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Bi-local holography in the SYK model: perturbations

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Abstract: We continue the study of the Sachdev-Ye-Kitaev model in the Large N limit. Following our formulation in terms of bi-local collective fields with dynamical reparametrization symmetry, we perform perturbative calculations around the conformal IR point. These are based on an ε expansion which allows for analytical evaluation of correlators and finite temperature quantities.

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Contents

1 Introduction

In this paper we continue the development of the Large N formulation of the Sachdev-Ye-Kitaev (SYK) model begun in our earlier work. The SYK model [5–8] and the earlier Sachdev-Ye (SY) model [1–4] represent valuable laboratories for understanding of holography and quantum features of black holes. They represent fermionic systems with quenched disorder with nontrivial properties [9–12] and gravity duals. In addition to models based on random matrices, they represent some of the simplest models of holography (see also [13]).

The framework for accessing the IR critical point and the corresponding AdS_2 dual can be provided by the large N expansion at strong coupling. In this limit, Kitaev [6] has demonstrated the chaotic behavior of the system in terms of the Lyapunov exponent and has exhibited elements of the dual black hole.

Recently, in-depth studies [14–16] have given large N correlations and spectrum of two-particle states of the model. In these (and earlier works [6, 7]), a notable feature is the emergence of reparametrization symmetry showing characteristic features of the dual AdS Gravity.

The present work continues the development of systematic Large N representation of the model given in [15] (which we will refer to as I), through a nonlinear bi-local collective field theory. This representation systematically incorporates arbitrary n-point bi-local correlators through a set of $1/N$ vertices and propagator(s) and as such gives the bridge to a dual description. It naturally provides a holographic interpretation along the lines proposed more generally in [22, 23], where the relative coordinate is seen to represents the radial AdS_2 coordinate z. The Large N SYK model represents a highly nontrivial nonlinear system. At the IR critical point (which is analytically accessible) there appears a zero mode problem which at the outset prevents a perturbative expansion. In (I), this is treated through introduction of collective 'time' coordinate as a dynamical variable as in quantization of extended systems [25]. Its Faddeev-Popov quantization was seen to systematically project out the zero modes, providing for a well defined propagator and expansion around the IR point. What one has is a fully nonlinear interacting system of bi-local matter with a discrete gravitational degree of freedom governed by a Schwarzian action. In [16] the zero modes were enhanced away from the IR defining a near critical theory, and correspondence. We will be able to demonstrate that the nonlinear treatment that we employ leads to very same effects ('big' contributions) at the linearized quadratic level, it is expected hoverer to be exact at all orders.

In the present work, we present perturbative calculations (around the IR point) using this collective formulation. A particular scheme that we employ for perturbative calculations is an ε expansion, where ε represents a deviation from the exactly solvable case. Using this scheme we are able to perform analytic calculations in powers of $1/J$ (where J represents the strong coupling constant). These calculations are compared with and are seen to be in agreement with numerical evaluations of [16]. The content of this paper is as follows: in the rest of section 1, we give a short summary of our formulation with the treatment of symmetry modes. In section 2, we perform a perturbative evaluation of the Large N classical background, to all orders in the inverse of the strong coupling defining the IR. In section 3, we discuss the two-point function in the leading and sub-leading order. In section 4, we deal with the finite temperature case and give the free energy to several orders. Comments are given in section 5.

1.1 The method

In this subsection, we will give a brief review of our formalism [15]. The Sachdev-Ye-Kitaev model [6] is a quantum mechanical many body system with all-to-all interactions on fermionic N sites ($N \gg 1$), represented by the Hamiltonian

$$H = \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \chi_i\, \chi_j\, \chi_k\, \chi_l\,, \tag{1.1}$$

where χ_i are Majorana fermions, which satisfy $\{\chi_i, \chi_j\} = \delta_{ij}$. The coupling constant J_{ijkl} are random with a Gaussian distribution. The original model is given by this four-point interaction; however, with a simple generalization to analogous q-point interacting model [6, 16]. In this paper, we follow the more general q model, unless otherwise specified. Nevertheless, our main interest represents the original $q = 4$ model. After the disorder averaging for the random coupling J_{ijkl}, there is only one effective coupling J and the effective action is written as

$$S_q = -\frac{1}{2} \int dt \sum_{i=1}^{N} \sum_{a=1}^{n} \chi_i^a \partial_t \chi_i^a - \frac{J^2}{2qN^{q-1}} \int dt_1 dt_2 \sum_{a,b=1}^{n} \left(\sum_{i=1}^{N} \chi_i^a(t_1) \chi_i^b(t_2) \right)^q, \tag{1.2}$$

where a, b are the replica indexes. Throughout this paper, we only use Euclidean time. We do not expect a spin glass state in this model [7] and we can restrict to replica diagonal subspace [15]. Therefore, introducing a (replica diagonal) bi-local collective field:

$$\Psi(t_1, t_2) \equiv \frac{1}{N} \sum_{i=1}^{N} \chi_i(t_1) \chi_i(t_2)\,, \tag{1.3}$$

the model is described by a path-integral

$$Z = \int \prod_{t_1, t_2} \mathcal{D}\Psi(t_1, t_2)\; \mu(\Psi)\, e^{-S_{\rm col}[\Psi]}\,, \tag{1.4}$$

with an appropriate measure μ and the collective action:

$$S_{\rm col}[\Psi] = \frac{N}{2} \int dt \left[\partial_t \Psi(t, t') \right]_{t'=t} + \frac{N}{2} \int dt \, \log \Psi(t, t) - \frac{J^2 N}{2q} \int dt_1 dt_2\, \Psi^q(t_1, t_2)\,. \tag{1.5}$$

This action being of order N gives a systematic $G = 1/N$ expansion, while the measure μ found as in [24] begins to contribute at one loop level (in $1/N$). Here the first linear term represents a conformal breaking term, while the other terms respect conformal invariance.[1] In the strong coupling limit $|t|J \gg 1$, the collective action is reduces to the critical action

$$S_{\rm c}[\Psi] = \frac{N}{2} \int dt \, \log \Psi(t, t) - \frac{J^2 N}{2q} \int dt_1 dt_2\, \Psi^q(t_1, t_2)\,, \tag{1.6}$$

which exhibits the emergent conformal reparametrization symmetry $t \to f(t)$ with

$$\Psi(t_1, t_2) \to \Psi_f(t_1, t_2) = \Big| f'(t_1) f'(t_2) \Big|^{\frac{1}{q}}\, \Psi(f(t_1), f(t_2))\,. \tag{1.7}$$

[1]Such linear breaking term was seen previously in [27].

The critical solution is given by

$$\Psi_{0,f}(t_1,t_2) \,=\, b\left(\frac{\sqrt{|f'(t_1)f'(t_2)|}}{|f(t_1)-f(t_2)|}\right)^{\frac{2}{q}}\,, \tag{1.8}$$

where b is a time-independent constant. This symmetry is responsible for the appearance of zero modes in the strict IR critical theory. This problem was addressed in [15] with analog of the quantization of extended systems with symmetry modes [25]. The above symmetry mode representing time reparametrization can be elevated to a dynamical variable introduced according to [26] through the Faddeev-Popov method which we summarize as follows: we insert into the partition function (1.4), the functional identity:

$$\int \prod_t \mathcal{D}f(t)\,\prod_t \delta\left(\int u\cdot\Psi_f\right)\left|\frac{\delta\left(\int u\cdot\Psi_f\right)}{\delta f}\right| \,=\, 1\,, \tag{1.9}$$

so that after an inverse change of the integration variable, it results in a combined representation

$$Z \,=\, \int \prod_t \mathcal{D}f(t)\prod_{t_1,t_2}\mathcal{D}\Psi(t_1,t_2)\;\mu(f,\Psi)\,\delta\left(\int u\cdot\Psi_f\right)\,e^{-S_{\rm col}[\Psi,f]}\,, \tag{1.10}$$

with an appropriate Jacobian. After separating the critical classical solution Ψ_0 from the bi-local field: $\Psi=\Psi_0+\overline{\Psi}$, the total action is now given by

$$S_{\rm col}[\Psi,f] \,=\, S[f] \,+\, \frac{N}{2}\int dt\Big[\partial_t\overline{\Psi}_f(t,t')\Big]_{t'=t} \,+\, S_{\rm c}[\Psi]\,, \tag{1.11}$$

where the action of the time collective coordinate is

$$S[f] \,=\, \frac{N}{2}\int dt_1\,\Big[\partial_1\Psi_{0,f}(t_1,t_2)\Big]_{t_2=t_1}\,. \tag{1.12}$$

We have given the explicit evaluation of the nonlinear action $S[f]$ for the case of $q=2$ in [15]. This evaluation is based on expanding the critical solution in the $t_1\to t_2$ limit and taking the derivative; the result producing the form of a Schwarzian derivative. For general q, and in particular $q=4$ which is our main interest we employ an ε-expansion with $q=2/(1-\varepsilon)$ in appendix A. So in general the action $S[f]$ always comes in the form of the Schwarzian derivative but with a constant overall coefficient, which is in ε-expansion found to be $\alpha=1-\varepsilon^2=4(q-1)/q^2$:

$$S[f] \,=\, -\,\frac{N\alpha}{24\pi J}\int dt\,\left[\frac{f'''(t)}{f'(t)} \,-\, \frac{3}{2}\left(\frac{f''(t)}{f'(t)}\right)^2\right]. \tag{1.13}$$

At the linearized level this action was deduced as the action of enhanced zero mode in [16] through numerical evaluation. Our value of α for $q=4$ is $\alpha=3/4$, which agrees very well with a numerical result found there, the corresponding numerical value being $\alpha\approx 0.756$. Summarizing the $f(t)$ is now introduced as a dynamical degree of freedom, with the combined action of eq. (1.11) showing interaction with the bi-local field and possessing

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a reparametrization symmetry which is now present at but also away from the IR point. The delta function condition can be understood as gauge fixing condition projecting out a state associated with wave function $u(t_1,t_2)$. This wave function is arbitrary (representing different gauges), it will be chosen to eliminate the troublesome zero mode of the IR. This formulation then allows systematic perturbative calculations around the IR point.

1.2 Relation to zero mode dynamics

Before we proceed with our perturbative calculations it is worth comparing the above exact treatment of the reparametrization mode (1.13) with a linearized determination of the zero mode dynamics, as considered in [16]. We will be able to see that the latter follows from the former.

