Published for SISSA by Springer

Received: July 29, 2016
Accepted: October 26, 2016
Published: November 9, 2016

Looijenga's weighted projective space, Tate's algorithm and Mordell-Weil lattice in F-theory and heterotic string theory

Shun'ya Mizoguchi[a,b] **and Taro Tani**[c]

[a] *Theory Center, Institute of Particle and Nuclear Studies, KEK, 1-1 Oho, Tsukuba, Ibaraki, 305-0801 Japan*
[b] *SOKENDAI (The Graduate University for Advanced Studies), 1-1 Oho, Tsukuba, Ibaraki, 305-0801 Japan*
[c] *National Institute of Technology, Kurume College, 1-1-1 Komorino, Kurume, Fukuoka, 830-8555 Japan*

E-mail: mizoguch@post.kek.jp, tani@kurume-nct.ac.jp

Abstract: It is now well known that the moduli space of a vector bundle for heterotic string compactifications to four dimensions is parameterized by a set of sections of a weighted projective space bundle of a particular kind, known as Looijenga's weighted projective space bundle. We show that the requisite weighted projective spaces and the Weierstrass equations describing the spectral covers for gauge groups E_N $(N = 4, \cdots, 8)$ and $\mathrm{SU}(n+1)$ $(n = 1, 2, 3)$ can be obtained systematically by a series of blowing-up procedures according to Tate's algorithm, thereby the sections of correct line bundles claimed to arise by Looijenga's theorem can be automatically obtained. They are nothing but the four-dimensional analogue of the set of independent polynomials in the six-dimensional F-theory parameterizing the complex structure, which is further confirmed in the constructions of D_4, A_5, D_6, E_3 and $\mathrm{SU}(2) \times \mathrm{SU}(2)$ bundles. We also explain why we can obtain them in this way by using the structure theorem of the Mordell-Weil lattice, which is also useful for understanding the relation between the singularity and the occurrence of chiral matter in F-theory.

Keywords: Superstrings and Heterotic Strings, F-Theory

ArXiv ePrint: 1607.07280

doi:10.1007/JHEP11(2016)053

JHEP11(2016)053

Contents

1 Introduction

It is now a well-known fact that holomorphic vector bundles on an elliptically fibered Calabi-Yau, needed for heterotic string compactifications to four dimensions, are constructed by considering spectral covers [1]. A spectral cover is basically a ramified n-fold cover (for an $\mathrm{SU}(n)$ bundle) of the base of the elliptic Calabi-Yau, representing the Wilson lines in each elliptic fiber as points on the elliptic fiber identified with its dual. One then introduces a twisting line bundle over the base whose first Chern class (the η class) is related to the number of instantons of the bundle. Once one has a spectral surface and a line bundle over it, one can construct a vector bundle via the Fourier-Mukai transform using the Poincare bundle, up to a so-called γ class corresponding to the G-flux in the F-theory dual. For more detail of the spectral cover construction, see e.g. [1–4].

It is also now well known that the moduli space of the vector bundle is parameterized by a set of sections of a weighted projective space bundle of a particular kind, known as Looijenga's weighted projective space bundle. Some time ago, for E_8, E_7 and E_6 bundles and other lower-rank ones Looijenga's theorem was confirmed [1] (except for some subtlety in E_8) by explicitly constructing spectral covers by using del Pezzo surfaces. Although this approach was enough to reveal the validity of the miraculous nature of Looijenga's theorem, the constructions of the bundles were done case by case and appear to be independent and unrelated with each other. In this paper we will show that the requisite weighted projective spaces and the Weierstrass equations describing the spectral covers for E_8 through A_1 bundles can be obtained systematically by a series of blowing up procedures according to the well-known Tate's algorithm, thereby the sections of correct line bundles claimed to arise by the theorem can be automatically obtained. We will also explain why we can obtain them in this way by using the structure theorem of the Mordell-Weil lattice [5].

We will also show that the structure theorem of the Mordell-Weil lattice is useful for understanding the relation between the singularity and the occurrence of chiral matter in F-theory. (This new role of the Mordell-Weil lattice in F-theory was already briefly discussed in [6].) In the literature the relation between sections and the appearance of chiral matter is somewhat indirect. That is, on the heterotic side one considers a vector bundle on a spectral cover and computes the cohomology by means of the Leray spectral sequence to find that the chiral matter is localized where one or some of the "matter curves" representing the Wilson lines go(es) to infinity (zero in the addition rule of the $\wp$ function). On the F-theory side, the del Pezzo surface (or rational elliptic surface) itself in which the spectral cover is defined becomes the fiber with an appropriate twist corresponding to the weighted projective space bundle of Looijenga, and matter arise where the singularity is enhanced [7–10] along the 7-brane. We will show that the structure of the Mordell-Weil lattice ensures the compatibility of these two pictures of chiral matter generations.

The plan of this paper is as follows. In section 2, we start from the same degree six equation in the weighted projective space $\mathrm{WP}^3_{(1,1,2,3)}$ given in [1] for E_8 bundles and review the construction of the spectral cover. Then we tune some of the sections to be in a special form so that the dP_9 develops a singularity. It turns out that, by blowing up the singularity, we are automatically led to the equation for E_7 bundles discussed in [1], where the relevant sections are precisely the ones constituting the correct weighted projective space of Looijenga. Repeating a similar procedure, we will find a series of spectral covers for the vector bundles from E_7 through A_1. In section 3 we will see that the sections parameterizing a Looijenga's weighted projective space are nothing but the four-dimensional analogue of the set of independent polynomials in the six-dimensional F-theory parameterizing the complex structure of the elliptic manifold with a singularity orthogonal to the gauge group of the vector bundle in the whole E_8. This fact is further confirmed in the constructions of D_4, A_5, D_6, E_3 and $\mathrm{SU}(2)\times\mathrm{SU}(2)$ bundles. In section 4 we discuss why this is possible by introducing the structure theorem of the Mordell-Weil lattice. The final section is devoted to conclusions.

2 Tate's algorithm and Looijenga's weighted projective space

2.1 E_8 bundles: the generic case

We start with the construction of E_8 bundles, following [1]. As pointed out there, it is well known that E_8 bundles have exceptional features but the construction is important because it is the starting point of all the constructions of the vector bundles for other gauge groups of lower ranks.

Let us first consider a generic degree six equation in $\mathrm{WP}^3_{(1,1,2,3)}$ [1] with homogeneous coordinates $(u,v,X,Y)\sim(\lambda u,\lambda v,\lambda^2X,\lambda^3Y)$ $(\lambda\in\mathbf{C})$:

$$\begin{aligned}0 = {} & Y^2+X^3-\frac{g_2}{4}Xv^4-\frac{g_3}{4}v^6\\&+(\beta_4u^4+\beta_3u^3v+\beta_2u^2v^2+\beta_1uv^3)X\\&+(\alpha_6u^6+\alpha_5u^5v+\alpha_4u^4v^2+\alpha_3u^3v^3+\alpha_1uv^5).\end{aligned}\tag{2.1}$$

In a patch $u \neq 0$, we define affine coordinates (z, x, y) by $(u, v, X, Y) \sim (1, \frac{v}{u}, \frac{X}{u^2}, \frac{Y}{u^3}) \equiv (1, z, x, y)$. Then we have

$$\begin{aligned} 0 = y^2 + x^3 - \frac{g_2}{4} x z^4 - \frac{g_3}{4} z^6 \\ +(\beta_4 + \beta_3 z + \beta_2 z^2 + \beta_1 z^3) x \\ +(\alpha_6 + \alpha_5 z + \alpha_4 z^2 + \alpha_3 z^3 + \alpha_1 z^5). \end{aligned} \tag{2.2}$$

(2.1) is a dP_8, and by blowing up at $u = v = 0$ it becomes a rational elliptic surface dP_9 with section [1]. Then it can be viewed as an elliptic fibration over $\mathbb{P}^1$ whose coordinates are $(u : v)$, the affine coordinate being z in the affine patch $u \neq 0$.

