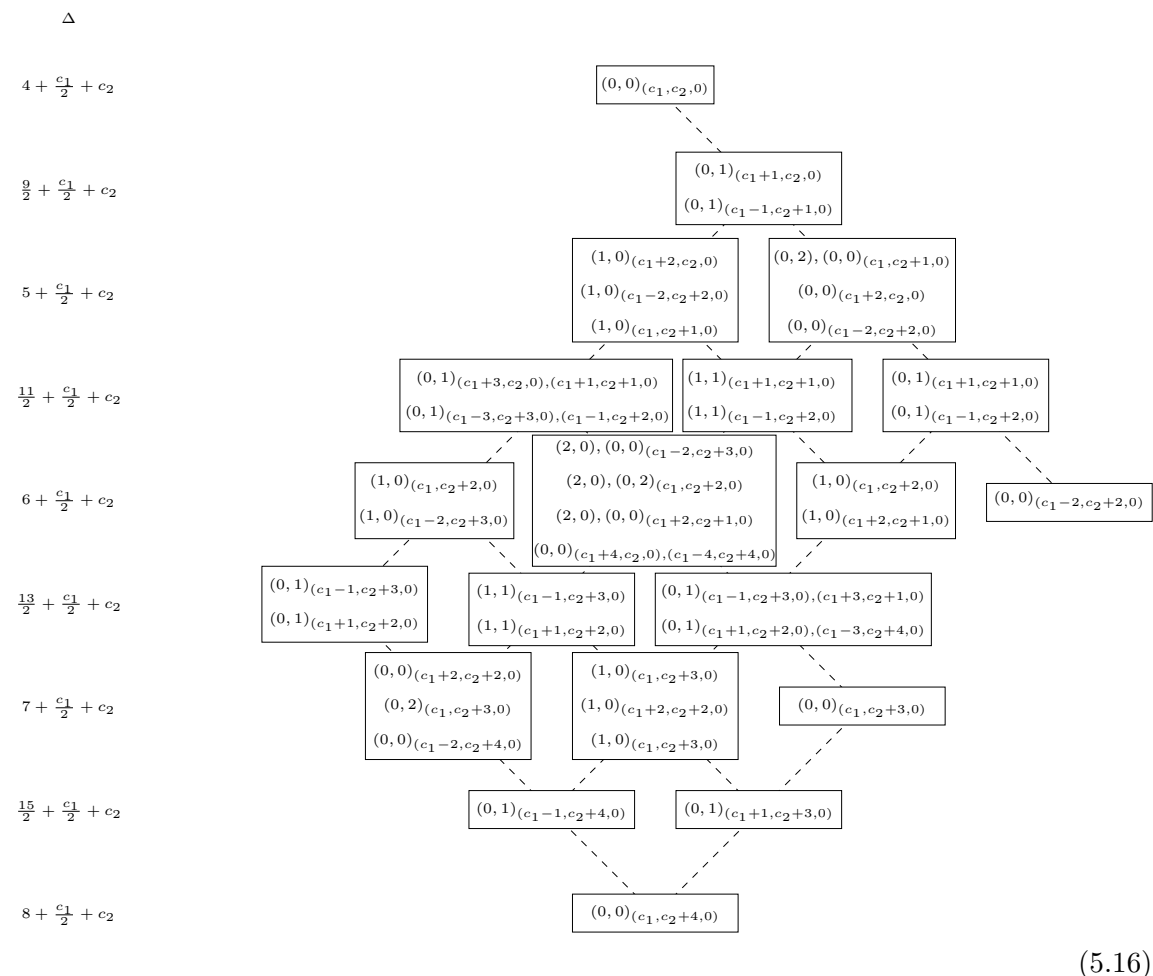
5.6.1 Higher-spin-current multiplets: $\mathcal{B}[c_1, c_2, 0; 0, 0]$

Recall that we are only constructing the auxiliary Verma module with the supercharges \mathcal{Q}_{A1} and \mathcal{Q}_{A2} . Their action on a superconformal primary with generic Dynkin labels $(d_1, d_2)_{(c_1, c_2, c_3)}$ are given by

$$\begin{aligned}
 (d_1, d_2)_{(c_1, c_2, c_3)} &\xrightarrow{\mathcal{Q}} (d_1 - 1, d_2 + 1), (d_1 + 1, d_2 - 1)_{(c_1+1, c_2, c_3), (c_1-1, c_2+1, c_3)}, \\
 &\xrightarrow{\mathcal{Q}^2} (d_1, d_2)_{(c_1+2, c_2, c_3), (c_1-2, c_2+2, c_3), (c_1, c_2+1, c_3)^2}, (d_1 - 2, d_2 + 2)_{(c_1, c_2+1, c_3)}, \\
 &\quad (d_1 + 2, d_2 - 2)_{(c_1, c_2+1, c_3)}, \\
 &\xrightarrow{\mathcal{Q}^3} (d_1 + 1, d_2 - 1), (d_1 - 1, d_2 + 1)_{(c_1+1, c_2+1, c_3), (c_1-1, c_2+2, c_3)}, \\
 &\xrightarrow{\mathcal{Q}^4} (d_1, d_2)_{(c_1, c_2+2, c_3)}, \\
 (d_1, d_2)_{(c_1, c_2, c_3)} &\xrightarrow{\tilde{\mathcal{Q}}} (d_1, d_2 + 1), (d_1, d_2 - 1)_{(c_1+1, c_2, c_3), (c_1-1, c_2+1, c_3)}, \\
 &\xrightarrow{\tilde{\mathcal{Q}}^2} (d_1, d_2)_{(c_1+2, c_2, c_3), (c_1-2, c_2+2, c_3), (c_1, c_2+1, c_3)^2}, (d_1, d_2 + 2)_{(c_1, c_2+1, c_3)}, \\
 &\quad (d_1, d_2 - 2)_{(c_1, c_2+1, c_3)}, \\
 &\xrightarrow{\tilde{\mathcal{Q}}^3} (d_1, d_2 - 1), (d_1, d_2 + 1)_{(c_1+1, c_2+1, c_3), (c_1-1, c_2+2, c_3)}, \\
 &\xrightarrow{\tilde{\mathcal{Q}}^4} (d_1, d_2)_{(c_1, c_2+2, c_3)}. \tag{5.15}
 \end{aligned}$$

The full representation is then built from these chains of supercharges. Clearly since $d_1 = d_2 = 0$ we will only be using the $\mathfrak{so}(5)$ Weyl reflections in the implementation of the RS algorithm. Denoting the action of $Q = (\mathcal{Q}_{2a}, \mathcal{Q}_{3a})$ as moving *southwest* on the diagram

and $\tilde{Q} = (Q_{1a}, Q_{4a})$ as moving *southeast*, we can represent this multiplet as:



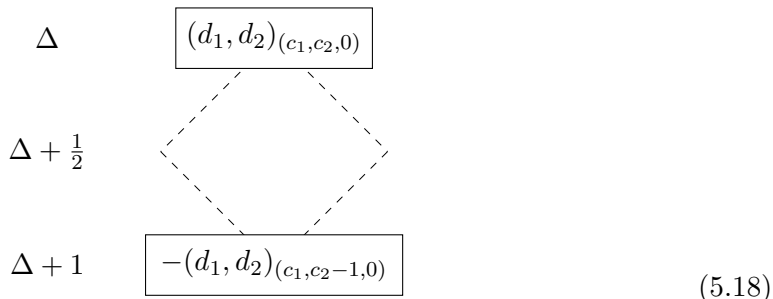
(5.16)

This is the reduced spectrum because, as predicted from the discussion around (5.13), there are no negative-multiplicity states. In order to restore them we use the character relation

$$\hat{\chi}[\Delta; c_1, c_2, c_3; d_1, d_2] = \chi[\Delta; c_1, c_2, c_3; d_1, d_2] - \chi[\Delta + 1; c_1, c_2 - 1, c_3; d_1, d_2], \quad (5.17)$$

where the hat denotes a character over the reduced (i.e. \mathcal{P}_6 -removed) Verma module.

Since there are many states in the reduced spectrum, reconstructing the full multiplet with the operator constraints included would be rather unwieldy. We will however note that states with $c_2 \neq 0$ will pair up with their conservation equation according to



(5.18)

This leads to the observation that, for arbitrary values of c_1 and c_2 , we have an infinite family of conserved currents which have higher spin.

A subset of these conserved higher-spin currents are the ones belonging to the multiplet $\mathcal{B}[0, c_2, 0; 0, 0]$. This multiplet has a superconformal primary in the rank- c_2 symmetric traceless representation of $\mathfrak{su}(4)$, which corresponds to the higher-spin currents that one expects to find in the free 6D (2, 0) theory.

For example, we may take

$$\mathcal{O}_{\mu_1 \dots \mu_{c_2}} = \sum_{\mathbf{I}} \Phi^{\mathbf{I}} \overset{\leftrightarrow}{\partial}_{\mu_1} \dots \overset{\leftrightarrow}{\partial}_{\mu_{c_2}} \Phi^{\mathbf{I}}, \tag{5.19}$$

where $\mathbf{I} = 1, \dots, 5$ is an $\mathfrak{so}(5)_R$ vector index, and $\Phi^{\mathbf{I}}$ is a free-tensor primary. Therefore, this object satisfies the conservation equation

$$\partial^\mu \mathcal{O}_{\mu \mu_2 \dots \mu_{c_2-1}} = 0. \tag{5.20}$$

For generic c_1, c_2 , the Schur index for this type of multiplet is given by

$$\mathcal{I}_{\mathcal{B}[c_1, c_2, 0; 0, 0]}^{\text{Schur}}(q, s) = (-1)^{c_1} \frac{q^{\frac{c_1}{2} + c_2 + 4}}{1 - q} \chi_{c_1+1}(s). \tag{5.21}$$

5.7 C-type multiplets

From table 3 the two distinct \mathcal{C} -type multiplets are $\mathcal{C}[c_1, 0, 0; d_1, d_2]$ and $\mathcal{C}[0, 0, 0; d_1, d_2]$. Upon repeating the null-state analysis, one finds that for generic values of d_1, d_2 one is required to remove \mathcal{Q}_{1a} for $a \neq 1$ and $\mathcal{Q}_{1a} \mathcal{Q}_{1b}$ respectively from the basis of auxiliary Verma-module generators.

One also obtains additional absent auxiliary Verma-module generators for certain values of d_1 and d_2 . The procedure for identifying these is the same as the one presented in section 5.6, so we simply summarise the additional set of \mathcal{Q} s that are to be removed:

$$\begin{aligned} \mathcal{C}[c_1, 0, 0; d_1, 2] &: \mathcal{Q}_{22} \mathcal{Q}_{23} \mathcal{Q}_{24}, \\ \mathcal{C}[c_1, 0, 0; d_1, 1] &: \mathcal{Q}_{2a} \mathcal{Q}_{2b} \quad \text{for } a \neq b \neq 1, \\ \mathcal{C}[c_1, 0, 0; d_1, 0] &: \mathcal{Q}_{2a} \quad \text{for } a \neq 1, \\ \mathcal{C}[c_1, 0, 0; 0, 0] &: \mathcal{Q}_{Aa} \quad \text{for } a \neq 1. \end{aligned} \tag{5.22}$$

We provide the spectra for the above in appendix E.3.

There are three combinations of d_1, d_2 for which the multiplet contains operator constraints. The first two $\mathcal{C}[c_1, 0, 0; 1, 0]$, $\mathcal{C}[c_1, 0, 0; 0, 1]$ appear in appendix E.3. The case of $\mathcal{C}[c_1, 0, 0; 0, 0]$ will be discussed separately below.

The only non-vanishing Schur indices for the \mathcal{C} -multiplets are

$$\begin{aligned} \mathcal{I}_{\mathcal{C}[c_1, 0, 0; d_1, 1]}^{\text{Schur}}(q, s) &= (-1)^{c_1+1} \frac{q^{\frac{7}{2} + d_1 + \frac{c_1}{2}}}{1 - q} \chi_{c_1+1}(s), \\ \mathcal{I}_{\mathcal{C}[c_1, 0, 0; d_1, 0]}^{\text{Schur}}(q, s) &= (-1)^{c_1} \frac{q^{2 + d_1 + \frac{c_1}{2}}}{1 - q} \chi_{c_1+2}(s). \end{aligned} \tag{5.23}$$

5.7.1 $\mathcal{C}[c_1, 0, 0; 0, 0]$

Recall from (5.22) that the associated Verma module is constructed by the action of \mathcal{Q}_{A1} . Since we are removing all other supercharges, we also have to remove several momenta from the basis of generators of the Verma module. These are

$$\begin{aligned} \mathcal{P}_{23} &= \{Q_{12}, Q_{43}\} = \{Q_{22}, Q_{33}\} \propto \mathcal{P}_3, \\ \mathcal{P}_{24} &= \{Q_{12}, Q_{44}\} = \{Q_{22}, Q_{34}\} \propto \mathcal{P}_5, \\ \mathcal{P}_{34} &= \{Q_{13}, Q_{44}\} = \{Q_{23}, Q_{34}\} \propto \mathcal{P}_6, \end{aligned} \tag{5.24}$$

where in the last column we have converted to Lorentz vector indices. Thus for generic Dynkin labels the actions of the supercharges lead to

$$\begin{aligned} (d_1, d_2)_{(c_1, c_2, c_3)} &\xrightarrow{Q} (d_1 - 1, d_2 + 1), (d_1 + 1, d_2 - 1)_{(c_1+1, c_2, c_3)}, \\ &\xrightarrow{Q^2} (d_1, d_2)_{(c_1+2, c_2, c_3)}, \\ (d_1, d_2)_{(c_1, c_2, c_3)} &\xrightarrow{\tilde{Q}} (d_1, d_2 + 1), (d_1, d_2 - 1)_{(c_1+1, c_2, c_3)}, \\ &\xrightarrow{\tilde{Q}^2} (d_1, d_2)_{(c_1+2, c_2, c_3)}. \end{aligned} \tag{5.25}$$

Upon replacing the actual values of the Dynkin labels and running the RS algorithm, the physical spectrum is given by

Δ	
$2 + \frac{c_1}{2}$	$(0, 0)_{(c_1, 0, 0)}$
$\frac{5}{2} + \frac{c_1}{2}$	$(0, 1)_{(c_1+1, 0, 0)}$
$3 + \frac{c_1}{2}$	$(1, 0)_{(c_1+2, 0, 0)}$ $(0, 0)_{(c_1+2, 0, 0)}$
$\frac{7}{2} + \frac{c_1}{2}$	$(0, 1)_{(c_1+3, 0, 0)}$
$4 + \frac{c_1}{2}$	$(0, 0)_{(c_1+4, 0, 0)}$

(5.26)

As in the $\mathcal{B}[c_1, c_2, 0; 0, 0]$ case, we can restore the negative-multiplicity states by making use of the character relation

$$\begin{aligned} \hat{\chi}[\Delta; c_1, 0, 0; d_1, d_2] &= \chi[\Delta; c_1, 0, 0; d_1, d_2] - \chi[\Delta + 1; c_1 - 1, 0, 1; d_1, d_2] \\ &\quad + \chi[\Delta + 2; c_1 - 2, 1, 0; d_1, d_2] - \chi[\Delta + 3; c_1 - 2, 0, 0; d_1, d_2], \end{aligned} \tag{5.27}$$

where a hat denotes the \mathcal{P} -reduced character. This will lead to a set of equations of motion.

The index over this multiplet in the Schur limit is

$$\mathcal{I}_{\mathcal{C}[c_1,0,0;0,0]}^{\text{Schur}}(q, s) = (-1)^{c_1} \frac{q^{\frac{c_1}{2}+2}}{1-q} \chi_{c_1+2}(s). \quad (5.28)$$

5.8 \mathcal{D} -type multiplets

These multiplets, summarised in table 3, are the smallest of the $\mathfrak{osp}(8^*|4)$ algebra. The associated null state is $A^1\Psi = 0$, which implies that $\mathcal{Q}_{1a}\Psi_{\text{aux}} = 0$ and the multiplet is thus $\frac{1}{4}$ -BPS.

For $d_2 \leq 3$, one also needs to remove the following combinations of supercharges from the basis of auxiliary Verma-module generators:

$$\begin{aligned} \mathcal{D}[0, 0, 0; d_1, 3] &: \mathcal{Q}_{21} \mathcal{Q}_{22} \mathcal{Q}_{23} \mathcal{Q}_{24}, \\ \mathcal{D}[0, 0, 0; d_1, 2] &: \mathcal{Q}_{2a} \mathcal{Q}_{2b} \mathcal{Q}_{2c} \\ \mathcal{D}[0, 0, 0; d_1, 1] &: \mathcal{Q}_{2a} \mathcal{Q}_{2b}, \\ \mathcal{D}[0, 0, 0; d_1, 0] &: \mathcal{Q}_{2a}. \end{aligned} \quad (5.29)$$

When $d_1 = d_2 = 0$ this corresponds to the vacuum, because all supercharges annihilate the superconformal primary.

For generic d_1, d_2 we have the Q -chain

$$\begin{aligned} (d_1, d_2)_{(c_1, c_2, c_3)} &\xrightarrow{Q} (d_1, d_2 - 1)_{(c_1+1, c_2, c_3), (c_1-1, c_2+1, c_3), (c_1, c_2-1, c_3+1), (c_1, c_2, c_3-1)}, \\ &\xrightarrow{Q^2} (d_1, d_2 - 2)_{(c_1, c_2+1, c_3), (c_1+1, c_2-1, c_3+1), (c_1+1, c_2, c_3-1), (c_1-1, c_2, c_3+1)}, \\ &\quad (d_1, d_2 - 2)_{(c_1-1, c_2+1, c_3-1), (c_1, c_2-1, c_3)}, \\ &\xrightarrow{Q^3} (d_1, d_2 - 3)_{(c_1, c_2, c_3+1), (c_1, c_2+1, c_3-1), (c_1+1, c_2-1, c_3), (c_1-1, c_2, c_3)}, \\ &\xrightarrow{Q^4} (d_1, d_2 - 4)_{(c_1, c_2, c_3)}. \end{aligned} \quad (5.30)$$

The action of the \tilde{Q} supercharges will be dealt with on a case-by-case basis as we dial d_2 , and provided in appendix E.4. The only exception is the case $d_2 = 0$, which will be detailed below.

The non-zero contributions to the Schur limit of the index are given by

$$\begin{aligned} \mathcal{I}_{\mathcal{D}[0,0,0;d_1,2]}^{\text{Schur}}(q, s) &= \frac{q^{3+d_1}}{1-q} && \text{for } d_1 \geq 0, \\ \mathcal{I}_{\mathcal{D}[0,0,0;d_1,1]}^{\text{Schur}}(q, s) &= -\frac{q^{\frac{3}{2}+d_1}}{1-q} \chi_1(s) && \text{for } d_1 \geq 0, \\ \mathcal{I}_{\mathcal{D}[0,0,0;d_1,0]}^{\text{Schur}}(q, s) &= \frac{q^{d_1}}{1-q} && \text{for } d_1 > 0. \end{aligned} \quad (5.31)$$

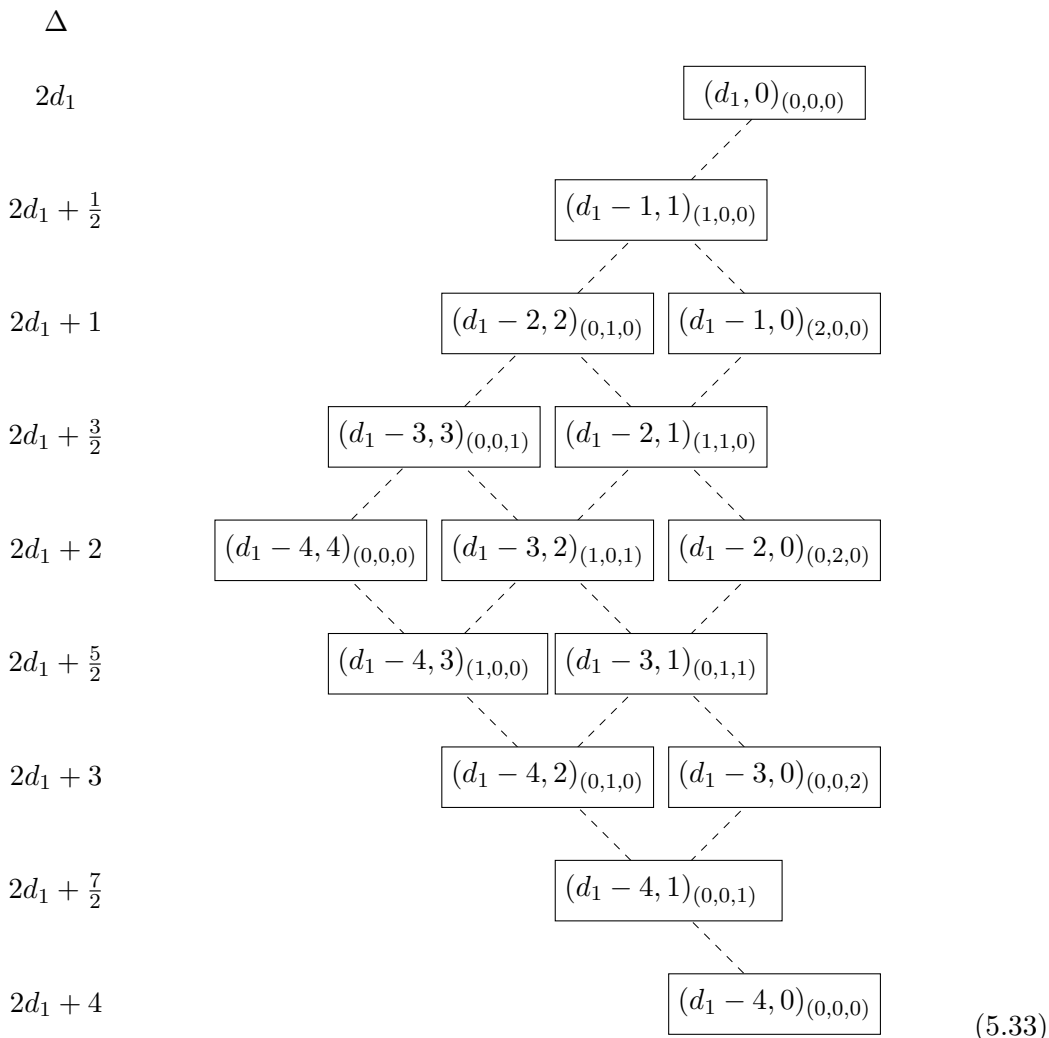
The \mathcal{D} -type multiplets contain negative-multiplicity states for $d_1 + d_2 \leq 2$. These include the free-tensor and stress-tensor multiplet.

5.8.1 Half-BPS multiplets: $\mathcal{D}[0, 0, 0; d_1, 0]$

Recall that in this case we are prescribed to remove \mathcal{Q}_{2a} from the basis of auxiliary Verma-module generators. This is because $A^1\Psi = \mathcal{Q}_{11}\Psi = 0$ and using R -symmetry lowering operators we find that all $\mathcal{Q}_{1a}, \mathcal{Q}_{2a}$ annihilate the primary. As a result, the multiplet is $\frac{1}{2}$ -BPS. The set of \tilde{Q} s consist entirely of \mathcal{Q}_{3a} supercharges. Acting on a generic superconformal primary state $(d_1, d_2)_{(c_1, c_2, c_3)}$ with the \tilde{Q} s yields

$$\begin{aligned}
 (d_1, d_2)_{(c_1, c_2, c_3)} &\xrightarrow{\tilde{Q}} (d_1 - 1, d_2 + 1)_{(c_1+1, c_2, c_3), (c_1-1, c_2+1, c_3), (c_1, c_2-1, c_3+1), (c_1, c_2, c_3-1)}, \\
 &\xrightarrow{\tilde{Q}^2} (d_1 - 2, d_2 + 2)_{(c_1, c_2+1, c_3), (c_1+1, c_2-1, c_3+1), (c_1+1, c_2, c_3-1), (c_1, c_2-1, c_3)}, \\
 &\quad (d_1 - 2, d_2 + 2)_{(c_1-1, c_2+1, c_3-1), (c_1-1, c_2, c_3+1)}, \\
 &\xrightarrow{\tilde{Q}^3} (d_1 - 3, d_2 + 3)_{(c_1, c_2, c_3+1), (c_1, c_2+1, c_3-1), (c_1+1, c_2-1, c_3), (c_1-1, c_2, c_3)}, \\
 &\xrightarrow{\tilde{Q}^4} (d_1 - 4, d_2 + 4)_{(c_1, c_2, c_3)}. \tag{5.32}
 \end{aligned}$$

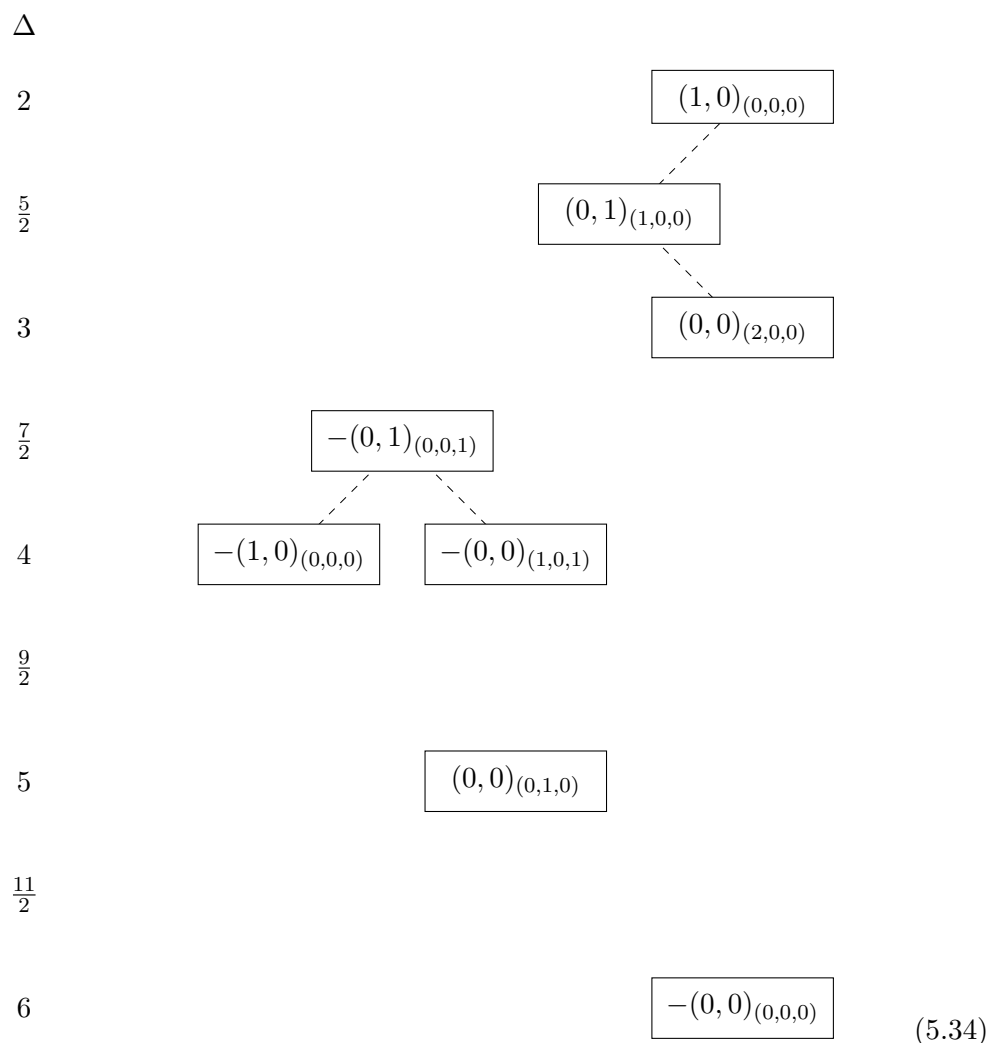
The Verma module is then built out of $\mathcal{Q}_{3a}, \mathcal{Q}_{4a}$ and we obtain



5.8.2 The free-tensor multiplet: $\mathcal{D}[0, 0, 0; 1, 0]$

This spectrum can be obtained by substituting $d_1 = 1$ into that of $\mathcal{D}[0, 0, 0; d_1, 0]$ and running the RS algorithm. The resulting states match a scalar $\Phi^{\mathbf{I}}$, $\mathbf{I} = 1, \dots, 5$, a fermion $\lambda_a^{\mathbf{A}}$ and a selfdual tensor $H_{[\mu\nu\rho]}^+$. The negative-multiplicity representations are naturally matched with the equations of motion $\partial^2 \Phi^{\mathbf{I}} = 0$, $\not{\partial}^{\dot{a}b} \tilde{\lambda}_b^{\mathbf{A}} = 0$ and the Bianchi identity for the selfdual tensor $\partial_{[\mu} H_{\nu\rho\sigma]}^+ = 0$.