Expanding the critical action around the critical saddle-point solution Ψ_0, we have in I generated [15], the quadratic kernel (which defines the propagator) and a sequence of higher vertices. This expansion is schematically written as

$$S_{\rm c}\Big[\Psi_0+\sqrt{2/N}\,\eta\Big] \;=\; N\,S_{\rm c}[\Psi_0]\;+\;\frac{1}{2}\int\eta\cdot\mathcal{K}\cdot\eta\;+\;\frac{1}{\sqrt{N}}\int\mathcal{V}_{(3)}\cdot\eta\eta\eta\;+\;\cdots\,, \tag{1.14}$$

where the kernel is

$$\begin{aligned}\mathcal{K}(t_1,t_2;t_3,t_4) \;&=\; \frac{\delta^2 S_c[\Psi_0]}{\delta\Psi_0(t_1,t_2)\delta\Psi_0(t_3,t_4)}\\ &=\; \Psi_0^{-1}(t_1,t_3)\Psi_0^{-1}(t_2,t_4)\;+\;(q-1)J^2\,\delta(t_{13})\delta(t_{24})\,\Psi_0^{q-2}(t_1,t_2)\,,\end{aligned} \tag{1.15}$$

with $t_{ij}=t_i-t_j$. For other detail of the expansion, please refer to [15]. Then, the bi-local propagator $\mathcal{D}$ is determined as a solution of the following Green's equation:

$$\int dt_3dt_4\,\mathcal{K}(t_1,t_2;t_3,t_4)\,\mathcal{D}(t_3,t_4;t_5,t_6)\;=\;\delta(t_{15})\delta(t_{26})\,. \tag{1.16}$$

In order to inverse the kernel $\mathcal{K}$ in the Green's equation (1.16) and determine the bi-local propagator, let us first consider an eigenvalue problem of the kernel $\mathcal{K}$:

$$\int dt_3dt_4\,\mathcal{K}(t_1,t_2;t_3,t_4)\,u_{n,t}(t_3,t_4)\;=\;k_{n,t}\,u_{n,t}(t_1,t_2)\,, \tag{1.17}$$

where n and t are labels to distinguish the eigenfunctions. The zero mode, whose eigenvalue is $k_0=0$ is given by

$$u_{0,t}(t_1,t_2)\;=\;\frac{\delta\Psi_{0,f}(t_1,t_2)}{\delta f(t)}\bigg|_{f(t)=t}\,. \tag{1.18}$$

Now, we consider the zero mode quantum fluctuation around a shifted classical background

$$\Psi(t_1,t_2)\;=\;\Psi_{\rm cl}(t_1,t_2)\;+\;\int dt'\,\varepsilon(t')\,u_{0,t'}(t_1,t_2)\,, \tag{1.19}$$

with $\Psi_{\rm cl}=\Psi_0+\Psi^{(1)}$ where $\Psi^{(1)}$ is a shift of the classical field from the critical point. Then, the quadratic action of ε in the first order of the shift is given by expanding $S_{\rm c}[\Psi_{\rm cl}+\varepsilon\cdot u_0]$. This quadratic action can be written in terms of the shift of the kernel $\delta\mathcal{K}$ as

$$S_2[\varepsilon]\;=\;-\frac{N}{4}\int dtdt'\,\varepsilon(t)\,\varepsilon(t')\int dt_1dt_2dt_3dt_4\,u_{0,t}(t_1,t_2)\,\delta\mathcal{K}(t_1,t_2;t_3,t_4)\,u_{0,t'}(t_3,t_4)\,, \tag{1.20}$$

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where

$$\delta\mathcal{K}(t_1,t_2;t_3,t_4) \;=\; \int dt_5 dt_6\, \frac{\delta^3 S_c[\Psi_0]}{\delta\Psi_0(t_1,t_2)\delta\Psi_0(t_3,t_4)\delta\Psi_0(t_5,t_6)}\,\Psi_{(1)}(t_5,t_6)\,. \tag{1.21}$$

Let us formally denote the t_1 - t_4 integrals in eq. (1.20) by

$$\delta k_t\,\delta(t-t') \;=\; \int dt_1 dt_2 dt_3 dt_4\, u_{0,t}(t_1,t_2)\,\delta\mathcal{K}(t_1,t_2;t_3,t_4)\,u_{0,t'}(t_3,t_4)\,, \tag{1.22}$$

because this is related to the eigenvalue shift due to $\delta\mathcal{K}$ up to normalization. Then, we can write the quadratic action (1.20) as

$$S_2[\varepsilon] \;=\; -\frac{N}{4}\int dt\,\delta k_t\,\varepsilon^2(t)\,. \tag{1.23}$$

We now give a formal proof that the quadratic action (1.23) is equivalent to the quadratic action of eq. (1.13). To show this, we need the following identity:

$$\begin{aligned}\int dt_1 dt_2 dt_3 dt_4\; u_{0,t}(t_1,t_2)\,&\frac{\delta^3 S_c[\Psi_0]}{\delta\Psi_0(t_1,t_2)\delta\Psi_0(t_3,t_4)\delta\Psi_0(t_5,t_6)}\,u_{0,t'}(t_3,t_4)\\ &= -\int dt_3 dt_4\;\mathcal{K}(t_3,t_4;t_5,t_6)\,\frac{\delta^2\Psi_{0,f}(t_3,t_4)}{\delta f(t)\delta f(t')}\bigg|_{f(t)=t}\,.\end{aligned} \tag{1.24}$$

This identity can be derived as follows. In the zero mode equation $\int\mathcal{K}\cdot u_0 = 0$, rewriting the kernel as derivatives of S_c as in the first line of eq. (1.15), and taking a derivative of this equation respect to $f(t')$, one finds

$$\begin{aligned}0 = \int dt_1 dt_2 dt_3 dt_4\, &\frac{\delta\Psi_{0,f}(t_1,t_2)}{\delta f(t)}\bigg|_{f(t)=t}\cdot\frac{\delta^3 S_c[\Psi_{0,f}]}{\delta\Psi_{0,f}(t_1,t_2)\delta\Psi_{0,f}(t_3,t_4)\delta\Psi_{0,f}(t_5,t_6)}\cdot\frac{\delta\Psi_{0,f}(t_3,t_4)}{\delta f(t')}\bigg|_{f(t')=t'}\\ &+\int dt_3 dt_4\,\frac{\delta^2 S_c[\Psi_{0,f}]}{\delta\Psi_{0,f}(t_3,t_4)\delta\Psi_{0,f}(t_5,t_6)}\cdot\frac{\delta^2\Psi_{0,f}(t_3,t_4)}{\delta f(t)\delta f(t')}\bigg|_{f(t)=t}\,,\end{aligned} \tag{1.25}$$

where we used the zero mode expression (1.18). Since S_c is invariant under the reparametrization, we can change the argument of S_c from $\Psi_{0,f}$ to Ψ_0. Then, we get the identity (1.24). Now for the zero mode eigenvalue shift (1.22), rewriting the kernel shift $\delta\mathcal{K}$ as in eq. (1.21) and using the above identity, one can show that δk is given as

$$\delta k_t\,\delta(t-t') \;=\; -\int dt_3 dt_4 dt_5 dt_6\,\mathcal{K}(t_3,t_4;t_5,t_6)\,\Psi^{(1)}(t_1,t_2)\,\frac{\delta^2\Psi_{0,f}(t_3,t_4)}{\delta f(t)\delta f(t')}\bigg|_{f(t)=t}\,. \tag{1.26}$$

Next we use the equation of motion of $\Psi^{(1)}$:

$$\int dt_3 dt_4\,\mathcal{K}(t_1,t_2;t_3,t_4)\Psi^{(1)}(t_3,t_4) \;=\; \partial_1\delta(t_{12})\,. \tag{1.27}$$

Then, one finds

$$\delta k_t \;=\; \int dt_1\,\partial_1\left[\frac{\delta^2\Psi_{0,f}(t_1,t_2)}{\delta f^2(t)}\bigg|_{f(t)=t}\right]_{t_2=t_1}\,, \tag{1.28}$$

and from eq. (1.23)

$$S_2[\varepsilon] \;=\; -\frac{N}{4}\int dt\int dt_1\,\partial_1\left[\frac{\delta^2\Psi_{0,f}(t_1,t_2)}{\delta f^2(t)}\bigg|_{f(t)=t}\right]_{t_2=t_1}\varepsilon^2(t)\,. \tag{1.29}$$

This agrees with the quadratic action of eq. (1.12).

2 Shift of the classical solution

In large N limit, the exact classical solution $\Psi_{\rm cl}$ is given by the solution of the saddle-point equation of the collective action (1.5). This classical solution corresponds to the one-point function:

$$\big\langle \Psi(t_1,t_2) \big\rangle \;=\; \Psi_{\rm cl}(t_1,t_2)\,. \tag{2.1}$$

At the strict strong coupling limit, the classical solution is given by the critical solution Ψ_0, which is a solution of the saddle-point equation of the critical action (1.6). Now we would like to consider a first order shift $\Psi_{(1)}$ of the classical solution from the critical solution induced by the kinetic term. Substituting $\Psi_{\rm cl} = \Psi_0 + \Psi_{(1)}$ into the collective action $S_{\rm col}$ (1.5) and expanding it up to the first order of the shift, one finds

$$\int dt_3 dt_4\, \mathcal{K}(t_1,t_2;t_3,t_4)\Psi_{(1)}(t_3,t_4) \;=\; \partial_1\delta(t_{12})\,, \tag{2.2}$$

where the kernel is given in eq. (1.15). This is the equation which determines $\Psi_{(1)}$ with the delta function source.

In the following, we will consider even integer q and perform this perturbative evaluation for the corrections of the classical field.

2.1 Inhomogeneous solution $\Psi_{(1)}$

In this subsection, we will determine $\Psi_{(1)}$ from the eq. (2.2). For explicit evaluations, it is actually useful to separate the J dependence from the bi-local field by

$$\Psi_{\rm cl}(t_1,t_2) \;=\; J^{-\frac{2}{q}}\,\Psi_0(t_1,t_2) \,+\, \cdots\,, \tag{2.3}$$

where we separated J dependence from the critical solution Ψ_0, which now reads

$$\Psi_0(t_1,t_2) \;=\; b\,\frac{{\rm sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}}}\,, \tag{2.4}$$

with

$$b \;=\; -\left[\frac{\tan\left(\frac{\pi}{q}\right)}{2\pi}\left(1-\frac{2}{q}\right)\right]^{\frac{1}{q}}\,. \tag{2.5}$$

By excluding J dependence in this way, the kernel (1.15) does not have the explicit J^2 factor in the second term, and we will refer this new kernel as $\mathcal{K}$ in the rest of the paper. Since

$$\Psi_0^{-1}(t_1,t_2) \;=\; -\,b^{q-1}\,\frac{{\rm sgn}(t_{12})}{|t_{12}|^{2-\frac{2}{q}}}\,, \tag{2.6}$$

we know the kernel has dimension $\mathcal{K}\sim|t|^{-4+4/q}$. Therefore, from dimension analysis $\Psi_{(1)}$ has to be the form of

$$\Psi_{(1)}(t_1,t_2) \;=\; A\,\frac{{\rm sgn}(t_{12})}{|t_{12}|^{\frac{4}{q}}}\,, \tag{2.7}$$

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where A is a t-independent coefficient. Now we are going to check this ansatz actually solves eq. (2.2) and fix the coefficient A. The integral for the first term of the l.h.s. of eq. (2.2) is given by

$$A b^{2q-2} \int dt_3 dt_4 \, \frac{\operatorname{sgn}(t_{13}) \operatorname{sgn}(t_{24}) \operatorname{sgn}(t_{34})}{|t_{13}|^{2-\frac{2}{q}} \, |t_{24}|^{2-\frac{2}{q}} \, |t_{34}|^{\frac{4}{q}}} \,. \tag{2.8}$$

