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Color-factor symmetry and BCJ relations for QCD amplitudes

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Abstract: Tree-level n-point gauge-theory amplitudes with $n-2k$ gluons and k pairs of (massless or massive) particles in the fundamental (or other) representation of the gauge group are invariant under a set of symmetries that act as momentum-dependent shifts on the color factors in the cubic decomposition of the amplitude. These symmetries lead to gauge-invariant constraints on the kinematic numerators. They also directly imply the BCJ relations among the Melia-basis primitive amplitudes previously obtained by Johansson and Ochirov.

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Contents

1 Introduction

Bern, Carrasco, and Johansson discovered that the color-ordered amplitudes of tree-level n-gluon amplitudes obey a set of helicity-independent relations, linear relations whose coefficients depend only on Lorentz-invariant combinations of the momenta of the particles [1]. They inferred these relations by imposing color-kinematic duality on the kinematic numerators appearing in a cubic decomposition of the amplitude. These BCJ relations have been proven through a variety of approaches: string theory [2, 3], on-shell BCFW recursion [4, 5], the Cachazo-He-Yuan representation [6–8], and most recently by using the invariance of the n-gluon amplitude under a momentum-dependent shift of the color factors appearing in the cubic decomposition [9].

Several authors have turned their attention to the question of color-kinematic duality in gauge-theory amplitudes containing (massless or massive) quarks or other particles in the fundamental representation of the gauge group [10–18]. Melia characterized an independent basis of primitive amplitudes for tree-level amplitudes that involve gluons and an arbitrary number of pairs of differently flavored fundamentals [19–21]. By imposing color-kinematic duality, Johansson and Ochirov showed that these Melia amplitudes satisfy a set of BCJ relations that are formally identical[1] to a subset of the n-gluon BCJ relations [13]. The same relations had previously been established, also by assuming color-kinematic duality, for amplitudes with gluons and a single pair of massive fundamentals [11]. These relations were subsequently proven using BCFW on-shell recursion by de la Cruz, Kniss, and Weinzierl [15].

In this paper, we demonstrate that all the BCJ relations found in ref. [13] are a consequence of the color-factor symmetry possessed by the amplitude. This symmetry, recently

[1] I.e., when written in terms of $k_a \cdot k_b$, where k_a is the momentum of one of the gluons.

established in ref. [9], states that the amplitude is invariant under certain momentum-dependent shifts of the color factors. We show that for the n-point amplitude $\mathcal{A}_{n,k}$, with k pairs of fundamentals and $n-2k$ gluons, there is an $(n-3)!/k!$-parameter family of color-factor shifts associated with each external gluon in the amplitude. The total number of independent color-factor shifts is given by $(n-2k)(n-3)!/k!$ for $k \geq 2$, and $(n-3)(n-3)!$ for $k=0$ and 1. We show that the color factors introduced by Johansson and Ochirov [13] transform in a particularly simple way under a color-factor shift. Consequently, the BCJ relations among the primitive amplitudes follow immediately from the invariance under color-factor shifts of the Melia-Johansson-Ochirov proper decomposition of the amplitude.

Johansson and Ochirov found that the BCJ relations satisfied by the Melia primitives are directly tied to the presence of external gluons in the amplitude. Each set of fundamental relations corresponds to one of the external gluons, and when the amplitude contains no external gluons, no relations exist among the primitive amplitudes. The link shown in this paper between the color-factor symmetry and BCJ relations makes sense of this result, since the color-factor shifts are associated with external gluons in the amplitude. The number of independent BCJ relations is precisely equal to the dimension of the color-factor group, and the absence of color-factor symmetry for amplitudes with no gluons explains the absence of relations among the primitive amplitudes.

We also describe the cubic vertex expansion for $\mathcal{A}_{n,k}$ that was introduced in ref. [9], and show that the color-factor symmetry leads to gauge-invariant constraints on the kinematic numerators. These constraints are less stringent than the kinematic Jacobi relations but are nonetheless sufficient to imply the validity of the BCJ relations.

The contents of this paper are as follows. In section 2, we describe the Melia basis of primitive amplitudes for $\mathcal{A}_{n,k}$ and the corresponding color factors found by Johansson and Ochirov. In section 3, we review the BCJ relations for the primitive amplitudes of $\mathcal{A}_{n,k}$. In section 4, we describe the color-factor symmetry and determine the shifts of the Johansson-Ochirov color factors. In section 5, we review the proof of the invariance of $\mathcal{A}_{n,k}$ under the color-factor shift, and show that the BCJ relations as well as constraints on kinematic numerators follow directly from this invariance. Section 6 contains our conclusions.

2 Proper decompositions of gauge-theory amplitudes

In this section, we describe proper decompositions of gauge-theory amplitudes with particles in the adjoint and fundamental representations; that is, decompositions in terms of an independent set of color factors and gauge-invariant primitive amplitudes. These include the Del Duca-Dixon-Maltoni decomposition of n-gluon amplitudes and the recently developed Melia-Johansson-Ochirov decomposition of amplitudes with $n-2k$ gluons and k pairs of fundamentals.

Consider a tree-level n-point gauge-theory amplitude $\mathcal{A}_{n,k}$ with $n-2k$ gluons and k pairs of (massless or massive) particles ψ and $\bar{\psi}$ in the fundamental (and antifundamental) representation[2] of the gauge group, with arbitrary spin ≤ 1. This amplitude may be written

[2] The particles ψ can actually be in any representation, but we refer to the fundamental for convenience.

in a cubic decomposition [1]

$$\mathcal{A}_{n,k} \;=\; \sum_i \frac{c_i\, n_i}{d_i} \tag{2.1}$$

namely, as a sum over diagrams i consisting only of trivalent vertices. The color factor c_i associated with each diagram is obtained by sewing together ggg vertices f_{abc} and $\bar{\psi}g\psi$ vertices $(T^{\mathsf{a}})^{\mathsf{i}}{}_{\mathsf{j}}$, where T^{a} denote generators in the fundamental (or other) representation. The kinematic numerator n_i carries information about the spin state of the particles. The denominator d_i consists of the product of the inverse propagators associated with the diagram. Contributions from Feynman diagrams with quartic vertices (either $gggg$ or $\bar{\psi}gg\psi$ in the case of a scalar or vector ψ) are parceled out among the cubic diagrams.

