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On butterfly effect in higher derivative gravities

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ABSTRACT: We study butterfly effect in D -dimensional gravitational theories containing terms quadratic in Ricci scalar and Ricci tensor. One observes that due to higher order derivatives in the corresponding equations of motion there are two butterfly velocities. The velocities are determined by the dimension of operators whose sources are provided by the metric. The three dimensional TMG model is also studied where we get two butterfly velocities at generic point of the moduli space of parameters. At critical point two velocities coincide.

KEYWORDS: Black Holes, Conformal Field Theory, Gauge-gravity correspondence

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1 Introduction

In the context of gauge/gravity duality a thermal system may be described by a gravitational theory on an AdS black hole solution [1] which can be used to explore different aspects of a thermal system such as chaos. Indeed it was shown [2–5] that chaos in a thermal CFT may be described by shock waves near the horizon of an AdS black hole. More precisely, the black hole geometry may be perturbed by a small perturbation and due to the back-reaction the perturbation grows in time resulting to a geometry which is given by a shock wave propagating on the horizon of the black hole. In other words, holographically the propagation of the shock wave on the horizon would provide a description of butterfly effect in the dual field theory.

On the other hand in the field theory side the butterfly effect may be diagnosed by out-of-time order four-point function between pairs of local operators

$$\langle V_x(0)W_y(t)V_x(0)W_y(t) \rangle_\beta \tag{1.1}$$

where β indicates a thermal expectation value. In terms of this correlation function, the butterfly effect may be seen by a sudden decay after the *scrambling time*, t_* ,

$$\frac{\langle V_x(0)W_y(t)V_x(0)W_y(t) \rangle_\beta}{\langle V_x(0)V_x(0) \rangle_\beta \langle W_y(t)W_y(t) \rangle_\beta} \sim 1 - e^{\lambda_L \left(t - t_* - \frac{|x-y|}{v_B} \right)}, \tag{1.2}$$

where λ_L is the Lyapunov exponent and v_B is butterfly velocity. From gravity point of view, this four-point function can be holographically computed from the certain component of the back-reacted metric [2] and thus the butterfly velocity should be identified with the velocity of shock wave by which the perturbation spreads in the space. The Lyapunov exponent is given in terms of the Hawking temperature, $\lambda_L = \frac{2\pi}{\beta}$.

The aim of this paper is to further study butterfly effect in gravitational theories containing higher derivative terms. We note, however, that butterfly effect for Gauss-Bonnet action and an action containing quadratic terms have been partially studied in literature (see e.g. [5–7]). In the present paper we would like to extend these works in more details. In particular, we will show that for theories whose gravitational dual are

provided by the Einstein gravity modified by terms quadratic in Ricci scalar and Ricci tensor, one generally finds two butterfly velocities which are given by graviton excitations on the boundary. This is also the case for three dimensional TMG model.

Actually this is a generic feature of higher derivative gravity whose equations of motion are higher order differential equations. The precise number of butterfly velocities are given by the number of boundary conditions needed to fix the metric.

This may be understood as follows: indeed it was shown [8] that in any holographic CFTs whose gravitational description is provided by the Einstein gravity, the butterfly velocity is determined by the spin-2 operator of lowest twist that is the energy-momentum tensor of dual boundary theory. On the other hand, from holographic renormalization [9] it is known that the boundary value of the metric provides a source for the energy-momentum tensor. When the action contains higher derivative terms, generally the corresponding equations of motion are higher order differential equations. Therefore to fix the metric one needs more than one boundary value. This, in turn, indicates that boundary values of metric provide sources for more than one operator.

On the other hand since, generically, by tuning the parameters of the model one can make the dimensions of these extra operators as closed as that of energy-momentum tensor, their contributions could be as important as the energy-momentum tensor. Therefore for each of these operators one has a butterfly velocity that is given in terms of its dimension.

The paper is organized as follows: in the next section, we study shock wave in D -dimensional gravity where we find that at a generic point there are two butterfly velocities, while at the critical point two velocities coincide. In section three, we will redo the same computations for TMG model where, again, we get two butterfly velocities. In this case, we will also reproduce the resultant velocities from the dual 2D conformal field theory where one shows that the butterfly velocity is given in terms of dimension of operators dual to the perturbation of metric. The last section is devoted to conclusions.

2 Shock wave in higher derivative gravity

In this section, we would like to study butterfly effect in D -dimensional gravitational theories consisting of Einstein gravity modified by certain R -squared terms. The action we will be considering is

$$I = \frac{1}{\kappa} \int d^D x \sqrt{-g} \left[R + \frac{(D-1)(D-2)}{\ell_0^2} + \alpha_1 R^2 + \alpha_2 R^{\mu\nu} R_{\mu\nu} \right], \quad (2.1)$$

where α_1 and α_2 are free parameters, ℓ_0 is a length scale. This model, has been studied in the literature (see .e.g.[10–12]) where it was shown that at a generic point of moduli space of parameters, excitations above an AdS vacuum contain scalar ghost, massive and massless spin-2 gravitons. Nevertheless, it is possible to remove the scalar ghost by tuning parameters α_1 and α_2 [13, 14]. Moreover at *critical points*, the massive spin-2 becomes massless leading to a logarithmic mode [11].