This type of integral is already evaluated in appendix A of [14]. In general, the result is

$$\begin{aligned}\int dt_3 dt_4 \, \frac{\operatorname{sgn}(t_{13}) \operatorname{sgn}(t_{24}) \operatorname{sgn}(t_{34})}{|t_{13}|^{2\Delta} \, |t_{24}|^{2\Delta} \, |t_{34}|^{2\alpha}} &= -\pi^2 \left[\frac{\sin(2\pi\alpha) \, + \, 2\sin(2\pi(\alpha+\Delta)) \, + \, \sin(2\pi(\alpha+2\Delta))}{\sin(2\pi\alpha)\sin(2\pi\Delta)\sin(2\pi(\alpha+\Delta))\sin(2\pi(\alpha+2\Delta))} \right] \\ &\quad \times \frac{\Big[\sin(2\pi\Delta) + \sin(2\pi(\alpha+\Delta)) \Big] \Gamma(1-2\Delta)}{\Gamma(2\alpha)\Gamma(2\Delta)\Gamma(3-2\alpha-4\Delta)} \, \frac{\operatorname{sgn}(t_{12})}{|t_{12}|^{2\alpha+4\Delta-2}} \,. \end{aligned} \tag{2.9}$$

Our interest is $\Delta = 1 - 1/q$. For this case, the result is inversely proportional to $\Gamma(4/q - 2\alpha - 1)$. If we plug $\alpha = 2/q$ into this equation, we can see that the Gamma function in the denominator gives infinity: $\Gamma(4/q - 2\alpha - 1) = \Gamma(-1) = \infty$, while other part is finite. Therefore, the first term of the l.h.s. of eq. (2.2) vanishes. The second term is trivial to evaluate and the equation is now reduced to

$$(q-1) A b^{q-2} \, \frac{\operatorname{sgn}(t_{12})}{|t_{12}|^2} \; = \; \partial_1 \delta(t_{12}) \,. \tag{2.10}$$

In order to determine the coefficient A, let us use the Fourier transform of the both hand sides. For the l.h.s., we use for example eq. (2.11) of [16]. However, the result is proportional to $\Gamma(0) = \infty$, and the Fourier transform of the right-hand side of eq. (2.10) is just $-i\omega$. Therefore, $A = 0$. Since we saw that from dimensional analysis, only possible solution was $\Psi_{(1)} \sim |t_{12}|^{-4/q}$ form, the conclusion is

$$\Psi_{(1)}(t_1, t_2) \; = \; 0 \,. \tag{2.11}$$

Here we have concluded that $A = 0$ started from the ansatz (2.7). However, for $q = 2$ case we could take another type of ansatz: $\Psi_{(1)}(t_1, t_2) = \widetilde{A} \; \delta'(t_{12})$, where $\widetilde{A}$ is a t-independent coefficient. This ansatz is antisymmetric and has the correct dimension when $q = 2$. Indeed, this is the correct solution for $\Psi_{(1)}$ when $q = 2$, and we will present a detail analysis of this ansatz in appendix B.2. Similarly, one might expect an ansatz $\Psi_{(1)}(t_{12}) \sim \delta(t_{12})$ when $q = 4$, because this has the correct dimension. To make this ansatz antisymmetric, one has to multiply $\operatorname{sng}(t_{12})$. However, now $\Psi_{(1)}(t_{12}) \sim \operatorname{sgn}(t_{12})\delta(t_{12})$, and this function is essentially zero for all value of t_{12}. Hence, we do not have any delta function type of ansatz for $q = 4$.

2.2 Homogeneous solution Ψ_1

In the previous subsection, we concluded that $\Psi_{(1)} = 0$. However, we recall that a general solution of a non-homogeneous differential equation is given by a linear combination of a specific non-homogeneous solution and a general corresponding homogeneous solution. Therefore, a general solution for the non-homogeneous differential equation (2.2) is given by $\Psi^{(1)} = \Psi_{(1)} + \Psi_1$, where $\Psi_{(1)}$ is a specific non-homogeneous solution, which we concluded

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$\Psi_{(1)} = 0$ in the previous subsection. In this subsection, we consider the corresponding homogeneous differential equation:

$$\int dt_3 dt_4\, \mathcal{K}(t_1, t_2; t_3, t_4)\Psi_1(t_3, t_4) \,=\, 0\,, \tag{2.12}$$

to determine Ψ_1. This equation looks like a zero-mode equation. However, we can find another mode which satisfies this equation. Since the r.h.s. of the equation (2.12) is zero, we cannot determine the scaling dimension of Ψ_1 a priori. Nevertheless, the dimension of Ψ_1 should be less than the scaling dimension of Ψ_0. Hence, we use a general ansatz for Ψ_1:

$$\Psi_1(t_1, t_2) \,=\, B_1\, \frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+2s}}\,, \tag{2.13}$$

where B_1 is a t-independent coefficient and $s > 0$. Now we are going to fix s by requiring this ansatz solves the homogenous equation (2.12). The integral of the first term of l.h.s. of eq. (2.12) is evaluated from eq. (2.9) with $\Delta = 1 - 1/q$ and $\alpha = s + 1/q$ as

$$\frac{B_1\, b^{2q-2}\, \pi^2 \cot\left(\frac{\pi}{q}\right) \Gamma\left(\frac{2}{q}-1\right)}{\sin\left(\pi\left(\frac{1}{q}+s\right)\right) \cos\left(\pi\left(s-\frac{1}{q}\right)\right) \Gamma\left(\frac{2}{q}+2s\right) \Gamma\left(2-\frac{2}{q}\right) \Gamma\left(\frac{2}{q}-2s-1\right)}\, \frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{2-\frac{2}{q}+2s}}\,. \tag{2.14}$$

Hence, after a slight manipulation the l.h.s. of eq. (2.12) becomes

$$\begin{aligned}
&\int dt_3 dt_4\, \mathcal{K}(t_1, t_2; t_3, t_4)\Psi_1(t_3, t_4) \\
&= (q-1) B_1 b^{q-2} \left[1 - \frac{\pi\, \Gamma\left(\frac{2}{q}\right)}{q \sin\left(\pi\left(\frac{1}{q}+s\right)\right) \cos\left(\pi\left(s-\frac{1}{q}\right)\right) \Gamma\left(\frac{2}{q}+2s\right) \Gamma\left(3-\frac{2}{q}\right) \Gamma\left(\frac{2}{q}-2s-1\right)}\right] \frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{2-\frac{2}{q}+2s}}\,,
\end{aligned} \tag{2.15}$$

where we used eq. (2.5). Therefore, in order to determine s, we need to solve the equation obtained by setting the inside of the bracket in r.h.s. to zero. A solution of this equation is given by $s = 1/2$. We note that since the r.h.s. of the eq. (2.12) is zero, we cannot determine the coefficient B_1 from the equation. However, we will fix this coefficient B_1 in section 2.5 by relating Ψ_1 to the dynamical collective coordinate action (1.13) found in the previous section as we did in section 1.2. Hence, the expansion of the classical solution is now given by

$$\Psi_{\mathrm{cl}}(t_1, t_2) \,=\, J^{-\frac{2}{q}} \Big[\Psi_0(t_1, t_2) \,+\, J^{-1}\, \Psi_1(t_1, t_2) \,+\, \cdots \Big]\,, \tag{2.16}$$

where

$$\Psi_0(t_1, t_2) \,=\, b\, \frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}}}\,, \qquad \Psi_1(t_1, t_2) \,=\, B_1\, \frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+1}}\,. \tag{2.17}$$

2.3 Evaluation of Ψ_2

Now we would like to go further higher order term in the expansion of the classical solution. This term is given by

$$\Psi_{\mathrm{cl}}(t_1, t_2) \,=\, J^{-\frac{2}{q}} \Big[\Psi_0(t_1, t_2) \,+\, J^{-1}\, \Psi_1(t_1, t_2) \,+\, J^{-2}\, \Psi_2(t_1, t_2) \,+\, \cdots \Big]\,, \tag{2.18}$$

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with

$$\Psi_2(t_1,t_2) \,=\, B_2\, \frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+2}}\,, \tag{2.19}$$

where B_2 is a t-independent coefficient. The dimension of Ψ_2 is already fixed by Ψ_1, so what we need to do is just to fix the coefficient B_2. Substituting the above expansion of the classical field into the critical action $S_{\rm c}$ (1.6) and expanding it, one finds that the equation determining Ψ_2 is given by

$$\begin{aligned} &\int dt_3 dt_4\, \mathcal{K}(t_1,t_2;t_3,t_4)\Psi_2(t_3,t_4) \qquad\qquad (2.20)\\ &\quad = -\,[\Psi_0^{-1}\star\Psi_1\star\Psi_0^{-1}\star\Psi_1\star\Psi_0^{-1}](t_1,t_2) \,-\, \frac{(q-1)(q-2)}{2}\,\Psi_0^{q-3}(t_1,t_2)\,\Psi_1^2(t_1,t_2)\,, \end{aligned}$$

where the star product is defined by $[A\star B](t_1,t_2) \equiv \int dt_3\, A(t_1,t_3)B(t_3,t_2)$. Now, we are going to evaluate each term of this equation. For the first term in the l.h.s. is again given by eq. (2.9) with $\Delta = 1-1/q$ and $\alpha = 1/q+1$ as

$$(\text{l.h.s. 1st}) \,=\, 2\pi\, B_2\, b^{2q-2}\, \frac{q(q-1)(3q-2)}{(q^2-4)\tan(\frac{\pi}{q})}\, \frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{4-\frac{2}{q}}}\,. \tag{2.21}$$

For the first term of the r.h.s. , we need to use eq. (2.9) twice. First for the middle of the term: $\Psi_1\star\Psi_0^{-1}\star\Psi_1$, and then for the result sandwiched by the remaining Ψ_0^{-1}'s. Then, we have

$$(\text{r.h.s. 1st}) \,=\, -\,B_1^2\, b^{3(q-1)}\, \frac{2\pi^2 q^2(q-1)(3q-2)}{(q-2)^2}\, \frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{4-\frac{2}{q}}}\,. \tag{2.22}$$

The second terms in the l.h.s. and r.h.s. are trivially evaluated. Therefore, now one can see that all terms have the same t_{12} dependence. Then, comparing their coefficients, we finally fix B_2 as

$$B_2 \,=\, -\,\frac{B_1^2}{b}\left(\frac{q+2}{8q}\right)\left[(q-2)+(3q-2)\tan^2\left(\frac{\pi}{q}\right)\right]. \tag{2.23}$$

2.4 All order evaluation in $q>2$

In this subsection, we extend our previous perturbative expansion of the classical solution to all order contributions in the $1/J$ expansion. Because of the dimension of Ψ_1 (2.17), the time-dependence is already fixed for all order as in eq. (2.28). Therefore, we only need to determine the coefficient B_n, and in this subsection we will give a recursion relation which fixes the coefficients. However, we will not use this subsection's result in the rest of the paper, so readers who are interested only in the first few terms in the $1/J$ expansion (2.18) may skip this subsection and move on to section 2.5. As we saw in section 2.2 and appendix B.2, the structure of the classical solution in $q=2$ model is different from $q>2$ case. In this subsection, we focus on $q>2$ case.