The full multiplet is given by

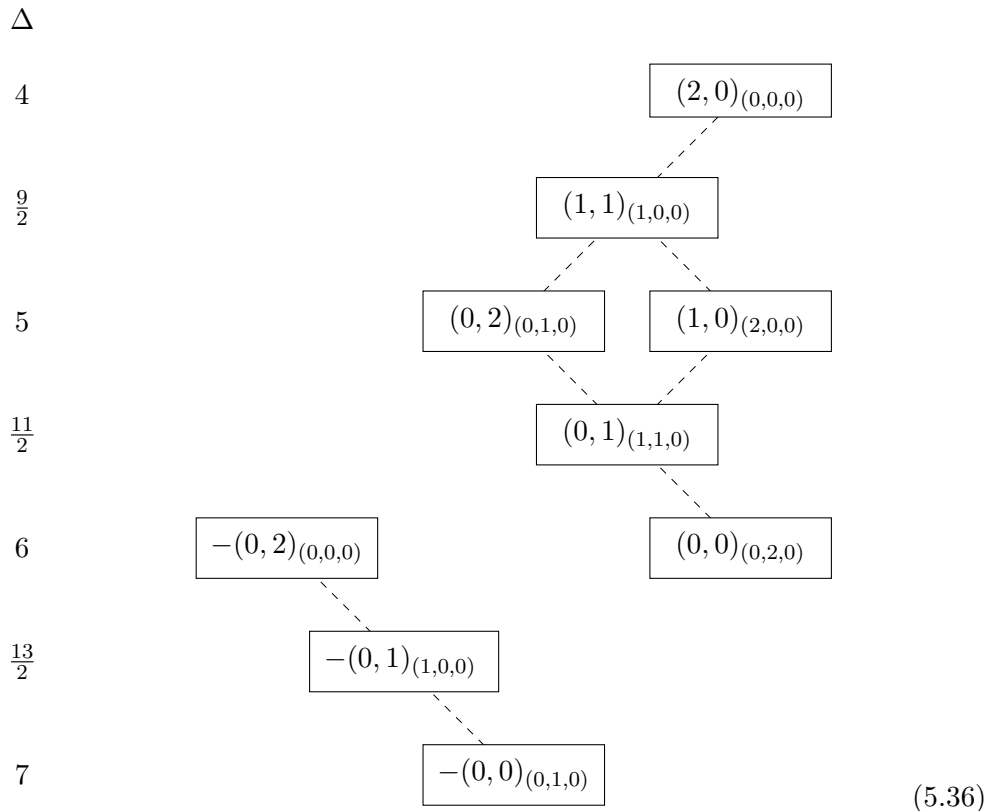


The refined index over this multiplet is compact enough to be presented in full

$$\mathcal{I}_{\mathcal{D}[0,0,0;1,0]}(p, q, s, t) = \frac{qt^{-1} + q^2 p^2 t - q^2 p(s + p + s^{-1}) + q^3 p^2}{(1 - q)(1 - qps)(1 - qps^{-1})}. \quad (5.35)$$

5.8.3 The stress-tensor multiplet: $\mathcal{D}[0, 0, 0; 2, 0]$

This spectrum can be obtained by substituting $d_1 = 2$ into that of $\mathcal{D}[0, 0, 0; d_1, 0]$ and running the RS algorithm. The resulting representation is



The superconformal primary is now the diboson

$$\mathcal{O}^{\mathbf{IJ}} := :\Phi^{\mathbf{I}}\Phi^{\mathbf{J}}: . \quad (5.37)$$

We can match the remaining states with the following currents and their conservation equations

$$\begin{aligned} [5; 0, 1, 0; 0, 2] : J_{\mu}^{(\mathbf{AB})} , & \quad -[6; 0, 0, 0; 0, 2] : \partial^{\mu} J_{\mu}^{(\mathbf{AB})} = 0 , \\ [11/2; 1, 1, 0; 0, 1] : S_{\mu a}^{\mathbf{A}} , & \quad -[13/2; 1, 0, 0; 0, 1] : \partial^{\mu} S_{\mu a}^{\mathbf{A}} = 0 , \\ [6; 0, 2, 0; 0, 0] : \Theta_{\mu\nu} , & \quad -[7; 0, 1, 0; 0, 0] : \partial^{\mu} \Theta_{\mu\nu} = 0 , \end{aligned} \quad (5.38)$$

namely the R -symmetry current, supersymmetry current and stress tensor respectively. We also have three states with no associated conservation equations. These are

$$\begin{aligned} [4; 0, 0, 0; 2, 0] : \Phi^{\mathbf{I}}\Phi^{\mathbf{J}} , \\ [9/2; 1, 0, 0; 1, 1] : \Phi^{\mathbf{I}}\lambda_a^{\mathbf{A}} , \\ [5; 2, 0, 0; 1, 0] : H_{[\mu\nu\rho]}^{+}\Phi^{\mathbf{I}} . \end{aligned} \quad (5.39)$$

The refined index for this multiplet is calculated to be

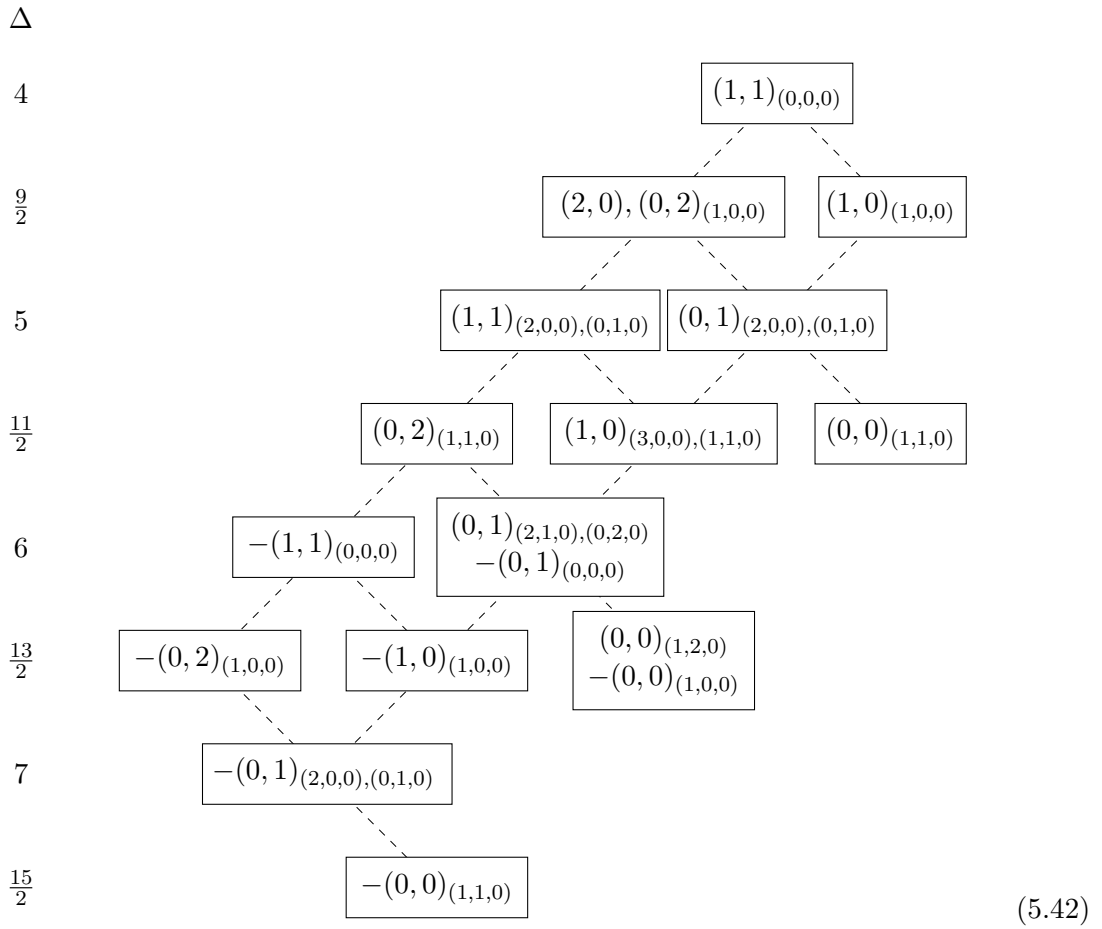
$$\begin{aligned} \mathcal{I}_{\mathcal{D}[0,0,0;2,0]}(p, q, s, t) = & -\frac{pq^3(st^{-1} + s^{-1}t^{-1} + pt^{-1} + p^3qt + p^2qst + p^2qs^{-1}t)}{(1-q)(1-pps)(1-pps^{-1})} \\ & + \frac{q^2t^{-2} + p^2q^3(1 + p^2qt^2) + p^2q^4(1 + ps + ps^{-1})}{(1-q)(1-pps)(1-pps^{-1})}. \end{aligned} \quad (5.40)$$

5.8.4 $\mathcal{D}[0, 0, 0; 1, 1]$

For this multiplet we are prescribed to remove the combinations $\mathcal{Q}_{2a}\mathcal{Q}_{2b}$ from the basis of \mathcal{Q} -generators of the Verma module. Acting on the superconformal primary with the remaining set of \tilde{Q} supercharges gives rise to

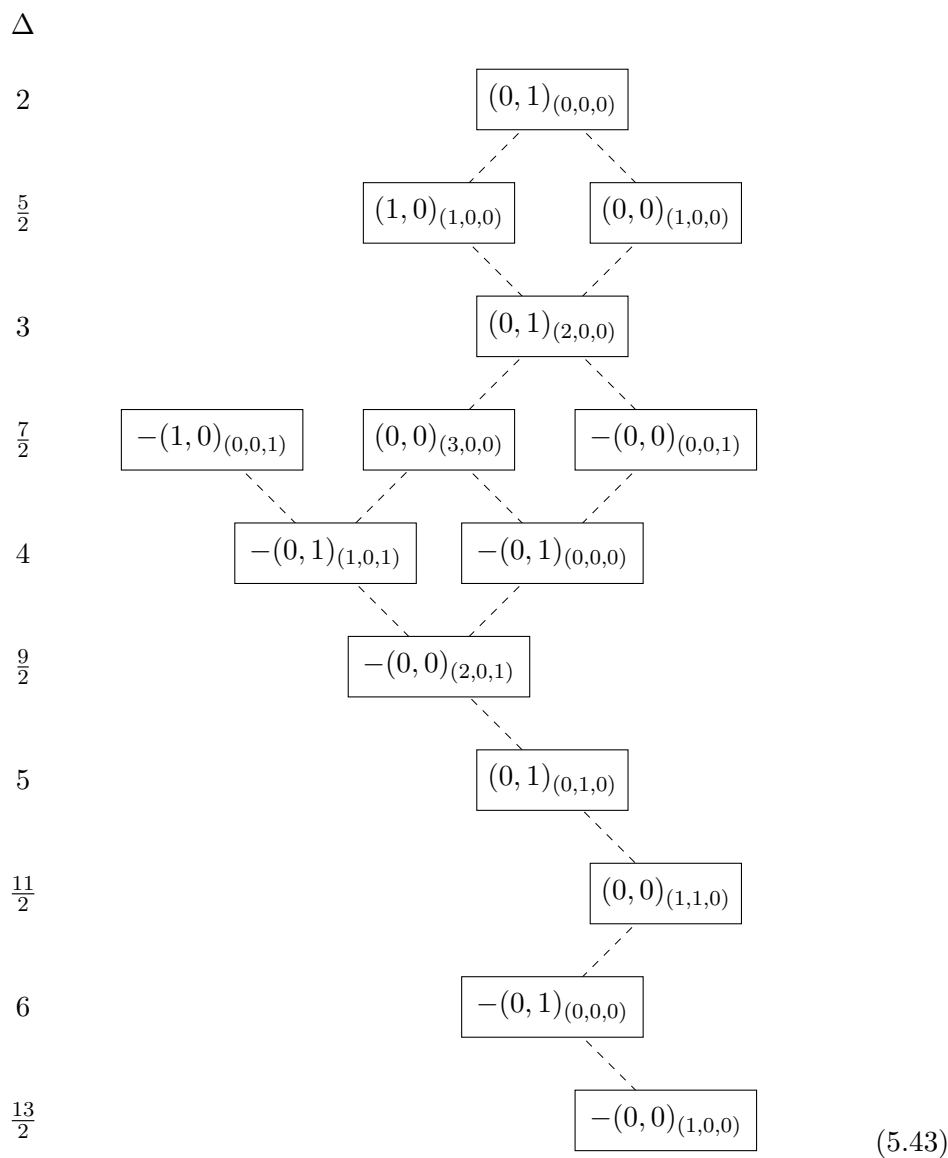
$$\begin{aligned} & (d_1, d_2)_{(c_1, c_2, c_3)} \\ & \xrightarrow{\tilde{Q}} (d_1 - 1, d_2 + 1), (d_1 + 1, d_2 - 1)_{(c_1+1, c_2, c_3), (c_1-1, c_2+1, c_3), (c_1, c_2-1, c_3+1), (c_1, c_2, c_3-1)}, \\ & \xrightarrow{\tilde{Q}^2} (d_1 - 2, d_2 + 2), (d_1, d_2)_{(c_1, c_2+1, c_3), (c_1+1, c_2-1, c_3+1), (c_1+1, c_2, c_3-1), (c_1-1, c_2, c_3+1)}, \\ & \quad (d_1 - 2, d_2 + 2), (d_1, d_2)_{(c_1-1, c_2+1, c_3-1), (c_1, c_2-1, c_3)}, \\ & \quad (d_1, d_2)_{(c_1+2, c_2, c_3), (c_1-2, c_2+2, c_3), (c_1, c_2-2, c_3+2), (c_1, c_2, c_3-2)}, \\ & \xrightarrow{\tilde{Q}^3} (d_1 - 3, d_2 + 3), (d_1 - 1, d_2 + 1)_{(c_1, c_2, c_3+1), (c_1, c_2+1, c_3-1), (c_1+1, c_2-1, c_3), (c_1-1, c_2, c_3)}, \\ & \quad (d_1 - 1, d_2 + 1)_{(c_1+1, c_2+1, c_3), (c_1-1, c_2+2, c_3), (c_1+2, c_2-1, c_3+1), (c_1-2, c_2+1, c_3+1)}, \\ & \quad (d_1 - 1, d_2 + 1)_{(c_1+2, c_2, c_3-1), (c_1+1, c_2, c_3-2), (c_1-2, c_2+2, c_3-1), (c_1-1, c_2+1, c_3-2)}, \\ & \quad (d_1 - 1, d_2 + 1)_{(c_1, c_2-1, c_3-1), (c_1-1, c_2-1, c_3+2), (c_1, c_2-2, c_3+1)}, \\ & \xrightarrow{\tilde{Q}^4} (d_1 - 4, d_2 + 4), (d_1 - 2, d_2 + 2)_{(c_1, c_2, c_3)}, \\ & \quad (d_1 - 2, d_2 + 2)_{(c_1+1, c_2, c_3+1), (c_1-1, c_2+1, c_3+1), (c_1, c_2-1, c_3+2), (c_1+1, c_2+1, c_3-1)}, \\ & \quad (d_1 - 2, d_2 + 2)_{(c_1, c_2+1, c_3-2), (c_1+2, c_2-1, c_3), (c_1, c_2-1, c_3+2), (c_1+1, c_2-1, c_3-1)}, \\ & \quad (d_1 - 2, d_2 + 2)_{(c_1-1, c_2-1, c_3+1), (c_1-1, c_2, c_3-1), (c_1-1, c_2+2, c_3-1), (c_1-2, c_2+1, c_3)}, \\ & \xrightarrow{\tilde{Q}^5} (d_1 - 3, d_2 + 3)_{(c_1+1, c_2, c_3), (c_1-1, c_2+1, c_3), (c_1, c_2-1, c_3+1), (c_1, c_2, c_3-1)}. \end{aligned} \quad (5.41)$$

The action of the Q supercharges is still the one presented in (5.30). Combining the two, the full module is given by



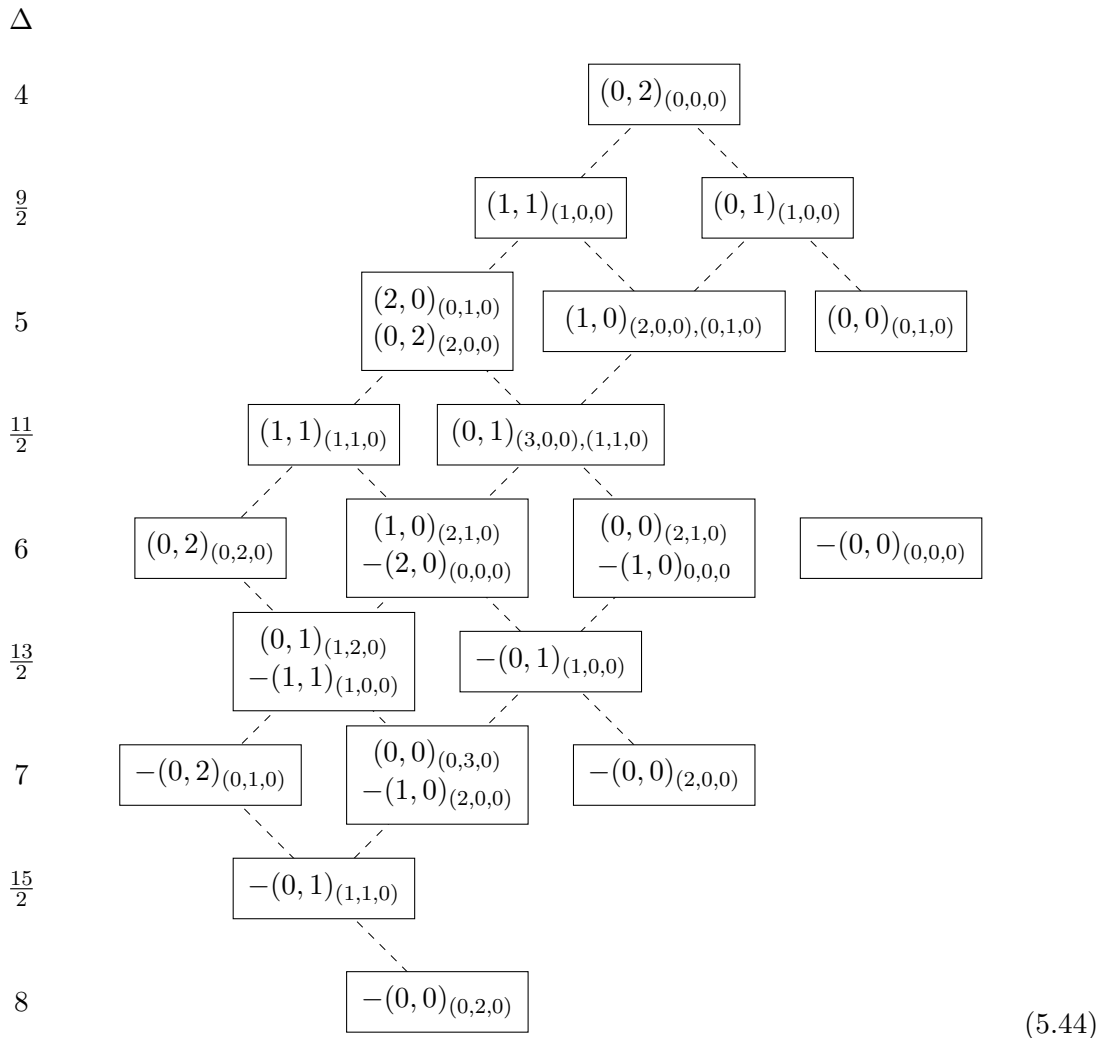
5.8.5 $\mathcal{D}[0, 0, 0; 0, 1]$

The shortening conditions for this multiplet are similar to the $\mathcal{D}[0, 0, 0; 1, 1]$ case; they follow from (5.41) by setting $d_1 = 0$. The full spectrum of states, including those corresponding to operator constraints, is given by the diagram:



5.8.6 $\mathcal{D}[0, 0, 0; 0, 2]$

The Verma module is obtained through the set of Q -actions of (5.30) along with the set of \tilde{Q} -actions of table 16; the latter can be found in appendix E.4. Using these one has



This concludes our discussion of 6D (2,0) superconformal multiplets.

6 Some initial applications

We finally mention some brief applications of the results that we have presented thus far, while leaving a more in-depth exploration for future work. We study aspects of flavour symmetries and flavour anomalies. We also rule out the presence of certain multiplets in free theories and provide evidence that certain 6D (1,0) multiplets can consistently combine to form 6D (2,0) multiplets.

6.1 Flavour symmetries

Both the 5D $\mathcal{N} = 1$ and 6D $(1,0)$ algebras admit flavour symmetries.²⁴ Therefore it is interesting to ask which representations are allowed to transform nontrivially under these symmetries.

For example, in 4D $\mathcal{N} = 2$ theories, it is known that $\mathcal{N} = 2$ chiral operators do not transform under flavour symmetries [59, 60]. Since scalar $\mathcal{N} = 2$ chiral operators parametrise the Coulomb branch (when it exists), this statement is consistent with the fact that these theories do not have any massless matter (besides free vector multiplets) at generic points on the Coulomb branch. Below, we will explore similar physical constraints in 5D and 6D.

Let us begin by considering 6D $(1,0)$ theories. Many of these theories have tensor branches. Along these moduli spaces, the $\mathfrak{sp}(1)_R$ and flavour symmetries are unbroken, and free-tensor multiplets play an important role. It is therefore reasonable to assume that there are flavour and $\mathfrak{sp}(1)_R$ -neutral superconformal primaries in short multiplets that (partially) describe the physics on this branch of the moduli space.

To understand the last point in more detail, let us study short multiplets with an $\mathfrak{sp}(1)_R$ -neutral primary, \mathcal{O}_I . Let us further suppose that \mathcal{O}_I transforms linearly under a representation, $R_{\mathcal{O}}$, of the theory's flavour symmetry group, G . We will study the conditions under which superconformal representation theory forces this representation to be trivial, i.e. $R_{\mathcal{O}} = \mathbf{1}$. One crucial observation for us is that flavour Ward identities require the following leading OPE between the flavour symmetry current, j_{ab}^α , associated with G and \mathcal{O}_I :

$$j_{ab}^\alpha(x)\mathcal{O}_I(0) \sim -it_{\mathcal{O}}^\alpha \frac{J^{xab}}{x^6} \mathcal{O}_I + \dots \tag{6.1}$$

In this equation, the ellipsis contains less singular terms, and $t_{\mathcal{O}}^\alpha$ is the matrix for the representation, $R_{\mathcal{O}}$, that \mathcal{O}_I transforms under. For now we will assume \mathcal{O}_I is a Lorentz scalar, but we will relax this condition later. One important point that will come into play below is that j_{ab}^α in (6.1) is a level-two superconformal descendant of the $\text{Sp}(1)_R$ spin-1 moment map primary

$$j_{ab}^\alpha(x) = \mathcal{Q}_{Aa} \mathcal{Q}_{Bb} (J^{\alpha\mathbf{AB}}(x)) \tag{6.2}$$

To understand why certain operators are forbidden by the superconformal algebra from transforming linearly under flavour symmetry, it will be important to relate (6.1) to an OPE of superconformal primaries. In particular, we will need the fact that

$$\begin{aligned} \mathcal{S}_{\mathbf{A}}^a (J^{\alpha\mathbf{AB}}(x)) &\equiv [\mathcal{S}_{\mathbf{A}}^a, J^{\alpha\mathbf{AB}}(x)] = -ix^{bc} (\delta_b^a \mathcal{Q}_{Ac} - \delta_c^a \mathcal{Q}_{Ab}) (J^{\alpha\mathbf{AB}}(x)) \\ &= -2ix^{ab} \mathcal{Q}_{Ab} (J^{\alpha\mathbf{AB}}(x)) \end{aligned} \tag{6.3}$$

where we have used the superconformal algebra and the form of the translated operator

$$J^{\alpha\mathbf{AB}}(x) = e^{-iP \cdot x} J^{\alpha\mathbf{AB}}(0) e^{iP \cdot x} \tag{6.4}$$

²⁴Here we define such symmetries to be those that commute with the superconformal algebra. See the discussion around (2.32) for an explanation of why flavour symmetries do not sit in multiplets with higher-spin symmetries.

Using (6.3), we can also deduce that

$$\mathcal{S}_{\mathbf{B}}^c \mathcal{S}_{\mathbf{A}}^a (J^{\alpha\mathbf{AB}}(x)) = -2ix^{ab} \mathcal{Q}_{\mathbf{Ab}} \mathcal{S}_{\mathbf{B}}^c (J^{\alpha\mathbf{AB}}(x)) = -4x^{ab} x^{cd} j_{bd}^\alpha(x). \quad (6.5)$$

From this discussion, it follows that we can extract the OPE in (6.1) by acting with the special supercharges on the superconformal primary OPE

$$x^{ab} x^{cd} j_{bd}^\alpha(x) \mathcal{O}_I(0) = \mathcal{S}_{\mathbf{B}}^c \mathcal{S}_{\mathbf{A}}^a (J^{\alpha\mathbf{AB}}(x) \mathcal{O}_I(0)) \sim -it_{\mathcal{O}_I}^\alpha J \frac{x^{ab} x^{cd} x_{bd}}{x^6} \mathcal{O}_J(0) + \dots \quad (6.6)$$

In order to produce the correct leading singularity in (6.6), the superconformal primary OPE must then contain terms of the form

$$J^{\alpha\mathbf{AB}}(x) \mathcal{O}_I(0) \supset -it_{\mathcal{O}_I}^\alpha J \frac{x^{ab}}{x^4} \mathcal{Q}_a^{\mathbf{A}} \mathcal{Q}_b^{\mathbf{B}} \mathcal{O}_J(0) + \dots \quad (6.7)$$

This discussion implies the following results:

- It is straightforward to check that in the $\mathcal{C}[0, 0, 0; 0]$ multiplet there are no operators of the type written on the r.h.s. of (6.7) since there are no $\mathfrak{sp}(1)_R$ spin-1 descendants at level two. Therefore, these multiplets cannot transform linearly under any flavour symmetries in an SCFT. Since adding spin does not allow for such a descendant, we conclude that all the $\mathcal{C}[c_1, 0, 0; 0]$ multiplets cannot transform under nontrivial linear representations of the flavour symmetry.
- Note that the absence of a conformal primary with the quantum numbers of the operator on the r.h.s. of (6.7) is a sufficient but not necessary condition for the multiplet to be flavour neutral. For example, the stress-tensor multiplet contains an $\mathfrak{sp}(1)_R$ current which has the correct quantum numbers. However, in this case, since the stress tensor generates translations, it must be flavour neutral. Still, in free theories, with a collection of hypermultiplets, $\phi^{i\mathbf{A}} \in \mathcal{D}[0, 0, 0; 1]$ (here $i = 1, \dots, N$ indexes the hypermultiplets) we can construct $\mathcal{B}[0, 0, 0; 0]$ multiplets that are not flavour neutral as follows

$$\mathcal{O}^{ij} = \epsilon_{\mathbf{AB}} \phi^{i\mathbf{A}} \phi^{j\mathbf{B}} \in \mathcal{B}[0, 0, 0; 0]. \quad (6.8)$$

- We can also play the same game in 5D, since the R symmetry is the same, and the flavour moment maps are again superconformal primaries of $\mathfrak{su}(2)_R$ spin-1 multiplets containing the flavour currents. It is straightforward to check that the above argument does not rule out flavour transformations for any unitary representations.