In general, the color factors c_i are not independent, but obey a set of Jacobi relations of the form

$$c_i + c_j + c_k \;=\; 0 \tag{2.2}$$

by virtue of the group theory identities $f_{\mathsf{abe}}f_{\mathsf{cde}} + f_{\mathsf{ace}}f_{\mathsf{dbe}} + f_{\mathsf{ade}}f_{\mathsf{bce}} = 0$ and $f_{\mathsf{abc}}(T^{\mathsf{c}})^{\mathsf{i}}{}_{\mathsf{j}} = [T^{\mathsf{a}}, T^{\mathsf{b}}]^{\mathsf{i}}{}_{\mathsf{j}}$. Because of this, the kinematic numerators n_i are not well-defined, but may undergo generalized gauge transformations [22, 23] that leave eq. (2.1) unchanged. In principle, the Jacobi relations for the color factors may be solved in terms of an independent set of color factors C_j, and the amplitude written in a *proper* color decomposition [21]

$$\mathcal{A}_{n,k} \;=\; \sum_j C_j A_j \,. \tag{2.3}$$

The coefficients A_j, referred to as primitive amplitudes, receive contributions from several different terms in eq. (2.1). Because the C_j are independent, the primitive amplitudes will be well-defined (gauge invariant).

The primitive amplitudes may be chosen to be an independent subset of color-ordered amplitudes. The color-ordered amplitude $A(\alpha)$ is computed, using color-ordered Feynman rules [24], from the sum of planar Feynman diagrams whose external legs are in the order specified by the permutation α. The color-ordered amplitudes are not independent, but obey various relations such as cyclicity and reflection invariance [25] and the Kleiss-Kuijf relations [26]. For pure gluon amplitudes ($k = 0$), an independent set of $(n-2)!$ color-ordered amplitudes are those belonging to the Kleiss-Kuijf basis $A(1, \gamma(2), \cdots, \gamma(n-1), n)$, in which the positions of two of the gluons are fixed and γ is a permutation of $\{2, \cdots, n-1\}$. The corresponding color factors C_j are the $(n-2)!$ half-ladder diagrams

$$\mathbf{c}_{1\gamma n} \;=\; \sum_{\mathsf{b}_1,\ldots,\mathsf{b}_{n-3}} f_{\mathsf{a}_1\mathsf{a}_{\gamma(2)}\mathsf{b}_1} f_{\mathsf{b}_1\mathsf{a}_{\gamma(3)}\mathsf{b}_2} \cdots f_{\mathsf{b}_{n-3}\mathsf{a}_{\gamma(n-1)}\mathsf{a}_n}\,, \qquad \gamma \in S_{n-2} \tag{2.4}$$

and the proper decomposition

$$\mathcal{A}_{n,0}(g_1, g_2, \cdots, g_n) \;=\; \sum_{\gamma\in S_{n-2}} \mathbf{c}_{1\gamma n}\, A(1, \gamma(2), \cdots, \gamma(n-1), n) \tag{2.5}$$

is known as the Del Duca-Dixon-Maltoni decomposition [27].

The n-point gauge-theory amplitude with $n-2$ gluons and one pair of fundamentals has a similar proper decomposition [28, 29]

$$\mathcal{A}_{n,1}(\bar{\psi}_1, \psi_2, g_3, \cdots, g_n) = \sum_{\gamma \in S_{n-2}} C_{1\gamma 2}\, A(1, \gamma(3), \cdots, \gamma(n), 2) \tag{2.6}$$

where the independent color factors are given by

$$C_{1\gamma 2} = (T^{\mathsf{a}_{\gamma(3)}} T^{\mathsf{a}_{\gamma(4)}} \cdots T^{\mathsf{a}_{\gamma(n)}})^{\mathsf{i}_1}{}_{\mathsf{i}_2} \qquad \text{for} \quad k=1 \tag{2.7}$$

with γ a permutation of $\{3, \cdots, n\}$.

Finding a proper decomposition for amplitudes with more than one pair of fundamentals is a more subtle problem, but was recently solved by Melia [19, 20] and Johansson and Ochirov [13]. Consider an n-point amplitude of the form

$$\mathcal{A}_{n,k}(\bar{\psi}_1, \psi_2, \bar{\psi}_3, \psi_4, \cdots, \bar{\psi}_{2k-1}, \psi_{2k}, g_{2k+1}, \cdots, g_n) \tag{2.8}$$

where particles ψ in the fundamental representation have even labels, and particles $\bar{\psi}$ in the antifundamental representation have odd labels.[3] We assume that the $\psi_{2\ell}$ all have different flavors (and possibly different masses), with $\bar{\psi}_{2\ell-1}$ having the corresponding antiflavor (and equal mass) to $\psi_{2\ell}$. This assumption entails no loss of generality since amplitudes with multiple pairs of fundamentals with the same flavor and mass can be obtained by setting flavors and masses equal and summing over permutations. As shown in ref. [19], many of the color-ordered amplitudes $A(\alpha)$ associated with this amplitude vanish because no planar Feynman diagram with external legs in the order α can be drawn that does not violate flavor conservation. Moreover, the non-vanishing color-ordered amplitudes satisfy further relations in addition to the Kleiss-Kuijf relations. Melia identified a subset of $(n-2)!/k!$ color-ordered amplitudes that form an independent set.

To describe the Melia basis of primitive amplitudes, we must recall the definition of a Dyck word [19]. A Dyck word of length $2r$ is a string composed of r letters $\bar{\psi}$ and r letters ψ such that the number of $\bar{\psi}$'s preceding any point in the string is greater than the number of preceding ψ's. An easy way to understand this is to visualize $\bar{\psi}$ as a left bracket { and ψ as a right bracket }, in which case a Dyck word corresponds to a well-formed set of brackets. The number of such words is $(2r)!/(r+1)!r!$, the rth Catalan number. For example for $r=1$ there is only one Dyck word: {}, for $r=2$ there are two: {}{} and {{}}, and for $r=3$ there are five: {}{}{}, {}{{}}, {{}}{}, {{}{}}, and {{{}}}.

The Melia basis is the set of color-ordered amplitudes $A(1, \gamma(3), \cdots, \gamma(n), 2)$, where γ is any permutation of $\{3, \cdots, n\}$ such that the set of $k-1$ $\bar{\psi}$ and $k-1$ ψ in γ form a Dyck word of length $2k-2$. The gluons may be distributed anywhere among the $\bar{\psi}$ and ψ in γ. The number of distinct allowed patterns of $\bar{\psi}$, ψ, and g is given by the number of Dyck words of length $2k-2$ times the number of ways of distributing $n-2k$ gluons among the letters of the Dyck word

$$\frac{(2k-2)!}{k!(k-1)!} \times \binom{n-2}{2k-2}. \tag{2.9}$$

[3] In refs. [13, 19, 20], ψ and $\bar{\psi}$ are referred to as quarks and antiquarks, but they could just as easily be scalar or vector particles.