To serve as a part of compactifications of F-theory [11] to lower dimensions, this $\mathbb{P}^1$ must be further fibered over some base space $\mathcal{B}$, where the coefficients α_i $(1, 3, \ldots, 6)$ and β_j $(j = 1, \ldots, 4)$ as well as the coordinates are promoted to sections of some appropriate line bundles over the $\mathbb{P}^1$ fibration. More precisely, we regard this dP_9 as a part of an elliptic $K3$ in the stable degeneration limit, which itself is fibered over $\mathcal{B}$ in such a way that the total space is an elliptic Calabi-Yau $\mathcal{Y}$, whose base $\mathcal{W}$ itself is a $\mathbb{P}^1$ fibration over $\mathcal{B}$. We take

$$\begin{aligned} X &\in \Gamma((\mathcal{L} \otimes \mathcal{N})^2), \\ Y &\in \Gamma((\mathcal{L} \otimes \mathcal{N})^3), \\ v &\in \Gamma(\mathcal{N}), \\ u &\in \Gamma(\mathcal{L}^6), \end{aligned} \tag{2.3}$$

where Γ denotes the space of the sections, $\mathcal{L}$ is the anti-canonical line bundle of the base $\mathcal{B}$, and $\mathcal{N}$ is the "twisting" line bundle over $\mathcal{B}$,[1] characterizing the vector bundle of the dual heterotic string theory compactified on an elliptic Calabi-Yau $\mathcal{Z}$ of complex dimension one less, whose complex structure is identical to that of the elliptic fibration at $z = \infty$. At the same time, the fiber of this elliptic fibration at infinity also plays the role of the "dual" torus, at which the values of rational sections of (2.1) describe the spectral surface [1], i.e. the holonomies of the flat connections, and hence the moduli space of the heterotic vector bundle V.[2] Then the affine coordinates (z, x, y) transform as

$$\begin{aligned} z &\in \Gamma(\mathcal{M}), \\ x &\in \Gamma((\mathcal{L} \otimes \mathcal{M})^2), \\ y &\in \Gamma((\mathcal{L} \otimes \mathcal{M})^3) \end{aligned} \tag{2.4}$$

as sections of line bundles over $\mathcal{B}$, where $\mathcal{M} \equiv \mathcal{L}^{-6} \otimes \mathcal{N}$. They are also sections of some line bundles over the $\mathbb{P}^1$ with $(u : v)$ being its coordinates. Due to the Calabi-Yau condition for $\mathcal{Y}$, $\mathcal{W}$ is required to be such that the base $\mathcal{B}$ part of the anti-canonical class of $\mathcal{W}$ coincides with $\mathcal{L} \otimes \mathcal{M}$ (2.4).

For example, if we take $\mathcal{B}$ to be $\mathbb{P}^1$ and $\mathcal{Z}$ to be an elliptic $K3$, then $\mathcal{L}$ is an $\mathcal{O}(2)$ bundle whose sections are described by quadratic polynomials of the affine coordinate z' of the

[1] The first Chern class of $\mathcal{N}$ is customarily referred to as "the η class".

[2] We assume $c_1(V) = 0$ throughout this paper.

base $\mathbb{P}^1$. Also $\mathcal{N}$ is chosen to be an $\mathcal{O}(12+n) = \mathcal{L}^6 \otimes \mathcal{L}^{\frac{n}{2}}$ bundle, corresponding to $12+n$ instantons for one of the two E_8 gauge groups of the six-dimensional heterotic string theory. $\mathcal{M}$ is $\mathcal{L}^{\frac{n}{2}}$. The corresponding dual F-theory description chooses [7, 8] $\mathcal{W}$ to be a Hirzebruch surface $\mathbb{F}_n$ so that the base $\mathbb{P}^1$ part of the anti-canonical class is $\mathcal{L} \otimes \mathcal{M} = \mathcal{O}(2+n)$ (or $\mathcal{O}(2-n)$ depending on the choice of the divisor "at infinity"), geometrically realizing the twisting of the spectral cover of the heterotic dual. Then we are led to the well-known Weierstrass equation on a Hirzebruch surface [7, 8]

$$0 = y^2 + x^3 + f(z,z')x + g(z,z'), \tag{2.5}$$

$$\begin{aligned} f(z,z') &= f_{8+4n} + z f_{8+3n} + z^2 f_{8+2n} + z^3 f_{8+n} + \cdots, \\ g(z,z') &= g_{12+6n} + z g_{12+5n} + z^2 g_{12+4n} + z^3 g_{12+3n} + \cdots, \end{aligned} \tag{2.6}$$

where $f_{8+(4-i)n}$ and $g_{12+(6-j)n}$ $(i,j = 0,1,2,\ldots)$ are polynomials of z' with subscripts being their degrees in z'.[3]

In the general case, we write (2.1) in the Neron-Tate's form:

$$\begin{aligned} 0 = y^2 + x^3 &+ (a_{1,0} + a_{1,1}z + a_{1,2}z^2 + \cdots)xy \\ &+ (a_{2,0} + a_{2,1}z + a_{2,2}z^2 + \cdots)x^2 \\ &+ (a_{3,0} + a_{3,1}z + a_{3,2}z^2 + \cdots)y \\ &+ (a_{4,0} + a_{4,1}z + a_{4,2}z^2 + \cdots)x \\ &+ a_{6,0} + a_{6,1}z + a_{6,2}z^2 + \cdots. \end{aligned} \tag{2.7}$$

The coefficients must be

$$\begin{aligned} f_{8+4n} &= \beta_4 = a_{4,0} \in \Gamma(\mathcal{L}^{-20} \otimes \mathcal{N}^4) = \Gamma(\mathcal{L}^4 \otimes \mathcal{M}^4) \\ f_{8+3n} &= \beta_3 = a_{4,1} \in \Gamma(\mathcal{L}^{-14} \otimes \mathcal{N}^3) = \Gamma(\mathcal{L}^4 \otimes \mathcal{M}^3) \\ f_{8+2n} &= \beta_2 = a_{4,2} \in \Gamma(\mathcal{L}^{-8} \otimes \mathcal{N}^2) = \Gamma(\mathcal{L}^4 \otimes \mathcal{M}^2) \\ f_{8+n} &= \beta_1 = a_{4,3} \in \Gamma(\mathcal{L}^{-2} \otimes \mathcal{N}^1) = \Gamma(\mathcal{L}^4 \otimes \mathcal{M}^1) \\ g_{12+6n} &= \alpha_6 = a_{6,0} \in \Gamma(\mathcal{L}^{-30} \otimes \mathcal{N}^6) = \Gamma(\mathcal{L}^6 \otimes \mathcal{M}^6) \\ g_{12+5n} &= \alpha_5 = a_{6,1} \in \Gamma(\mathcal{L}^{-24} \otimes \mathcal{N}^5) = \Gamma(\mathcal{L}^6 \otimes \mathcal{M}^5) \\ g_{12+4n} &= \alpha_4 = a_{6,2} \in \Gamma(\mathcal{L}^{-18} \otimes \mathcal{N}^4) = \Gamma(\mathcal{L}^6 \otimes \mathcal{M}^4) \\ g_{12+3n} &= \alpha_3 = a_{6,3} \in \Gamma(\mathcal{L}^{-12} \otimes \mathcal{N}^3) = \Gamma(\mathcal{L}^6 \otimes \mathcal{M}^3) \\ (g_{12+2n} &= \alpha_2 = a_{6,4} \in \Gamma(\mathcal{L}^{-6} \otimes \mathcal{N}^2) = \Gamma(\mathcal{L}^6 \otimes \mathcal{M}^2)) \\ g_{12+n} &= \alpha_1 = a_{6,5} \in \Gamma(\mathcal{L}^{-0} \otimes \mathcal{N}^1) = \Gamma(\mathcal{L}^6 \otimes \mathcal{M}^1). \end{aligned} \tag{2.8}$$

In the above equations we have also displayed on the leftmost side the corresponding coefficient polynomials of the Weierstrass form for the well-studied six-dimensional compactification.

Now Looijenga's theorem states that the moduli space of the vector bundle is parameterized by the sections

$$a_k = \Gamma(\mathcal{L}^{-d_k} \otimes \mathcal{N}^{s_k}) \qquad (k = 0, \ldots, \operatorname{rank} G), \tag{2.9}$$

[3] In the *four*-dimensional compactifications of F-theory, one also needs to specify the so-called γ class (G-flux), but it is irrelevant for the discussion here.

where d_k is 0 $(k = 0)$ or the degree of the independent Casimir of G, and s_k is 1 $(k = 0)$ or the coefficient of the k-th coroot when the lowest root $-\theta$ is expanded.

In the present case, the minus of the powers of $\mathcal{L}$ in the middle column of (2.8) read

$$0, \quad 2, \quad 8, \quad 12, \quad 14, \quad 18, \quad 20, \quad 24, \quad 30, \tag{2.10}$$

which precisely coincide with ("0" and) the set of degrees of independent Casimirs of E_8, while the powers of $\mathcal{N}$ are *close to* identical to the coefficients of the (co)root (E_8 is simply laced) expansion:

$$-\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8, \tag{2.11}$$

except that the power for $g_{12+n} = \alpha_1 = a_{6,5}$, which should be 2, is 1. If $g_{12+2n} = \alpha_2 = a_{6,4}$ were taken instead, then the power would become 2 which is correct, but then the power of $\mathcal{L}^{-1}$ would be 6 which does not agree with Looijenga's statement. Thus in this E_8 case, we have obtained the weighted projective space $\mathrm{W}\mathbb{P}^8_{(1,2,2,3,3,4,4,5,6)}$ but (2.9) is not completely true [1].