Note that the above discussion on the triviality of certain linear representations does not preclude the possibility that certain multiplets transform non-linearly when we turn on background gauge fields for global symmetries. Indeed, the free-tensor fields, $\varphi^I \in \mathcal{C}[0, 0, 0; 0]$, often must transform non-linearly in order to match nontrivial 't Hooft anomalies for RG flows onto the tensor branch of certain interacting (1,0) theories [16, 61, 62] (our discussion below mostly follows [61, 62]). For example, the φ^I multiplets can be used to match a discrepancy in the UV and IR anomalies of the form

$$\Delta I_8 = \frac{1}{2\pi} \Omega_{IJ} X_4^I \wedge X_4^J. \quad (6.9)$$

In this equation, ΔI_8 is the naive change in the anomaly polynomial eight-form between the UV and the IR, Ω_{IJ} a positive-definite symmetric matrix, and the four-form, X_4^I , has a contribution from the flavour symmetry background fields, F_i ,

$$X_4^I \supset \sum_i n^i c_2(F_i), \tag{6.10}$$

where c_2 is a Chern class for the background flavour symmetry. In order to make up for the discrepancy in (6.9) there must be a coupling of the tensor field two-form, B_2^I , to X_4^J in the IR effective action

$$\delta \mathcal{L} \sim \Omega_{IJ} B_2^I \wedge X_4^J, \tag{6.11}$$

where the necessary anomalous variation is generated by requiring that $\delta B_2^I = -\pi X_2^I$, where

$$\delta X_3^I = dX_2^I, \quad X_4^I = dX_3^I. \tag{6.12}$$

In particular, we see that the $\mathcal{C}[0, 0, 0; 0]$ multiplet transforms since

$$\delta H_3 = -\pi \delta X_3^I \neq 0. \tag{6.13}$$

This discussion does not contradict our previous statements because the $\mathcal{C}[0, 0, 0; 0] \ni H_3^I$ representations do not transform linearly under the flavour symmetry (in fact, they do not transform at all if we turn off the background flavour gauge fields).

6.2 Supersymmetry enhancement of 6D (1, 0) multiplets

An important consistency condition for our 6D $\mathcal{N} = 1$ multiplets is that they can combine to form $\mathcal{N} = 2$ multiplets. For instance, one should be able to construct the additional supersymmetry and R -symmetry currents of the $\mathcal{N} = 2$ SCA from the multiplets of section 4.

It will be instructive to first show that two hypermultiplets and a tensor multiplet of the 6D $\mathcal{N} = 1$ SCA combine to form a single tensor multiplet in $\mathcal{N} = 2$. We need only consider the R symmetry, since the Lorentz quantum numbers are the same for both cases. We identify the $\mathfrak{su}(2)_R \subset \mathfrak{so}(5)_R$ by choosing $K = d_1$; note that as a result distinct states for $\mathcal{N} = 2$ can give rise to the same state for $\mathcal{N} = 1$. The free tensor in $\mathcal{N} = 2$ has the R -symmetry highest weight (1, 0), which corresponds to a representation with Dynkin values (1, 0), (-1, 2), (0, 0), (1, -2) and (-1, 0). Reducing this to $\mathfrak{su}(2)_R$ results in the states $2 \times (1)$, (0) and $2 \times (-1)$. Another way of writing this is in terms of three modules, two with highest weights labelled by (1) and one labelled by (0). Thus the superconformal primary of $\mathcal{D}[0, 0, 0; 1, 0]$ can be identified with two primaries from the $\mathcal{N} = 1$ hypermultiplet and one from the $\mathcal{N} = 1$ tensor multiplet. Indeed one can repeat this for every state in $\mathcal{D}[0, 0, 0; 1, 0]$ and the result is that the entire multiplet is written as

$$\mathcal{D}[0, 0, 0; 1, 0] \simeq 2\mathcal{D}[0, 0, 0; 1] \oplus \mathcal{C}[0, 0, 0; 0], \tag{6.14}$$

where the equivalence is up to the identification $d_1 = K$.

One can similarly rewrite the stress-tensor multiplet of $\mathcal{N} = 2$, as a collection of $\mathcal{N} = 1$ multiplets. The result is that

$$\mathcal{D}[0, 0, 0; 2, 0] \simeq 3\mathcal{D}[0, 0, 0; 2] \oplus 2\mathcal{C}[0, 0, 0; 1] \oplus \mathcal{B}[0, 0, 0; 0], \tag{6.15}$$

where the equivalence is again up to the identification $d_1 = K$, confirming the original expectation.²⁵

Acknowledgments

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A The superconformal algebra in 5D and 6D

In this appendix we collect our notation and conventions for the 5D and 6D SCAs. For a fully detailed account of conformal and superconformal algebras in various dimensions we refer the interested reader to [20, 40].

A.1 The conformal algebra in D dimensions

The conformal group in D dimensions is locally isomorphic to $SO(D, 2)$ and generated by $\frac{1}{2}D(D - 1)$ Lorentz generators $M_{\mu\nu}$, D momenta \mathcal{P}_μ and special conformal generators \mathcal{K}_μ , and the conformal Hamiltonian H . The associated Lie algebra is defined by the commutation relations

$$\begin{aligned}
 [M_{\mu\nu}, M_{\kappa\lambda}] &= i(\delta_{\mu\kappa}M_{\nu\lambda} - \delta_{\mu\lambda}M_{\nu\kappa} - \delta_{\nu\kappa}M_{\mu\lambda} + \delta_{\nu\lambda}M_{\mu\kappa}), \\
 [M_{\mu\nu}, \mathcal{P}_\kappa] &= i(\delta_{\mu\kappa}\mathcal{P}_\nu - \delta_{\nu\kappa}\mathcal{P}_\mu), \\
 [M_{\mu\nu}, \mathcal{K}_\kappa] &= i(\delta_{\mu\kappa}\mathcal{K}_\nu - \delta_{\nu\kappa}\mathcal{K}_\mu), \\
 [\mathcal{K}_\mu, \mathcal{P}_\nu] &= -2iM_{\mu\nu} + 2\delta_{\mu\nu}H, \\
 [H, \mathcal{P}_\mu] &= \mathcal{P}_\mu, \\
 [H, \mathcal{K}_\mu] &= -\mathcal{K}_\mu,
 \end{aligned}
 \tag{A.1}$$

with all other commutators vanishing.

We note here that the subsequent analysis is performed in the orthogonal basis of quantum numbers, since the Lorentz raising/lowering operators, as well as \mathcal{P}_μ and \mathcal{K}_μ have very natural representations in orthogonal root space.

²⁵For example, note that we find precisely the expected decomposition of the $\mathfrak{so}(5)_R$ currents (recall that these are level two superconformal descendants of the $\mathcal{D}[0, 0, 0; 2, 0]$ multiplet) in terms of representations of $\mathfrak{su}(2)_R$: $\mathbf{10} = 3 \times \mathbf{1} + 2 \times \mathbf{2} + 1 \times \mathbf{3}$.

A.2 The 5D superconformal algebra

Extending the 5D conformal algebra to include supersymmetry is achieved by adding to our existing set (now with $D = 5$) the generators of supertranslations $\mathcal{Q}_{\mathbf{A}a}$ and superconformal translations $\mathcal{S}_{\mathbf{A}a}$. These are equipped with a Lorentz spinor index $a = 1, \dots, 4$ and an $\mathfrak{su}(2)_R$ index $\mathbf{A} = 1, 2$. Their associated Clifford algebras are generated by Γ_μ and $\tilde{\Gamma}_\mu$, respectively.²⁶ The collection of bosonic and fermionic generators build the $F(4)$ superconformal algebra, the bosonic part of which is $\mathfrak{so}(5, 2) \oplus \mathfrak{su}(2)_R$.

First, it will be useful to relate $M_{\mu\nu}$ to the Lorentz raising (lowering) operators \mathcal{M}_i^\pm , associated with the positive (negative) simple roots, and the Cartans \mathcal{H}_i (the eigenvalues of which are l_i), where $i = 1, 2$. We do so using the relations

$$\begin{aligned}\mathcal{M}_1^\pm &= M_{13} \pm iM_{23} \mp iM_{14} + M_{24}, \\ \mathcal{M}_2^\pm &= M_{35} \pm iM_{45}, \\ \mathcal{H}_i &= M_{2i-1\ 2i}, \quad \text{such that } [\mathcal{H}_i, \mathcal{H}_j] = 0.\end{aligned}\tag{A.2}$$

The algebra is now extended to include the following commutation relations — see e.g. [20]:

$$\begin{aligned}[M_{\mu\nu}, \mathcal{Q}_{\mathbf{A}a}] &= \frac{i}{4}[\Gamma_\mu, \Gamma_\nu]_a{}^b \mathcal{Q}_{\mathbf{A}b}, & [M_{\mu\nu}, \mathcal{S}_{\mathbf{A}a}] &= \frac{i}{4}[\tilde{\Gamma}_\mu, \tilde{\Gamma}_\nu]_a{}^b \mathcal{S}_{\mathbf{A}b}, \\ [H, \mathcal{Q}_{\mathbf{A}a}] &= \frac{1}{2}\mathcal{Q}_{\mathbf{A}a}, & [H, \mathcal{S}_{\mathbf{A}a}] &= -\frac{1}{2}\mathcal{S}_{\mathbf{A}a}, \\ [\mathcal{P}_\mu, \mathcal{Q}_{\mathbf{A}a}] &= 0, & [\mathcal{P}_\mu, \mathcal{S}_{\mathbf{A}a}] &= i(\tilde{\Gamma}_\mu \tilde{\Gamma}_5)_a{}^b \mathcal{Q}_{\mathbf{A}b}, \\ [\mathcal{K}_\mu, \mathcal{Q}_{\mathbf{A}a}] &= i(\Gamma_\mu \Gamma_5)_a{}^b \mathcal{S}_{\mathbf{A}b}, & [\mathcal{K}_\mu, \mathcal{S}_{\mathbf{A}a}] &= 0.\end{aligned}\tag{A.3}$$

The generators of the $\mathfrak{su}(2)_R$ algebra are denoted T_m for $m = 1, 2, 3$. They act on the supercharges according to

$$\begin{aligned}[T_m, \mathcal{Q}_{\mathbf{A}a}] &= \left(\frac{\sigma_m}{2}\right)_{\mathbf{A}}{}^{\mathbf{B}} \mathcal{Q}_{\mathbf{B}a}, \\ [T_m, \mathcal{S}_{\mathbf{A}a}] &= \left(\frac{\sigma_m}{2}\right)_{\mathbf{A}}{}^{\mathbf{B}} \mathcal{S}_{\mathbf{B}a},\end{aligned}\tag{A.4}$$

where σ_m are the Pauli matrices. In fact we may compactly write these R -symmetry generators as

$$[R_{\mathbf{A}}{}^{\mathbf{B}}] = [(T^m \sigma_m)_{\mathbf{A}}{}^{\mathbf{B}}] = \begin{pmatrix} \hat{\mathcal{R}} & \mathcal{R}^+ \\ \mathcal{R}^- & -\hat{\mathcal{R}} \end{pmatrix},\tag{A.5}$$

with the algebra

$$[R_{\mathbf{A}}{}^{\mathbf{B}}, R_{\mathbf{C}}{}^{\mathbf{D}}] = \delta_{\mathbf{C}}^{\mathbf{B}} R_{\mathbf{A}}{}^{\mathbf{D}} - \delta_{\mathbf{A}}^{\mathbf{D}} R_{\mathbf{C}}{}^{\mathbf{B}}.\tag{A.6}$$

From the above, we may infer that

$$\begin{aligned}[\mathcal{R}^+, \mathcal{Q}_{1a}] &= 0, & [\mathcal{R}^-, \mathcal{Q}_{1a}] &= \mathcal{Q}_{2a}, & [\hat{\mathcal{R}}, \mathcal{Q}_{1a}] &= \frac{1}{2}\mathcal{Q}_{1a}, \\ [\mathcal{R}^+, \mathcal{S}_{1a}] &= 0, & [\mathcal{R}^-, \mathcal{S}_{1a}] &= \mathcal{S}_{2a}, & [\hat{\mathcal{R}}, \mathcal{S}_{1a}] &= \frac{1}{2}\mathcal{S}_{1a}.\end{aligned}\tag{A.7}$$

²⁶See appendix A.3 for our gamma-matrix conventions.

We denote the eigenvalue of $\hat{\mathcal{R}}$ in the orthogonal basis to be k . The odd elements of the 5D superconformal algebra satisfy

$$\begin{aligned} \{\mathcal{Q}_{\mathbf{A}a}, \mathcal{Q}_{\mathbf{B}b}\} &= (\Gamma^\mu \mathcal{P}_\mu K)_{ab} \epsilon_{\mathbf{AB}}, \\ \{\mathcal{S}_{\mathbf{A}a}, \mathcal{S}_{\mathbf{B}b}\} &= (\tilde{\Gamma}^\mu \mathcal{K}_\mu K)_{ab} \epsilon_{\mathbf{AB}}, \\ \{\mathcal{Q}_{\mathbf{A}a}, \mathcal{S}_{\mathbf{C}c}\} &= \left[\delta_{\mathbf{A}}^{\mathbf{B}} M_a{}^b + \delta_{\mathbf{A}}^{\mathbf{B}} \delta_a^b H - 3R_{\mathbf{A}}{}^{\mathbf{B}} \delta_a^b \right] (i\epsilon_{\mathbf{BC}} \Gamma^5 K)_{bc}, \end{aligned} \quad (\text{A.8})$$

where K is the charge-conjugation matrix and $\epsilon_{\mathbf{AB}}$ is the antisymmetric 2×2 matrix such that $\epsilon_{12} = 1$. It is straightforward to check that these matrices are given by

$$\begin{aligned} [(\Gamma^\mu \mathcal{P}_\mu K)_{ab}] &= \begin{pmatrix} 0 & \mathcal{P}_1 & i\mathcal{P}_2 & -\mathcal{P}_3 \\ -\mathcal{P}_1 & 0 & -\mathcal{P}_3 & i\mathcal{P}_4 \\ -i\mathcal{P}_2 & \mathcal{P}_3 & 0 & \mathcal{P}_5 \\ \mathcal{P}_3 & -i\mathcal{P}_4 & -\mathcal{P}_5 & 0 \end{pmatrix}, \\ [(\tilde{\Gamma}^\mu \mathcal{K}_\mu K)_{ab}] &= \begin{pmatrix} 0 & \mathcal{K}_1 & i\mathcal{K}_2 & \mathcal{K}_3 \\ -\mathcal{K}_1 & 0 & \mathcal{K}_3 & i\mathcal{K}_4 \\ -i\mathcal{K}_2 & -\mathcal{K}_3 & 0 & \mathcal{K}_5 \\ -\mathcal{K}_3 & -i\mathcal{K}_4 & -\mathcal{K}_5 & 0 \end{pmatrix}. \end{aligned} \quad (\text{A.9})$$

We have also used the matrix $M_a{}^b$, which is defined by

$$\begin{aligned} [M_a{}^b] &= \frac{i}{4} [\Gamma^\mu, \Gamma^\nu]_a{}^b M_{\mu\nu} \\ &= \begin{pmatrix} \mathcal{H}_1 + \mathcal{H}_2 & \mathcal{M}_2^+ & -\frac{1}{2}[\mathcal{M}_1^+, \mathcal{M}_2^+] & -\frac{1}{2}[[\mathcal{M}_1^+, \mathcal{M}_2^+], \mathcal{M}_2^+] \\ \mathcal{M}_2^- & \mathcal{H}_1 - \mathcal{H}_2 & \mathcal{M}_1^+ & \frac{1}{2}[\mathcal{M}_1^+, \mathcal{M}_2^+] \\ \frac{1}{2}[\mathcal{M}_1^-, \mathcal{M}_2^-] & \mathcal{M}_1^- & -\mathcal{H}_1 + \mathcal{H}_2 & \mathcal{M}_2^+ \\ -\frac{1}{2}[[\mathcal{M}_1^-, \mathcal{M}_2^-], \mathcal{M}_2^-] & -\frac{1}{2}[\mathcal{M}_1^-, \mathcal{M}_2^-] & \mathcal{M}_2^- & -\mathcal{H}_1 - \mathcal{H}_2 \end{pmatrix}. \end{aligned} \quad (\text{A.10})$$

The supercharges have the following conjugation relations [20]

$$\mathcal{Q}_{\mathbf{A}a} = i\epsilon_{\mathbf{AB}} (K\Gamma_5^T)_{ab} \mathcal{S}^{\mathbf{B}b}, \quad \mathcal{S}_{\mathbf{A}a} = -i\epsilon_{\mathbf{AB}} (K\tilde{\Gamma}_5^T)_{ab} \mathcal{Q}^{\mathbf{B}b}, \quad (\text{A.11})$$

which allows us to rewrite the $\{\mathcal{Q}, \mathcal{S}\}$ anticommutator as

$$\{\mathcal{Q}_{\mathbf{A}a}, \mathcal{Q}^{\mathbf{B}b}\} = \delta_{\mathbf{A}}^{\mathbf{B}} M_a{}^b + \delta_{\mathbf{A}}^{\mathbf{B}} \delta_a^b H - 3R_{\mathbf{A}}{}^{\mathbf{B}} \delta_a^b. \quad (\text{A.12})$$

A.3 Gamma-matrix conventions in 5D

Here we collect our gamma-matrix conventions. For Euclidean $\mathfrak{so}(5)$ spinors we have

$$\begin{aligned} \Gamma^1 &= \begin{pmatrix} 0 & \mathbb{1}_{2 \times 2} \\ \mathbb{1}_{2 \times 2} & 0 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 0 & i\mathbb{1}_{2 \times 2} \\ -i\mathbb{1}_{2 \times 2} & 0 \end{pmatrix}, \quad \Gamma^3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \\ \Gamma^4 &= \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \end{aligned} \quad (\text{A.13})$$

These are supplemented by the charge conjugation matrix

$$K = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}. \quad (\text{A.14})$$

Note that $\tilde{\Gamma}^{1,2,3,4} = \Gamma^{1,2,3,4}$, while $\tilde{\Gamma}^5 = -\Gamma^5$. Each set generates the Euclidean Clifford algebra in five dimensions, that is they satisfy the relations

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}, \quad (\Gamma^\mu)^\dagger = \Gamma^\mu, \quad \Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5 = \mathbb{1}_{4\times 4}, \quad K^T = -K, \quad (K\Gamma^\mu)^T = -K\Gamma^\mu \quad (\text{A.15})$$

and similarly for the $\tilde{\Gamma}$ matrices.

A.4 The Λ -basis of generators

There exists a natural basis for constructing the UIRs of the 5D SCA, in which particular combinations of supercharges map highest weights of the maximal compact subalgebra, $\mathfrak{so}(5) \oplus \mathfrak{so}(2) \oplus \mathfrak{su}(2)_R$, to other highest weights. We define

$$\Lambda_1^a := \sum_{b=1}^a \mathcal{Q}_{1b} \lambda_b^a \quad \text{and} \quad \Lambda_2^a := \sum_{b=1}^a \mathcal{Q}_{2b} \lambda_b^a - \sum_{b=1}^a \mathcal{Q}_{1b} \mathcal{R}^- \lambda_b^a \frac{1}{2\mathcal{R}}. \quad (\text{A.16})$$

The λ_b^a are given by

$$\begin{aligned} \lambda_a^a &= 1, \quad \text{where } a \text{ is not summed over,} \\ \lambda_1^2 &= -\mathcal{M}_2^- \frac{1}{2\mathcal{H}_2}, \\ \lambda_1^3 &= \left(-\mathcal{M}_1^- \mathcal{M}_2^- + \mathcal{M}_2^- \mathcal{M}_1^- \frac{(\mathcal{H}_1 - \mathcal{H}_2 + 1)}{(\mathcal{H}_1 - \mathcal{H}_2)} \right) \frac{1}{4(\mathcal{H}_1 + 1)}, \\ \lambda_2^3 &= -\mathcal{M}_1^- \frac{1}{2(\mathcal{H}_1 - \mathcal{H}_2)}, \\ \lambda_1^4 &= \left(-\mathcal{M}_2^- \mathcal{M}_1^- \mathcal{M}_2^- \frac{(2 + \mathcal{H}_1 + 5\mathcal{H}_2 + 4\mathcal{H}_1 \mathcal{H}_2)}{(\mathcal{H}_1 + 1)\mathcal{H}_2} + \mathcal{M}_1^- (\mathcal{M}_2^-)^2 \frac{(2\mathcal{H}_2 + 1)}{\mathcal{H}_2} \right. \\ &\quad \left. + (\mathcal{M}_2^-)^2 \mathcal{M}_1^- \frac{(2\mathcal{H}_1 + 3)}{(\mathcal{H}_1 + 1)} \right) \frac{1}{8(\mathcal{H}_1 + \mathcal{H}_2 + 1)}, \\ \lambda_2^4 &= \left(-\mathcal{M}_2^- \mathcal{M}_1^- + \mathcal{M}_1^- \mathcal{M}_2^- \frac{(\mathcal{H}_2 + 1)}{\mathcal{H}_2} \right) \frac{1}{4(\mathcal{H}_1 + 1)}, \\ \lambda_3^4 &= -\mathcal{M}_2^- \frac{1}{2\mathcal{H}_2}. \end{aligned} \quad (\text{A.17})$$

With the help of the above it is straightforward to calculate all level-one norms:

$$\begin{aligned}
\|\Lambda_{\mathbf{1}}^4 |\Delta; l_1, l_2; k\rangle^{\text{hw}}\|^2 &= (\Delta - 3k - l_1 - l_2 - 4) \frac{(2l_1 + 3)(l_1 + l_2 + 2)(2l_2 + 1)}{4(l_1 + 1)l_2(l_1 + l_2 + 1)}, \\
\|\Lambda_{\mathbf{1}}^3 |\Delta; l_1, l_2; k\rangle^{\text{hw}}\|^2 &= (\Delta - 3k - l_1 + l_2 - 3) \frac{(2l_1 + 3)(l_1 - l_2 + 1)}{2(l_1 + 1)(l_1 - l_2)}, \\
\|\Lambda_{\mathbf{1}}^2 |\Delta; l_1, l_2; k\rangle^{\text{hw}}\|^2 &= (\Delta - 3k + l_1 - l_2 - 1) \frac{(2l_2 + 1)}{2l_2}, \\
\|\Lambda_{\mathbf{1}}^1 |\Delta; l_1, l_2; k\rangle^{\text{hw}}\|^2 &= (\Delta - 3k + l_1 + l_2) \\
\|\Lambda_{\mathbf{2}}^4 |\Delta; l_1, l_2; k\rangle^{\text{hw}}\|^2 &= (\Delta + 3k - l_1 - l_2 - 1) \frac{(2l_1 + 3)(l_1 + l_2 + 2)(2l_2 + 1)}{4(l_1 + 1)l_2(l_1 + l_2 + 1)} \left(1 + \frac{1}{2k}\right), \\
\|\Lambda_{\mathbf{2}}^3 |\Delta; l_1, l_2; k\rangle^{\text{hw}}\|^2 &= (\Delta + 3k - l_1 + l_2) \frac{(2l_1 + 3)(l_1 - l_2 + 1)}{2(l_1 + 1)(l_1 - l_2)} \left(1 + \frac{1}{2k}\right), \\
\|\Lambda_{\mathbf{2}}^2 |\Delta; l_1, l_2; k\rangle^{\text{hw}}\|^2 &= (\Delta + 3k + l_1 - l_2 + 2) \frac{(2l_2 + 1)}{2l_2} \left(1 + \frac{1}{2k}\right), \\
\|\Lambda_{\mathbf{2}}^1 |\Delta; l_1, l_2; k\rangle^{\text{hw}}\|^2 &= (\Delta + 3k + l_1 + l_2 + 3) \left(1 + \frac{1}{2k}\right), \tag{A.18}
\end{aligned}$$

where we have normalised $\| |\Delta; l_1, l_2; k\rangle^{\text{hw}} \|^2 = 1$.

A.5 The 6D superconformal algebras

The superconformal algebra in 6D is $\mathfrak{osp}(8^*|2\mathcal{N})$, the bosonic part of which is $\mathfrak{so}(6, 2) \oplus \mathfrak{sp}(\mathcal{N})_R$.²⁷ The set of conformal generators is extended to include the generators of supersymmetry $\mathcal{Q}_{\mathbf{A}a}$ and superconformal translations $\mathcal{S}_{\mathbf{A}\dot{a}}$. The Lorentz spinor index ranges from $a, \dot{a} = 1, \dots, 4$ (the dotted spinor index refers to the fact that it is an antifundamental index), while the $\mathfrak{sp}(\mathcal{N})_R$ index ranges from $\mathbf{A} = 1, \dots, 2\mathcal{N}$. Their associated Clifford algebras are generated by Γ_μ and $\tilde{\Gamma}_\mu$, respectively, the conventions of which are detailed in appendix A.6.

Since their algebras are very similar, we first list the features that are shared by both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases. Due to the fact that spinors in 6D are pseudo-real, they satisfy the reality conditions

$$\mathcal{Q}_{\mathbf{A}a} = i\Omega_{\mathbf{AB}}(K\Gamma_6^T)_{ab}\mathcal{S}^{\dagger\mathbf{B}b}, \quad \mathcal{S}_{\mathbf{A}\dot{a}} = -i\Omega_{\mathbf{AB}}(K\tilde{\Gamma}_6^T)_{\dot{a}b}\mathcal{Q}^{\dagger\mathbf{B}b}, \tag{A.19}$$

where $\Omega_{\mathbf{AB}}$ is the appropriate antisymmetric matrix in $2\mathcal{N}$ dimensions.²⁸ The commutation

²⁷Note that we will carry out this discussion using the $\mathfrak{so}(6)$ algebra in the orthogonal basis, instead of $\mathfrak{su}(4)$ algebra in the Dynkin basis. The dictionary between the two will be provided at the end.