We generalize the expansion (2.18) to all order by

$$\Psi_{\rm cl}(t_1,t_2) \,=\, J^{-\frac{2}{q}}\sum_{m=0}^{\infty} J^{-m}\,\Psi_m(t_1,t_2)\,. \tag{2.24}$$

Now, we substitute this expansion into the critical action S_c (1.6). As we saw before, the kinetic term does not contribute to the perturbative analysis when $q > 2$; therefore, we discard the kinetic term here. The contribution of the kinetic term will be recovered in the full classical solution with correct UV boundary conditions. Hence, the saddle-point equation is now formally written as

$$0 = \left[\sum_{m=0}^{\infty} J^{-m}\,\Psi_m(t_1,t_2)\right]^{-1} + \left[\sum_{m=0}^{\infty} J^{-m}\,\Psi_m(t_1,t_2)\right]^{q-1} . \tag{2.25}$$

Using the multinomial theorem, each term can be reduced to polynomials of Ψ_m's. Substituting these results into eq. (2.25) leads the saddle-point equation written in terms polynomials with all order of $1/J$ expansion. From this equation, one can further pick up order $\mathcal{O}(J^{-n})$ terms. For $n = 0$, it is the equation of Ψ_0. Therefore, we consider $n \geq 1$ case, which is given by

$$\begin{aligned}
0 =& \sum_{k_1+2k_2+\cdots=n} (-1)^{k_1+k_2+\cdots}\frac{(k_1+k_2+\cdots)!}{k_1!k_2!k_3!\cdots} \times \left[\Psi_0^{-1}\star\left(\Psi_1\star\Psi_0^{-1}\right)^{k_1}\star\left(\Psi_2\star\Psi_0^{-1}\right)^{k_2}\star\cdots\right](t_1,t_2)\\
&+\sum_{k_1+2k_2+\cdots=n}\frac{(q-1)!}{k_0!k_1!k_2!\cdots}\times\Psi_0^{k_0}(t_1,t_2)\,\Psi_1^{k_1}(t_1,t_2)\,\Psi_2^{k_2}(t_1,t_2)\cdots ,
\end{aligned} \tag{2.26}$$

with $k_0 = q - (1 + k_1 + \cdots + k_{n-1})$. Let us consider this order $\mathcal{O}(J^{-n})$ equation more. Because of the constraint $k_1 + 2k_2 + \cdots = n$, we know that $k_{n+1} = k_{n+2} = \cdots = 0$. Also the same constraint implies that $k_n = 0$ or 1, and when $k_n = 1$, then $k_1 = k_2 = \cdots = k_{n-1} = 0$. Therefore, it is useful to separate $k_n = 1$ terms from $k_n = 0$ ones. After this separation, the order $\mathcal{O}(J^{-n})$ equation is reduced to a more familiar form:

$$\begin{aligned}
&\int dt_3dt_4\,\mathcal{K}(t_1,t_2;t_3,t_4)\Psi_n(t_3,t_4)\\
&\quad= -\sum_{k_1+2k_2+\cdots+(n-1)k_{n-1}=n}(-1)^{k_1+\cdots+k_{n-1}}\frac{(k_1+\cdots+k_{n-1})!}{k_1!\cdots k_{n-1}!}\\
&\qquad\qquad\times\left[\Psi_0^{-1}\star\left(\Psi_1\star\Psi_0^{-1}\right)^{k_1}\star\cdots\star\left(\Psi_{n-1}\star\Psi_0^{-1}\right)^{k_{n-1}}\right](t_1,t_2)\\
&\qquad-\sum_{k_1+2k_2+\cdots+(n-1)k_{n-1}=n}\frac{(q-1)!}{k_0!k_1!\cdots k_{n-1}!}\times\Psi_0^{k_0}(t_1,t_2)\,\Psi_1^{k_1}(t_1,t_2)\,\cdots\,\Psi_{n-1}^{k_{n-1}}(t_1,t_2)\,,
\end{aligned} \tag{2.27}$$

where $k_0 = q - (1 + k_1 + \cdots + k_{n-1})$. This is the equation which determines Ψ_n from $\{\Psi_0, \Psi_1, \cdots, \Psi_{n-1}\}$ sources. However, we already know the t_{12} dependence of $\Psi_n(t_1,t_2)$. Namely,

$$\Psi_n(t_1,t_2) = B_n\,\frac{\operatorname{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+n}} . \tag{2.28}$$

Therefore, we only need to determine the coefficient B_n. Probably it is hard to evaluate the star products in the r.h.s. of eq. (2.27) by direct integrations of t's, and it is better to use momentum space representations.

$$\Psi_m(t_1,t_2) = B_m \int \frac{d\omega}{2\pi}\,e^{-i\omega t_{12}}\,\Psi_m(\omega)\,, \tag{2.29}$$

where we excluded the coefficient B_m from $\Psi_m(\omega)$ for later convenience, and $\Psi_m(\omega) = C_m\, |\omega|^{\frac{2}{q}+m-1}\,\mathrm{sgn}(\omega)$, with

$$C_m \;\equiv\; i\, 2^{1-m-\frac{2}{q}}\sqrt{\pi}\,\frac{\Gamma(1-\frac{1}{q}-\frac{m}{2})}{\Gamma(\frac{1}{q}+\frac{m}{2}+\frac{1}{2})}\,. \tag{2.30}$$

With this definition of C_m, we can write the inverse of the critical solution as

$$\Psi_0^{-1}(t_1,t_2) \;=\; \int\frac{d\omega}{2\pi}\, e^{-i\omega t_{12}}\,\Psi_0^{-1}(\omega) \;=\; -\,b^{q-1}\,C_{2-\frac{4}{q}}\int\frac{d\omega}{2\pi}\,e^{-i\omega t_{12}}\,|\omega|^{1-\frac{2}{q}}\,\mathrm{sgn}(\omega)\,. \tag{2.31}$$

Now, we can evaluate each term in eq. (2.27) using these Fourier transforms. Then, every term has the same ω integral; therefore, comparing the coefficients, one obtains

$$\begin{aligned}
&b^{q-2}\Big[(q-1)\,C_{2+n-\frac{4}{q}} - b^q\,C^2_{2-\frac{4}{q}}\,C_n\Big]B_n\\
&\quad= -\sum_{k_1+2k_2+\cdots+(n-1)k_{n-1}=n}(-1)^{k_1+\cdots+k_{n-1}}\,\frac{(k_1+\cdots+k_{n-1})!}{k_1!\cdots k_{n-1}!}\\
&\qquad\qquad\times\Big(-\,b^{q-1}\,C_{2-\frac{4}{q}}\Big)^{k_1+\cdots+k_{n-1}+1}\Big(B_1\,C_1\Big)^{k_1}\cdots\Big(B_{n-1}\,C_{n-1}\Big)^{k_{n-1}}\\
&\qquad-\sum_{k_1+2k_2+\cdots+(n-1)k_{n-1}=n}\frac{(q-1)!}{k_0!k_1!\cdots k_{n-1}!}\times b^{k_0}B_1^{k_1}\cdots B_{n-1}^{k_{n-1}}\,C_{2+n-\frac{4}{q}}\,,
\end{aligned} \tag{2.32}$$

with $k_0 = q-(1+k_1+\cdots+k_{n-1})$. This is the recursion relation which determines B_n from $\{B_1,B_2,\cdots,B_{n-1}\}$. Note that C_m's are a priori known numbers as defined in eq. (2.30).

2.5 B_1 from consistency condition

Now we can fix the so-far-unfixed coefficient B_1 of Ψ_1 from a consistency condition of the equivalence of the two methods shown in section 1.2. This is done by evaluating the zero mode eigenvalue shift explicitly. We will also give a comparison of our result of B_1 with the numerically approximated result found in [16].

Since the evaluation of the zero mode eigenvalue shift (1.22) is slightly technical, we present a derivation in appendix C. Here, we simply give the result:

$$\delta k(t_a,t_b) \;=\; \gamma\,B_1\,\partial_a^2\,\partial_b^2\,\delta(t_{ab})\,, \tag{2.33}$$

with

$$\gamma \;=\; -\,\frac{\tan(\frac{\pi}{q})}{12\pi b q}\left[\frac{2\pi(q-1)(q-2)}{q\sin(\frac{2\pi}{q})} - (q^2-6q+6)\right]. \tag{2.34}$$

Now we can fix B_1 by equating the two results of the quadratic action of the collective coordinates. From the result of δk, eq. (1.23) is reduced to

$$S_2[\varepsilon] \;=\; -\,\frac{NB_1\gamma}{4J}\int dt\,\Big(\varepsilon''(t)\Big)^2. \tag{2.35}$$

On the other hand, in eq. (1.13), taking $f(t)=t+\varepsilon(t)$ and from quadratic order of ε, we find

$$S_2[\varepsilon] \;=\; -\,\frac{\alpha N}{48\pi J}\int dt\,\Big(\varepsilon''(t)\Big)^2. \tag{2.36}$$

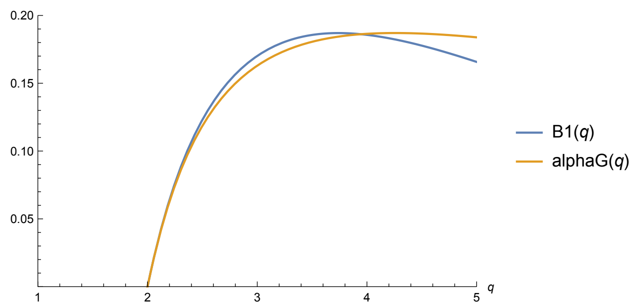

Figure 1. The red line represents α_G and the blue line represents the right hand side of eq. (2.40).

Equating the coefficients of the above two results, we obtain

$$B_1 = \frac{\alpha}{12\pi\gamma} = \frac{8b(q-1)\cos^2(\frac{\pi}{q})}{2\pi(q-1)(q-2) - q(q^2-6q+6)\sin(\frac{2\pi}{q})}, \tag{2.37}$$

where we used $\alpha = 1 - \varepsilon^2 = 4(q-1)/q^2$. We note that for $q = 2$, $B_1 = 0$ and then the recursion relation (2.32) implies $B_n = 0$ for all order. Therefore, the series of the $q = 2$ classical solution is triggered by a different solution as we discussed in appendix B.2.

Finally, we compare our result for B_1 with the numerical result found in [16] . Their α_G is related to our B_1 in the following way

$$\frac{\alpha_G}{\mathcal{J}} = \frac{B_1}{bJ}, \tag{2.38}$$

where their numerical approximated value of α_G is

$$\alpha_G \approx \frac{2(q-2)}{16/\pi + 6.18(q-2) + (q-2)^2}. \tag{2.39}$$

Since $\mathcal{J} = \frac{\sqrt{q}}{2^{\frac{q-1}{1}}} J$, we need to compare

$$\alpha_G = \frac{\sqrt{q}}{2^{\frac{q-1}{2}}} \frac{B_1}{b}. \tag{2.40}$$

The figure 1 shows the both hand sides of this equation. We can see that they agree very well from $q = 2$ to $q = 4$.

3 Two-point function

In this section, we consider the bi-local two-point function:

$$\Big\langle \Psi(t_1,t_2)\Psi(t_3,t_4) \Big\rangle, \tag{3.1}$$

where the expectation value is evaluated by the path integral (1.4). After the Faddeev-Popov prosedure and changing the integration variable as we discussed in section 1, this two-point function becomes

$$\Big\langle \Psi_f(t_1,t_2)\Psi_f(t_3,t_4) \Big\rangle, \tag{3.2}$$

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where now the expectation value is evaluated by the gauged path integral (1.10).