The corresponding equations of motion are given by $\mathcal{E}_{\mu\nu} = 0$ with [15]

$$\begin{aligned} \mathcal{E}_{\mu\nu} = & R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{(D-1)(D-2)}{2\ell_0^2}g_{\mu\nu} + 2\alpha_1 \left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R + g_{\mu\nu}\square - \nabla_\mu\nabla_\nu \right) R \\ & + \alpha_2 \left[(g_{\mu\nu}\square - \nabla_\mu\nabla_\nu) R + \square \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \right) + 2 \left(R_{\mu\sigma\nu\rho} - \frac{1}{4}g_{\mu\nu}R_{\sigma\rho} \right) R^{\sigma\rho} \right]. \end{aligned} \quad (2.2)$$

For generic values of the parameters ℓ_0 , α_1 and α_2 the model has two distinct vacua such that $R_{\mu\nu} = \frac{D-1}{\ell^2}g_{\mu\nu}$, with ℓ being a root of the following equation [12]¹

$$\ell^2(\ell^2 - \ell_0^2) + \frac{(D-4)(D-1)}{D-2}(D\alpha_1 + \alpha_2)\ell_0^2 = 0. \quad (2.3)$$

Then it is straightforward to show that the above equations of motion admit asymptotically AdS black brane solution

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + \frac{r^2}{\ell^2}d\vec{x}^2, \quad f(r) = \frac{r^2}{\ell^2} \left(1 - \frac{r_h^{D-1}}{r^{D-1}} \right), \quad (2.4)$$

where r_h is the radius of horizon.

Now the aim is to study a shock wave solution of this model when the above black hole solution is perturbed by injecting a small amount of energy. To proceed, it is useful to re-write the black brane solution in the Kruskal coordinates

$$u = \exp\left[\frac{2\pi}{\beta}(r_* - t)\right], \quad v = -\exp\left[\frac{2\pi}{\beta}(r_* + t)\right], \quad (2.5)$$

where $\beta = 4\pi/f'(r_h)$ is the inverse of the temperature and $dr_* = dr/f(r)$ is the tortoise coordinate whose near horizon expression is

$$\begin{aligned} r_* = & \frac{\beta}{4\pi} \left[\log \frac{r - r_h}{2r_h} - \log c + \frac{D-4}{2r_h}(r - r_h) + \frac{(D-14)D + 36}{24r_h^2}(r - r_h)^2 \right. \\ & \left. - \frac{(D-6)(5D-16)}{72r_h^3}(r - r_h)^3 + \mathcal{O}((r - r_h)^4) \right], \end{aligned} \quad (2.6)$$

and c is a positive number to be fixed latter. By making use of this coordinate system, the metric can be recast into the following form

$$ds^2 = 2A(w)dudv + B(w)d\vec{x}^2. \quad (2.7)$$

Here $A(w)$ and $B(w)$ are two functions, implicitly, given by the component of the black brane metric f , whose near horizon expansions are

$$\begin{aligned} A(x) = & -\frac{4c\ell^2}{D-1} \left(1 + 2c(D-4)x + c^2(4D^2 - 29D + 54)x^2 \right. \\ & \left. + \frac{4}{9}c^3(18D^3 - 185D^2 + 646D - 768)x^3 + \dots \right), \\ B(x) = & \frac{r_h^2}{\ell^2} \left(1 - 4cx - 4c^2(D-5)x^2 - \frac{1}{3}4c^3(4D^2 - 35D + 78)x^3 + \dots \right). \end{aligned}$$

¹In four dimensions or for the case of $D\alpha_1 + \alpha_2 = 0$ the equation has a single solution $\ell^2 = \ell_0^2$. Note that for generic values of parameters, it is always possible to tune the parameters such that at least one of the vacua to be an AdS_D geometry.

Actually since we are going to study the back-reacted geometry near the horizon, the above expressions are sufficient to study the shock wave solution.

Now let us consider an injection of a small amount of energy from boundary towards the horizon at time $-t_w$. This will cross the $t = 0$ time slice while it is red shifted. Therefore the equations of motion should be deformed as

$$\mathcal{E}_{\mu\nu} = \kappa T_{\mu\nu}^S, \quad (2.8)$$

where the energy-momentum tensor associated with the energy injection which has only uu component is given by

$$T_{uu}^S = \ell E e^{2\pi t_w/\beta} \delta(u) \delta^{D-2}(\vec{x}). \quad (2.9)$$