2.2 E_7 bundles by blowing up (A_1 singularity)

In [1] it was shown that E_7 bundles can be constructed in terms of a degree-4 equation in $\mathrm{W}\mathbb{P}^3_{(1,1,1,2)}$. Sections of Looijenga's weighted projective bundle are similarly parameterized by the coefficients of the Weierstrass equation, which themselves are sections of a particular set of line bundles specified by Looijenga's theorem (2.9). In this section we will show that these setups naturally arise by blowing up the singularity on the degree-6 equation in $\mathrm{W}\mathbb{P}^3_{(1,1,2,3)}$ for E_8 discussed in the previous section, according to a well-known procedure known as Tate's algorithm.

Physically, an E_7 bundle implies an SU(2) unbroken gauge symmetry of one of the two E_8 of heterotic string theory. Mathematically, this is a reflection of the structure of the Mordell-Weil lattice [5] stating the complementarity in E_8 of the sections and the singularities of a rational elliptic surface dP_9.

Below we use in the process of blowing up, *even in the general case not restricted to the six-dimensional case*, the notation:

$$f_{8+(4-i)n} := a_{4,i}, \tag{2.12}$$

$$g_{12+(6-j)n} := a_{6,j}, \tag{2.13}$$

by using the dictionary (2.8).[4] In that case, n no longer has the meaning of the number of instantons, but is rather just a dummy variable with its coefficient specifying the powers of the twisting line bundle to which the sections belong. This notation is intended for the convenience of, and will be particularly useful to, the readers who are familiar with the well-known six-dimensional F-theory compactification [7–9]. This enables us to easily recognize that the sections parameterizing a Looijenga's weighted projective space are nothing but

[4] More generally, a degree $(an + b)$ polynomial in z' in the 6D F-theory compactification corresponds to a section of $\mathcal{L}^{-d} \otimes \mathcal{N}^s$ with $d = 6a - \frac{b}{2}$ and $s = a$.

the set of independent polynomials parameterizing the complex structure of the elliptic manifold in F-theory with a singularity, which is the orthogonal complement in E_8 of the gauge group of the vector bundle. Why they are the orthogonal complement of each other will be explained in section 4.

In order to have an $\mathrm{SU}(2) = A_1 = I_2$ singularity, we assume that the coefficients f_{8+4n}, g_{12+6n} and g_{12+5n} in (2.6) can be written in term of some $h_{2n+4} \in \Gamma(\mathcal{L}^2 \otimes \mathcal{M}^2)$ as [9]

$$\begin{aligned} f_{8+4n} &= -3h_{2n+4}^2, \\ g_{12+6n} &= 2h_{2n+4}^3, \\ g_{12+5n} &= -h_{2n+4} f_{8+3n}. \end{aligned} \tag{2.14}$$

Then the discriminant

$$\Delta = 4f(z,z')^3 + 27g(z,z')^2 \tag{2.15}$$

becomes $O(z^2)$ or higher at $z = 0$, implying an A_1 singularity. The location of the singularity is $y = z = 0$ but $x = h_{2n+4}$, so it is not at the origin in general. So we define

$$x_{\text{new}} \equiv x - h_{2n+4}, \tag{2.16}$$

then (2.5) becomes

$$\begin{aligned} 0 = {} & y^2 + x_{\text{new}}^3 + 3h_{2n+4}x_{\text{new}}^2 + (f_{8+3n}z + f_{8+2n}z^2 + f_{8+n}z^3 + f_8 z^4)x_{\text{new}} \\ & + (h_{2n+4}f_{8+2n} + g_{12+4n})z^2 + (h_{2n+4}f_{8+n} + g_{12+3n})z^3 \\ & + (h_{2n+4}f_8 + g_{12+2n})z^4 + g_{12+n}z^5 + g_{12}z^6. \end{aligned} \tag{2.17}$$

By construction it has a singularity at $x_{\text{new}} = y = z = 0$ so we blow it up at $(x_{\text{new}}, y, z) = (0,0,0) \in \mathbf{C}^3$ by defining

$$\tilde{\mathbf{C}}^3 = \left\{ ((x_{\text{new}}, y, z), (\xi : \eta : \zeta)) \in \mathbf{C}^3 \times \mathbb{P}^2 | (x_{\text{new}}, y, z) \in (\xi : \eta : \zeta) \right\}, \tag{2.18}$$

where $(x_{\text{new}}, y, z) \in (\xi : \eta : \zeta)$ means that (x_{new}, y, z) and $(\xi : \eta : \zeta)$ are parallel to each other.

Let $x' \equiv \frac{\xi}{\zeta}$, $y' \equiv \frac{\eta}{\zeta}$ in the affine patch with $\zeta \neq 0$, then

$$(x_{\text{new}}, y, z) = (x'z, y'z, z). \tag{2.19}$$

Plugging this into (2.17) and dividing it by z^2, we have

$$\begin{aligned} 0 = {} & y'^2 + x'^3 z + 3h_{2n+4}x'^2 + (f_{8+3n} + f_{8+2n}z + f_{8+n}z^2 + f_8 z^3)x' \\ & + (h_{2n+4}f_{8+2n} + g_{12+4n}) + (h_{2n+4}f_{8+n} + g_{12+3n})z \\ & + (h_{2n+4}f_8 + g_{12+2n})z^2 + g_{12+n}z^3 + g_{12}z^4. \end{aligned} \tag{2.20}$$

One can show that this is a smooth curve *unless*

$$f_{8+3n}^2 - 12h_{2n+4}(h_{2n+4}f_{8+2n} + g_{12+4n}) = 0 \tag{2.21}$$

JHEP11(2016)053

is satisfied. In fact, the left hand side of (2.21) is the coefficient of the $O(z^2)$ term of the discriminant Δ, so that (2.21) implies $\mathrm{ord}(\Delta) \geq 3$ at $z = 0$. Note that this is a necessary condition for the curve to be singular, and even if (2.21) holds, (2.20) still remains regular unless some additional conditions are satisfied, as we see in the next section.

Now we observe that (2.20) is nothing but a degree-4 equation in $\mathrm{W}\mathbb{P}^3_{(1,1,1,2)}$ $((u, v, X', Y') \sim (\lambda u, \lambda v, \lambda X', \lambda^2 Y'))$:

$$\begin{aligned} 0 &= Y'^2 + X'^3 v + a_{2,0} X'^2 u^2 \\ &\quad +(a_{4,1}u^3 + a_{4,2}u^2 v + a_{4,3}uv^2 + a_{4,4}v^3)X' \\ &\quad +(a_{6,2}u^4 + a_{6,3}u^3 v + a_{6,4}u^2 v^2 + a_{6,5}uv^3 + a_{6,6}v^4) \end{aligned} \tag{2.22}$$

expressed in the affine patch with $u \neq 0$ in terms of the affine coordinates $(u, v, X', Y') \sim (1, \frac{v}{u}, \frac{X'}{u}, \frac{Y'}{u^2}) \equiv (1, z, x', y')$:

$$\begin{aligned} 0 &= y'^2 + x'^3 z + a_{2,0} x'^2 \\ &\quad +(a_{4,1} + a_{4,2}z + a_{4,3}z^2 + a_{4,4}z^3)x' \\ &\quad +(a_{6,2} + a_{6,3}z + a_{6,4}z^2 + a_{6,5}z^3 + a_{6,6}z^4) \end{aligned} \tag{2.23}$$

with

$$\begin{aligned} a_{2,0} &= 3h_{2n+4} \\ a_{4,1} &= f_{8+3n} \\ a_{4,2} &= f_{8+2n} \\ a_{4,3} &= f_{8+n} \\ a_{4,4} &= f_8 \\ a_{6,2} &= h_{2n+4}f_{8+2n} + g_{12+4n} \\ a_{6,3} &= h_{2n+4}f_{8+n} + g_{12+3n} \\ a_{6,4} &= h_{2n+4}f_8 + g_{12+2n} \\ a_{6,5} &= g_{12+n} \\ a_{6,6} &= g_{12}. \end{aligned} \tag{2.24}$$

The degree-4 equation in $\mathrm{W}\mathbb{P}^3_{(1,1,1,2)}$ (2.22) is precisely the one found in [1] for E_7 bundles. Thus we see that the set up for the construction of E_7 bundles in [1] is naturally obtained by blowing up the singularity of the Weierstrass equation in $\mathrm{W}\mathbb{P}^3_{(1,1,2,3)}$ for E_8 bundles.

Since

$$a_{i,j} \in \mathcal{L}^i \otimes \mathcal{M}^{i-j} \;=\; \mathcal{L}^{6j-5i} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^{i-j}, \tag{2.25}$$

we have

$$\begin{aligned} a_{2,0} &\in \Gamma(\mathcal{L}^{-10} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\ a_{4,1} &\in \Gamma(\mathcal{L}^{-14} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^3) \end{aligned}$$

$$
\begin{aligned}
a_{4,2} &\in \Gamma(\mathcal{L}^{-8} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\
a_{4,3} &\in \Gamma(\mathcal{L}^{-2} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\
a_{6,2} &\in \Gamma(\mathcal{L}^{-18} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^4) \\
a_{6,3} &\in \Gamma(\mathcal{L}^{-12} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^3) \\
a_{6,4} &\in \Gamma(\mathcal{L}^{-6} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\
a_{6,5} &\in \Gamma(\mathcal{L}^{-0} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1).
\end{aligned} \tag{2.26}
$$

Note that $a_{4,4} \in \Gamma(\mathcal{L}^4)$ or $a_{6,6} \in \Gamma(\mathcal{L}^6)$ does not have information of the vector bundles but describes the complex structure of the Calabi-Yau on the heterotic side, and that of the elliptic fibration connecting the two dP_9 fibrations on the F-theory side.