For each allowed pattern, there are $(n-2k)!$ distinct choices for the gluon labels, and $(k-1)!$ choices for the $\bar{\psi}$ labels. The label on each ψ is then fixed: it must have the flavor of the nearest unpaired $\bar{\psi}$ to its left. Thus, for example, for $\mathcal{A}_{6,3}$ the allowed permutations γ are $\bar{\psi}_3\psi_4\bar{\psi}_5\psi_6$, $\bar{\psi}_5\psi_6\bar{\psi}_3\psi_4$, $\bar{\psi}_3\bar{\psi}_5\psi_6\psi_4$, and $\bar{\psi}_5\bar{\psi}_3\psi_4\psi_6$, whereas for $\mathcal{A}_{5,2}$ the allowed permutations are $\bar{\psi}_3\psi_4 g_5$, $\bar{\psi}_3 g_5\psi_4$, and $g_5\bar{\psi}_3\psi_4$. The multiplicity of the Melia basis is given by

$$\frac{(2k-2)!}{k!(k-1)!} \times \binom{n-2}{2k-2} \times (n-2k)! \times (k-1)! \;=\; \frac{(n-2)!}{k!} \tag{2.10}$$

as found in ref. [20].

Having chosen the color-ordered amplitudes in the Melia basis $A(1,\gamma,2)$ to be the primitive amplitudes in a proper decomposition (2.3), one may ask what is the corresponding set of independent color factors $C_{1\gamma 2}$? Johansson and Ochirov (JO) posed and solved this problem in ref. [13]. For $k=1$, all permutations γ of $\{3,\cdots,n\}$ are allowed, and the corresponding color factors are given by eq. (2.7), which we can conveniently denote as $\{1|\gamma(3)\cdots\gamma(n)|2\}$. For $k>1$, the JO color factors $C_{1\gamma 2}$ are given by linear combinations of cubic color factors c_i. To obtain these linear combinations, one starts with $\{1|\gamma(3)\cdots\gamma(n)|2\}$ and then replaces each $\bar{\psi}_a$ appearing in γ with the expression $\{a|\, T^{\mathsf{b}} \otimes \Xi^{\mathsf{b}}_{l-1}$, each ψ_a with the expression $|a\}$, and each gluon g_a with the operator $\Xi^{\mathsf{a}_a}_l$, where

$$\Xi^{\mathsf{a}}_l \;=\; \sum_{s=1}^{l} \underbrace{1\otimes\cdots\otimes 1\otimes \overbrace{T^{\mathsf{a}}\otimes 1\otimes\cdots\otimes 1}^{s}}_{l}\,. \tag{2.11}$$

The integer l denotes the level of bracket "nestedness" at the point where $\Xi^{\mathsf{a}_a}_l$ is inserted, i.e., the number of left brackets minus the number of right brackets to the left of the operator. The operator Ξ^{a}_l is a tensor product of l copies of the Lie algebra, where the sum runs over each position s in the tensor product. Each copy of the Lie algebra representation corresponds to a particular nestedness level, starting from level l (the leftmost copy) down to level one (the rightmost copy). The $\{a|$ and $|a\}$ act only on the copy of the Lie algebra at their corresponding nestedness level l. The operator Ξ^{a}_l is conveniently represented by figure 1, in which the open circles represent summation over the possible locations where the gluon line can attach. For $k=1$, $\Xi^{\mathsf{a}}_1 = T^{\mathsf{a}}$ so the JO prescription simply reduces to eq. (2.7). For $k>1$, we end up with (a linear combination of) k strings of generators of the form (2.7). For a more detailed description and justification of the JO color factors and many illuminating examples, we refer the reader to ref. [13].

Having defined the primitive amplitudes and corresponding color factors, we can write the proper (Melia-Johansson-Ochirov) decomposition of the amplitude $\mathcal{A}_{n,k}$ as

$$\mathcal{A}_{n,k}(\bar{\psi}_1,\psi_2,\bar{\psi}_3,\psi_4,\cdots,\bar{\psi}_{2k-1},\psi_{2k},g_{2k+1},\cdots,g_n) \;=\; \sum_{\gamma\in \text{Melia basis}} C_{1\gamma 2} A(1,\gamma(3),\cdots,\gamma(n),2). \tag{2.12}$$

This expression is equivalent to eq. (2.1), as was explicitly verified for $n\le 8$ (and for $n=9$, $k=4$) in ref. [13], and was proven for all n in ref. [21]. Noting the similarity to eq. (2.6), the JO color factors $C_{1\gamma 2}$ can be considered the natural generalization of the half-ladder color factors eq. (2.7). We will see more evidence of this correspondence in section 4.

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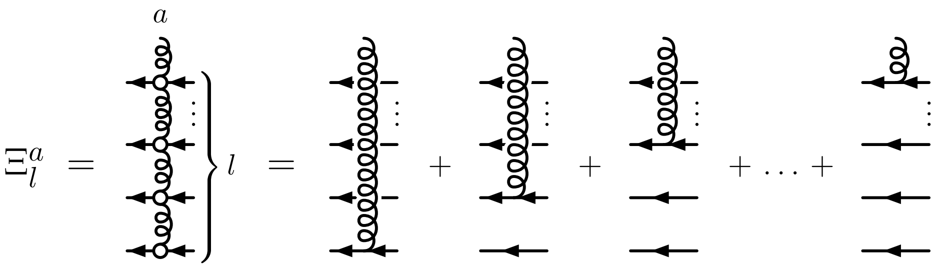

Figure 1. Diagrammatic form of the operator Ξ^{a}_{l}.

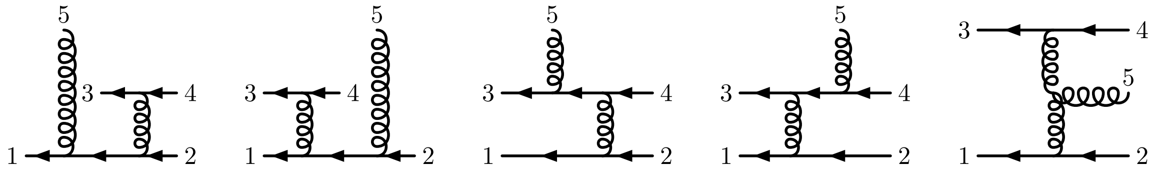

Figure 2. Color factors c_1 through c_5 for $\mathcal{A}_{5,2}$.

