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Newton-Cartan supergravity with torsion and Schrödinger supergravity

Eric Bergshoeff,[a] **Jan Rosseel**[b,c] **and Thomas Zojer**[a]

[a] *Van Swinderen Institute for Particle Physics and Gravity, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands*
[b] *Institute for Theoretical Physics, Vienna University of Technology, Wiedner Hauptstr. 8-10/136, A-1040 Vienna, Austria*
[c] *Albert Einstein Center for Fundamental Physics, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland*

E-mail: e.a.bergshoeff@rug.nl, rosseelj@hep.itp.tuwien.ac.at, t.zojer@rug.nl

Abstract: We derive a torsionfull version of three-dimensional $\mathcal{N} = 2$ Newton-Cartan supergravity using a non-relativistic notion of the superconformal tensor calculus. The "superconformal" theory that we start with is Schrödinger supergravity which we obtain by gauging the Schrödinger superalgebra. We present two non-relativistic $\mathcal{N} = 2$ matter multiplets that can be used as compensators in the superconformal calculus. They lead to two different off-shell formulations which, in analogy with the relativistic case, we call "old minimal" and "new minimal" Newton-Cartan supergravity. We find similarities but also point out some differences with respect to the relativistic case.

Keywords: Gauge Symmetry, Supergravity Models, Holography and condensed matter physics (AdS/CMT), Classical Theories of Gravity

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1 Introduction

Recent applications in condensed matter physics and gauge-gravity duality have led to a renewed interest in the question of how to consistently couple non-relativistic field theories to arbitrary non-relativistic space-time backgrounds. As in the relativistic case, a consistent coupling of a field theory to arbitrary geometric background data allows one to covariantly define currents such as the energy-momentum tensor and to study linear response. This geometric approach has been used in condensed matter physics recently, as a means to construct effective field theories that capture universal properties of the fractional quantum Hall effect [1–4]. It also plays a prominent role in recent applications of gauge-gravity duality to condensed matter physics, such as Lifshitz and Schrödinger holography [5–7]. Here, one views non-relativistic conformal field theories as living on the boundary of a higher-dimensional space-time with non-relativistic isometries, that is a vacuum solution of a suitable dual gravitational theory. The partition function of the field theory can then be calculated holographically as the partition function of the dual gravitational theory, in which all fields are subject to well-prescribed fall-off conditions towards the boundary. The asymptotic values of the fields of the gravitational dual correspond to sources for operators in the conformal field theory and play the role of arbitrary geometric background data to which the field theory couples.

In both condensed matter and gauge-gravity duality applications, it has been argued that the correct geometric framework to specify the background data is given by Newton-Cartan geometry with torsion [4, 8–18]. Newton-Cartan geometry was first introduced in the context of Newton-Cartan gravity [19–21], as the differential geometry necessary to cast Newtonian gravity in a covariant form akin to General Relativity. Even though Newton-Cartan geometry was originally formulated in a metric-like fashion, recent advances and applications have focused more on an equivalent vielbein formulation, in both torsionless and torsionfull cases. In this vielbein formulation one introduces temporal and spatial vielbeins that transform under local spatial rotations and Galilean boosts, as well as spin connections for spatial rotations and Galilean boosts. Crucially, one also includes an extra gauge field that is associated to particle number conservation. In the torsionless case, the vielbein formulation of Newton-Cartan geometry can be constructed by gauging the Bargmann algebra, i.e. the central extension of the Galilei algebra [22, 23], where the central charge corresponds to particle number. Similarly, it is possible to obtain particular torsionfull Newton-Cartan geometries by gauging the conformal extension of the Bargmann algebra, namely the Schrödinger algebra [14].

An interesting question is whether Newton-Cartan geometry and Newton-Cartan gravity can be made compatible with supersymmetry, i.e. whether one can construct Newton-Cartan supergravity theories. Such theories can be relevant for the construction of supersymmetric non-relativistic field theories, coupled to arbitrary backgrounds, that could e.g. be used as toy models to study exact results in non-relativistic quantum field theory. Relatedly, one might use Newton-Cartan supergravity theories to see whether localization techniques, that have proved useful to obtain exact results for relativistic supersymmetric theories on curved backgrounds [24–26], can be extended to non-relativistic theories.

The first example of a Newton-Cartan supergravity theory was obtained in [23] and corresponds to three-dimensional, $\mathcal{N}=2$, on-shell, pure Newton-Cartan supergravity with zero torsion. The independent gauge fields of this theory are given by[1]

$$\text{non-relativistic on-shell}: \quad \left(\tau_\mu, e_\mu{}^a, m_\mu, \psi_{\mu\pm}\right). \tag{1.1}$$

Initially, this theory was constructed via a gauging of the $d=3$, $\mathcal{N}=2$ Bargmann superalgebra; it was recently revisited in [27], where it was re-obtained from relativistic $d=3$, $\mathcal{N}=2$ supergravity via a procedure that corresponds to properly taking the non-relativistic limit while keeping an arbitrary frame formulation. This limiting procedure was then subsequently used to obtain an *off-shell*, pure $d=3$, $\mathcal{N}=2$ Newton-Cartan supergravity theory. Even though these examples show that Newton-Cartan geometry and gravity can be appropriately supersymmetrized, for practical purposes it is desirable to construct more elaborate examples than the pure, torsionless supergravities just mentioned. In particular, in view of the above mentioned condensed matter and gauge-gravity duality applications

[1]We use the same notation and conventions as in [23].

one would like to obtain Newton-Cartan supergravity theories that include non-trivial torsion as well as matter couplings. Such theories can generically not be obtained by applying the simple gauging procedure that led to the on-shell theory of [23], as not all fields will correspond to gauge fields of an underlying superalgebra. Since taking a proper and consistent non-relativistic limit can be rather cumbersome, new techniques are thus required to obtain such torsionfull and/or matter-coupled Newton-Cartan supergravity theories.

A very useful way to construct relativistic supergravity theories is offered by the superconformal tensor calculus (see [28] for an introduction and references). In relativistic superconformal tensor calculus, one obtains Poincaré supergravity theories by starting from a gauge theory of the superconformal algebra. In particular, one starts from a so-called 'Weyl multiplet', that realizes the superconformal algebra and contains its gauge fields (either as independent or as dependent ones). In a next step, one couples the Weyl multiplet to a 'compensator multiplet', whose role is to gauge fix the superconformal symmetries that are not part of the Poincaré superalgebra. As a concrete example, we remind how this procedure is applied to obtain $d = 4$, $\mathcal{N} = 1$ 'old minimal' supergravity. In this case, the $d = 4$, $\mathcal{N} = 1$ Weyl multiplet contains the vielbein $E_\mu{}^A$, gravitino Ψ_μ, R-symmetry gauge field A_μ and dilatation gauge field b_μ as independent fields. One can gauge fix the special conformal transformations by putting b_μ to zero. As a compensator multiplet, one takes a chiral multiplet that comprises two complex scalars Φ and F and a spinor χ. To derive a Poincaré multiplet from the Weyl multiplet one gauge fixes dilatations, R-symmetry and conformal S-supersymmetry. As gauge fixing conditions, one can choose:

$$\begin{aligned} \Phi = 1 \ : & \qquad \text{fixes dilatations and } R\text{-symmetry}\,, \\ \chi = 0 \ : & \qquad \text{fixes conformal } S\text{-supersymmetry}\,. \end{aligned} \tag{1.2}$$

In this way, one obtains the old minimal Poincaré multiplet which comprises $(E_\mu{}^A, \Psi_\mu, A_\mu, F)$. Alternatively, one may also use a tensor multiplet $(\phi, \lambda, B_{\mu\nu})$ as a compensator multiplet where ϕ is a real scalar, λ a spinor and $B_{\mu\nu}$ a 2-form gauge field. Imposing the gauge fixing conditions

$$\begin{aligned} \phi = 1 \ : & \qquad \text{fixes dilatations}\,, \\ \lambda = 0 \ : & \qquad \text{fixes conformal } S\text{-supersymmetry}\,, \end{aligned} \tag{1.3}$$

one then obtains the new minimal Poincaré multiplet with the fields $(E_\mu{}^A, \Psi_\mu, A_\mu, B_{\mu\nu})$. This theory still enjoys a local U(1)-symmetry.

In this paper, we will show that superconformal techniques can also be used to construct non-relativistic Newton-Cartan supergravity theories. We will in particular use a non-relativistic analogue of the superconformal tensor calculus to construct off-shell formulations of $d = 3$, $\mathcal{N} = 2$ pure Newton-Cartan supergravity. The non-relativistic superconformal algebra we will start from is the Schrödinger superalgebra. This algebra contains the Bargmann superalgebra as a subalgebra (hence our interest in it) and extends it with a dilatation generator, a single special conformal generator, an extra bosonic R-symmetry generator and a single fermionic S-supersymmetry generator. We will then construct a non-relativistic Schrödinger supergravity multiplet[2] that realizes the Schrödinger superalgebra and contains its gauge fields. The independent fields of Schrödinger supergravity are a temporal vielbein τ_μ, a spatial vielbein $e_\mu{}^a$, a central charge gauge field m_μ, a R-symmetry gauge field r_μ and two gravitini $\psi_{\mu\pm}$. The Schrödinger supergravity multiplet also contains an extra independent field b, that corresponds to the time-like component of the dilatation gauge field and that can be put to zero by gauge fixing the special conformal transformation.

In a next step, we will couple the Schrödinger supergravity multiplet to a compensator multiplet, that as in the relativistic case can be used to gauge fix superfluous superconformal symmetries. We will consider two different choices of compensator multiplet. The first choice is given by a non-relativistic $d = 3, \mathcal{N} = 2$ scalar multiplet and this will lead to a non-relativistic analog of old minimal supergravity with independent fields (see subsection 4.1)

$$\text{non-relativistic old minimal :} \qquad \big(\tau_\mu, e_\mu{}^a, m_\mu, r_\mu, \psi_{\mu\pm}, \chi_-, F_1, F_2\big)\,. \tag{1.4}$$

The second compensator multiplet we will consider consists of a scalar ϕ, a spinor λ and an extra bosonic field S, that transforms non-trivially under Galilean boosts. It can be obtained as a truncation of the non-relativistic limit of a vector multiplet. The fields ϕ and λ can then be used to gauge fix dilatations and S-supersymmetry, so that one ends up with a non-relativistic analogue of new minimal supergravity whose independent fields are given by (see subsection 4.2)

$$\text{non-relativistic new minimal :} \qquad \big(\tau_\mu, e_\mu{}^a, m_\mu, r_\mu, \psi_{\mu\pm}, S\big)\,. \tag{1.5}$$

As was shown in [14], the gauging of the Schrödinger algebra naturally leads to Newton-Cartan geometry with torsion. The torsion is provided by the spatial components of the dilatation gauge field, that are dependent on the other fields. This feature remains in the construction of the Schrödinger supergravity multiplet and our non-relativistic superconformal tensor calculus therefore naturally leads to *torsionfull* Newton-Cartan supergravity theories. In this way, we are thus able to extend the constructions of [23, 27] to the torsionfull case. The torsionless case can be retrieved by putting the torsion to zero. As the torsion is provided by gauge field components that depend on the other fields in the supergravity multiplet, this truncation is non-trivial and its consistency has to be examined. We will study this truncation in the case of non-relativistic new minimal supergravity and we will show that this truncation leads to the off-shell $d = 3$, $\mathcal{N} = 2$ theory of [27].

The organization of this paper is as follows. In section 2, we discuss the gauging of a suitable Schrödinger superalgebra and the ensuing construction of the $d = 3$, $\mathcal{N} = 2$ Schrödinger supergravity theory. Section 3 is devoted to a discussion of the matter multiplets that we will consider as compensator multiplets. We will show how these multiplets can be obtained as non-relativistic limits of a relativistic scalar and vector multiplet and

[2] We prefer to reserve the name non-relativistic "conformal" supergravity multiplet for the multiplet that realizes the gauging of the Galilean Conformal Superalgebra [29–31]. The reason for this is that the Schrödinger superalgebra, with only a single special conformal generator, allows a mass parameter while the Galilean Conformal Superalgebra does not. We thank Jerzy Lukierski for a discussion on this point.

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how they can be coupled to the Weyl multiplet. The construction of torsionfull old minimal and new minimal Newton-Cartan supergravity will be performed in section 4, whereas section 5 will be devoted to the truncation to the torsionless case. Finally, we conclude and give an outlook on future work in section 6.

2 Schrödinger supergravity

In this section we discuss the gauging of superconformal extensions of the Bargmann algebra, the so-called Schrödinger superalgebras. This is done in several steps. First, in section 2.1 we write down the transformation rules of all gauge fields, as determined by the algebra. Then we solve for some of the gauge fields in terms of others, using so-called conventional curvature constraints. The full set of curvature constraints is discussed in detail in subsection 2.2. Once the dependent gauge fields are expressed in terms of independent ones, their transformation rules do not necessarily coincide with those given by the structure constants of the algebra. The final transformation rules of the dependent gauge fields thus need to be re-evaluated and this is done in subsection 2.3. Having determined the transformations of all fields, one can check whether the set of curvature constraints is a consistent one. This analysis is given in subsection 2.2 for ease of presentation. Note however that checking consistency of the constraints constitutes the last step of the analysis and relies on the transformation rules determined in subsection 2.3.

2.1 The Schrödinger superalgebra and transformation rules

Schrödinger superalgebras were first found in [32] as the symmetry group of a spinning particle. However, this leads to an algebra with a Grassmann valued vector charge, instead of a spinor (Q_- in our notation). Because we are mainly interested in extensions of the Bargmann superalgebra with two spinorial supercharges we prefer that our Schrödinger superalgebra also contains such operators. For this reason, and because we work in three space-time dimensions, we will work with the superalgebra of [33].

For the purpose of this work we restrict ourselves to using $z = 2$ Schrödinger algebras. This algebra, as well as its supersymmetric extension, is similar to the Bargmann algebra in that it allows for the same central extension in the commutator of spatial translations and Galilean boosts. This is important because it enables us to solve for the non-relativistic spin- and Galilean boost-connections and thus the gauging works in the same way as e.g. in [14, 22, 23].

To be concrete, we use the following set of commutators. The bosonic commutation relations of the Bargmann algebra ($a = 1, 2$)

$$\begin{aligned} \big[P_a, J_{bc}\big] &= 2\,\delta_{a[b}\,P_{c]}\,, & \big[H, G_a\big] &= P_a\,, \\ \big[G_a, J_{bc}\big] &= 2\,\delta_{a[b}\,G_{c]}\,, & \big[P_a, G_b\big] &= \delta_{ab}\,Z\,, \end{aligned} \tag{2.1}$$

are supplemented by the action of the dilatation operator D and special conformal transformations K as follows:

$$\begin{aligned} \big[D, H\big] &= -2\,H\,, & \big[H, K\big] &= D\,, & \big[D, K\big] &= 2\,K\,, \\ \big[D, P_a\big] &= -P_a\,, & \big[D, G_a\big] &= G_a\,, & \big[K, P_a\big] &= -G_a\,. \end{aligned} \tag{2.2}$$

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Here H, P_a, J_{ab}, G_a and Z are the generators corresponding to time translations, spatial translations, spatial rotations, Galilean boosts and central charge transformations, respectively.