We see in (2.26) that the minus of the powers of $\mathcal{L}$ read

$$
0, \qquad 2, \qquad 6, \qquad 8, \qquad 10, \qquad 12, \qquad 14, \qquad 18, \tag{2.27}
$$

which coincides with the set of degrees of independent Casimirs of E_7 (including 0), and the powers of $\mathcal{N}$ are the coefficients of the expansion of the highest root of E_7:

$$
-\theta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 1\alpha_7, \tag{2.28}
$$

being (this time) in complete agreement with Looijenga.

2.3 E_6 bundles (A_2 singularity)

Next we suppose that (2.21) is satisfied. Since the first term is a square of a section, so must be the second term. This is achieved by requiring [9]

$$
\begin{aligned}
h_{2n+4} &= h_{n+2}^2 \\
f_{8+3n} &= 12 h_{n+2} H_{2n+6} \\
g_{12+4n} &= 12 H_{2n+6}^2 - h_{n+2}^2 f_{8+2n}
\end{aligned} \tag{2.29}
$$

for some h_{n+2} and H_{2n+6}. These conditions are the ones for the exceptional curve to factorize into two lines, and the singularity becomes I_3 of the Kodaira classification.[5] This is smooth unless

$$
-2H_{2n+6}(h_{n+2}^2 f_{8+2n} + 4H_{2n+6}^2) + h_{n+2}^3 (h_{n+2}^2 f_{8+n} + g_{12+3n}) = 0 \tag{2.30}
$$

is satisfied, in which the order of the discriminant would become higher than 3 and we would need a further blow-up. Plugging (2.29) into (2.17), we find

$$
\begin{aligned}
0 = y^2 &+ x_{\text{new}}^3 + 3h_{n+2}^2 x_{\text{new}}^2 + (12 h_{n+2} H_{2n+6} z + f_{8+2n} z^2 + f_{8+n} z^3 + f_8 z^4) x_{\text{new}} \\
&+ 12 H_{2n+6}^2 z^2 + (h_{n+2}^2 f_{8+n} + g_{12+3n}) z^3 + (h_{n+2}^2 f_8 + g_{12+2n}) z^4 + g_{12+n} z^5 + g_{12} z^6.
\end{aligned} \tag{2.31}
$$

[5] The eqs. (2.29) as well as (2.42) and (2.54) are the conditions for a so-called "split" singularity of the corresponding type [9] (see also [12]).

We can further rewrite it in terms of

$$y_{\rm new} \equiv y - \sqrt{3}i(h_{n+2}x_{\rm new} + 2H_{2n+6}z) \tag{2.32}$$

as

$$\begin{aligned}0 = y_{\rm new}^2 + x_{\rm new}^3 + 2\sqrt{3}ih_{n+2}x_{\rm new}y_{\rm new} + 4\sqrt{3}iH_{2n+6}zy_{\rm new} + (f_{8+2n}z^2 + f_{8+n}z^3 + f_8z^4)x_{\rm new} \\ +(h_{n+2}^2 f_{8+n} + g_{12+3n})z^3 + (h_{n+2}^2 f_8 + g_{12+2n})z^4 + g_{12+n}z^5 + g_{12}z^6.\end{aligned} \tag{2.33}$$

Note that $x_{\rm new} = y = 0 \Leftrightarrow x_{\rm new} = y_{\rm new} = 0$ at $z = 0$. Similarly to (2.19) in (2.17), we set

$$(x_{\rm new}, y_{\rm new}, z) = (x'z, y'z, z) \tag{2.34}$$

to find

$$\begin{aligned}0 = y'^2 + x'^3z + 2\sqrt{3}ih_{n+2}x'y' + 4\sqrt{3}iH_{2n+6}y' + (f_{8+2n}z + f_{8+n}z^2 + f_8z^3)x' \\ +(h_{n+2}^2 f_{8+n} + g_{12+3n})z + (h_{n+2}^2 f_8 + g_{12+2n})z^2 + g_{12+n}z^3 + g_{12}z^4.\end{aligned} \tag{2.35}$$

Again, this is a fourth-order equation in $\mathrm{W}\mathbb{P}^3_{(1,1,1,2)}$ $((u, v, X', Y') \sim (\lambda u, \lambda v, \lambda X', \lambda^2 Y'))$:

$$\begin{aligned}0 &= Y'^2 + X'^3v + a_{1,0}X'Y'u + a_{3,1}Y'u^2 \\ &\quad +(a_{4,2}u^2v + a_{4,3}uv^2 + a_{4,4}v^3)X' \\ &\quad +a_{6,3}u^3v + a_{6,4}u^2v^2 + a_{6,5}uv^3 + a_{6,6}v^4\end{aligned} \tag{2.36}$$

expressed in terms of the affine coordinates $(u, v, X', Y') \sim (1, \frac{v}{u}, \frac{X'}{u}, \frac{Y'}{u^2}) \equiv (1, z, x', y')$ in the patch $u \neq 0$:

$$\begin{aligned}0 &= y'^2 + x'^3z + a_{1,0}x'y' + a_{3,1}y' \\ &\quad +(a_{4,2}z + a_{4,3}z^2 + a_{4,4}z^3)X' \\ &\quad +a_{6,3}z + a_{6,4}z^2 + a_{6,5}z^3 + a_{6,6}z^4\end{aligned} \tag{2.37}$$

with

$$\begin{aligned}a_{1,0} &= 2\sqrt{3}ih_{n+2} \\ a_{3,1} &= 4\sqrt{3}iH_{2n+6} \\ a_{4,2} &= f_{8+2n} \\ a_{4,3} &= f_{8+n} \\ a_{4,4} &= f_8 \\ a_{6,3} &= h_{2n+4}f_{8+n} + g_{12+3n} \\ a_{6,4} &= h_{2n+4}f_8 + g_{12+2n} \\ a_{6,5} &= g_{12+n} \\ a_{6,6} &= g_{12}.\end{aligned} \tag{2.38}$$

In this case we have

$$
\begin{aligned}
a_{1,0} &\in \Gamma(\mathcal{L}^{-5} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\
a_{3,1} &\in \Gamma(\mathcal{L}^{-9} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\
a_{4,2} &\in \Gamma(\mathcal{L}^{-8} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\
a_{4,3} &\in \Gamma(\mathcal{L}^{-2} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\
a_{6,3} &\in \Gamma(\mathcal{L}^{-12} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^3) \\
a_{6,4} &\in \Gamma(\mathcal{L}^{-6} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\
a_{6,5} &\in \Gamma(\mathcal{L}^{-0} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1),
\end{aligned} \tag{2.39}
$$

which is consistent with the facts that the degrees of the independent Casimirs of E_6 (including 0) are

$$
0, \qquad 2, \qquad 5, \qquad 6, \qquad 8, \qquad 9, \qquad 12 \tag{2.40}
$$

and the (co)root expansion of the highest root is

$$
-\theta = 1\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 1\alpha_6. \tag{2.41}
$$

Thus we have seen that not only E_7 bundles but E_6 bundles can also be constructed from $\mathrm{W}\mathbb{P}^3_{(1,1,1,2)}$. In contrast, instead of $\mathrm{W}\mathbb{P}^3_{(1,1,1,2)}$, $\mathrm{W}\mathbb{P}^3_{(1,1,1,1)}$ was used in FMW [1], which can be obtained by a further blow-up as we will see in the next section. Note, however, that in the case of the $I_3 = A_2$ singularity the exceptional curve arising in the $I_2 = A_1$ singularity simply splits into to two lines, in which no additional blow-ups are needed, and therefore $\mathrm{W}\mathbb{P}^3_{(1,1,1,2)}$ suffice. Of course, one is free to blow it up so it is not a contradiction.