Now the aim is to solve the equations of motion near the horizon to find the shock wave solution. To proceed, by making use of the step function $\Theta(x)$ one may consider the following ansatz for the back-reacted geometry

$$ds^2 = 2A(UV) dU dV + B(UV) dx^2 - 2A(UV) h(x) \delta(U) dU^2, \quad (2.10)$$

where the new coordinates U and V are defined by

$$U \equiv u, \quad V \equiv v + h(\vec{x})\Theta(u). \quad (2.11)$$

Here $h(x)$ is a function to be found by the equations of motion (2.8). Plugging the ansatz (2.10) into the above equations, near the horizon at the leading order one finds a fourth order differential equation for $h(x)$

$$\begin{aligned} & \left(\frac{\ell^2}{r_h^2} \partial_i \partial^i - \frac{(D-1)(4D\alpha_1 + (D+2)\alpha_2) - 2\ell^2}{2\alpha_2 \ell^2} \right) \left(\frac{\ell^2}{r_h^2} \partial_i \partial^i - \frac{(D-1)(D-2)}{2\ell^2} \right) h(x^i) \\ & = -\frac{(D-1)}{4\alpha_2} \frac{1}{c \ell^2} \left[\kappa \ell E e^{2\pi t_w/\beta} \right] \delta^{D-2}(x^i), \end{aligned} \quad (2.12)$$

which can be reduced into two second order differential equations as follows

$$\left\{ \begin{aligned} (\partial_i \partial^i - a_1^2) q(x^i) &= \eta \delta^{D-2}(x^i) \end{aligned} \right. \quad (2.13a)$$

$$\left\{ \begin{aligned} (\partial_i \partial^i - a_2^2) h(x^i) &= q(x^i), \end{aligned} \right. \quad (2.13b)$$

where

$$\begin{aligned} a_1^2 &= \frac{(D-1)(4D\alpha_1 + (D+2)\alpha_2) - 2\ell^2}{2\alpha_2 \ell^2} \frac{r_h^2}{\ell^2}, & a_2^2 &= \frac{(D-1)(D-2)}{2\ell^2} \frac{r_h^2}{\ell^2}, \\ \eta &= -\frac{(D-1)}{4\alpha_2} \frac{r_h^4}{c \ell^6} \left[\kappa \ell E e^{2\pi t_w/\beta} \right]. \end{aligned} \quad (2.14)$$

To simplify the computations, it is useful to use the symmetry of the background to study a shock wave which is a plane-wave propagating in $x = x_1$ direction. This can be done

by injecting energy along x , leading to the energy-momentum $T_{uu}^S = \ell E e^{2\pi t_w/\beta} \delta(u) \delta(x)$ so that the equation (2.13a) reduces to $(\partial_i \partial^i - a_1^2) q(x) = \eta \delta(x)$ whose solution is

$$q(x) = -\frac{\eta}{2a_1} e^{-a_1|x|}, \tag{2.15}$$

where $|x|$ denotes the absolute value of x . From the equation (2.13b), it is clear that $q(x)$ should be thought of as a source for the function $h(x)$. Moreover taking into account that the Green's function of the equation (2.13b) has the same form as that of $q(x)$ one finds

$$h(x) = \frac{\eta}{4a_1 a_2} \int_{-\infty}^{\infty} dy e^{-a_1|y| - a_2|x-y|}. \tag{2.16}$$

It is now easy to evaluate this integral to find $h(x)$. To proceed one assumes that $x > 0$ (we get the same result for $x < 0$) in which the above expression reads

$$h(x) = \frac{\eta}{4a_1 a_2} \left[e^{-a_2 x} \int_{-\infty}^0 dy e^{(a_1+a_2)y} + e^{-a_2 x} \int_0^x dy e^{-(a_1-a_2)y} + e^{a_2 x} \int_x^{\infty} dy e^{-(a_1+a_2)y} \right]. \tag{2.17}$$

So that

$$h(x) = \frac{\eta}{2a_1 a_2} \frac{a_1 e^{-a_2 x} - a_2 e^{-a_1 x}}{a_1^2 - a_2^2}. \tag{2.18}$$

Using the explicit expressions of η , a_1 and a_2 and for an appropriate choice of c , one gets

$$h(x) = \frac{\ell^2 \sqrt{1/2(D-1)(D-2)}}{(\ell^2 - 2(D-1)(D\alpha_1 + \alpha_2))} \left[v_B^{(1)} e^{\frac{2\pi}{\beta} [(t_w - t_*) - |x|/v_B^{(1)}]} - v_B^{(2)} e^{\frac{2\pi}{\beta} [(t_w - t_*) - |x|/v_B^{(2)}]} \right], \tag{2.19}$$

where the scrambling time is defined by $t_* = -\frac{\beta}{2\pi} \log \frac{\kappa}{\ell^{d-2}}$. From this expression one can read two different butterfly velocities as follows.²

$$v_B^{(1)} = \frac{2\pi}{\beta a_2} = \sqrt{\frac{D-1}{2(D-2)}},$$

$$v_B^{(2)} = \frac{2\pi}{\beta a_1} = \sqrt{\frac{D-1}{2(D-2)}} \sqrt{\frac{(D-1)(D-2)\alpha_2}{(D-1)(4D\alpha_1 + (D+2)\alpha_2) - 2\ell^2}}. \tag{2.20}$$