²⁸For $\mathcal{N} = 1$ one has $\Omega_{\mathbf{AB}} = \epsilon_{\mathbf{AB}}$, the 2D antisymmetric matrix. For $\mathcal{N} = 2$, $\Omega_{\mathbf{AB}}$ is the 4D symplectic matrix with $\Omega_{14} = -\Omega_{41} = \Omega_{23} = -\Omega_{32} = 1$ and all other components vanishing.

relations of the \mathcal{Q} and \mathcal{S} with the 6D conformal algebra are given by

$$\begin{aligned}
 [M_{\mu\nu}, \mathcal{Q}_{\mathbf{A}a}] &= \frac{i}{4} [\Gamma_\mu, \Gamma_\nu]_a{}^b \mathcal{Q}_{\mathbf{A}b}, & [M_{\mu\nu}, \mathcal{S}_{\mathbf{A}\dot{a}}] &= \frac{i}{4} [\tilde{\Gamma}_\mu, \tilde{\Gamma}_\nu]_{\dot{a}}{}^{\dot{b}} \mathcal{S}_{\mathbf{A}\dot{b}}, \\
 [H, \mathcal{Q}_{\mathbf{A}a}] &= \frac{1}{2} \mathcal{Q}_{\mathbf{A}a}, & [H, \mathcal{S}_{\mathbf{A}\dot{a}}] &= -\frac{1}{2} \mathcal{S}_{\mathbf{A}\dot{a}}, \\
 [\mathcal{P}_\mu, \mathcal{Q}_{\mathbf{A}a}] &= 0, & [\mathcal{P}_\mu, \mathcal{S}_{\mathbf{A}\dot{a}}] &= i(\tilde{\Gamma}_\mu \tilde{\Gamma}_6)_{\dot{a}}{}^{\dot{b}} \mathcal{Q}_{\mathbf{A}b}, \\
 [\mathcal{K}_\mu, \mathcal{Q}_{\mathbf{A}a}] &= i(\Gamma_\mu \Gamma_6)_a{}^b \mathcal{S}_{\mathbf{A}\dot{b}}, & [\mathcal{K}_\mu, \mathcal{S}_{\mathbf{A}\dot{a}}] &= 0.
 \end{aligned} \tag{A.20}$$

It is helpful to define the Lorentz raising/lowering operators \mathcal{M}_i^\pm and Cartans \mathcal{H}_i — the eigenvalues of which are h_i in the orthogonal basis of $\mathfrak{so}(6)$ — in terms of $M_{\mu\nu}$. The relations are provided below:

$$\begin{aligned}
 \mathcal{M}_1^\pm &= M_{13} \pm iM_{23} \mp iM_{14} + M_{24}, \\
 \mathcal{M}_2^\pm &= M_{35} \pm iM_{45} \mp iM_{36} + M_{46}, \\
 \mathcal{M}_3^\pm &= M_{35} \pm iM_{45} \pm iM_{36} - M_{46}, \\
 \mathcal{H}_i &= M_{2i-1\ 2i}, \quad \text{such that } [\mathcal{H}_i, \mathcal{H}_j] = 0.
 \end{aligned} \tag{A.21}$$

For a superconformal algebra, all supercharges must have the same chirality [20], which we choose to be positive. Therefore we define the projector $P_+ = \frac{1}{2}(1 + \Gamma_7)$ and have that

$$\begin{aligned}
 \{\mathcal{Q}_{\mathbf{A}a}, \mathcal{Q}_{\mathbf{B}b}\} &= (P_+ \Gamma^\mu \mathcal{P}_\mu K)_{ab} \Omega_{\mathbf{A}\mathbf{B}}, \\
 \{\mathcal{S}_{\mathbf{A}\dot{a}}, \mathcal{S}_{\mathbf{B}\dot{b}}\} &= (P_+ \tilde{\Gamma}^\mu \mathcal{K}_\mu K)_{\dot{a}\dot{b}} \Omega_{\mathbf{A}\mathbf{B}}.
 \end{aligned} \tag{A.22}$$

We may also use the projector P_+ to define $M_a{}^b$ as

$$\begin{aligned}
 [M_a{}^b] &= -\frac{i}{4} (P_+)_a{}^c [\Gamma^\mu, \Gamma^\nu]_c{}^b M_{\mu\nu} \\
 &= \begin{pmatrix} \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 & i\mathcal{M}_3^- & -\frac{1}{2}[\mathcal{M}_1^-, \mathcal{M}_3^-] & \frac{1}{4}[\mathcal{M}_1^-, [\mathcal{M}_2^-, \mathcal{M}_3^-]] \\ -i\mathcal{M}_3^+ & \mathcal{H}_1 - \mathcal{H}_2 - \mathcal{H}_3 & i\mathcal{M}_1^- & \frac{1}{2}[\mathcal{M}_1^-, \mathcal{M}_2^-] \\ \frac{1}{2}[\mathcal{M}_1^+, \mathcal{M}_3^+] & -i\mathcal{M}_1^+ & -\mathcal{H}_1 + \mathcal{H}_2 - \mathcal{H}_3 & i\mathcal{M}_2^- \\ \frac{1}{4}[\mathcal{M}_1^+, [\mathcal{M}_2^+, \mathcal{M}_3^+]] & -\frac{1}{2}[\mathcal{M}_1^+, \mathcal{M}_2^+] & -i\mathcal{M}_2^+ & -\mathcal{H}_1 - \mathcal{H}_2 + \mathcal{H}_3 \end{pmatrix}.
 \end{aligned} \tag{A.23}$$

A.5.1 $\mathcal{N} = 1$

For this case, the R -symmetry algebra is $\mathfrak{sp}(1)_R \simeq \mathfrak{su}(2)_R$. Conveniently, all the information we require about $\mathfrak{su}(2)_R$ has already been provided in the 5D $\mathcal{N} = 1$ discussion, specifically (A.4) onwards. We may therefore use the previously-defined $R_{\mathbf{A}\mathbf{B}}$ and write the $\{\mathcal{Q}, \mathcal{Q}^\dagger\}$ anticommutator as

$$\{\mathcal{Q}_{\mathbf{A}a}, \mathcal{Q}^{\dagger\mathbf{B}b}\} = \delta_{\mathbf{A}}^{\mathbf{B}} M_a{}^b + \delta_{\mathbf{A}}^{\mathbf{B}} (P_+)_a{}^b H - 4R_{\mathbf{A}}{}^{\mathbf{B}} (P_+)_a{}^b. \tag{A.24}$$

The expression (4.6) we used for the 6D (1, 0) index is then, in the orthogonal basis,

$$\delta := \{\mathcal{Q}_{14}, \mathcal{S}_{2\dot{4}}\} = \{\mathcal{Q}_{14}, \mathcal{Q}^{\dagger 1\dot{4}}\} = \Delta - 4k - (h_1 + h_2 - h_3). \tag{A.25}$$

A.5.2 $\mathcal{N} = 2$

For this case, the R -symmetry algebra is $\mathfrak{sp}(2)_R \simeq \mathfrak{so}(5)_R$. The matrix $R_{\mathbf{A}}^{\mathbf{B}}$ for $\mathfrak{so}(5)_R$ is

$$[R_{\mathbf{A}}^{\mathbf{B}}] = \begin{pmatrix} \mathcal{J}_1 + \mathcal{J}_2 & \mathcal{R}_2^+ & -\frac{1}{2}[\mathcal{R}_1^+, \mathcal{R}_2^+] & -\frac{1}{2}[[\mathcal{R}_1^+, \mathcal{R}_2^+], \mathcal{R}_2^+] \\ \mathcal{R}_2^- & \mathcal{J}_1 - \mathcal{J}_2 & \mathcal{R}_1^+ & \frac{1}{2}[\mathcal{R}_1^+, \mathcal{R}_2^+] \\ \frac{1}{2}[\mathcal{R}_1^-, \mathcal{R}_2^-] & \mathcal{R}_1^- & -\mathcal{J}_1 + \mathcal{J}_2 & \mathcal{R}_2^+ \\ -\frac{1}{2}[[\mathcal{R}_1^-, \mathcal{R}_2^-], \mathcal{R}_2^-] & -\frac{1}{2}[\mathcal{R}_1^-, \mathcal{R}_2^-] & \mathcal{R}_2^- & -\mathcal{J}_1 - \mathcal{J}_2 \end{pmatrix}, \quad (\text{A.26})$$

where the \mathcal{J}_i are the Cartans of $\mathfrak{so}(5)_R$ —the eigenvalues of which are j_i —and \mathcal{R}_i^\pm are the raising/lowering operators. This allows the $\{\mathcal{Q}, \mathcal{Q}^\dagger\}$ anticommutator to be written as [20]

$$\{\mathcal{Q}_{\mathbf{A}a}, \mathcal{Q}^{\dagger \mathbf{B}b}\} = \delta_{\mathbf{A}}^{\mathbf{B}} M_a^b + \delta_{\mathbf{A}}^{\mathbf{B}} (P_+)_a^b H - 2R_{\mathbf{A}}^{\mathbf{B}} (P_+)_a^b. \quad (\text{A.27})$$

Hence the δ we used for the 6D (2,0) index is in the orthogonal basis

$$\delta := \{\mathcal{Q}_{24}, \mathcal{S}_{34}, \} = \{\mathcal{Q}_{24}, \mathcal{Q}^{\dagger 24}\} = \Delta - 2j_1 + 2j_2 - (h_1 + h_2 - h_3). \quad (\text{A.28})$$

To make contact with the main part of this paper, we convert from the orthogonal bases to the Dynkin bases using the following expressions for the $\mathfrak{so}(6)$ —to $\mathfrak{su}(4)$ Dynkin—, $\mathfrak{so}(5)$ and $\mathfrak{su}(2)$ orthogonal Cartans respectively:

$$\begin{aligned} h_1 &= \frac{1}{2}(c_1 + 2c_2 + c_3), & h_2 &= \frac{1}{2}(c_1 + c_3), & h_3 &= \frac{1}{2}(c_1 - c_3), \\ j_1 &= d_1 + \frac{1}{2}d_2, & j_2 &= \frac{1}{2}d_2, \\ k &= \frac{1}{2}K. \end{aligned} \quad (\text{A.29})$$

A.6 Gamma-matrix conventions in 6D

Here we collect our gamma-matrix conventions. For Euclidean $\mathfrak{so}(6)$ spinors we have

$$\begin{aligned} \Gamma^1 &= \sigma_1 \otimes \mathbb{1}_{2 \times 2} \otimes \mathbb{1}_{2 \times 2}, & \Gamma^2 &= \sigma_2 \otimes \mathbb{1}_{2 \times 2} \otimes \mathbb{1}_{2 \times 2}, & \Gamma^3 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1}_{2 \times 2}, \\ \Gamma^4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1}_{2 \times 2}, & \Gamma^5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1, & \Gamma^6 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_2. \end{aligned} \quad (\text{A.30})$$

We also have $\Gamma^7 = i\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\Gamma^6$, which can be equivalently defined as

$$\Gamma^7 = \sigma_3 \otimes \sigma_3 \otimes \sigma_3. \quad (\text{A.31})$$

Since we are in even dimensions, we have a choice between which charge conjugation matrices to use, $K_{(\pm)}$. These have the properties $K_{(\pm)}^T = \mp K_{(\pm)}$ and $K_{(\pm)}^2 = \mp 1$. We choose $K_{(-)}$, but to lighten the notation, we will omit the subscript. We define our $K_{(-)} = K$ as

$$K = (-i\sigma_2) \otimes \sigma_1 \otimes (-i\sigma_2), \quad (\text{A.32})$$

Note that $\Gamma^{1,2,3,4,5} = \tilde{\Gamma}^{1,2,3,4,5}$, while $\tilde{\Gamma}^6 = -\Gamma^6$. They each generate the Euclidean Clifford algebra in five dimensions. The Γ s satisfy the relations

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}, \quad (\Gamma^\mu)^\dagger = \Gamma^\mu, \quad K(\Gamma^\mu)^T K^{-1} = -\Gamma^\mu. \quad (\text{A.33})$$

and similarly for the $\tilde{\Gamma}$ matrices.

B Multiplet supercharacters

Consider a representation of a Lie algebra \mathfrak{g} with highest weight λ . The Weyl character formula is given by [63]

$$\chi_\lambda = \frac{\sum_{w \in W} \text{sgn}(w) e^{w(\lambda + \rho)}}{e^\rho \prod_{\alpha \in \Phi_-} (1 - e^\alpha)}, \quad (\text{B.1})$$

where W is the Weyl group of the Lie algebra root system and ρ is the half sum of the positive roots Φ_+ . Note that $\text{sgn}(w) = (-1)^{l(w)}$ where $l(w)$ is the length of the Weyl group element, i.e. how many simple reflections it is comprised of.

One can alternatively obtain this formula using a Verma-module construction: decompose the algebra \mathfrak{g} as

$$\mathfrak{g} = \Phi_+ \oplus \mathfrak{h} \oplus \Phi_-, \quad (\text{B.2})$$

where \mathfrak{h} corresponds to the Cartan subalgebra and Φ_- (Φ_+) are the negative (positive) roots. We construct the Verma module \mathcal{V} corresponding to some highest (lowest) weight $|\lambda\rangle$ by considering the space comprised of the states $f(\Phi_-)|\lambda\rangle$ ($f(\Phi_+)|\lambda\rangle$), where f is any polynomial of the negative (positive) roots modulo algebraic relations. The character of this module is defined to be [63]

$$\chi_{\mathcal{V}} = \frac{e^\lambda}{\prod_{\alpha \in \Phi_-} (1 - e^\alpha)}. \quad (\text{B.3})$$

The character of the representation labelled by Λ is recovered by summing over the Weyl group action on the roots

$$\chi_\Lambda = \sum_{w \in W} w(\chi_{\mathcal{V}}) = \sum_{w \in W} \frac{e^{w(\lambda)}}{\prod_{\alpha \in \Phi_-} (1 - e^{w(\alpha)})}. \quad (\text{B.4})$$

We can utilise the identity $w(e^{-\rho-\lambda}\chi_{\mathcal{V}}) = \text{sgn}(w)e^{-\rho-\lambda}\chi_{\mathcal{V}}$ along with the fact that w acts naturally — i.e. we may take $w(e^{-\rho-\lambda}\chi_{\mathcal{V}}) = w(e^{-\rho-\lambda})w(\chi_{\mathcal{V}})$ [63]—to show that

$$\chi_\Lambda = \sum_{w \in W} w(\chi_{\mathcal{V}}) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda + \rho)}}{e^{\rho + \lambda}} \chi_{\mathcal{V}} = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda + \rho)}}{e^\rho \prod_{\alpha \in \Phi_-} (1 - e^\alpha)}. \quad (\text{B.5})$$

The formulation of the Weyl character formula (B.4) is particularly useful in the context of UIRs of the SCA [14].

B.1 Characters of 5D $\mathcal{N} = 1$ multiplets

We are now in a position to compute the characters of $F(4)$.²⁹ Let us consider a representation the highest weight of which has conformal dimension Δ with $\mathfrak{so}(5)$ Lorentz quantum numbers (d_1, d_2) and $\mathfrak{su}(2)_R$ quantum numbers K . Note that these are expressed in the Dynkin basis and hence are integer. The highest weight can be decomposed as

$$\lambda = \omega_1^\beta d_1 + \omega_2^\beta d_2 + \omega^\alpha K, \quad (\text{B.6})$$

²⁹A summary of this discussion for the case of the (2,0) SCA can be found in appendix C of [14].

where ω_i^β ($i = 1, 2$) are the fundamental weights associated with the $\mathfrak{so}(5)$ simple roots β_i , while ω^α is the fundamental weight associated with the $\mathfrak{su}(2)_R$ simple root α . We may in turn express the fundamental weights in terms of the simple roots (for reasons which will become apparent) by using the Cartan matrix A_{ij}

$$\begin{aligned}\omega_i^\beta &= (A_{B_2}^{-1})_{ij}\beta_j = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \\ \omega^\alpha &= \frac{1}{2}\alpha.\end{aligned}\tag{B.7}$$

Let us next consider e^λ by defining the fugacities

$$b_1 = e^{\beta_1 + \beta_2}, \quad b_2 = e^{\frac{1}{2}(\beta_1 + 2\beta_2)}, \quad a = e^{\frac{1}{2}\alpha},\tag{B.8}$$

hence

$$e^\lambda = b_1^{d_1} b_2^{d_2} a^K.\tag{B.9}$$

The character of a particular representation \mathcal{R} is then defined to be

$$\chi_{\mathcal{R}}(a, \mathbf{b}, q) = \text{Tr}_{\mathcal{R}}(q^\Delta b_1^{d_1} b_2^{d_2} a^K),\tag{B.10}$$

where we have included the $\mathfrak{so}(2)$ Cartan, Δ . For example we can read off the character for the supercharges $\mathcal{Q}_{\mathbf{A}a}$ as (recall that we are in the Dynkin basis)

$$\sum_{a=1}^4 \chi(\mathcal{Q}_{1a}) = ab_2 q^{\frac{1}{2}} + \frac{ab_1 q^{\frac{1}{2}}}{b_2} + \frac{ab_2 q^{\frac{1}{2}}}{b_1} + \frac{aq^{\frac{1}{2}}}{b_2}.\tag{B.11}$$

We can then apply this to specific representations of the SCA.

B.1.1 Long representations

For the long representation $\mathcal{L}_{[\Delta; d_1, d_2; K]}$ we construct the superconformal multiplet by acting on the highest weight state $|\Delta; d_1, d_2; K\rangle^{\text{hw}}$ with momentum operators and supercharges as in eq. (2.1). Thus for a generic long representation we can decompose the character of the superconformal Verma module using (B.3) as

$$\chi_{\mathcal{L}}(a, \mathbf{b}, q) = q^\Delta \chi_{[d_1, d_2]}(\mathbf{b}) \chi_{[K]}(a) f(a, \mathbf{b}, q).\tag{B.12}$$

The polynomial appearing above can be decomposed as $f(a, \mathbf{b}, q) = Q(a, \mathbf{b}, q)P(\mathbf{b}, q)$ since the momentum operators and supercharges commute. Explicitly these functions are

$$\begin{aligned}Q(a, \mathbf{b}, q) &= \prod_{\mathbf{A}, a} (1 + \chi(\mathcal{Q}_{\mathbf{A}a})), \\ P(\mathbf{b}, q) &= \prod_{\mu=1}^5 (1 - \chi(\mathcal{P}_\mu))^{-1}.\end{aligned}\tag{B.13}$$

The characters $\chi_{[d_1, d_2]}(\mathbf{b})$ and $\chi_{[K]}(a)$ can be obtained through their Weyl orbits. As a result one has

$$\chi_{[d_1, d_2]}(\mathbf{b}) = \sum_{w \in W_{\text{SO}(5)}} w(b_1)^{d_1} w(b_2)^{d_2} M(w(\mathbf{b})), \quad \chi_{[K]}(a) = \sum_{w \in W_{\text{SU}(2)}} w(a)^K R(w(a)), \quad (\text{B.14})$$

where the $M(\mathbf{b})$ and $R(a)$ are the products of the characters of negative roots, explicitly

$$M(\mathbf{b}) = \frac{1}{\left(1 - \frac{1}{b_1}\right) \left(1 - \frac{1}{b_2}\right) \left(1 - \frac{b_1}{b_2}\right) \left(1 - \frac{b_2}{b_1}\right)},$$

$$R(a) = \frac{a^2}{a^2 - 1}. \quad (\text{B.15})$$

As an aside it will be worthwhile to explicitly demonstrate the Weyl group actions appearing in (B.14). In the orthogonal basis the generators of $W_{\text{SO}(5)} = \mathcal{S}_2 \times (\mathbb{Z}_2)^2$ and $W_{\text{SU}(2)} = \mathcal{S}_2$ have the form

$$w_1^B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad w_2^B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad w^A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (\text{B.16})$$

There are eight elements in $\mathcal{S}_2 \times (\mathbb{Z}_2)^2$ and two in \mathcal{S}_2 . These act on the simple roots as

$$w_i^B \beta_j = f(\beta_1, \beta_2), \quad w^A \alpha = g(\alpha), \quad (\text{B.17})$$

where the r.h.s. is a combination of simple roots depending on the particular example. In fact, the simple reflections on the simple roots will generate every other root minus the Cartans of the algebra. For example, the simple roots of $\mathfrak{so}(5)$ in the orthogonal basis are $\beta_1 = (1, -1)$ and $\beta_2 = (0, 1)$. Acting on them with w_2^B produces

$$w_2^B \beta_1 = (1, 1) = \beta_1 + 2\beta_2,$$

$$w_2^B \beta_2 = (0, -1) = -\beta_2. \quad (\text{B.18})$$

Similarly, acting with w_1^B will produce $-\beta_1$ and $\beta_1 + \beta_2$ respectively. Furthermore, since the Weyl group has a natural action on e^λ (i.e. $w(e^\lambda) = e^{w(\lambda)}$), this action can be directly translated to the fugacities. Following the same example

$$w_2^B(b_1) = e^{w_2^B(\beta_1)} = e^{(\beta_1 + \beta_2)} = b_1,$$

$$w_2^B(b_2) = e^{\frac{1}{2}w_2^B(\beta_1 + 2\beta_2)} = e^{\frac{1}{2}\beta_1} = \frac{b_1}{b_2}. \quad (\text{B.19})$$

Combining the Weyl groups leads to $W_{\text{SO}(5) \times \text{SU}(2)}$, which has sixteen elements acting on a and b_i . The $Q(a, \mathbf{b}, q)$ and $P(\mathbf{b}, q)$ are both invariant under the action of any element of this combined Weyl group. Using this fact, the character for long representations can be rewritten in terms of

$$\chi_{\mathcal{L}}(a, \mathbf{b}, q) = \llbracket q^\Delta b_1^{d_1} b_2^{d_2} a^K M(\mathbf{b}) R(a) P(\mathbf{b}, q) Q(a, \mathbf{b}, q) \rrbracket_W, \quad (\text{B.20})$$

where $\llbracket \dots \rrbracket_W$ is shorthand for the Weyl symmetriser.

B.1.2 Short representations

Consider now the short multiplets of table 1. In order to calculate their characters, one is instructed [13, 40, 49] to remove certain combinations of \mathcal{Q} s and \mathcal{P} s from the expressions $Q(a, \mathbf{b}, q)$ and $P(\mathbf{b}, q)$ given in (B.13).³⁰ We explicitly consider a few examples to elucidate this point:

- a. Take the most basic short multiplet, $\mathcal{A}[d_1, d_2; K]$, with $d_1, d_2, K > 0$. Its superconformal primary is annihilated by the supercharge \mathcal{Q}_{14} and the associated character would be

$$\chi_{\mathcal{A}}(a, \mathbf{b}, q) = \left[\left[q^{\Delta} b_1^{d_1} b_2^{d_2} a^K M(\mathbf{b}) R(a) P(\mathbf{b}, q) Q(a, \mathbf{b}, q) \left(1 + \frac{aq^{\frac{1}{2}}}{b_2} \right)^{-1} \right] \right]_W, \quad (\text{B.21})$$

with $\Delta = 4 + 3K + d_1 + d_2$. Notice that this includes the character for the product over $\mathcal{Q}_{\mathbf{A}a}$ but now with the \mathcal{Q}_{14} contribution removed; hence the Weyl symmetrisation removes descendant states associated with the action of \mathcal{Q}_{14} .

- b. Take the $\mathcal{B}[d_1, 0; K]$ multiplet with $K \neq 0$. One is instructed to remove \mathcal{Q}_{13} and \mathcal{Q}_{14} and the supercharacter is

$$\chi_{\mathcal{B}}(a, \mathbf{b}, q) = \left[\left[q^{\Delta} b_1^{d_1} a^K M(\mathbf{b}) R(a) P(\mathbf{b}, q) Q(a, \mathbf{b}, q) \left(1 + \frac{ab_2q^{\frac{1}{2}}}{b_1} \right)^{-1} \left(1 + \frac{aq^{\frac{1}{2}}}{b_2} \right)^{-1} \right] \right]_W, \quad (\text{B.22})$$

with $\Delta = 3 + 3K + d_1$.

- c. Suppose now we consider the multiplet $\mathcal{B}[d_1, 0; 0]$. In this case the R -symmetry lowering operator in the Dynkin basis \mathcal{R}^- also annihilates the superconformal primary and two additional shortening conditions are generated

$$\begin{aligned} \mathcal{R}^- \mathcal{Q}_{13} \Psi_{\text{aux}} &= \mathcal{Q}_{23} \Psi_{\text{aux}} = 0, \\ \mathcal{R}^- \mathcal{M}_2^- \mathcal{Q}_{13} \Psi_{\text{aux}} &= \mathcal{Q}_{24} \Psi_{\text{aux}} = 0, \end{aligned} \quad (\text{B.23})$$

where we remind the reader that \mathcal{M}_2^- is a Lorentz lowering operator in the Dynkin basis. Therefore, for the purposes of building the Verma module we can remove both of these from the basis of Verma-module generators. As a result, the modified product over supercharges, now indicated by $\hat{Q}(\mathbf{a}, \mathbf{b}, q)$, is

$$\hat{Q}(a, \mathbf{b}, q) = Q(a, \mathbf{b}, q) \left(1 + \frac{ab_2q^{\frac{1}{2}}}{b_1} \right)^{-1} \left(1 + \frac{aq^{\frac{1}{2}}}{b_2} \right)^{-1} \left(1 + \frac{b_2q^{\frac{1}{2}}}{b_1a} \right)^{-1} \left(1 + \frac{q^{\frac{1}{2}}}{b_2a} \right)^{-1}. \quad (\text{B.24})$$

The last thing to take into account is the possible removal of \mathcal{P} s from $P(\mathbf{b}, q)$ when some components of the multiplet correspond to operator constraints. This will be discussed at length in appendix C.

³⁰There is a subtlety with removing momentum operators, which will be addressed in the following section.

B.1.3 The 5D superconformal index

The supercharacter for a given multiplet can be readily converted into the superconformal index. The five-dimensional superconformal index, as we have previously defined it in the Dynkin basis in section 2.4, is given by³¹

$$\mathcal{I}(x, y) = \text{Tr}_{\mathcal{H}}(-1)^F e^{-\beta\delta} x^{\frac{2}{3}\Delta + \frac{1}{3}(d_1+d_2)} y^{d_1}. \quad (\text{B.25})$$

The states that are counted satisfy $\delta = 0$, where

$$\delta := \{\mathcal{S}_{21}, \mathcal{Q}_{14}\} = \Delta - \frac{3}{2}K - d_1 - d_2. \quad (\text{B.26})$$

In order to make contact between the character of a 5D superconformal representation and this index, one can simply make the following fugacity reparametrisations

$$q \rightarrow x^{2/3}, \quad b_1 \rightarrow x^{1/3}y, \quad b_2 \rightarrow x^{1/3} \quad (\text{B.27})$$

and introduce a factor of $(-1)^F$. The resulting object is precisely the index since every state without $\delta = 0$ pairwise cancels.

B.2 Characters of 6D $(\mathcal{N}, 0)$ multiplets

Consider a representation, the highest weight of which has conformal dimension Δ with $\mathfrak{su}(4)$ quantum numbers (c_1, c_2, c_3) and R -symmetry quantum numbers \mathbf{R} . For $\mathcal{N} = 1$ we have that $\mathbf{R} = K$, the Dynkin label of $\mathfrak{su}(2)_R$, while for $\mathcal{N} = 2$ the Dynkin labels of $\mathfrak{so}(5)_R$ are $\mathbf{R} = (d_1, d_2)$. The highest weight can be decomposed as

$$\lambda = \omega_1^\alpha c_1 + \omega_2^\alpha c_2 + \omega_3^\alpha c_3 + \omega_i^R R^i, \quad (\text{B.28})$$

where the index in $\omega_i^R R^i$ is summed over. The ω_i^α are the fundamental weights associated with the $\mathfrak{su}(4)$ simple roots α_i and $\vec{\omega}^R$ are the fundamental weights associated with the simple roots of the R -symmetry algebra, $\vec{\beta}$.³² For $\mathfrak{su}(2)_R$ $\vec{\beta}$ has only one component, while for $\mathfrak{so}(5)_R$ $\vec{\beta}$ has two components, (β_1, β_2) . Again, we can express the fundamental weights in terms of the simple roots by using the Cartan matrix A_{ij}

$$\omega_i^\alpha = (A_{A_3}^{-1})_{ij} \alpha_j = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \quad (\text{B.29})$$

alongside the expressions for $\mathfrak{su}(2)_R$ and $\mathfrak{so}(5)_R$ given in (B.7).

This allows for a rewriting of e^λ by defining the $\mathfrak{su}(4)$ fugacities

$$a_1 = e^{\frac{1}{4}(3\alpha_1+2\alpha_2+\alpha_3)}, \quad a_2 = e^{\frac{1}{2}(\alpha_1+2\alpha_2+\alpha_3)}, \quad a_3 = e^{\frac{1}{4}(\alpha_1+2\alpha_2+3\alpha_3)}. \quad (\text{B.30})$$

³¹The fermion number in this case is $F = 2d_1 + d_2 \simeq d_2$, since d_1 is always integer.

³²Recall that in the previous section, we had defined the simple root of $\mathfrak{su}(2)_R$ as α . In order to avoid confusion with the $\mathfrak{su}(4)$ Lorentz-algebra simple roots, we label all simple roots associated with the 6D R -symmetry algebras as $\vec{\beta}$.