2.4 D_5 bundles (A_3 singularity)

In this section we consider the case in which (2.30) is satisfied and the curve in the previous section becomes singular. Then the discriminant Δ bedomes $\mathrm{ord}(\Delta) \geq 4$. In this case we require [9]

$$
H_{2n+6} = h_{n+2} H_{n+4} \tag{2.42}
$$

for some H_{n+4}. Due to (2.30), we need to have

$$
g_{12+3n} = 2H_{n+4}(f_{8+2n} + 4H_{n+4}^2) - h_{n+2}^2 f_{8+n}. \tag{2.43}
$$

Then (2.35) becomes singular at $x' = -2H_{n+4}$, $y' = z = 0$. To resolve this singularity we define

$$
x'_{\mathrm{new}} \equiv x' + 2H_{n+4}, \tag{2.44}
$$

then

$$
\begin{aligned}
0 = {} & y'^2 + {x'_{\mathrm{new}}}^3 z + 2i\sqrt{3} h_{n+2} x'_{\mathrm{new}} y' - 6zH_{n+4} {x'_{\mathrm{new}}}^2 \\
& + x'_{\mathrm{new}} \left(z \left(f_{8+2n} + 12H_{n+4}^2 \right) + f_{8+n} z^2 + f_8 z^3 \right) \\
& + z^2 \left(h_{n+2}^2 f_8 - 2H_{n+4} f_{8+n} + g_{12+2n} \right) \\
& + z^3 \left(-2H_{n+4} f_8 + g_{12+n} \right) + g_{12} z^4.
\end{aligned} \tag{2.45}
$$

The singularity is located at $x'_{\rm new} = y' = z = 0$, so defining

$$y' = \tilde{y}' z \tag{2.46}$$

and factoring z out, we derive

$$\begin{aligned} 0 = &\tilde{y}'^2 z + {x'_{\rm new}}^3 + f_8 x'_{\rm new} z^2 + g_{12} z^3 \\ &+2i\sqrt{3} h_{n+2} x'_{\rm new} \tilde{y}' - 6 H_{n+4} {x'_{\rm new}}^2 \\ &+x'_{\rm new} \left((f_{8+2n} + 12 H_{n+4}^2) + f_{8+n} z \right) \\ &+z \left(h_{n+2}^2 f_8 - 2 H_{n+4} f_{8+n} + g_{12+2n} \right) \\ &+z^2 \left(-2 H_{n+4} f_8 + g_{12+n} \right). \end{aligned} \tag{2.47}$$

Rewriting this equation as

$$\begin{aligned} 0 = &\tilde{y}'^2 z + {x'_{\rm new}}^3 + a_{4,4} x'_{\rm new} z^2 + a_{6,6} z^3 \\ &+a_{1,0} x'_{\rm new} \tilde{y}' + a_{2,1} {x'_{\rm new}}^2 \\ &+x'_{\rm new} \left(a_{4,2} + a_{4,3} z \right) \\ &+a_{6,4} z + a_{6,5} z^2, \end{aligned} \tag{2.48}$$

we see that this is a third-order equation in $\mathrm{W}\mathbb{P}^3_{(1,1,1,1)}$ $((u, v, X', \tilde{Y}') \sim (\lambda u, \lambda v, \lambda X', \lambda \tilde{Y}'))$:

$$\begin{aligned} 0 = &\tilde{Y}'^2 v + X'^3 + a_{4,4} X' v^2 + a_{6,6} v^3 \\ &+u^2 \left(a_{4,2} X' + a_{6,4} v \right) \\ &+u \left(a_{1,0} X' \tilde{Y}' + a_{2,1} X'^2 + a_{4,3} X' v + a_{6,5} v^2 \right), \end{aligned} \tag{2.49}$$

expressed in terms of the affine coordinates $(u, v, X', \tilde{Y}') \sim (1, \frac{v}{u}, \frac{X'}{u}, \frac{\tilde{Y}'}{u}) \equiv (1, z, x'_{\rm new}, \tilde{y}')$ in the patch $u \neq 0$, with the identifications

$$\begin{aligned} a_{1,0} &= 2\sqrt{3} i h_{n+2} \\ a_{2,1} &= -6 H_{n+4} \\ a_{4,2} &= f_{8+2n} + 12 H_{n+4}^2 \\ a_{4,3} &= f_{8+n} \\ a_{6,4} &= h_{n+2}^2 f_8 - 2 H_{n+4} f_{8+n} + g_{12+2n} \\ a_{6,5} &= -2 H_{n+4} f_8 + g_{12+n} \end{aligned} \tag{2.50}$$

$(a_{4,4} = f_8,\ a_{6,6} = g_{12})$.

The relevant sections are

$$\begin{aligned} a_{1,0} &\in \Gamma(\mathcal{L}^{-5} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\ a_{2,1} &\in \Gamma(\mathcal{L}^{-4} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\ a_{4,2} &\in \Gamma(\mathcal{L}^{-8} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\ a_{4,3} &\in \Gamma(\mathcal{L}^{-2} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\ a_{6,4} &\in \Gamma(\mathcal{L}^{-6} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\ a_{6,5} &\in \Gamma(\mathcal{L}^{-0} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1), \end{aligned} \tag{2.51}$$

JHEP11(2016)053

which agree with the degrees of Casimirs of D_5 with 0:

$$0, \qquad 2, \qquad 4, \qquad 5, \qquad 6, \qquad 8 \tag{2.52}$$

and the coroot expansion:

$$-\theta = 1\alpha_1 + 2\alpha_2 + 2\alpha_3 + 1\alpha_4 + 1\alpha_5. \tag{2.53}$$

Thus we have derived $\mathrm{W}\mathbb{P}^5_{(1,1,1,1,2,2)}$ from a third-order equation in $\mathrm{W}\mathbb{P}^3_{(1,1,1,1)}$. This construction of D_5 bundles was not explicitly mentioned in FMW.

2.5 A_4 bundles (A_4 singularity)

If we further assume [9]

$$\begin{aligned} f_{8+2n} &= -12H_{n+4}^2 + 12h_{n+2}p_{n+6}, \\ g_{12+2n} &= 12p_{n+6}^2 + 2f_{8+n}H_{n+4} - f_8 h_{n+2}^2 \end{aligned} \tag{2.54}$$

for some p_{n+6} in (2.45), we have $\mathrm{ord}(\Delta) \geq 5$ and the exceptional curve again splits into two lines. In this case, unlike the case for E_6 bundles, the singularity of (2.45) is not resolved by (2.46) but we also need to scale x'_{new}. This can be done, but we can still use (2.49) to see which sections are independent. Then (2.50) reads

$$\begin{aligned} a_{1,0} &= 2\sqrt{3}ih_{n+2} \\ a_{2,1} &= -6H_{n+4} \\ a_{4,2} &= 12h_{n+2}p_{n+6} \\ a_{4,3} &= f_{8+n} \\ a_{6,4} &= 12p_{n+6}^2 \\ a_{6,5} &= -2H_{n+4}f_8 + g_{12+n}, \end{aligned} \tag{2.55}$$

where we see that $a_{4,2}$ and $a_{6,4}$ are simplified. They are the coefficients of the u^2 term in (2.49) so using

$$a_{3,2} = 4\sqrt{3}ip_{n+6}, \qquad \tilde{Y}'_{\mathrm{new}} = \tilde{Y}' - 2\sqrt{3}iup_{n+6} \tag{2.56}$$

we have

$$\begin{aligned} 0 = &\, \tilde{Y}'^2_{\mathrm{new}}v + a_{3,2}\tilde{Y}'_{\mathrm{new}}uv + X'^3 + a_{4,4}X'v^2 + a_{6,6}v^3 \\ &+u\left(a_{1,0}X'\tilde{Y}' + a_{2,1}X'^2 + a_{4,3}X'v + a_{6,5}v^2\right). \end{aligned} \tag{2.57}$$

$a_{4,2}$ and $a_{6,4}$ in (2.49) are thus eliminated. In this way, for A_4 bundles, we have obtained a third-order equation in $\mathrm{W}\mathbb{P}^3_{(1,1,1,1)}$ (which is singular but can be smooth by a blow up) with

$$\begin{aligned} a_{1,0} &\in \Gamma(\mathcal{L}^{-5} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\ a_{2,1} &\in \Gamma(\mathcal{L}^{-4} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\ a_{3,2} &\in \Gamma(\mathcal{L}^{-3} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\ a_{4,3} &\in \Gamma(\mathcal{L}^{-2} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\ a_{6,5} &\in \Gamma(\mathcal{L}^{-0} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1). \end{aligned} \tag{2.58}$$

JHEP11(2016)053

Again this agrees with the set of Casimirs of A_4 with degrees (with 0):

$$0, \qquad 2, \qquad 3, \qquad 4, \qquad 5 \tag{2.59}$$

and the expansion

$$-\theta = 1\alpha_1 + 1\alpha_2 + 1\alpha_3 + 1\alpha_4. \tag{2.60}$$

2.6 A_3, A_2, A_1 bundles (D_5, E_6, E_7 singularity)

So far we have considered bundles for the E series up to $E_4 = A_4$. Since E_3 or E_2 is not a simple Lie algebra, we need a separate discussion for them. Instead, however, A_3, A_2 and A_1 bundles can be similarly constructed by setting h_{n+2}, H_{n+4} and p_{n+6} to zero in this order. In either case, one can show that there is an agreement between the powers of the line bundles and the degrees of the independent Casimirs and the expansion coefficients of the highest weight. Note that in these cases there is still a singularity at $z = 0$ to be further blown up.

