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Higher spins in AdS$_5$ at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT

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Abstract: We consider general-symmetry higher spin fields in AdS$_5$ and derive the expressions for their one-loop corrections to vacuum energy E_c and the associated 4d boundary conformal anomaly a-coefficient. We propose a similar expression for the second conformal anomaly c-coefficient. We show that all the three quantities $(E_c, \mathrm{a}, \mathrm{c})$ computed for $\mathcal{N} = 8$ gauged 5d supergravity are equal to $-\frac{1}{2}$ of their values for $\mathcal{N} = 4$ conformal 4d supergravity and also to twice the values for $\mathcal{N} = 4$ Maxwell multiplet. This gives a 5d derivation of the fact that the system of $\mathcal{N} = 4$ conformal supergravity and four $\mathcal{N} = 4$ Maxwell multiplets is anomaly free. The values of $(E_c, \mathrm{a}, \mathrm{c})$ for the states at level p of Kaluza-Klein tower of 10d type IIB supergravity compactified on S^5 turn out to be equal to those for p copies of $\mathcal{N} = 4$ Maxwell multiplets. This may be related to the fact that these states appear in the tensor product of p superdoubletons. Under a natural regularization of the sum over p, the full 10d supergravity contribution is then minus that of one Maxwell multiplet, in agreement with the standard adjoint AdS/CFT duality (SU(N) SYM contribution is $N^2 - 1$ times that of one Maxwell multiplet). We also verify the matching of $(E_c, \mathrm{a}, \mathrm{c})$ for spin 0 and $\frac{1}{2}$ boundary theory cases of vectorial AdS/CFT duality. The consistency conditions for vectorial AdS/CFT turn out to be equivalent to the cancellation of anomalies in the closely related 4d conformal higher spin theories. In addition, we study novel example of the vectorial AdS/CFT duality when the boundary theory is described by free spin 1 fields and is dual to a particular higher spin theory in AdS$_5$ containing fields in mixed-symmetry representations. We also discuss its supersymmetric generalizations.

Keywords: Higher Spin Symmetry, AdS-CFT Correspondence, Anomalies in Field and String Theories

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Contents

1 Introduction

$\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ framework leads to interesting connections between properties of conformal fields in dimension d and their counterparts in $d+1$. In particular, there are "kinematic" relations based on symmetries and special properties of AdS type spaces. One set of such relations involves singlet sector of free CFT_d, dual higher spin theory in AdS_{d+1} and "shadow" conformal higher spin theory in d dimensions (see, e.g., [1–7] for some recent discussions related to the topic of this paper). Here we will be interested in the case of $d = 4$.

Starting, e.g., with a free massless complex scalar theory $\int d^4x\, \Phi_r^* \partial^2 \Phi_r$ one gets a tower of conserved symmetric traceless higher spin currents $J_s \sim \Phi_r^* \partial^s \Phi_r$ which are primary conformal fields of dimension $\Delta = 2 + s \equiv \Delta_+$. Adding these currents to the action with the source or shadow fields $\varphi_s(x)$ one observes that ϕ_s has the same dimension $\Delta_- = 4 - \Delta_+ = 2 - s$ and effectively the same algebraic and gauge symmetries as (in general, non-unitary) conformal higher spin (CHS) fields.

Integrating out the free fields Φ_r gives an "induced" action for φ_s with the kinetic term $\mathrm{K}(x, x') \sim \langle J_s(x) J_s(x')\rangle$. The leading (logarithmically divergent) local part of this action is the same as the CHS action $\int d^4x\, \varphi_s \partial^{2s} \varphi_s + \ldots$ (with $s = 1$ being Maxwell vector, $s = 2$ being Weyl graviton, etc.). From the dual AdS_5 perspective (implying matching between the correlators of currents and amplitudes for dual AdS fields ϕ_s) this induced action can be found upon the substitution of the solution of the Dirichlet problem with $\phi_s\big|_\partial = \varphi_s$ into the classical 5d action for a massless spin s field ϕ_s.

In addition to this "tree-level" relation between 5d fields ϕ_s and and 4d conformal higher spin fields φ_s (or shadow counterparts of the conserved currents J_s) there is also a relation between the corresponding one-loop partition functions, i.e. between the determinant of the 4d kinetic operator $\mathrm{K} \sim \partial^{2s}\delta(x, x')$ and the ratio of determinants of 2nd-order 5d operators for the field ϕ_s with Neumann-type (Δ_-) and Dirichlet-type (Δ_+) boundary conditions. This relation has essentially a "kinematic" origin belonging to a general class of bulk-boundary relations discussed in [8]; for scalar operators it was also implicit in mathematics literature as discussed in [9, 10]. In the context of AdS/CFT it appeared in the context of the discussion of the bulk counterpart of a "double trace" deformation of the boundary CFT (see [11–14, 9, 10, 1]).

The generalization to higher symmetric tensors was made explicit in [1, 3, 4]). In the case when the 4d boundary is a sphere S^4 this leads to an expression for the conformal anomaly a-coefficient of the 4d CHS field in terms of the properties of the AdS_5 determinants [1].[1] In the case of the $\mathbb{R} \times S^3$ boundary one gets a relation for the AdS_5 vacuum energy or the Casimir energy on S^3 for totally symmetric CHS fields [7]. For a more general curved 4d boundary one should be able to obtain also a 5d expression for the second conformal anomaly coefficient c.

The point which will be important below is that instead of a 4d CHS field we may consider a generic primary 4d conformal field that will be associated to a particular (in

[1]From the AdS/CFT point of view this is related, at the same time, to the change of the a-coefficient under the RG flow induced by double-trace deformation.

general, massive or massless higher spin) field in AdS_5 which will effectively encode its quantum characteristics. Dimension 4 is the first case when the conformal fields and the dual higher spin fields in AdS_5 are not only totally symmetric, but may also appear in mixed-symmetry representations (described by SO(4) