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One loop mass renormalization of unstable particles in superstring theory

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Abstract: Most of the massive states in superstring theory are expected to undergo mass renormalization at one loop order. Typically these corrections should contain imaginary parts, indicating that the states are unstable against decay into lighter particles. However in such cases, direct computation of the renormalized mass using superstring perturbation theory yields divergent result. Previous approaches to this problem involve various analytic continuation techniques, or deforming the integral over the moduli space of the torus with two punctures into the complexified moduli space near the boundary. In this paper we use insights from string field theory to describe a different approach that gives manifestly finite result for the mass shift satisfying unitarity relations. The procedure is applicable to all states of (compactified) type II and heterotic string theories. We illustrate this by computing the one loop correction to the mass of the first massive state on the leading Regge trajectory in SO(32) heterotic string theory.

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Contents

1 Introduction and summary

The world-sheet formulation of superstring perturbation theory gives an elegant expression for scattering amplitudes, expressing the amplitude at any given order in perturbation theory as a single integral over the moduli space of a Riemann surface with punctures. This expression is manifestly free from ultraviolet divergences. However superstring perturbation theory shares all the usual infrared divergence problems in quantum field theory, but unlike in the case of quantum field theories, there is no systematic way of dealing with these divergences within the frame-work of the world-sheet formalism.

Superstring field theory provides a solution to this problem. By construction, the Feynman rules of superstring field theory reproduce the amplitude given by the world-sheet description when the latter gives finite result, but the existence of the underlying quantum field theory allows us to deal with the infrared divergence problems when they arise.

In this paper we shall use the insight from superstring field theory to address a related problem that arises in the world-sheet description of superstring perturbation theory. String theory has many massive states in its spectrum, but most of them are unstable against decay to lighter states. Therefore one expects that when quantum corrections to the masses are taken into account, the mass^2 of an unstable particle should receive correction that contains an imaginary part (and also possibly a real part). Now while higher loop mass renormalization requires full use of string field theory — because one needs to subtract the one particle reducible (1PR) contributions from the two point function — one would expect that the one loop contribution to the shift in mass^2 should be given by the on-shell two point function on the torus, and hence should be straightforward to compute using the usual world-sheet formalism. However when one tries to repeat this computation for an unstable state, one finds a divergent answer [1–3].

Intuitively the reason for this divergence is as follows [1–8]. In quantum field theory, while computing the mass renormalization of a particle that can decay into two or more particles, one finds that there are Feynman diagrams for which one or more internal propagators have negative denominator $(k^2 + m^2)$ for some region of internal loop momentum integration, and there is no way to deform the integration contours of loop momenta that can make all denominators have positive real parts everywhere along the contour. This means that the Schwinger parameter representation of this propagator breaks down, — if we try to replace $(k^2+m^2)^{-1}$ by $\int_0^\infty ds\, e^{-s(k^2+m^2)}$ then the integration over s encounters a divergence from infinity. On the other hand, integration over the moduli space of Riemann surfaces directly gives the result in the Schwinger parameter representation. Therefore the issue shows up as a divergence in the integration over the moduli space of Riemann surfaces.

It is also possible to argue that a finite result would necessarily have led to a contradiction. The loop correction to mass^2 of an unstable particle is expected to have an imaginary part, but straightforward world-sheet computation in string theory gives real results for all amplitudes. Therefore the only way an imaginary part can arise is if the naive world-sheet description gives divergent answer. In that case one might hope that by defining the amplitude for unphysical external momenta where the result is finite and then analytically continuing the result to on-shell external momenta, we may get an imaginary part. Early attempts to implement this achieved only partial success [1, 2]. A systematic method of dealing with this was suggested in [4–6] (see also [9, 10]). This was achieved by considering a four point amplitude with external momenta chosen in appropriate range where the integrals are well defined, then analytically continuing the result to the physical region where we expect a pole due to the massive particle of interest, and finally finding the shift in mass^2 from the location of the pole. Alternative approaches to analytic continuation, working directly with two point function, can be found in [11–13]. The imaginary part of the shift, which is related to the decay rate, is relatively easier to compute, and various other methods for computing this can be found in [11–22].[1]

One disadvantage of the analytic continuation procedure is that it has to be done on a case by case basis, and may not provide a systematic procedure to deal with all cases. For example not every massive state may appear as an intermediate state in the four point amplitude of massless external states. Also at higher mass levels there will be mixing between different states, leading to a renormalized mass^2 matrix with both real and imaginary parts, and it may not be easy to extract this matrix from the four point function of massless states. Finally, lack of a general procedure makes it difficult to prove general properties like unitarity that relates the imaginary part of the mass shift to the decay rate — except by explicit computation in each case. For these reasons, it will clearly be useful to develop a systematic procedure for computing string theory amplitudes that directly gives a finite result instead of having to define the amplitudes via analytic continuation. This

[1]In a quantum field theory the imaginary part is determined by unitarity relation. On the other hand the real part is ultraviolet divergent. This has to be removed by a counterterm and hence has to be taken as an input parameter of the theory. In string theory both parts are finite and computable.

JHEP11(2016)050

will be the analog of the $i\epsilon$ prescription in quantum field theory, — instead of defining the amplitudes as the analytic continuations of Euclidean Green's functions, one can write down the expression for the Green's functions with Lorentzian external momenta as integrals over loop momenta, but one needs the $i\epsilon$ prescription for regulating the poles of the propagator. Proposals for generalizing this to string theory was given in [7, 8]. These approaches involve deforming the integration over the moduli space of Riemann surfaces — that appear in the expression for the loop amplitudes — into the complexified moduli space. In terms of the Schwinger parameter representation of the propagators, this corresponds to taking the upper limit of s integration to be $i\infty$ instead of ∞, and at the same time supplying a small damping factor that represent the effect of replacing m^2 by $m^2 - i\epsilon$ as in a conventional quantum field theory.

In this paper we suggest a different approach to this problem by directly drawing insight from string field theory. In any quantum field theory, writing down the expression for a loop amplitude is quite straightforward if the Feynman rules are known, but typically it suffers from ultraviolet divergence. In string field theory there are no ultraviolet divergences since the vertices fall off exponentially for large space-like external momenta. However in the conventional formulation of string field theory, the vertices grow exponentially for large time-like momenta. Due to this property, while computing Feynman amplitudes by integrating over internal momenta, we cannot take the integral over internal energies along the real axis — the ends of the integration contour have to be tied to $\pm i\infty$ [23]. However in the interior of the complex plane the contour has to be deformed appropriately away from the imaginary axis following the algorithm described in [23]. With this prescription we get finite results for all loop corrections except where there are physical infrared divergences involving one or more divergent propagators — e.g. mass renormalization diagrams if we fail to take into account the shift of mass due to quantum corrections, or tadpole divergences if the original perturbative vacuum is destabilized by quantum corrections. In the absence of such divergences, we should get finite results. This includes the one loop two point function that is needed for computing the renormalized mass — both its real and the imaginary parts.

One could wonder how the results in string field theory are related to those of other approaches — e.g. analytic continuation. To this end we note that the string field theory amplitudes, constructed using the procedure mentioned above, are automatically analytic functions of external momenta. Therefore by the uniqueness of analytic continuation, string field theory results must agree with those computed using analytic continuation. However what string field theory achieves is that it expresses the result as a (contour) integral over momenta that is manifestly finite without any need for analytic continuation. Therefore this automatically gives the analytically continued result that we would have gotten from the usual world-sheet approach. Another bonus of this approach is that the amplitudes defined this way automatically satisfies the Cutkosky cutting rules [23]. While for general amplitudes one still needs few more steps to prove unitarity from the cutting rules by showing that the contribution to the cut diagrams from unphysical intermediate states cancel, for diagrams involving one loop mass renormalization this can be shown explicitly. Therefore the imaginary parts of the mass shifts computed using this approach are automatically consistent with unitarity.[2]

While string field theory is essential for carrying out this computation to higher loop order, for one loop correction to the masses one does not require the full power of string field theory. The reason has already been mentioned earlier: one loop mass renormalization can be computed from one loop two point function of external states that satisfy tree level on-shell condition. No subtraction is necessary, unlike in the case of higher loop two point functions from which the contribution from 1PR graphs have to be subtracted. Nevertheless since this one loop two point function diverges due to the reasons mentioned above, we need a way to deal with this divergence. The strategy we follow is to isolate the divergent part and reinterpret this as coming from a specific Feynman diagram of string field theory. If we try to express this as integration over Schwinger parameters, we get back the expression that we have in the world-sheet description, and it is divergent. But we can directly evaluate this Feynman diagram by performing integration over loop momenta following the prescription of [23] and this yields a finite answer. The difference between the two can be traced to the fact that the Schwinger parameter representation of the internal propagators breaks down for certain range of momentum integration. Since from the point of view of string field theory, the Feynman diagrams are more fundamental, the procedure of evaluating the Feynman diagrams directly is the correct one, even when its Schwinger parameter representation fails.

The rest of the paper is organized as follows. In section 2 we introduce a toy quantum field theory that shares some essential properties of string theory. We compute one loop mass renormalization of an unstable particle in this theory and show that we get a finite answer. On the other hand if we try to evaluate the same expression by using Schwinger parameter representation of the propagators, we get a divergent result. The divergence can be traced to the breakdown of the Schwinger parameter representation of the propagator. In section 3 we compute one loop mass renormalization of the lowest massive string state of ten dimensional heterotic string theory on the leading Regge trajectory. The answer, expressed as an integral over the moduli space of a torus with two punctures, has certain divergences from the boundary of the moduli space. We isolate the divergent piece, and by comparing it with the result of section 2 in the Schwinger parameter representation of the propagator, identify the divergent piece as the contribution from a specific Feynman diagram of string field theory. This Feynman diagram is then evaluated using direct momentum space integration, leading to finite answer. Our final result is expressed as a sum of three terms, given in (3.12), (3.25) and (3.26), each of which is manifestly finite. We discuss extension of this analysis to general external states in section 4 where we also give a justification of the procedure from string field theory and show that the results for the renormalized mass obtained this way agree for different versions of string field theory. We

[2]Since the approach of [7] was motivated from light-cone string field theory [24, 25], one could ask if we can directly work with the light-cone string field theory and impose the $i\epsilon$ prescription there. This would make the proof of unitarity more straightforward. However light-cone superstring field theory suffers from contact term divergences which have not yet been understood fully [26–29]. A way to circumvent this has been suggested in [30].

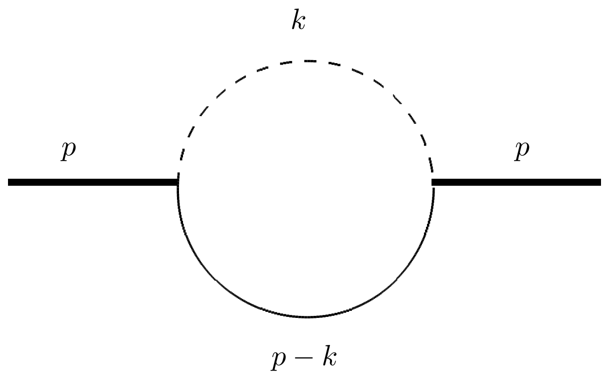

Figure 1. One loop mass renormalization diagram of a heavy state, labelled by a thick line, due to a loop of light particles, labelled by thin lines. The dashed line corresponds to a light particle of mass m_1 carrying momentum k and the continuous thin line corresponds to a light particle of mass m_2 carrying momentum $(p-k)$. All momenta flow from left to right.

also describe how our analysis can be easily extended to compactified string theories. In section 5 we show that the imaginary part of the mass^2 computed using our approach is manifestly consistent with unitarity. In appendix A we show the equivalence between the $i\epsilon$ prescription of [7, 8] and our prescription of section 2 in the context of one loop two point functions. In appendix B we analyze in detail the 'stringy contribution' to mass renormalization given by (3.12) and show explicitly that this gives a finite contribution.

2 Toy model

Let us consider a quantum field theory in D space-time dimensions with three particles of masses M, m_1 and m_2 respectively, with $M > m_1 + m_2$, in which there is a three point vertex that couples the three particles. Our goal will be to analyze the one loop mass renormalization diagram shown in figure 1. Inspired by string field theory, we shall assume that the vertex contains a factor of $\exp[-\frac{1}{2}A\{k^2+m_1^2\} - \frac{1}{2}A\{(p-k)^2+m_2^2\}]$ for some positive constant A that makes the diagram ultraviolet (UV) finite [23]. In that case the contribution of this diagram to mass^2 of the heavy particle can be expressed as

$$\delta M^2 = i\,B \int \frac{d^Dk}{(2\pi)^D}\,\exp[-A\{k^2+m_1^2\} - A\{(p-k)^2+m_2^2\}]\,\{k^2+m_1^2\}^{-1}\{(p-k)^2+m_2^2\}^{-1}, \tag{2.1}$$

where B is another positive constant that includes multiplicative constant contributions to the vertices, and p is an on-shell external momentum satisfying $p^2 = -M^2$. In general we could include factors involving polynomials in the momenta in the vertices without affecting the UV finiteness, but we have not included them to keep the analysis simple. Later we shall consider the effect of including such interactions.

2.1 Direct evaluation

Using $k^2 = -(k^0)^2 + \vec{k}^2$ where $\vec{k}$ denotes $(D-1)$-dimensional spatial momenta, we see that the exponential factor falls off exponentially as $|\vec{k}| \to \infty$ but grows exponentially as

JHEP11(2016)050

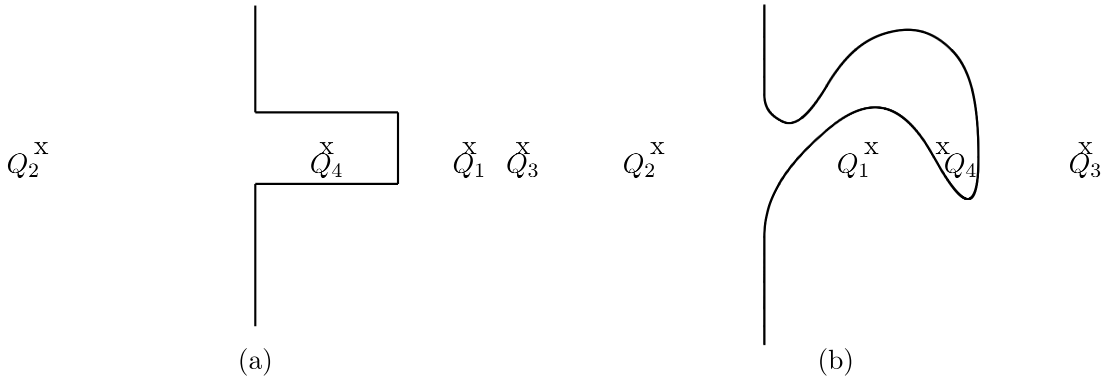

Figure 2. The integrations contours in the k^0 plane.

$k^0 \to \pm\infty$. This shows that we cannot take the k^0 integral to run along the real axis. This issue was discussed in detail in [23] where we proposed that the ends of the k^0 integral must always be at $\pm i\infty$ to ensure convergence of the integral, but the integration contour may take complicated form in the interior of the complex k^0 plane to avoid poles of the propagator. This is done as follows: begin with imaginary p^0 for which the k^0 contour is taken along the imaginary axis and then deform p^0 to the physical real value staying in the first quadrant of the complex p^0 plane, simultaneously deforming the k^0 contour appropriately to always stay away from the poles. In particular (2.1) was analyzed in detail in [23] using this prescription. Here we shall review some of the important details of that analysis.

The integrand of (2.1) has poles in the k^0 plane at

$$Q_1 \equiv \sqrt{\vec{k}^2 + m_1^2}, \qquad Q_2 \equiv -\sqrt{\vec{k}^2 + m_1^2},$$
$$Q_3 \equiv p^0 + \sqrt{(\vec{p} - \vec{k})^2 + m_2^2}, \qquad Q_4 \equiv p^0 - \sqrt{(\vec{p} - \vec{k})^2 + m_2^2}\,. \tag{2.2}$$

For imaginary p^0, and k^0 contour running along the imaginary axis from $-i\infty$ to $i\infty$, the poles Q_1 and Q_3 are to the right of the integration contour whereas the poles Q_2 and Q_4 are to the left of the integration contour. When p^0 is continued to the real axis along the first quadrant, the contour needs to be deformed appropriately so that Q_1 and Q_3 continue to lie on the right and Q_2 and Q_4 continue to lie on the left. There are different possible configurations depending on the value of $\vec{k}$.

As long as $p^0 < \sqrt{\vec{k}^2 + m_1^2} + \sqrt{(\vec{p} - \vec{k})^2 + m_2^2}$, Q_4 lies to the left of Q_1 and the contour can be taken as shown in figure 2(a). On the other hand for $p^0 > \sqrt{\vec{k}^2 + m_1^2} + \sqrt{(\vec{p} - \vec{k})^2 + m_2^2}$, Q_4 is to the right of Q_1 and the deformed contour takes the form shown in figure 2(b). In drawing this we have used the fact that when p^0 lies in the first quadrant, Q_4 remains above Q_1 as it passes Q_1 and that during this process the contour needs to be deformed continuously without passing through a pole. At the boundary between these two regions Q_4 approaches Q_1. In this case we have to use a limiting procedure to determine the contour, and the correct procedure will be to take p^0 in the first quadrant, evaluate the integral and then take the limit of real p^0. This in particular means that Q_4 approaches Q_1 from above in this limit.

In order to evaluate the integral, in both cases we deform the k^0 contour to be a sum of a contour along the imaginary axis and an anti-clockwise contour around the pole at Q_4. We shall choose, for convenience,

$$p = (M, \vec{0}) \,. \tag{2.3}$$

In this case the contribution from the first contour, after relabeling k^0 as $i\,u$, takes the form

$$\begin{aligned} I_1 = -B \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \int_{-\infty}^{\infty} \frac{du}{2\pi} \exp\left[-A\left\{u^2+\vec{k}^2+m_1^2\right\} - A\left\{(u+iM)^2+\vec{k}^2+m_2^2\right\}\right] \\ \left(u^2+\vec{k}^2+m_1^2\right)^{-1}\left\{(u+iM)^2+\vec{k}^2+m_2^2\right\}^{-1} . \end{aligned} \tag{2.4}$$

On the other hand the contribution from the residue at Q_4 gives

$$\begin{aligned} I_2 = -B \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \exp\left[A\left(M-\sqrt{\vec{k}^2+m_2^2}\right)^2 - A(\vec{k}^2+m_1^2)\right] \Theta\left(M-\sqrt{\vec{k}^2+m_2^2}\right) \\ \left(2\sqrt{\vec{k}^2+m_2^2}\right)^{-1}\left\{M+\sqrt{\vec{k}^2+m_1^2}-\sqrt{\vec{k}^2+m_2^2}\right\}^{-1} \\ \left\{\sqrt{\vec{k}^2+m_1^2}+\sqrt{\vec{k}^2+m_2^2}-M-i\epsilon\right\}^{-1} . \end{aligned} \tag{2.5}$$

In this expression Θ denotes the Heaviside function and reflects that this contribution is present only when Q_4 is to the right of the imaginary axis. The $i\epsilon$ in the arguments of the last term represents that we need to take the limit $p^0 \to M$ from the first quadrant, i.e. set p^0 to $M + i\epsilon$ and then take the $\epsilon \to 0^+$ limit. Defining $v = |\vec{k}|$ and doing the angular integration, I_1 and I_2 may be rewritten as

$$\begin{aligned} I_1 = -B\,(2\pi)^{-D}\Omega_{D-2} \int_0^{\infty} dv \int_{-\infty}^{\infty} du\, v^{D-2} \exp\Big[- A\left\{u^2+v^2+m_1^2\right\} \\ - A\left\{(u+iM)^2+v^2+m_2^2\right\}\Big] \left(u^2+v^2+m_1^2\right)^{-1}\left\{(u+iM)^2+v^2+m_2^2\right\}^{-1} , \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} I_2 = -B\,(2\pi)^{-(D-1)}\Omega_{D-2} \int_0^{\sqrt{M^2-m_2^2}} dv\, v^{D-2} \exp\left[A\left(M-\sqrt{v^2+m_2^2}\right)^2 - A(v^2+m_1^2)\right] \\ \left(2\sqrt{v^2+m_2^2}\right)^{-1}\left\{M+\sqrt{v^2+m_1^2}-\sqrt{v^2+m_2^2}\right\}^{-1} \\ \left\{\sqrt{v^2+m_1^2}+\sqrt{v^2+m_2^2}-M-i\epsilon\right\}^{-1} , \end{aligned} \tag{2.7}$$

where Ω_{D-2} is the volume of the unit $(D-2)$ sphere. Due to the exponential suppression factors and/or limits of integration, neither I_1 nor I_2 has any divergence from the large u

JHEP11(2016)050

or large v region. Even though as $\epsilon \to 0$ the integrand of I_2 has a pole on the real v axis from the last term, the contour is not pinched there. Hence we can define the integral by deforming the v integration contour below the real axis, getting a finite result. Therefore both I_1 and I_2 are manifestly finite (and in particular can be evaluated using numerical integration).

The analysis given above can be easily generalized to the case where the integrand in (2.1) is multiplied by an additional polynomial in momenta coming from the vertices and/or the propagators. Using rotational invariance we can always replace this by a polynomial Q in k^0 and $\vec{k}^2$. The result will still be given by the sum of two terms like (2.6) and (2.7). The integrand in (2.6) will now be multiplied by the polynomial Q with k^0 replaced by $i\,u$ and $\vec{k}^2$ replaced by v^2. On the other hand the integrand in (2.7) will be multiplied by the polynomial Q with k^0 replaced by $M-\sqrt{v^2+m_2^2}$ and $\vec{k}^2$ replaced by v^2.

2.2 Schwinger parameter representation

We shall now try to evaluate (2.1) by representing the propagators as integrals over Schwinger parameters. For this we write

$$\begin{aligned}
(k^2+m_1^2)^{-1} &= \int_0^\infty ds_1\, \exp\left[-s_1(k^2+m_1^2)\right], \\
\{(p-k)^2+m_2^2\}^{-1} &= \int_0^\infty ds_2\, \exp\left[-s_2\{(p-k)^2+m_2^2\}\right],
\end{aligned} \tag{2.8}$$

and substitute into (2.1). This give

$$\delta M^2 = iB\int_0^\infty ds_1 \int_0^\infty ds_2 \int \frac{d^Dk}{(2\pi)^D} \exp\left[-(A+s_1)\{k^2+m_1^2\}-(A+s_2)\{(p-k)^2+m_2^2\}\right]. \tag{2.9}$$

After performing integral over k, pretending that the k^0 integral runs along the imaginary axis and is convergent, and defining new variables

$$t_1 = s_1 + A, \quad t_2 = s_2 + A\,, \tag{2.10}$$

we get

$$\delta M^2 = -B\,(4\pi)^{-D/2} \int_A^\infty dt_1 \int_A^\infty dt_2\, (t_1+t_2)^{-D/2} \exp\left[\frac{t_1 t_2}{t_1+t_2}\, M^2 - (t_1 m_1^2 + t_2 m_2^2)\right]. \tag{2.11}$$

This expression has no UV divergence, i.e. divergence from the small t_i region, since the lower limits of t_i integrals are shifted to positive values A. However it is easy to see that this integral diverges from the region $t_1, t_2 \to \infty$ if

$$M > m_1 + m_2\,. \tag{2.12}$$

This divergence can be traced to the fact that for $M > m_1 + m_2$, it is not possible to choose the k^0 integration contour in a way that keeps the real parts of both $k^2 + m_1^2$ and $(p-k)^2 + m_2^2$ positive. As a result the Schwinger parameter representation (2.8) breaks

JHEP11(2016)050

down. However note that we can get finite results by taking the upper limits of the t_i integrals to be $i\infty$ instead of ∞ [7, 8]. We have shown in appendix A that this gives the same result as what we would obtain by following the prescription of section 2.1 for evaluating (2.1).

Since string world-sheet description of the S-matrix elements naturally gives the amplitudes in the Schwinger parameter representation, we shall see that the world-sheet description of one loop mass renormalization in string theory encounters similar divergences. Our strategy will be to use the insight gain from our analysis above to convert this to a momentum space integral of the form given in (2.1) and extract finite answers. For this we shall need a generalization of the analysis given above, where the integrand in (2.1) has an additional multiplicative factor given by some polynomial in the momenta $\{k^\mu\}$. We shall first discuss a few examples. The first example we consider is when the integrand in (2.1) has an additional factor of $(k^0)^2$. In this case it is easy to see that the integrand in (2.11) will be multiplied by an additional factor of

$$-\frac{1}{2(t_1+t_2)}+\frac{t_2^2}{(t_1+t_2)^2}M^2\,. \tag{2.13}$$

Next we consider the case where the integrand in (2.1) has a multiplicative factor of k^0. In this case the integrand in (2.11) is multiplied by an additional factor of

$$\frac{t_2}{t_1+t_2}M\,. \tag{2.14}$$

If we consider the case where the integrand in (2.1) has an additional multiplicative factor of k^ik^j with $1\le i,j\le(D-1)$, then we get an additional multiplicative factor of

$$\delta_{ij}\,\frac{1}{2(t_1+t_2)} \tag{2.15}$$

in (2.11). Finally if the integrand has an additional factor of $k^ik^jk^mk^n$ then we get an additional multiplicative factor of

$$\frac{1}{4(t_1+t_2)^2}\left(\delta_{ij}\delta_{mn}+\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}\right)\,. \tag{2.16}$$

It is clear that given any polynomial in $\{k^\mu\}$ inserted into (2.9), we can find the corresponding insertion in the integrand of the Schwinger parameter representation (2.11) by formally carrying out the integration over momenta using the rules of gaussian integration, pretending that the integral is convergent. An interesting question is whether the reverse is true: given any polynomial P in $1/(t_1+t_2)$ and $t_2/(t_1+t_2)$, can we find a function Q of momenta such that the following holds?

$$\begin{aligned} &i\int\frac{d^Dk}{(2\pi)^D}\exp\left[-t_1\{k^2+m_1^2\}-t_2\{(p-k)^2+m_2^2\}\right]Q(k)\\ &\quad=-(4\pi)^{-D/2}\exp\left[\frac{t_1t_2}{t_1+t_2}\,M^2-(t_1m_1^2+t_2m_2^2)\right]P\left(\frac{1}{t_1+t_2},\frac{t_2}{t_1+t_2}\right)\,. \end{aligned} \tag{2.17}$$

JHEP11(2016)050

It is clear that due to rotational invariance of the problem Q cannot be unique — e.g. $(k^1)^2$, $(k^2)^2$ and $\vec{k}^2/(D-1)$ will all generate the same expression after momentum integration. However they will also give the same result if we insert $Q(k)$ into the integrand in (2.1) and carry out the momentum integration directly using the procedure described in section 2.1. Therefore we can easily resolve this ambiguity in the form of Q by restricting Q to be a polynomial in k^0 and $\vec{k}^2$. In that case we can construct a unique Q from a given P as follows. We can start from the terms in P with the highest power of $t_2/(t_1+t_2)$, and among these the term with highest power of $1/(t_1+t_2)$. If this has the form $\{t_2/(t_1+t_2)\}^n\{1/(t_1+t_2)\}^m$, then we need a term Q_1 in Q proportional to $(k^0)^n(\vec{k}^2)^m$ to generate this. Let P_1 be the polynomial in $1/(t_1+t_2)$ and $t_2/(t_1+t_2)$ obtained by replacing Q, P by Q_1, P_1 in (2.17). Besides containing the term proportional to $\{t_2/(t_1+t_2)\}^n\{1/(t_1+t_2)\}^m$ appearing in P, P_1 will generically also contain terms with lower powers of $t_2/(t_1+t_2)$. We now repeat the analysis for $P-P_1$, by identifying the terms in $P-P_1$ with highest power of $t_2/(t_1+t_2)$, and among them the term with highest power of $1/(t_1+t_2)$. Proceeding this way till we have exhausted all the terms in P, we can find the polynomial $Q = Q_1 + Q_2 + \cdots$ that, when inserted into the left hand side of (2.17), will produce the desired P on the right hand side.

The effect of inserting P in the integrand of (2.11) can now be represented by insertion of $Q(k)$ in the integrand of (2.9) and hence of (2.1). Since Q is a polynomial in $\{k^\mu\}$, there will be no difficulty in carrying out the momentum integration in (2.1) directly following the procedure described in section 2.1 to get a finite result. This way any integral of the form (2.11), with arbitrary polynomial of $1/(t_1+t_2)$ and $t_2/(t_1+t_2)$ inserted in the integrand, can be interpreted as a finite momentum space integral.

3 One loop mass renormalization of an unstable state in string theory

We shall now use the insight gained from the analysis of section 2 to compute one loop mass renormalization in string theory. In this section we shall consider a specific example, leaving the general analysis to section 4. We consider the lowest massive state on the leading Regge trajectory in the SO(32) heterotic string theory.[3] The one loop correction to the mass^2 of this state can be computed from the on-shell two point function of the corresponding vertex operators on the torus. If we define

$$X^\pm = (X^1 \pm iX^2), \quad \psi^\pm = (\psi^1 \pm i\psi^2)\,, \tag{3.1}$$

where X^μ are the world-sheet scalars and ψ^μ are the right-moving world-sheet fermions, then the -1 picture unintegrated vertex operators of the states whose two point function on the torus we need to compute are:

$$\bar{c}\,c\,e^{-\phi}\psi^+\partial X^+(\bar\partial X^+)^2 e^{ik^0X^0} \quad \text{and} \quad \bar{c}\,c\,e^{-\phi}\psi^-\partial X^-(\bar\partial X^-)^2 e^{-ik^0X^0}\,, \tag{3.2}$$

[3]The advantage of working with states on the leading Regge trajectory is that they do not mix with any other state at the same mass level. This simplifies our analysis, but the method that we shall describe is valid for arbitrary states.

up to overall normalization constants. Here ϕ is the world-sheet scalar that originates from bosonizing the β-γ ghost system, and c, $\bar{c}$ are the usual ghost fields associated with diffeomorphism invariance on the world-sheet. Now it was argued in [31] that all the states at the first massive level which differ from each other by different right-moving excitations are related by space-time supersymmetry and hence will have the same mass renormalization. Using this we can instead consider the vertex operators

$$\bar{c}\,c\,e^{-\phi}\psi^1\psi^2\psi^3(\bar{\partial}X^+)^2 e^{ik^0X^0} \quad \text{and} \quad \bar{c}\,c\,e^{-\phi}\psi^1\psi^2\psi^3(\bar{\partial}X^-)^2 e^{-ik^0X^0}\,. \tag{3.3}$$

The reason for doing this is that with this choice the right-moving parts of the vertex operators become identical to those used in [31] and we can make use of the results of [31].[4] In this case the only difference between the vertex operators used in [31] and those used here is that the left-moving part of the vertex operators used in [31] were $\bar{S}_\alpha$ — the spin fields of the left-moving world-sheet fermions responsible for the SO(32) gauge group — instead of $(\bar{\partial}X^{\pm})^2$. Therefore if we want to compute the two point correlation function of the vertex operators (3.3) inserted at 0 and z on a torus with modular parameter τ, all we need to do is to replace, in the result of [31], the normalized two point function $\langle \bar{S}_\alpha(\bar{z})\bar{S}_\beta(0)\rangle$ by the normalized two point function $\langle(\bar{\partial}X^+(\bar{z}))^2(\bar{\partial}X^-(0))^2\rangle$. Normalizing both correlators so that as $\bar{z}\to 0$ they go as $1/\bar{z}^4$, we have

$$\langle \bar{S}_\alpha(\bar{z})\bar{S}_\beta(0)\rangle = \delta_{\alpha\beta}\left(\sum_\nu \overline{\vartheta_\nu(z/2)}^{16}\right)\left(\sum_\nu \overline{\vartheta_\nu(0)}^{16}\right)^{-1}\left(\overline{\vartheta_1'(0)}\right)^4\left(\overline{\vartheta_1(z)}\right)^{-4}\,, \tag{3.4}$$

and

$$\langle(\bar{\partial}X^+(\bar{z}))^2(\bar{\partial}X^-(0))^2\rangle = \left[\left(\frac{\overline{\vartheta_1'(z)}}{\overline{\vartheta_1(z)}}\right)^2 - \frac{\overline{\vartheta_1''(z)}}{\overline{\vartheta_1(z)}} - \frac{\pi}{\tau_2}\right]^2\,, \tag{3.5}$$

where ϑ_ν for $1\le\nu\le 4$ denotes Jacobi theta function of spin structure ν, with ϑ_1 being the Jacobi theta function with odd spin structure, and τ_1,τ_2,z_1,z_2 are defined via

$$\tau = \tau_1 + i\tau_2, \qquad z = z_1 + iz_2\,. \tag{3.6}$$

Therefore, to compute δM^2 we have to multiply the integrand obtained in [31] by the ratio of (3.5) and (3.4). This gives, from eq. (4.16), (4.17) of [31]:

$$\begin{aligned}
\delta M^2 &= -\frac{1}{32\,\pi}\,M^2\,g^2\int d^2\tau\int d^2z\,F(z,\bar{z},\tau,\bar{\tau})\,,\\
F(z,\bar{z},\tau,\bar{\tau}) &\equiv \left\{\sum_\nu\overline{\vartheta_\nu(0)^{16}}\right\}(\overline{\eta(\tau)})^{-18}(\eta(\tau))^{-6}(\overline{\vartheta_1'(0)})^{-4}\left(\vartheta_1(z)\overline{\vartheta_1(z)}\right)^2\\
&\quad\left[\left(\frac{\overline{\vartheta_1'(z)}}{\overline{\vartheta_1(z)}}\right)^2 - \frac{\overline{\vartheta_1''(z)}}{\overline{\vartheta_1(z)}} - \frac{\pi}{\tau_2}\right]^2 \exp[-4\pi\,z_2^2/\tau_2]\,(\tau_2)^{-5}\,,
\end{aligned} \tag{3.7}$$

[4] In [31] we converted both vertex operators to zero picture vertex operators for carrying out the computation. This does not satisfy the correct factorization condition when two vertex operators approach each other, and in some cases, can give erroneous results [32, 33]. However for flat space-time background, including toroidal compactification, the difference between the correct result and the one obtained using zero picture vertex operators can be computed using the analysis given in [33] and can be shown to vanish.

where g is the string coupling constant, normalized as in [31]. The integration over τ runs over the fundamental region and that over z runs over the whole torus. The sum over ν in (3.7) comes from the sum over spin structures in the left-moving sector of the world-sheet — the sum over spin structures in the right-moving sector have already been performed [31] in arriving at (3.7). Analogous expression for arbitrary state on the leading Regge trajectory in type II string theory can be found in [1, 13].

We shall now try to analyze possible divergences in this integral. It is easy to see that the integral has no divergence from the $z \to 0$ region, and is in fact finite for all finite values of τ and z. Since z_1 and τ_1 integrals are restricted to the range (0,1) and the z_2 integral is restricted to the range $0 \leq z_2 < \tau_2$, possible divergences come from the region of large τ_2 and possibly large z_2. In particular we can remove the $\tau_2 < 1$ region from our consideration, since this is a finite region with bounded integrand. For $\tau_2 \geq 1$ the τ_1 and z_1 integrals run over the entire range between 0 and 1. While evaluating these integrals we need to first integrate over z_1 and τ_1 for fixed z_2 and τ_2, and then integrate over z_2 and τ_2. A justification for this from string field theory will be given in section 4. Therefore if we expand the integrand in this region in powers of $e^{2\pi i\tau}$, $e^{-2\pi i\bar{\tau}}$, $e^{2\pi iz}$ and $e^{-2\pi i\bar{z}}$, all terms with non-zero powers of $e^{2\pi i\tau_1}$ or $e^{2\pi iz_1}$ will integrate to zero, and only the τ_1 and z_1 independent terms will survive.

We shall first consider the large τ_2 but finite z_2 region. For this we define finite z_2 region to be the region $z_2 < \Lambda$ for some fixed positive number $\Lambda \leq \tau_2$. In this region $F(z, \bar{z}, \tau, \bar{\tau})$ has the form

$$F(z,\bar{z},\tau,\bar{\tau}) = \exp[-4\pi z_2^2/\tau_2](\tau_2)^{-5}\left[2\pi^{-4}e^{2\pi i\bar{\tau}}|\sin(\pi z)|^4\left(\pi^2\cot^2(\pi\bar{z})+\pi^2-\frac{\pi}{\tau_2}\right)^2+\mathcal{O}(1)\right], \tag{3.8}$$

where the $\mathcal{O}(1)$ term is finite for any finite z, τ and approaches a fixed finite function of z for $\tau_2 \to \infty$ and finite z.[5] Therefore for $z_2 < \Lambda$, the $\mathcal{O}(1)$ term inside the square bracket can be bounded from above by a positive number Δ, and after integration over z and τ restricted to the region $z_2 < \Lambda \leq \tau_2$, $\tau_2 \geq 1$, its contribution to $\int d^2\tau d^2z\, F$ will be bounded from above by

$$\Lambda\,\Delta \int_1^\infty d\tau_2\, \tau_2^{-5} = \Lambda\,\Delta/4\,. \tag{3.9}$$

On the other hand the term proportional to $e^{2\pi i\bar{\tau}}$ inside the square bracket in (3.8) gives vanishing contribution after the τ_1 integration. This shows that the integral does not receive any divergent contribution from the $z_2 < \Lambda$ and large τ_2 region.