As we have already mentioned, the model under consideration given by the action (2.1) above its AdS vacuum has different propagating modes including massive and massless spin-2 modes. The mass of the massive graviton is also given by

$$M^2 = \frac{2(D-1)(D\alpha_1 + \alpha_2) - \ell^2}{\alpha_2 \ell^2}. \tag{2.21}$$

²As we already mentioned butterfly velocity for higher derivative gravity has been also studied in appendix B of [6]. There the authors have also realized that there are two butterfly velocities though they have only taken one of them. Of course their results are consistent with ours, though as we will argue one should take both velocities. This is also necessary to understand the situation at the critical point.

It is then interesting to re-write the second butterfly velocity in the equation (2.20) in terms of the mass M^2 ,

$$v_B^{(2)} = \sqrt{\frac{D-1}{2(D-2)}} \frac{1}{\sqrt{1 + \frac{2\ell^2}{(D-1)(D-2)}M^2}}. \quad (2.22)$$

It is worth mentioning that in the expression of $h(x)$ there are two terms with different signs though one would expect to get positive $h(x)$. Actually it can be seen that it is always positive for $\alpha_2 < 0$. Indeed this is the case for critical gravities studied on their AdS vacua [10–12].³

From (2.22) it is clear that at the critical point where the massive graviton degenerates with the massless graviton, $M^2 = 0$, two velocities coincide resulting to one butterfly velocity. In this case the model exhibits a logarithmic mode [11] and therefore the expression of metric perturbation $h(x)$ gets modified as follows

$$h(x) = \left(\frac{\ell^2 v_B (v_B + 2\pi|x|/\beta)}{-(D-1)^2(D-2)\alpha_2} \right) e^{\frac{2\pi}{\beta}[(t_w - t_*) - |x|/v_B]}, \quad (2.23)$$

where $v_B = v_B^{(1)}$ is the butterfly velocity at the critical point. Note that at the critical point one has $\alpha_2 = -\frac{2\ell^2}{(D-2)^2}$ [11], therefore the above equation reads⁴

$$h(x) = \left(\frac{(D-2)v_B(v_B + 2\pi|x|/\beta)}{2(D-1)^2} \right) e^{\frac{2\pi}{\beta}[(t_w - t_*) - |x|/v_B]}. \quad (2.24)$$

These results may be understood as follows. Actually in the context of AdS/CFT correspondence there is a correspondence between bulk fields and boundary operators in the sense that the boundary value of the bulk field provides a source for the boundary operator. In particular the energy-momentum tensor in the boundary theory is sourced by the metric. We note, however, that when the equations of motion of a bulk field contains higher order derivatives, the corresponding field could provide sources for different operators on the boundary.

In the present case where we are dealing with higher derivative gravity the equations of motion of the metric are fourth order and the metric provides two sources for two operators corresponding to massless and massive gravitons. Each operators results to a butterfly velocity which is given in terms of its dimension. In other words, when we are perturbing the bulk metric by injecting energy, the boundary values of the metric get changed that would excite both operators on the boundary. This is the reason that we get two butterfly velocities in this case. Of course at the critical point where both operators have the same dimension we get one butterfly velocity. We will make this point more precise in the next section where we are considering the three dimensional TMG theory.

³Actually for $M^2 > 0$ one has $v_B^{(1)} > v_B^{(2)}$ and therefore the term with positive sign dominates, though for $M^2 < 0$ where $v_B^{(1)} < v_B^{(2)}$ the term with negative sign dominates, but in this case the overall factor is negative as well. Therefore altogether we get positive $h(x)$. We would like to thank the referee for his/her comment on this point.

⁴Actually to remove the scalar ghost from the spectrum one should set $4(D-1)\alpha_1 + D\alpha_2 = 0$ [10–12] which together with the condition $M = 0$ can fix the parameters α_1 and α_2 at the critical point.

To explore the role of boundary excitations, it is illustrating to study $D \geq 5$ dimensional gravities modified by Gauss-Bonnet terms. These models have been studied in [5] and in what follows we use the results of this paper and study butterfly velocity at a distinguished point. Let us consider the following action

$$I = \frac{1}{\kappa} \int d^D x \sqrt{-g} \left[R + \frac{(D-1)(D-2)}{\ell_0^2} + \gamma (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2) \right]. \quad (2.25)$$