Now, we expand the bi-local field around the shifted background classical solution $\Psi_{\rm cl} = \Psi_0 + J^{-1}\Psi_1$. Namely,

$$\Psi(t_1,t_2) \;=\; \Psi_0(t_1,t_2) \,+\, \frac{1}{J}\,\Psi_1(t_1,t_2) \,+\, \sqrt{\frac{2}{N}}\;\overline{\eta}(t_1,t_2)\,, \tag{3.3}$$

where we have rescaled the entire field Ψ by $J^{2/q}$, and $\overline{\eta}$ is a quantum fluctuation, but the zero mode is eliminated from its Hilbert space. Therefore, the two-point function is now decomposed as

$$\Big\langle \Psi_f(t_1,t_2)\Psi_f(t_3,t_4)\Big\rangle \;=\; \Big\langle \Psi_{{\rm cl},f}(t_1,t_2)\Psi_{{\rm cl},f}(t_3,t_4)\Big\rangle \,+\, \frac{2}{N}\Big\langle \overline{\eta}(t_1,t_2)\overline{\eta}(t_3,t_4)\Big\rangle\,. \tag{3.4}$$

The second term in the r.h.s. is the bi-local propagator $\mathcal{D}$ determined by eq. (1.16), which was already evaluated in I for $q=4$ (and also in [14, 16]) as

$$\begin{aligned}\mathcal{D}(t_1,t_2;t_3,t_4) \;=\; -\,{\rm sgn}(t_- t'_-)\,\frac{8}{N\sqrt{\pi}}\sum_{m=1}^{\infty}\int d\omega\,\frac{e^{-i\omega(t_+-t'_+)}}{\sin(\pi p_m)}\,\frac{p_m^2}{p_m^2+(3/2)^2}\\ \times\left[J_{-p_m}(|\omega t_-|) \,+\, \frac{p_m+\frac{3}{2}}{p_m-\frac{3}{2}}\,J_{p_m}(|\omega t_-|)\right]J_{p_m}(|\omega t'_-|)\,,\end{aligned} \tag{3.5}$$

where p_m are the solutions of $2p_m/3 = -\tan(\pi p_m/2)$, and $t_\pm = (t_1 \pm t_2)/2$ and $t'_\pm = (t_3 \pm t_4)/2$.

Therefore, in this section let us focus on the first term in the r.h.s. of eq. (3.4). Expanding the classical field up to the second order, one has

$$\begin{aligned}&\Big\langle \Psi_{{\rm cl},f}(t_1,t_2)\Psi_{{\rm cl},f}(t_3,t_4)\Big\rangle\\ &\quad= \Big\langle \Psi_{0,f}(t_1,t_2)\Psi_{0,f}(t_3,t_4)\Big\rangle \,+\, \frac{1}{J}\left[\Big\langle \Psi_{0,f}(t_1,t_2)\Psi_{1,f}(t_3,t_4)\Big\rangle \,+\, \begin{pmatrix} t_1\leftrightarrow t_3\\ t_2\leftrightarrow t_4\end{pmatrix}\right] \,+\, \cdots\,,\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}\Psi_{0,f}(t_1,t_2) \;&=\; \Big|f'(t_1)f'(t_2)\Big|^{\frac{1}{q}}\;\Psi_0(f(t_1),f(t_2))\,,\\ \Psi_{1,f}(t_1,t_2) \;&=\; \Big|f'(t_1)f'(t_2)\Big|^{\frac{1}{q}+\frac{1}{2}}\;\Psi_1(f(t_1),f(t_2))\,.\end{aligned} \tag{3.7}$$

Now, we consider an infinitesimal reparametrization $f(t) = t + \varepsilon(t)$. Then, the classical fields are expanded as

$$\begin{aligned}\Psi_{0,f}(t_1,t_2) \;&=\; \Psi_0(t_1,t_2) \,+\, \int dt\,\varepsilon(t)\,u_{0,t}(t_1,t_2) \,+\, \cdots\,,\\ \Psi_{1,f}(t_1,t_2) \;&=\; \Psi_1(t_1,t_2) \,+\, \int dt\,\varepsilon(t)\,u_{1,t}(t_1,t_2) \,+\, \cdots\,,\end{aligned} \tag{3.8}$$

where

$$u_{0,t}(t_1,t_2) \;\equiv\; \frac{\partial\Psi_{0,f}(t_1,t_2)}{\partial f(t)}\bigg|_{f(t)=t}\,,\qquad u_{1,t}(t_1,t_2) \;\equiv\; \frac{\partial\Psi_{1,f}(t_1,t_2)}{\partial f(t)}\bigg|_{f(t)=t}\,. \tag{3.9}$$

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Therefore, in the quadratic order of ε, the classical field two-point function is now written in term of the two-point function of ε. For later convenience, it is better to write down this as momentum space integral as

$$\begin{aligned}&\Big\langle \Psi_{\mathrm{cl},f}(t_1,t_2)\Psi_{\mathrm{cl},f}(t_3,t_4)\Big\rangle \\ &\quad = \int \frac{d\omega}{2\pi}\,\langle \varepsilon(\omega)\varepsilon(-\omega)\rangle \bigg[u^*_{0,\omega}(t_1,t_2)u_{0,\omega}(t_3,t_4) + \frac{1}{J}\bigg(u^*_{0,\omega}(t_1,t_2)u_{1,\omega}(t_3,t_4) + \begin{pmatrix} t_1 \leftrightarrow t_3 \\ t_2 \leftrightarrow t_4 \end{pmatrix}\bigg) + \cdots \bigg].\end{aligned} \tag{3.10}$$

Let us first evaluate the ε two-point function. The collective coordinate action is given in eq. (1.13). Expanding $f(t) = t + \varepsilon(t)$, the quadratic action of ε is given by eq. (2.36). Hence, the two-point function in momentum space is

$$\langle \varepsilon(\omega)\varepsilon(-\omega)\rangle = \frac{24\pi J}{\alpha N}\,\frac{1}{\omega^4}\,. \tag{3.11}$$

One can also Fourier transform back to the time representation to get

$$\langle \varepsilon(t_1)\varepsilon(t_2)\rangle = \frac{2\pi J}{\alpha N}\,|t_{12}|^3\,. \tag{3.12}$$

Next, we evaluate u_0 and u_1. Taking the derivative respect to $f(t)$, one obtains

$$\begin{aligned} u_{0,t}(t_1,t_2) &= \frac{1}{q}\left[\delta'(t_1-t) + \delta'(t_2-t) - 2\left(\frac{\delta(t_1-t)-\delta(t_2-t)}{t_1-t_2}\right)\right] \Psi_0(t_1,t_2)\,, \\ u_{1,t}(t_1,t_2) &= \frac{2+q}{2q}\left[\delta'(t_1-t) + \delta'(t_2-t) - 2\left(\frac{\delta(t_1-t)-\delta(t_2-t)}{t_1-t_2}\right)\right] \Psi_1(t_1,t_2) \\ &= \frac{(2+q)B_1}{2b}\,\frac{u_{0,t}(t_1,t_2)}{|t_{12}|}\,. \end{aligned} \tag{3.13}$$

After some manipulation, one can show that the momentum space expressions are given by

$$\begin{aligned} u_{0,\omega}(t_1,t_2) &= -\frac{ib\sqrt{\pi}}{q}\,\frac{|\omega|^{\frac{3}{2}}\,\mathrm{sgn}(\omega t_-)}{|2\,t_-|^{\frac{2}{q}-\frac{1}{2}}}\,e^{i\omega t_+}\,J_{\frac{3}{2}}(|\omega t_-|)\,, \\ u_{1,\omega}(t_1,t_2) &= \frac{(2+q)B_1}{4b}\,\frac{u_{0,\omega}(t_1,t_2)}{|t_-|}\,. \end{aligned} \tag{3.14}$$

Using the two-point function of ε and above u_0 and u_1 expressions, finally the two-point function (3.4) up to order J^0 is given by

$$\begin{aligned} \Big\langle \Psi_f(t_1,t_2)\Psi_f(t_3,t_4)\Big\rangle &= \frac{3q^2}{(q-1)N}\left[J + \frac{(2+q)B_1}{4b}\left(\frac{1}{|t_-|}+\frac{1}{|t'_-|}\right)\right]\int \frac{d\omega}{\omega^4}\,u^*_{0,\omega}(t_1,t_2)u_{0,\omega}(t_3,t_4) \\ &\quad + \mathcal{D}(t_1,t_2;t_3,t_4)\,. \end{aligned} \tag{3.15}$$

What we have established therefore is the following. What one has is first the leading "classical" contribution to the bi-local two-point function which usually factorizes, due to the dynamics of the reparametrization symmetry mode. It now represents the leading 'big' contribution, as in [16], and a sub-leading one. This is followed by the matter fluctuations given by the zero mode projected propagator of I [15].

4 Finite temperature classical solution

Up to here, we have been considering only zero-temperature solutions in the SYK model. In this section, we will determine the finite-temperature solutions $\Psi_{1,\beta}$ and $\Psi_{2,\beta}$ and evaluate their contributions to the tree-level free energy.

As we saw in section 2, the $1/J$ expansion of the classical solution in the strongly coupling region is given by

$$\Psi_{\rm cl}(t_1,t_2) \;=\; J^{-\frac{2}{q}}\Big[\Psi_0(t_1,t_2) \,+\, J^{-1}\,\Psi_1(t_1,t_2) \,+\, J^{-2}\,\Psi_2(t_1,t_2) \,+\, \cdots\Big]\,, \tag{4.1}$$

where

$$\Psi_0(t_1,t_2) \;=\; b\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}}}\,,\quad \Psi_1(t_1,t_2) \;=\; B_1\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+1}}\,,\quad \Psi_2(t_1,t_2) \;=\; B_2\,\frac{\mathrm{sgn}(t_{12})}{|t_{12}|^{\frac{2}{q}+2}}\,. \tag{4.2}$$

In order to evaluate tree-level free energy, we first need finite-temperature versions of these classical solutions. Ψ_0 is the solution of the strict strong coupling limit, where the model exhibits an emergent conformal reparametrization symmetry: $t\to f(t)$ with the Ψ_0 transformation (1.7). Therefore, to obtain the finite-temperature version of Ψ_0, we just need to use $f(t)=\frac{\beta}{\pi}\tan(\frac{\pi t}{\beta})$ with the above transformation [6]. This map maps the infinitely long zero-temperature time to periodic thermal circle. Thus, this gives us

$$\Psi_{0,\beta}(t_1,t_2) \;=\; b\,\left[\frac{\pi}{\beta\sin(\frac{\pi t_{12}}{\beta})}\right]^{\frac{2}{q}}\,\mathrm{sgn}(t_{12})\,. \tag{4.3}$$

Since Ψ_1 and Ψ_2 are the shifts of the classical solution from the strict IR limit, they do not enjoy the reparametrization symmetry. Therefore, we cannot use the above method to get their finite-temperature counterparts. However, we can obtain finite-temperature solutions by mapping the zero-temperature solutions onto a thermal circle and summing over all image charges:

$$\Psi_\beta(t_{12}) \;=\; \sum_{m=-\infty}^{\infty}(-1)^m\,\Psi_{\beta=\infty}(t_{12}+\beta m)\,. \tag{4.4}$$

By defining the finite-temperature solution by this way, the thermal two-point function (in terms of the fundamental fermions) trivially satisfies the KMS condition. Of course, this method works order by order in the $1/J$ expansion. Therefore, after separating positive m and negative m and changing the labeling, one finds

$$\Psi_{1,\beta}(t_{12}) \;=\; B_1\left[\sum_{m=0}^{\infty}\frac{(-1)^m}{(\beta m+t_{12})^{\frac{2}{q}+1}} \,-\, \sum_{m=1}^{\infty}\frac{(-1)^m}{(\beta m-t_{12})^{\frac{2}{q}+1}}\right]\,. \tag{4.5}$$