Next we examine the region of integration where both τ_2 and z_2 are large. Note that due to the reflection symmetry $z \to \tau - z$ of the integrand, there is also no divergence from the region where τ_2 and z_2 are large with $\tau_2 - z_2$ finite; so we focus on the region where z_2 and $\tau_2 - z_2$ are both large. Expanding the integrand in powers of $e^{2\pi i\tau}$, $e^{2\pi iz}$

[5]For $z = 0$ the $\mathcal{O}(1)$ term has a phase ambiguity since both the function F and the first term inside the square bracket in (3.8) is proportional to $(z/\bar{z})^2$ for small z. But the integral of this term over any finite neighborhood of $z = 0$ is unambiguous and finite for all τ inside the fundamental domain, as well as in the $\tau \to i\infty$ limit. The analogous expression in type II string theory will not have any such phase ambiguity.

and their complex conjugates, and throwing away all terms which have non-zero powers of $e^{2\pi i z_1}$ and/or $e^{2\pi i \tau_1}$ since they vanish after integration over z_1 and τ_1, we find that the part of $F(z,\bar{z})$ that can give divergent contribution to (3.7) takes the form

$$2\,(2\pi)^{-4}\left(32\pi^4 - 32\frac{\pi^3}{\tau_2} + 512\,\frac{\pi^2}{\tau_2^2}\right)\exp[4\pi z_2 - 4\pi z_2^2/\tau_2]\,\tau_2^{-5}\,. \tag{3.10}$$

Based on the above understanding of the possible sources of divergence, we shall now give a systematic procedure for isolating and dealing with the potentially divergent part. Using (3.7) we can write

$$\delta M^2 = J_1 + J_2\,, \tag{3.11}$$

where

$$\begin{aligned} J_1 = -\frac{1}{32\,\pi}\,M^2\,g^2 \int d^2\tau \int d^2 z\,\Big[F(z,\bar{z},\tau,\bar{\tau}) \qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.12)\\ -\,\Theta(\tau_2-z_2-\Lambda)\Theta(z_2-\Lambda)2(2\pi)^{-4}\left(32\pi^4 - 32\frac{\pi^3}{\tau_2} + 512\frac{\pi^2}{\tau_2^2}\right)\exp[4\pi z_2 - 4\pi z_2^2/\tau_2]\tau_2^{-5}\Big], \end{aligned}$$

and

$$\begin{aligned} J_2 = -\frac{1}{32\pi}M^2 g^2 \int d^2\tau \int d^2 z 2(2\pi)^{-4}\left(32\pi^4 - 32\frac{\pi^3}{\tau_2} + 512\frac{\pi^2}{\tau_2^2}\right)\exp[4\pi z_2 - 4\pi z_2^2/\tau_2]\tau_2^{-5}\\ \Theta(\tau_2 - z_2 - \Lambda)\,\Theta(z_2-\Lambda)\,, \end{aligned} \tag{3.13}$$

where Λ is an arbitrary positive constant larger than 1, and Θ denotes Heaviside step function.

First let us analyze J_1. For this it will be convenient to define the variable

$$w = \tau - z = w_1 + i w_2\,, \tag{3.14}$$

and divide the integration region into four parts. The region $z_2 < \Lambda$, $w_2 < \Lambda$ has finite size, and the integrand F is bounded. Hence there is no divergence from this region. In the region $z_2 < \Lambda$, $w_2 \geq \Lambda$ the integrand is F and by our previous argument that there is no divergence from the finite z_2, large τ_2 region, this integral is also finite. The region $z_2 \geq \Lambda$, $w_2 < \Lambda$ is related to the one just described by the $z \leftrightarrow w$, or equivalently $z \to \tau - z$ symmetry, and gives finite result. This leaves us with the region $z_2 \geq \Lambda$, $w_2 \geq \Lambda$. In this region the term proportional to the Heaviside functions in (3.12) subtracts the leading divergent piece. A careful analysis (see appendix B) shows that after throwing away all terms carrying non-zero powers of $e^{2\pi i z_1}$ and $e^{2\pi i w_1}$, we get finite result for J_1 from the $z_2 \geq \Lambda$, $w_2 \geq \Lambda$ region. Therefore there are no divergences in J_1 from any part of the region of integration.

Next we turn to the analysis of J_2 which only receives contribution from the $z_2 \geq \Lambda$, $w_2 \geq \Lambda$ region. We can bring J_2 to a more recognizable form by performing integrations over z_1 and τ_1 and defining the variables

$$t_1 = \pi\, z_2, \qquad t_2 = \pi w_2 = \pi(\tau_2 - z_2)\,. \tag{3.15}$$

In terms of these variables J_2 takes the form

$$J_2 = -2^{-3}\pi^2 M^2 g^2 \int_{\pi\Lambda}^{\infty} dt_1 \int_{\pi\Lambda}^{\infty} dt_2 (t_1+t_2)^{-5} \exp\left[4\frac{t_1 t_2}{t_1+t_2}\right] \left\{1-\frac{1}{(t_1+t_2)} + 16\frac{1}{(t_1+t_2)^2}\right\}. \tag{3.16}$$

The integral has apparent divergence from the large t_1, t_2 region. However we shall now try to interpret it as a finite momentum space integral by comparing this with (2.11) for $D = 10$. Comparing the overall normalization and the argument of the exponential we get[6]

$$B = (2\pi)^7 M^2 g^2, \quad A = \pi\Lambda, \quad M = 2, \quad m_1 = 0, \quad m_2 = 0\,. \tag{3.17}$$

Matching the rest of the integrand in (3.16) with what appears in (2.11) for $D = 10$, we see that we have an extra insertion of a factor of

$$\left(1 - \frac{1}{(t_1+t_2)} + 16\,\frac{1}{(t_1+t_2)^2}\right). \tag{3.18}$$

Using (2.15), (2.16) this can be identified as the effect of inserting a factor of

$$(1 - 2\,(k^1)^2 + 64\,(k^1)^2 (k^2)^2)\,. \tag{3.19}$$

in the integrand in the momentum space.[7] Combining this with (2.1), we can express J_2 as a momentum space integral

$$J_2 = i\,(2\pi)^7 M^2 g^2 \int \frac{d^{10}k}{(2\pi)^{10}} \exp[-\pi\Lambda k^2 - \pi\Lambda(p-k)^2]\,(k^2)^{-1}\{(p-k)^2\}^{-1}$$
$$\{1 - 2\,(k^1)^2 + 64\,(k^1)^2(k^2)^2\}\,. \tag{3.20}$$

This of course gives a finite contribution and can be evaluated using the method described in section 2.1.

Therefore we see that J_2 can be identified as the contribution from the Feynman diagram of the form shown in figure 1 with the parameters given in (3.17), and extra momentum dependent insertion in the integrand given in (3.19). In the $\alpha' = 1$ unit that we have been working in, $M = 2$ is the correct mass of the external state. The result $m_1 = m_2 = 0$ in (3.17) indicates that for this state the only source of divergence comes from the graphs where the intermediate states are massless. J_1 can be regarded as the contribution from the Feynman diagrams of figure 1 with other massive string states propagating in the loop and from other Feynman diagrams, including the elementary two point vertex. Note the dependence of J_1 and J_2 on the arbitrary parameter Λ; this represents the freedom of changing the interaction vertices of string field theory by ‘adding

[6] The peculiar factor of $(2\pi)^7$ in the expression for B can be traced to the fact that the heterotic string coupling g_H is related to the coupling g used here by the relation $g_H = (2\pi)^{7/2} g$ [31].

[7] Note that knowing the integrand in the Schwinger parameter representation does not fix the form in momentum space completely, e.g. the multiplicative factor could also have been $(1-(k^2)^2+64(k^3)^2(k^4)^2)$, or averages of various factors of this form. If we had started from string field theory, then Feynman diagrams would lead to a specific form. However for evaluation of the integral the detailed form is not necessary since due to rotation symmetry all of them lead to the same value of the integral.

JHEP11(2016)050

stubs', and can be compensated for by a redefinition of the string fields [34]. We shall show later that $J_1 + J_2$ is independent of Λ.

Manipulating (3.20) as in section 2.1 with $m_1 = m_2 = 0$, we can express this as

$$J_2 = I_1 + I_2 \,, \tag{3.21}$$

where

$$\begin{aligned} I_1 = -(2\pi)^7 M^2 g^2 \int \frac{d^9k}{(2\pi)^9} \int_{-\infty}^{\infty} \frac{du}{2\pi} \exp\left[-\pi\Lambda\left\{u^2+\vec{k}^2\right\} - \pi\Lambda\left\{(u+iM)^2+\vec{k}^2\right\}\right] \\ \times \{1-2\,(k^1)^2 + 64\,(k^1)^2(k^2)^2\} \left(u^2+\vec{k}^2\right)^{-1} \left\{(u+iM)^2+\vec{k}^2\right\}^{-1} , \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} I_2 = -(2\pi)^7 M^2 g^2 \int \frac{d^9k}{(2\pi)^9} \exp\left[\pi\Lambda\left(M-|\vec{k}|\right)^2 - \pi\Lambda\,\vec{k}^2\right] \Theta\left(M-|\vec{k}|\right) \\ \times \{1-2\,(k^1)^2 + 64\,(k^1)^2(k^2)^2\} \left(2M|\vec{k}|\right)^{-1} \left\{2|\vec{k}|-M-i\epsilon\right\}^{-1} . \end{aligned} \tag{3.23}$$

We can simplify both expressions by noting that due to rotational invariance the insertions of k^ik^j and $k^ik^jk^mk^n$ must give contributions proportional to

$$\delta_{ij} \quad \text{and} \quad \delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} \,, \tag{3.24}$$

respectively. This allows us to replace the insertion of $(k^1)^2$ by $\vec{k}^2/9$ and $(k^1)^2(k^2)^2$ by $(\vec{k}^2)^2/99$. Defining $v = |\vec{k}|$ we can write

$$\begin{aligned} I_1 = -(2\pi)^{-3} M^2 g^2\,\Omega_8 \int_0^{\infty} dv \int_{-\infty}^{\infty} du\, v^8 \exp\left[-\pi\Lambda\left\{u^2+v^2\right\} - \pi\Lambda\left\{(u+iM)^2+v^2\right\}\right] \\ \left(1-\frac{2}{9}\,v^2 + \frac{64}{99}\,v^4\right) \left(u^2+v^2\right)^{-1} \left\{(u+iM)^2+v^2\right\}^{-1} . \end{aligned} \tag{3.25}$$

On the other hand I_2 takes the form:

$$\begin{aligned} I_2 = -(2\pi)^{-2} M^2 g^2\,\Omega_8 \int_0^{M} dv\, v^8 \exp\left[\pi\Lambda\,(M-v)^2 - \pi\Lambda\,v^2\right] \\ \left(1-\frac{2}{9}\,v^2 + \frac{64}{99}\,v^4\right) (2Mv)^{-1} \left\{2v-M-i\epsilon\right\}^{-1} . \end{aligned} \tag{3.26}$$

I_1 is manifestly finite. I_2 is also manifestly finite if we deform the integration contour to avoid the pole at $v = (M+i\epsilon)/2$ by taking it to lie below the real axis. This gives a completely finite result for δM^2, given by the sum of J_1, I_1 and I_2.

It is easy to see that I_1 is real. We can also see the reality of J_1 given in (3.12) by observing that

$$(F(z,\bar{z},\tau,\bar{\tau}))^* = F(-\bar{z},-z,-\bar{\tau},-\tau) \,, \tag{3.27}$$

and that the integration domain and the integration measure are invariant under $(z \leftrightarrow -\bar{z}, \tau \leftrightarrow -\bar{\tau})$. Therefore the imaginary part of the amplitude comes only from I_2. This can be isolated by replacing the last factor in (3.26) by a sum of the principal value and a delta function and noting that the imaginary part comes from the delta function. This gives

$$\begin{aligned}
\mathrm{Im}\,\left(\delta M^2\right) &= -\frac{1}{4\pi} M^2 g^2 \Omega_8 \int_0^M dv\, v^8\, \exp\left[\pi\Lambda\,(M-v)^2 - \pi\Lambda\, v^2\right] \\
&\qquad \left(1 - \frac{2}{9}\, v^2 + \frac{64}{99}\, v^4\right)\, (2Mv)^{-1}\, \delta(2v - M) \\
&= -\frac{1}{8\pi}\Omega_8\, g^2 \left(\frac{M}{2}\right)^8 \left(1 - \frac{1}{18} M^2 + \frac{4}{99} M^4\right) = -\frac{47}{264\,\pi}\,\Omega_8\, g^2\,, \qquad (3.28)
\end{aligned}$$

where in the last step we have used $M = 2$. In section 5 we shall argue that this result is consistent with unitarity.