We conclude this section with a concrete example of the ideas discussed above, the five-point amplitude with one gluon and two pairs of fundamentals $\mathcal{A}_{5,2}(\bar{\psi}_1, \psi_2, \bar{\psi}_3, \psi_4, g_5)$. Five cubic diagrams contribute to this amplitude (see figure 2)

$$\mathcal{A}_{5,2} \;=\; \sum_{i=1}^{5} \frac{c_i\; n_i}{d_i} \tag{2.13}$$

where the color factors and denominators have the form [13]

$$\begin{aligned}
c_1 &= (T^{\mathsf{a}_5}T^{\mathsf{b}})^{\mathsf{i}_1}{}_{\mathsf{i}_2}(T^{\mathsf{b}})^{\mathsf{i}_3}{}_{\mathsf{i}_4}\,, && d_1 = (s_{15}-m_1^2)s_{34} = 2s_{34}\,k_1\cdot k_5\;,\\
c_2 &= (T^{\mathsf{b}}T^{\mathsf{a}_5})^{\mathsf{i}_1}{}_{\mathsf{i}_2}(T^{\mathsf{b}})^{\mathsf{i}_3}{}_{\mathsf{i}_4}\,, && d_2 = (s_{25}-m_1^2)s_{34} = 2s_{34}\,k_2\cdot k_5\;,\\
c_3 &= (T^{\mathsf{b}})^{\mathsf{i}_1}{}_{\mathsf{i}_2}(T^{\mathsf{a}_5}T^{\mathsf{b}})^{\mathsf{i}_3}{}_{\mathsf{i}_4}\,, && d_3 = s_{12}(s_{35}-m_3^2) = 2s_{12}\,k_3\cdot k_5\;,\\
c_4 &= (T^{\mathsf{b}})^{\mathsf{i}_1}{}_{\mathsf{i}_2}(T^{\mathsf{b}}T^{\mathsf{a}_5})^{\mathsf{i}_3}{}_{\mathsf{i}_4}\,, && d_4 = s_{12}(s_{45}-m_3^2) = 2s_{12}\,k_4\cdot k_5\;,\\
c_5 &= f^{\mathsf{a}_5\mathsf{bc}}(T^{\mathsf{b}})^{\mathsf{i}_1}{}_{\mathsf{i}_2}(T^{\mathsf{c}})^{\mathsf{i}_3}{}_{\mathsf{i}_4}\,, && d_5 = s_{12}s_{34}
\end{aligned} \tag{2.14}$$

with $s_{ij} = (k_i + k_j)^2$. The kinematic numerators n_i depend on the spin of the fundamentals; explicit expressions for spin one-half fundamentals are given in ref. [13]. By virtue of $[T^{\mathsf{a}}, T^{\mathsf{b}}]^{\mathsf{i}}{}_{\mathsf{j}} = f_{\mathsf{abc}}(T^{\mathsf{c}})^{\mathsf{i}}{}_{\mathsf{j}}$, the color factors obey two Jacobi relations

$$c_1 - c_2 + c_5 \;=\; 0\,, \qquad c_3 - c_4 - c_5 \;=\; 0\,. \tag{2.15}$$

Thus the proper decomposition contains three terms. The Melia basis primitive amplitudes are

$$A(1,5,3,4,2) = \frac{n_1}{d_1} - \frac{n_3}{d_3} - \frac{n_5}{d_5}, \quad A(1,3,5,4,2) = \frac{n_3}{d_3} + \frac{n_4}{d_4}, \quad A(1,3,4,5,2) = \frac{n_2}{d_2} - \frac{n_4}{d_4} + \frac{n_5}{d_5} \tag{2.16}$$

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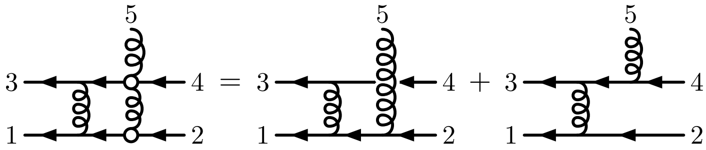

Figure 3. Color factor C_{13542} for the amplitude $\mathcal{A}_{5,2}$.

and the corresponding JO color factors are

$$\begin{aligned}
C_{15342} &= \{1|T^{\mathsf{a}_5}\{3|T^{\mathsf{b}} \otimes T^{\mathsf{b}}|4\}|2\} \;=\; c_1\,, \\
C_{13542} &= \{1|\{3|(T^{\mathsf{b}} \otimes T^{\mathsf{b}})\Xi_2^{a_5}|4\}|2\} \;=\; c_2 + c_4\,, \\
C_{13452} &= \{1|\{3|T^{\mathsf{b}} \otimes T^{\mathsf{b}}|4\}T^{\mathsf{a}_5}|2\} \;=\; c_2
\end{aligned} \tag{2.17}$$

where C_{13542} is represented graphically in figure 3. It is straightforward to check that the proper decomposition

$$\mathcal{A}_{5,2} \;=\; C_{15342}\,A(1,5,3,4,2) \;+\; C_{13542}\,A(1,3,5,4,2) \;+\; C_{13452}\,A(1,3,4,5,2) \tag{2.18}$$

is equal to eq. (2.13).

3 BCJ relations for QCD amplitudes

In refs. [10, 13], Johansson and Ochirov explored whether tree-level QCD amplitudes with both gluons and quarks obey color-kinematic duality; that is, whether there exists a generalized gauge in which the kinematic numerators n_i in the cubic decomposition (2.1) satisfy the same algebraic relations as do the color factors c_i for amplitudes with particles in the adjoint and fundamental representations (see also ref. [14]). Color-kinematic duality is well-established for pure gluon amplitudes [1, 23, 30], for massless particles in supersymmetric Yang-Mills multiplets that contain gluons [22, 31–38], or only matter [10, 39, 40] (see also refs. [41–44] for a string-theoretic approach). The Cachazo-He-Yuan (CHY) representation for gauge theory amplitudes naturally encodes color-kinematic duality [8]. In ref. [13], Johansson and Ochirov established color-kinematic duality for amplitudes $\mathcal{A}_{n,k}$ with k pairs of quarks and $n-2k$ gluons through explicit calculations for $n \leq 8$. For certain low-multiplicity amplitudes (specifically, $\mathcal{A}_{4,1}$, $\mathcal{A}_{5,2}$, and $\mathcal{A}_{6,3}$) they found that the numerators derived from Feynman rules automatically satisfy the kinematic Jacobi relations. For higher-multiplicity amplitudes, a generalized gauge transformation is required to bring the kinematic numerators into a form that is manifestly color-kinematic dual.

As discussed in the introduction, color-kinematic duality implies new relations among color-ordered amplitudes for tree-level n-gluon amplitudes [1]. These were subsequently proven using string theory [2, 3] and on-shell recursion [4, 5], and necessarily hold for massless CHY amplitudes [6, 7]. BCJ relations have also been established for amplitudes containing gluons and a pair of massive scalars in the fundamental representation by expressing these amplitudes in a CHY representation [11]. See ref. [12] for earlier work on a CHY representation for amplitudes containing gluons and massless quark-antiquark pairs.