The summations of m can be evaluated to give the Hurwitz zeta functions. However, this form is more convenient for later evaluations of tree level free energy, so we stop here. In the same way, we obtain finite-temperature Ψ_2 as

$$\Psi_{2,\beta}(t_{12}) \;=\; B_2\left[\sum_{m=0}^{\infty}\frac{(-1)^m}{(\beta m+t_{12})^{\frac{2}{q}+2}} \,-\, \sum_{m=1}^{\infty}\frac{(-1)^m}{(\beta m-t_{12})^{\frac{2}{q}+2}}\right]\,. \tag{4.6}$$

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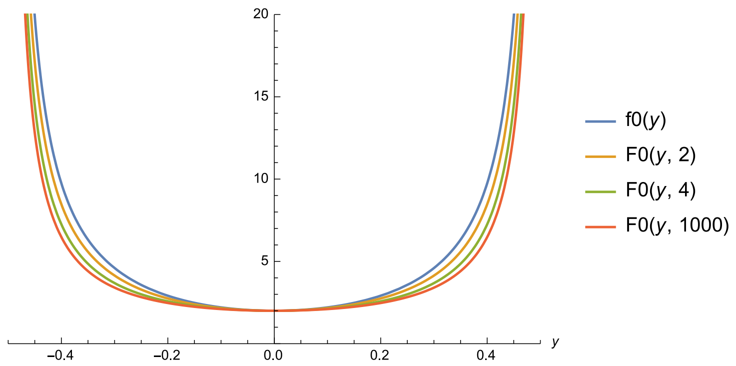

Figure 2. $f_0(y)$ and $F_0(y,q)$ with $q = 2, 4, 1000$ in the range of $-\frac{1}{2} \leq y \leq \frac{1}{2}$.

In [16], Maldacena and Stanford found a first order shift of the classical solution in finite-temperature through a numerical solution of the exact Schwinger-Dyson equation. Therefore, let us compare our result of $\Psi_{1,\beta}$ with their result before we consider free energy. The solution of [16] is shown in their eq. (3.122) reading:

$$\frac{\delta G(t_1,t_2)}{G_c(t_1,t_2)} = -\frac{\alpha_G}{\beta \mathcal{J}} f_0(t_{12}) \,, \qquad f_0(t_{12}) = 2 + \frac{\pi - \frac{2\pi |t_{12}|}{\beta}}{\tan |\frac{\pi t_{12}}{\beta}|} \,. \tag{4.7}$$

with the notation, $G_c = \Psi_{0,\beta}$ and $\delta G = \Psi_{1,\beta}$. This thermal two-point function does not satisfy the KMS condition; but as we will see below it gives a pretty good approximation. It is more convenient to introduce a new variable

$$y \equiv \frac{|t_{12}|}{\beta} - \frac{1}{2} \,. \qquad \left(-\frac{1}{2} \leq y \leq \frac{1}{2}\right) \tag{4.8}$$

Then, we have

$$f_0(y) = 2 + 2\pi y \tan(\pi y) \,. \tag{4.9}$$

Now, we can see that $\delta G(y)$ is even function of y. This can be understood as a combination of the following two anti-symmetries. (i) the two-point function is anti-symmetric under $t_{12} \to t_{21}$. (ii) the two-point function is anti-symmetric under $t_1 \to t_1 + \beta$, (or $t_2 \to t_2 + \beta$). We also note that

$$G_c(t_1,t_2) = \Psi_{0,\beta}(t_1,t_2) = b \left[\frac{\pi}{\beta \cos \pi y}\right]^{\frac{2}{q}} \,. \tag{4.10}$$

Next, we consider our result of $\Psi_{1,\beta}$. We can rewrite eq. (4.5) in the form of

$$\Psi_{1,\beta}(t_{12}) = \frac{B_1}{\beta^{\frac{2}{q}+1}} \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{(m + \frac{t_{12}}{\beta})^{\frac{2}{q}+1}} + \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1-\frac{t_{12}}{\beta})^{\frac{2}{q}+1}} \right] \,. \tag{4.11}$$

Using an integral representation of the Hurwitz Zeta Function (for example, see 25.11.35 of [28]), one can see that indeed our result is also even function of y. We can also perform

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the summation in eq. (4.11) directly to get

$$\Psi_{1,\beta}(t_{12})=\frac{B_1}{(2\beta)^{\frac{2}{q}+1}}\left[\zeta\left(\frac{2}{q}+1,\frac{1}{4}+\frac{y}{2}\right)+\zeta\left(\frac{2}{q}+1,\frac{1}{4}-\frac{y}{2}\right)-\zeta\left(\frac{2}{q}+1,\frac{3}{4}+\frac{y}{2}\right)-\zeta\left(\frac{2}{q}+1,\frac{3}{4}-\frac{y}{2}\right)\right]. \tag{4.12}$$

Therefore, together with eq. (4.10), we have

$$\frac{\Psi_{1,\beta}(t_{12})}{\Psi_{0,\beta}(t_{12})} = \frac{B_1}{2(2\pi)^{\frac{2}{q}}b\beta}\left[\zeta\left(\frac{2}{q}+1,\frac{1}{4}\right) - \zeta\left(\frac{2}{q}+1,\frac{3}{4}\right)\right] \times F_0(y,q)\,, \tag{4.13}$$

where

$$F_0(y,q) \equiv (\cos\pi y)^{\frac{2}{q}}\left[\frac{\zeta\left(\frac{2}{q}+1,\frac{1}{4}+\frac{y}{2}\right)+\zeta\left(\frac{2}{q}+1,\frac{1}{4}-\frac{y}{2}\right)-\zeta\left(\frac{2}{q}+1,\frac{3}{4}+\frac{y}{2}\right)-\zeta\left(\frac{2}{q}+1,\frac{3}{4}-\frac{y}{2}\right)}{\zeta\left(\frac{2}{q}+1,\frac{1}{4}\right)-\zeta\left(\frac{2}{q}+1,\frac{3}{4}\right)}\right]. \tag{4.14}$$

Here, we adjusted the normalization of F_0 so that $F_0(y=0,q) = 2 = f_0(y=0)$. A numerical plots are given in figure 2, where we plotted $f_0(y)$ and $F_0(y,q)$ with $q = 2, 4, 1000$. We can see that for any value of q, F_0 is pretty close to f_0 in all range of y.

4.1 Tree-level free energy

Now we use our finite-temperature solutions to determine their contributions to the tree-level free energy. The order $(\beta J)^0$ contribution to the tree-level free energy, which comes from $S_c[\Psi_{0,\beta}]$, was already evaluated in [6, 16, 18]. Therefore in this section, we will evaluate higher order contributions of the $1/\beta J$ expansion to the tree-level free energy.

4.1.1 Contribution from $S[f]$

The action of the collective time coordinate was evaluated in appendix A by using ε-expansion with $q = 2/(1-\varepsilon)$. The result is given by eq. (1.13). Now, we use the classical solution: $f(t) = \frac{\beta}{\pi}\tan(\frac{\pi t}{\beta})$. Then, the integral can be evaluated to give $2\pi^2/\beta$. Therefore, the $S[f]$ contribution to the tree-level free energy is

$$\beta F_{(0)} = -\frac{\pi(1-\varepsilon^2)}{12}\frac{N}{\beta J}\,. \tag{4.15}$$

Equivalently, this can be written as

$$\log Z_{(0)} = -\beta F_{(0)} = \frac{c}{2\beta}\,, \qquad \text{where} \qquad c \equiv \frac{\pi(1-\varepsilon^2)}{6}\frac{N}{J}\,. \tag{4.16}$$

For $q = 4$ ($\varepsilon = 1/2$), we find $c = (\pi/8) \times (N/J) \approx 0.393 \times N/J$, which agrees very well with the value found in [16].

4.1.2 Contribution from $\Psi_{1,\beta}$

Now we evaluate the contribution to tree-level free energy from the kinetic term of $\Psi_{1,\beta}$:

$$S[\Psi_{1,\beta}] = \frac{N}{2J^2}\int_0^\beta dt_1 \left[\partial_1\Psi_{1,\beta}(t_1,t_2)\right]_{t_2\to t_1}. \tag{4.17}$$

Substituting the solution (4.5) into this kinetic term, the contribution to free energy is given by

$$\beta F_{(0)} = \frac{NB_1}{2J^2}\left[\sum_{m=0}^{\infty}(-1)^m\int_0^\beta dt_1\left[\partial_1\frac{1}{(\beta m+t_{12})^{2-\varepsilon}}\right]_{t_2\to t_1}-\sum_{m=1}^{\infty}(-1)^m\int_0^\beta dt_1\left[\partial_1\frac{1}{(\beta m-t_{12})^{2-\varepsilon}}\right]_{t_2\to t_1}\right], \tag{4.18}$$

where we used $q = 2/(1-\varepsilon)$. Then, we use ε-expansion to evaluate the limit and integral. First, to make sure the contribution has the correct dimension, we rewrite the integrand as

$$\frac{1}{(\beta m \pm t_{12})^{2-\varepsilon}} \approx \frac{1}{\beta^2 m^2}\frac{1}{(1\pm\frac{t_{12}}{\beta m})^{2-\varepsilon}}\,. \tag{4.19}$$

We are interested in low temperature expansion ($\beta \gg t_{12}$). Therefore, together with ε-expansion, we have

$$\frac{1}{(1\pm\frac{t_{12}}{\beta m})^{2-\varepsilon}} = \frac{1}{(1\pm\frac{t_{12}}{\beta m})^{2}}\times\exp\left[\varepsilon\log\left(1\pm\frac{t_{12}}{\beta m}\right)\right] = 1\mp\frac{(2-\varepsilon)}{\beta m}t_{12}+\cdots\,. \tag{4.20}$$

Therefore, now the limit and derivative in the free energy can be evaluated to lead

$$\beta F_{(0)} = \frac{NB_1(2-\varepsilon)}{(\beta J)^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^3} = \frac{3(2-\varepsilon)B_1\zeta(3)}{4}\frac{N}{(\beta J)^2}\,. \tag{4.21}$$

We note that for $q = 2$, $B_1 = 0$. Therefore, this agrees with the result (B.14) in appendix B.3. The delta function type of solution for $\Psi_{(1)}$ in $q = 2$ model does not have any non-zero finite contribution to the free energy.

Now we evaluate the contribution to tree-level free energy from the critical action $S_c[\Psi_{1,\beta}]$. The critical action is given by eq. (1.6). Substituting the expansion of the classical field (4.1) into this critical action, one can find two terms in $\mathcal{O}(J^{-1})$ and four terms in $\mathcal{O}(J^{-2})$. The order $\mathcal{O}(J^{-1})$ contribution is zero, due to the equation of motion of Ψ_0. Next, among the order $\mathcal{O}(J^{-2})$ contributions, two terms proportional to Ψ_2 are canceled each other due to the Ψ_0 equation of motion again. The other terms can be written as

$$-\frac{N}{4J^2}\int dt_1dt_2dt_3dt_4\,\Psi_1(t_1,t_2)\,\mathcal{K}(t_1,t_2;t_3,t_4)\,\Psi_1(t_3,t_4) = 0\,. \tag{4.22}$$

This contribution is again zero due to the equation of motion of Ψ_1.