The extension to supersymmetry is done by adding two fermionic supersymmetry generators Q_+, Q_- and one so-called "special" supersymmetry generator S. We also have to add one more bosonic so-called R-symmetry generator R which, however, does not contribute to the commutation relations (2.1) and (2.2). This leads to the superalgebra that was found in [33], see also [34, 35]. In this way the commutators of the Bargmann superalgebra,

$$\begin{aligned}
&[J_{ab}, Q_\pm] = -\frac{1}{2}\,\gamma_{ab} Q_\pm\,, && [G_a, Q_+] = -\frac{1}{2}\,\gamma_{a0} Q_-\,,\\
&\{Q_+, Q_+\} = -\gamma^0 C^{-1}\,H\,, && \{Q_+, Q_-\} = -\gamma^a C^{-1}\,P_a\,,\\
&\{Q_-, Q_-\} = -2\,\gamma^0 C^{-1}\,Z\,,
\end{aligned} \tag{2.3}$$

are augmented by the following commutators that involve the extra bosonic and fermionic operators of the Schrödinger superalgebra:

$$\begin{aligned}
&[D, Q_+] = -Q_+\,, \quad [D, S] = S\,, \quad [R, Q_\pm] = \pm\gamma_0 Q_\pm\,, \quad [R, S] = \gamma_0 S\\
&[J_{ab}, S] = -\frac{1}{2}\,\gamma_{ab} S\,, \quad [S, H] = Q_+\,, \quad [S, P_a] = \frac{1}{2}\,\gamma_{a0} Q_-\,, \quad [K, Q_+] = S\,,\\
&\{S, S\} = -\gamma^0 C^{-1}\,K\,, \qquad \{S, Q_-\} = \gamma^a C^{-1}\,G_a\,,\\
&\{S, Q_+\} = \frac{1}{2}\,\gamma^0 C^{-1}\,D + \frac{1}{4}\,\gamma^{0ab} C^{-1}\,J_{ab} + \frac{3}{4}\,C^{-1}\,R\,.
\end{aligned} \tag{2.4}$$

According to [34] this algebra is of a special kind that only exists in odd dimensions. Nevertheless, it will serve our purpose to construct a non-relativistic Schrödinger supergravity theory in three dimensions.

After imposing the conventional constraints we will find that the gauge fields ${\omega_\mu}^{ab}$, ${\omega_\mu}^a$, f_μ and ϕ_μ of spatial rotations, Galilean boosts, special conformal transformations and S-supersymmetry transformations, respectively, together with the spatial components $b_a = {e^\mu}_a b_\mu$ of the dilatation gauge field b_μ are dependent. The time-component $b = \tau^\mu b_\mu$ of b_μ will turn out to be a Stückelberg field for special conformal transformations, just like in the bosonic case [14]. Eventually, we will use this to set b to zero, gauge fixing special conformal transformations. For notational purposes though, it is easier to keep the full b_μ.

We start with the transformations of the independent bosonic fields under the bosonic symmetries. They are

$$\begin{aligned}
\delta\tau_\mu &= 2\,\Lambda_D\,\tau_\mu\,,\\
\delta {e_\mu}^a &= {\lambda^a}_b\,{e_\mu}^b + \lambda^a\,\tau_\mu + \Lambda_D\,{e_\mu}^a\,,\\
\delta m_\mu &= \partial_\mu\sigma + \lambda^a\,{e_\mu}^a\,,\\
\delta b_\mu &= \partial_\mu\Lambda_D + \Lambda_K\,\tau_\mu\,,\\
\delta r_\mu &= \partial_\mu\rho\,.
\end{aligned} \tag{2.5}$$

For the fermionic fields we find

$$\begin{aligned}\delta\psi_{\mu+} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu+} + \Lambda_D\,\psi_{\mu+} - \gamma_0\psi_{\mu+}\,\rho\,,\\ \delta\psi_{\mu-} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu-} - \frac{1}{2}\,\lambda^{a}\gamma_{a0}\psi_{\mu+} + \gamma_0\psi_{\mu-}\,\rho\,.\end{aligned} \tag{2.6}$$

Here $\lambda^a{}_b, \lambda^a, \Lambda_D$ and ρ are the parameters of spatial rotations, Galilean boosts, dilatations and R-symmetry transformations, respectively.

The fermionic symmetries act on the bosonic fields as follows:

$$\begin{aligned}\delta\tau_\mu &= \frac{1}{2}\,\bar\epsilon_+\gamma^0\psi_{\mu+}\,,\\ \delta e_\mu{}^a &= \frac{1}{2}\,\bar\epsilon_+\gamma^a\psi_{\mu-} + \frac{1}{2}\,\bar\epsilon_-\gamma^a\psi_{\mu+}\,,\\ \delta m_\mu &= \bar\epsilon_-\gamma^0\psi_{\mu-}\,,\\ \delta b_\mu &= -\frac{1}{4}\,\bar\epsilon_+\gamma^0\phi_\mu - \frac{1}{4}\,\bar\eta\,\gamma^0\psi_{\mu+}\,,\\ \delta r_\mu &= -\frac{3}{8}\,\bar\epsilon_+\phi_\mu + \frac{3}{8}\,\bar\eta\,\psi_{\mu+}\,,\end{aligned} \tag{2.7}$$

where $\epsilon_\pm$ are the two Q-supersymmetry parameters while η is the single S-supersymmetry parameter. Under these fermionic symmetries the fermionic fields transform as follows:

$$\begin{aligned}\delta\psi_{\mu+} &= D_\mu\epsilon_+ - b_\mu\,\epsilon_+ + r_\mu\,\gamma_0\epsilon_+ - \tau_\mu\,\eta\,,\\ \delta\psi_{\mu-} &= D_\mu\epsilon_- - r_\mu\,\gamma_0\epsilon_- + \frac{1}{2}\,\omega_\mu{}^a\gamma_{a0}\epsilon_+ + \frac{1}{2}\,e_\mu{}^a\gamma_{a0}\eta\,.\end{aligned} \tag{2.8}$$

Since we expect the transformation rules of the dependent gauge fields to change when we solve for them we will not denote them here. Rather, we will first solve for the gauge fields $\omega_\mu{}^{ab}$, $\omega_\mu{}^a$, b_a, f_μ and ϕ_μ, using conventional curvature constraints. The following subsection is devoted to a discussion of all curvature constraints of the Schrödinger supergravity theory.

2.2 Curvature constraints

While gauging the Schrödinger superalgebra we impose several curvature constraints. These follow mostly from requiring the correct transformation properties under diffeomorphisms. At the same time they allow us to solve for some of the gauge fields in terms of the remaining independent ones. According to the Schrödinger superalgebra the curvatures of the independent gauge fields are given by

$$\begin{aligned}\mathcal{R}_{\mu\nu}(H) &= 2\,\partial_{[\mu}\tau_{\nu]} - 4\,b_{[\mu}\tau_{\nu]} - \frac{1}{2}\,\bar\psi_{[\mu+}\gamma^0\psi_{\nu]+}\,,\\ \mathcal{R}_{\mu\nu}{}^a(P) &= 2\,\partial_{[\mu}e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}e_{\nu]}{}^b - 2\,\omega_{[\mu}{}^a\tau_{\nu]} - 2\,b_{[\mu}e_{\nu]}{}^a - \bar\psi_{[\mu+}\gamma^a\psi_{\nu]-}\,,\\ \mathcal{R}_{\mu\nu}(Z) &= 2\,\partial_{[\mu}m_{\nu]} - 2\,\omega_{[\mu}{}^a e_{\nu]}{}^a - \bar\psi_{[\mu-}\gamma^0\psi_{\nu]-}\,,\\ \mathcal{R}_{\mu\nu}(D) &= 2\,\partial_{[\mu}b_{\nu]} - 2\,f_{[\mu}\tau_{\nu]} + \frac{1}{2}\,\bar\psi_{[\mu+}\gamma^0\phi_{\nu]}\,,\\ \mathcal{R}_{\mu\nu}(R) &= 2\,\partial_{[\mu}r_{\nu]} + \frac{3}{4}\,\bar\psi_{[\mu+}\phi_{\nu]}\,,\end{aligned} \tag{2.9}$$

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and

$$\begin{aligned}
\hat{\Psi}_{\mu\nu+}(Q_+) &= 2\,\partial_{[\mu}\psi_{\nu]+} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]+} - 2\,b_{[\mu}\psi_{\nu]+} + 2\,r_{[\mu}\gamma_0\psi_{\nu]+} - 2\,\tau_{[\mu}\phi_{\nu]}\,, \\
\hat{\Psi}_{\mu\nu-}(Q_-) &= 2\,\partial_{[\mu}\psi_{\nu]-} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]-} - 2\,r_{[\mu}\gamma_0\psi_{\nu]-} + \omega_{[\mu}{}^{a}\gamma_{a0}\psi_{\nu]+} + e_{[\mu}{}^{a}\gamma_{a0}\phi_{\nu]}\,.
\end{aligned} \tag{2.10}$$

The covariant curvatures $\mathcal{R}$ of the dependent gauge fields are not a priori given by the "curvatures" R that follow from the structure constants of the Schrödinger superalgebra since the transformation rules of the dependent gauge fields are not necessarily equal to the ones that follow from the structure constants of the algebra, see e.g. the fermionic transformation rules given in eqs. (2.34). For the following discussion we will need the curvatures of spatial rotations, Galilean boosts and S-supersymmetry. In the case of spatial rotations the full curvature coincides with the expression that follows from the structure constants, i.e. $\mathcal{R}(J) = R(J)$, but in the other two cases there are additional terms in $\mathcal{R}$ since the fermionic transformation rules of those gauge fields contain extra terms beyond those that are determined by the structure constants, see eq. (2.34). We therefore have that

$$\mathcal{R}_{\mu\nu}{}^{ab}(J) = 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - \frac{1}{2}\,\bar{\phi}_{[\mu}\gamma^{0ab}\psi_{\nu]+}\,, \tag{2.11}$$

but that

$$\mathcal{R}_{\mu\nu}{}^{a}(G) = R_{\mu\nu}{}^{a}(G) + \text{additional terms}\,, \tag{2.12}$$

with the structure constant dependent part $R_{\mu\nu}{}^{a}(G)$ given by

$$R_{\mu\nu}{}^{a}(G) = 2\,\partial_{[\mu}\omega_{\nu]}{}^{a} - 2\,\omega_{[\mu}{}^{ab}\omega_{\nu]}{}^{b} - 2\,\omega_{[\mu}{}^{a}b_{\nu]} - 2\,f_{[\mu}e_{\nu]}{}^{a} + \bar{\phi}_{[\mu}\gamma^{a}\psi_{\nu]-}\,. \tag{2.13}$$

We will not need the 'additional terms' in $\mathcal{R}(G)$ except for a special trace combination in which case the full expression for $\mathcal{R}(G)$ is given by

$$\mathcal{R}_{0a}{}^{a}(G) = R_{0a}{}^{a}(G) - e^{\mu}{}_{a}\,\bar{\psi}_{\mu-}\gamma^{0}\hat{\Psi}_{a0-}(Q_-)\,. \tag{2.14}$$

Using the same notation we find that the curvature of the gauge field of S-supersymmetry is given by

$$\begin{aligned}
\mathcal{R}_{\mu\nu}(S) = {}& 2\,\partial_{[\mu}\phi_{\nu]} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\phi_{\nu]} + 2\,b_{[\mu}\,\phi_{\nu]} + 2\,r_{[\mu}\,\gamma_0\phi_{\nu]} + 2\,f_{[\mu}\,\psi_{\nu]+} \\
& + 2\,\gamma^{0}\psi_{[\mu+}\left[\frac{1}{4}\,\varepsilon^{ab}\,\mathcal{R}_{\nu]0}{}^{ab}(J) - \mathcal{R}_{\nu]0}(R)\right] - 2\,\gamma^{c}\psi_{[\mu-}\left[\frac{1}{4}\,\varepsilon^{ab}\,\mathcal{R}_{\nu]c}{}^{ab}(J) + \mathcal{R}_{\nu]c}(R)\right],
\end{aligned} \tag{2.15}$$

where the first line comprises all terms that follow from the structure constants.

In the following subsection we will solve for the gauge fields $\omega_\mu{}^{ab}$, $\omega_\mu{}^{a}$, b_a, f_μ and ϕ_μ in terms of the independent ones using the following set of conventional constraints:

$$\begin{aligned}
&\mathcal{R}_{\mu\nu}{}^{a}(P) = 0\,, && \mathcal{R}_{\mu\nu}(Z) = 0\,, && \mathcal{R}_{a0}(H) = 0\,, \\
&\hat{\Psi}_{a0+}(Q_+) = 0\,, && \gamma^{a}\hat{\Psi}_{a0-}(Q_-) = 0\,, && \\
&\mathcal{R}_{a0}(D) = 0\,, && \mathcal{R}_{0a}{}^{a}(G) = 0\,. &&
\end{aligned} \tag{2.16}$$

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Note that the last constraint involves the curvature of the dependent Galilean boost gauge field whose definition in terms of the part of the curvature that is determined by the structure constants is given in eq. (2.14). Since the conventional constraints are used to solve for some of the gauge fields their supersymmetry transformations do not lead to new constraints. We note that, imposing constraints on the curvatures, the Bianchi identities generically imply further constraints on the curvatures, which holds for the constraints in (2.16) and those to be discussed below.

Besides the conventional constraints we also impose the foliation constraint

$$\mathcal{R}_{\mu\nu}(H) = 0\,. \tag{2.17}$$

The time-space component of this constraint is conventional but the space-space part is not. Its Q_+-supersymmetry transformation leads to

$$\hat{\Psi}_{\mu\nu+}(Q_+) = 0\,, \tag{2.18}$$

where, again, only the space-space part is a new, un-conventional constraint. The constraints (2.17) and (2.18) lead to

$$\mathcal{R}_{ab}(D) = 0\,, \tag{2.19}$$

as a consequence of a Bianchi identity. We now consider supersymmetry transformations of the un-conventional constraint $\hat{\Psi}_{ab+}(Q_+) = 0$. A Q_--variation enforces[3]

$$\hat{\Psi}_{ab-}(Q_-) = 0\,. \tag{2.20}$$

Upon use of all known constraints and Bianchi identities, we find that the only non-trivial variation of (2.20) is its Q_--variation which we combine with a Q_+-variation of (2.18) to find

$$\mathcal{R}_{ab}(R) = 0\,, \qquad\qquad \mathcal{R}_{ab}{}^{cd}(J) = 0\,. \tag{2.21}$$

At this point we have checked the symmetry variations of all constraints except the last two, i.e. (2.21). Before we go on determining the implications of their transformations we note that using all constraints so far we find the Bianchi identity

$$\mathcal{R}_{ab}(S) = 0\,. \tag{2.22}$$

The only non-trivial transformation of $\mathcal{R}_{ab}(R) = 0$ then leads to [3]

$$\frac{3}{4}\,\varepsilon^{ab}\,\mathcal{R}_{\mu\nu}{}^{ab}(J) = \mathcal{R}_{\mu\nu}(R)\,. \tag{2.23}$$

[3]One might wonder how the supersymmetry transformation of a fermionic [bosonic] constraint can lead to another fermionic [bosonic] constraint. It is true that this is not possible when following generic transformation rules of covariant quantities. However, those rules only apply if we already know the full set of constraints and the commutator algebra closes precisely because some constraints are needed to eliminate apparently non-covariant terms. Hence, we can certainly take guidance from those covariant rules, but when we use them too naively we might miss some constraints.