We shall now show that although each of the quantities J_1, I_1 and I_2 depends on the arbitrary parameter Λ, their sum does not depend on Λ. For this, note that from (3.12) we get

$$\begin{aligned}
\frac{d}{d\Lambda} J_1 &= -\frac{1}{16\,\pi}\, M^2\, g^2 \int d^2\tau \int d^2 z\, 2\, (2\pi)^{-4} \left(32\pi^4 - 32\frac{\pi^3}{\tau_2} + 512\,\frac{\pi^2}{\tau_2^2}\right) \\
&\qquad \exp[4\pi z_2 - 4\pi z_2^2/\tau_2]\, \tau_2^{-5}\, \Theta(\tau_2 - z_2 - \Lambda)\delta(z_2 - \Lambda)\,, \\
&= -2^{-7}\pi^{-5} M^2 g^2 \int_{2\Lambda}^{\infty} d\tau_2 \exp[4\pi\Lambda - 4\pi\Lambda^2/\tau_2]\tau_2^{-5}\left(32\pi^4 - 32\frac{\pi^3}{\tau_2} + 512\frac{\pi^2}{\tau_2^2}\right), \qquad (3.29)
\end{aligned}$$

where in the first step we have used the $z \to \tau - z$ symmetry to combine two terms into a single term. On the other hand from (3.20), (3.21) we get

$$\frac{d}{d\Lambda}(I_1 + I_2) = -i\, 2^8\, \pi^8\, M^2\, g^2 \int \frac{d^{10}k}{(2\pi)^{10}} \exp[-\pi\Lambda k^2 - \pi\Lambda(p-k)^2]\, (k^2)^{-1}(1 - 2\, k_1^2 + 64\, k_1^2 k_2^2) \qquad (3.30)$$

where again we have exploited the $k \to (p-k)$ symmetry to combine two terms into a single term. Once the pole associated with $\{(p-k)^2\}^{-1}$ has been removed, there is no obstruction to taking the k^0 integration contour to lie along the imaginary axis, and representing $(k^2)^{-1}$ as $\int_0^\infty ds e^{-sk^2}$. Carrying out the integration over k^μ using the rules of gaussian integration, and defining $\tau_2 = (s + 2\pi\Lambda)/\pi$, we get

$$\frac{d}{d\Lambda}(I_1 + I_2) = 2^{-2}\pi^{-1} M^2 g^2 \int_{2\Lambda}^{\infty} d\tau_2 \exp[4\pi\Lambda - 4\pi\Lambda^2/\tau_2]\tau_2^{-5}\left(1 - \frac{1}{\pi\tau_2} + \frac{16}{\pi^2\tau_2^2}\right). \qquad (3.31)$$

Using (3.29) and (3.31) we get

$$\frac{d}{d\Lambda}(J_1 + I_1 + I_2) = 0\,. \qquad (3.32)$$

4 Generalizations and justification using string field theory

The procedure described in the previous section can be used to compute the renormalized mass of any massive state in heterotic or type II string theory. For general physical states, at one loop order one has to consider the possibility of mixing with other physical states at the same mass level, but not with pure gauge or unphysical states [35], or with states at different mass level. If we denote by δM^2 the one loop two point function of physical states — typically a matrix with both real and imaginary parts — then the one loop propagator will be proportional to $(k^2 + M^2 + \delta M^2)^{-1}$, and its poles will be at places where $\det(k^2 + M^2 + \delta M^2)$ vanishes.

The general strategy for computing the matrix δM^2 will be as follows. The two point function of general on-shell external states of mass M can be brought to the form

$$\delta M^2 = \int d^2\tau\, d^2z\, F \tag{4.1}$$

where F is some function of $z, \bar{z}, \tau, \bar{\tau}$ describing the two point function of the corresponding vertex operators on the torus. Let us define z_1, z_2, w_1, w_2 via

$$z = z_1 + iz_2, \quad w = \tau - z \equiv w_1 + iw_2\,. \tag{4.2}$$

The potential divergence in (4.1) comes from the region of large z_2 and w_2. If we denote by F_0 the part of F that can give divergent contribution, then F_0 has the general form

$$F_0 = \tau_2^{-5}\, \exp[\pi M^2 z_2 w_2/\tau_2] \sum_{m,n} e^{2\pi i m z_1 + 2\pi i n w_1} e^{2\pi z_2 + 2\pi w_2}\, A_{m,n}(z_2, w_2)\,, \tag{4.3}$$

where the sum over m, n runs over a finite set of integers, and $A_{m,n}$ is a function of z_2, w_2 that involves a finite sum of products of non-negative powers of $e^{-2\pi z_2}$, $e^{-2\pi w_2}$, and polynomial of $1/\tau_2$ and z_2/τ_2. In defining F_0 we shall include in $e^{2\pi z_2 + 2\pi w_2} A_{m,n}$ a term proportional to $e^{-2\pi p z_2 - 2\pi q w_2}$ if and only if either p or q is negative, or $\sqrt{2p} + \sqrt{2q} < M$, since these are the terms that can cause potential divergence in (4.1) from the large z_2 and large w_2 region. In (4.3) the $\tau_2^{-5} \exp[\pi w_2 z_2 M^2/\tau_2]$ factor comes from the non-holomorphically factorized part of the correlation function of $e^{\pm ik.X}$ factors in the vertex operators — expressed in $z \to \tau - z$ invariant form. The factors of $e^{2\pi i m z_1 + 2\pi i n w_1}$, $e^{2\pi z_2 + 2\pi w_2}$, and the powers of $e^{-2\pi z_2}$, $e^{-2\pi w_2}$ hidden in the definition of $A_{m,n}$ come from the expansion of the holomorphically factorized pieces in the correlation function for large z_2 and w_2. Finally the polynomials of $1/\tau_2$ and z_2/τ_2 in the expansion of $A_{m,n}$ come from the derivatives of the term proportional to $(z_2 - w_2)^2/\tau_2$ in the Green's function $\langle X^\mu(z, \bar{z}) X^\nu(w, \bar{w})\rangle$. The presence of the explicit factor of $e^{2\pi z_2}$ and $e^{2\pi w_2}$ is a reflection of the presence of the tachyon in the left-moving sector before level matching.[8] However a term proportional to $e^{2\pi z_2}$ (resp. $e^{2\pi w_2}$) in the expression for F_0 appears only when accompanied by a factor of $e^{2\pi i z_1}$ (resp. $e^{2\pi i w_1}$), i.e. $A_{p,q}$ will have its expansion beginning with the power of $e^{-2\pi z_2}$ (resp. $e^{-2\pi w_2}$) except for $p = 1$ (resp. $q = 1$). Therefore for $\tau_2 \geq 1$, the contribution from terms proportional to $e^{2\pi z_2}$ (resp. $e^{2\pi w_2}$) disappears after integration over z_1 (resp. w_1). More generally, for $\tau_2 \geq 1$ integration over z_1 and w_1 will make the integral (4.3) vanish unless $m = n = 0$, but we shall continue to display them for reasons that will become clear later.

Using (4.1), (4.3) we can write

$$\delta M^2 = J_1 + J_2\,, \tag{4.4}$$

[8] These factors will be absent in type II string theories.

where

$$J_1 = \int d^2w\, d^2z\, [F - F_0\, \Theta(w_2 - \Lambda)\, \Theta(z_2 - \Lambda)]\,, \tag{4.5}$$

$$\begin{aligned} J_2 &= \int d^2w\, d^2z\, F_0(z_1, z_2, w_1, w_2)\, \Theta(w_2 - \Lambda)\, \Theta(z_2 - \Lambda) \\ &= \int d^2w\, d^2z\, \Theta(w_2 - \Lambda)\Theta(z_2 - \Lambda)\, \tau_2^{-5}\; e^{\pi M^2 z_2 w_2/\tau_2} e^{2\pi z_2 + 2\pi w_2} \\ &\quad \times \sum_{m,n} e^{2\pi i m z_1 + 2\pi i n w_1}\, A_{m,n}(z_2, w_2)\,, \end{aligned} \tag{4.6}$$

and Λ is an arbitrary constant, which we shall take to be larger than 1. J_1 can be shown to be finite following the strategy used in appendix B. Our strategy for evaluation of J_2 will be to drop all terms with non zero m, n since they vanish by integration over z_1 and w_1, and for the $m = n = 0$ term, expand $e^{2\pi z_2 + 2\pi w_2} A_{0,0}$ in a power series

$$e^{2\pi z_2 + 2\pi w_2} A_{0,0}(z_2, w_2) = \sum_{\substack{p,q \ge 0 \\ 2\sqrt{p} + 2\sqrt{q} < M}} e^{-4\pi p z_2 - 4\pi q w_2} P_{p,q}(z_2, w_2)\,, \tag{4.7}$$

where $P_{p,q}$ is a polynomial in $1/(z_2 + w_2)$ and $z_2/(z_2 + w_2)$. Note that once we have focussed on terms independent of z_1 and w_1, the series expansion is in powers of $e^{2\pi i(z - \bar{z})} = e^{-4\pi z_2}$ and $e^{2\pi i(w - \bar{w})} = e^{-4\pi w_2}$. By making the substitution[9]

$$\pi\, z_2 = t_1, \quad \pi\, w_2 = t_2, \quad \pi\tau_2 = t_1 + t_2\,, \tag{4.8}$$

and using (4.7), we can now express J_2 as

$$J_2 = \pi^3 \int_{\pi\Lambda}^{\infty} dt_1 \int_{\pi\Lambda}^{\infty} dt_2\, (t_1 + t_2)^{-5} \sum_{\substack{p,q \ge 0 \\ 2\sqrt{p} + 2\sqrt{q} < M}} \exp\left[M^2 t_1 t_2/(t_1 + t_2) - 4\, p\, t_1 - 4\, q\, t_2\right] P_{p,q}\,. \tag{4.9}$$

This integral diverges for $t_1, t_2 \to \infty$, but we can replace this by a momentum space integral by comparing with the results of section 2.2. Once we have made the replacement, the integration over k^0 has to be interpreted as a contour integral following the procedure described in section 2.1, while integration over $\vec{k}$ can be regarded as ordinary integrals running along the real axes. This gives finite result due to exponential suppression factor in the integrand for large space-like momenta.

Note that this method is applicable for all massive states, including the ones that do not appear as intermediate states in the scattering of massless external states, e.g. massive states in SO(32) heterotic string theory carrying SO(32) spinor representation. For such states the method of [4, 5] based on factorization of four point function of massless states is not directly applicable. Furthermore, since this method allows us to directly compute the one loop two point function of two arbitrary physical states at the same mass level, we do

[9]The scale factor π is fixed as follows. The Schwinger parameters t_1 and t_2 introduced in section 2.2 appears in the exponent multiplied by a factor of $k^2 + m^2$. On the other hand z_2 and w_2 appear in the exponent multiplied by a factor of $2\pi(L_0 + \bar{L}_0) = \pi(k^2 + m^2)$.

not have to make the effort of disentangling the contributions from different intermediate states to the four point function.

There is however a possible subtlety with this procedure arising out of the following consideration. If we compute the one loop two point amplitude in string field theory, then, for sufficiently large Λ, the contribution J_2 comes from the sum of Feynman diagrams of the type shown in figure 1 with different states propagating in the loop. If we represent the Siegel gauge propagator as

$$b_0\,\bar{b}_0\,(L_0+\bar{L}_0)^{-1}\delta_{L_0,\bar{L}_0} = 2\pi\, b_0\,\bar{b}_0 \int_0^\infty d\xi_2 \int_0^1 d\xi_1\, e^{-2\pi\xi_2(L_0+\bar{L}_0)} e^{2\pi i\xi_1(L_0-\bar{L}_0)}\,, \qquad (4.10)$$

then for the two internal propagators of figure 1 we have two complex variables ξ and ζ — the analog of the variable $\xi_1+i\xi_2$ in (4.10). Now if ξ and ζ could be identified as the moduli parameters z and w, then replacing the right hand side of (4.10) by the left hand side is equivalent to the prescription for doing the integration in the way we have suggested — i.e. first integrate over z_1 and w_1 at fixed z_2 and w_2, and then replace the integration over z_2 and w_2 by momentum space integrals. However the parameters z and w are not directly the variables ξ and ζ of the string field theory — they are given by some functions of ξ and ζ. Therefore it is not *a priori* guaranteed that first performing the integration over the real parts of z and w, and then treating the imaginary parts of z and w as Schwinger parameters to translate the amplitude to a momentum space integral is a valid procedure. The correct procedure will be to first express the amplitude as integrals over the variables ξ and ζ, carry out the integrations over the real parts of ξ and η, and then interpret the expression as coming from momentum space integrals treating the imaginary parts of ξ and ζ as Schwinger parameters. We shall now argue that this does not change the result.