Similarly, we define the R -symmetry fugacities \mathbf{b} using (B.8). For $\mathcal{N} = 1$ we have that

$$\mathbf{b} = b = e^{\frac{1}{2}\beta}, \quad (\text{B.31})$$

while for $\mathcal{N} = 2$ we have $\mathbf{b} = (b_1, b_2)$ with

$$b_1 = e^{\beta_1 + \beta_2}, \quad b_2 = e^{\frac{1}{2}(\beta_1 + 2\beta_2)}. \quad (\text{B.32})$$

hence

$$e^\lambda = a_1^{c_1} a_2^{c_2} a_3^{c_3} \prod_i b_i^{R^i}. \quad (\text{B.33})$$

The character of a particular representation \mathcal{R} is then given by

$$\chi_{\mathcal{R}}(\mathbf{a}, \mathbf{b}, q) = \text{Tr}_{\mathcal{R}} \left(a_1^{c_1} a_2^{c_2} a_3^{c_3} \prod_i b_i^{R^i} \right) \quad (\text{B.34})$$

and for $\mathcal{N} = 1, 2$ respectively we have:

$$\begin{aligned} \chi_{\mathcal{R}}^{(1,0)}(\mathbf{a}, b, q) &= \text{Tr}_{\mathcal{R}} \left(a_1^{c_1} a_2^{c_2} a_3^{c_3} b^K \right), \\ \chi_{\mathcal{R}}^{(2,0)}(\mathbf{a}, \mathbf{b}, q) &= \text{Tr}_{\mathcal{R}} \left(a_1^{c_1} a_2^{c_2} a_3^{c_3} b_1^{d_1} b_2^{d_2} \right). \end{aligned} \quad (\text{B.35})$$

We can then apply this to specific irreducible representations of the 6D $(\mathcal{N}, 0)$ SCA.

B.2.1 Long representations

For the long representation $\mathcal{L}_{[\Delta; c_1, c_2, c_3; \mathbf{R}]}$ we construct the superconformal multiplet by acting on the highest weight state $|\Delta; c_1, c_2, c_3; \mathbf{R}\rangle^{\text{hw}}$ with momentum operators and supercharges $f(\mathcal{Q}, \mathcal{P})|\Delta; c_1, c_2, c_3; \mathbf{R}\rangle^{\text{hw}}$. This polynomial can be factorised as $f(\mathbf{a}, \mathbf{b}, q) = Q(\mathbf{a}, \mathbf{b}, q)P(\mathbf{a}, q)$, since the momentum operators and supercharges commute. Explicitly these functions are

$$\begin{aligned} Q(\mathbf{a}, \mathbf{b}, q) &= \prod_{\mathbf{A}, a} (1 + \chi(\mathcal{Q}_{\mathbf{A}a})), \\ P(\mathbf{a}, q) &= \prod_{\mu=1}^6 (1 - \chi(\mathcal{P}_\mu))^{-1}, \end{aligned} \quad (\text{B.36})$$

where the range of the sum over \mathbf{A} depends on the amount of supersymmetry.

Thus for a generic long representation we can decompose the character of the superconformal Verma module using (B.3) as

$$\chi_{\mathcal{L}}(\mathbf{a}, \mathbf{b}, q) = q^\Delta \chi_{[c_1, c_2, c_3]}(\mathbf{a}) \chi_{[d_1, d_2]}(\mathbf{b}) P(\mathbf{a}, q) Q(\mathbf{a}, \mathbf{b}, q). \quad (\text{B.37})$$

The characters $\chi_{[c_1, c_2, c_3]}(\mathbf{a})$ and $\chi_{[d_1, d_2]}(\mathbf{b})$ can in turn be obtained through their Weyl orbits. The relevant Weyl groups are $W_{\text{SU}(4)} = \mathcal{S}_4$, $W_{\text{SO}(5)} = \mathcal{S}_2 \times (\mathbb{Z}_2)^2$ and $W_{\text{SU}(2)} = \mathbb{Z}_2$ so one has

$$\chi_{[c_1, c_2, c_3]}(\mathbf{a}) = \sum_{w \in W_{\text{SU}(4)}} w(a_1)^{c_1} w(a_2)^{c_2} w(a_3)^{c_3} M(w(\mathbf{a})), \quad (\text{B.38})$$

$$\chi_{[\mathbf{R}]}(\mathbf{b}) = \sum_{w \in W_{\mathbf{R}}} \left(\prod_i w(b_i)^{R^i} \right) R^{(\mathcal{N}, 0)}(w(\mathbf{b})). \quad (\text{B.39})$$

We use W_R to indicate the Weyl group appropriate for the R symmetry of the $(\mathcal{N}, 0)$ SCA. The $M(\mathbf{a})$ and $R^{(\mathcal{N}, 0)}(\mathbf{b})$ are the products of characters of negative roots as defined in (B.3); explicitly

$$\begin{aligned}
 M(\mathbf{a}) &= \frac{1}{\left(1 - \frac{a_2}{a_1^2}\right) \left(1 - \frac{a_2}{a_3^2}\right) \left(1 - \frac{1}{a_1 a_3}\right) \left(1 - \frac{a_1}{a_2 a_3}\right) \left(1 - \frac{a_1 a_3}{a_2^2}\right) \left(1 - \frac{a_3}{a_1 a_2}\right)}, \\
 R^{(2,0)}(\mathbf{b}) &= \frac{1}{\left(1 - \frac{1}{b_1}\right) \left(1 - \frac{1}{b_2^2}\right) \left(1 - \frac{b_1}{b_2}\right) \left(1 - \frac{b_2^2}{b_1}\right)}, \\
 R^{(1,0)}(b) &= \frac{b^2}{b^2 - 1}.
 \end{aligned}
 \tag{B.40}$$

Again, we note that both $Q(\mathbf{a}, \mathbf{b}, q)$ and $P(\mathbf{a}, q)$ are invariant under the appropriate Weyl symmetrisations and one can write for $\mathcal{N} = 2$

$$\chi_{\mathcal{L}}^{(2,0)}(\mathbf{a}, \mathbf{b}, q) = \llbracket q^\Delta a_1^{c_1} a_2^{c_2} a_3^{c_3} b_1^{d_1} b_2^{d_2} M(\mathbf{a}) R^{(2,0)}(\mathbf{b}) P(\mathbf{a}, q) Q(\mathbf{a}, \mathbf{b}, q) \rrbracket_W,
 \tag{B.41}$$

matching [14], while for $\mathcal{N} = 1$

$$\chi_{\mathcal{L}}^{(1,0)}(\mathbf{a}, b, q) = \llbracket q^\Delta a_1^{c_1} a_2^{c_2} a_3^{c_3} b^K M(\mathbf{a}) R^{(1,0)}(b) P(\mathbf{a}, q) Q(\mathbf{a}, b, q) \rrbracket_W,
 \tag{B.42}$$

where $\llbracket \dots \rrbracket_W$ denotes the Weyl symmetriser.

B.2.2 Short representations

Consider now the short multiplets of table 2 for $\mathcal{N} = 1$ or table 3 for $\mathcal{N} = 2$. To calculate their characters we remove certain combinations of Q s from the expressions $Q(\mathbf{a}, \mathbf{b}, q)$ given in (B.36). This discussion is completely analogous to section B.1.2. There are a few cases when one is also prescribed to remove certain \mathcal{P}_μ from $P(\mathbf{a}, q)$. This is discussed at length in appendix C.

B.2.3 The 6D (1, 0) superconformal index

Once we have obtained the full supercharacter we can readily covert it into the superconformal index. For $\mathcal{N} = 1$ the index as defined in (4.3) is given by³³

$$\mathcal{I}(p, q, s) = \text{Tr}_{\mathcal{H}} (-1)^F e^{-\beta \delta} q^{\Delta - \frac{1}{2}K} p^{c_2} s^{c_1}.
 \tag{B.43}$$

The states that are counted satisfy $\delta = 0$, where

$$\delta = \Delta - 2K - \frac{1}{2}(c_1 + 2c_2 + 3c_3).
 \tag{B.44}$$

We can therefore write the character of a representation as an index via the following fugacity reparametrisations

$$a_1 \rightarrow s, \quad a_2 \rightarrow p, \quad a_3 \rightarrow 1, \quad b \rightarrow \frac{1}{q^{\frac{1}{2}}}
 \tag{B.45}$$

and inserting $(-1)^F$. The resulting object is precisely the index since every state without $\delta = 0$ pairwise cancels.

³³The fermion number in this case is $F = c_1 + c_3$.

B.2.4 The 6D (2, 0) superconformal index

The 6D (2, 0) superconformal index, as previously defined in (5.3), is given by

$$\mathcal{I}(p, q, s, t) = \text{Tr}_{\mathcal{H}}(-1)^F e^{-\beta\delta} q^{\Delta-d_1-\frac{1}{2}d_2} p^{c_2+c_3-d_2} t^{-d_1-d_2} s^{c_1+c_2}. \quad (\text{B.46})$$

The states that are counted satisfy $\delta = 0$, where

$$\delta = \Delta - 2d_1 - \frac{1}{2}c_1 - c_2 - \frac{3}{2}c_3. \quad (\text{B.47})$$

In order to make contact between the character and this index, we make the following fugacity reparametrisations

$$a_1 \rightarrow s, \quad a_2 \rightarrow ps, \quad a_3 \rightarrow p, \quad b_1 \rightarrow \frac{1}{qt}, \quad b_2 \rightarrow \frac{q^{\frac{1}{2}}}{pt} \quad (\text{B.48})$$

and insert $(-1)^F$. The resulting object is precisely the index since every state previously counted by the character without $\delta = 0$ pairwise cancels.

C The Racah-Speiser algorithm and operator constraints

In this appendix we provide the details needed to carry out the RS algorithm for the 5D $\mathcal{N} = 1$, 6D (1, 0) and 6D (2, 0) SCAs. Fortunately, we need only discuss three different algebras, $\mathfrak{so}(5)$, $\mathfrak{su}(4)$ and $\mathfrak{su}(2)$. We will assume some familiarity with the description of the RS algorithm from appendix B of [39], the notation of which we use.

First, let us consider $\mathfrak{so}(5)$. This is the Lorentz Lie algebra for 5D $\mathcal{N} = 1$ and the R -symmetry Lie algebra for 6D (2, 0). The highest weight (λ) identifications resulting from Weyl reflections (σ) are given by

$$\begin{aligned} [d_1, d_2] &= -[-d_1 - 2, 2d_1 + d_2 + 2], \\ &= -[d_1 + d_2 + 1, -d_2 - 2], \\ &= -[-d_1 - d_2 - 3, d_2], \\ &= -[d_1, -4 - 2d_1 - d_2]. \end{aligned} \quad (\text{C.1})$$

We see that $\lambda^\sigma = \lambda$ with $\text{sign}(\sigma) = -$ under the following conditions:

$$d_1 = -1, \quad d_2 = -1, \quad 2d_1 + d_2 = -3, \quad d_1 + d_2 = -2. \quad (\text{C.2})$$

Therefore representations which satisfy any one of these conditions are labelled by a highest weight on the boundary of the Weyl chamber, in which case they have zero multiplicity and need to be removed. Notice that the conditions (C.2) correspond to the zeros of the Weyl dimension formula for irreducible representations of $\mathfrak{so}(5)$:

$$d(d_1, d_2) = \frac{1}{6}(d_1 + 1)(d_2 + 1)(d_1 + d_2 + 2)(2d_1 + d_2 + 3). \quad (\text{C.3})$$

Next we consider $\mathfrak{su}(4)$, the Lorentz Lie algebra of 6D theories. The highest weight identifications resulting from Weyl reflections are

$$\begin{aligned}
 [c_1, c_2, c_3] &= -[-c_1 - 2, c_1 + c_2 + 1, c_3], \\
 &= -[c_1 + c_2 + 1, -c_2 - 2, c_2 + c_3 + 1], \\
 &= -[c_1, c_2 + c_3 + 1, -c_3 - 2], \\
 &= -[-c_2 - 2, -c_1 - 2, c_1 + c_2 + c_3 + 2], \\
 &= -[c_1 + c_2 + c_3 + 2, -c_3 - 2, -c_2 - 2], \\
 &= -[-c_2 - c_3 - 3, c_2, -c_1 - c_2 - 3].
 \end{aligned} \tag{C.4}$$

The representations we delete are the ones where the following conditions are met:

$$c_1 = -1, \quad c_2 = -1, \quad c_3 = -1, \quad c_1 + c_2 = -2, \quad c_2 + c_3 = -2, \quad c_1 + c_2 + c_3 = -3. \tag{C.5}$$

Again, these correspond to the zeros of the Weyl dimension formula for $\mathfrak{su}(4)$:

$$d(c_1, c_2, c_3) = \frac{1}{12}(c_1+1)(c_2+1)(c_3+1)(c_1+c_2+2)(c_2+c_3+2)(c_1+c_2+c_3+3). \tag{C.6}$$

Lastly, the R -symmetry Lie algebra for the minimally supersymmetric theories is $\mathfrak{su}(2)$, which involves identifying highest weights via the reflection:

$$[K] = -[-K - 2]. \tag{C.7}$$

We therefore see that the representations we delete are the ones where the following condition is met:

$$K = -1. \tag{C.8}$$

Clearly this corresponds to the zero for the Weyl dimension formula of $\mathfrak{su}(2)$

$$d(K) = K + 1. \tag{C.9}$$

We may now simply combine the set of Weyl reflections appropriate for our SCA in order to dictate which states survive when building representations. Our method will involve generating all possible highest weight states, even in cases where the Dynkin labels become negative, and then performing the RS algorithm.

It is interesting to note that after implementing the RS algorithm one often encounters pairs of superconformal descendants with exactly the same Dynkin labels but opposite multiplicities. These cancel out to leave behind a much simpler set of superconformal representations with only positive multiplicities. If after performing this step there are also negative representations that have not been cancelled, they are interpreted as constraints for operators in the multiplet [39].

There are instances when this general procedure leads to ambiguities, i.e. there is more than one choice for performing the cancellations; see also [18]. However, these can be resolved uniquely by the requirement that all physical states should be reached by the successive action of allowed supercharges on the superconformal primary. For the examples that we investigate in this article, this phenomenon only appears in the (2,0) SCA ($\mathcal{B}[c_1, c_2, 0; 0, 0]$, $\mathcal{C}[c_1, 0, 0; 0, 0]$), in which case we have constructed the multiplet spectra by also using successive \mathcal{Q} -actions.

C.1 Operator constraints through Racah-Speiser

We now further explore this concept. Since R -symmetry quantum numbers will not play a role in this analysis we will simplify our discussion by denoting all of their quantum numbers by R . The only distinction we need to make is between the $\mathfrak{so}(5)$ and $\mathfrak{su}(4)$ Lorentz Lie algebras. It will be instructive to proceed by first providing an example and then the result in full generality.

Consider the Lorentz vector representation in 6D, $[\Delta; 0, 1, 0; R]$, corresponding to an operator \mathcal{O}_μ . In terms of quantum numbers, its components are given by

$$\begin{aligned} \mathcal{O}_1 &\sim [0, 1, 0], & \mathcal{O}_2 &\sim [1, -1, 1], & \mathcal{O}_3 &\sim [-1, 0, 1], \\ \mathcal{O}_4 &\sim [1, 0, -1], & \mathcal{O}_5 &\sim [-1, 1, -1], & \mathcal{O}_6 &\sim [0, -1, 0]. \end{aligned} \quad (\text{C.10})$$

One can envisage a conservation-equation state for this object being $-\Delta + 1; 0, 0, 0; R]$, that is we pick components of \mathcal{P}_μ and \mathcal{O}_μ such that their combination has Lorentz quantum numbers $[0, 0, 0]$. This combination is

$$\mathcal{P}_6\mathcal{O}_1 + \mathcal{P}_5\mathcal{O}_2 + \mathcal{P}_4\mathcal{O}_3 + \mathcal{P}_3\mathcal{O}_4 + \mathcal{P}_2\mathcal{O}_5 + \mathcal{P}_1\mathcal{O}_6 = 0, \quad (\text{C.11})$$

which can be concisely interpreted as the conservation equation³⁴

$$\partial^\mu \mathcal{O}_\mu = 0. \quad (\text{C.12})$$

This is precisely what we see e.g. for the R -symmetry currents of the $(1, 0)$ and $(2, 0)$ stress-tensor multiplets, $\mathcal{B}[0, 0, 0; 0]$ and $\mathcal{D}[0, 0, 0; 2, 0]$.

More generally, there are only two ways in which operator constraints manifest themselves in 6D using the approach employed in this paper. These are

$$\begin{aligned} [\Delta; c_1, c_2, c_3; R] &\rightarrow -[\Delta + 1; c_1, c_2 - 1, c_3; R], \\ [\Delta; c_1, 0, 0; R] &\rightarrow -[\Delta + 1; c_1 - 1, 0, 1; R] + [\Delta + 2; c_1 - 2, 1, 0; R] \\ &\quad - [\Delta + 3, c_1 - 2, 0, 0; R]. \end{aligned} \quad (\text{C.13})$$

The first equation is a contraction of a vector index; this is a conservation equation. The other requires that $c_2 = c_3 = 0$. These are the “generalised Dirac equations” or “generalised equations of motion” mentioned in [40]. When $c_1 = 0$ the primary is a scalar $[2; 0, 0, 0; R]$, and the only state that survives the RS algorithm is $-[4; 0, 0, 0; R]$, namely a Klein-Gordon equation. When $c_1 = 1$ the primary is a fermion $[5/2; 1, 0, 0; R]$, with the only state surviving RS being a Dirac equation $-[7/2; 0, 0, 1; R]$. For $c_1 \geq 2$, all states survive RS, leaving a Bianchi identity for a higher p -form field; this is usually endowed with additional constraints, i.e. self-duality in the free-tensor case, $c_1 = 2$.

The analogous states in 5D are given by

$$\begin{aligned} [\Delta; d_1, d_2; R] &\rightarrow -[\Delta + 1; d_1 - 1, d_2; R], \\ [\Delta; 0, d_2; R] &\rightarrow -[\Delta + 1; 0, d_2; R] + [\Delta + 2; 1, d_2 - 2; R] - [\Delta + 3, 0, d_2 - 2; R]. \end{aligned} \quad (\text{C.14})$$

Again, the first expression is the contraction of a vector index and the second expression recovers the Klein-Gordon equation ($d_2 = 0$), the Dirac equation ($d_2 = 1$) and the Bianchi identity ($d_2 \geq 2$).

³⁴One has that $(\mathcal{P}_\mu)^\dagger = \mathcal{P}^\mu$. Then it can be straightforwardly checked that $\mathcal{P}^\mu = \mathcal{P}_{7-\mu}$.

C.2 The dictionary between Racah-Speiser and momentum-null states

The expressions (C.13) and (C.14) can also be understood from a different perspective. When the superconformal primary saturates both a conformal and a superconformal bound, certain components in the multiplet satisfy operator constraints. One is then instructed to remove appropriate \mathcal{P}_μ generators from the auxiliary Verma-module basis [40]. We call the set contained in the resulting module “reduced states”.

Following [40], in 5D $\mathcal{N} = 1$ we are instructed to remove \mathcal{P}_5 for the multiplets $\mathcal{B}[0, 0; 0]$ and $\mathcal{D}[0, 0; 2]$. For $\mathcal{D}[0, 0; 1]$ one removes $\mathcal{P}_3, \mathcal{P}_4$ and \mathcal{P}_5 . In 6D, where $R = K$ for $\mathcal{N} = 1$ and $R = d_1 + d_2$ for $\mathcal{N} = 2$, one removes \mathcal{P}_6 from $\mathcal{B}[c_1, c_2, 0; R = 0]$, $\mathcal{C}[c_1, 0, 0; R = 1]$ and $\mathcal{D}[0, 0, 0; R = 2]$. Likewise we remove $\mathcal{P}_3, \mathcal{P}_5$ and \mathcal{P}_6 from $\mathcal{C}[c_1, 0, 0; R = 0]$ and $\mathcal{D}[0, 0, 0; R = 1]$.

The connection between the momentum-null states and the negative-multiplicity representations is clear when considering the supercharacter: taking the character of all states — including the negative-multiplicity representations — produces the same result as doing so for all reduced states with the appropriate \mathcal{P} s removed from the basis of generators. Hence it is not appropriate to do both. This is reflected in the fact that the following character identity holds in 6D:

$$\hat{\chi}[\Delta; c_1, c_2, c_3; R] \equiv \chi[\Delta; c_1, c_2, c_3; R] - \chi[\Delta + 1; c_1, c_2 - 1, c_3; R], \quad (\text{C.15})$$

where a hat denotes using the P -polynomial from (B.36) after having removed \mathcal{P}_6 ; the operator constraints that one recovers in this fashion are conservation equations. Likewise when we remove $\mathcal{P}_3, \mathcal{P}_5$ and \mathcal{P}_6 we obtain:

$$\begin{aligned} \hat{\chi}[\Delta; c_1, 0, 0; R] &\equiv \chi[\Delta; c_1, 0, 0; R] - \chi[\Delta + 1; c_1 - 1, 0, 1; R] + \chi[\Delta + 2; c_1 - 2, 1, 0; R] \\ &\quad - \chi[\Delta + 3; c_1 - 2, 0, 0; R]. \end{aligned} \quad (\text{C.16})$$

The operator constraints recovered in this case are equations of motion.

In 5D one can make the analogous identifications:

$$\hat{\chi}[\Delta; d_1, d_2; R] \equiv \chi[\Delta; d_1, d_2; R] - \chi[\Delta + 1; d_1 - 1, d_2; R], \quad (\text{C.17})$$

where we have removed \mathcal{P}_5 from $\hat{\chi}$. Likewise, when we remove $\mathcal{P}_3, \mathcal{P}_4$ and \mathcal{P}_5 we obtain:

$$\begin{aligned} \hat{\chi}[\Delta; 0, d_2; R] &\equiv \chi[\Delta; 0, d_2; R] - \chi[\Delta + 1; 0, d_2; R] + \chi[\Delta + 2; 1, d_2 - 2; R] \\ &\quad - \chi[\Delta + 3; 0, d_2 - 2; R]. \end{aligned} \quad (\text{C.18})$$

This effectively endows us with a choice for how to treat the operator constraints: we can either take the character of all *reduced* states — not including states associated with constraints — with the appropriate \mathcal{P}_μ removed, or we can take the character of all states (including negative-multiplicity representations) without removing any \mathcal{P}_μ .

Note that if one has to remove momenta from the basis of generators for a given multiplet then not every component contained within need have an associated operator constraint. The stress-tensor multiplets in 6D $\mathcal{B}[0, 0, 0; 0]$ and $\mathcal{D}[0, 0, 0; 2, 0]$ are examples

of such superconformal representations: they contain superconformal descendants which do not contain operators satisfying constraints. However, according to the arguments of the previous subsection, we are still instructed to remove \mathcal{P}_6 in one approach of evaluating the supercharacter. At first glance this is perplexing.

The resolution to this small puzzle is that the associated characters are actually invariant under the removal of \mathcal{P}_6 . Hence such states satisfy

$$\hat{\chi}[\Delta; c_1, c_2, c_3; R] = \chi[\Delta; c_1, c_2, c_3; R], \tag{C.19}$$

in 6D, or in 5D

$$\hat{\chi}[\Delta; d_1, d_2; R] = \chi[\Delta; d_1, d_2; R]. \tag{C.20}$$

The method for evaluating characters of conformal UIRs presented in [40] makes no distinction between states that do/do not obey constraints due to the above invariance.

This knowledge is useful when the absent generators of the Verma module are such that the removed \mathcal{Q} s anticommute into \mathcal{P} s; cf. section 2.7.1, section 4.6.1 and section 5.6.1. In that case, one is effectively projecting out the states associated with operator constraints from the very beginning. Therefore what is generated under these circumstances is the spectrum of *reduced* states. It is still possible to reconstruct the full multiplet spectrum using character relations, as e.g. in (C.15). This lets us recover all the negative-multiplicity states.

D Summary of superconformal indices

In this appendix we list the superconformal indices for multiplets in 5D $\mathcal{N} = 1$ and 6D (1, 0), alongside the Schur limit of the indices for multiplets in 6D (2, 0). All the indices, including the fully refined indices for the (2,0) multiplets, are given in the accompanying Mathematica file. We use the $\mathfrak{su}(2)$ character for the spin- $\frac{1}{2}$ representation in our expressions:

$$\chi_l(y) = \frac{y^{l+1} - y^{-l-1}}{y - y^{-1}}. \tag{D.1}$$

D.1 5D $\mathcal{N} = 1$

The distinct indices are provided below:

- $\mathcal{A}[d_1, d_2; K]$ (for $d_1, d_2 \geq 0, K \geq 0$):

$$\mathcal{I}(x, y) = (-1)^{d_2+1} \frac{x^{d_1+d_2+K+4}}{(1-xy^{-1})(1-xy)} \chi_{d_1}(y). \tag{D.2}$$

- $\mathcal{B}[d_1, 0; K]$ (for $d_1 \geq 0, K \geq 0$):

$$\mathcal{I}(x, y) = \frac{x^{d_1+K+3}}{(1-xy^{-1})(1-xy)} \chi_{d_1+1}(y). \tag{D.3}$$

- $\mathcal{D}[0, 0; K]$ (for $K \geq 1$):

$$\mathcal{I}(x, y) = \frac{x^K}{(1-xy^{-1})(1-xy)}. \tag{D.4}$$

D.2 6D (1, 0)

The distinct indices are provided below:

- $\mathcal{A}[c_1, c_2, c_3; K]$ (for $c_1, c_2, c_3 \geq 0, K \geq 0$):

$$\mathcal{I}(p, q, s) = (-1)^{c_1+c_3} q^{6+\frac{3K}{2}+\frac{1}{2}(c_1+2c_2+3c_3)} \left\{ \frac{p^{-c_2+1} s^{c_2+4} (p^{-c_1} - p s^{c_1+1}) - p^{-c_1+2} s^{2-c_2}}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(p-qs)} + \frac{p^{c_2+4} s^{c_1+4} + p^{c_1+4} s^{-c_1+1} (s^{-c_2} - s p^{c_2+1})}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(p-qs)} \right\}. \quad (\text{D.5})$$

- $\mathcal{B}[c_1, c_2, 0; K]$ (for $c_1, c_2 \geq 0, K \geq 0$):

$$\mathcal{I}(p, q, s) = (-1)^{c_1+1} q^{4+\frac{3K}{2}+\frac{1}{2}(c_1+2c_2)} \left\{ \frac{p^{-c_2} s^{c_2+5} (p^{-c_1} - p s^{c_1+1}) - p^{-c_1+2} s^{1-c_2}}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(p-qs)} + \frac{p^{c_2+5} s^{c_1+4} + p^{c_1+4} s^{-c_1} (s^{-c_2} - s^2 p^{c_2+2})}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(p-qs)} \right\}. \quad (\text{D.6})$$

- $\mathcal{C}[c_1, 0, 0; K]$ (for $c_1 \geq 0, K \geq 1$):

$$\mathcal{I}(p, q, s) = (-1)^{c_1} q^{2+\frac{3K}{2}+\frac{c_1}{2}} \frac{p^{c_1+5} s^{-c_1} (ps-1) + p^2 s^{c_1+5} (s-p^2) + p^{-c_1} s^2 (p-s^2)}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(qs-p)}. \quad (\text{D.7})$$

- $\mathcal{C}[c_1, 0, 0; 0]$ (for $c_1 \geq 0$):

$$\mathcal{I}(p, q, s) = (-1)^{c_1} q^{2+\frac{c_1}{2}} \left\{ \frac{p^{c_1+6} s^{1-c_1} - p^{c_1+5} s^{-c_1} + s^2 p^{-c_1} (p-s^2) - p^4 s^{c_1+5} + p^2 s^{c_1+6}}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(qs-p)} + \frac{p^{c_1+5} s^{-c_1} (ps-1) + p^2 s^{c_1+5} (s-p^2) + p^{-c_1} s^2 (p-s^2)}{(pq-1)(p^2-s)(ps-1)(p-s^2)(q-s)(qs-p)} \right\}. \quad (\text{D.8})$$

- $\mathcal{D}[0, 0, 0; K]$ (for $K \geq 1$):

$$\mathcal{I}(p, q, s) = \frac{q^{\frac{3K}{2}}}{(1-pq)(1-qs^{-1})(1-p^{-1}qs)}. \quad (\text{D.9})$$

D.3 6D (2, 0)

The non-vanishing Schur indices are valid for all c_i and d_1 values. These are provided below:

- $\mathcal{B}[c_1, c_2, 0; d_1, 0]$:

$$\mathcal{I}^{\text{Schur}}(q, s) = (-1)^{c_1} \frac{q^{4+d_1+\frac{c_1}{2}+c_2}}{1-q} \chi_{c_1}(s). \quad (\text{D.10})$$

- $\mathcal{C}[c_1, 0, 0; d_1, 1]$:

$$\mathcal{I}^{\text{Schur}}(q, s) = (-1)^{c_1+1} \frac{q^{\frac{7}{2}+d_1+\frac{c_1}{2}}}{1-q} \chi_{c_1+1}(s). \quad (\text{D.11})$$

- $\mathcal{C}[c_1, 0, 0; d_1, 0]$:

$$\mathcal{I}^{\text{Schur}}(q, s) = (-1)^{c_1} \frac{q^{2+d_1+\frac{c_1}{2}}}{1-q} \chi_{c_1+2}(s). \quad (\text{D.12})$$

- $\mathcal{D}[0, 0, 0; d_1, 2]$:

$$\mathcal{I}^{\text{Schur}}(q, s) = \frac{q^{3+d_1}}{1-q}. \quad (\text{D.13})$$

- $\mathcal{D}[0, 0, 0; d_1, 1]$:

$$\mathcal{I}^{\text{Schur}}(q, s) = -\frac{q^{\frac{3}{2}+d_1}}{1-q} \chi_1(s). \quad (\text{D.14})$$

- $\mathcal{D}[0, 0, 0; d_1, 0]$:

$$\mathcal{I}^{\text{Schur}}(q, s) = \frac{q^{d_1}}{1-q}. \quad (\text{D.15})$$

The refined versions of the above indices, $\mathcal{I}(p, q, s, t)$, are provided in the accompanying Mathematica notebook.