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Since (2.23) essentially identifies $\mathcal{R}(J)$ with $\mathcal{R}(R)$ we have derived all consequences of (2.21). The constraint (2.23) itself is inert under all symmetries and hence we have derived the full set of un-conventional constraints that follow from (2.17).

In summary, the set of constraints comprises the following chain of un-conventional constraints:

$$\begin{aligned} &\mathcal{R}_{ab}(H)=0 \quad \xrightarrow{Q_+} \quad \hat{\Psi}_{ab+}=0 \quad \xrightarrow{Q_-} \quad \hat{\Psi}_{ab-}=0 \quad \longrightarrow \\ &\left.\begin{array}{l}\hat{\Psi}_{ab+}=0\\ \hat{\Psi}_{ab-}=0\end{array}\right\} \quad \xrightarrow{Q_\pm} \quad \mathcal{R}_{ab}(R)=0 \quad \xrightarrow{Q_+} \quad \frac{3}{4}\,\varepsilon^{ab}\,\mathcal{R}_{\mu\nu}{}^{ab}(J)=\mathcal{R}_{\mu\nu}(R)\,. \end{aligned} \tag{2.24}$$

The Bianchi identities that feature in the discussion above are given by

$$\mathcal{R}_{ab}(D)=0\,,\qquad \mathcal{R}_{0[a}{}^{b]}(G)=0\,,\qquad \mathcal{R}_{ab}{}^{c}(G)=2\,\mathcal{R}_{0[a}{}^{b]c}(J)\,,\qquad \mathcal{R}_{ab}(S)=0\,. \tag{2.25}$$

2.3 The dependent gauge fields

Let us now determine the expressions of the dependent gauge fields. We first determine the spatial component of b_μ. Using $\mathcal{R}_{a0}(H)=0$ we find

$$b_a=e^\mu{}_a b_\mu=\frac{1}{2}\,e^\mu{}_a\tau^\nu\left(2\,\partial_{[\mu}\tau_{\nu]}-\frac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^0\psi_{\nu]+}\right). \tag{2.26}$$

The (independent) scalar $b=\tau^\mu b_\mu$ is a Stückelberg field for special conformal transformations:

$$\begin{aligned} \delta b=\Lambda_K+\tau^\mu\partial_\mu\Lambda_D-2\,\Lambda_D\,b-\lambda^a\,b_a-\frac{1}{4}\,\tau^\mu\left(\bar{\epsilon}_+\gamma^0\phi_\mu+\bar{\eta}\,\gamma^0\psi_{\mu+}\right)\\ -\frac{1}{2}\,b\,\bar{\epsilon}_+\gamma^0\psi_{\rho+}\tau^\rho-\frac{1}{2}\,b_a\,\tau^\rho\left(\bar{\epsilon}_+\gamma^a\psi_{\rho-}+\bar{\epsilon}_-\gamma^a\psi_{\rho+}\right). \end{aligned} \tag{2.27}$$

Thus, we could choose to set $b=0$. This would induce the compensating transformation

$$\begin{aligned} \Lambda_K=-\tau^\mu\partial_\mu\Lambda_D+\lambda^a\,b_a+\frac{1}{4}\,\tau^\mu\left(\bar{\epsilon}_+\gamma^0\phi_\mu+\bar{\eta}\,\gamma^0\psi_{\mu+}\right)\\ +\frac{1}{2}\,b_a\,\tau^\rho\left(\bar{\epsilon}_+\gamma^a\psi_{\rho-}+\bar{\epsilon}_-\gamma^a\psi_{\rho+}\right). \end{aligned} \tag{2.28}$$

In the following we will keep $b\neq 0$. In any case, since no independent field transforms under special conformal transformations there is in essence no effect from this gauge fixing.

We proceed with determining the other dependent gauge fields. The gauge fields $\omega_\mu{}^{ab}$ of spatial rotations and $\omega_\mu{}^a$ of Galilean boosts are solved for using the conventional constraints $\mathcal{R}_{\mu\nu}{}^a(P)=0$ and $\mathcal{R}_{\mu\nu}(Z)=0$. We find the following expressions:

$$\begin{aligned} \omega_\mu{}^{ab}&=2\,e^{\nu[a}\left(\partial_{[\nu}e_{\mu]}{}^{b]}-\frac{1}{2}\,\psi_{[\nu+}\gamma^{b]}\psi_{\mu]-}-b_{[\nu}\,e_{\mu]}{}^{b]}\right) \\ &\quad+e_\mu{}^c e^{\rho a}e^{\nu b}\left(\partial_{[\rho}e_{\nu]}{}^c-\frac{1}{2}\,\psi_{[\rho+}\gamma^c\psi_{\nu]-}-b_{[\rho}\,e_{\nu]}{}^c\right)-\tau_\mu e^{\rho a}e^{\nu b}\left(\partial_{[\rho}m_{\nu]}-\frac{1}{2}\,\psi_{[\rho-}\gamma^0\psi_{\nu]-}\right), \end{aligned} \tag{2.29}$$

$$\begin{aligned} \omega_\mu{}^a&=-\tau^\nu\left(\partial_{[\nu}e_{\mu]}{}^a-\frac{1}{2}\,\psi_{[\nu+}\gamma^a\psi_{\mu]-}-b_{[\nu}\,e_{\mu]}{}^a\right) \\ &\quad+e_\mu{}^c e^{\rho a}\tau^\nu\left(\partial_{[\rho}e_{\nu]}{}^c-\frac{1}{2}\,\psi_{[\rho+}\gamma^c\psi_{\nu]-}-b_{[\rho}\,e_{\nu]}{}^c\right) \\ &\quad+e^{\nu a}\left(\partial_{[\mu}m_{\nu]}-\frac{1}{2}\,\psi_{[\mu-}\gamma^0\psi_{\nu]-}\right)-\tau_\mu e^{\rho a}\tau^\nu\left(\partial_{[\rho}m_{\nu]}-\frac{1}{2}\,\psi_{[\rho-}\gamma^0\psi_{\nu]-}\right). \end{aligned} \tag{2.30}$$

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The S-supersymmetry gauge field ϕ_μ is determined through the conventional constraints $\hat{\Psi}_{a0+}(Q_+) = 0$ and $\gamma^a \hat{\Psi}_{a0-}(Q_-) = 0$, which lead to the following expression:

$$\begin{aligned}\phi_\mu = & -\tau^\nu \left(2\,\partial_{[\mu}\psi_{\nu]+} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]+} - 2\,b_{[\mu}\psi_{\nu]+} + 2\,r_{[\mu}\gamma_0\psi_{\nu]+}\right) \\ & + \tau_\mu\tau^\rho e^\nu{}_c\gamma^{0c}\left(2\,\partial_{[\rho}\psi_{\nu]-} - \frac{1}{2}\,\omega_{[\rho}{}^{ab}\gamma_{ab}\psi_{\nu]-} + \omega_{[\rho}{}^{a}\gamma_{a0}\psi_{\nu]+} - 2\,r_{[\mu}\gamma_0\psi_{\nu]-}\right).\end{aligned} \tag{2.31}$$

Finally, to solve for the special conformal boost gauge field f_μ we use the conventional constraints $\mathcal{R}_{a0}(D) = 0$ and $\mathcal{R}_{0a}{}^a(G) = 0$. In this way we find that

$$\begin{aligned}f_\mu = & \,\tau^\nu \left(2\,\partial_{[\mu}b_{\nu]} + \frac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^0\phi_{\nu]}\right) \\ & + \frac{1}{2}\,\tau_\mu\tau^\rho e^\nu{}_a\left(2\,\partial_{[\rho}\omega_{\nu]}{}^a - 2\,\omega_{[\rho}{}^{ab}\omega_{\nu]}{}^b - 2\,\omega_{[\rho}{}^a b_{\nu]} + \bar{\phi}_{[\rho}\gamma^a\psi_{\nu]-}\right) \\ & - \frac{1}{2}\,\tau_\mu e^\rho{}_a\,\bar{\psi}_{\rho-}\gamma^0\hat{\Psi}_{a0-}(Q_-)\,.\end{aligned} \tag{2.32}$$

At this point we have solved for all the dependent gauge fields in terms of the independent ones. Using their expressions in terms of the independent gauge fields, we find that they transform under the bosonic Schrödinger transformations as follows:

$$\begin{aligned}\delta\omega_\mu{}^{ab} &= \partial_\mu\lambda^{ab}\,, \\ \delta\omega_\mu{}^a &= \partial_\mu\lambda^a - \omega_\mu{}^a{}_b\lambda^b + b_\mu\lambda^a + \lambda^a{}_b\,\omega_\mu{}^b - \Lambda_D\,\omega_\mu{}^a + \Lambda_K\,e_\mu{}^a\,, \\ \delta f_\mu &= \partial_\mu\Lambda_K + 2\,\Lambda_K\,b_\mu - 2\,\Lambda_D\,f_\mu - \tau_\mu\,\lambda^b\,\mathcal{R}_{0a}{}^{ab}(J)\,, \\ \delta\phi_\mu &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\phi_\mu - \Lambda_D\,\phi_\mu - \Lambda_K\,\psi_{\mu+} - \gamma_0\phi_\mu\,\rho\,.\end{aligned} \tag{2.33}$$

These are precisely the transformation rules that follow from the structure constants of the Schrödinger algebra except for the curvature term in the transformation rule of the special conformal boost gauge field f_μ. In [14] this was circumvented by redefining f_μ by adding terms with m_μ and $\mathcal{R}_{\mu\nu}{}^{ab}(J)$ in the conventional constraint $\mathcal{R}_{0a}{}^a(G) = 0$ that is used to solve for f_μ. However, then the field acquired a non-trivial transformation under the central charge symmetry. We will not perform any redefinition of that kind here.

Concerning the fermionic symmetries, we find that the Q and S-transformations of the dependent gauge fields fields $\omega_\mu{}^{ab}, \omega_\mu{}^a$ and ϕ_μ are given by

$$\begin{aligned}\delta\omega_\mu{}^{ab} &= -\frac{1}{4}\,\bar{\epsilon}_+\gamma^{ab0}\phi_\mu + \frac{1}{4}\,\bar{\eta}\,\gamma^{ab0}\psi_{\mu+}\,, \\ \delta\omega_\mu{}^a &= \bar{\epsilon}_-\gamma^0\hat{\Psi}_\mu{}^a{}_-(Q_-) - \frac{1}{2}\,\bar{\epsilon}_-\gamma^a\phi_\mu + \frac{1}{4}\,e_{\mu b}\,\bar{\epsilon}_+\gamma^b\hat{\Psi}^a{}_{0-}(Q_-) \\ &\quad + \frac{1}{4}\,\bar{\epsilon}_+\gamma^a\hat{\Psi}_{\mu 0-}(Q_-) - \frac{1}{2}\,\bar{\eta}\,\gamma^a\psi_{\mu-}\,, \\ \delta\phi_\mu &= D_\mu\eta + b_\mu\,\eta + r_\mu\,\gamma_0\eta + f_\mu\,\epsilon_+ \\ &\quad + \gamma_0\epsilon_+\left[\frac{1}{4}\,\varepsilon^{ab}\,\mathcal{R}_{\mu 0}{}^{ab}(J) - \mathcal{R}_{\mu 0}(R)\right] + \gamma^c\epsilon_-\left[\frac{1}{4}\,\varepsilon^{ab}\,\mathcal{R}_{\mu c}{}^{ab}(J) + \mathcal{R}_{\mu c}(R)\right].\end{aligned} \tag{2.34}$$

The above bosonic and fermionic transformations allow us to explicitly check that the commutator algebra of two supersymmetries is realized by the formula

$$\begin{aligned}\big[\delta(Q_1,S_1),\delta(Q_2,S_2)\big] &= \delta_{\text{g.c.t.}}\big(\Xi^\mu\big)+\delta_J\big(\Lambda^{ab}\big)+\delta_G\big(\Lambda^a\big)+\delta_Z\big(\Sigma\big)+\delta_D\big(\lambda_D\big)\\ &\quad+\delta_K\big(\lambda_K\big)+\delta_{Q_+}\big(\Upsilon_+\big)+\delta_{Q_-}\big(\Upsilon_-\big)+\delta_S\big(\eta\big)+\delta_R\big(\rho_R\big)\,,\end{aligned} \tag{2.35}$$

where the parameters are given by

$$\begin{aligned}
\Xi^\mu &= \frac{1}{2}\,\bar\epsilon_{2+}\gamma^0\epsilon_{1+}\,\tau^\mu+\frac{1}{2}\big(\bar\epsilon_{2+}\gamma^a\epsilon_{1-}+\bar\epsilon_{2-}\gamma^a\epsilon_{1+}\big)e^\mu{}_a\,,\\
\Lambda^{ab} &= -\Xi^\mu\omega_\mu{}^{ab}+\frac{1}{4}\big(\bar\epsilon_{1+}\gamma^{0ab}\eta_2-\bar\eta_1\,\gamma^{0ab}\epsilon_{2+}\big)\,, &\qquad \Upsilon_\pm &= -\Xi^\mu\psi_{\mu\pm}\,,\\
\Lambda^a &= -\Xi^\mu\omega_\mu{}^a-\frac{1}{2}\big(\bar\epsilon_{1-}\gamma^a\eta_2+\bar\eta_1\,\gamma^a\epsilon_{2-}\big)\,, & \lambda_K &= -\Xi^\mu f_\mu+\frac{1}{2}\,\bar\eta_2\,\gamma^0\eta_1\,,\\
\Sigma &= -\Xi^\mu m_\mu+\bar\epsilon_{2-}\gamma^0\epsilon_{1-}\,, & \rho_R &= -\Xi^\mu r_\mu+\frac{3}{8}\big(\bar\epsilon_{1+}\eta_2-\bar\eta_1\,\epsilon_{2+}\big)\,,\\
\lambda_D &= -\Xi^\mu b_\mu+\frac{1}{4}\big(\bar\epsilon_{1+}\gamma^0\eta_2+\bar\eta_1\,\gamma^0\epsilon_{2+}\big)\,, & \eta &= -\Xi^\mu\phi_\mu\,.
\end{aligned} \tag{2.36}$$

This finishes the discussion of the Schrödinger supergravity theory.