The butterfly velocity for this model is [5]

$$v_B = \sqrt{\frac{1 + \sqrt{1 - 4\lambda_{\text{GB}}}}{2}} \sqrt{\frac{D-1}{2(D-2)}}, \quad (2.26)$$

where

$$\lambda_{\text{GB}} = \frac{(D-3)(D-4)}{\ell_0^2} \gamma. \quad (2.27)$$

Note that although the action contains higher derivative terms, the resultant equations of motion are still second order and therefore we find only one butterfly velocity. We note, however, that although the equations are second order, the model in general could have two AdS vacuum solutions whose radius are given by the following algebraic equation

$$\ell^4 - \ell^2 \ell_0^2 + \lambda_{\text{GB}} \ell_0^4 = 0. \quad (2.28)$$

Interestingly enough, when the above equation degenerates the model does not have local propagating graviton. This occurs at $4\lambda_{\text{GB}} = 1$ [16]. In this case the butterfly velocity reads

$$v_B = \sqrt{\frac{1}{2}} \sqrt{\frac{D-1}{2(D-2)}}. \quad (2.29)$$

It is important to note that in this case although the model does not have propagating gravitons, due to boundary gravitons the butterfly velocity is non-zero. This is indeed very similar to what happens in three dimensional Einstein gravity.

3 Shock waves in 3D TMG model

In the previous section, we have studied butterfly effete in D -dimensional massive gravities that also includes $D = 3$ where we get the New Massive Gravity (NMG) [17]. In this section we would like to study butterfly effect for yet another interesting three dimensional gravity; Topologically Massive Gravity (TMG) [18].

The TMG model is a three dimensional gravity whose action contains the Einstein-Hilbert action and the three dimensional gravitational Chern-Simons term

$$S_{\text{TMG}} = \frac{1}{16\pi G_N} \left[\int d^3 x (R - 2\Lambda) + S_{\text{CS}} \right], \quad (3.1)$$

with

$$S_{\text{CS}} = \frac{1}{4\mu} \int d^3 x \epsilon^{\mu\nu\rho} \left[R_{ab\ \mu\nu} \omega_{,\rho}^{ab} + \frac{2}{3} \omega_{b,\mu}^a \omega_{c,\nu}^b \omega_{a,\rho}^c \right], \quad (3.2)$$

where $\omega^a_{b,\mu}$ is the spin connection whose inner Lorentz indices are denoted by a, b, \dots while the space-time indices are denoted by μ, ν, \dots .

For a generic value of μ this model admits an AdS vacuum solution. It is conjectured that the TMG model on an asymptotically locally AdS solution with a proper boundary condition would provide a gravitational dual for a two dimensional CFT with the following central charges

$$c_L = \frac{3\ell}{2G_N} \left(1 - \frac{1}{\mu\ell}\right), \quad c_R = \frac{3\ell}{2G_N} \left(1 + \frac{1}{\mu\ell}\right). \quad (3.3)$$

The model has a critical point at $\mu\ell = 1$ where the left central charge vanishes and the equations of motion degenerate leading to a log-gravity whose dual theory is a LCFT [19] (see also [20, 21]).

The equations of motion obtained from the above action are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \quad (3.4)$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad C_{\mu\nu} = \epsilon_{\mu}^{\alpha\beta} \nabla_{\alpha} \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right), \quad (3.5)$$

are Einstein and the Cotton tensors, respectively. AdS_3 black brane is a solution of the equations of motion,

$$ds^2 = -\frac{r^2 - r_h^2}{\ell^2} dt^2 + \frac{\ell^2}{r^2 - r_h^2} dr^2 + \frac{r^2}{\ell^2} dx^2, \quad \Lambda = -\frac{1}{\ell^2}. \quad (3.6)$$

In order to study the shock wave solution in the AdS_3 black brane background, one may go through the same procedure considered in the previous section.⁵ To do so, one should write the above black brane metric in the Kruskal coordinates,

$$ds^2 = 2A(uv) du dv + B(uv) dx^2, \quad (3.7)$$

where

$$A(uv) = -\frac{2c\ell^2}{(1 + cuv)^2}, \quad B(uv) = \frac{r_h^2}{\ell^2} \left(\frac{1 - cuv}{1 + cuv} \right)^2. \quad (3.8)$$

Moreover in this case the tortoise coordinate r_* is

$$r_*(r) = \ell^2 \int \frac{dr}{r^2 - r_h^2} = \frac{\beta}{4\pi} \left[\log \left(\frac{r - r_h}{r + r_h} \right) - \log c \right], \quad (3.9)$$

where c is an arbitrary constant. Now, the aim is to study the back-reaction on the metric (3.7) when we inject a small amount of energy towards the horizon so that the equations of motion should be modified as follows

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = \kappa T_{\mu\nu}^S, \quad (3.10)$$

⁵The shock wave solution in Mankowski space background for TMG (and NMG) is studied in [22].