Since different string field theories (related by field redefinition) lead to different plumbing fixture variables, instead of focussing on any particular string field theory we shall consider the effect of a general parameter redefinition of the form

$$z = f(\xi,\zeta), \quad w = g(\xi,\zeta)\,. \qquad (4.11)$$

In order to get some insight into the form of the functions f and g, it will be useful to recall the geometric interpretation of the parameters ξ and ζ. In string field theory the Feynman diagram of figure 1 will represent the effect of sewing two three punctured spheres. If the first one has punctures P_1, P_2, P_3 with local coordinates y_1, y_2 and y_3, and the second one has punctures $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ with local coordinates $\widetilde{y}_1$, $\widetilde{y}_2$ and $\widetilde{y}_3$, then the sewing is done via the relations

$$y_2\,\widetilde{y}_2 = e^{2\pi i\xi}, \quad y_3\,\widetilde{y}_3 = e^{2\pi i\zeta}\,. \qquad (4.12)$$

The external states are inserted at the punctures P_1 and $\widetilde{P}_1$. Using this geometric interpretation of the parameters ξ and ζ it is easy to see that for large ξ_2 and ζ_2, we have $z\simeq\xi$ and $w\simeq\zeta$. Using this and the fact that z, w, ξ and ζ are periodic variables with period 1, we see that $z-\xi$ and $w-\zeta$ will have expansions in non-negative powers of $e^{2\pi i\xi}$ and $e^{2\pi i\zeta}$. Since such a redefinition of parameters can be built from successive infinitesimal deformations, we shall now focus on infinitesimal deformations of the form

$$z = \xi + a(\xi,\zeta), \quad w = \zeta + b(\xi,\zeta)\,, \qquad (4.13)$$

where a and b are infinitesimal functions admitting expansion in non-negative powers of $e^{2\pi i\xi}$ and $e^{2\pi i\zeta}$. If we can show that for general infinitesimal a and b, first expressing J_2 in the ξ, ζ variables and then mapping it to momentum space representation regarding ξ_2 and ζ_2 as Schwinger parameters, gives the same result as what we get by directly converting the original expression for J_2 to momentum space integral treating z_2 and w_2 as Schwinger parameters, then we would have proven a similar result for finite redefinitions relating z and w to ξ and ζ. This is what we shall now show.

Taking real and imaginary parts of (4.13) we write

$$\begin{aligned} z_1 &= \xi_1 + a_1(\xi_1, \xi_2, \zeta_1, \zeta_2), \quad & z_2 &= \xi_2 + a_2(\xi_1, \xi_2, \zeta_1, \zeta_2), \\ w_1 &= \zeta_1 + b_1(\xi_1, \xi_2, \zeta_1, \zeta_2), \quad & w_2 &= \zeta_2 + b_2(\xi_1, \xi_2, \zeta_1, \zeta_2)\,, \end{aligned} \tag{4.14}$$

where a_i and b_i are periodic functions of ξ_1 and ζ_1 with period 1. Under this change of variables, we get

$$J_2 = \widetilde{J}_2 + \delta J_2 \tag{4.15}$$

where

$$\widetilde{J}_2 = \int d^2\xi\, d^2\zeta\, F_0(\xi_1, \xi_2, \zeta_1, \zeta_2)\, \Theta(\xi_2 - \Lambda)\, \Theta(\zeta_2 - \Lambda) \tag{4.16}$$

and

$$\begin{aligned} \delta J_2 = & \int d^2\xi\, d^2\zeta\, \Theta(\xi_2 - \Lambda)\Theta(\zeta_2 - \Lambda) \sum_{i=1}^{2} \left[\frac{\partial}{\partial \xi_i} \{a_i\, F_0\} + \frac{\partial}{\partial \zeta_i} \{b_i\, F_0\} \right] \\ & + \int d^2\xi\, d^2\zeta\, \left[\delta(\xi_2 - \Lambda)\Theta(\zeta_2 - \Lambda)\, a_2 + \Theta(\xi_2 - \Lambda)\delta(\zeta_2 - \Lambda)\, b_2\right] F_0\,. \end{aligned} \tag{4.17}$$

The arguments of a_i, b_i and F_0 in (4.17) are $\xi_1, \xi_2, \zeta_1, \zeta_2$. Now J_2 evaluated by regarding z_2 and w_2 as Schwinger parameters is identical to $\widetilde{J}_2$ evaluated by regarding ξ_2 and ζ_2 as Schwinger parameters. Therefore we need to show that δJ_2 evaluated by regarding ξ_2 and ζ_2 as Schwinger parameters vanish.

Since the integration rules involve carrying out integration over ξ_1 and ζ_1 first at fixed ξ_2 and ζ_2 and then integrating over ξ_2 and ζ_2, the derivatives with respect to ξ_1 and ζ_1 vanish after integration due to the periodicity of the functions a_i, b_i and F_0 in the ξ_1 and ζ_1 variables. Since the rest of the terms admit expansion in powers of $e^{2\pi i\xi_1}$ and $e^{2\pi i\zeta_1}$, only the ξ_1 and ζ_1 independent terms can contribute, — the other terms will vanish after integration over ξ_1 and ζ_1. Therefore we can write

$$\delta J_2 = \int_\Lambda^\infty d\xi_2 \int_\Lambda^\infty d\zeta_2 \left[\frac{\partial \widetilde{a}(\xi_2, \zeta_2)}{\partial \xi_2} + \frac{\partial \widetilde{b}(\xi_2, \zeta_2)}{\partial \zeta_2} \right] + \int_\Lambda^\infty d\zeta_2\, \widetilde{a}(\Lambda, \zeta_2) + \int_\Lambda^\infty d\xi_2\, \widetilde{b}(\xi_2, \Lambda)\,, \tag{4.18}$$

where[10]

$$\widetilde{a}(\xi_2,\zeta_2) \equiv \int_0^1 d\xi_1 \int_0^1 d\zeta_1\, a_2(\xi_1,\xi_2,\zeta_1,\zeta_2)\, F_0(\xi_1,\xi_2,\zeta_1,\zeta_2),$$
$$\widetilde{b}(\xi_2,\zeta_2) \equiv \int_0^1 d\xi_1 \int_0^1 d\zeta_1\, b_2(\xi_1,\xi_2,\zeta_1,\zeta_2)\, F_0(\xi_1,\xi_2,\zeta_1,\zeta_2)\,. \tag{4.19}$$

From the form of F_0 given in (4.3) we see that $\widetilde{a}$ and $\widetilde{b}$ will have expansions of the form

$$\begin{pmatrix} \widetilde{a} \\ \widetilde{b} \end{pmatrix} = (\xi_2+\zeta_2)^{-5} \exp[\pi M^2 \xi_2\zeta_2/(\xi_2+\zeta_2)] \sum_{m,n\geq 0} e^{-4\pi m\xi_2 - 4\pi n\zeta_2} \begin{pmatrix} C^a_{m,n} \\ C^b_{m.n}\,, \end{pmatrix} \tag{4.20}$$

where $C^a_{m,n}$ and $C^b_{m,n}$ are polynomials in $1/(\xi_2+\zeta_2)$ and $\xi_2/(\xi_2+\zeta_2)$. Formally the right hand side of (4.18) vanishes by integration by parts. However we have to remember that these are divergent integrals and in order to make sense of them we have to replace them by momentum space integrals following the dictionary given in section 2. Therefore we shall now replace each of the terms in the expression (4.18) by momentum space integrals, and then ask if the total contribution vanishes.

We proceed as follows. Using the algorithm described in section 2.2 we first express $\widetilde{a}(\xi_2,\zeta_2)$ and $\widetilde{b}(\xi_2,\zeta_2)$ given in (4.20) in the form

$$\begin{pmatrix} \widetilde{a}(\xi_2,\zeta_2) \\ \widetilde{b}(\xi_2,\zeta_2) \end{pmatrix} = \sum_{m,n\geq 0} \int \frac{d^{10}k}{(2\pi)^{10}} e^{-\pi\xi_2(k^2+4m)-\pi\zeta_2((p-k)^2+4n)} \begin{pmatrix} f_{a,m,n}(k) \\ f_{b,m,n}(k)\,, \end{pmatrix} \tag{4.21}$$

where $f_{a,m,n}(k)$ and $f_{b,m,n}(k)$ is some polynomial in k^0 and $\vec{k}^2$, and we have $p^2=-M^2$. In that case $\partial\widetilde{a}/\partial\xi_2$ will have the expression of the form

$$\frac{\partial\widetilde{a}}{\partial\xi_2} = -\pi \sum_{m,n\geq 0} \int \frac{d^{10}k}{(2\pi)^{10}} e^{-\pi\xi_2(k^2+4m)-\pi\zeta_2((p-k)^2+4n)} (k^2+4m) f_{a,m,n}(k)\,. \tag{4.22}$$

Now the replacement rule says that after substituting the expressions given above into the integrals appearing in (4.18), we make the replacements

$$\int_\Lambda^\infty d\xi_2 e^{-\pi\xi_2(k^2+4m)} \to \frac{1}{\pi} \exp\left[-\pi\Lambda(k^2+4m)\right] \, (k^2+4m)^{-1}\,, \tag{4.23}$$

and

$$\int_\Lambda^\infty d\zeta_2 e^{-\pi\zeta_2((p-k)^2+4n)} \to \frac{1}{\pi} \exp\left[-\pi\Lambda\{(p-k)^2+4n\}\right] \, \{(p-k)^2+4n\}^{-1}\,, \tag{4.24}$$

and then interpret the integration over k^0 as a contour integration of the kind described in section 2.1, and the integration over $\vec{k}$ as ordinary $(D-1)$ dimensional integral along real

[10] Note that part of the contribution comes from the terms carrying powers of $e^{2\pi i z_1}$ and $e^{2\pi i w_1}$ in the original expression for $F_0(z_1,z_2,w_1,w_2)$, since such terms, after combining with the ξ_1 and ζ_1 dependent terms in a_2 and b_2, can give rise to ξ_1 and ζ_1 independent terms in a_2F_0 and b_2F_0. This is the reason we had kept such terms in the expression for F_0.

axis. This makes the replacement rules (4.23) and (4.24) only formal, since the integration over k^0 can run over domains in which k^2+4m or $(p-k)^2+4n$ may turn negative making the left hand sides diverge. Using these rules, we get

$$\int_\Lambda^\infty d\xi_2 \int_\Lambda^\infty d\zeta_2 \frac{\partial \widetilde{a}(\xi_2,\zeta_2)}{\partial \xi_2} \to -\frac{1}{\pi} \sum_{m,n\geq 0} \int \frac{d^{10}k}{(2\pi)^{10}} \exp\left[-\pi\Lambda(k^2+4m) - \pi\Lambda\{(p-k)^2+4n\}\right] \times \{(p-k)^2+4n\}^{-1} f_{a,m,n}(k)\,. \tag{4.25}$$

Note that the $(k^2+4m)^{-1}$ factor of (4.23) has been cancelled by the explicit (k^2+4m) factor produced in (4.22) by the $\partial/\partial\xi_2$ operation. The right hand side of this expression is finite, while the individual terms contributing to the left hand side can be infinite for $M > \sqrt{4m}+\sqrt{4n}$. The rules we have proposed uses the right hand side as the definition of the left hand side. On the other hand we have

$$\int_\Lambda^\infty d\zeta_2\, \widetilde{a}(\Lambda,\zeta_2) = \frac{1}{\pi} \sum_{m,n\geq 0} \int \frac{d^{10}k}{(2\pi)^{10}} \exp\left[-\pi\Lambda(k^2+4m) - \pi\Lambda\{(p-k)^2+4n\}\right] \times \{(p-k)^2+4n\}^{-1} f_{a,m,n}(k)\,. \tag{4.26}$$

Note that this is an equality — both the left and the right hand sides are finite since the integral of an expression of the form given in (4.20) is finite if either ξ_2 or ζ_2 is fixed. Therefore we can use either description to evaluate this contribution. We now see that the right hand sides of (4.25) and (4.26) cancel. A similar analysis shows that the other two terms in (4.18) also cancel.

This shows that δJ_2 vanishes. Therefore J_2 takes the same value irrespective of whether we use its expression in the w, z coordinate and express it as momentum space integral by regarding z_2 and w_2 as Schwinger parameters, or whether we take its expression in the ξ, ζ coordinate and express it as momentum space integral by regarding ξ_2 and ζ_2 as Schwinger parameters. Integrating this result to generate finite deformations, we see that the result remains the same irrespective of whether we use the z, w variables or the sewing parameters of a string field theory to generate the momentum space representation. Besides justifying the use of z, w variables to generate momentum space representation, this analysis also shows that the result is independent of which string field theory we use to generate the momentum space representation.