Note that our analysis of the Schrödinger theory is not fully complete, since we did not derive the variation of the dependent field f_μ under fermionic symmetries. Even so, this was not needed to show that the set of constraints (2.24) is a consistent one and that the commutator algebra closes on all independent fields.

3 Matter multiplets

In this section we present matter multiplets that realize the same commutators corresponding to the Schrödinger superalgebra as we derived for the Schrödinger supergravity multiplet in the previous section. These multiplets will be used as compensator multiplets in the next section to derive off-shell formulations of Newton-Cartan supergravity.

One such off-shell formulation already exists in the literature [27]. It was obtained by taking a non-relativistic limit of the three-dimensional $\mathcal{N}=2$ new minimal Poincaré multiplet [36]. The new minimal Poincaré multiplet follows from superconformal techniques using a compensating (relativistic) vector multiplet. Hence, in order to derive its non-relativistic analog we should use as a compensator a non-relativistic vector multiplet. This is one of the two non-relativistic matter multiplets which we derive in this section. The other one is the scalar multiplet which we shall later use to derive a new off-shell formulation of Newton-Cartan supergravity.

It would be very efficient if we could derive the matter multiplets coupled to Schrödinger supergravity by applying the non-relativistic limiting procedure of [27]. However we cannot, because the Schrödinger superalgebra does not follow from the contraction of any relativistic superalgebra and the same applies to the corresponding Schrödinger supergravity theory. Instead, we shall start from the rigid version of a relativistic matter multiplet that realizes the Poincaré superalgebra. First, we use that as a starting point to

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derive a non-relativistic matter multiplet that realizes the rigid Bargmann superalgebra.[4] The important thing is that we have now derived the field content of the non-relativistic multiplet. It turns out that the same multiplet also provides a representation of the rigid Schrödinger superalgebra. Therefore, once we have obtained this non-relativistic multiplet, we can couple it to the fields of Schrödinger supergravity, thereby realizing the commutator algebra derived in the previous section, in the standard way.

3.1 The scalar multiplet

In this subsection we construct the non-relativistic scalar multiplet. We start with the three-dimensional rigid relativistic $\mathcal{N} = 2$ scalar multiplet which comprises two complex scalars and two spinors. In real notation we are thus left with the fields $(\varphi_1, \varphi_2, \chi_1, \chi_2, F_1, F_2)$:

$$\begin{aligned}
\delta\varphi_1 &= \bar{\eta}_1\chi_1 + \bar{\eta}_2\chi_2\,,\\
\delta\varphi_2 &= \bar{\eta}_1\chi_2 - \bar{\eta}_2\chi_1\,,\\
\delta\chi_1 &= \frac{1}{4}\,\gamma^\mu\partial_\mu\varphi_1\,\eta_1 - \frac{1}{4}\,\gamma^\mu\partial_\mu\varphi_2\,\eta_2 - \frac{1}{4}\,F_1\,\eta_1 - \frac{1}{4}\,F_2\,\eta_2\,,\\
\delta\chi_2 &= \frac{1}{4}\,\gamma^\mu\partial_\mu\varphi_2\,\eta_1 + \frac{1}{4}\,\gamma^\mu\partial_\mu\varphi_1\,\eta_2 - \frac{1}{4}\,F_2\,\eta_1 + \frac{1}{4}\,F_1\,\eta_2\,,\\
\delta F_1 &= -\bar{\eta}_1\gamma^\mu\partial_\mu\chi_1 + \bar{\eta}_2\gamma^\mu\partial_\mu\chi_2\,,\\
\delta F_2 &= -\eta_1\gamma^\mu\partial_\mu\chi_2 - \eta_2\gamma^\mu\partial_\mu\chi_1\,.
\end{aligned} \tag{3.1}$$

To take the non-relativistic limit we use a contraction parameter ω which we will send to infinity. The rescaling of the symmetry parameters follows from the Inönü-Wigner contraction of the related symmetry generators, see [27]. This means for example that we will require

$$\epsilon_\pm = \frac{\omega^{\mp 1/2}}{\sqrt{2}}\,\big(\eta_1 \pm \gamma_0\eta_2\big)\,. \tag{3.2}$$

It remains to find the scalings of all other fields. It turns out that, in order to avoid terms that diverge in the limit $\omega \to \infty$, we need to use

$$\chi_\pm = \frac{\omega^{-1\pm 1/2}}{\sqrt{2}}\,(\chi_1 \pm \gamma_0\chi_2) \tag{3.3}$$

for the two spinors, while for the scalings of the bosons we need to take

$$\tilde{\varphi}_i = \frac{1}{\omega}\,\varphi_i\,, \qquad\qquad \tilde{F}_i = -\frac{1}{\omega}\,F_i\,. \tag{3.4}$$

[4]This (rigid) limit coincides with the non-relativistic limit performed in [37].

After calculating the transformation rules in the limit $\omega \to \infty$ we drop the tildes and find

$$
\begin{aligned}
\delta\varphi_1 &= \bar\epsilon_+\chi_+ + \bar\epsilon_-\chi_- \,,\\
\delta\varphi_2 &= \bar\epsilon_+\gamma^0\chi_+ - \bar\epsilon_-\gamma^0\chi_- \,,\\
\delta\chi_+ &= \frac{1}{4}\,\gamma^0\epsilon_+\,\partial_t\varphi_1 + \frac{1}{4}\,\epsilon_+\,\partial_t\varphi_2 + \frac{1}{4}\,\gamma^i\epsilon_-\,\partial_i\varphi_1 + \frac{1}{4}\,\gamma^{i0}\epsilon_-\,\partial_i\varphi_2 + \frac{1}{4}\,\epsilon_-\,F_1 + \frac{1}{4}\,\gamma_0\epsilon_-\,F_2 \,,\\
\delta\chi_- &= \frac{1}{4}\,\gamma^i\epsilon_+\,\partial_i\varphi_1 - \frac{1}{4}\,\gamma^{i0}\epsilon_+\,\partial_i\varphi_2 + \frac{1}{4}\,\epsilon_+\,F_1 - \frac{1}{4}\,\gamma_0\epsilon_+\,F_2 \,,\\
\delta F_1 &= \bar\epsilon_+\gamma^i\partial_i\chi_+ + \bar\epsilon_+\gamma^0\partial_t\chi_- + \bar\epsilon_-\gamma^i\partial_i\chi_- \,,\\
\delta F_2 &= \bar\epsilon_+\gamma^{i0}\partial_i\chi_+ + \bar\epsilon_+\partial_t\chi_- - \bar\epsilon_-\gamma^{i0}\partial_i\chi_- \,.
\end{aligned}
\tag{3.5}
$$

Together with the bosonic transformation rules, which we refrain from giving here but which can be obtained easily by similar techniques, the transformation rules (3.5) realize the rigid Bargmann superalgebra. Next, we promote this multiplet to a representation of the rigid Schrödinger superalgebra by assigning transformations under the Schrödinger transformations that are not contained in the Bargmann superalgebra. After that we couple the multiplet to the fields of Schrödinger supergravity. Following standard techniques of coupling matter to supergravity we find for the bosonic transformations

$$
\begin{aligned}
\delta\varphi_1 &= w\,\Lambda_D\varphi_1 + \frac{2w}{3}\,\rho\,\varphi_2 \,,\\
\delta\varphi_2 &= w\,\Lambda_D\varphi_2 - \frac{2w}{3}\,\rho\,\varphi_1 \,,\\
\delta\chi_+ &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\chi_+ - \frac{1}{2}\,\lambda^a\gamma_{a0}\chi_- + (w-1)\,\Lambda_D\chi_+ - \left(\frac{2w}{3}+1\right)\gamma_0\chi_+\,\rho \,,\\
\delta\chi_- &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\chi_- + w\,\Lambda_D\chi_- + \left(\frac{2w}{3}+1\right)\gamma_0\chi_-\,\rho \,,\\
\delta F_1 &= (w-1)\,\Lambda_D F_1 + 2\left(\frac{w}{3}+1\right)\rho\,F_2 \,,\\
\delta F_2 &= (w-1)\,\Lambda_D F_2 - 2\left(\frac{w}{3}+1\right)\rho\,F_1 \,,
\end{aligned}
\tag{3.6}
$$

while for the fermionic transformation rules we find the following expressions:

$$
\begin{aligned}
\delta\varphi_1 &= \bar\epsilon_+\chi_+ + \bar\epsilon_-\chi_- \,,\\
\delta\varphi_2 &= \bar\epsilon_+\gamma^0\chi_+ - \bar\epsilon_-\gamma^0\chi_- \,,\\
\delta\chi_+ &= \frac{1}{4}\,\gamma^0\epsilon_+\,\tau^\mu\hat D_\mu\varphi_1 + \frac{1}{4}\,\epsilon_+\,\tau^\mu\hat D_\mu\varphi_2 + \frac{1}{4}\,\gamma^a\epsilon_-\,e^\mu{}_a\hat D_\mu\varphi_1 + \frac{1}{4}\,\gamma^{a0}\epsilon_-\,e^\mu{}_a\hat D_\mu\varphi_2\\
&\quad + \frac{1}{4}\,\epsilon_-\,F_1 + \frac{1}{4}\,\gamma_0\epsilon_-\,F_2 - \frac{w}{4}\,\gamma^0\eta\,\varphi_1 - \frac{w}{4}\,\eta\,\varphi_2 \,,\\
\delta\chi_- &= \frac{1}{4}\,\gamma^a\epsilon_+\,e^\mu{}_a\hat D_\mu\varphi_1 - \frac{1}{4}\,\gamma^{a0}\epsilon_+\,e^\mu{}_a\hat D_\mu\varphi_2 + \frac{1}{4}\,\epsilon_+\,F_1 - \frac{1}{4}\,\gamma_0\epsilon_+\,F_2 \,,\\
\delta F_1 &= \bar\epsilon_+\gamma^a e^\mu{}_a\hat D_\mu\chi_+ + \bar\epsilon_+\gamma^0\tau^\mu\hat D_\mu\chi_- + \bar\epsilon_-\gamma^a e^\mu{}_a\hat D_\mu\chi_- - (w+1)\,\bar\eta\,\gamma^0\chi_- \,,\\
\delta F_2 &= \bar\epsilon_+\gamma^{a0}e^\mu{}_a\hat D_\mu\chi_+ + \bar\epsilon_+\tau^\mu\hat D_\mu\chi_- - \bar\epsilon_-\gamma^{a0}e^\mu{}_a\hat D_\mu\chi_- - (w+1)\,\bar\eta\,\chi_- \,.
\end{aligned}
\tag{3.7}
$$

The covariant derivatives that appear in (3.7) can be deduced from the transformation rules (3.6) and (3.7). For the bosonic fields they are given by

$$
\begin{aligned}
\hat{D}_\mu\varphi_1 &= \partial_\mu\varphi_1 - w\,b_\mu\,\varphi_1 - \frac{2w}{3}\,r_\mu\,\varphi_2 - \bar{\psi}_{\mu+}\chi_+ - \bar{\psi}_{\mu-}\chi_- \,, \\
\hat{D}_\mu\varphi_2 &= \partial_\mu\varphi_2 - w\,b_\mu\,\varphi_2 + \frac{2w}{3}\,r_\mu\,\varphi_1 - \bar{\psi}_{\mu+}\gamma^0\chi_+ + \bar{\psi}_{\mu-}\gamma^0\chi_- \,, \\
\hat{D}_\mu F_1 &= \partial_\mu F_1 - (w-1)\,b_\mu\,F_1 - 2\left(\frac{w}{3}+1\right) r_\mu\,F_2 \\
&\quad - \bar{\psi}_{\mu+}\gamma^a e^\rho{}_a \hat{D}_\rho\chi_+ - \bar{\psi}_{\mu+}\gamma^0\tau^\rho\hat{D}_\rho\chi_- - \bar{\psi}_{\mu-}\gamma^a e^\rho{}_a\hat{D}_\rho\chi_- + (w+1)\,\bar{\phi}_\mu\gamma^0\chi_- \,, \\
\hat{D}_\mu F_2 &= \partial_\mu F_2 - (w-1)\,b_\mu\,F_2 + 2\left(\frac{w}{3}+1\right) r_\mu\,F_1 \\
&\quad - \bar{\psi}_{\mu+}\gamma^{a0} e^\rho{}_a \hat{D}_\rho\chi_+ - \bar{\psi}_{\mu+}\tau^\rho\hat{D}_\rho\chi_- + \bar{\psi}_{\mu-}\gamma^{a0} e^\rho{}_a\hat{D}_\rho\chi_- + (w+1)\,\bar{\phi}_\mu\chi_- \,,
\end{aligned}
\tag{3.8}
$$

while for the covariant derivatives of the fermions we find the following expressions:

$$
\begin{aligned}
\hat{D}_\mu\chi_+ &= D_\mu\chi_+ + \frac{1}{2}\,\omega_\mu{}^a\gamma_{a0}\chi_- - (w-1)\,b_\mu\chi_+ + \left(\frac{2w}{3}+1\right) r_\mu\,\gamma_0\chi_+ \\
&\quad - \frac{1}{4}\,\gamma^0\psi_{\mu+}\,\tau^\rho\hat{D}_\rho\varphi_1 - \frac{1}{4}\,\psi_{\mu+}\,\tau^\rho\hat{D}_\rho\varphi_2 - \frac{1}{4}\,\gamma^a\psi_{\mu-}\,e^\rho{}_a\hat{D}_\rho\varphi_1 \\
&\quad - \frac{1}{4}\,\gamma^{a0}\psi_{\mu-}\,e^\rho{}_a\hat{D}_\rho\varphi_2 - \frac{1}{4}\,\psi_{\mu-}\,F_1 - \frac{1}{4}\,\gamma_0\psi_{\mu-}\,F_2 + \frac{w}{4}\,\gamma^0\phi_\mu\,\varphi_1 + \frac{w}{4}\,\phi_\mu\,\varphi_2 \,, \\
\hat{D}_\mu\chi_- &= D_\mu\chi_- - w\,b_\mu\,\chi_- - \left(\frac{2w}{3}+1\right) r_\mu\,\gamma_0\chi_- \\
&\quad - \frac{1}{4}\,\gamma^a\psi_{\mu+}\,e^\rho{}_a\hat{D}_\rho\varphi_1 + \frac{1}{4}\,\gamma^{a0}\psi_{\mu+}\,e^\rho{}_a\hat{D}_\rho\varphi_2 - \frac{1}{4}\,\psi_{\mu+}\,F_1 + \frac{1}{4}\,\gamma_0\psi_{\mu+}\,F_2 \,.
\end{aligned}
\tag{3.9}
$$

This completes our derivation of the non-relativistic scalar multiplet. In section 4 we will use this scalar multiplet to derive a new off-shell formulation of Newton-Cartan supergravity.