E 6D (2, 0) spectra

We will finally provide the spectra for short multiplets in 6D (2,0) with generic quantum numbers. Since long representations are cumbersome and standard to obtain we will not list them here. Moreover, we will only write down *distinct* spectra, that cannot be obtained by fixing c_i or d_i and performing the RS algorithm. For book-keeping purposes we have grouped the supercharges into $Q = (\mathcal{Q}_{2a}, \mathcal{Q}_{3a})$ and $\tilde{Q} = (\mathcal{Q}_{1a}, \mathcal{Q}_{4a})$, with $a = 1, \dots, 4$.

E.1 \mathcal{A} -type multiplets

The superconformal primary null-state condition for a generic \mathcal{A} -type multiplet is $A^4\Psi = 0$. This corresponds to removing \mathcal{Q}_{14} from the basis of auxiliary Verma-module generators (5.1). Starting from a superconformal primary given by $[c_1, c_2, c_3; d_1, d_2]$ with conformal dimension $\Delta = 6 + 2(d_1 + d_2) + \frac{1}{2}(c_1 + 2c_2 + 3c_3)$, we will provide the states at each level l ; the conformal dimension of the superconformal descendants will be equal to $\Delta + \frac{l}{2}$. Thus the spectrum for $\mathcal{A}[c_1, c_2, c_3; d_1, d_2]$ is given by acting with all Q s and \tilde{Q} s, resulting in the two chains, table 4 and table 5 respectively. Note that this is a complete description for generic d_1 and d_2 . The reason being that, as described in section 5.5, once we have $\mathcal{A}[c_1, c_2, c_3; d_1, d_2]$ we can obtain the spectra for $\mathcal{A}[c_1, c_2, 0; d_1, d_2]$, $\mathcal{A}[c_1, 0, 0; d_1, d_2]$ and $\mathcal{A}[0, 0, 0; d_1, d_2]$ by dialling c_i appropriately and running the RS algorithm.

It turns out that the same also holds for dialling d_1 and d_2 to zero. Thus the complete spectra for all \mathcal{A} -type multiplets can be obtained exclusively by considering the set of Q and \tilde{Q} actions (table 4 and table 5) and then substituting in the desired quantum numbers followed by implementing the RS algorithm.

E.2 \mathcal{B} -type multiplets

For generic quantum numbers, the superconformal primary for this multiplet type obeys the null-state condition $A^3\Psi = 0$. This corresponds to removing \mathcal{Q}_{13} and \mathcal{Q}_{14} from the auxiliary Verma-module basis (5.1). Starting from a generic primary $[c_1, c_2, c_3; d_1, d_2]$ with conformal dimension $\Delta = 4 + 2(d_1 + d_2) + \frac{1}{2}(c_1 + 2c_2)$, the action of the \tilde{Q} supercharges remains the same as in table 5, with the difference that one needs to apply RS after setting $c_3 = 0$. The Q -chain, however, is significantly shorter and given in table 6.

E.2.1 $\mathcal{B}[c_1, c_2, 0; d_1, 1]$

In this case the action of the Q supercharges is the same as in table 6. However, recall from section 5.6 that we also have the null-state condition $(\mathcal{R}_2^-)^2 A^3 A^4 \Psi = 0$. Thus we are also prescribed to remove $\mathcal{Q}_{23} \mathcal{Q}_{24}$ from the \tilde{Q} -spectrum. The action of the \tilde{Q} supercharges is therefore adjusted and described in table 7.

E.2.2 $\mathcal{B}[c_1, c_2, 0; d_1, 0]$

For this case the action of the Q supercharges is the same as in table 6, after replacing $d_2 = 0$ and running the RS algorithm. We are also prescribed to remove \mathcal{Q}_{23} and \mathcal{Q}_{24} from the \tilde{Q} -spectrum. The action of the \tilde{Q} supercharges is therefore adjusted and described by table 8.

E.3 \mathcal{C} -type multiplets

For generic quantum numbers the superconformal primary for this multiplet obeys the null-state condition $A^2\Psi = 0$. This corresponds to removing \mathcal{Q}_{12} , \mathcal{Q}_{13} and \mathcal{Q}_{14} from the basis of auxiliary Verma-module generators. Starting from a generic primary $[c_1, c_2, c_3; d_1, d_2]$ with conformal dimension $\Delta = 2 + 2(d_1 + d_2) + \frac{c_1}{2}$, the action of the \tilde{Q} supercharges remains the same as in table 5, with the difference that one needs to apply RS after setting $c_2 = c_3 = 0$. The action of the Q supercharges is given in table 9.

E.3.1 $\mathcal{C}[c_1, 0, 0; d_1, 2]$

Recall that in this case we should also remove $\mathcal{Q}_{21} \mathcal{Q}_{22} \mathcal{Q}_{23}$ from the basis of auxiliary Verma-module generators. The resulting set of \tilde{Q} actions are given in table 10.

E.3.2 $\mathcal{C}[c_1, 0, 0; d_1, 1]$

For this multiplet we also need to remove $\mathcal{Q}_{2a} \mathcal{Q}_{2b}$ for $a \neq b \neq 1$. The resulting set of \tilde{Q} actions are given in table 11.

E.3.3 $\mathcal{C}[c_1, 0, 0; d_1, 0]$

For this multiplet we also need to remove \mathcal{Q}_{2a} for $a \neq 1$. The resulting set of \tilde{Q} actions are given in table 12.

E.3.4 $\mathcal{C}[c_1, 0, 0; 1, 0]$

This multiplet has the same Q and \tilde{Q} actions as $\mathcal{C}[c_1, 0, 0; d_1, 0]$ with the difference that one needs to apply RS after setting $d_1 = 1$. This multiplet contains generalised conservation equations and is small enough for us to detail its entire spectrum in table 13.

E.3.5 $\mathcal{C}[c_1, 0, 0; 0, 1]$

For this case we can simply substitute $d_1 = 0$ into the spectrum of $\mathcal{C}[c_1, 0, 0; d_1, 1]$ and run the RS algorithm to get the spectrum of table 14. This multiplet also contains generalised conservation equations.

E.4 \mathcal{D} -type multiplets

Since the action of the Q supercharges have been given in section 5.8, we need only provide the \tilde{Q} chains for specific d_2 values.

E.4.1 $\mathcal{D}[0, 0, 0; d_1, 3]$

Recall that in this case we remove $Q_{21}Q_{22}Q_{23}Q_{24}$. We summarise the actions of the \tilde{Q} supercharges in table 15.

E.4.2 $\mathcal{D}[0, 0, 0; d_1, 2]$

For this multiplet we remove $Q_{2a}Q_{2b}Q_{2c}$ from the basis of auxiliary Verma-module generators. We summarise the actions of the \tilde{Q} supercharges in table 16.

Q actions for $\mathcal{A}[c_1, c_2, c_3; d_1, d_2]$
Level: 0
$[c_1, c_2, c_3; d_1, d_2]$
Level: 1
$[c_1 + 1, c_2, c_3; d_1, d_2 + 1], [c_1 + 1, c_2, c_3; d_1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3; d_1, d_2 + 1], [c_1 - 1, c_2 + 1, c_3; d_1, d_2 - 1],$ $[c_1, c_2 - 1, c_3 + 1; d_1, d_2 + 1], [c_1, c_2 - 1, c_3 + 1; d_1, d_2 - 1], [c_1, c_2, c_3 - 1; d_1, d_2 - 1]$
Level: 2
$[c_1 + 2, c_2, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1, d_2 + 2], 2 [c_1, c_2 + 1, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1, d_2 - 2],$ $[c_1 - 2, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2 + 2], 2 [c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2], [c_1 - 1, c_2, c_3 + 1; d_1, d_2 + 2],$ $2 [c_1 - 1, c_2, c_3 + 1; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2 - 2], [c_1 - 1, c_2, c_3 + 1; d_1, d_2 - 2], [c_1, c_2 - 2, c_3 + 2; d_1, d_2],$ $[c_1 + 1, c_2, c_3 - 1; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1, d_2 - 2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2 - 2],$ $[c_1, c_2 - 1, c_3; d_1, d_2], [c_1, c_2 - 1, c_3; d_1, d_2 - 2]$
Level: 3
$[c_1 + 1, c_2 + 1, c_3; d_1, d_2 + 1], [c_1 + 1, c_2 + 1, c_3; d_1, d_2 - 1], [c_1 - 1, c_2 + 2, c_3; d_1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3; d_1, d_2 - 1],$ $[c_1 + 2, c_2 - 1, c_3 + 1; d_1, d_2 + 1], [c_1, c_2, c_3 + 1; d_1, d_2 + 3], 3 [c_1, c_2, c_3 + 1; d_1, d_2 + 1], 3 [c_1, c_2, c_3 + 1; d_1, d_2 - 1],$ $[c_1 - 2, c_2 + 1, c_3 + 1; d_1, d_2 + 1], [c_1 + 2, c_2 - 1, c_3 + 1; d_1, d_2 - 1], [c_1, c_2, c_3 + 1; d_1, d_2 - 3], [c_1 - 2, c_2 + 1, c_3 + 1; d_1, d_2 - 1],$ $[c_1 + 1, c_2 - 2, c_3 + 2; d_1, d_2 + 1], [c_1 + 1, c_2 - 2, c_3 + 2; d_1, d_2 - 1], [c_1 - 1, c_2 - 1, c_3 + 2; d_1, d_2 + 1], [c_1, c_2 - 2, c_3 + 1; d_1, d_2 - 1]$ $[c_1 + 2, c_2, c_3 - 1; d_1, d_2 - 1], [c_1, c_2 + 1, c_3 - 1; d_1, d_2 + 1], 2 [c_1, c_2 + 1, c_3 - 1; d_1, d_2 - 1], [c_1, c_2 + 1, c_3 - 1; d_1, d_2 - 3],$ $[c_1 - 2, c_2 + 2, c_3 - 1; d_1, d_2 - 1], [c_1 + 1, c_2 - 1, c_3; d_1, d_2 + 1], 2 [c_1 + 1, c_2 - 1, c_3; d_1, d_2 - 1], [c_1 - 1, c_2, c_3; d_1, d_2 + 1],$ $2 [c_1 - 1, c_2, c_3; d_1, d_2 - 1], [c_1 + 1, c_2 - 1, c_3; d_1, d_2 - 3], [c_1 - 1, c_2, c_3; d_1, d_2 - 3], [c_1 - 1, c_2 - 1, c_3 + 2; d_1, d_2 - 1],$
Level: 4
$[c_1, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2, c_3 + 1; d_1, d_2 + 2], 2 [c_1 + 1, c_2, c_3 + 1; d_1, d_2], [c_1 - 1, c_2 + 1, c_3 + 1; d_1, d_2 + 2],$ $2 [c_1 - 1, c_2 + 1, c_3 + 1; d_1, d_2], [c_1 + 1, c_2, c_3 + 1; d_1, d_2 - 2], [c_1 - 1, c_2 + 1, c_3 + 1; d_1, d_2 - 2], [c_1 + 2, c_2 - 2, c_3 + 2; d_1, d_2],$ $[c_1, c_2 - 1, c_3 + 2; d_1, d_2 + 2], 2 [c_1, c_2 - 1, c_3 + 2; d_1, d_2], [c_1, c_2 - 1, c_3 + 2; d_1, d_2 - 2], [c_1 - 2, c_2, c_3 + 2; d_1, d_2],$ $[c_1 + 1, c_2 + 1, c_3 - 1; d_1, d_2], [c_1 + 1, c_2 + 1, c_3 - 1; d_1, d_2 - 2], [c_1 - 1, c_2 + 2, c_3 - 1; d_1, d_2], [c_1 - 1, c_2 + 2, c_3 - 1; d_1, d_2 - 2],$ $[c_1 + 2, c_2 - 1, c_3; d_1, d_2], [c_1, c_2, c_3; d_1, d_2 + 2], 3 [c_1, c_2, c_3; d_1, d_2], 3 [c_1, c_2, c_3; d_1, d_2 - 2],$ $[c_1 - 2, c_2 + 1, c_3; d_1, d_2], [c_1 + 2, c_2 - 1, c_3; d_1, d_2 - 2], [c_1, c_2, c_3; d_1, d_2 - 4], [c_1 - 2, c_2 + 1, c_3; d_1, d_2 - 2],$ $[c_1 + 1, c_2 - 2, c_3 + 1; d_1, d_2], [c_1 + 1, c_2 - 2, c_3 + 1; d_1, d_2 - 2], [c_1 - 1, c_2 - 1, c_3 + 1; d_1, d_2], [c_1 - 1, c_2 - 1, c_3 + 1; d_1, d_2 - 2]$
Level: 5
$[c_1, c_2 + 1, c_3 + 1; d_1, d_2 + 1], [c_1, c_2 + 1, c_3 + 1; d_1, d_2 - 1], [c_1 + 1, c_2 - 1, c_3 + 2; d_1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3 + 2; d_1, d_2 - 1],$ $[c_1 - 1, c_2, c_3 + 2; d_1, d_2 + 1], [c_1 - 1, c_2, c_3 + 2; d_1, d_2 - 1], [c_1, c_2 + 2, c_3 - 1; d_1, d_2 - 1], [c_1 + 1, c_2, c_3; d_1, d_2 + 1],$ $2 [c_1 + 1, c_2, c_3; d_1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3; d_1, d_2 + 1], 2 [c_1 - 1, c_2 + 1, c_3; d_1, d_2 - 1], [c_1 + 1, c_2, c_3; d_1, d_2 - 3],$ $[c_1 - 1, c_2 + 1, c_3; d_1, d_2 - 3], [c_1 + 2, c_2 - 2, c_3 + 1; d_1, d_2 - 1], [c_1, c_2 - 1, c_3 + 1; d_1, d_2 + 1], 2 [c_1, c_2 - 1, c_3 + 1; d_1, d_2 - 1],$ $[c_1, c_2 - 1, c_3 + 1; d_1, d_2 - 3], [c_1 - 2, c_2, c_3 + 1; d_1, d_2 - 1]$
Level: 6
$[c_1, c_2, c_3 + 2; d_1, d_2], [c_1, c_2 + 1, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1, d_2 - 2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2],$ $[c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2 - 2], [c_1 - 1, c_2, c_3 + 1; d_1, d_2], [c_1 - 1, c_2, c_3 + 1; d_1, d_2 - 2]$
Level: 7
$[c_1, c_2, c_3 + 1; d_1, d_2 - 1]$

Table 4. The spectrum of $\mathcal{A}[c_1, c_2, c_3; d_1, d_2]$ associated with the action of Q s.

\tilde{Q} actions for $\mathcal{A}[c_1, c_2, c_3; d_1, d_2]$
Level: 0
$[c_1, c_2, c_3; d_1, d_2]$
Level: 1
$[c_1 + 1, c_2, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1],$ $[c_1, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1, c_2, c_3 - 1; d_1 - 1, d_2 + 1]$
Level: 2
$[c_1 + 2, c_2, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 + 2, d_2 - 2], 2[c_1, c_2 + 1, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 - 2, d_2 + 2],$ $[c_1 - 2, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2], [c_1 - 1, c_2, c_3 + 1; d_1 + 2, d_2 - 2],$ $2[c_1 - 1, c_2, c_3 + 1; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1, c_2 - 2, c_3 + 2; d_1, d_2],$ $[c_1 + 1, c_2, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2, c_3 - 1; d_1, d_2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2],$ $[c_1, c_2 - 1, c_3; d_1 + 2, d_2 - 2], 2[c_1, c_2 - 1, c_3; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2],$ $[c_1, c_2 - 1, c_3; d_1 - 2, d_2 + 2], [c_1, c_2, c_3 - 2; d_1, d_2]$
Level: 3
$[c_1 + 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 2, c_3; d_1 - 1, d_2 + 1],$ $[c_1 + 2, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2, c_3 + 1; d_1 + 3, d_2 - 3], 3[c_1, c_2, c_3 + 1; d_1 + 1, d_2 - 1], 3[c_1, c_2, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1 - 2, c_2 + 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1 + 2, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 + 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2 + 1, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1 + 1, c_2 - 2, c_3 + 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 2, c_3 + 2; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 - 1, c_3 + 2; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 - 1, c_3 + 2; d_1 - 1, d_2 + 1],$ $[c_1, c_2 - 1, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 + 3, d_2 - 3], 3[c_1, c_2 + 1, c_3 - 1; d_1 + 1, d_2 - 1], 3[c_1, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1 - 2, c_2 + 2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 1, c_3; d_1 + 3, d_2 - 3], 3[c_1 - 1, c_2, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2, c_3; d_1 + 3, d_2 - 3],$ $3[c_1 - 1, c_2, c_3; d_1 + 1, d_2 - 1], 3[c_1 + 1, c_2 - 1, c_3; d_1 - 1, d_2 + 1], 3[c_1 - 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1, c_2 - 2, c_3 + 1; d_1 + 1, d_2 - 1],$ $[c_1 + 2, c_2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2 + 2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3; d_1 - 3, d_2 + 3],$ $[c_1 - 1, c_2, c_3; d_1 - 3, d_2 + 3], [c_1, c_2 - 2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2, c_3 - 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2, c_3 - 2; d_1 - 1, d_2 + 1],$ $[c_1 - 1, c_2 + 1, c_3 - 2; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3 - 2; d_1 - 1, d_2 + 1], [c_1, c_2 - 1, c_3 - 1; d_1 + 1, d_2 - 1],$ $[c_1 + 2, c_2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 - 1, c_3 + 2; d_1 - 1, d_2 + 1],$
Level: 4
$[c_1, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2, c_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2, c_3 + 1; d_1, d_2], [c_1 - 1, c_2 + 1, c_3 + 1; d_1 + 2, d_2 - 2],$ $2[c_1 - 1, c_2 + 1, c_3 + 1; d_1, d_2], [c_1 + 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 + 2, c_2 - 2, c_3 + 2; d_1, d_2],$ $[c_1, c_2 - 1, c_3 + 2; d_1 + 2, d_2 - 2], 2[c_1, c_2 - 1, c_3 + 2; d_1, d_2], [c_1, c_2 - 1, c_3 + 2; d_1 - 2, d_2 + 2], [c_1 - 2, c_2, c_3 + 2; d_1, d_2],$ $[c_1 + 1, c_2 + 1, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 + 1, c_3 - 1; d_1, d_2], a[c_1 - 1, c_2 + 2, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 - 1, c_2 + 2, c_3 - 1; d_1, d_2],$ $[c_1 + 2, c_2 - 1, c_3; d_1 + 2, d_2 - 2], [c_1, c_2, c_3; d_1 + 4, d_2 - 4], 4[c_1, c_2, c_3; d_1 + 2, d_2 - 2], 6[c_1, c_2, c_3; d_1, d_2],$ $[c_1 - 2, c_2 + 1, c_3; d_1 + 2, d_2 - 2], 2[c_1 + 2, c_2 - 1, c_3; d_1, d_2], 4[c_1, c_2, c_3; d_1 - 2, d_2 + 2], 2[c_1 - 2, c_2 + 1, c_3; d_1, d_2],$ $[c_1 + 1, c_2 - 2, c_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 - 2, c_3 + 1; d_1, d_2], [c_1 - 1, c_2 - 1, c_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 - 1, c_2 - 1, c_3 + 1; d_1, d_2],$ $[c_1 + 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 + 2, c_2 - 1, c_3; d_1 - 2, d_2 + 2], [c_1, c_2, c_3; d_1 - 4, d_2 + 4],$ $[c_1 - 2, c_2 + 1, c_3; d_1 - 2, d_2 + 2], [c_1 + 1, c_2 - 2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2],$ $[c_1 + 2, c_2, c_3 - 2; d_1, d_2], [c_1, c_2 + 1, c_3 - 2; d_1 + 2, d_2 - 2], 2[c_1, c_2 + 1, c_3 - 2; d_1, d_2], [c_1, c_2 + 1, c_3 - 2; d_1 - 2, d_2 + 2],$ $[c_1 - 2, c_2, c_3 - 2; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 - 1, c_3 - 1; d_1, d_2], [c_1 - 1, c_2, c_3 - 1; d_1 + 2, d_2 - 2],$ $2[c_1 - 1, c_2, c_3 - 1; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1, c_2 - 2, c_3; d_1, d_2]$
Level: 5
$[c_1, c_2 + 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2 + 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3 + 2; d_1 + 1, d_2 - 1],$ $[c_1 - 1, c_2, c_3 + 2; d_1 + 1, d_2 - 1], [c_1 - 1, c_2, c_3 + 2; d_1 - 1, d_2 + 1], [c_1, c_2 + 2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1 + 1, c_2, c_3; d_1 + 3, d_2 - 3],$ $3[c_1 + 1, c_2, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3; d_1 + 3, d_2 - 3], 3[c_1 - 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], 3[c_1 + 1, c_2, c_3; d_1 - 1, d_2 + 1],$ $3[c_1 - 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1 + 2, c_2 - 2, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2 - 1, c_3 + 1; d_1 + 3, d_2 - 3], 3[c_1, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1],$ $3[c_1, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 - 2, c_2, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2 + 2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2, c_3; d_1 - 3, d_2 + 3],$ $[c_1 - 1, c_2 + 1, c_3; d_1 - 3, d_2 + 3], [c_1 + 2, c_2 - 2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2 - 1, c_3 + 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1 + 1, c_2 + 1, c_3 - 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 + 1, c_3 - 2; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3 - 2; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 2, c_3 - 2; d_1 - 1, d_2 + 1],$ $[c_1 + 2, c_2 - 1, c_3 - 1; d_1 + 1, d_2 - 1], [c_1, c_2, c_3 - 1; d_1 + 3, d_2 - 3], 3[c_1, c_2, c_3 - 1; d_1 + 1, d_2 - 1], 3[c_1, c_2, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1 - 2, c_2 + 1, c_3 - 1; d_1 + 1, d_2 - 1], [c_1 + 2, c_2 - 1, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 - 1; d_1 - 3, d_2 + 3],$ $[c_1 + 1, c_2 - 2, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 2, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 - 1, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 - 1, c_3; d_1 - 1, d_2 + 1],$ $[c_1 - 1, c_2 + 2, c_3 - 2; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3 + 2; d_1 - 1, d_2 + 1], [c_1 - 2, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1],$
Level: 6
$[c_1, c_2, c_3 + 2; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 + 2, d_2 - 2], 2[c_1, c_2 + 1, c_3; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 + 2, d_2 - 2],$ $2[c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2], [c_1 - 1, c_2, c_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 - 1, c_2, c_3 + 1; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 - 2, d_2 + 2],$ $[c_1 + 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1, c_2 + 2, c_3 - 2; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1 + 2, d_2 - 2],$ $2[c_1 + 1, c_2, c_3 - 1; d_1, d_2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2],$ $[c_1 - 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 + 2, c_2 - 2, c_3; d_1, d_2], [c_1, c_2 - 1, c_3; d_1 + 2, d_2 - 2], 2[c_1, c_2 - 1, c_3; d_1, d_2],$ $[c_1, c_2 - 1, c_3; d_1 - 2, d_2 + 2], [c_1 - 2, c_2, c_3; d_1, d_2]$
Level: 7
$[c_1, c_2, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 + 1, d_2 - 1], [c_1, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1 + 1, c_2 - 1, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 1, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2, c_3; d_1 - 1, d_2 + 1]$
Level: 8
$[c_1, c_2, c_3; d_1, d_2]$

Table 5. The spectrum of $\mathcal{A}[c_1, c_2, c_3; d_1, d_2]$ associated with the action of \tilde{Q} s.

Q actions for $\mathcal{B}[c_1, c_2, 0; d_1, d_2]$
Level: 0
$[c_1, c_2, c_3; d_1, d_2]$
Level: 1
$[c_1 + 1, c_2, c_3; d_1, d_2 + 1], [c_1 + 1, c_2, c_3; d_1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3; d_1, d_2 + 1], [c_1 - 1, c_2 + 1, c_3; d_1, d_2 - 1],$ $[c_1, c_2 - 1, c_3 + 1; d_1, d_2 - 1], [c_1, c_2, c_3 - 1; d_1, d_2 - 1]$
Level: 2
$[c_1 + 2, c_2, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1, d_2 + 2], 2[c_1, c_2 + 1, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1, d_2 - 2], [c_1, c_2 - 1, c_3; d_1, d_2 - 2],$ $[c_1 - 2, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2 - 2], [c_1 - 1, c_2, c_3 + 1; d_1, d_2],$ $[c_1 - 1, c_2, c_3 + 1; d_1, d_2 - 2], [c_1 + 1, c_2, c_3 - 1; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1, d_2 - 2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2],$ $[c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2 - 2]$
Level: 3
$[c_1 + 1, c_2 + 1, c_3; d_1, d_2 + 1], [c_1 + 1, c_2 + 1, c_3; d_1, d_2 - 1], [c_1 - 1, c_2 + 2, c_3; d_1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3; d_1, d_2 - 1],$ $[c_1, c_2, c_3 + 1; d_1, d_2 + 1], 2[c_1, c_2, c_3 + 1; d_1, d_2 - 1], [c_1, c_2, c_3 + 1; d_1, d_2 - 3], [c_1 - 2, c_2 + 1, c_3 + 1; d_1, d_2 - 1],$ $[c_1 + 2, c_2, c_3 - 1; d_1, d_2 - 1], [c_1, c_2 + 1, c_3 - 1; d_1, d_2 + 1], 2[c_1, c_2 + 1, c_3 - 1; d_1, d_2 - 1], [c_1, c_2 + 1, c_3 - 1; d_1, d_2 - 3],$ $[c_1 + 1, c_2 - 1, c_3; d_1, d_2 - 1], [c_1 + 1, c_2 - 1, c_3; d_1, d_2 - 3], [c_1 - 1, c_2, c_3; d_1, d_2 - 1], [c_1 - 1, c_2, c_3; d_1, d_2 - 3],$ $[c_1 + 2, c_2 - 1, c_3 + 1; d_1, d_2 - 1], [c_1 - 2, c_2 + 2, c_3 - 1; d_1, d_2 - 1]$
Level: 4
$[c_1, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2, c_3 + 1; d_1, d_2], [c_1 + 1, c_2, c_3 + 1; d_1, d_2 - 2], [c_1 - 1, c_2 + 1, c_3 + 1; d_1, d_2], [c_1, c_2, c_3; d_1, d_2 - 4],$ $[c_1 - 1, c_2 + 1, c_3 + 1; d_1, d_2 - 2], [c_1 + 1, c_2 + 1, c_3 - 1; d_1, d_2], [c_1 + 1, c_2 + 1, c_3 - 1; d_1, d_2 - 2], [c_1 - 1, c_2 + 2, c_3 - 1; d_1, d_2],$ $[c_1 - 1, c_2 + 2, c_3 - 1; d_1, d_2 - 2], [c_1 + 2, c_2 - 1, c_3; d_1, d_2 - 2], [c_1, c_2, c_3; d_1, d_2], 2[c_1, c_2, c_3; d_1, d_2 - 2],$ $[c_1 - 2, c_2 + 1, c_3; d_1, d_2 - 2]$
Level: 5
$[c_1, c_2 + 1, c_3 + 1; d_1, d_2 - 1], [c_1, c_2 + 2, c_3 - 1; d_1, d_2 - 1], [c_1 + 1, c_2, c_3; d_1, d_2 - 1], [c_1 + 1, c_2, c_3; d_1, d_2 - 3],$ $[c_1 - 1, c_2 + 1, c_3; d_1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3; d_1, d_2 - 3]$
Level: 6
$[c_1, c_2 + 1, c_3; d_1, d_2 - 2]$

Table 6. The spectrum of $\mathcal{B}[c_1, c_2, 0; d_1, d_2]$ associated with the action of Q s.