4.1.3 Contribution from $\Psi_{2,\beta}$

Now, we consider the kinetic term of $\Psi_{2,\beta}$:

$$S[\Psi_{2,\beta}] = \frac{N}{2J^3}\int_0^\beta dt_1\left[\partial_1\Psi_{2,\beta}(t_1,t_2)\right]_{t_2\to t_1}\,. \tag{4.23}$$

This evaluation is completely parallel to the one with $\Psi_{1,\beta}$. The contribution to free energy is

$$\beta F_{(0)} = \frac{NB_2}{2J^3}\left[\sum_{m=0}^{\infty}(-1)^m\int_0^\beta dt_1\left[\partial_1\frac{1}{(\beta m+t_{12})^{3-\varepsilon}}\right]_{t_2\to t_1}-\sum_{m=1}^{\infty}(-1)^m\int_0^\beta dt_1\left[\partial_1\frac{1}{(\beta m-t_{12})^{3-\varepsilon}}\right]_{t_2\to t_1}\right]. \tag{4.24}$$

Again, we expand the integrand as in eqs. (4.19) and (4.20), and the free energy is now given by

$$\beta F_{(0)} = \frac{NB_2(3-\varepsilon)}{(\beta J)^3}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^4} = \frac{7(3-\varepsilon)\pi^4 B_2}{720}\frac{N}{(\beta J)^3}\,. \tag{4.25}$$

Now we evaluate the contribution to tree-level free energy from the critical action $S_c[\Psi_{2,\beta}]$: we again expand $S_c[\Psi_{cl}]$ with the expansion of the classical solution (4.1). For the order $\mathcal{O}(J^{-3})$, one finds six terms. Two terms proportional to Ψ_3 are cancels each other due to the Ψ_0 equation of motion. Using the equation of motion for Ψ_2, the other four terms can be combined as

$$\beta F_{(0)} = \frac{2N}{3J^3}\int_0^\beta dt_1dt_2dt_3dt_4\,\Psi_2(t_1,t_2)\,\mathcal{K}_\beta(t_1,t_2;t_3,t_4)\,\Psi_1(t_3,t_4)\,. \tag{4.26}$$

For $q>2$ case, this is zero due to the Ψ_1 equation of motion. However, when $q=2$, the equation of Ψ_1 has the delta function source term. Therefore, using this we can rewrite the contribution as

$$\beta F_{(0)} = -\frac{2N}{3J^3}\int_0^\beta dt_1\left[\partial_1\Psi_{2,\beta}(t_1,t_2)\right]_{t_2\to t_1} = -\frac{7\pi^4 B_2}{180}\frac{N}{(\beta J)^3}\,. \qquad (q=2,\ \varepsilon=0) \tag{4.27}$$

In the last step, we used the result of the kinetic term of $\Psi_{2,\beta}$. Combining with the kinetic term contribution, for $q=2$ the total contribution to order $\mathcal{O}(J^{-3})$ free energy is

$$\beta F_{(0)} = -\frac{7\pi^4 B_2}{720}\frac{N}{(\beta J)^3} = \frac{7\pi^3}{2880}\frac{N}{(\beta J)^3}\,, \qquad (q=2) \tag{4.28}$$

where we used $B_2=-1/4\pi$ for $q=2$. This result completely agrees with eq. (B.14).

4.1.4 Summary

Up to here, we have the following perturbative result for tree-level free energy.

$$\beta F_{(0)}/N = -\frac{\pi}{12\beta J} + \frac{7\pi^3}{2880}\frac{1}{(\beta J)^3} + \cdots\,, \qquad (q=2) \tag{4.29}$$

$$\beta F_{(0)}/N = -\frac{(1-\varepsilon^2)\pi}{12\beta J} + \frac{3(2-\varepsilon)B_1\zeta(3)}{4}\frac{1}{(\beta J)^2} + \frac{7(3-\varepsilon)\pi^4 B_2}{720}\frac{1}{(\beta J)^3} + \cdots\,. \qquad (q>2) \tag{4.30}$$

5 Conclusion

In the present paper we have in the framework of the formulation given in (I) performed perturbative calculations in the SYK model around the conformal IR point. These calculations are systematic in the inverse of the strong coupling J. We are able to present analytical calculations through the use of a suitably defined ε expansion representing a perturbation around the exactly solvable $q=2$ case. It turned out that in a number of quantities (most notably the coefficient of the Schwarzian action $S[f]$) this expansion truncates. Our analytical calculations, for all quantities considered, agreed within the margin of error with the numerical evaluations of [16]. It will be interesting to perform further

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analytical calculations in this strong coupling expansion, with further comparison with improved numerical calculations. Also, the present calculations are done at tree level in $1/N$. The formalism that we have given allows for loop level calculations with no difficulty, due to projection of the zero mode the perturbation expansion is well defined, while the Jacobian(s) of the changes of variables provide exact counter terms which are expected to cancel infinities appearing in loop diagrams.

These higher order calculations and further detailed study of the model will be of definite usefulness regarding the question of the exact AdS_2 Gravity dual representing this theory. A class of dilation Gravities related to the models developed by Almheiri and Polchinski [17] shows features contained in SYK model [18–21]. The representation that we have given with exact action featuring interaction between the dynamical (time) coordinate and bi-local matter is the system that one might hope to recover from the corresponding AdS_2 theory.

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We acknowledge useful conversation with Robert de Melo Koch and Sumit Das on the topics of this paper. This work is supported by the Department of Energy under contract DE-SC0010010.

A ϵ-expansion for general q

In this appendix, we will give a derivation of the Schwarzian action (1.13) for general q. This is done by using ε-expansion with $q = 2/(1-\varepsilon)$ and treating ε as a small parameter. We note that for any q in the range of $2 \leq q \leq \infty$, the value of ε is $0 \leq \varepsilon \leq 1$. Therefore, the convergence of this ϵ-expansion is guaranteed. Even though we use the ε-expansion, we can nevertheless calculate all order contributions of ε as we will see below. We first rewrite the critical solution in the following way:

$$\begin{aligned}\Psi_{0,f}(t_1,t_2) \;=&\; -\frac{1}{\pi J}\left(\frac{\sqrt{|f'(t_1)f'(t_2)|}}{|f(t_1)-f(t_2)|}\right) \\ &\times\left[1-\varepsilon\,\log\left(\frac{\sqrt{|f'(t_1)f'(t_2)|}}{|f(t_1)-f(t_2)|}\right)+\frac{\varepsilon^2}{2}\left(\log\frac{\sqrt{|f'(t_1)f'(t_2)|}}{|f(t_1)-f(t_2)|}\right)^2+\cdots\right],\end{aligned} \tag{A.1}$$

where the first term is the contribution from $q = 2$ case, which leads to the result eq. (1.13) with $\alpha = 1$. To evaluate higher order ε contributions, we use the following expansions of the logarithm in the $t_1 \to t_2$ limit:

$$\log\left(\frac{\sqrt{|f'(t_1)f'(t_2)|}}{|f(t_1)-f(t_2)|}\right) \;=\; -\,\log|t_1-t_2| - \frac{1}{8}\frac{|f''(t_2)|^2}{|f'(t_2)|^2}|t_1-t_2|^2 + \frac{1}{12}\frac{|f'''(t_2)|}{|f'(t_2)|}|t_1-t_2|^2 + \cdots. \tag{A.2}$$

The first log term gives an f-independent divergent term which we will eliminate in the following. One also expands the factor representing $q = 2$ reparametrized critical solution and then one finds $\mathcal{O}(\varepsilon) = 0$. For order $\mathcal{O}(\varepsilon^2)$ contribution, from eq. (A.2), one can find

$$\mathcal{O}(\varepsilon^2) = -\frac{N\varepsilon^2}{4\pi J}\int dt_1\,\partial_1\left[\left(\frac{1}{4}\frac{|f''(t_2)|^2}{|f'(t_2)|^2} - \frac{1}{6}\frac{|f'''(t_2)|}{|f'(t_2)|}\right)|t_1-t_2|\,\log|t_1-t_2|\right]_{t_2\to t_1}$$
$$= \frac{N\varepsilon^2}{24\pi J}\int dt_1\left[\frac{f'''(t_1)}{f'(t_1)} - \frac{3}{2}\left(\frac{f''(t_1)}{f'(t_1)}\right)^2\right], \tag{A.3}$$

where we again eliminated the divergence term and used integration by parts. Hence, the total contribution up to $\mathcal{O}(\varepsilon^2)$ for $q = 2/(1-\varepsilon)$ action is given by

$$S[f] = -\frac{N\alpha}{24\pi J}\int dt\left[\frac{f'''(t)}{f'(t)} - \frac{3}{2}\left(\frac{f''(t)}{f'(t)}\right)^2\right], \tag{A.4}$$

where

$$\alpha(\varepsilon) = 1 - \varepsilon^2 + \mathcal{O}(\varepsilon^3). \tag{A.5}$$

In fact, there is no higher order contributions from $\mathcal{O}(\varepsilon^3)$, and the expression for α in eq. (A.5) is exact for all order of ε. Namely, $\alpha(\varepsilon) = 1 - \varepsilon^2$. This can be seen from an expansion

$$\left(\log\frac{\sqrt{|f'(t_1)f'(t_2)|}}{|f(t_1)-f(t_2)|}\right)^n \tag{A.6}$$
$$= \Big(-\log|t_1-t_2|\Big)^n - n\left(\frac{1}{8}\frac{|f''(t_2)|^2}{|f'(t_2)|^2} - \frac{1}{12}\frac{|f'''(t_2)|}{|f'(t_2)|}\right)|t_1-t_2|^2\Big(-\log|t_1-t_2|\Big)^{n-1} + \cdots.$$

This expansion together with the expansion of $q = 2$ reparametrized critical solution does not give any non-zero finite contribution to the action after the limit when $n \geq 3$. Namely, the $(\log|t_1-t_2|)^n$ factor gives a strong divergence when n is large. However, if one wants to lower the power of this logarithm, then one gets a higher power of $|t_1-t_2|^n$, which strongly vanishes after setting $t_2 = t_1$. Therefore, we don't have $\mathcal{O}(\varepsilon^3)$ order contributions and $\alpha = 1-\varepsilon^2$ is exact. Finally, we comment that our exact analytical results, for example, $\alpha = 1$ $(q = 2)$ and $\alpha = 3/4$ $(q = 4)$ agree very well with the numerical results found in [16].

B $q = 2$ model

B.1 Exact classical solution

In section 2, we considered a shift of the classical solution from the critical IR point for a general even integer q case. However, for $q = 2$ case the problem becomes very easy and we can indeed obtain the exact classical solution as discussed in [16], which is valid for any region from UV to IR.