3.2 The vector multiplet

The $\mathcal{N}=2$ vector multiplet in three dimensions contains a vector, a physical scalar, two spinors and an auxiliary scalar $(C_\mu, \rho, \lambda_i, D)$. Using the three-dimensional epsilon symbol we can define a new "dual" vector $V_\mu = \varepsilon_\mu{}^{\nu\rho}\,\partial_\nu C_\rho$ with

$$
\partial^\mu V_\mu = 0\,, \tag{3.10}
$$

which has the dimension of an auxiliary field. In terms of $(\rho, \lambda_i, V_\mu, D)$ we have the following transformation rules:

$$
\begin{aligned}
\delta\rho &= \varepsilon^{ij}\,\bar{\eta}_i\lambda_j \,, \\
\delta\lambda_i &= -\frac{1}{2}\,\gamma^\mu\eta_i\,V_\mu - \frac{1}{2}\,\varepsilon^{ij}\eta_j\,D - \frac{1}{4}\,\gamma^\mu\varepsilon^{ij}\eta_j\,\partial_\mu\rho \,, \\
\delta D &= \frac{1}{2}\,\varepsilon^{ij}\,\bar{\eta}_i\,\gamma^\mu\partial_\mu\lambda_j \,, \\
\delta V_\mu &= \frac{1}{2}\,\delta^{ij}\,\bar{\eta}_i\,\gamma_\mu{}^\nu\partial_\nu\lambda_j \,.
\end{aligned}
\tag{3.11}
$$

Next, we perform the non-relativistic limiting procedure. First, we have to find the scalings of the fields, starting with the scalings of the supersymmetry parameters given in eq. (3.2). We define new spinors

$$\lambda_\pm = \frac{\omega^{-1\pm 1/2}}{\sqrt{2}}\left(\lambda_1 \pm \gamma_0\lambda_2\right), \tag{3.12}$$

and the bosonic field

$$\phi = \frac{\rho}{\omega}. \tag{3.13}$$

Furthermore, we find it useful to introduce the new fields

$$S = -\frac{1}{\omega}\,V_0 - D\,, \qquad F = \frac{1}{\omega^3}\,V_0 - \frac{1}{\omega^2}\,D\,, \qquad C_i = \frac{1}{\omega}\left(V_i + \frac{1}{2}\,\varepsilon^{ij}\,\partial_j\rho\right). \tag{3.14}$$

In the limit $\omega \to \infty$ this leads to the following supersymmetry transformations:

$$\begin{aligned}
\delta\phi &= \bar{\epsilon}_+\gamma^0\lambda_+ - \bar{\epsilon}_-\gamma^0\lambda_-\,,\\
\delta\lambda_+ &= \frac{1}{4}\,\epsilon_+\,\partial_t\phi - \frac{1}{2}\,\gamma_0\epsilon_+\,S + \frac{1}{2}\,\gamma^{i0}\epsilon_-\,\partial_i\phi - \frac{1}{2}\,\gamma^i\epsilon_-\,C_i\,,\\
\delta S &= \frac{1}{2}\,\bar{\epsilon}_+\,\partial_t\lambda_+ - \bar{\epsilon}_-\gamma^{i0}\partial_i\lambda_+ - \frac{1}{2}\,\bar{\epsilon}_-\,\partial_t\lambda_-\,,\\
\delta C_i &= \bar{\epsilon}_-\gamma^{ij}\partial_j\lambda_- + \frac{1}{2}\,\bar{\epsilon}_+\gamma^{i0}\partial_t\lambda_-\,,\\
\delta\lambda_- &= -\frac{1}{2}\,\gamma^i\epsilon_+\,C_i + \frac{1}{2}\,\gamma_0\epsilon_-\,F\,,\\
\delta F &= \bar{\epsilon}_+\gamma^{i0}\partial_i\lambda_-\,.
\end{aligned} \tag{3.15}$$

To prove closure one has to use the constraint

$$\partial^i C_i = \frac{1}{2}\,\partial_t F\,, \tag{3.16}$$

which follows from inserting the definitions (3.14) in the relativistic constraint (3.10) and sending $\omega \to \infty$.

An effect of taking the non-relativistic limit is that there exists a consistent truncation of this multiplet. We can impose

$$C_i = 0\,, \qquad F = 0\,, \qquad \lambda_- = 0\,, \tag{3.17}$$

which results into

$$\begin{aligned}
\delta\phi &= \bar{\epsilon}_+\gamma^0\lambda_+\,,\\
\delta\lambda_+ &= \frac{1}{4}\,\epsilon_+\,\partial_t\phi - \frac{1}{2}\,\gamma_0\epsilon_+\,S + \frac{1}{2}\,\gamma^{i0}\epsilon_-\,\partial_i\phi\,,\\
\delta S &= \frac{1}{2}\,\bar{\epsilon}_+\,\partial_t\lambda_+ - \bar{\epsilon}_-\gamma^{i0}\partial_i\lambda_+\,.
\end{aligned} \tag{3.18}$$

While this multiplet looks like a scalar multiplet and appears to be simpler than the scalar multiplet given in (3.5) its relation to the relativistic vector multiplet manifests itself in the following way. Due to the redefinition (3.14) the auxiliary field S is related to the zero component of the vector field. As a consequence of this the auxiliary field transforms non-trivially under Galilean boosts. This can already be seen in the rigid transformations but we will only give the bosonic transformations when we couple (3.18) to Schrödinger supergravity.

After coupling to supergravity the bosonic transformations read

$$
\begin{aligned}
\delta\phi &= w\,\Lambda_D\phi\,,\\
\delta\lambda &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\lambda + (w-1)\,\Lambda_D\,\lambda - \rho\,\gamma_0\lambda\,,\\
\delta S &= (w-2)\,\Lambda_D\,S - \frac{1}{2}\,\varepsilon^{ab}\lambda^a\,e^\mu{}_b\hat{D}_\mu\phi\,,
\end{aligned}
\tag{3.19}
$$

while the fermionic ones take the form

$$
\begin{aligned}
\delta\phi &= \bar{\epsilon}_+\gamma^0\lambda\,,\\
\delta\lambda &= \frac{1}{4}\,\epsilon_+\,\tau^\mu\hat{D}_\mu\phi + \frac{1}{2}\,\gamma^{a0}\epsilon_-\,e^\mu{}_a\hat{D}_\mu\phi - \frac{1}{2}\,\gamma_0\epsilon_+\,S - \frac{w}{4}\,\eta\,\phi\,,\\
\delta S &= \frac{1}{2}\,\bar{\epsilon}_+\tau^\mu\hat{D}_\mu\lambda - \bar{\epsilon}_-\gamma^{a0}e^\mu{}_a\hat{D}_\mu\lambda - \frac{w-1}{2}\,\bar{\eta}\,\lambda\,.
\end{aligned}
\tag{3.20}
$$

Note the non-trivial transformation of S under local Galilean boosts, see eq. (3.19). This makes clear the vector multiplet origin of (3.19) and (3.20). In the formulas above we use the covariant derivatives

$$
\begin{aligned}
\hat{D}_\mu\phi &= \partial_\mu\phi - \bar{\psi}_{\mu+}\gamma^0\lambda - w\,b_\mu\,\phi\,,\\
\hat{D}_\mu\lambda &= \partial_\mu\lambda - \frac{1}{4}\,\omega_\mu{}^{ab}\gamma_{ab}\lambda - (w-1)\,b_\mu\lambda + r_\mu\,\gamma_0\lambda + \frac{w}{4}\,\phi_\mu\,\phi\\
&\quad - \frac{1}{4}\,\psi_{\mu+}\,\tau^\nu\hat{D}_\nu\phi - \frac{1}{2}\,\gamma^{a0}\psi_{\mu-}\,e^\nu{}_a\hat{D}_\nu\phi + \frac{1}{2}\,\gamma_0\psi_{\mu+}\,S\,,\\
\hat{D}_\mu S &= \partial_\mu S + 2\,b_\mu\,S - \frac{1}{2}\,\bar{\psi}_{\mu+}\tau^\rho\hat{D}_\rho\lambda + \bar{\psi}_{\mu-}\gamma^{a0}e^\rho{}_a\hat{D}_\rho\lambda\\
&\quad + \frac{1}{2}\,\varepsilon^{ab}\,\omega_\mu{}^a\,e^\rho{}_b\hat{D}_\rho\phi + \frac{w-1}{2}\,\bar{\lambda}\,\phi_\mu\,.
\end{aligned}
\tag{3.21}
$$

This finishes our derivation of the non-relativistic vector multiplet. In the following section we will use the non-relativistic scalar and vector multiplets to derive two inequivalent off-shell formulations of Newton-Cartan supergravity with torsion. Before doing so we will give a brief overview of the multiplets that we have discussed so far and which provide the basis of a non-relativistic superconformal tensor calculus, see table 1.

Note that if we were to add another column to this table for the central charge weight (Z-weight) we would have only zeros. We will come back to this in the conclusion section.

4 Newton-Cartan supergravity with torsion

At this point we have at our disposal a "conformal" Schrödinger supergravity theory and two matter multiplets which we can use to fix some of the gauge symmetries. This en-

Overview of non-relativistic multiplets				
multiplet	field	type	D-weight	R-weight
Schrödinger	τ_μ	time-like vielbein	2	0
	${e_\mu}^a$	spatial vielbein	1	0
	m_μ	Z gauge field	0	0
	r_μ	R gauge field	0	0
	b	"D gauge field"	-2	0
	$\psi_{\mu+}$	Q_+ gravitino	1	-1
	$\psi_{\mu-}$	Q_- gravitino	0	1
Scalar	φ_1	physical scalar	w	$\frac{2w}{3}$
	φ_2	physical scalar	w	$-\frac{2w}{3}$
	χ_+	spinor	$w-1$	$-\frac{2w}{3}-1$
	χ_-	spinor	w	$\frac{2w}{3}+1$
	F_1	auxiliary scalar	$w-1$	$\frac{2w}{3}+2$
	F_2	auxiliary scalar	$w-1$	$-\frac{2w}{3}-2$
Vector	ϕ	physical scalar	w	0
	λ	spinor	$w-1$	-1
	S	auxiliary	$w-2$	0

Table 1. Properties of three-dimensional non-relativistic multiplets.

ables us to use superconformal techniques to derive off-shell non-relativistic supergravity multiplets. The superconformal tensor calculus naturally leads to a Newton-Cartan supergravity with non-zero torsion, i.e. the curl of the gauge field τ_μ of local time translations is non-zero, see also [14] for a discussion of the bosonic case. The origin of the torsion is the spatial part b_a of the dilatation gauge field. Unlike in the relativistic case, this spatial part cannot be shifted away by a special conformal transformation. Instead, it is a dependent gauge field whose presence leads to torsion.

In this section we show how the extra symmetries of the Schrödinger superalgebra that are not contained in the Bargmann superalgebra, i.e. dilatations D, special conformal transformations K, S-supersymmetry and possibly R-symmetry, can be eliminated by using a compensator matter multiplet. First, we eliminate the special conformal transformations by setting

$$b = \tau^\mu b_\mu = 0\,. \tag{4.1}$$

The induced compensating transformation is given in eq. (2.28). This step is the same independent of which compensator multiplet we use. In the following we shall use both, the scalar and the vector multiplet from the previous section. In analogy to the relativistic case we refer to the resulting off-shell formulations as the "old minimal" one when we use a compensator scalar multiplet and the "new minimal" formulation when the compensator multiplet is the vector multiplet.

4.1 The "old minimal" formulation

In this subsection we choose the scalar multiplet whose transformation rules can be found in eqs. (3.6) and (3.7) as the compensator multiplet. Like in the relativistic case we eliminate both physical scalars thus gauge fixing the dilatations and the local U(1) R-symmetry. One of the fermions is used to get rid of the special conformal S-supersymmetry:

$$\left.\begin{aligned}\varphi_1 = 1 &:\\ \varphi_2 = 0 &:\end{aligned}\right\} \qquad \text{fixes dilatations and } R\text{-symmetry}\,, \tag{4.2}$$

$$\chi_+ = 0 : \qquad \text{fixes special conformal } S\text{-supersymmetry}\,. \tag{4.3}$$

The compensating transformations are given by

$$\Lambda_D = -\frac{1}{w}\,\bar\epsilon_-\chi_-\,, \qquad \rho = -\frac{3}{2w}\,\bar\epsilon_-\gamma^0\chi_-\,, \tag{4.4}$$

and

$$\begin{aligned}\eta = &-\frac{1}{w}\,\epsilon_+\,\tau^\mu\bar\psi_{\mu-}\chi_- + \gamma_0\epsilon_+\,\tau^\mu\left(\frac{2}{3}\,r_\mu + \frac{1}{w}\,\bar\psi_{\mu-}\gamma^0\chi_-\right) - \gamma^{a0}\epsilon_-\left(b_a + \frac{1}{w}\,e^\mu{}_a\,\bar\psi_{\mu-}\chi_-\right)\\ &-\gamma^a\epsilon_-\,e^\mu{}_a\left(\frac{2}{3}\,r_\mu + \frac{1}{w}\,\bar\psi_{\mu-}\gamma^0\chi_-\right) + \frac{1}{w}\,\gamma_0\epsilon_-\,F_1 - \frac{1}{w}\,\epsilon_-\,F_2 - \frac{2}{w}\,\lambda^a\gamma_a\chi_-\,. \end{aligned}\tag{4.5}$$

We thus end up with the field content given in eq. (1.4) of the "old minimal" Newton-Cartan supergravity theory that realizes the Bargmann superalgebra off-shell. The transformation rules of all fields can be easily constructed using those of Schrödinger supergravity, see section 2, and those of the scalar multiplet, see eqs. (3.6) and (3.7), together with the compensating transformations given in eqs. (2.28), (4.4) and (4.5). Given the lengthy nature of the final transformation rules we have moved the explicit expressions to appendix A.