The analysis has a straightforward generalization to compactified heterotic and type II string theories described by general superconformal world-sheet theories. If we consider a vacuum with D non-compact space-time dimensions, then the overall multiplicative factor of τ_2^{-5} in (4.3) will be replaced by $\tau_2^{-D/2}$. The other difference will be that the coefficients $A_{m,n}$ will not only have integer powers of $e^{-2\pi z_2}$, $e^{-2\pi w_2}$, $e^{2\pi i z_1}$, $e^{2\pi i w_1}$, but also fractional powers of $e^{-2\pi z_2}$ and $e^{-2\pi w_2}$. For example for compactification on a circle of radius R, $A_{p,q}$ will contain factors of

$$\exp\left[-\frac{\pi i \bar{z}}{2}\left(\frac{n}{R}+mR\right)^2 + \frac{\pi i z}{2}\left(\frac{n}{R}-mR\right)^2\right], \tag{4.27}$$

and

$$\exp\left[-\frac{\pi i \bar{w}}{2}\left(\frac{n}{R}+mR\right)^2 + \frac{\pi i w}{2}\left(\frac{n}{R}-mR\right)^2\right]. \tag{4.28}$$

Here n, m are integers labelling the momentum and winding numbers along the circle. The rest of the analysis can be carried out as before by converting each term into momentum space integrals.

5 Unitarity

In this section we shall show that the result for one loop contribution to mass^2 computed using our method is consistent with unitarity. The general analysis of [23] already shows that the result satisfies Cutkosky rules. This would prove unitarity if in the Siegel gauge all states with $L_0 = \bar{L}_0 = 0$ had been physical states. However in general there will also be unphysical and pure gauge states. Hence we need to show that their contribution to the cut diagram vanishes.

While for a general amplitude establishing this requires some effort [36], for the one loop two point function the analysis can be carried out as follows. Let us focus on states with $L_0 = \bar{L}_0 = 0$ and annihilated by b_0 and $\bar{b}_0$, since these are the states that are associated with a cut propagator in the Siegel gauge. We choose a basis of states such that unphysical states — those not annihilated by the BRST charge Q_B — are labelled as $|\phi_s\rangle$, and physical states — annihilated by Q_B but not pure gauge — are labelled as $|\chi_a\rangle$. In this basis we do not need to introduce separately the basis of pure gauge states — they can be taken to be $Q_B|\phi_s\rangle$. Pure gauge states have non-zero inner product only with unphysical states, while physical states can have non-zero inner product with unphysical and physical states. Using the fact that the BPZ inner product is non-degenerate, one can argue that it is possible to choose a basis in which unphysical states have non-zero inner product only with pure gauge states and physical states have non-zero inner product only with physical states. We denote by $|\phi_s^c\rangle$ and $|\chi_a^c\rangle$ another basis of unphysical and physical states, also annihilated by b_0, $\bar{b}_0$, and satisfying

$$\begin{aligned} \langle\phi_s^c|c_0^- c_0^+ Q_B|\phi_r\rangle = \delta_{rs}, \quad \langle\phi_s^c|c_0^- c_0^+|\phi_r\rangle = 0, \quad \langle\chi_b^c|c_0^- c_0^+|\chi_a\rangle = \delta_{ab}, \\ \langle\phi_s^c|c_0^- c_0^+|\chi_a\rangle = 0, \qquad \langle\chi_b^c|c_0^- c_0^+|\phi_r\rangle = 0\,, \end{aligned} \tag{5.1}$$

where

$$c_0^\pm = \frac{1}{2}(c_0 \pm \bar{c}_0), \quad b_0^\pm = b_0 \pm \bar{b}_0, \quad L_0^\pm = L_0 \pm \bar{L}_0\,. \tag{5.2}$$

From (5.1) we get

$$n_s + n_s^c = 3\,, \tag{5.3}$$

where n_s and n_s^c are the ghost numbers of ϕ_s and ϕ_s^c respectively.

Since Siegel gauge propagator is proportional to $b_0^+ b_0^- (L_0^+)^{-1}\delta_{L_0,\bar{L}_0}$, a cut propagator in the Siegel gauge will be proportional to $b_0^+ b_0^- \delta(L_0^+)\delta_{L_0^-,0}$. It is easy to see that in the $L_0^\pm = 0$ subspace, $b_0^+ b_0^-$ may be decomposed as

$$b_0^+ b_0^- = |\phi_r\rangle\langle\phi_r^c|Q_B + Q_B|\phi_r\rangle\langle\phi_r^c| + |\chi_a\rangle\langle\chi_a^c|\,. \tag{5.4}$$

Now consider the diagram of figure 1 but interpret this as a string theory diagram with all string states propagating in the internal lines. A cut passing through both internal propagators will insert a factor of (5.4) for each propagator. Let us denote the first one by (5.4) and the second one by

$$|\phi_s\rangle\langle\phi_s^c|Q_B + Q_B|\phi_s\rangle\langle\phi_s^c| + |\chi_b\rangle\langle\chi_b^c|\,. \tag{5.5}$$

We shall assume that the ket is inserted on the vertex to the left and the bra is inserted on the vertex to the right. Now since the external state in each vertex is physical, and since the three point function on the sphere of two physical states and one pure gauge state vanishes, it is easy to see that many of the contributions vanish. For example, for the combination

$$Q_B|\phi_s\rangle\langle\phi_s^c| \otimes |\chi_a\rangle\langle\chi_a^c| \tag{5.6}$$

the left vertex will represent the three point function on the sphere of $Q_B\phi_s$, χ_a and the external state. Since χ_a and the external state are BRST invariant, this amplitude vanishes by standard argument involving deformation of the BRST contour. Only the following combination survives from the tensor product of (5.4) and (5.5) inserted at the vertices:

$$|\chi_a\rangle\langle\chi_a^c| \otimes |\chi_b\rangle\langle\chi_b^c| + |\phi_r\rangle\langle\phi_r^c|Q_B \otimes Q_B|\phi_s\rangle\langle\phi_s^c| + Q_B|\phi_r\rangle\langle\phi_r^c| \otimes |\phi_s\rangle\langle\phi_s^c|Q_B\,. \tag{5.7}$$

Of these the first term gives the desired contribution — the sum over physical states. Therefore we need to show that the contribution from the other two terms cancel. Consider the second term. For this the left vertex has the insertion of $|\phi_r\rangle$, $Q_B|\phi_s\rangle$ and the BRST invariant external state. We can now use the usual argument involving deformation of the BRST contour to put Q_B on the $|\phi_r\rangle$ at the cost of getting an extra minus sign and whatever other sign we get for passing Q_B through the grassmann odd operators. Similarly for the last term in (5.7), the right vertex has the insertion of $\langle\phi_r^c|$, $\langle\phi_s^c|Q_B$ and the external state, and we move Q_B from ϕ_s^c to ϕ_r^c. This brings (5.7) to

$$|\chi_a\rangle\langle\chi_a^c| \otimes |\chi_b\rangle\langle\chi_b^c| - Q_B|\phi_r\rangle\langle\phi_r^c|Q_B \otimes |\phi_s\rangle\langle\phi_s^c| + Q_B|\phi_r\rangle\langle\phi_r^c|Q_B \otimes |\phi_s\rangle\langle\phi_s^c|\,. \tag{5.8}$$

The minus sign in the second term is due to the reversal of the orientation of the BRST contour. No further minus signs appear since here Q_B has to pass through $|\phi_r\rangle\langle\phi_r^c|Q_B$ which is grassmann even due to (5.3). On the other hand in going from the last term in (5.7) to the the last term in (5.8), Q_B has to pass through the grassmann odd combination $|\phi_s\rangle\langle\phi_s^c|$ that gives an extra minus sign and cancels the minus sign coming from the reversal of orientation of the BRST contour. We now see that the last two terms in (5.8) cancel, leaving behind the contribution from only the physical intermediate states in the Cutkosky rules. This proves unitarity of the one loop two point function.

Note that the cancelation described above involves loops carrying states of different ghost numbers — the ghost numbers of the states $Q_B\phi_s$ and ϕ_s in (5.7) differ by 1. This is a generalization of the results in ordinary gauge theories where the proof of unitarity in the Feynman gauge involves cancelation between unphysical states in the matter sector and the ghost states propagating in the loop.

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JHEP11(2016)050

A Equivalence to the $i\epsilon$ prescription

In section 2.1 we described a specific choice of contour that can be used to evaluate (2.1). An alternative prescription, known as the $i\epsilon$ prescription, is to take the expression (2.11) and define the integration over t_1, t_2 by taking the upper limits of integration to be $t_0+i\infty$ instead of ∞ where t_0 is some fixed positive number [7, 8]. The question that we would like to address in this appendix is: are these two prescriptions equivalent?

As pointed out in section 2.2, the failure of the Schwinger parameter representation is in the use of (2.8), i.e. it is not possible to choose the k^0 contour shown in figure 2 such that $k^2+m_1^2$ and $(p-k)^2+m_2^2$ always have positive real parts so that the integrals (2.8) converge. With the new prescription of turning the contours of t_1 and t_2 (equivalently of s_1 and s_2 in (2.8)) towards $t_0+i\infty$, the relevant question becomes: is it possible to deform the k^0 integration contours in figure 2 to a form such that $k^2+m_1^2$ and $(p-k)^2+m_2^2$ always have negative imaginary parts? If this is the case then the integrals in (2.8) — with the new upper limits $t_0+i\infty$ — converge and the use of these equations will be justified.

Now since the imaginary parts of $k^2+m_1^2$ and $(p-k)^2+m_2^2$ come respectively from the $-(k^0)^2$ and the $-(p^0-k^0)^2$ terms, in order to satisfy the requirement described above we need k^0 and p^0-k^0 to lie either in the first quadrant or in the third quadrant. At the same time we must ensure that the poles Q_1 and Q_3 lie to the right of the contour and the poles Q_2 and Q_4 lie to the left of the contour as in figure 2. It is easy to see that the contour shown in figure 3 satisfies these requirements. In drawing this we have used that we need to take the limit of p^0 approaching the real axis from the first quadrant, and have consequently taken p^0 to have a small positive imaginary part.

Note that unlike the contours shown in figure 2, the contour shown in figure 3 does not approach $\pm i\infty$ at the two ends. Instead it approaches $\pm i\infty$ plus finite real parts. It is easy to see however that the integrand in (2.1) decays exponentially as $k^0\to A\pm i\infty$ for any finite real A and hence the contour shown in figure 3 can be deformed to the ones in figure 2 without changing the value of the integral (2.1).

This shows that at least for the one loop two point function the prescription of [7, 8] agrees with the prescription of [23] that we have used in section 2.1. Whether the two prescriptions agree for general amplitudes is not known to us at present.

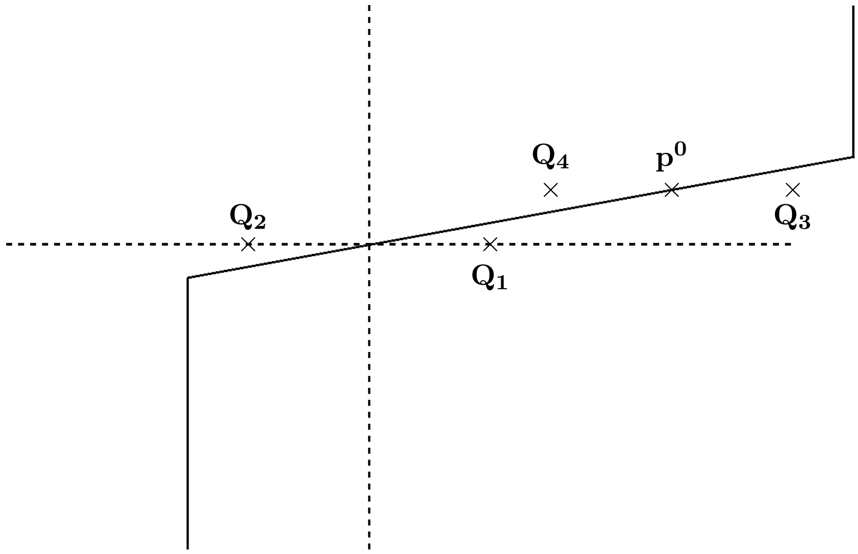

Figure 3. Choice of integration contour in the complex k^0 plane that can be used to prove equivalence between our prescription and the $i\epsilon$ prescription.