The exact classical solution is determined by the saddle-point equation of the collective action (1.5):

$$\partial_1\,\delta(t_{12}) = \Psi_{\rm cl}^{-1}(t_1,t_2) + J^2\,\Psi_{\rm cl}^{q-1}(t_1,t_2). \tag{B.1}$$

This equation can be solved exactly when $q = 2$. Using the Fourier transform defined as in eq. (2.29), the exact solution is given by

$$\Psi_{\rm cl}(\omega) = \frac{-i\omega + i\,{\rm sgn}(\omega)\sqrt{\omega^2+4J^2}}{2J^2} = -\frac{2}{i\omega + i\,{\rm sgn}(\omega)\sqrt{4J^2+\omega^2}}. \tag{B.2}$$

The expansion of this exact solution in the strong coupling region $J/\omega \gg 1$ is given by

$$\Psi_{\rm cl}(\omega) \;=\; -\frac{i\omega}{2J^2} + \frac{i\,{\rm sgn}(\omega)}{J}\left(1 + \frac{\omega^2}{8J^2} + \cdots\right). \tag{B.3}$$

We can also go back to the bi-local time representation by the Fourier inverse transformation as

$$\Psi_{\rm cl}(t_1,t_2) \;=\; \frac{\delta'(t_{12})}{2J^2} + \frac{1}{\pi J t_{12}} - \frac{1}{4\pi (J t_{12})^3} + \cdots. \tag{B.4}$$

B.2 Perturbative classical solution

Even though we know the exact solution for $q = 2$ model, we can also obtain the expansion (B.4) of the exact solution by the perturbative analysis we did in section 2, and this is what we will do in this subsection.

As we described in section 2.2, $\Psi_{(1)}(t_{12}) \sim |t_{12}|^{-4/q}$ type of solution does not exist for any q. However when $q = 2$, we could have another type of ansatz:

$$\Psi_{(1)}(t_1,t_2) \;=\; \widetilde{A}_1\; \delta'(t_{12})\,, \tag{B.5}$$

where $\widetilde{A}_1$ is a t-independent coefficient. This ansatz is antisymmetric and has the correct dimension when $q = 2$. Therefore, let us start to analyze whether this ansatz satisfies eq. (2.2):

$$\int dt_3 dt_4\, \mathcal{K}(t_1,t_2;t_3,t_4)\Psi_{(1)}(t_3,t_4) \;=\; \partial_1\delta(t_{12})\,, \tag{B.6}$$

with the kernel

$$\mathcal{K}(t_1,t_2;t_3,t_4) \;=\; \Psi_0^{-1}(t_1,t_3)\Psi_0^{-1}(t_2,t_4) \,+\, \delta(t_{13})\delta(t_{24})\,. \tag{B.7}$$

For this purpose, it is convenient to use momentum space representation. The Fourier transforms give us

$$\int dt_3 dt_4\, \mathcal{K}(t_1,t_2;t_3,t_4)\Psi_{(1)}(t_3,t_4) \;=\; -i\int \frac{d\omega}{2\pi}\, e^{-i\omega t_{12}}\, 2\widetilde{A}_1\omega\,. \tag{B.8}$$

Also expressing the r.h.s. of eq. (B.6) in momentum representation, one finds $\widetilde{A}_1 = 1/2$. This agrees with the expansion of the exact solution (B.4).

Let us keep going this perturbative evaluation. For $\Psi_{(2)}$ we have an ansatz:

$$\Psi_{(2)}(t_1,t_2) \;=\; \widetilde{A}_2\, \frac{{\rm sgn}(t_{12})}{|t_{12}|^3}\,, \tag{B.9}$$

where $\widetilde{A}_2$ is a t-independent coefficient. Then, we study eq. (2.20) to fix the coefficient $\widetilde{A}_2$. Note that the second term in the r.h.s. is absent for $q = 2$. To evaluate the equation, we again use Fourier transform. Then, we can obtain

$$\int dt_3 dt_4\, \mathcal{K}(t_1,t_2;t_3,t_4)\Psi_{(2)}(t_3,t_4) \;=\; -i\pi\widetilde{A}_2 \int \frac{d\omega}{2\pi}\, e^{-i\omega t_{12}}\, \omega^2\, {\rm sgn}(\omega)\,, \tag{B.10}$$

and

$$-\,[\Psi_0^{-1}\star\Psi_{(1)}\star\Psi_0^{-1}\star\Psi_{(1)}\star\Psi_0^{-1}](t_1,t_2) \;=\; \frac{i}{4}\int\frac{d\omega}{2\pi}\,e^{-i\omega t_{12}}\,\omega^2\,\mathrm{sgn}(\omega)\,. \tag{B.11}$$

Therefore, now the coefficient is fixed as $\widetilde{A}_2=-1/4\pi$. This again agrees with the expansion of the exact solution (B.4). We expect that higher order perturbative calculation results will also agree with the expansion of the exact solution. As we explained in section 2.2, the delta function type ansatz for $\Psi_{(1)}$ is only available when $q=2$. Also from the analysis in section 2.5, we saw that $B_1=0$ when $q=2$, which implies $B_n=0$ for all order. Therefore, in this sense the expansion of the classical solution of $q=2$ model is a different series from that of $q>2$ model.

B.3 Tree-level free energy

In [16], Maldacena and Stanford computed the exact tree-level free energy of $q=2$ model using the free fermion picture. The result is given in eq. (2.34) of [16]. We want to give an expression for the low temperature ($\beta J\gg1$) expansion of this free energy. This can be done as follows. First, we expand the logarithm by Taylor series, because $\beta J\gg1$. Then, the θ integral is now given by the modified Struve function

$$\log Z/N \;=\; -\frac{1}{2\beta J}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\,\mathbf{M}_1(2\beta Jn)\,. \tag{B.12}$$

Using the large argument expansion of the modified Struve function (for example, see 11.6.2 of [28]), one finds

$$\log Z/N \;=\; \frac{1}{\pi\beta J}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}-\frac{1}{4\pi(\beta J)^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}-\frac{3}{16\pi(\beta J)^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^6}+\mathcal{O}\big((\beta J)^{-7}\big)\,. \tag{B.13}$$

After evaluating the summations, the low temperature expansion of the free energy is given by

$$\log Z/N \;=\; \frac{\pi}{12\beta J}-\frac{7\pi^3}{2880}\frac{1}{(\beta J)^3}-\frac{31\pi^5}{161280}\frac{1}{(\beta J)^5}+\mathcal{O}\Big((\beta J)^{-7}\Big)\,. \tag{B.14}$$

C Explicit integrations of $\boldsymbol{\delta k}$

In this subsection, we explicitly evaluate the integrals of δk in eq. (1.22):

$$\delta k_t\,\delta(t-t') \;=\; \int dt_1dt_2dt_3dt_4\,u_{0,t}(t_1,t_2)\,\delta\mathcal{K}(t_1,t_2;t_3,t_4)\,u_{0,t'}(t_3,t_4)\,, \tag{C.1}$$

where the zero mode $u_{0,t}$ is given in eq. (3.13). For the integrals in eq. (C.1), it is more convenient to use the momentum representation of this zero mode:

$$u_{0,t}(t_1,t_2) \;=\; \int\frac{d\omega}{2\pi}\,e^{-i\omega t}\,u_{0,\omega}(t_1,t_2)\,, \tag{C.2}$$

where

$$u_{0,\omega}(t_1,t_2) \;=\; \frac{1}{q}\left[i\omega\Big(e^{i\omega t_1}\,+\,e^{i\omega t_2}\Big)\,-\,2\left(\frac{e^{i\omega t_1}\,-\,e^{i\omega t_2}}{t_1-t_2}\right)\right]\;\Psi_0(t_1,t_2)\,. \tag{C.3}$$

After some manipulation, one can obtain the expression given in eq. (3.14), which we will use in the following calculation. Next, we consider the shift of the kernel $\delta\mathcal{K}$ defined in eq. (1.21). The explicit form is given by

$$\begin{aligned}\delta\mathcal{K}(t_1,t_2;t_3,t_4) \;=&\; -\int dt_a dt_b\;\Psi_0^{-1}(t_1,t_a)\,\Psi_{(1)}(t_a,t_b)\,\Psi_0^{-1}(t_b,t_3)\,\Psi_0^{-1}(t_2,t_4)\\ &-\int dt_a dt_b\;\Psi_0^{-1}(t_1,t_3)\,\Psi_0^{-1}(t_2,t_a)\,\Psi_{(1)}(t_a,t_b)\,\Psi_0^{-1}(t_b,t_4)\\ &+\,(q-1)(q-2)\,\delta(t_{13})\,\delta(t_{24})\,\Psi_0^{q-3}(t_1,t_2)\,\Psi_{(1)}(t_1,t_2)\,. \end{aligned} \tag{C.4}$$

It's useful to perform the t_a and t_b integrals first of all. For these integrals, we again use eq. (2.9). Then, now we can write the shift of the kernel as

$$\begin{aligned}\delta\mathcal{K}(t_1,t_2;t_3,t_4) \;=&\; -\frac{2\pi q(q-1)B_1 b^{3q-3}}{(q-2)\tan(\frac{\pi}{q})}\left[\frac{\mathrm{sgn}(t_{13})\,\mathrm{sgn}(t_{24})}{|t_{13}|^{3-\frac{2}{q}}\,|t_{24}|^{2-\frac{2}{q}}}\,+\,(t_{13}\leftrightarrow t_{24})\right]\\ &-B_1 b^{q-3}(q-1)(q-2)\delta(t_{13})\delta(t_{24})\,\frac{1}{|t_{12}|^{3-\frac{4}{q}}}\,. \end{aligned} \tag{C.5}$$

We denote the first line in the r.h.s. as $\delta\mathcal{K}_{(1)}$ and the second line as $\delta\mathcal{K}_{(2)}$. These are analog of what called “rail” and “rung” in [16], respectively. The contribution to δk from the second line is easily evaluated. Using the Fourier transform for the zero modes, we have

$$\delta k_{(2)}(t_a,t_b) \;=\; -B_1 b^{q-3}(q-1)(q-2)\int\frac{d\omega d\omega'}{(2\pi)^2}\,e^{i\omega t_a-i\omega' t_b}\int dt_1dt_2\,\frac{u^*_{0,\omega}(t_1,t_2)u_{0,\omega'}(t_1,t_2)}{|t_{12}|^{3-\frac{4}{q}}}\,. \tag{C.6}$$

The t_1,t_2 integrals are evaluated by changing the integral variables to $t_\pm=(t_1\pm t_2)/2$ as follows.

$$\begin{aligned}\int dt_1dt_2\,\frac{u^*_{0,\omega}(t_1,t_2)u_{0,\omega'}(t_1,t_2)}{|t_{12}|^{3-\frac{4}{q}}} \;&=\; \frac{\pi^2b^2}{q^2}\,|\omega|^3\,\delta(\omega-\omega')\int\frac{dt_-}{|t_-|^2}\;J_{\frac{3}{2}}(|\omega t_-|)\,J_{\frac{3}{2}}(|\omega t_-|)\\ &=\;\frac{\pi b^2}{q^2}\,\omega^4\,\delta(\omega-\omega')\,. \end{aligned} \tag{C.7}$$

Substituting this result into eq. (C.6), one finds

$$\delta k_{(2)}(t_a,t_b) \;=\; -\,\frac{B_1 b^{q-1}(q-1)(q-2)}{2q^2}\;\partial_a^2\,\partial_b^2\,\delta(t_{ab})\,. \tag{C.8}$$

The contribution from the first line is more involved, but can be evaluated (for example, see appendix.E of [16]). After evaluating the integrals, the total contribution is given by

$$\delta k(t_a,t_b) \;=\; \gamma\,B_1\,\partial_a^2\,\partial_b^2\,\delta(t_{ab})\,, \tag{C.9}$$

with

$$\gamma = -\frac{\tan(\frac{\pi}{q})}{12\pi b q}\left[\frac{2\pi(q-1)(q-2)}{q\sin(\frac{2\pi}{q})} - (q^2-6q+6)\right]. \tag{C.10}$$

This result is used in section 2.5 to fix the coefficient B_1 of Ψ_1 together with the consistency condition.

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