4.2 The "new minimal" formulation

In this subsection we choose the vector multiplet, see eqs. (3.19) and (3.20), as the compensator multiplet. The gauge fixing of dilatations and the special conformal S-supersymmetry is done by imposing the conditions

$$\begin{aligned}\phi = 1 &: \qquad \text{fixes dilatations}\,,\\ \lambda = 0 &: \qquad \text{fixes } S\text{-supersymmetry}\,,\end{aligned}\tag{4.6}$$

and the resulting compensating gauge transformations are given by

$$\Lambda_D = 0\,, \qquad \eta = -\frac{2}{w}\,\gamma_0\epsilon_+\,S - 2\,\gamma^{a0}\epsilon_-\,b_a\,. \tag{4.7}$$

At this point we are left with the symmetries of the Bargmann superalgebra, see eqs. (2.1) and (2.3), plus an extra U(1) R-symmetry. These symmetries are realized on the set of independent fields of the "new minimal" Newton-Cartan supergravity theory given in eq. (1.5). This theory is the non-relativistic version of the three-dimensional $\mathcal{N} = (2,0)$ new minimal Poincaré supergravity theory. The bosonic transformations of the different fields are given by

$$\begin{aligned}
\delta\tau_\mu &= 0\,,\\
\delta e_\mu{}^a &= \lambda^a{}_b\, e_\mu{}^b + \tau_\mu\, \lambda^a\,,\\
\delta m_\mu &= \partial_\mu\sigma + \lambda^a e_\mu{}^a\,,\\
\delta r_\mu &= \partial_\mu\rho\,,\\
\delta S &= -\frac{1}{2}\,\varepsilon^{ab}\lambda^a b_b\,,
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
\delta\psi_{\mu+} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu+} - \gamma_0\psi_{\mu+}\,\rho\,,\\
\delta\psi_{\mu-} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu-} - \frac{1}{2}\,\lambda^a\gamma_{a0}\psi_{\mu+} + \gamma_0\psi_{\mu+}\,\rho\,.
\end{aligned} \tag{4.9}$$

Note that S transforms non-trivially under a Galilean boost transformation which is proportional to b_a, i.e. to torsion, see eq. (5.2). The fermionic transformations including the compensating terms that follow from eq. (4.7) are given by

$$\begin{aligned}
\delta\tau_\mu &= \frac{1}{2}\,\bar\epsilon_+\gamma^0\psi_{\mu+}\,,\\
\delta e_\mu{}^a &= \frac{1}{2}\,\bar\epsilon_+\gamma^a\psi_{\mu-} + \frac{1}{2}\,\bar\epsilon_-\gamma^a\psi_{\mu+}\,,\\
\delta m_\mu &= \bar\epsilon_-\gamma^0\psi_{\mu-}\,,\\
\delta r_\mu &= \frac{3}{4}\,\bar\epsilon_-\gamma^{a0}\psi_{\mu+}\,b_a - \frac{3}{8}\,\bar\epsilon_+\phi_\mu - \frac{3}{4w}\,\bar\epsilon_+\gamma^0\psi_{\mu+}\,S\,,\\
\delta S &= \frac{w}{8}\,\bar\epsilon_+\phi_\mu\,\tau^\mu + \frac{w}{4}\,\bar\epsilon_+\gamma^{a0}\psi_{\mu-}\,\tau^\mu\,b_a - \frac{1}{4}\,\bar\epsilon_+\gamma^0\psi_{\mu+}\,S\\
&\quad - \frac{w}{4}\,\bar\epsilon_-\gamma^{a0}\phi_\mu\,e^\mu{}_a - \frac{w}{2}\,\bar\epsilon_-\gamma^a\gamma^b\psi_{\mu-}\,e^\mu{}_a\,b_b - \frac{1}{2}\,\bar\epsilon_-\gamma^a\psi_{\mu+}\,e^\mu{}_a\,S\,,
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
\delta\psi_{\mu+} &= D_\mu\epsilon_+ - \epsilon_+\,e_\mu{}^a\,b_a + \gamma_0\epsilon_+\,r_\mu + \frac{2}{w}\,\gamma_0\epsilon_+\,\tau_\mu\,S + 2\,\gamma^{a0}\epsilon_-\,\tau_\mu\,b_a\,,\\
\delta\psi_{\mu-} &= D_\mu\epsilon_- - \gamma_0\epsilon_-\,r_\mu + \gamma^a\gamma^b\epsilon_-\,e_\mu{}^a\,b_b + \frac{1}{2}\,\gamma_{a0}\epsilon_+\,\omega_\mu{}^a + \frac{1}{w}\gamma_a\epsilon_+\,e_\mu{}^a\,S\,.
\end{aligned} \tag{4.11}$$

The transformation rules of the dependent gauge fields can be found in appendix A.

5 Truncation to zero torsion

In the previous section we derived a Newton-Cartan supergravity theory with non-zero torsion. This needs to be contrasted with the Newton-Cartan supergravity theories constructed in [23, 27] that have zero torsion. To see the difference, it is instructive to compare the curvature of local time translations for the theories with and without torsion. Indicating the curvature of the torsionfull theory with $\mathcal{R}(H)$ and the one of the zero-torsion theory with $\hat{R}(H)$ we have

$$\begin{aligned}\mathcal{R}_{\mu\nu}(H) &= 2\,\partial_{[\mu}\tau_{\nu]} - 4\,b_{[\mu}\tau_{\nu]} - \frac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^0\psi_{\nu]+}\,,\\ \hat{R}_{\mu\nu}(H) &= 2\,\partial_{[\mu}\tau_{\nu]} - \frac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^0\psi_{\nu]+}\,.\end{aligned} \tag{5.1}$$

Note that the space-space components of both curvatures are the same. The difference is in the time-space component. In the torsionfull case, setting the time-space component to zero, is a conventional constraint that is used to solve for the spatial part b_a of the dilatation gauge field whereas in the torsionless case it represents an un-conventional constraint. Indeed, we have

$$b_a = \frac{1}{2}\,\hat{R}_{a0}(H)\,, \tag{5.2}$$

and therefore setting the torsion to zero, i.e.

$$b_a = 0\,, \tag{5.3}$$

leads to the un-conventional constraint $\hat{R}_{a0}(H)$ in the torsionless theory.

This points us to an interesting observation: the existence of a non-trivial truncation of the old minimal and new minimal Newton-Cartan supergravity multiplets constructed in section 4. Indeed, we shall show in this section how we can reduce the new minimal torsionfull theory constructed in subsection 4.2 to the known new minimal torsionless Newton-Cartan supergravity theory constructed in [23, 27].

We now investigate the consequences of imposing the zero-torsion constraint (5.3). It is convenient to use the explicit expression for the S-supersymmetry gauge field field ϕ_μ, which simplifies to

$$\phi_\mu = \gamma^{a0}\hat{\psi}_{a\mu-} - \frac{2}{w}\,\gamma_0\psi_{\mu+}\,S\,, \tag{5.4}$$

when we use the curvatures and constraints that we introduce below. The only dependent gauge fields of the Newton-Cartan supergravity theory are the connection fields for spatial rotations and Galilean boosts. For the supersymmetry rules of the independent gauge fields we find

$$\begin{aligned}\delta\tau_\mu &= \frac{1}{2}\,\bar{\epsilon}_+\gamma^0\psi_{\mu+}\,,\\ \delta e_\mu{}^a &= \frac{1}{2}\,\bar{\epsilon}_+\gamma^a\psi_{\mu-} + \frac{1}{2}\,\bar{\epsilon}_-\gamma^a\psi_{\mu+}\,,\\ \delta m_\mu &= \bar{\epsilon}_-\gamma^0\psi_{\mu-}\,,\\ \delta r_\mu &= -\frac{3}{8}\,\bar{\epsilon}_+\gamma^{a0}\hat{\psi}_{a\mu-} - \frac{3}{2w}\,\bar{\epsilon}_+\gamma^0\psi_{\mu+}\,S\,,\\ \delta S &= \frac{w}{8}\,\bar{\epsilon}_+\gamma^{a0}\hat{\psi}_{a0-}\,,\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\delta\psi_{\mu+} &= D_\mu\epsilon_+ + \gamma_0\epsilon_+ r_\mu + \frac{2}{w}\gamma_0\epsilon_+\tau_\mu S\,,\\ \delta\psi_{\mu-} &= D_\mu\epsilon_- - \gamma_0\epsilon_- r_\mu + \frac{1}{2}\gamma_{a0}\epsilon_+\omega_\mu{}^a + \frac{1}{w}\gamma_a\epsilon_+ e_\mu{}^a S\,.\end{aligned} \tag{5.6}$$

The curvatures and derivatives of the new minimal torsionless Newton-Cartan supergravity theory are now given by (5.1) and

$$\begin{aligned}\hat{R}_{\mu\nu}{}^a(P) &= 2\,\partial_{[\mu}e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}e_{\nu]}{}^b - 2\,\omega_{[\mu}{}^a\tau_{\nu]} - \bar{\psi}_{[\mu+}\gamma^a\psi_{\nu]-}\,,\\ \hat{R}_{\mu\nu}(Z) &= 2\,\partial_{[\mu}m_{\nu]} - \bar{\psi}_{[\mu-}\gamma^0\psi_{\nu]-}\,,\\ \hat{R}_{\mu\nu}(R) &= 2\,\partial_{[\mu}r_{\nu]} + \frac{3}{2w}\bar{\psi}_{[\mu+}\gamma^0\psi_{\nu]+}S + \frac{3}{4}\bar{\psi}_{[\mu+}\gamma^{a0}\hat{\psi}_{a\nu]-}\,,\\ \hat{D}_\mu S &= \partial_\mu S - \frac{w}{8}\bar{\psi}_{\mu+}\gamma^{a0}\hat{\psi}_{a0-}\,,\\ \hat{\psi}_{\mu\nu+} &= 2\,\partial_{[\mu}\psi_{\nu]+} - \frac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]+} - 2\,\gamma_0\psi_{[\mu+}r_{\nu]} - \frac{4}{w}\gamma_0\psi_{[\mu+}\tau_{\nu]}S\,,\\ \hat{\psi}_{\mu\nu-} &= 2\,\partial_{[\mu}\psi_{\nu]-} - \frac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]-} + 2\,\gamma_0\psi_{[\mu-}r_{\nu]} + \omega_{[\mu}{}^a\gamma_{a0}\psi_{\nu]+} - \frac{2}{w}\gamma_a\psi_{[\mu+}e_{\nu]}{}^a S\,.\end{aligned} \tag{5.7}$$

As we explained at the beginning of this section, the zero-torsion constraint (5.3) may convert a conventional constraint into an un-conventional one. If this happens we have to check if the supersymmetry variation of this un-conventional constraint leads to further constraints. To perform this check we need the transformation rules of the dependent connection gauge fields which reduce to

$$\begin{aligned}\delta\omega_\mu{}^{ab} &= -\frac{1}{2}\bar{\epsilon}_+\gamma^{[a}\hat{\psi}^{b]}{}_{\mu-} + \frac{1}{w}\bar{\epsilon}_+\gamma^{ab}\psi_{\mu+}S\,,\\ \delta\omega_\mu{}^a &= \bar{\epsilon}_-\gamma^0\hat{\psi}_\mu{}^a{}_- + \frac{1}{4}e_\mu{}^b\bar{\epsilon}_+\gamma^b\hat{\psi}^a{}_{0-} + \frac{1}{4}\bar{\epsilon}_+\gamma^a\hat{\psi}_{\mu0-} - \frac{1}{w}\bar{\epsilon}_+\gamma^{a0}\psi_{\mu-}S - \frac{1}{w}\bar{\epsilon}_-\gamma^{a0}\psi_{\mu+}S\,.\end{aligned} \tag{5.8}$$

The corresponding curvatures are given by

$$\begin{aligned}\hat{R}_{\mu\nu}{}^{ab}(J) &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} + \bar{\psi}_{[\mu+}\gamma^{[a}\hat{\psi}^{b]}{}_{\nu]-} - \frac{1}{w}\bar{\psi}_{[\mu+}\gamma^{ab}\psi_{\nu]+}S\,,\\ \hat{R}_{\mu\nu}{}^a(G) &= 2\,\partial_{[\mu}\omega_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}\omega_{\nu]}{}^b - 2\,\bar{\psi}_{[\mu-}\gamma^0\hat{\psi}_{\nu]}{}^a{}_- - \frac{1}{2}e_{[\nu}{}^b\bar{\psi}_{\mu]+}\gamma^b\hat{\psi}^a{}_{0-}\\ &\quad - \frac{1}{2}\bar{\psi}_{[\mu+}\gamma^a\hat{\psi}_{\nu]0-} + \frac{2}{w}\bar{\psi}_{[\mu+}\gamma^{a0}\psi_{\nu]-}S\,.\end{aligned} \tag{5.9}$$

We are now ready to discuss the constraint structure of the truncated theory. Some of the curvatures did not change, hence we can immediately infer, e.g., that

$$\hat{R}_{ab}(R) = 0\,, \qquad \frac{3}{4}\varepsilon^{ab}\,\hat{R}_{\mu\nu}{}^{ab}(J) = \hat{R}_{\mu\nu}(R)\,. \tag{5.10}$$

The constraints $\hat{R}_{\mu\nu}{}^a(P) = 0$ and $\hat{R}_{\mu\nu}(Z) = 0$ are identities when we insert the expressions for the connection gauge fields, i.e. they are conventional constraints. More importantly though, we find new constraints. This is due to the fact that we imposed $\hat{R}_{a0}(H) = 0$ which is an example of a conventional constraint (necessary to solve for the spatial part b_a of the dilatation gauge field) that gets converted into an un-conventional constraint. Together with the constraint $\hat{R}_{ab}(H) = 0$ which reads the same in the torsionfull as well as in the torsionless case, we find $\hat{R}_{\mu\nu}(H) = 0$. Supersymmetry variations of this constraint reveal the following additional constraints:

$$\xrightarrow{Q_-} \qquad \hat{\psi}_{ab-} = 0 \tag{5.11}$$

$$\hat{R}_{\mu\nu}(H) = 0 \quad \xrightarrow{Q_+} \quad \hat{\psi}_{\mu\nu+} = 0 \quad \xrightarrow{Q_+} \quad \hat{R}_{\mu\nu}{}^{ab}(J) = \frac{4}{w}\,\varepsilon^{ab}\,\tau_{[\mu}\hat{D}_{\nu]}\,S\,. \tag{5.12}$$

Further transformations only lead to Bianchi identities. By combining the constraints (5.12) with (5.10) we furthermore derive that

$$-\frac{6}{w}\,\hat{D}_{[\mu}\big(\tau_{\nu]}\,S\big) = 2\,\hat{D}_{[\mu}r_{\nu]}\,. \tag{5.13}$$

This constraint implies that up to an arbitrary constant the R-symmetry gauge field r_μ is determined by τ_μ and S. In fact, when we set

$$r_\mu = -\frac{3}{w}\,\tau_\mu\,S\,, \tag{5.14}$$

the truncated theory leads to the off-shell Newton-Cartan multiplet that was presented in [27]. Furthermore, by making the redefinition

$$r_\mu = -V_\mu - \frac{1}{w}\,\tau_\mu\,S\,, \tag{5.15}$$

one obtains precisely the off-shell multiplet that is obtained when taking the limit of the new minimal Poincaré multiplet as described in [27].

6 Conclusions and outlook

In this paper we have discussed extensions of non-relativistic supergravity to include conformal symmetries. As an example we have constructed a three-dimensional theory of Schrödinger supergravity, i.e. a theory that realizes a Schrödinger superalgebra, and we have successfully constructed two matter multiplets. These results are summarized in table 1. We have then introduced a non-relativistic version of the superconformal tensor calculus and used it to construct two inequivalent off-shell formulations, called the old minimal and new minimal formulation, of a three-dimensional non-relativistic Newton-Cartan supergravity multiplet with torsion.