\tilde{Q} actions for $\mathcal{B}[c_1, c_2, 0; d_1, 1]$
Level: 0
$[c_1, c_2, 0; d_1, 1]$
Level: 1
$[c_1 + 1, c_2, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1],$ $[c_1, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1, c_2, c_3 - 1; d_1 - 1, d_2 + 1]$
Level: 2
$[c_1 + 2, c_2, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 + 2, d_2 - 2], 2[c_1, c_2 + 1, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 - 2, d_2 + 2],$ $[c_1 - 2, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2], [c_1 - 1, c_2, c_3 + 1; d_1 + 2, d_2 - 2],$ $2[c_1 - 1, c_2, c_3 + 1; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1, c_2 - 2, c_3 + 2; d_1, d_2],$ $[c_1 + 1, c_2, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2, c_3 - 1; d_1, d_2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2],$ $2[c_1, c_2 - 1, c_3; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1, c_2 - 1, c_3; d_1 - 2, d_2 + 2],$ $[c_1, c_2, c_3 - 2; d_1, d_2]$
Level: 3
$[c_1 + 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 2, c_3; d_1 - 1, d_2 + 1],$ $[c_1 + 2, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2, c_3 + 1; d_1 + 3, d_2 - 3], 3[c_1, c_2, c_3 + 1; d_1 + 1, d_2 - 1], 3[c_1, c_2, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1 - 2, c_2 + 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1 + 2, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 + 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2 + 1, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1 + 1, c_2 - 2, c_3 + 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 2, c_3 + 2; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 - 1, c_3 + 2; d_1 + 1, d_2 - 1],$ $[c_1 + 2, c_2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1, c_2 + 1, c_3 - 1; d_1 + 3, d_2 - 3], 3[c_1, c_2 + 1, c_3 - 1; d_1 + 1, d_2 - 1], 3[c_1, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1 - 2, c_2 + 2, c_3 - 1; d_1 + 1, d_2 - 1], 2[c_1 + 1, c_2 - 1, c_3; d_1 + 1, d_2 - 1], 3[c_1 + 1, c_2 - 1, c_3; d_1 - 1, d_2 + 1], 2[c_1 - 1, c_2, c_3; d_1 + 1, d_2 - 1],$ $3[c_1 - 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1 + 2, c_2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2 + 2, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1 + 1, c_2 - 1, c_3; d_1 - 3, d_2 + 3], [c_1 - 1, c_2, c_3; d_1 - 3, d_2 + 3], [c_1, c_2 - 2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2, c_3 - 2; d_1 + 1, d_2 - 1],$ $[c_1 + 1, c_2, c_3 - 2; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 1, c_3 - 2; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3 - 2; d_1 - 1, d_2 + 1], [c_1, c_2 - 1, c_3 - 1; d_1 - 1, d_2 + 1]$ $[c_1 - 1, c_2 - 1, c_3 + 2; d_1 - 1, d_2 + 1]$
Level: 4
$[c_1, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2, c_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2, c_3 + 1; d_1, d_2], [c_1 - 1, c_2 + 1, c_3 + 1; d_1 + 2, d_2 - 2],$ $2[c_1 - 1, c_2 + 1, c_3 + 1; d_1, d_2], [c_1 + 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 + 2, c_2 - 2, c_3 + 2; d_1, d_2],$ $[c_1, c_2 - 1, c_3 + 2; d_1 + 2, d_2 - 2], 2[c_1, c_2 - 1, c_3 + 2; d_1, d_2], [c_1, c_2 - 1, c_3 + 2; d_1 - 2, d_2 + 2], [c_1 - 2, c_2, c_3 + 2; d_1, d_2],$ $[c_1 + 1, c_2 + 1, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 + 1, c_3 - 1; d_1, d_2], [c_1 - 1, c_2 + 2, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 - 1, c_2 + 2, c_3 - 1; d_1, d_2],$ $2[c_1 + 2, c_2 - 1, c_3; d_1, d_2], 2[c_1, c_2, c_3; d_1 + 2, d_2 - 2], 5[c_1, c_2, c_3; d_1, d_2], 4[c_1, c_2, c_3; d_1 - 2, d_2 + 2],$ $2[c_1 - 2, c_2 + 1, c_3; d_1, d_2], [c_1 + 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 + 2, c_2 - 1, c_3; d_1 - 2, d_2 + 2],$ $[c_1, c_2, c_3; d_1 - 4, d_2 + 4], [c_1 - 2, c_2 + 1, c_3; d_1 - 2, d_2 + 2], [c_1 + 1, c_2 - 2, c_3 + 1; d_1, d_2], [c_1 + 1, c_2 - 2, c_3 + 1; d_1 - 2, d_2 + 2],$ $[c_1 - 1, c_2 - 1, c_3 + 1; d_1, d_2], [c_1 - 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 + 2, c_2, c_3 - 2; d_1, d_2], [c_1, c_2 + 1, c_3 - 2; d_1 + 2, d_2 - 2],$ $2[c_1, c_2 + 1, c_3 - 2; d_1, d_2], [c_1, c_2 + 1, c_3 - 2; d_1 - 2, d_2 + 2], [c_1 - 2, c_2 + 2, c_3 - 2; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 - 1; d_1, d_2],$ $[c_1 + 1, c_2 - 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 - 1; d_1, d_2], [c_1 - 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2]$
Level: 5
$[c_1, c_2 + 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2 + 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3 + 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 1, c_3 + 2; d_1 - 1, d_2 + 1],$ $[c_1 - 1, c_2, c_3 + 2; d_1 + 1, d_2 - 1], [c_1 - 1, c_2, c_3 + 2; d_1 - 1, d_2 + 1], [c_1, c_2 + 2, c_3 - 1; d_1 + 1, d_2 - 1], 2[c_1 + 1, c_2, c_3; d_1 + 1, d_2 - 1],$ $3[c_1 + 1, c_2, c_3; d_1 - 1, d_2 + 1], 2[c_1 - 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], 3[c_1 - 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1, c_2 + 2, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1 + 1, c_2, c_3; d_1 - 3, d_2 + 3], [c_1 - 1, c_2 + 1, c_3; d_1 - 3, d_2 + 3], [c_1 + 2, c_2 - 2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1],$ $2[c_1, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2 - 1, c_3 + 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 + 1, c_3 - 2; d_1 + 1, d_2 - 1],$ $[c_1 + 1, c_2 + 1, c_3 - 2; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3 - 2; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 2, c_3 - 2; d_1 - 1, d_2 + 1],$ $[c_1, c_2, c_3 - 1; d_1 + 1, d_2 - 1], 2[c_1, c_2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 - 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1 + 2, c_2 - 1, c_3 - 1; d_1 - 1, d_2 + 1]$
Level: 6
$[c_1, c_2, c_3 + 2; d_1, d_2], 2[c_1, c_2 + 1, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 - 2, d_2 + 2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2],$ $[c_1 + 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 + 1; d_1, d_2], [c_1 - 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1, c_2 + 2, c_3 - 2; d_1, d_2],$ $[c_1 + 1, c_2, c_3 - 1; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2]$
Level: 7
$[c_1, c_2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1]$

Table 7. The spectrum of $\mathcal{B}[c_1, c_2, 0; d_1, 1]$ associated with the action of \tilde{Q} s.

\tilde{Q} actions for $\mathcal{B}[c_1, c_2, 0; d_1, 0]$	
Level: 0	
	$[c_1, c_2, c_3; d_1, d_2]$
Level: 1	
	$[c_1 + 1, c_2, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1],$ $[c_1, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 - 1; d_1 - 1, d_2 + 1]$
Level: 2	
	$[c_1 + 2, c_2, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 + 2, d_2 - 2], 2[c_1, c_2 + 1, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 - 2, d_2 + 2],$ $[c_1 - 2, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 + 1; d_1, d_2],$ $[c_1 - 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 + 1, c_2, c_3 - 1; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2],$ $[c_1 - 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1, c_2 - 1, c_3; d_1 - 2, d_2 + 2]$
Level: 3	
	$[c_1 + 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 2, c_3; d_1 - 1, d_2 + 1],$ $[c_1 + 2, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 + 1; d_1 + 1, d_2 - 1], 2[c_1, c_2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 + 1; d_1 - 3, d_2 + 3],$ $[c_1 - 2, c_2 + 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 2, c_2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 + 1, d_2 - 1], 2[c_1, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1, c_2 + 1, c_3 - 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2 + 2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3; d_1 - 3, d_2 + 3],$ $[c_1 - 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2, c_3; d_1 - 3, d_2 + 3]$
Level: 4	
	$[c_1, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2, c_3 + 1; d_1, d_2], [c_1 + 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 + 1; d_1, d_2],$ $[c_1 - 1, c_2 + 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 + 1, c_2 + 1, c_3 - 1; d_1, d_2], [c_1 + 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 2, c_3 - 1; d_1, d_2],$ $[c_1 - 1, c_2 + 2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 + 2, c_2 - 1, c_3; d_1 - 2, d_2 + 2], [c_1, c_2, c_3; d_1, d_2], 2[c_1, c_2, c_3; d_1 - 2, d_2 + 2],$ $[c_1, c_2, c_3; d_1 - 4, d_2 + 4], [c_1 - 2, c_2 + 1, c_3; d_1 - 2, d_2 + 2]$
Level: 5	
	$[c_1, c_2 + 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1 + 1, c_2, c_3; d_1 - 3, d_2 + 3],$ $[c_1 - 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 1, c_3; d_1 - 3, d_2 + 3]$
Level: 6	
	$[c_1, c_2 + 1, c_3; d_1 - 2, d_2 + 2]$

Table 8. The spectrum of $\mathcal{B}[c_1, c_2, 0; d_1, 0]$ associated with the action of \tilde{Q} s.

Q actions for $\mathcal{C}[c_1, 0, 0; d_1, d_2]$	
Level: 0	
	$[c_1, c_2, c_3; d_1, d_2]$
Level: 1	
	$[c_1 + 1, c_2, c_3; d_1, d_2 + 1], [c_1 + 1, c_2, c_3; d_1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3; d_1, d_2 - 1], [c_1, c_2 - 1, c_3 + 1; d_1, d_2 - 1],$ $[c_1, c_2, c_3 - 1; d_1, d_2 - 1]$
Level: 2	
	$[c_1 + 2, c_2, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1, d_2 - 2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2], [c_1, c_2 - 1, c_3; d_1, d_2 - 2],$ $[c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2 - 2], [c_1 - 1, c_2, c_3 + 1; d_1, d_2 - 2], [c_1 + 1, c_2, c_3 - 1; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1, d_2 - 2],$ $[c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2 - 2]$
Level: 3	
	$[c_1 + 1, c_2 + 1, c_3; d_1, d_2 - 1], [c_1 + 2, c_2 - 1, c_3 + 1; d_1, d_2 - 1], [c_1, c_2, c_3 + 1; d_1, d_2 - 1], [c_1, c_2, c_3 + 1; d_1, d_2 - 3],$ $[c_1 + 2, c_2, c_3 - 1; d_1, d_2 - 1], [c_1, c_2 + 1, c_3 - 1; d_1, d_2 - 1], [c_1, c_2 + 1, c_3 - 1; d_1, d_2 - 3], [c_1 + 1, c_2 - 1, c_3; d_1, d_2 - 1],$ $[c_1 + 1, c_2 - 1, c_3; d_1, d_2 - 3], [c_1 - 1, c_2, c_3; d_1, d_2 - 3]$
Level: 4	
	$[c_1 + 1, c_2, c_3 + 1; d_1, d_2 - 2], [c_1 + 1, c_2 + 1, c_3 - 1; d_1, d_2 - 2], [c_1 + 2, c_2 - 1, c_3; d_1, d_2 - 2], [c_1, c_2, c_3; d_1, d_2 - 2], [c_1, c_2, c_3; d_1, d_2 - 4]$
Level: 5	
	$[c_1 + 1, c_2, c_3; d_1, d_2 - 3]$

Table 9. The spectrum of $\mathcal{C}[c_1, 0, 0; d_1, d_2]$ associated with the action of Q s.

\tilde{Q} actions for $\mathcal{C}[c_1, 0, 0; d_1, 2]$
Level: 0
$[c_1, c_2, c_3; d_1, d_2]$
Level: 1
$[c_1 + 1, c_2, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1],$ $[c_1, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1, c_2, c_3 - 1; d_1 - 1, d_2 + 1]$
Level: 2
$[c_1 + 2, c_2, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 + 2, d_2 - 2], 2[c_1, c_2 + 1, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 - 2, d_2 + 2],$ $[c_1 - 2, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2], [c_1 - 1, c_2, c_3 + 1; d_1 + 2, d_2 - 2],$ $2[c_1 - 1, c_2, c_3 + 1; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1, c_2 - 2, c_3 + 2; d_1, d_2],$ $[c_1 + 1, c_2, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2, c_3 - 1; d_1, d_2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2],$ $[c_1, c_2 - 1, c_3; d_1 + 2, d_2 - 2], 2[c_1, c_2 - 1, c_3; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2],$ $[c_1, c_2 - 1, c_3; d_1 - 2, d_2 + 2], [c_1, c_2, c_3 - 2; d_1, d_2]$
Level: 3
$[c_1 + 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 2, c_3; d_1 - 1, d_2 + 1],$ $[c_1 + 2, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2, c_3 + 1; d_1 + 3, d_2 - 3], 3[c_1, c_2, c_3 + 1; d_1 + 1, d_2 - 1], 3[c_1, c_2, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1 - 2, c_2 + 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1 + 2, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 + 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2 + 1, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1 + 1, c_2 - 2, c_3 + 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 2, c_3 + 2; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 - 1, c_3 + 2; d_1 + 1, d_2 - 1],$ $[c_1, c_2 - 1, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 + 3, d_2 - 3], 3[c_1, c_2 + 1, c_3 - 1; d_1 + 1, d_2 - 1], 3[c_1, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1 - 2, c_2 + 2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 1, c_3; d_1 + 3, d_2 - 3], 3[c_1 + 1, c_2 - 1, c_3; d_1 + 1, d_2 - 1],$ $3[c_1 - 1, c_2, c_3; d_1 + 1, d_2 - 1], 3[c_1 + 1, c_2 - 1, c_3; d_1 - 1, d_2 + 1], 3[c_1 - 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1, c_2 - 2, c_3 + 1; d_1 + 1, d_2 - 1],$ $[c_1 + 2, c_2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2 + 2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3; d_1 - 3, d_2 + 3],$ $[c_1 - 1, c_2, c_3; d_1 - 3, d_2 + 3], [c_1, c_2 - 2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2, c_3 - 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2, c_3 - 2; d_1 - 1, d_2 + 1],$ $[c_1 - 1, c_2 + 1, c_3 - 2; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3 - 2; d_1 - 1, d_2 + 1], [c_1, c_2 - 1, c_3 - 1; d_1 + 1, d_2 - 1],$ $[c_1 + 2, c_2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 - 1, c_3 + 2; d_1 - 1, d_2 + 1],$
Level: 4
$[c_1, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2, c_3 + 1, d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2, c_3 + 1, d_1, d_2], [c_1 - 1, c_2 + 1, c_3 + 1, d_1 + 2, d_2 - 2],$ $2[c_1 - 1, c_2 + 1, c_3 + 1, d_1, d_2], [c_1 + 1, c_2, c_3 + 1, d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 + 1, d_1 - 2, d_2 + 2], [c_1 + 2, c_2 - 2, c_3 + 2; d_1, d_2],$ $[c_1, c_2 - 1, c_3 + 2; d_1 + 2, d_2 - 2], 2[c_1, c_2 - 1, c_3 + 2; d_1, d_2], [c_1, c_2 - 1, c_3 + 2; d_1 - 2, d_2 + 2], [c_1 - 2, c_2, c_3 + 2; d_1, d_2],$ $[c_1 + 1, c_2 + 1, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 + 1, c_3 - 1; d_1, d_2], [c_1 - 1, c_2 + 2, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 - 1, c_2 + 2, c_3 - 1; d_1, d_2],$ $[c_1 + 2, c_2 - 1, c_3; d_1 + 2, d_2 - 2], 3[c_1, c_2, c_3; d_1 + 2, d_2 - 2], 6[c_1, c_2, c_3; d_1, d_2], 2[c_1 + 2, c_2 - 1, c_3; d_1, d_2],$ $4[c_1, c_2, c_3; d_1 - 2, d_2 + 2], 2[c_1 - 2, c_2 + 1, c_3; d_1, d_2], [c_1 + 1, c_2 - 2, c_3 + 1, d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 - 2, c_3 + 1, d_1, d_2],$ $2[c_1 - 1, c_2 - 1, c_3 + 1, d_1, d_2], [c_1 + 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 + 2, c_2 - 1, c_3; d_1 - 2, d_2 + 2],$ $[c_1, c_2, c_3; d_1 - 4, d_2 + 4], [c_1 - 2, c_2 + 1, c_3; d_1 - 2, d_2 + 2], [c_1 + 1, c_2 - 2, c_3 + 1, d_1 - 2, d_2 + 2], [c_1 - 1, c_2 - 1, c_3 + 1, d_1 - 2, d_2 + 2],$ $[c_1 + 2, c_2, c_3 - 2; d_1, d_2], [c_1, c_2 + 1, c_3 - 2; d_1 + 2, d_2 - 2], 2[c_1, c_2 + 1, c_3 - 2; d_1, d_2], [c_1, c_2 + 1, c_3 - 2; d_1 - 2, d_2 + 2],$ $[c_1 - 2, c_2 + 2, c_3 - 2; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 - 1, c_3 - 1; d_1, d_2], 2[c_1 - 1, c_2, c_3 - 1; d_1, d_2],$ $[c_1 + 1, c_2 - 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1, c_2 - 2, c_3; d_1, d_2]$
Level: 5
$[c_1, c_2 + 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2 + 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3 + 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 1, c_3 + 2; d_1 - 1, d_2 + 1],$ $[c_1 - 1, c_2, c_3 + 2; d_1 + 1, d_2 - 1], [c_1 - 1, c_2, c_3 + 2; d_1 - 1, d_2 + 1], [c_1, c_2 + 2, c_3 - 1; d_1 + 1, d_2 - 1], 3[c_1 + 1, c_2, c_3; d_1 + 1, d_2 - 1],$ $3[c_1 + 1, c_2, c_3; d_1 - 1, d_2 + 1], 2[c_1 - 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], 3[c_1 - 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1 + 2, c_2 - 2, c_3 + 1; d_1 + 1, d_2 - 1],$ $2[c_1, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1], 3[c_1, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2, c_3; d_1 - 3, d_2 + 3],$ $[c_1 - 1, c_2 + 1, c_3; d_1 - 3, d_2 + 3], [c_1 + 2, c_2 - 2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2 - 1, c_3 + 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1 + 1, c_2 + 1, c_3 - 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 + 1, c_3 - 2; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3 - 2; d_1 + 1, d_2 - 1],$ $[c_1 + 2, c_2 - 1, c_3 - 1; d_1 + 1, d_2 - 1], 2[c_1, c_2, c_3 - 1; d_1 + 1, d_2 - 1], 3[c_1, c_2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 2, c_2 - 1, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1, c_2, c_3 - 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 2, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 2, c_3; d_1 - 1, d_2 + 1],$ $[c_1 - 1, c_2 - 1, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3 - 2; d_1 - 1, d_2 + 1]$
Level: 6
$[c_1, c_2, c_3 + 2; d_1, d_2], 2[c_1, c_2 + 1, c_3; d_1, d_2], 2[c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 - 2, d_2 + 2],$ $[c_1 + 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 + 1; d_1, d_2], [c_1 - 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1, c_2 + 2, c_3 - 2; d_1, d_2],$ $2[c_1 + 1, c_2, c_3 - 1; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2],$ $[c_1 + 2, c_2 - 2, c_3; d_1, d_2], [c_1, c_2 - 1, c_3; d_1, d_2], [c_1, c_2 - 1, c_3; d_1 - 2, d_2 + 2]$
Level: 7
$[c_1, c_2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3; d_1 - 1, d_2 + 1]$

Table 10. The spectrum of $\mathcal{C}[c_1, 0, 0; d_1, 2]$ associated with the action of \tilde{Q} s.

\tilde{Q} actions for $\mathcal{C}[c_1, 0, 0; d_1, 1]$
Level: 0
$[c_1, c_2, c_3; d_1, d_2]$
Level: 1
$[c_1 + 1, c_2, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1],$ $[c_1, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1, c_2, c_3 - 1; d_1 - 1, d_2 + 1]$
Level: 2
$[c_1 + 2, c_2, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 + 2, d_2 - 2], 2[c_1, c_2 + 1, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 - 2, d_2 + 2], [c_1, c_2, c_3 - 2; d_1, d_2],$ $[c_1 - 2, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2], [c_1, c_2 - 1, c_3; d_1 - 2, d_2 + 2],$ $2[c_1 - 1, c_2, c_3 + 1; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1, c_2 - 2, c_3 + 2; d_1, d_2],$ $[c_1 + 1, c_2, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2, c_3 - 1; d_1, d_2], 2[c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2],$ $2[c_1, c_2 - 1, c_3; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2]$
Level: 3
$[c_1 + 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 2, c_3; d_1 - 1, d_2 + 1],$ $[c_1 + 2, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1], 2[c_1, c_2, c_3 + 1; d_1 + 1, d_2 - 1], 3[c_1, c_2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 2, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1, c_2, c_3 + 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2 + 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 2, c_3 + 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 2, c_3 + 2; d_1 - 1, d_2 + 1],$ $[c_1 - 1, c_2 - 1, c_3 + 2; d_1 - 1, d_2 + 1], [c_1 + 2, c_2, c_3 - 1; d_1 + 1, d_2 - 1], 2[c_1, c_2 + 1, c_3 - 1; d_1 + 1, d_2 - 1], 3[c_1, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1 + 2, c_2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2 + 2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3; d_1 - 3, d_2 + 3],$ $[c_1 - 1, c_2, c_3; d_1 - 3, d_2 + 3], [c_1, c_2 - 2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2, c_3 - 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2, c_3 - 2; d_1 - 1, d_2 + 1],$ $[c_1 - 1, c_2 + 1, c_3 - 2; d_1 - 1, d_2 + 1], [c_1, c_2 - 1, c_3 - 1; d_1 - 1, d_2 + 1] 2[c_1 + 1, c_2 - 1, c_3; d_1 + 1, d_2 - 1], 3[c_1 + 1, c_2 - 1, c_3; d_1 - 1, d_2 + 1],$ $3[c_1 - 1, c_2, c_3; d_1 - 1, d_2 + 1]$
Level: 4
$[c_1, c_2 + 2, c_3; d_1, d_2], 2[c_1 + 1, c_2, c_3 + 1; d_1, d_2], [c_1 + 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 + 1; d_1, d_2],$ $[c_1 - 1, c_2 + 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 + 2, c_2 - 2, c_3 + 2; d_1, d_2], [c_1, c_2 - 1, c_3 + 2; d_1, d_2], [c_1, c_2 - 1, c_3 + 2; d_1 - 2, d_2 + 2],$ $2[c_1 + 1, c_2 + 1, c_3 - 1; d_1, d_2], 2[c_1 + 2, c_2 - 1, c_3; d_1, d_2], 3[c_1, c_2, c_3; d_1, d_2], 4[c_1, c_2, c_3; d_1 - 2, d_2 + 2], [c_1 + 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2],$ $[c_1 - 1, c_2 + 2, c_3 - 1; d_1, d_2], [c_1 - 1, c_2 + 2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 + 2, c_2 - 1, c_3; d_1 - 2, d_2 + 2], [c_1, c_2, c_3; d_1 - 4, d_2 + 4],$ $[c_1 - 2, c_2 + 1, c_3; d_1 - 2, d_2 + 2], [c_1 + 1, c_2 - 2, c_3 + 1; d_1, d_2], [c_1 + 1, c_2 - 2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2],$ $[c_1 + 2, c_2, c_3 - 2; d_1, d_2], [c_1, c_2 + 1, c_3 - 2; d_1, d_2], [c_1, c_2 + 1, c_3 - 2; d_1 - 2, d_2 + 2], [c_1 + 1, c_2 - 1, c_3 - 1; d_1, d_2],$ $[c_1 + 1, c_2 - 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2]$
Level: 5
$[c_1, c_2 + 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3 + 2; d_1 - 1, d_2 + 1], 3[c_1 + 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1, c_2 + 2, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1 + 1, c_2, c_3; d_1 - 3, d_2 + 3], [c_1 - 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 1, c_3; d_1 - 3, d_2 + 3], [c_1 + 2, c_2 - 2, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2 - 1, c_3 + 1; d_1 - 3, d_2 + 3], [c_1 + 1, c_2 + 1, c_3 - 2; d_1 - 1, d_2 + 1],$ $[c_1, c_2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 - 1; d_1 - 3, d_2 + 3], [c_1 + 2, c_2 - 1, c_3 - 1; d_1 - 1, d_2 + 1]$
Level: 6
$[c_1, c_2 + 1, c_3; d_1 - 2, d_2 + 2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 + 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2]$

Table 11. The spectrum of $\mathcal{C}[c_1, 0, 0; d_1, 1]$ associated with the action of \tilde{Q} s.

\tilde{Q} actions for $\mathcal{C}[c_1, 0, 0; d_1, 0]$
Level: 0
$[c_1, c_2, c_3; d_1, d_2]$
Level: 1
$[c_1 + 1, c_2, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1, c_2, c_3 - 1; d_1 - 1, d_2 + 1]$
Level: 2
$[c_1 + 2, c_2, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 - 2, d_2 + 2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2],$ $[c_1 + 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 + 1, c_2, c_3 - 1; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2],$ $[c_1 - 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1, c_2 - 1, c_3; d_1 - 2, d_2 + 2]$
Level: 3
$[c_1 + 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1 + 2, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 + 1; d_1 - 3, d_2 + 3],$ $[c_1 + 2, c_2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 - 3, d_2 + 3], [c_1 + 1, c_2 - 1, c_3; d_1 - 1, d_2 + 1],$ $[c_1 + 1, c_2 - 1, c_3; d_1 - 3, d_2 + 3], [c_1 - 1, c_2, c_3; d_1 - 3, d_2 + 3]$
Level: 4
$[c_1 + 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 + 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 + 2, c_2 - 1, c_3; d_1 - 2, d_2 + 2], [c_1, c_2, c_3; d_1 - 2, d_2 + 2],$ $[c_1, c_2, c_3; d_1 - 4, d_2 + 4]$
Level: 5
$[c_1 + 1, c_2, c_3; d_1 - 3, d_2 + 3]$

Table 12. The spectrum of $\mathcal{C}[c_1, 0, 0; d_1, 0]$ associated with the action of \tilde{Q} s.