with $T_{uu}^S = \ell E e^{2\pi t_w/\beta} \delta(u) \delta(x)$. Following the procedure presented in the previous section, we will consider the following ansatz for the back-reacted metric

$$ds^2 = 2A(UV) dU dV + B(UV) dx^2 - 2A(UV) h(x) \delta(U) dU^2, \quad (3.11)$$

where $U = u$, $V = v + h(x)\Theta(u)$. Plugging this ansatz into the modified equations of motion, one arrives at

$$\left(\partial_x + \mu \frac{r_h}{\ell}\right) \left(\partial_x^2 - \frac{r_h^2}{\ell^2}\right) h(x) = -\frac{r_h^3 \mu}{2c \ell^5} \left[\kappa \ell E e^{2\pi t_w/\beta}\right] \delta(x), \quad (3.12)$$

which can be decomposed into two differential equations as follows

$$\begin{cases} q'(x) + a_1 q(x) = \eta \delta(x) & (3.13a) \\ h''(x) - a_2^2 h(x) = q(x), & (3.13b) \end{cases}$$

where

$$a_1 = \frac{r_h \mu}{\ell}, \quad a_2 = \frac{r_h}{\ell^2}, \quad \eta = -\frac{r_h^3 \mu}{2c \ell^5} \left[\kappa \ell E e^{2\pi t_w/\beta}\right]. \quad (3.14)$$

The equation (3.13a) is indeed Green's function equation whose solution for $x > 0$ that falls off at infinity is⁶

$$q(x) = \eta \Theta(x) e^{-a_1 x}. \quad (3.15)$$

Treating the function $q(x)$ as a source for function $h(x)$ and taking into account that the Green's function of the equation (3.13b) is given by (3.15) one arrives at

$$h(x) = -\frac{\eta}{2a_2} \int_{-\infty}^{\infty} dy \Theta(y) e^{-a_1 y - a_2 |x-y|}. \quad (3.16)$$

It is then straightforward to perform the integral. Indeed for $a_1 \neq a_2$ one gets

$$\begin{aligned} h(x) &= -\frac{\eta}{2a_2} \left[e^{-a_2 x} \int_0^x dy e^{-(a_1-a_2)y} + e^{a_2 x} \int_x^{\infty} dy e^{-(a_1+a_2)y} \right] \\ &= -\frac{\eta}{2a_2} \left[\frac{e^{-a_2 x}}{a_1 - a_2} - \frac{2a_2 e^{-a_1 x}}{a_1^2 - a_2^2} \right], \end{aligned} \quad (3.17)$$

while at the special case of $a_1 = a_2$, which corresponds to the critical point of the model, one gets

$$h(x) = -\frac{\eta}{2a_2} \left(x + \frac{1}{2a_2}\right) e^{-a_2 x}, \quad (3.18)$$

indicating that the logarithmic mode appears in the spectrum of the model.

Using the explicit expressions for the parameters η, a_1, a_2 and with a proper choice of c , one can read the scrambling time and butterfly velocities as follows

$$t_* = -\frac{\beta}{2\pi} \log \frac{\kappa}{\ell}, \quad v_B^{(1)} = \frac{2\pi}{\beta a_2} = 1, \quad \hat{v}_B^{(2)} = \frac{2\pi}{\beta a_1} = \frac{1}{\mu \ell}. \quad (3.19)$$

⁶There could be an extra constant in the solution, though it does not change the results and therefore we have set it to zero.

One observes that due to higher derivative terms in the equations of motion, there are two butterfly velocities for left moving sector. On the other hand, at the critical point where $a_1 = a_2$ the dimensions of both operators become the same resulting to one butterfly velocity, $v_B^{(1)} = 1$.

As we have already mentioned in the previous section, the butterfly velocities are given by the dimension of operators sourced by metric. To explore this point better let us consider butterfly effect from the dual 2D CFT.

To proceed, we recall that to diagnose quantum chaos it is useful to study out-of-time order four-point correlation function between pairs of local operators

$$\langle W(t)VW(t)V \rangle_\beta, \tag{3.20}$$

which should be thought of as averaging in the thermal state $|\beta\rangle$. In the present case in order to compute this correlation function one may take advantage of 2D CFT to map the above four-point correlation function to a four-point function in a vacuum state. More precisely, using the transformation

$$z(x, t) = e^{\frac{2\pi}{\beta}(x+t)}, \quad \bar{z}(x, t) = e^{\frac{2\pi}{\beta}(x-t)}, \tag{3.21}$$

one leads to compute the four-point function $\langle W(z_1, \bar{z}_1)V(z_2, \bar{z}_2)W(z_3, \bar{z}_3)V(z_4, \bar{z}_4) \rangle_{vac}$. Actually by making use of 2D conformal symmetry one has

$$\frac{\langle W(z_1, \bar{z}_1)V(z_2, \bar{z}_2)W(z_3, \bar{z}_3)V(z_4, \bar{z}_4) \rangle}{\langle W(z_1, \bar{z}_1)W(z_3, \bar{z}_3) \rangle \langle V(z_2, \bar{z}_2)V(z_4, \bar{z}_4) \rangle} = f(z, \bar{z}), \tag{3.22}$$

where $f(z, \bar{z})$ is an arbitrary function and

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}. \tag{3.23}$$

There is a well-known procedure to compute the function $f(z, \bar{z})$ (see e.g. [23]). In fact in our case the result is [8]