The appearance of torsion is one of the points where our analysis differs from the relativistic one. In the relativistic case the full gauge field of dilatations b_μ is a Stückelberg field for special conformal transformations and the theory is by construction torsionless. In contrast, in the non-relativistic case only the time component b is a Stückelberg field for the single (scalar) special conformal transformation of the Schrödinger superalgebra. The spatial components b_a on the other hand are dependent gauge field components and

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they are proportional to torsion. Thus, unless we set set $b_a = 0$ as we did in section 5, the superconformal approach always leads to torsionfull theories in the non-relativistic setting.

It would be interesting to see how one can go on-shell in the presence of torsion. This is not a straightforward thing to do since to our knowledge even in the bosonic case the equations of motion describing Newton-Cartan gravity with torsion have not been written down so far.[5] Even in the absence of torsion the equations of motion have only been written down under the assumption that the curvature of spatial rotations is zero [22]. It is not difficult to write down the equations of motion for the case that this curvature is nonzero but the price one has to pay is that one has to add extra terms to the equation of motion proposed in [22] that break the invariance under central charge transformations [14]. In the bosonic case this extended equation of motion can be understood by applying a conformal tensor calculus at the level of the equations of motion (without the need to write down an action) using a single compensator scalar transforming under dilatations. The situation gets more intricate when one introduces non-zero torsion because in that case a second compensating scalar is needed that transforms non-trivially under central charge transformations. This second compensating scalar should therefore be part of a different multiplet than the scalar and vector multiplets we considered in this work. The construction of such a multiplet is different from our investigations in section 3 and goes beyond the scope of this paper. We hope to return to the issue of how to go on-shell with a non-flat foliation space and in the presence of torsion in a future work.

Perhaps we can get some inspiration from a similar problem in the relativistic case. In the four-dimensional $\mathcal{N} = 2$ off-shell formulation one also has to use two compensator multiplets in order to be able to write down an action [39]. The first compensator multiplet fixes dilatations, S-supersymmetry and a chiral U(1) symmetry. The second compensator multiplet fixes a remaining local chiral SU(2) symmetry and it is needed only to be able to write down an action. In our analogy this would correspond to fixing central charge symmetry. Maybe a non-relativistic matter multiplet with a scalar field that has a non-trivial central charge transformation could be found as a non-relativistic (three-dimensional) analogue of one of the three compensator multiplets used in [39].

In this paper we only considered the construction of pure Newton-Cartan supergravity. A natural generalization of our work would be to consider general non-relativistic matter-coupled Newton-Cartan supergravity theories with simple or extended supersymmetry. This would answer the question of what the non-relativistic analogue is of the geometries that one encounters in the relativistic matter-coupled supergravity theories. For example, it would be interesting to find out what the non-relativistic analogue is of a Kähler target space.

Finally, it would be very interesting to find higher-dimensional analogues of our results on Newton-Cartan supergravity. So far, the use of gauging techniques has failed to lead to e.g. a four-dimensional theory of Newton-Cartan supergravity. It is a priori not clear what auxiliary fields are needed to close the supersymmetry algebra. Presumably, similar obstacles are encountered if one were to try to gauge a four-dimensional Schrödinger superalge-

[5]A systematic approach to construct such an equation of motion will be given in [38].

bra. The limiting procedure discussed in [27], if its application is equally straightforward in higher dimensions, might be the simplest way to find a four-dimensional Newton-Cartan supergravity theory.

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A Details on the off-shell multiplets

This appendix contains more details about the two off-shell formulations of torsional Newton-Cartan supergravity that feature in the main text. In particular, we give the transformation rules of all independent fields of the old minimal formulation in appendix A.1. Those for the new minimal formulation were given in section 4.2. In appendix A.2 we give the transformation rules of the dependent gauge fields of the new minimal formulation which are needed to show that the commutator algebra closes.

A.1 "Old minimal" formulation

We collect here the transformation rules of the independent gauge fields of the old minimal formulation. We find that the bosonic gauge fields transform as follows under the bosonic transformations

$$\begin{aligned}
\delta\tau_\mu &= 0\,,\\
\delta e_\mu{}^a &= \lambda^a{}_b\, e_\mu{}^b + \tau_\mu\, \lambda^a \quad ,\\
\delta m_\mu &= \partial_\mu\sigma + \lambda^a e_\mu{}^a\,,\\
\delta r_\mu &= -\frac{3}{4w}\,\lambda^a\,\bar{\psi}_{\mu+}\gamma^a\chi_-\,,\\
\delta F_1 &= 0\,,\\
\delta F_2 &= 0\,,
\end{aligned} \tag{A.1}$$

while the fermionic gauge fields transform as

$$\begin{aligned}
\delta\psi_{\mu+} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu+} + \frac{2}{w}\,\tau_\mu\,\lambda^a\gamma_a\chi_-\,,\\
\delta\psi_{\mu-} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu-} - \frac{1}{2}\,\lambda^a\gamma_{a0}\psi_{\mu+} + \frac{1}{w}\,e_\mu{}^a\,\lambda^b\,\gamma_a\gamma_{b0}\chi_-\,,\\
\delta\chi_- &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\chi_-\,.
\end{aligned} \tag{A.2}$$

Note the non-trivial Galilean boost transformation of the R-symmetry gauge field r_μ in (A.1). The supersymmetry transformations are given by

$$\begin{aligned}
\delta\tau_\mu &= \frac{1}{2}\,\bar\epsilon_+\gamma^0\psi_{\mu+}\,,\\
\delta e_\mu{}^a &= \frac{1}{2}\,\bar\epsilon_+\gamma^a\psi_{\mu-} + \frac{1}{2}\,\bar\epsilon_-\gamma^a\psi_{\mu+}\,,\\
\delta m_\mu &= \bar\epsilon_-\gamma^0\psi_{\mu-}\,,
\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}
\delta\psi_{\mu+} =\ & D_\mu\epsilon_+ - e_\mu{}^a\, b_a\,\epsilon_+ + \left(r_\mu - \frac{2}{3}\,\tau_\mu\tau^\rho r_\rho\right)\gamma_0\epsilon_+ + \frac{2}{3}\,\gamma^a\epsilon_-\,\tau_\mu\, e^\rho{}_a\, r_\rho + \gamma^{a0}\epsilon_-\,\tau_\mu\, b_a\\
& - \frac{1}{w}\,\gamma_0\epsilon_-\,\tau_\mu\, F_1 + \frac{1}{w}\,\epsilon_-\,\tau_\mu\, F_2 - \frac{1}{w}\,\psi_{\mu+}\,\bar\epsilon_-\chi_- + \frac{3}{2w}\,\gamma_0\psi_{\mu+}\,\bar\epsilon_-\gamma^0\chi_-\\
& + \frac{1}{w}\,\tau_\mu\,\gamma^a\chi_-\,\bar\epsilon_+\gamma^a\psi_{\rho-}\,\tau^\rho - \frac{1}{w}\,\tau_\mu\,\gamma^{a0}\chi_-\,\bar\epsilon_-\psi_{\rho-}\, e^\rho{}_a + \frac{1}{w}\,\tau_\mu\,\gamma^a\chi_-\,\bar\epsilon_-\gamma^0\psi_{\rho-}\, e^\rho{}_a\,,\\
\delta\psi_{\mu-} =\ & D_\mu\epsilon_- - r_\mu\,\gamma_0\epsilon_- + \frac{1}{2}\,\omega_\mu{}^a\gamma_{a0}\epsilon_+ - \frac{1}{3}\,\gamma^a\epsilon_+\, e_\mu{}^a\tau^\rho\, r_\rho - \frac{1}{3}\,\gamma^a\gamma^{b0}\epsilon_+\, e_\mu{}^a e^\rho{}_b\, r_\rho\\
& + \frac{1}{2}\,\gamma^a\gamma^b\epsilon_-\, e_\mu{}^a\, b_b - \frac{1}{2w}\,\gamma^a\epsilon_-\, e_\mu{}^a\, F_1 - \frac{1}{2w}\,\gamma_{a0}\epsilon_-\, e_\mu{}^a\, F_2 - \frac{3}{2w}\,\gamma_0\psi_{\mu-}\,\bar\epsilon_-\gamma^0\chi_-\\
& - \frac{1}{2w}\,\gamma^a\gamma^{b0}\chi_-\,\bar\epsilon_+\gamma^b\psi_{\rho-}\, e_\mu{}^a\,\tau^\rho - \frac{1}{2w}\,\gamma^a\gamma^b\chi_-\,\bar\epsilon_-\psi_{\rho-}\, e_\mu{}^a e^\rho{}_b\\
& - \frac{1}{2w}\,\gamma^a\gamma^{b0}\chi_-\,\bar\epsilon_-\gamma^0\psi_{\rho-}\, e_\mu{}^a e^\rho{}_b\,,\\
\delta\chi_- =\ & -\frac{w}{6}\,\gamma^{a0}\epsilon_+\, e^\mu{}_a\, r_\mu - \frac{w}{4}\,\gamma^a\epsilon_+\, b_a - \frac{1}{3w}\,\epsilon_-\,\bar\chi_-\chi_- + \frac{1}{4}\,\epsilon_+\, F_1 - \frac{1}{4}\,\gamma_0\epsilon_+\, F_2\\
& - \frac{1}{4}\,\gamma^a\gamma^b\chi_-\,\bar\epsilon_+\gamma^b\psi_{\mu-}\, e^\mu{}_a\,.
\end{aligned} \tag{A.4}$$

Finally, for the R-symmetry gauge field r_μ and the auxiliary scalars F_1 and F_2 we find the following transformations:

$$\begin{aligned}
\delta r_\mu =\ & -\frac{3}{8}\,\bar\epsilon_+\phi_\mu + \frac{1}{4}\,\bar\epsilon_+\gamma^0\psi_{\mu+}\,\tau^\rho r_\rho + \frac{1}{4}\,\bar\epsilon_-\gamma^{a0}\psi_{\mu+}\, e^\rho{}_a r_\rho + \frac{3}{8}\,\bar\epsilon_-\gamma^{a0}\psi_{\mu+}\, b_a\\
& + \frac{3}{8w}\,\bar\epsilon_-\gamma^0\psi_{\mu+}\, F_1 - \frac{3}{8w}\,\bar\epsilon_-\psi_{\mu+}\, F_2 - \frac{3}{8w}\,\bar\epsilon_+\gamma^a\psi_{\rho-}\,\tau^\rho\,\bar\psi_{\mu+}\gamma^a\chi_-\\
& + \frac{3}{8w}\,\bar\epsilon_-\psi_{\rho-}\, e^\rho{}_a\,\bar\psi_{\mu+}\gamma^{a0}\chi_- - \frac{3}{8w}\,\bar\epsilon_-\gamma^0\psi_{\rho-}\, e^\rho{}_a\,\bar\psi_{\mu+}\gamma^a\chi_-\,,\\
\delta F_1 =\ & \bar\epsilon_+\gamma^0\tau^\mu\hat D_\mu\chi_- + \bar\epsilon_-\gamma^a e^\mu{}_a\hat D_\mu\chi_- + \frac{2}{w}\,\bar\epsilon_-\chi_-\, F_1 - \frac{2}{w}\,\bar\epsilon_-\gamma^0\chi_-\, F_2\\
& - \frac{1}{4}\,\bar\epsilon_+\gamma^a\psi_{\mu-}\, e^\mu{}_a\, F_1 + \frac{1}{4}\,\bar\epsilon_+\gamma^{a0}\psi_{\mu-}\, e^\mu{}_a\, F_2 + \frac{w}{4}\,\bar\epsilon_+\gamma^{a0}\phi_\mu\, e^\mu{}_a\\
& + \frac{1}{2}\,\bar\epsilon_+\gamma^a\gamma_{b0}\chi_-\, e^\mu{}_a\,\omega_\mu{}^b - \frac{w}{6}\,\bar\epsilon_+\gamma^a\psi_{\mu+}\, e^\mu{}_a\tau^\rho\, r_\rho - \frac{w}{6}\,\bar\epsilon_+\gamma^a\gamma^{b0}\psi_{\mu-}\, e^\mu{}_a e^\rho{}_b\, r_\rho\\
& + \frac{2(w+1)}{3}\,\bar\epsilon_+\chi_-\,\tau^\mu\, r_\mu - \frac{2(w+1)}{3}\,\bar\epsilon_-\gamma^{a0}\chi_-\, e^\mu{}_a\, r_\mu + \frac{w}{4}\,\bar\epsilon_+\gamma^a\gamma^b\psi_{\mu-}\, e^\mu{}_a\, b_b\\
& + (w+1)\,\bar\epsilon_-\gamma^a\chi_-\, b_a + \frac{1}{4}\,\bar\epsilon_+\gamma^a\gamma^{b0}\chi_-\,\bar\psi_{\rho-}\gamma^b\psi_{\mu+}\, e^\mu{}_a\tau^\rho - \frac{1}{4}\,\bar\epsilon_+\chi_-\,\bar\psi_{\rho-}\psi_{\mu-}\, e^\mu{}_a e^\rho{}_a\\
& + \frac{1}{4}\,\bar\epsilon_+\gamma^{ab0}\chi_-\,\bar\psi_{\rho-}\gamma^0\psi_{\mu-}\, e^\mu{}_a e^\rho{}_b\,,
\end{aligned} \tag{A.5}$$

and

$$
\begin{aligned}
\delta F_2 = {} & \bar\epsilon_+\gamma^0\tau^\mu\hat D_\mu\chi_- - \bar\epsilon_-\gamma^{a0}e^\mu{}_a\hat D_\mu\chi_- + \frac{2}{w}\,\bar\epsilon_-\chi_-\,F_2 + \frac{2}{w}\,\bar\epsilon_-\gamma^0\chi_-\,F_1 \\
& - \frac14\,\bar\epsilon_+\gamma^{a0}\psi_{\mu-}\,e^\mu{}_a\,F_1 - \frac14\,\bar\epsilon_+\gamma^a\psi_{\mu-}\,e^\mu{}_a\,F_2 - \frac{w}{4}\,\bar\epsilon_+\gamma^0\phi_\mu\,e^\mu{}_a \\
& - \frac12\,\bar\epsilon_+\gamma^a\gamma^b\chi_-\,e^\mu{}_a\omega_\mu{}^b - \frac{w}{6}\,\bar\epsilon_+\gamma^{a0}\psi_{\mu+}\,e^\mu{}_a\tau^\rho\,r_\rho - \frac{w}{6}\,\bar\epsilon_+\gamma^a\gamma^b\psi_{\mu+}\,e^\mu{}_a e^\rho{}_b\,r_\rho \\
& - \frac{2(w+1)}{3}\,\bar\epsilon_+\gamma^0\chi_-\,\tau^\mu\,r_\mu - \frac{2(w+1)}{3}\,\bar\epsilon_-\gamma^a\chi_-\,e^\mu{}_a\,r_\mu - \frac{w}{4}\,\bar\epsilon_+\gamma^a\gamma^{b0}\psi_{\mu-}\,e^\mu{}_a\,b_b \\
& - (w+1)\,\bar\epsilon_-\gamma^{a0}\chi_-\,b_a + \frac14\,\bar\epsilon_+\gamma^a\gamma^b\chi_-\,\bar\psi_{\rho-}\gamma^b\psi_{\mu+}\,e^\mu{}_a\tau^\rho \\
& + \frac14\,\bar\epsilon_+\gamma^0\chi_-\,\bar\psi_{\rho-}\psi_{\mu-}\,e^\mu{}_a e^\rho{}_a + \frac14\,\bar\epsilon_+\gamma^{ab}\chi_-\,\bar\psi_{\rho-}\gamma^0\psi_{\mu-}\,e^\mu{}_a e^\rho{}_b\,.
\end{aligned}
\tag{A.6}
$$

These are only the transformations of the independent fields. Those of the dependent gauge fields $\omega_\mu{}^{ab}$, $\omega_\mu{}^a$, f_μ, b_a and ϕ_μ would be even longer, which is why we refrain from giving them here. They can be derived easily from eqs. (2.5), (2.7), (2.33) and (2.34). Note that in the transformations of $\omega_\mu{}^a$ and ϕ_μ one should also take into account the new expressions for curvatures of the gravitini $\psi_{\mu-}$ and of r_μ, see also the next section were we do work out those transformations for the dependent fields.