Full spectrum for $\mathcal{C}[c_1, 0, 0; 1, 0]$	
Level: 0	
	$[c_1, 0, 0; 1, 0]$
Level: 1	
	$[c_1 + 1, 0, 0; 0, 1], [c_1 - 1, 1, 0; 0, 1], [c_1 + 1, 0, 0; 1, 1]$
Level: 2	
	$[c_1 + 2, 0, 0; 0, 0], [c_1, 1, 0; 0, 0], [c_1 - 2, 2, 0; 0, 0], [c_1 + 2, 0, 0; 1, 0], [c_1, 1, 0; 1, 0], [c_1 + 2, 0, 0; 2, 0], [c_1 + 2, 0, 0; 0, 2], [c_1, 1, 0; 0, 2]$
Level: 3	
	$-[c_1 - 1, 0, 0; 0, 1], [c_1 + 3, 0, 0; 0, 1], 2[c_1 + 1, 1, 0; 0, 1], [c_1 - 1, 2, 0; 0, 1], [c_1 + 3, 0, 0; 1, 1], [c_1 + 1, 1, 0; 1, 1]$
Level: 4	
	$-[c_1, 0, 0; 0, 0], -[c_1 - 2, 1, 0; 0, 0], [c_1 + 2, 1, 0; 0, 0], [c_1, 2, 0; 0, 0], -[c_1, 0, 0; 1, 0], [c_1 + 4, 0, 0; 1, 0], [c_1 + 2, 1, 0; 1, 0], [c_1, 2, 0; 1, 0], -[c_1, 0, 0; 0, 2], [c_1 + 2, 1, 0; 0, 2]$
Level: 5	
	$-2[c_1 + 1, 0, 0; 0, 1], -[c_1 - 1, 1, 0; 0, 1], [c_1 + 3, 1, 0; 0, 1], [c_1 + 1, 2, 0; 0, 1], -[c_1 + 1, 0, 0; 1, 1]$
Level: 6	
	$-[c_1 + 2, 0, 0; 0, 0], -[c_1, 1, 0, 0, 0], [c_1 + 2, 2, 0, 0, 0], -[c_1 + 2, 0, 0, 1, 0], -[c_1, 1, 0, 1, 0], -[c_1 + 2, 0, 0, 0, 2]$
Level: 7	
	$-[c_1 + 3, 0, 0; 0, 1], -[c_1 + 1, 1, 0; 0, 1]$
Level: 8	
	$-[c_1 + 2, 1, 0; 0, 0]$

Table 13. The spectrum of $\mathcal{C}[c_1, 0, 0; 1, 0]$.

Full spectrum for $\mathcal{C}[c_1, 0, 0; 0, 1]$	
Level: 0	
	$[c_1, 0, 0; 0, 1]$
Level: 1	
	$[c_1 + 1, 0, 0; 0, 0], [c_1 - 1, 1, 0; 0, 0], [c_1 + 1, 0, 0; 1, 0], [c_1 - 1, 1, 0; 1, 0], [c_1 + 1, 1, 0; 0, 2]$
Level: 2	
	$2[c_1 + 2, 0, 0; 0, 1], 2[c_1, 1, 0; 0, 1], [c_1 - 2, 2, 0; 0, 1], [c_1 + 2, 0, 0; 1, 1], [c_1, 1, 0; 1, 1]$
Level: 3	
	$-[c_1 - 1, 0, 0; 0, 0], [c_1 + 3, 0, 0; 0, 0], 2[c_1 + 1, 1, 0; 0, 0], [c_1 - 1, 2, 0; 0, 0], [c_1 - 3, 3, 0; 0, 0], -[c_1 - 1, 0, 0; 1, 0], [c_1 + 3, 0, 0; 1, 0], 2[c_1 + 1, 1, 0; 1, 0], [c_1 - 1, 2, 0; 1, 0], [c_1 + 1, 1, 0; 2, 0], [c_1 + 3, 0, 0; 0, 2], [c_1 + 1, 1, 0; 0, 2], [c_1 - 1, 2, 0; 0, 2]$
Level: 4	
	$-2[c_1, 0, 0; 0, 1], [c_1 + 4, 0, 0; 0, 1], -[c_1 - 2, 1, 0; 0, 1], 2[c_1 + 2, 1, 0; 0, 1], 2[c_1, 2, 0; 0, 1], [c_1 - 2, 3, 0; 0, 1], -[c_1, 0, 0; 1, 1], [c_1 + 2, 1, 0; 1, 1], [c_1, 2, 0; 1, 1]$
Level: 5	
	$-2[c_1 + 1, 0, 0; 0, 0], -[c_1 - 1, 1, 0; 0, 0], [c_1 + 3, 1, 0; 0, 0], -[c_1 - 3, 2, 0; 0, 0], [c_1 + 1, 2, 0; 0, 0], [c_1 - 1, 3, 0; 0, 0], -2[c_1 + 1, 0, 0; 1, 0], -[c_1 - 1, 1, 0; 1, 0], [c_1 + 3, 1, 0; 1, 0], [c_1 + 1, 2, 0; 1, 0], [c_1 - 1, 3, 0; 1, 0], -[c_1 + 1, 0, 0; 2, 0], -[c_1 + 1, 0, 0; 0, 2], -[c_1 - 1, 1, 0; 0, 2], [c_1 + 1, 2, 0; 0, 2]$
Level: 6	
	$-2[c_1 + 2, 0, 0; 0, 1], -2[c_1, 1, 0; 0, 1], -[c_1 - 2, 2, 0; 0, 1], [c_1 + 2, 2, 0; 0, 1], [c_1, 3, 0; 0, 1], -[c_1 + 2, 0, 0; 1, 1], -[c_1, 1, 0; 1, 1]$
Level: 7	
	$-[c_1 + 3, 0, 0; 0, 0], -[c_1 + 1, 1, 0; 0, 0], -[c_1 - 1, 2, 0; 0, 0], [c_1 + 1, 3, 0; 0, 0], -[c_1 + 3, 0, 0; 1, 0], -[c_1 + 1, 1, 0; 1, 0], -[c_1 - 1, 2, 0; 1, 0], -[c_1 + 1, 1, 0; 0, 2]$
Level: 8	
	$-[c_1 + 2, 1, 0; 0, 1], -[c_1, 2, 0; 0, 1]$
Level: 9	
	$-[c_1 + 1, 2, 0; 0, 0]$

Table 14. The spectrum of $\mathcal{C}[c_1, 0, 0; 0, 1]$.

\tilde{Q} actions for $\mathcal{D}[0, 0, 0; d_1, 3]$
Level: 0
$[c_1, c_2, c_3; d_1, d_2]$
Level: 1
$[c_1 + 1, c_2, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1],$ $[c_1, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1, c_2, c_3 - 1; d_1 - 1, d_2 + 1]$
Level: 2
$[c_1 + 2, c_2, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 + 2, d_2 - 2], 2[c_1, c_2 + 1, c_3; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 - 2, d_2 + 2],$ $[c_1 - 2, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2], [c_1 - 1, c_2, c_3 + 1; d_1 + 2, d_2 - 2],$ $2[c_1 - 1, c_2, c_3 + 1; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1, c_2 - 2, c_3 + 2; d_1, d_2],$ $[c_1 + 1, c_2, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2, c_3 - 1; d_1, d_2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2],$ $[c_1, c_2 - 1, c_3; d_1 + 2, d_2 - 2], 2[c_1, c_2 - 1, c_3; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2],$ $[c_1, c_2 - 1, c_3; d_1 - 2, d_2 + 2], [c_1, c_2, c_3 - 2; d_1, d_2]$
Level: 3
$[c_1 + 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 2, c_3; d_1 - 1, d_2 + 1],$ $[c_1 + 2, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2, c_3 + 1; d_1 + 3, d_2 - 3], 3[c_1, c_2, c_3 + 1; d_1 + 1, d_2 - 1], 3[c_1, c_2, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1 - 2, c_2 + 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1 + 2, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 + 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2 + 1, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1 + 1, c_2 - 2, c_3 + 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 2, c_3 + 2; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 - 1, c_3 + 2; d_1 + 1, d_2 - 1],$ $[c_1, c_2 - 1, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 + 3, d_2 - 3], 3[c_1, c_2 + 1, c_3 - 1; d_1 + 1, d_2 - 1], 3[c_1, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1 - 2, c_2 + 2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 1, c_3; d_1 + 3, d_2 - 3], 3[c_1 + 1, c_2 - 1, c_3; d_1 + 1, d_2 - 1], [c_1 - 1, c_2, c_3; d_1 + 3, d_2 - 3],$ $3[c_1 - 1, c_2, c_3; d_1 + 1, d_2 - 1], 3[c_1 + 1, c_2 - 1, c_3; d_1 - 1, d_2 + 1], 3[c_1 - 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1, c_2 - 2, c_3 + 1; d_1 + 1, d_2 - 1],$ $[c_1 + 2, c_2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2 + 2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3; d_1 - 3, d_2 + 3],$ $[c_1 - 1, c_2, c_3; d_1 - 3, d_2 + 3], [c_1, c_2 - 2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2, c_3 - 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2, c_3 - 2; d_1 - 1, d_2 + 1],$ $[c_1 - 1, c_2 + 1, c_3 - 2; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 + 1, c_3 - 2; d_1 - 1, d_2 + 1], [c_1, c_2 - 1, c_3 - 1; d_1 + 1, d_2 - 1],$ $[c_1 + 2, c_2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 - 1, c_3 + 2; d_1 - 1, d_2 + 1],$
Level: 4
$[c_1, c_2 + 2, c_3; d_1, d_2], [c_1 + 1, c_2, c_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2, c_3 + 1; d_1, d_2], [c_1 - 1, c_2 + 1, c_3 + 1; d_1 + 2, d_2 - 2],$ $2[c_1 - 1, c_2 + 1, c_3 + 1; d_1, d_2], [c_1 + 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 + 2, c_2 - 2, c_3 + 2; d_1, d_2],$ $[c_1, c_2 - 1, c_3 + 2; d_1 + 2, d_2 - 2], 2[c_1, c_2 - 1, c_3 + 2; d_1, d_2], [c_1, c_2 - 1, c_3 + 2; d_1 - 2, d_2 + 2], [c_1 - 2, c_2, c_3 + 2; d_1, d_2],$ $[c_1 + 1, c_2 + 1, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 + 1, c_3 - 1; d_1, d_2], a[c_1 - 1, c_2 + 2, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 - 1, c_2 + 2, c_3 - 1; d_1, d_2],$ $[c_1 - 2, c_2 - 1, c_3; d_1 + 2, d_2 - 2], 4[c_1, c_2, c_3; d_1 + 2, d_2 - 2], 6[c_1, c_2, c_3; d_1, d_2], [c_1 - 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1, c_2 - 2, c_3; d_1, d_2],$ $[c_1 - 2, c_2 + 1, c_3; d_1 + 2, d_2 - 2], 2[c_1 + 2, c_2 - 1, c_3; d_1, d_2], 4[c_1, c_2, c_3; d_1 - 2, d_2 + 2], 2[c_1 - 2, c_2 + 1, c_3; d_1, d_2],$ $[c_1 + 1, c_2 - 2, c_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 - 2, c_3 + 1; d_1, d_2], [c_1 - 1, c_2 - 1, c_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 - 1, c_2 - 1, c_3 + 1; d_1, d_2],$ $[c_1 + 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 + 2, c_2 - 1, c_3; d_1 - 2, d_2 + 2], [c_1, c_2, c_3; d_1 - 4, d_2 + 4],$ $[c_1 - 2, c_2 + 1, c_3; d_1 - 2, d_2 + 2], [c_1 + 1, c_2 - 2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2],$ $[c_1 + 2, c_2, c_3 - 2; d_1, d_2], [c_1, c_2 + 1, c_3 - 2; d_1 + 2, d_2 - 2], 2[c_1, c_2 + 1, c_3 - 2; d_1, d_2], [c_1, c_2 + 1, c_3 - 2; d_1 - 2, d_2 + 2],$ $[c_1 - 2, c_2 + 2, c_3 - 2; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, c_2 - 1, c_3 - 1; d_1, d_2], [c_1 - 1, c_2, c_3 - 1; d_1 + 2, d_2 - 2],$ $2[c_1 - 1, c_2, c_3 - 1; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 - 1; d_1 - 2, d_2 + 2]$
Level: 5
$[c_1, c_2 + 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2 + 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3 + 2; d_1 + 1, d_2 - 1], [c_1 - 2, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1]$ $[c_1 - 1, c_2, c_3 + 2; d_1 + 1, d_2 - 1], [c_1 - 1, c_2, c_3 + 2; d_1 - 1, d_2 + 1], [c_1, c_2 + 2, c_3 - 1; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 1, c_3 + 2; d_1 - 1, d_2 + 1]$ $3[c_1 + 1, c_2, c_3; d_1 + 1, d_2 - 1], 3[c_1 - 1, c_2 + 1, c_3; d_1 + 1, d_2 - 1], 3[c_1 + 1, c_2, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3 - 2; d_1 - 1, d_2 + 1],$ $3[c_1 - 1, c_2 + 1, c_3; d_1 - 1, d_2 + 1], [c_1 + 2, c_2 - 2, c_3 + 1; d_1 + 1, d_2 - 1], 3[c_1, c_2 - 1, c_3 + 1; d_1 + 1, d_2 - 1], [c_1 - 1, c_2 - 1, c_3; d_1 - 1, d_2 + 1],$ $3[c_1, c_2 - 1, c_3 + 1; d_1 - 1, d_2 + 1], [c_1 - 2, c_2, c_3 + 1; d_1 + 1, d_2 - 1], [c_1, c_2 + 2, c_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2, c_3; d_1 - 3, d_2 + 3],$ $[c_1 - 1, c_2 + 1, c_3; d_1 - 3, d_2 + 3], [c_1 + 2, c_2 - 2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2 - 1, c_3 + 1; d_1 - 3, d_2 + 3], [c_1 - 2, c_2, c_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1 + 1, c_2 + 1, c_3 - 2; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 + 1, c_3 - 2; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 + 2, c_3 - 2; d_1 + 1, d_2 - 1],$ $[c_1 + 2, c_2 - 1, c_3 - 1; d_1 + 1, d_2 - 1], 3[c_1, c_2, c_3 - 1; d_1 + 1, d_2 - 1], 3[c_1, c_2, c_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1 - 2, c_2 + 1, c_3 - 1; d_1 + 1, d_2 - 1], [c_1 + 2, c_2 - 1, c_3 - 1; d_1 - 1, d_2 + 1], [c_1, c_2, c_3 - 1; d_1 - 3, d_2 + 3],$ $[c_1 + 1, c_2 - 2, c_3; d_1 + 1, d_2 - 1], [c_1 + 1, c_2 - 2, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2 - 1, c_3; d_1 + 1, d_2 - 1]$
Level: 6
$[c_1, c_2, c_3 + 2; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 + 2, d_2 - 2], 2[c_1, c_2 + 1, c_3; d_1, d_2], [c_1 + 1, c_2 - 1, c_3 + 1; d_1 + 2, d_2 - 2], [c_1 - 2, c_2, c_3; d_1, d_2]$ $2[c_1 + 1, c_2 - 1, c_3 + 1; d_1, d_2], 2[c_1 - 1, c_2, c_3 + 1; d_1, d_2], [c_1, c_2 + 1, c_3; d_1 - 2, d_2 + 2], [c_1, c_2 - 1, c_3; d_1 - 2, d_2 + 2], 2[c_1, c_2 - 1, c_3; d_1, d_2]$ $[c_1 + 1, c_2 - 1, c_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2, c_3 + 1; d_1 - 2, d_2 + 2], [c_1, c_2 + 2, c_3 - 2; d_1, d_2], [c_1 + 2, c_2 - 2, c_3; d_1, d_2]$ $2[c_1 + 1, c_2, c_3 - 1; d_1, d_2], 2[c_1 - 1, c_2 + 1, c_3 - 1; d_1, d_2], [c_1 + 1, c_2, c_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, c_2 + 1, c_3 - 1; d_1 - 2, d_2 + 2]$
Level: 7
$[c_1, c_2, c_3 + 1; d_1 - 1, d_2 + 1], [c_1, c_2 + 1, c_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, c_2 - 1, c_3; d_1 - 1, d_2 + 1], [c_1 - 1, c_2, c_3; d_1 - 1, d_2 + 1]$

Table 15. The spectrum of $\mathcal{D}[0, 0, 0; d_1, 3]$ associated with the action of \tilde{Q} s.

\tilde{Q} actions for $\mathcal{D}[0, 0, 0; d_1, 2]$
Level: 0
$[c_1, e_2, e_3; d_1, d_2]$
Level: 1
$[c_1 + 1, e_2, e_3; d_1 + 1, d_2 - 1], [c_1 + 1, e_2, e_3; d_1 - 1, d_2 + 1], [c_1 - 1, e_2 + 1, e_3; d_1 + 1, d_2 - 1], [c_1 - 1, e_2 + 1, e_3; d_1 - 1, d_2 + 1],$ $[c_1, e_2 - 1, e_3 + 1; d_1 + 1, d_2 - 1], [c_1, e_2 - 1, e_3 + 1; d_1 - 1, d_2 + 1], [c_1, e_2, e_3 - 1; d_1 + 1, d_2 - 1], [c_1, e_2, e_3 - 1; d_1 - 1, d_2 + 1]$
Level: 2
$[c_1 + 2, e_2, e_3; d_1, d_2], [c_1, e_2 + 1, e_3; d_1 + 2, d_2 - 2], 2[c_1, e_2 + 1, e_3; d_1, d_2], [c_1, e_2 + 1, e_3; d_1 - 2, d_2 + 2],$ $[c_1 - 2, e_2 + 2, e_3; d_1, d_2], [c_1 + 1, e_2 - 1, e_3 + 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, e_2 - 1, e_3 + 1; d_1, d_2], [c_1 - 1, e_2, e_3 + 1; d_1 + 2, d_2 - 2],$ $2[c_1 - 1, e_2, e_3 + 1; d_1, d_2], [c_1 + 1, e_2 - 1, e_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, e_2, e_3 + 1; d_1 - 2, d_2 + 2], [c_1, e_2 - 2, e_3 + 2; d_1, d_2],$ $[c_1 + 1, e_2, e_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 + 1, e_2, e_3 - 1; d_1, d_2], [c_1 - 1, e_2 + 1, e_3 - 1; d_1 + 2, d_2 - 2], 2[c_1 - 1, e_2 + 1, e_3 - 1; d_1, d_2],$ $[c_1, e_2 - 1, e_3; d_1 + 2, d_2 - 2], 2[c_1, e_2 - 1, e_3; d_1, d_2], [c_1 + 1, e_2, e_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, e_2 + 1, e_3 - 1; d_1 - 2, d_2 + 2],$ $[c_1, e_2 - 1, e_3; d_1 - 2, d_2 + 2], [c_1, e_2, e_3 - 2; d_1, d_2]$
Level: 3
$[c_1 + 1, e_2 + 1, e_3; d_1 + 1, d_2 - 1], [c_1 + 1, e_2 + 1, e_3; d_1 - 1, d_2 + 1], [c_1 - 1, e_2 + 2, e_3; d_1 + 1, d_2 - 1], [c_1 - 1, e_2 + 2, e_3; d_1 - 1, d_2 + 1],$ $[c_1 + 2, e_2 - 1, e_3 + 1; d_1 + 1, d_2 - 1], 3[c_1, e_2, e_3 + 1; d_1 + 1, d_2 - 1], 3[c_1, e_2, e_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1 - 2, e_2 + 1, e_3 + 1; d_1 + 1, d_2 - 1], [c_1 + 2, e_2 - 1, e_3 + 1; d_1 - 1, d_2 + 1], [c_1, e_2, e_3 + 1; d_1 - 3, d_2 + 3],$ $[c_1 + 1, e_2 - 2, e_3 + 2; d_1 + 1, d_2 - 1], [c_1 + 1, e_2 - 2, e_3 + 2; d_1 - 1, d_2 + 1], [c_1 - 1, e_2 - 1, e_3 + 2; d_1 + 1, d_2 - 1],$ $[c_1, e_2 - 1, e_3 - 1; d_1 - 1, d_2 + 1], 3[c_1, e_2 + 1, e_3 - 1; d_1 + 1, d_2 - 1], 3[c_1, e_2 + 1, e_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1 - 2, e_2 + 2, e_3 - 1; d_1 + 1, d_2 - 1], 3[c_1 + 1, e_2 - 1, e_3; d_1 + 1, d_2 - 1], [c_1 - 2, e_2 + 1, e_3 + 1; d_1 - 1, d_2 + 1],$ $3[c_1 - 1, e_2, e_3; d_1 + 1, d_2 - 1], 3[c_1 + 1, e_2 - 1, e_3; d_1 - 1, d_2 + 1], 3[c_1 - 1, e_2, e_3; d_1 - 1, d_2 + 1], [c_1, e_2 - 2, e_3 + 1; d_1 + 1, d_2 - 1],$ $[c_1 + 2, e_2, e_3 - 1; d_1 - 1, d_2 + 1], [c_1, e_2 + 1, e_3 - 1; d_1 - 3, d_2 + 3], [c_1 - 2, e_2 + 2, e_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, e_2 - 1, e_3; d_1 - 3, d_2 + 3],$ $[c_1 - 1, e_2, e_3; d_1 - 3, d_2 + 3], [c_1, e_2 - 2, e_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, e_2, e_3 - 2; d_1 + 1, d_2 - 1], [c_1 + 1, e_2, e_3 - 2; d_1 - 1, d_2 + 1],$ $[c_1 - 1, e_2 + 1, e_3 - 2; d_1 + 1, d_2 - 1], [c_1 - 1, e_2 + 1, e_3 - 2; d_1 - 1, d_2 + 1], [c_1, e_2 - 1, e_3 - 1; d_1 + 1, d_2 - 1],$ $[c_1 + 2, e_2, e_3 - 1; d_1 + 1, d_2 - 1], [c_1 - 1, e_2 - 1, e_3 + 2; d_1 - 1, d_2 + 1]$
Level: 4
$[c_1, e_2 + 2, e_3; d_1, d_2], 2[c_1 + 1, e_2, e_3 + 1; d_1, d_2], 2[c_1 - 1, e_2 + 1, e_3 + 1; d_1, d_2], [c_1 + 1, e_2, e_3 + 1; d_1 - 2, d_2 + 2],$ $[c_1 + 2, e_2 - 2, e_3 + 2; d_1, d_2], 2[c_1, e_2 - 1, e_3 + 2; d_1, d_2], [c_1, e_2 - 1, e_3 + 2; d_1 - 2, d_2 + 2], [c_1 - 2, e_2, e_3 + 2; d_1, d_2],$ $2[c_1 + 1, e_2 + 1, e_3 - 1; d_1, d_2], 2[c_1 - 1, e_2 + 2, e_3 - 1; d_1, d_2], 6[c_1, e_2, e_3; d_1, d_2], 2[c_1 + 2, e_2 - 1, e_3; d_1, d_2], 4[c_1, e_2, e_3; d_1 - 2, d_2 + 2],$ $2[c_1 - 2, e_2 + 1, e_3; d_1, d_2], 2[c_1 + 1, e_2 - 2, e_3 + 1; d_1, d_2], 2[c_1 - 1, e_2 - 1, e_3 + 1; d_1, d_2], [c_1 + 1, e_2 + 1, e_3 - 1; d_1 - 2, d_2 + 2],$ $[c_1 - 1, e_2 + 2, e_3 - 1; d_1 - 2, d_2 + 2], [c_1 + 2, e_2 - 1, e_3; d_1 - 2, d_2 + 2], [c_1, e_2, e_3; d_1 - 4, d_2 + 4], [c_1 - 2, e_2 + 1, e_3; d_1 - 2, d_2 + 2],$ $[c_1 + 1, e_2 - 2, e_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, e_2 - 1, e_3 + 1; d_1 - 2, d_2 + 2], [c_1 + 2, e_2, e_3 - 2; d_1, d_2], 2[c_1, e_2 + 1, e_3 - 2; d_1, d_2],$ $[c_1, e_2 + 1, e_3 - 2; d_1 - 2, d_2 + 2], [c_1 - 2, e_2 + 2, e_3 - 2; d_1, d_2], 2[c_1 + 1, e_2 - 1, e_3 - 1; d_1, d_2], 2[c_1 - 1, e_2, e_3 - 1; d_1, d_2],$ $[c_1 + 1, e_2 - 1, e_3 - 1; d_1 - 2, d_2 + 2], [c_1 - 1, e_2, e_3 - 1; d_1 - 2, d_2 + 2], [c_1, e_2 - 2, e_3; d_1, d_2], [c_1 - 1, e_2 + 1, e_3 + 1; d_1 - 2, d_2 + 2]$
Level: 5
$[c_1, e_2 + 1, e_3 + 1; d_1 - 1, d_2 + 1], [c_1 + 1, e_2 - 1, e_3 + 2; d_1 - 1, d_2 + 1], [c_1 - 1, e_2, e_3 + 2; d_1 - 1, d_2 + 1], 3[c_1 + 1, e_2, e_3; d_1 - 1, d_2 + 1],$ $3[c_1 - 1, e_2 + 1, e_3; d_1 - 1, d_2 + 1], 3[c_1, e_2 - 1, e_3 + 1; d_1 - 1, d_2 + 1], [c_1, e_2 + 2, e_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, e_2, e_3; d_1 - 3, d_2 + 3],$ $[c_1 - 1, e_2 + 1, e_3; d_1 - 3, d_2 + 3], [c_1 + 2, e_2 - 2, e_3 + 1; d_1 - 1, d_2 + 1], [c_1, e_2 - 1, e_3 + 1; d_1 - 3, d_2 + 3], [c_1 - 2, e_2, e_3 + 1; d_1 - 1, d_2 + 1],$ $[c_1 + 1, e_2 + 1, e_3 - 2; d_1 - 1, d_2 + 1], [c_1 - 1, e_2 + 2, e_3 - 2; d_1 - 1, d_2 + 1], 3[c_1, e_2, e_3 - 1; d_1 - 1, d_2 + 1],$ $[c_1, e_2, e_3 - 1; d_1 - 3, d_2 + 3], [c_1 - 2, e_2 + 1, e_3 - 1; d_1 - 1, d_2 + 1], [c_1 + 1, e_2 - 2, e_3; d_1 - 1, d_2 + 1], [c_1 - 1, e_2 - 1, e_3; d_1 - 1, d_2 + 1],$ $[c_1 + 2, e_2 - 1, e_3 - 1; d_1 - 1, d_2 + 1]$
Level: 6
$[c_1, e_2 + 1, e_3; d_1 - 2, d_2 + 2], [c_1 + 1, e_2 - 1, e_3 + 1; d_1 - 2, d_2 + 2], [c_1 - 1, e_2, e_3 + 1; d_1 - 2, d_2 + 2], [c_1 + 1, e_2, e_3 - 1; d_1 - 2, d_2 + 2],$ $[c_1 - 1, e_2 + 1, e_3 - 1; d_1 - 2, d_2 + 2], [c_1, e_2 - 1, e_3; d_1 - 2, d_2 + 2]$

Table 16. The spectrum of $\mathcal{D}[0, 0, 0; d_1, 2]$ associated with the action of \tilde{Q} s.

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