$$f(z, \bar{z}) = 2\pi i \sum_{\mathcal{O}(\Delta, s)} \alpha_{\mathcal{O}}^2 \frac{\Gamma(\Delta + s)\Gamma(\Delta + s - 1)}{\Gamma^4(\frac{\Delta+s}{2})} z^{1-s} \eta^{\frac{\Delta-s}{2}}, \tag{3.24}$$

where the sum runs over the conformal primary operators \mathcal{O} whose dimension and spin are given by Δ and s , respectively and, $\eta = \frac{\bar{z}}{z}$. Moreover, $\alpha_{\mathcal{O}}^2 = \alpha_{WW\mathcal{O}}\alpha_{VVO}$, with e.g. $\alpha_{WW\mathcal{O}}$ is the OPE coefficient in WW operator product. By making use of the definition of cross ratios (3.23) and with the desired time-ordering that fixes the expressions of z and \bar{z} [8], one finds

$$f(z, \bar{z}) \approx 2\pi i \sum_{\mathcal{O}(\Delta, s)} \alpha_{\mathcal{O}}^2 \frac{\Gamma(\Delta + s)\Gamma(\Delta + s - 1)}{\Gamma^4(\frac{\Delta+s}{2})(-\epsilon_{12}^*\epsilon_{34})^{s-1}} e^{\frac{2\pi}{\beta}(s-1)[t - \frac{\Delta-1}{s-1}x]},$$

where $\epsilon_{ij} = i(e^{\frac{2\pi}{\beta}i\epsilon_i} - e^{\frac{2\pi}{\beta}i\epsilon_j})$ with $i\epsilon_i$ being an infinitesimal Euclidean time associated to each of four operators.⁷ Using this expression, one can read the Lyapunov exponent and

⁷Note that in order to get the right thermal averaging four point correlation one should choose $\epsilon_1 < \epsilon_2 < \epsilon_3 < \epsilon_4$.

butterfly velocity as follows (see also [24])

$$\lambda_L = \frac{2\pi}{\beta}(s-1), \quad v_B(\Delta, s) = \frac{s-1}{\Delta-1}. \quad (3.25)$$

It was shown in [8] that for a CFT whose gravitational dual is provided by Einstein gravity the main contribution to $f(z, \bar{z})$ comes from spin-2 operator of the lowest twist that is energy-momentum tensor. On the other hand since in the present case where the metric sources two operators whose dimensions can be taken as close as we want by tuning the parameters of the model, one would expect that the operator which is dual to the massive spin-2 mode should also contribute to the four-point function. Therefore we arrive at two butterfly velocities which are given in terms of spin and dimension of operators as (3.25).

Let us now apply this result to our cases. For TMG model where the spectrum contains massive and massless gravitons one gets two spin-2 operators with dimensions $\Delta^{(1)} = 2$ and $\Delta^{(2)} = 1 + \sqrt{1 + \ell^2 M^2}$ with $M^2 = \mu^2 \ell^2 - 1$. Plugging these expressions into the equation (3.25) one arrives at

$$v_B(\Delta^{(1)}, 2) = 1, \quad v_B(\Delta^{(2)}, 2) = \frac{1}{\mu\ell}, \quad (3.26)$$

in agreement with (3.19). On the other hand for the NMG model one gets

$$v_B(\Delta^{(1)}, 2) = 1, \quad v_B(\Delta^{(2)}, 2) = \frac{1}{\sqrt{1 + \ell^2 M^2}}, \quad (3.27)$$

which is the same as that we have found in the previous section for $D = 3$.

We have also seen that at the critical point where the massive spin-2 degenerate with the massless graviton leading to the log-gravity, two butterfly velocities coincide. It is also illustrative to see this effect from proper conformal block decomposition approach. Actually in this case, the conformal block decomposition (3.24) should be substituted with [25]⁸

$$f(z, \bar{z}) = 2\pi i \left(\alpha_{T\tau}^2 + \alpha_{\tau T}^2 + \alpha_{\tau\tau}^2 \frac{\partial}{\partial \Delta} \right) \frac{\Gamma(\Delta+2)\Gamma(\Delta+1)}{\Gamma^4(\frac{\Delta+2}{2})} z^{-1} \eta^{\frac{\Delta-2}{2}} \Big|_{\Delta \rightarrow 2}, \quad (3.28)$$

where $T_{\mu\nu}$ and $\tau_{\mu\nu}$ are the energy-momentum tensor and its logarithmic counterpart, respectively. By making use of the proper cross ratios one arrives at

$$f(t, x) \approx 2\pi i \frac{4(3\alpha_{T\tau}^2 + 3\alpha_{\tau T}^2 + 4\alpha_{\tau\tau}^2)}{(-\epsilon_{12}^* \epsilon_{34})} \left[1 - \frac{3\alpha_{\tau\tau}^2}{(3\alpha_{T\tau}^2 + 3\alpha_{\tau T}^2 + 4\alpha_{\tau\tau}^2)} 2\pi x / \beta \right] e^{\frac{2\pi}{\beta}(t-x)}, \quad (3.29)$$

which has the same structure as that of (2.23) and (3.18).⁹ More precisely to reproduce these equations one should set

$$\begin{aligned} \text{NMG :} \quad & \alpha_{T\tau}^2 + \alpha_{\tau T}^2 = -\frac{7}{3}\alpha_{\tau\tau}^2 \\ \text{TMG :} \quad & \alpha_{T\tau}^2 + \alpha_{\tau T}^2 = -\frac{11}{6}\alpha_{\tau\tau}^2. \end{aligned} \quad (3.30)$$