Also as noted earlier, Johansson and Ochirov derived relations among the primitive amplitudes in the Melia basis for $\mathcal{A}_{n,k}$ that follow from color-kinematic duality; these are the $k \geq 1$ analogs of the BCJ relations [13]. They proceeded by expressing the Melia primitive amplitudes as linear combinations of n_i/d_i, and then imposing the Jacobi relations on the kinematic numerators. Inverting these equations, they obtained equations for a subset of the n_i in terms of the primitive amplitudes, together with (for $n > 2k$) equations among the primitive amplitudes. Among their findings are that there are no relations among the primitive amplitudes for amplitudes containing no gluons ($n = 2k$). For amplitudes containing gluons, they found that the primitive amplitudes obey relations such as

$$\sum_{b=3}^{n} \left(k_n \cdot k_1 + \sum_{c=3}^{b-1} k_n \cdot k_{\sigma(c)} \right) A(1, \sigma(3), \cdots, \sigma(b-1), n, \sigma(b), \cdots, \sigma(n-1), 2) \;=\; 0 \quad (3.1)$$

where n denotes a gluon, and σ is a permutation of $\{3, \cdots, n-1\}$. Equation (3.1) has exactly the same form (when expressed in terms of invariants $k_a \cdot k_b$ where k_a is the momentum of a gluon) as one of the fundamental BCJ relations [2, 4, 45] for an n-gluon amplitude. Johansson and Ochirov found that relations of the form (3.1) are satisfied when n is replaced by the label of any of the other gluons in the amplitude, but not when n is replaced by the label of a fundamental or antifundamental particle. Thus, the relations that hold for $k \geq 2$ are a proper subset of the BCJ relations for $k = 0$ or $k = 1$. These relations were obtained by Johansson and Ochirov from explicit calculations for $n \leq 8$ and for $n = 9$, $k = 4$, and were proved for all n using BCFW on-shell recursion by de la Cruz, Kniss, and Weinzierl [15]. The latter authors also presented a CHY representation for these amplitudes [16] (see also ref. [18]).

We will see in the next two sections that these results have a natural explanation in terms of the color-factor symmetry possessed by the amplitude. There is a set of color-factor shifts associated with each external gluon in the amplitude and these give rise to the corresponding BCJ relations. The absence of color-factor symmetry for amplitudes with no gluons explains the absence of relations among the Melia primitive amplitudes.

For amplitudes containing only gluons, or amplitudes with gluons and one pair of fundamentals, the fundamental BCJ relations allow one to express the $(n-2)!$ amplitudes of the Kleiss-Kuijf basis in terms of a smaller basis of $(n-3)!$ amplitudes [1]. Johansson and Ochirov found that for $k \geq 2$, the BCJ relations allow one to express the $(n-2)!/k!$ amplitudes of the Melia basis in terms of a reduced basis of $(n-3)!(2k-2)/k!$ amplitudes [13]. The difference between $(n-2)!/k!$ and $(n-3)!(2k-2)/k!$ is precisely equal to $(n-2k)(n-3)!/k!$, which as we will see is the dimension of the color-factor group for $k \geq 2$.

4 Color-factor shifts

In ref. [9], we introduced the color-factor symmetry, and proved that gauge-theory amplitudes containing gluons and massless or massive particles in an arbitrary representation of the gauge group and with arbitrary spin ≤ 1 are invariant under certain momentum-dependent shifts of the color factors. In this section, we review the definition of these shifts, and determine how they act on the Johansson-Ochirov color factors described in section 2.

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Associated with each external gluon in the amplitude is a set of symmetries that act as momentum-dependent shifts of the color factors appearing in the cubic decomposition (2.1). Consider a tree-level color factor c_i for an amplitude with an external gluon a. The gluon leg divides the diagram in two at its point of attachment. Denote by $S_{a,i}$ the subset of the remaining legs on one side of this point; it does not matter which side. The action of the shift $\delta_a c_i$ is constrained by two requirements: (I) that it preserve all the Jacobi relations satisfied by c_i, and (II) that it satisfy

$$\delta_a c_i \propto \sum_{c \in S_{a,i}} k_a \cdot k_c \tag{4.1}$$

where all momenta are outgoing. (Choosing to sum over the complement of $S_{a,i}$ gives the same result up to sign due to momentum conservation.) In particular, if gluon a is attached to an external leg b with momentum k_b, the shift is proportional to $k_a \cdot k_b$.

Let $\{c_i\}$ be the set of n-point color factors, and consider the subset of them obtained from a fixed $(n-1)$-point cubic diagram by attaching gluon a in all possible ways. One of these has gluon a attached to external leg 1 of the $(n-1)$-point diagram; define its shift to be $\alpha\, k_a \cdot k_1$. Then the conditions (I) and (II) above uniquely fix the coefficients of the shifts of all the other color factors in this subset. For example, in the five-point example discussed in the previous section (see figure 2), we define the color-factor shift associated with gluon 5 (the only external gluon in the amplitude) to act on c_1 as

$$\delta_5 c_1 \equiv \alpha\ k_5 \cdot k_1\,. \tag{4.2}$$

Then using eq. (2.15), we find the shifts of the other four color factors to be

$$\delta_5 c_2 = -\alpha\ k_5 \cdot k_2\,, \qquad \delta_5 c_3 = \alpha\ k_5 \cdot k_3\,, \qquad \delta_5 c_4 = -\alpha\ k_5 \cdot k_4\,, \qquad \delta_5 c_5 = -\alpha\ k_5 \cdot (k_1 + k_2)\,. \tag{4.3}$$

The shifts of the JO color factors (2.17) associated with this amplitude are then

$$\delta_5 C_{15342} \ = \ \alpha\ k_1 \cdot k_5\,, \qquad \delta_5 C_{13542} \ = \ \alpha\ (k_1 + k_3) \cdot k_5\,, \qquad \delta_5 C_{13452} \ = \ \alpha\ (k_1 + k_3 + k_4) \cdot k_5 \tag{4.4}$$

where a clear pattern emerges: the shift $\delta_a C_{\cdots a \cdots}$ depends on the sum of momenta of the particles whose labels appear to the left of a in $C_{\cdots a \cdots}$.