3 Relation to the independent polynomials characterizing the complex structure

In the preceding sections we have seen that for E_7, E_6, D_5, A_4, A_3, A_2 and A_1 bundles (besides E_8 bundles which are exceptional) the necessary sections which constitute the corresponding weighted projective space stated in Looijenga's theorem are naturally obtained by a series of singularity enhancements of the elliptic manifold followed by the blowing-up procedure. We can now notice that they are nothing but the four-dimensional analogue of the set of independent polynomials in the six-dimensional F-theory [7–9] parameterizing the complex structure of the elliptic manifold. The type of the singularity is always the one orthogonal to the gauge group of the vector bundle in the whole E_8. Indeed, as shown in table 1, there is a perfect correspondence between the set of independent polynomials describing the complex structure in 6D and the set of numbers d and s characterizing the sections required by Looijenga's theorem, for all the cases of the bundle groups discussed in the preceding section, as well as the other cases for simple, simply-laced gauge groups listed in [9]. As we already noted in the previous footnote 4, a degree $(an + b)$ polynomial in z' corresponds to a section of $\mathcal{L}^{-d} \otimes \mathcal{N}^s$ with $d = 6a - \frac{b}{2}$ and $s = a$.

For D_4 bundles, which are not discussed in the previous section, we consider curves with a D_4 singularity. This can be obtained by restricting h_{n+2} and H_{2n+6} to be zero in the A_2 curve (used for E_6 bundles) and requiring the sixth-order term of the discriminant to be of the form [9, 13]

$$4f_{2n+8}^3 + 27g_{3n+12}^2 = j_{n+4}^2 k_{n+4}^2 (j_{n+4}^2 + k_{n+4}^2) \tag{3.1}$$

for some j_{n+4} and k_{n+4}, which are precisely the polynomials with correct degrees needed to constitute the weighted projective space.

JHEP11(2016)053

Bundle gauge group ($=H$)	Singularity ($=G$)	6D neutral matter	Independent polynomial	d	s
E_7	A_1	$(18n+83)\mathbf{1}$	g_{12+n}	0	1
			f_{8+n}	2	1
			g_{12+2n}	6	2
			f_{8+2n}	8	2
			h_{2n+4}	10	2
			g_{12+3n}	12	3
			f_{8+3n}	14	3
			g_{12+4n}	18	4
E_6	A_2	$(12n+66)\mathbf{1}$	g_{12+n}	0	1
			f_{8+n}	2	1
			h_{n+2}	5	1
			g_{12+2n}	6	2
			f_{8+2n}	8	2
			H_{2n+6}	9	2
			g_{12+3n}	12	3
D_5	A_3	$(8n+51)\mathbf{1}$	g_{12+n}	0	1
			f_{8+n}	2	1
			H_{n+4}	4	1
			h_{n+2}	5	1
			g_{12+2n}	6	2
			f_{8+2n}	8	2
A_4	A_4	$(5n+36)\mathbf{1}$	g_{12+n}	0	1
			f_{8+n}	2	1
			p_{n+6}	3	1
			H_{n+4}	4	1
			h_{n+2}	5	1
A_3	D_5	$(4n+33)\mathbf{1}$	g_{n+12}	0	1
			f_{n+8}	2	1
			p_{n+6}	3	1
			H_{n+4}	4	1
A_2	E_6	$(3n+28)\mathbf{1}$	g_{n+12}	0	1
			f_{n+8}	2	1
			p_{n+6}	3	1
A_1	E_7	$(2n+21)\mathbf{1}$	g_{n+12}	0	1
			f_{n+8}	2	1
D_6	$A_1 \oplus A_1$	$(10n+54)\mathbf{1}$	w_{n+12}	0	1
			q_{n+8}	2	1
			h_{n+4}	4	1
			h_n	6	1

JHEP11(2016)053

(Cont'd)

			v_{2n+12} p_{2n+8} h_{2n+4}	6 8 10	2 2 2
A_5	$A_2 \oplus A_1$	$(6n+37)\mathbf{1}$	w_{n+12} q_{n+8} v_{n+6} h_{n+4} h_{n+2} h_n	0 2 3 4 5 6	1 1 1 1 1 1
D_4	D_4	$(6n+44)\mathbf{1}$	g_{n+12} f_{n+8} j_{n+4} k_{n+4} g_{2n+12}	0 2 4 4 6	1 1 1 1 2

Table 1. Independent polynomials as sections of weighted projective space bundles.

D_6 and A_5 bundles, which also do not appear in the previous section, are interesting because they are the cases where the singularity has two non-abelian factors. For D_6 bundles, one can show that the relevant curve is, again in the 6D notation,

$$\begin{aligned} 0 = &\, y^2 + x^3 + 3(h_{2n+4} + h_{n+4}z)x^2 \\ &+ z(z+h_n)(p_{2n+8} + q_{n+8}z + s_8 z^2)x \\ &+ z^2(z+h_n)^2(v_{2n+12} + w_{n+12}z + y_{12}z^2). \end{aligned} \tag{3.2}$$

This curve has an $A_1 \times A_1$ ($\mathrm{SU}(2)\times\mathrm{SU}(2)$) singularity. The lines $z=0$ and $z+h_n=0$ are the loci of the 7-branes responsible for the two unbroken SU(2) gauge symmetries. Indeed, the discriminant takes the forms

$$\begin{aligned} \Delta &= z^2 h_n^2 h_{2n+4}^2 K_{4n+16} + O(z^3) \\ &= \tilde{z}^2 h_n^2 \tilde{h}_{2n+4}^2 \tilde{K}_{4n+16} + O(\tilde{z}^3), \end{aligned} \tag{3.3}$$

where $\tilde{z} = z + h_n$ and $\tilde{h}_{2n+4} = h_{2n+4} - h_{n+4}h_n$. K_{4n+16} and $\tilde{K}_{4n+16}$ are given by

$$\begin{aligned} K_{4n+16} &= 9(12h_{2n+4}v_{2n+12} - p_{2n+8}^2), \\ \tilde{K}_{4n+16} &= 9(12\tilde{h}_{2n+4}\tilde{v}_{2n+12} - \tilde{p}_{2n+8}^2), \end{aligned} \tag{3.4}$$

where $\tilde{h}_{2n+4}$, $\tilde{v}_{2n+12}$ and $\tilde{p}_{2n+8}$ are the coefficient polynomials appearing when (3.2) is re-expressed in terms of $\tilde{z}$ as

$$\begin{aligned} 0 = &\, y^2 + x^3 + 3(\tilde{h}_{2n+4} + h_{n+4}\tilde{z})x^2 \\ &+ \tilde{z}(\tilde{z}-h_n)(\tilde{p}_{2n+8} + \tilde{q}_{n+8}\tilde{z} + s_8\tilde{z}^2)x \\ &+ \tilde{z}^2(\tilde{z}-h_n)^2(\tilde{v}_{2n+12} + \tilde{w}_{n+12}\tilde{z} + y_{12}\tilde{z}^2). \end{aligned} \tag{3.5}$$

JHEP11(2016)053

(3.3) is consistent with the fact that the 6D heterotic charged matter consists of n $(\mathbf{2},\mathbf{2})$ and $4n+16$ $((\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{2}))$ computed by the index theorem. Note that the loci of h_{2n+4} and $\tilde{h}_{2n+4}$ do not contribute to charged matter since the enhanced fiber type there is III in the Kodaira classification so the singularity type is unchanged. One can also verify that, in the six-dimensional case, the total number of degrees of freedom of these polynomials

$$(n+13)+(n+9)+(n+5)+(n+1)+(2n+13)+(2n+9)+(2n+5)-1 \tag{3.6}$$

is equal to $10n+54$ which precisely matches the number of neutral hypermultiplets. We can see that the sections w_{n+12}, q_{n+8}, h_{n+4}, h_n, v_{2n+12}, p_{2n+8} and h_{2n+4} are precisely the polynomials expected to arise by Looijenga's theorem as are shown in table 1.

Similarly, the curve for an A_5 bundle is given by

$$\begin{aligned}0 = {} & y^2+x^3+3(h_{n+2}^2+h_{n+4}z)x^2 \\ & +z(z+h_n)(12h_{n+2}v_{n+6}+q_{n+8}z+s_8z^2)x \\ & +z^2(z+h_n)^2(12v_{n+6}^2+w_{n+12}z+y_{12}z^2),\end{aligned} \tag{3.7}$$

which has an $E_3=A_2\times A_1$ (SU(3) $\times$ SU(2)) singularity. Here the $O(z^2)$ term in (3.3) vanishes ($K_{4n+16}=0$ in (3.4)) and the A_1 singularity at $z=0$ is enhanced to A_2. The discriminant in this case is

$$\begin{aligned}\Delta &= z^3h_n^2h_{n+2}^3K_{4n+18}+O(z^4) \\ &= \tilde{z}^2h_n^3\tilde{h}_{2n+4}^2\tilde{K}_{3n+16}+O(\tilde{z}^3),\end{aligned} \tag{3.8}$$

being in agreement with the fact that the 6D heterotic charged matter hypermultiplets are $\frac{n}{2}$ $((\mathbf{3},\mathbf{2})\oplus(\bar{\mathbf{3}},\mathbf{2}))$, $2n+9$ $((\mathbf{3},\mathbf{1})\oplus(\bar{\mathbf{3}},\mathbf{1}))$ and $3n+16$ $(\mathbf{1},\mathbf{2})$. The number of degrees of freedom of the polynomials also agrees with the number of neutral hypermultiplets $6n+37$. Again, the sections w_{n+12}, q_{n+8}, v_{n+6}, h_{n+4}, h_{n+2} and h_n have the desired set of d and s as are shown in table 1.