Note also that we have one butterfly velocity $v_B = 1$ as expected.

⁸Note that here one has $\alpha_{ab}^2 = \alpha_{W_a} \alpha_{V_b}$ [25].

⁹The logarithmic shock wave solution just for left moving sector in TMG might be understood from the non-parity invariant structure of OPE coefficients in this theory.

4 Conclusions

In this paper, we have studied butterfly effect in D -dimensional gravitational theory containing higher order derivatives. The higher order terms consist of Ricci scalar and Ricci tensor squared. For generic values of the parameters of the model we have found two butterfly velocities, though at the critical points where the equations of motion degenerate these two velocities coincide. The observation of our paper may be explored as follows.

From holographic renormalization [9] in the context of gauge/gravity duality we know that the boundary value of a bulk field (non-normalizable mode) should be identified with the source of the dual operator whose dimension and spin are fixed by the mass and the spin of the bulk field. In particular, for Einstein gravity the metric is dual to the energy-momentum tensor of the boundary theory.

Going to higher derivative gravities, typically the corresponding equations of motion consist of higher order differential equations so that the metric may be fixed by given several boundary conditions. The boundary condition (if corresponds to non-normalizable mode) might be identified with sources of dual operators all of which have spin-2, though their dimensions would be different.

In particular for the models we have considered in this paper, the excitation of the metric contains massive and massless gravitons so that the dual theory should have two spin-2 operators. When we are perturbing the bulk geometry, the boundary values of metric would also exciting the corresponding boundary operators. To each spin-2 operators, one may associate a butterfly velocity which is determined by the dimension of the corresponding operator.¹⁰

Actually this observation should be thought of as a generalization of the results presented in that [8] where it was shown that in any holographic CFT whose gravitational dual is provided by Einstein gravity the butterfly velocity is determined by the energy-momentum tensor (see also [24]).

To explore the role of the boundary value of the metric, we have also studied butterfly effect in D -dimensional gravitational theories corrected by Gauss-Bonnet term. In this case since the equations of motion are still second order one gets one butterfly velocity. There is, however, a point in the moduli space of the parameters of the model where the model does not have propagating gravitons on the bulk, though it still has boundary gravitons. In this case we still have non-zero butterfly velocity showing the importance of the boundary modes. Actually the situation is very similar to that of three dimensional gravity and indeed it can be seen that at this point the action reduces to five dimensional gravitational Chern-Simons action.

It is also interesting to compute butterfly velocity for a gravitational theory whose spectrum contains only a massive graviton (no massless graviton). Let us consider the following particular model (see [26])

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} \left(R + \frac{6}{L^2} - \alpha^2 [(\text{Tr } \mathcal{K})^2 - \text{Tr } \mathcal{K}^2] \right) - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} \right], \quad (4.1)$$

¹⁰Note that scalar field or vector field cannot lead to butterfly velocity.

where $\mathcal{K}^\mu{}_\nu = \sqrt{g^{\mu\lambda}f_{\lambda\nu}}$. It is straightforward to write the equations of motion of the above action. Then setting $f_{\mu\nu} = \text{diag}(0, 0, 1, 1)$ and for non-zero component of gauge field $A_0 = a(r)$, one finds the following black hole solution [27, 28],

$$ds^2 = \frac{L^2}{r^2} \left(-b(r) dt^2 + \frac{1}{b(r)} dr^2 + dx^2 + dy^2 \right), \quad A_t = \mu \left(1 - \frac{r}{r_h} \right), \quad (4.2)$$

where $b(r) = 1 - \alpha^2 r^2 - Mr^3 + \frac{\mu^2 r^4}{\gamma^2 r_h^2}$, with $\gamma^2 = \frac{2e^2 L^2}{\kappa^2}$. Going through the procedure we presented in the previous sections one can find the butterfly velocity as follows

$$v_B = \frac{1}{2} \sqrt{3 - \alpha^2 r_h^2 - \frac{\mu^2 r_h^2}{\gamma^2}}. \quad (4.3)$$

This should be compared with that of Einstein gravity which is given by setting $\alpha = 0$.

Of course in this paper we have just considered cases where the bulk equations of motion are at most fourth order and therefore we have obtained two butterfly velocities. Going beyond fourth order we may get more velocities.

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