B Finiteness of J_1

In this appendix we shall show that J_1 defined in (3.12) receives a finite contribution from the $z_2 \geq \Lambda$, $w_2 \equiv \tau_2 - z_2 \geq \Lambda$ region of integration. For this let us introduce variables

$$u = e^{2\pi i z}, \quad v = e^{2\pi i w} = e^{2\pi i(\tau - z)} . \tag{B.1}$$

In that case F given in (3.7) has the form

$$F(z, \bar{z}, \tau, \bar{\tau}) = \exp[4\pi w_2 z_2/(z_2 + w_2)]\, \tau_2^{-5}\, G(u, v)\,, \tag{B.2}$$

where

$$\begin{aligned} G(u, v) \equiv & \left\{\sum_\nu \overline{\vartheta_\nu(0)^{16}}\right\} (\overline{\eta(\tau)})^{-18} (\eta(\tau))^{-6} (\overline{\vartheta_1'(0)})^{-4} e^{2\pi i(z-\bar{z})} \left(\vartheta_1(z)\overline{\vartheta_1(z)}\right)^2 \\ & \times \left[\left(\frac{\overline{\vartheta_1'(z)}}{\overline{\vartheta_1(z)}}\right)^2 - \frac{\overline{\vartheta_1''(z)}}{\overline{\vartheta_1(z)}} - \frac{\pi}{\tau_2}\right]^2 \end{aligned} \tag{B.3}$$

can be organized as

$$G(u, v) = h(u, v) \sum_{i=0}^{2} \tau_2^{-i} \left[a^{(i)} \bar{u}^{-1} \bar{v}^{-1} + \bar{u}^{-1} f^{(i)}(\bar{v}) + \bar{v}^{-1} f^{(i)}(\bar{u}) + g^{(i)}(\bar{u}, \bar{v})\right] . \tag{B.4}$$

Here τ_2 has to be interpreted as $z_2 + w_2$, $a^{(i)}$'s are constants, and h, $f^{(i)}$ and $g^{(i)}$'s are holomorphic functions of their arguments in the domain $|u| < 1$, $|v| < 1$. The form of these functions can be easily read out from (B.3) and known expansions of the theta and eta functions, e.g. we have

$$
\begin{aligned}
h(u,v) &= e^{2\pi i z}\vartheta_1(z)^2/\eta(\tau)^6\,,\\
a^{(0)} &= 0, \qquad\qquad a^{(1)} = 0, \qquad a^{(2)} = -2^{-3}\pi^{-2}\,,\\
f^{(0)}(\bar u) &= -2\,\bar u\,(1-\bar u)^{-2}, \quad f^{(1)}(\bar u) = -\pi^{-1}, \quad f^{(2)}(\bar u) = 2^{-3}\pi^{-2}\,(2-\bar u)\,,\\
g^{(0)}(\bar u,\bar v) &= \left\{\sum_\nu \overline{\vartheta_\nu(0)^{16}}\right\}(\overline{\eta(\tau)})^{-18}(\overline{\vartheta_1'(0)})^{-4}e^{-2\pi i\bar z}\left(\overline{\vartheta_1(z)}\right)^2\left[\left(\frac{\overline{\vartheta_1'(z)}}{\overline{\vartheta_1(z)}}\right)^2-\frac{\overline{\vartheta_1''(z)}}{\overline{\vartheta_1(z)}}\right]^2\\
&\quad -\bar v^{-1}f^{(0)}(\bar u)-\bar u^{-1}f^{(0)}(\bar v)\,,\\
g^{(1)}(\bar u,\bar v) &= -2\pi\left\{\sum_\nu \overline{\vartheta_\nu(0)^{16}}\right\}(\overline{\eta(\tau)})^{-18}(\overline{\vartheta_1'(0)})^{-4}e^{-2\pi i\bar z}\left(\overline{\vartheta_1(z)}\right)^2\left[\left(\frac{\overline{\vartheta_1'(z)}}{\overline{\vartheta_1(z)}}\right)-\frac{\overline{\vartheta_1''(z)}}{\overline{\vartheta_1(z)}}\right]\\
&\quad -\bar v^{-1}f^{(1)}(\bar u)-\bar u^{-1}f^{(1)}(\bar v)\,,\\
g^{(2)}(\bar u,\bar v) &= \pi^2\left\{\sum_\nu \overline{\vartheta_\nu(0)^{16}}\right\}(\overline{\eta(\tau)})^{-18}(\overline{\vartheta_1'(0)})^{-4}e^{-2\pi i\bar z}\left(\overline{\vartheta_1(z)}\right)^2\\
&\quad -\bar u^{-1}\bar v^{-1}a^{(2)}-\bar v^{-1}f^{(2)}(\bar u)-\bar u^{-1}f^{(2)}(\bar v)\,.
\end{aligned}
\tag{B.5}
$$

Being holomorphic inside the disks $|u| < 1$, $|v| < 1$, these functions have Taylor series expansions of the form

$$
f^{(i)}(\bar u) = \sum_{m=0}^\infty f_m^{(i)}\bar u^m, \quad g^{(i)}(\bar u,\bar v) = \sum_{m,n=0}^\infty g_{m,n}^{(i)}\bar u^m\bar v^n, \quad h(u,v) = \sum_{m,n=0}^\infty h_{m,n}u^m v^n. \tag{B.6}
$$

Now after integration over z_1 and w_1, only those terms in the expression for F which carry equal powers of u and $\bar u$, and also equal powers of v and $\bar v$ will survive, This gives, using (B.2), (B.4) and (B.6):

$$
\int dz_1\int dw_1 F = \sum_{m,n\ge 0}\exp[4\pi w_2 z_2/(z_2+w_2)-4\pi m z_2-4\pi n w_2]\tau_2^{-5}\sum_{i=0}^2 A_{m,n}^{(i)}\tau_2^{-i}, \tag{B.7}
$$

where

$$
A_{m,n}^{(i)} = h_{m,n}g_{m,n}^{(i)}\,. \tag{B.8}
$$

The $(m,n) = (0,0)$ term in (B.7) is subtracted away from F in (3.12). It can be easily seen that the term in the argument of the exponential in (B.7) is always negative or 0 for $(m,n)\neq(0,0)$ and hence for each $(m,n)\neq(0,0)$ the integral converges due to the τ_2^{-5} factor. It will however be instructive to investigate the individual terms in some more detail. The $m=1$, $n=0$ term has an exponential factor $\exp[-4\pi z_2^2/(z_2+w_2)]$. If we first carry out the z_2 integral at fixed w_2, the leading contribution for large w_2 comes from the $z_2\sim w_2^{1/2}$ region, and, after carrying out the integration over z_2 the integrand falls off as $w_2^{-9/2}$ for large w_2. The subsequent integral over w_2 gives a finite result.[11] A similar

[11] If we had considered compactified string theory with D non-compact space-time dimensions then the τ_2^{-5} factor will be replaced by $\tau_2^{-D/2}$ and after integration over z_2 is performed, the integrand will fall off as $w_2^{-(D-1)/2}$. This integral diverges for $D\le 3$. This is related to an infrared divergence of the diagram of figure 1 for $m_1=0$ and $m_2=M$ in $D\le 3$.

contribution will arise from the $m = 0$, $n = 1$ term in (B.7). For terms with $m \geq 2$, $n = 0$ the z_2 integral yields a result of order w_2^{-5} for large w_2, and the result is finite after integration over w_2. Similar remark holds for the $m = 0$, $n \geq 2$ term. Finally for terms with $m, n \geq 1$, the integrand falls off exponentially for large z_2 and/or w_2 and the integral receives a finite contribution.

This shows that each term in the sum in (B.7) gives finite result after integration over z_2 and w_2, but one could still wonder if the sum over m, n could lead to divergence. For exploring this possibility we need to know the growth rate of $A^{(i)}_{m,n}$ for large m and/or n. For this recall that since h, $f^{(i)}$ and $g^{(i)}$ are holomorphic function of their arguments for $|u| < 1$, $|v| < 1$, the Taylor series expansions (B.6) should converge in this domain. This means that for any positive constant Λ_0, we can find another positive constant K such that

$$|f^{(i)}_m| < Ke^{2\pi\Lambda_0 m}, \quad |g^{(i)}_{m,n}| < Ke^{2\pi\Lambda_0(m+n)}, \quad |h_{m,n}| < Ke^{2\pi\Lambda_0(m+n)}\,. \tag{B.9}$$

(B.8) now gives

$$|A^{(i)}_{m,n}| < K^2\, e^{4\pi\Lambda_0(m+n)}\,. \tag{B.10}$$

We shall take $\Lambda_0 < \Lambda$. Using this we can put the following upper bound to the integral of the series expansion (B.7) without the $m = n = 0$ term:

$$\begin{aligned}
&\int_\Lambda^\infty dz_2 \int_\Lambda^\infty dw_2 \sum_{\substack{m,n\geq 0\\(m,n)\neq(0,0)}} \exp[4\pi w_2 z_2/(z_2+w_2) - 4\pi m z_2 - 4\pi n w_2] \sum_{i=0}^{2} A^{(i)}_{m,n}\tau_2^{-5-i}\\
&\leq \Delta_0 \sum_{\substack{m,n\geq 0\\(m,n)\neq(0,0)}} \int_\Lambda^\infty dz_2 \int_\Lambda^\infty dw_2\, \exp[4\pi w_2 z_2/(z_2+w_2) - 4\pi m z_2 - 4\pi n w_2]\, e^{4\pi\Lambda_0(m+n)}\tau_2^{-5}\,,\\
&\Delta_0 \equiv K^2\left(1 + \frac{1}{2}\Lambda^{-1} + \frac{1}{4}\Lambda^{-2}\right).
\end{aligned} \tag{B.11}$$

We now consider the following terms separately, leaving aside the $(m, n) = (0, 1)$ and $(1, 0)$ terms since their contribution has been analyzed separately anyway and found to be finite.

1. First consider the sum of all the terms with $m \geq 2$, $n = 0$. In this case the sum is bounded by

$$\begin{aligned}
&\Delta_0 \sum_{m=2}^\infty \int_\Lambda^\infty dz_2 \int_\Lambda^\infty dw_2\, \exp[4\pi w_2 z_2/(z_2+w_2) - 4\pi m z_2]\, e^{4\pi\Lambda_0 m}\,\tau_2^{-5}\\
&\leq \Delta_0 \sum_{m=2}^\infty \int_\Lambda^\infty dz_2 \int_\Lambda^\infty dw_2\, \exp[-4\pi(m-1)z_2]\, e^{4\pi\Lambda_0 m} w_2^{-5}\\
&= \frac{1}{16\pi}\,\Delta_0\, e^{4\pi\Lambda_0}\,\Lambda^{-4} \sum_{m=2}^\infty \frac{1}{m-1} e^{-4\pi(m-1)(\Lambda-\Lambda_0)}\,.
\end{aligned} \tag{B.12}$$

   Since $\Lambda > \Lambda_0$, this is a convergent sum. This shows that in the expression for J_1, the sum over m for $n = 0$ is convergent.

2. Sum over all terms with $m = 0$, $n \geq 2$ can be dealt with similarly.

3. Finally the sum over all terms in (B.11) with $m \geq 1$, $n \geq 1$, can be bounded as

$$\begin{aligned}
&\Delta_0 \sum_{m,n\geq 1} \int_\Lambda^\infty dz_2 \int_\Lambda^\infty dw_2 \, \exp[4\pi w_2 z_2/(z_2+w_2) - 4\pi m z_2 - 4\pi n w_2] \, e^{4\pi\Lambda_0(m+n)} \tau_2^{-5} \\
&\leq \Delta_0 \sum_{m,n\geq 1} \int_\Lambda^\infty dz_2 \int_\Lambda^\infty dw_2 \, \exp[-4\pi(m-1)z_2 - 4\pi n w_2] \, e^{4\pi\Lambda_0(m+n)} z_2^{-5} \\
&\leq \frac{1}{4\pi} \Delta_0 \int_\Lambda^\infty dz_2 \left[z_2^{-5} e^{4\pi\Lambda_0} + \sum_{m\geq 2} e^{-4\pi(m-1)z_2} e^{4\pi\Lambda_0 m} \Lambda^{-5} \right] \sum_{n=1}^\infty \frac{1}{n} e^{-4\pi\Lambda n} e^{4\pi\Lambda_0 n} \\
&= \frac{1}{4\pi} \Delta_0 e^{4\pi\Lambda_0} \left[\frac{1}{4\Lambda^4} + \frac{1}{4\pi\Lambda^5} \sum_{m\geq 2} \frac{1}{m-1} e^{-4\pi(\Lambda-\Lambda_0)(m-1)} \right] \sum_{n=1}^\infty \frac{1}{n} e^{-4\pi(\Lambda-\Lambda_0)n}. \qquad \text{(B.13)}
\end{aligned}$$

Since $\Lambda > \Lambda_0$, both sums on the right hand side of (B.13) are convergent. This shows that the sum on the left hand side of (B.13) is also convergent.

Combining all the results we conclude that the sum on the right hand side of (B.11) converges and hence the sum on the left hand side of (B.11) also converges. This in turn shows that the contribution to J_1 has no divergence from the sum over infinite set of terms.

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JHEP11(2016)050