Finally, let us consider $E_3=\mathrm{SU}(3)\times\mathrm{SU}(2)$ bundles and $\mathrm{SU}(2)\times\mathrm{SU}(2)$ bundles. These groups are the orthogonal complements of $A_5=\mathrm{SU}(6)$ and $D_6=\mathrm{SO}(12)$ in E_8. Although these are not simple groups (and hence outside the assumption of Looijenga's theorem), it is interesting to examine whether or not a similar characterization of the bundles is possible in these cases.[6]

For $E_3=\mathrm{SU}(3)\times\mathrm{SU}(2)$ bundles, we consider curves with a A_5 singularity. It is realized by further tuning the complex structure of the A_4 singularity (A_4 bundles) parametrized by the polynomials (2.55) to the following special forms:

$$\begin{aligned}h_{n+2} &= \tilde{h}_{n+2-r}t_r, \\ H_{n+4} &= \tilde{H}_{n+4-r}t_r, \\ p_{n+6} &= \tilde{h}_{n+2-r}u_{r+4}, \\ f_{n+8} &= \tilde{f}_{n+8-r}t_r-12\tilde{H}_{n+4-r}u_{r+4}, \\ g_{n+12} &= 2\tilde{f}_{n+8-r}u_{r+4}+2f_8H_{n+4}\end{aligned} \tag{3.9}$$

[6] E_2 contains U(1) and hence is beyond the scope of this paper.

for some h_{n+2-r}, t_r, $\tilde{H}_{n+4-r}$, u_{r+4} and $\tilde{f}_{n+8-r}$, which describes the heterotic configuration with $4+r$ of $12+n$ instantons are in SU(2) in E_3 and the remaining $8+n-r$ are in SU(3). Apparently, besides f_8 which describes the complex structure of the heterotic Calabi-Yau manifold, these five sections are needed to parametrize the moduli space of the bundle. However, defining

$$\begin{aligned} p_{n+6} &\equiv \tilde{h}_{n+2-r}u_{r+4}, \\ f^{(1)}_{n+8} &\equiv \tilde{f}_{n+8-r}t_r, \\ f^{(2)}_{n+8} &\equiv \tilde{H}_{n+4-r}u_{r+4}, \\ g'_{n+12} &\equiv 2\tilde{f}_{n+8-r}u_{r+4}, \end{aligned} \tag{3.10}$$

(3.9) can be formally written as

$$\begin{aligned} h_{n+2} &= \frac{2p_{n+6}f^{(1)}_{n+8}}{g'_{n+12}}, \\ H_{n+4} &= \frac{2f^{(1)}_{n+8}f^{(2)}_{n+8}}{g'_{n+12}}, \\ p_{n+6} &= p_{n+6}, \\ f_{n+8} &= f^{(1)}_{n+8} - 12f^{(2)}_{n+8}, \\ g_{n+12} &= g'_{n+12} + 2f_8H_{n+4} \end{aligned} \tag{3.11}$$

($2f_8H_{n+4}$ can be absorbed in g_{n+12} by redefinition). Therefore, provided that $2p_{n+6}f^{(1)}_{n+8}$ and $2f^{(1)}_{n+8}f^{(2)}_{n+8}$ are divisible by g'_{n+12}, they are parametrized by the four independent combinations p_{n+6}, $f^{(1)}_{n+8}$, $f^{(2)}_{n+8}$ and g'_{n+12}. The corresponding set of d and s are then 3, 2, 2, 0 and 1, 1, 1, 1, respectively. Thus we have seen that, though non-simple, the E_3 bundle is also parametrized by the sections specified by the Casimirs of $A_2 = \mathrm{SU}(3)$ and $A_1 = \mathrm{SU}(2)$, which are $\{3,2\}$ and $\{2\}$, and the coroot expansion coefficients $-\theta = \alpha_1 + \alpha_2$ and $-\theta = \alpha_1$.

For $\mathrm{SU}(2)\times\mathrm{SU}(2)$ bundles, the relevant curve is the one with a D_6 singularity. Such a curve is realized by setting

$$\tilde{h}_{n+2-r} = 0 \tag{3.12}$$

in the A_5 curve (3.9). Consequently, $p_{n+6} = 0$ in (3.10), so that the moduli space of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ bundle is parametrized by $f^{(1)}_{n+8}$, $f^{(2)}_{n+8}$ and g'_{n+12}. The corresponding set of d and s are 2, 2, 0 and 1, 1, 1, respectively. These agree with the Casimirs and the coroot expansion coefficients of the two SU(2)'s.

4 Why should this be so?: The Mordell-Weil lattice

In the previous sections we have seen that the sections of a particular set of line bundles coordinatizing Looijenga's weighted projective spaces can be automatically obtained as the

JHEP11(2016)053

coefficients of curves arising from a series of blow-ups in dP_9. They can be thought of as the four-dimensional analogue of the set of independent polynomials in the six-dimensional F-theory parameterizing the complex structure of the elliptic manifold, in which the gauge group of the bundle and the singularity are orthogonal to each other in E_8. In this section we explain why this is so.

As we stated in the previous section, the dP_9 we have blown up is supposed to be a half of a $K3$ in the stable degeneration limit, and the values of sections at infinity determine the spectral cover of the dual heterotic string theory.

Physically, a spectral cover describes the Wilson lines in the elliptic fibers of the heterotic Calabi-Yau over which the vector bundle is defined. Therefore, if the algebra of the Wilson lines is H, the Lie algebra of the unbroken gauge subgroup G is the commutant of H in E_8. Thus it is natural to derive H bundles when the singularity of dP_9 is G. This is a "physical" explanation, but there must also be a pure "mathematical" explanation which accounts for why the series of vector bundles are derived by the series of blow-ups, without referring to the heterotic/F-theory duality. What makes it possible is the structure theorem of the Mordell-Weil lattice.

The Mordell-Weil lattice [5] is the Mordell-Weil group [14, 15] equipped with a certain bilinear form. The Mordell-Weil group $E(K)$ of a rational elliptic surface ($=dP_9$) is defined as an Abelian group of rational sections of dP_9, where K is the field of rational functions of the coordinate z of the base $\mathbb{P}^1$ of dP_9. The addition of two sections is defined by the addition rule on an elliptic curve applied fiberwise, that is, as the addition of the two arguments of the $\wp$ (and also $\wp'$) function parameterizing the two sections. As is well known, the argument variable inside $\wp$ (and $\wp'$) is nothing but the complex coordinate itself if the fiber torus is expressed as a parallelogram with the two sets of sides identified.

$E(K)$ is called the Mordell-Weil lattice [16] if it is endowed with a bilinear form, or a height pairing, (P, Q) for sections $P, Q \in E(K)$ such that[7]

$$(P, Q) = P \cdot O + Q \cdot O - P \cdot Q + 1 - \sum_{v \in \text{singularities}} \text{contr}_v(P, Q), \tag{4.1}$$

$$(P, P) = 2 + 2P \cdot O - \sum_{v \in \text{singularities}} \text{contr}_v(P, P), \tag{4.2}$$

where $\cdot$ denotes the intersection pairing. For each singularity v, the function contr_v of a pair of sections P, Q is defined as

$$\text{contr}_v(P, Q) = \begin{cases} 0 & \text{if } i(P) = 0 \text{ or } i(Q) = 0, \\ (C_v^{-1})_{i(P), i(Q)} & \text{otherwise,} \end{cases} \tag{4.3}$$

where C_v is the Cartan matrix corresponding to the singularity v, and $i(P)$ ($i(Q)$) is either of $0, 1, \ldots, \operatorname{rank} C_v$ labeling the fiber component of v which (uniquely) intersects with the section P (Q). The fiber labeled as "the zeroth" ($i = 0$) is the one that intersects with the zero section.

[7]The fact that the arithmetic genus of dP_9 is one is taken into account here.