A.2 "New minimal" formulation

In the new minimal formulation the bosonic transformations of the dependent gauge fields $\omega_\mu{}^{ab}$, $\omega_\mu{}^a$, b_a and ϕ_μ are given by

$$
\begin{aligned}
\delta\omega_\mu{}^{ab} &= \partial_\mu\lambda^{ab}\,, \\
\delta\omega_\mu{}^a &= \partial_\mu\lambda^a - \omega_\mu{}^{ab}\lambda^b + \lambda^a\,e_\mu{}^b\,b_b + e_\mu{}^a\lambda^b\,b_b + \lambda^a{}_b\,\omega_\mu{}^b\,, \\
\delta b_a &= \lambda^a{}_b\,b_b\,, \\
\delta\phi_\mu &= \frac14\,\lambda^{ab}\gamma_{ab}\phi_\mu - \gamma_0\phi_\mu\,\rho - \psi_{\mu+}\,\lambda^a\,b_a\,,
\end{aligned}
\tag{A.7}
$$

while the fermionic transformations read

$$
\begin{aligned}
\delta\omega_\mu{}^{ab} &= -\frac14\,\bar\epsilon_+\gamma^{ab0}\phi_\mu + \frac{1}{2w}\,\bar\epsilon_+\gamma^{ab}\psi_{\mu+}\,S + \bar\epsilon_-\gamma^{[a}\psi_{\mu+}\,b^{b]}\,, \\
\delta\omega_\mu{}^a &= \bar\epsilon_-\gamma^0\hat\psi_\mu{}^a{}_- + \frac14\,e_\mu{}^b\,\bar\epsilon_+\gamma^b\hat\psi^a{}_{0-} + \frac14\,\bar\epsilon_+\gamma^a\hat\psi_{\mu 0-} - \frac1w\,\bar\epsilon_+\gamma^{a0}\psi_{\mu-}\,S - \frac1w\,\bar\epsilon_-\gamma^{a0}\psi_{\mu+}\,S \\
&\quad - 2\,\varepsilon^{ab}\,\bar\epsilon_-\psi_{\mu-}\,b_b + e_\mu{}^b e^\rho{}_a\,\bar\epsilon_-\gamma^0\gamma^b\gamma^c\psi_{\rho-}\,b_c - \frac12\,e_\mu{}^b e^\rho{}_a\,\bar\epsilon_-\gamma^b\left(\phi_\rho + \frac2w\,\gamma_0\psi_{\rho+}\,S\right) \\
&\quad + \frac12\,e_\mu{}^a\,\tau^\rho\,\bar\epsilon_+\gamma^b\psi_{\rho+}\,b_b\,, \\
\delta b_a &= -\frac12\,\bar\epsilon_+\gamma^b\psi_{\mu-}\,e^\mu{}_b\,b_a - \frac12\,\bar\epsilon_+\gamma^0\psi_{\mu+}\,\tau^\mu\,b_a - \frac14\,\bar\epsilon_+\gamma^0\phi_\mu\,e^\mu{}_a - \frac{1}{2w}\,\bar\epsilon_+\psi_{\mu+}\,e^\mu{}_a\,S\,, \\
\delta\phi_\mu &= \epsilon_+\,f_\mu - \frac23\,\gamma^0\epsilon_+\left[\hat R_{\mu 0}(R) + \frac32\,\tau^\nu\,\bar\psi_{[\mu-}\gamma^{a0}\psi_{\nu]+}\,b_a - \frac{3}{4w}\,\tau^\nu\,\bar\psi_{[\mu+}\gamma^0\psi_{\nu]+}\,S\right]
\end{aligned}
$$

$$
+ \frac{4}{3}\gamma^a \epsilon_- \left[\hat{R}_{\mu a}(R) + \frac{3}{2}\, e^\nu{}_a\, \bar{\psi}_{[\mu -}\gamma^{b0}\psi_{\nu]+}\, b_b - \frac{3}{4w}\, e^\nu{}_a\, \bar{\psi}_{[\mu +}\gamma^0\psi_{\nu]+}\, S \right]
$$

$$
- \left(D_\mu + e_\mu{}^a\, b_a + r_\mu\, \gamma_0\right)\left(\frac{2}{w}\,\gamma_0\epsilon_+\, S + 2\,\gamma^{b0}\epsilon_-\, b_b\right). \tag{A.8}
$$

Here, we used the covariant Newton-Cartan curvatures of the independent gauge fields $\psi_{\mu-}$ and r_μ, which are are given by

$$
\begin{aligned}
\hat{\psi}_{\mu\nu -} &= 2\,\partial_{[\mu}\psi_{\nu]-} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]-} - 2\, r_{[\mu}\,\gamma_0\psi_{\nu]-} + \omega_{[\mu}{}^a\,\gamma_{a0}\psi_{\nu]+} \\
&\quad + 2\,\gamma^a\gamma^b\psi_{[\nu -}\, e_{\mu]}{}^a b_b + \frac{2}{w}\,\gamma_a\psi_{[\nu +}\, e_{\mu]}{}^a\, S\,, \\
\hat{R}_{\mu\nu}(R) &= 2\,\partial_{[\mu}r_{\nu]} + \frac{3}{4}\,\bar{\psi}_{[\mu +}\phi_{\nu]} - \frac{3}{2}\,\bar{\psi}_{[\mu -}\gamma^{a0}\psi_{\nu]+}\, b_a + \frac{3}{4w}\,\bar{\psi}_{[\mu +}\gamma^0\psi_{\nu]+}\, S\,.
\end{aligned} \tag{A.9}
$$

Finally, the expression for the special conformal gauge field f_μ can be found in eq. (2.32). We did not derive the transformation rule of f_μ because no independent field transforms to f_μ. Therefore, its variation is not needed for any checks on the closure of the commutator algebra.

References

[1] C. Hoyos and D.T. Son, *Hall viscosity and electromagnetic response*, Phys. Rev. Lett. **108** (2012) 066805 [arXiv:1109.2651] [INSPIRE].

[2] D.T. Son, *Newton-Cartan geometry and the quantum Hall effect*, arXiv:1306.0638 [INSPIRE].

[3] A.G. Abanov and A. Gromov, *Electromagnetic and gravitational responses of two-dimensional noninteracting electrons in a background magnetic field*, Phys. Rev. **B 90** (2014) 014435 [arXiv:1401.3703] [INSPIRE].

[4] A. Gromov and A.G. Abanov, *Thermal Hall effect and geometry with torsion*, Phys. Rev. Lett. **114** (2015) 016802 [arXiv:1407.2908] [INSPIRE].

[5] S. Kachru, X. Liu and M. Mulligan, *Gravity duals of Lifshitz-like fixed points*, Phys. Rev. **D 78** (2008) 106005 [arXiv:0808.1725] [INSPIRE].

[6] K. Balasubramanian and J. McGreevy, *Gravity duals for non-relativistic CFTs*, Phys. Rev. Lett. **101** (2008) 061601 [arXiv:0804.4053] [INSPIRE].

[7] D.T. Son, *Toward an AdS/cold atoms correspondence: a geometric realization of the Schrödinger symmetry*, Phys. Rev. **D 78** (2008) 046003 [arXiv:0804.3972] [INSPIRE].

[8] M.H. Christensen, J. Hartong, N.A. Obers and B. Rollier, *Torsional Newton-Cartan geometry and Lifshitz holography*, Phys. Rev. **D 89** (2014) 061901 [arXiv:1311.4794] [INSPIRE].

[9] M.H. Christensen, J. Hartong, N.A. Obers and B. Rollier, *Boundary stress-energy tensor and Newton-Cartan geometry in Lifshitz holography*, JHEP **01** (2014) 057 [arXiv:1311.6471] [INSPIRE].

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[10] R. Banerjee, A. Mitra and P. Mukherjee, *Localization of the galilean symmetry and dynamical realization of Newton-Cartan geometry*, Class. Quant. Grav. **32** (2015) 045010 [arXiv:1407.3617] [INSPIRE].

[11] K. Jensen, *On the coupling of galilean-invariant field theories to curved spacetime*, arXiv:1408.6855 [INSPIRE].

[12] J. Hartong, E. Kiritsis and N.A. Obers, *Lifshitz space-times for Schrödinger holography*, Phys. Lett. **B 746** (2015) 318 [arXiv:1409.1519] [INSPIRE].

[13] J. Hartong, E. Kiritsis and N.A. Obers, *Schrödinger invariance from Lifshitz isometries in holography and field theory*, Phys. Rev. **D 92** (2015) 066003 [arXiv:1409.1522] [INSPIRE].

[14] E.A. Bergshoeff, J. Hartong and J. Rosseel, *Torsional Newton-Cartan geometry and the Schrödinger algebra*, Class. Quant. Grav. **32** (2015) 135017 [arXiv:1409.5555] [INSPIRE].

[15] K. Jensen and A. Karch, *Revisiting non-relativistic limits*, JHEP **04** (2015) 155 [arXiv:1412.2738] [INSPIRE].

[16] J. Hartong, E. Kiritsis and N.A. Obers, *Field theory on Newton-Cartan backgrounds and symmetries of the Lifshitz vacuum*, JHEP **08** (2015) 006 [arXiv:1502.00228] [INSPIRE].

[17] M. Geracie, K. Prabhu and M.M. Roberts, *Curved non-relativistic spacetimes, newtonian gravitation and massive matter*, J. Math. Phys. **56** (2015) 103505 [arXiv:1503.02682] [INSPIRE].

[18] M. Geracie, K. Prabhu and M.M. Roberts, *Fields and fluids on curved non-relativistic spacetimes*, JHEP **08** (2015) 042 [arXiv:1503.02680] [INSPIRE].

[19] E. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie)*, Ann. École Norm. Sup. **40** (1923) 325.

[20] E. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie) (suite)*, Ann. École Norm. Sup. **41** (1924) 1.

[21] C.W. Misner, K. Thorne and J. Wheeler, *Gravitation*, W.H. Freeman and company, San Francisco, U.S.A. (1973).

[22] R. Andringa, E. Bergshoeff, S. Panda and M. de Roo, *Newtonian gravity and the Bargmann algebra*, Class. Quant. Grav. **28** (2011) 105011 [arXiv:1011.1145] [INSPIRE].

[23] R. Andringa, E.A. Bergshoeff, J. Rosseel and E. Sezgin, *3D Newton-Cartan supergravity*, Class. Quant. Grav. **30** (2013) 205005 [arXiv:1305.6737] [INSPIRE].

[24] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, Commun. Math. Phys. **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[25] M. Mariño, *Lectures on localization and matrix models in supersymmetric Chern-Simons-matter theories*, J. Phys. **A 44** (2011) 463001 [arXiv:1104.0783] [INSPIRE].

[26] G. Festuccia and N. Seiberg, *Rigid supersymmetric theories in curved superspace*, JHEP **06** (2011) 114 [arXiv:1105.0689] [INSPIRE].

[27] E. Bergshoeff, J. Rosseel and T. Zojer, *Newton-Cartan (super)gravity as a non-relativistic limit*, Class. Quant. Grav. **32** (2015) 205003 [arXiv:1505.02095] [INSPIRE].

[28] D.Z. Freedman and A. Van Proeyen, *Supergravity*, Cambridge University Press, Cambridge U.K. (2012).

JHEP11(2015)180

[29] J.A. de Azcarraga and J. Lukierski, *Galilean superconformal symmetries*, Phys. Lett. **B 678** (2009) 411 [arXiv:0905.0141] [INSPIRE].

[30] M. Sakaguchi, *Super Galilean conformal algebra in AdS/CFT*, J. Math. Phys. **51** (2010) 042301 [arXiv:0905.0188] [INSPIRE].

[31] A. Bagchi and I. Mandal, *Supersymmetric extension of galilean conformal algebras*, Phys. Rev. **D 80** (2009) 086011 [arXiv:0905.0580] [INSPIRE].

[32] J.P. Gauntlett, J. Gomis and P.K. Townsend, *Supersymmetry and the physical phase space formulation of spinning particles*, Phys. Lett. **B 248** (1990) 288 [INSPIRE].

[33] M. Leblanc, G. Lozano and H. Min, *Extended superconformal Galilean symmetry in Chern-Simons matter systems*, Annals Phys. **219** (1992) 328 [hep-th/9206039] [INSPIRE].

[34] C. Duval and P.A. Horvathy, *On Schrödinger superalgebras*, J. Math. Phys. **35** (1994) 2516 [hep-th/0508079] [INSPIRE].

[35] M. Sakaguchi and K. Yoshida, *More super Schrödinger algebras from $PSU(2,2|4)$*, JHEP **08** (2008) 049 [arXiv:0806.3612] [INSPIRE].

[36] P.S. Howe, J.M. Izquierdo, G. Papadopoulos and P.K. Townsend, *New supergravities with central charges and Killing spinors in $(2+1)$-dimensions*, Nucl. Phys. **B 467** (1996) 183 [hep-th/9505032] [INSPIRE].

[37] J. Gomis, K. Kamimura and P.K. Townsend, *Non-relativistic superbranes*, JHEP **11** (2004) 051 [hep-th/0409219] [INSPIRE].

[38] H. Afshar, E.A. Bergshoeff, A. Mehra, P. Parekh and B. Rollier, *Hořava-Lifshitz gravity and Schrödinger scalar field theories*, work in progress, UG-15-55.

[39] B. de Wit, R. Philippe and A. Van Proeyen, *The improved tensor multiplet in $N = 2$ supergravity*, Nucl. Phys. **B 219** (1983) 143 [INSPIRE].