Now consider a general amplitude $\mathcal{A}_{n,k}$ with at least one gluon. There is a set of color-factor symmetries for each of the $n-2k$ gluons, but to simplify the presentation, we will focus on the shift associated with gluon n. Consider the set of JO color factors $C_{1n\sigma 2}$ where σ is a permutation of $\{3, \cdots, n-1\}$. Since σ is a permutation of $n-2k-1$ gluons, k $\bar{\psi}$'s, and k ψ's, where the $\bar{\psi}$'s and ψ's form a Dyck word of length $2k-2$, the number of allowed choices of σ is $(n-3)!/k!$. Each $C_{1n\sigma 2}$ is a linear combination of cubic color factors c_i, in each of which gluon n is attached to external leg 1, and which consequently have a shift proportional to $k_n \cdot k_1$. We therefore define the shift of $C_{1n\sigma 2}$ associated with gluon n to be

$$\delta_n\ C_{1n\sigma 2} \ \equiv \ \alpha_{n,\sigma}\, k_n \cdot k_1 \tag{4.5}$$

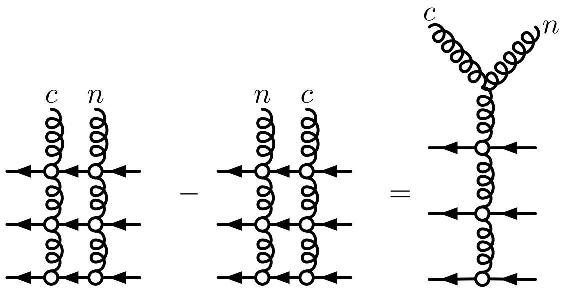

Figure 4. Commutator $C_{\cdots[cn]\cdots}$ for gluon g_c.

where $\alpha_{n,\sigma}$ are a set of $(n-3)!/k!$ independent arbitrary constants. Given eq. (4.5), the shifts $\delta_n c_i$ of all other color factors are then uniquely determined. The proof of this is as follows. The JO color factors form an independent basis in terms of which all the color factors c_i can be expressed. In particular, all color factors c_i with gluon n attached to external leg 1 can be expressed in terms of the JO color factors $C_{1n\sigma 2}$, and therefore their shifts under δ_n are determined by eq. (4.5). But we argued above that the shifts of all color factors are fixed once we know the shifts of the color factors that have gluon n attached to external leg 1.

Therefore we have shown that associated with each gluon is an $(n-3)!/k!$-parameter family of color-factor shifts. Including all the gluons, the color-factor shifts form an abelian group of dimension $(n-2k)(n-3)!/k!$ for $k \geq 2$. For $k=1$, the color-factor shift associated with one of the gluons is a linear combination of those associated with all the others [9] and thus in that case the dimension of the color-factor group is $(n-3)(n-3)!$.

We now show that, given eq. (4.5), the shifts of the rest of the JO color factors are particularly simple, viz.

$$\delta_n \; C_{1\sigma(3)\cdots\sigma(b-1)n\sigma(b)\cdots\sigma(n-1)2} \;=\; \alpha_{n,\sigma}\left(k_n\cdot k_1+\sum_{c=3}^{b-1}k_n\cdot k_{\sigma(c)}\right), \qquad b=3,\cdots,n\,. \tag{4.6}$$

We note that eq. (4.6) has exactly the same form as the shifts of the half-ladder color factors in ref. [9]; thus with respect to the color-factor symmetry, the JO color factors are the precise analog of the half-ladder color factors. To establish eq. (4.6), we consider the "commutator" of JO color factors

$$C_{1\sigma(3)\cdots\sigma(b-1)[\sigma(b)n]\sigma(b+1)\cdots\sigma(n-1)2} \;\equiv\; C_{1\sigma(3)\cdots\sigma(b)n\sigma(b+1)\cdots\sigma(n-1)2} - C_{1\sigma(3)\cdots\sigma(b-1)n\sigma(b)\cdots\sigma(n-1)2} \tag{4.7}$$

or more briefly $C_{\cdots[cn]\cdots}$, where we let $c=\sigma(b)$. We can most transparently compute this commutator using the inspired graphical notation of ref. [13].

First, let c label another gluon; then $C_{\cdots[cn]\cdots}$ is represented in figure 4. The two diagrams on the l.h.s. only fail to commute when the gluons are attached to the same line, in which case one can use $[T^{\mathsf{a}},T^{\mathsf{b}}]^{\mathsf{i}}{}_{\mathsf{j}} = f_{\mathsf{abc}}(T^{\mathsf{c}})^{\mathsf{i}}{}_{\mathsf{j}}$ to give the diagram on the right. Figure 4 is the graphical depiction of the identity [13]

$$\left[\Xi_l^{\mathsf{a}},\,\Xi_l^{\mathsf{b}}\right] \;=\; f_{\mathsf{abc}}\,\Xi_l^{\mathsf{c}}\,. \tag{4.8}$$

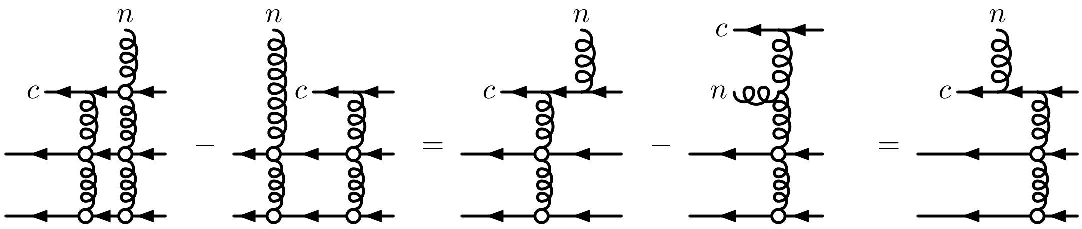

Figure 5. Commutator $C_{\cdots[cn]\cdots}$ for $\bar{\psi}_c$.

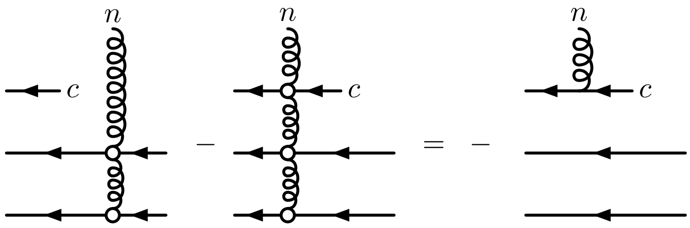

Figure 6. Commutator $C_{\cdots[cn]\cdots}$ for ψ_c.