One of the remarkable results of [5] is that then $E(K)$ is roughly the orthogonal complement of the singularity in the E_8 root lattice. More precisely [5],

$$E(K) \simeq L^* \otimes (T'/T), \tag{4.4}$$

where T is the singularity lattice embedded into the E_8 root lattice Λ_{E_8}, L is its orthogonal lattice with respect to the specified embedding into Λ_{E_8}, L^* is the dual of L, and

$$T' = T \otimes \mathbb{Q} \cap \Lambda_{E_8}. \tag{4.5}$$

This is a geometrical manifestation of the fact that if the instanton is in the group H, the unbroken gauge group is the commutant of H in E_8. By this theorem we can now explain why we could derive E_N bundles by blowing up the A_{9-N} singularities: as we mentioned earlier, an E_N bundle is constructed from the spectral cover, whose equation determines as the intersections with the elliptic fiber at infinity the Wilson lines of the vector bundle. As one can check explicitly, these intersection points are extended into sections in the dP_9 [4, 17], obtaining the E_N weight lattice generated by the sections. The structure theorem of the Mordell-Weil lattice then tells us that this occurs precisely when the singularity lattice is the orthogonal compliment of the E_N weight lattice, which is A_{9-N}.

We should mention that the rational elliptic surfaces with various sections and singularities are known to be identified [18–24] as the total spaces of Seiberg-Witten curves for the four-, five- and six-dimensional so-called E_N theories [25, 26], where the u parameter becomes the coordinate of the $\mathbb{P}^1$ base. Indeed, the curves we considered in section 2 are exactly the same as the ones found in [19, 21], although the line bundles of the sections and their relation to Looijenga's weighted projective spaces were not investigated there. We also note that the values of sections at infinity are known to determine the mass parameters of the gauge theory whose Seiberg-Witten curve (together with the u-plane ($\mathbb{P}^1$)) is a rational elliptic surface allowing those sections.

The Mordell-Weil lattice also provides us with an understanding of the relation between the singularity and the occurrence of chiral matter in F-theory. (This fact was already observed and briefly mentioned in [6].) In the standard explanation for the chiral matter generation [10], one considers an enhanced singularity [7–9], at which the light membrane (in the M-theory dual picture) wrapping the extra shrinking two-cycle is identified as the origin of the chiral matter. On the other hand, it was shown by using the Leray spectral sequence [2, 3] that chiral matter is localized where one or some of the sections of dP_9 goes to the zero section. Again, the relation between the two pictures of matter generation may also be understood as a consequence of the structure theorem of the Mordell-Weil lattice. Indeed, the theorem says if some of the sections disappear in dP_9, then the singularity lattice, which is the orthogonal complement in E_8, becomes larger, leading to a singularity enhancement. Also, in view of the isomorphism between the string junction algebra and the Picard lattice of a rational elliptic surface [22], it gives support to the understanding of matter generation in F-theory in terms of string junctions [6, 13, 27].

JHEP11(2016)053

5 Conclusions

We have shown that the holomorphic vector bundles for gauge groups E_N $(N = 4, \cdots, 8)$ and A_n $(n = 1, 2, 3)$ can be obtained systematically by a series of blowing-ups in the rational elliptic surface according to Tate's algorithm. The sections of correct line bundles claimed to arise by Looijenga's theorem have been found automatically by this procedure. We have also pointed out that the sections parameterizing a Looijenga's weighted projective space are nothing but the four-dimensional analogue of the set of independent polynomials in the six-dimensional F-theory parameterizing the complex structure of the elliptic manifold with a singularity orthogonal to the gauge group of the vector bundle in the whole E_8. We have explained the reason for this by using the structure theorem of the Mordell-Weil lattice. We have also used it to elucidate the relation between the singularity and the occurrence of chiral matter in F-theory.

The Mordell-Weil lattice is classified into 74 different patterns of decompositions of the E_8 root lattice, of which we have used only the ones with a simple Mordell-Weil group (Nos. 2, 3, 4, 5, 6, 8, 9, 16, 27 and 43 of [5]). The additional patterns not considered in this paper correspond to the cases where the gauge group of the bundle is non-simply-connected [14, 15] or a direct product of simple groups as we encountered in section 3 (Nos. 15 and 26). It would be interesting to extend the analysis to these cases and a thorough investigation of these types of curves will be reported elsewhere.

Acknowledgments

We thank H. Itoyama and K. Mohri for valuable discussions. The work of S.M. is supported by Grant-in-Aid for Scientific Research (C) #25400285, (C) #16K05337 and (A) #26247042 from The Ministry of Education, Culture, Sports, Science and Technology of Japan.

References

[1] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F-theory*, *Commun. Math. Phys.* **187** (1997) 679 [hep-th/9701162] [INSPIRE].

[2] G. Curio, *Chiral matter and transitions in heterotic string models*, *Phys. Lett.* **B 435** (1998) 39 [hep-th/9803224] [INSPIRE].

[3] D.-E. Diaconescu and G. Ionesei, *Spectral covers, charged matter and bundle cohomology*, *JHEP* **12** (1998) 001 [hep-th/9811129] [INSPIRE].

[4] R. Donagi and M. Wijnholt, *Model Building with F-theory*, *Adv. Theor. Math. Phys.* **15** (2011) 1237 [arXiv:0802.2969] [INSPIRE].

[5] K. Oguiso and T. Shioda, *The Mordell-Weil Lattice of a Rational Elliptic Surface*, *Comment. Math. Univ. St. Paul.* **40** (1991) 83.

[6] S. Mizoguchi and T. Tani, *Anomaly-free multiple singularity enhancement in F-theory*, *Prog. Theor. Exp. Phys.* **2016** (2016) 073B05 [arXiv:1508.07423] [INSPIRE].

[7] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. (I)*, *Nucl. Phys.* **B 473** (1996) 74 [hep-th/9602114] [INSPIRE].

[8] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. (II)*, *Nucl. Phys.* **B 476** (1996) 437 [hep-th/9603161] [INSPIRE].

[9] M. Bershadsky, K.A. Intriligator, S. Kachru, D.R. Morrison, V. Sadov and C. Vafa, *Geometric singularities and enhanced gauge symmetries*, *Nucl. Phys.* **B 481** (1996) 215 [hep-th/9605200] [INSPIRE].

[10] S.H. Katz and C. Vafa, *Matter from geometry*, *Nucl. Phys.* **B 497** (1997) 146 [hep-th/9606086] [INSPIRE].

[11] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [hep-th/9602022] [INSPIRE].

[12] S. Katz, D.R. Morrison, S. Schäfer-Nameki and J. Sully, *Tate's algorithm and F-theory*, *JHEP* **08** (2011) 094 [arXiv:1106.3854] [INSPIRE].

[13] T. Tani, *Matter from string junction*, *Nucl. Phys.* **B 602** (2001) 434 [INSPIRE].

[14] P.S. Aspinwall, *Aspects of the hypermultiplet moduli space in string duality*, *JHEP* **04** (1998) 019 [hep-th/9802194] [INSPIRE].

[15] P.S. Aspinwall and D.R. Morrison, *Nonsimply connected gauge groups and rational points on elliptic curves*, *JHEP* **07** (1998) 012 [hep-th/9805206] [INSPIRE].

[16] T. Shioda, *On the Mordell-Weil Lattices*, *Comment. Math. Univ. St. Paul.* **39** (1990) 211.

[17] G. Curio and R.Y. Donagi, *Moduli in $N = 1$ heterotic/F-theory duality*, *Nucl. Phys.* **B 518** (1998) 603 [hep-th/9801057] [INSPIRE].

[18] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099] [INSPIRE].

[19] W. Lerche, P. Mayr and N.P. Warner, *Noncritical strings, del Pezzo singularities and Seiberg-Witten curves*, *Nucl. Phys.* **B 499** (1997) 125 [hep-th/9612085] [INSPIRE].

[20] O. DeWolfe, A. Hanany, A. Iqbal and E. Katz, *Five-branes, seven-branes and five-dimensional E_n field theories*, *JHEP* **03** (1999) 006 [hep-th/9902179] [INSPIRE].

[21] Y. Yamada and S.-K. Yang, *Affine seven-brane backgrounds and five-dimensional E_N theories on S^1*, *Nucl. Phys.* **B 566** (2000) 642 [hep-th/9907134] [INSPIRE].

[22] M. Fukae, Y. Yamada and S.-K. Yang, *Mordell-Weil lattice via string junctions*, *Nucl. Phys.* **B 572** (2000) 71 [hep-th/9909122] [INSPIRE].

[23] T. Eguchi and K. Sakai, *Seiberg-Witten curve for the E string theory*, *JHEP* **05** (2002) 058 [hep-th/0203025] [INSPIRE].

[24] K. Mohri, *Exceptional string: Instanton expansions and Seiberg-Witten curve*, *Rev. Math. Phys.* **14** (2002) 913 [hep-th/0110121] [INSPIRE].

[25] D.R. Morrison and N. Seiberg, *Extremal transitions and five-dimensional supersymmetric field theories*, *Nucl. Phys.* **B 483** (1997) 229 [hep-th/9609070] [INSPIRE].

[26] M.R. Douglas, S.H. Katz and C. Vafa, *Small instantons, del Pezzo surfaces and type I′ theory*, *Nucl. Phys.* **B 497** (1997) 155 [hep-th/9609071] [INSPIRE].

[27] S. Mizoguchi, *F-theory Family Unification*, *JHEP* **07** (2014) 018 [arXiv:1403.7066] [INSPIRE].

JHEP11(2016)053