The diagram on the r.h.s. of figure 4 is a linear combination of color factors in which gluon n is attached to external gluon c, and which therefore undergo a shift proportional to $k_n \cdot k_c$ under the color-factor symmetry associated with gluon n. A little bit of thought shows that the coefficient of proportionality of the shift of the commutator is $\alpha_{n,\sigma}$ and therefore

$$\delta_n C_{1\sigma(3)\cdots\sigma(b-1)[\sigma(b)n]\sigma(b+1)\cdots\sigma(n-1)2} = \alpha_{n,\sigma} k_n \cdot k_{\sigma(b)} \,. \tag{4.9}$$

Next, let c be the label of an antifundamental $\bar{\psi}_c$; the commutator $C_{\cdots[cn]\cdots}$ is then represented by figure 5. The final diagram of the figure is a linear combination of color factors with gluon n attached to $\bar{\psi}_c$, whose shifts are proportional to $k_n \cdot k_c$. Again a bit of thought shows that the shift of this diagram is given by eq. (4.9).

Finally, consider the case where c labels a fundamental ψ_c; the commutator $C_{\cdots[cn]\cdots}$ is shown in figure 6. The r.h.s. of the figure is a linear combination of color factors with gluon n attached to ψ_c. Taking account of the minus sign, the shift of the commutator is given by eq. (4.9).

We can now apply eq. (4.9) recursively starting with eq. (4.5) to obtain eq. (4.6). The shift of the last JO factor is thus

$$\delta_n C_{1\sigma(3)\cdots\sigma(n-1)n2} = \alpha_{n,\sigma}\left(k_n \cdot k_1 + \sum_{c=3}^{n-1} k_n \cdot k_{\sigma(c)}\right) = -\,\alpha_{n,\sigma} k_n \cdot k_2 \tag{4.10}$$

which is consistent with the fact that $C_{1\sigma(3)\cdots\sigma(n-1)n2}$ represents a linear combination of color factors in which gluon n is attached to ψ_2.

In the next section we will use eq. (4.6) in the Melia-Johansson-Ochirov decomposition to obtain the BCJ relations for $\mathcal{A}_{n,k}$.

5 BCJ relations from color-factor symmetry

In this section, we establish that the BCJ relations obtained by Johansson and Ochirov for $\mathcal{A}_{n,k}$ are a direct consequence of the color-factor symmetry of the amplitude. We also show that the kinematic numerators for $\mathcal{A}_{n,k}$ obey a set of gauge-invariant constraints that are less stringent than the kinematic Jacobi relations, but which follow from the color-factor symmetry and are therefore sufficient to imply the BCJ relations.

It was shown in ref. [9] that gauge-theory amplitudes with gluons as well as massless or massive particles in an arbitrary representation of the gauge group and arbitrary spin ≤ 1, and therefore specifically $\mathcal{A}_{n,k}$, are invariant under the family of color-factor shifts described in section 4. The proof of this uses the radiation vertex expansion of the amplitude [46]. A full description of the radiation vertex expansion and the proof of color-factor symmetry is given in ref. [9], but the basic strategy is as follows. The radiation vertex expansion is a recursive approach that constructs an n-point amplitude by attaching a gluon in all possible ways to all possible diagrams that contribute to the $(n-1)$-point amplitude consisting of all the particles except for a chosen gluon a. We may attach the gluon to an external leg, an internal line, or to one of the cubic ggg or $\bar{\psi}g\psi$ vertices (for ψ a scalar or vector) to make a quartic vertex. Then all the contributions are reorganized into a sum over the legs of each of the vertices of each of the $(n-1)$-point diagrams. The next step is to consider the action of the color-factor shift associated with gluon a on the color factors appearing in the radiation vertex expansion. One proves that the sum over legs for each vertex is invariant under the shift of color factors. It immediately follows that the entire n-point amplitude is invariant under the color-factor symmetry associated with gluon a, viz., $\delta_a \mathcal{A}_{n,k} = 0$.

Now consider an amplitude $\mathcal{A}_{n,k}$ with at least one gluon n, and consider the effect of the color-factor shift δ_n associated with this gluon on the amplitude written in the Melia-Johansson-Ochirov proper decomposition (2.12). Since the action of the shift on the Johansson-Ochirov color factors $C_{1\gamma 2}$ is given by eq. (4.6), the shift acts on eq. (2.12) as

$$\delta_n \mathcal{A}_{n,k} = \sum_{\sigma \in \text{Melia basis}} \alpha_{n,\sigma} \sum_{b=3}^{n} \left(k_n \cdot k_1 + \sum_{c=3}^{b-1} k_n \cdot k_{\sigma(c)} \right) A(1,\sigma(3),\cdots,\sigma(b-1),n,\sigma(b),\cdots,\sigma(n-1),2). \tag{5.1}$$

and since $\alpha_{n,\sigma}$ are independent parameters, we conclude that

$$\sum_{b=3}^{n} \left(k_n \cdot k_1 + \sum_{c=3}^{b-1} k_n \cdot k_{\sigma(c)} \right) A(1,\sigma(3),\cdots,\sigma(b-1),n,\sigma(b),\cdots,\sigma(n-1),2) \;=\; 0 \tag{5.2}$$

precisely the fundamental BCJ relations obtained in ref. [13]. The BCJ relations with n replaced by another gluon a follow from the invariance of the amplitude under the color-factor shift associated with gluon a. There is no color-factor symmetry associated with gluonless amplitudes, and therefore no BCJ relations among the Melia primitive amplitudes are expected in that case, as was found in ref. [13]. In section 4, we showed that the dimension of color-factor group is $(n-2k)(n-3)!/k!$ for $k \geq 2$, which reduces the number of independent primitives from the Melia basis of $(n-2)!/k!$ to $(2k-2)(n-3)!/k!$

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as found in ref. [13]. (For $k = 1$, the color-factor group has dimension $(n-3)(n-3)!$ and thus reduces the number of independents from $(n-2)!$ to $(n-3)!$.)

Although the BCJ relations (3.1) were previously proven using on-shell BCFW recursion [16], the proof in this paper based on color-factor symmetry reveals a close connection between the BCJ relations and the symmetries of the Lagrangian formulation of gauge theory. This connection is made explicit in the radiation vertex expansion proof of color-factor symmetry given in ref. [9], and summarized in the discussion section of that paper, which we briefly recap here. The variation of the amplitude under the color-factor shift associated with gluon a can be separated into contributions that are constant, linear, and quadratic in the gluon momentum k_a. The $\mathcal{O}(k_a^0)$ term is proportional to $\sum_r \varepsilon_a \cdot K_r$, where K_r are the momenta flowing out of each vertex. This vanishes by $\varepsilon_a \cdot k_a = 0$ together with momentum conservation (a result of spacetime translation invariance of the Lagrangian). The $\mathcal{O}(k_a^1)$ term of the variation of the amplitude is given by a sum of angular momentum generators $J_r^{\alpha\beta}$, which act as a first-order Lorentz transformation on the relevant vertex factors. These terms vanish by Lorentz invariance of the Lagrangian. The vanishing of the $\mathcal{O}(k_a^2)$ term of the variation of the amplitude is more subtle, but relies on Poincaré invariance together with Yang-Mills gauge symmetry. Thus, the vanishing of the variation of the amplitude under the color-factor shift (and therefore the BCJ relations) is closely tied to (if not quite a direct consequence of) the gauge and Poincaré symmetries of the gauge theory.

We also introduced in ref. [9] the cubic vertex expansion of an amplitude containing at least one gluon. (This is related to, but distinct from, the radiation vertex expansion.) Consider the set of cubic diagrams I that contribute to the $(n-1)$-point amplitude of all the particles in $\mathcal{A}_{n,k}$ except for gluon a. For any a, the amplitude $\mathcal{A}_{n,k}$ can be written as a triple sum over the legs r of the vertices v of the cubic diagrams I:

$$\mathcal{A}_{n,k} \;=\; \sum_I \sum_v \frac{1}{\prod_{s=1}^3 d_{(a,I,v,s)}} \sum_{r=1}^3 \frac{c_{(a,I,v,r)} n_{(a,I,v,r)}}{2k_a \cdot K_{(a,I,v,r)}} \,. \tag{5.3}$$

Here $d_{(a,I,v,r)}$ is the product of propagators[4] that branch off from leg r of vertex v of diagram I, $K_{(a,I,v,r)}$ is the momentum flowing out of that leg, $c_{(a,I,v,r)}$ is the color factor of the n-point diagram obtained by attaching gluon a to leg r of vertex v of diagram I, and $n_{(a,I,v,r)}$ is the n-point kinematic numerator associated with that color factor. As a concrete example, consider the five-point amplitude discussed in section 2. Using the identity

$$\frac{1}{s_{12}s_{34}} \;=\; \frac{1}{2s_{34}(-k_1-k_2)\cdot k_5} \;+\; \frac{1}{2s_{12}(k_1+k_2)\cdot k_5} \tag{5.4}$$

it is straightforward to write eqs. (2.13) and (2.14) as

$$\mathcal{A}_{5,2} \;=\; \frac{1}{s_{34}}\left[\frac{c_1 n_1}{2k_1{\cdot}k_5}+\frac{c_2 n_2}{2k_2{\cdot}k_5}+\frac{c_5 n_5}{2(-k_1-k_2){\cdot}k_5}\right]+\frac{1}{s_{12}}\left[\frac{c_3 n_3}{2k_3{\cdot}k_5}+\frac{c_4 n_4}{2k_4{\cdot}k_5}+\frac{c_5 n_5}{2(k_1+k_2){\cdot}k_5}\right] \tag{5.5}$$

which is precisely of the form (5.3).

[4]If leg r is external, then $d_{(a,I,v,r)} = 1$.

The color factors appearing in eq. (5.3) obey $\delta_a\, c_{(a,I,v,r)} \;=\; \alpha_{(a,I,v)}\, k_a \cdot K_{(a,I,v,r)}$ under the shift associated with gluon a. Since $\delta_a \mathcal{A}_{n,k} = 0$, we may conclude from the cubic vertex expansion (5.3) that

$$\sum_I \sum_v \frac{\alpha_{(a,I,v)}}{\prod_{s=1}^{3} d_{(a,I,v,s)}} \sum_{r=1}^{3} n_{(a,I,v,r)} \;=\; 0\,. \tag{5.6}$$

Because the $\alpha_{(a,I,v)}$ are not independent[5] we may not draw the more stringent conclusion that $\sum_{r=1}^{3} n_{(a,I,v,r)} \;=\; 0$ for each vertex. For the five-point amplitude considered above, eq. (5.6) yields only one constraint

$$0 \;=\; \delta_5 \mathcal{A}_{5,2} \;=\; \frac{\alpha}{2}\left[\frac{n_1 - n_2 + n_5}{s_{34}} \;+\; \frac{n_3 - n_4 - n_5}{s_{12}}\right] \tag{5.7}$$

rather than the two kinematic Jacobi relations[6] $n_1 - n_2 + n_5 = 0$ and $n_3 - n_4 - n_5 = 0$. In general, color-kinematic duality states that a generalized gauge transformation exists such that the numerators *in that gauge* obey the kinematic Jacobi relations. The relation (5.6), however, holds for the kinematic numerators in any gauge, since it is invariant under generalized gauge transformations, as can be seen from its derivation.

6 Conclusions

In this paper, we have shown that BCJ relations [13] among the Melia-basis primitive amplitudes of $\mathcal{A}_{n,k}$ with $n - 2k$ gluons and k pairs of particles in the fundamental (or other) representation of the gauge group follow directly from the invariance of $\mathcal{A}_{n,k}$ under a set of color-factor shifts. We have also derived as a consequence of this symmetry a set of gauge-invariant constraints on the kinematic numerators of $\mathcal{A}_{n,k}$.

The tree-level color-factor symmetry has been proven for a wide class of gauge-theory amplitudes, including those with massless or massive particles with gauge-theory couplings in arbitrary representations of the gauge group and arbitrary spin ≤ 1 [9]. This is connected to the radiation symmetry coming out of theorems on photon radiation zeros in refs. [46, 47]. The color-factor symmetry also applies to theories with gauge bosons that become massive through spontaneous symmetry breaking (e.g., see refs. [17, 48, 49]). The only particles in the amplitude that need be massless are the gluons (or photons) with which the color-factor symmetries are associated. Thus it applies to standard-model gauge theory amplitudes as well as to many extensions thereof.

BCJ relations are constraints among gauge-invariant primitive amplitudes, the coefficients in a proper decomposition of the gauge-theory amplitude. Such decompositions have been identified for tree-level and one-loop n-gluon amplitudes in ref. [27] and for the

[5]The set of $\alpha_{(a,I,v)}$ for all the vertices of a given diagram I are equal (up to signs) because any two adjacent vertices share a common color factor. Moreover $\alpha_{(a,I,v)}$ must respect the Jacobi relations among the color factors for different diagrams I.

[6]For this five-point example with spin one-half fundamentals, Johansson and Ochirov found that the numerators derived from the Feynman rules automatically satisfy the kinematic Jacobi relations.

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tree-level amplitudes considered in this paper in refs. [13, 19–21]. Once proper decompositions for more general amplitudes have been identified, relations among their primitive amplitudes will follow as a consequence of color-factor symmetry.

Finally, it was shown in ref. [9] that one-loop amplitudes that have color-kinematic-dual representations are invariant under a loop-level generalization of the color-factor symmetry, although a proof based on Lagrangian methods is still lacking. One can legitimately hope that color-factor symmetry will soon lead to many new insights into gauge theories in general and color-kinematic duality in particular.

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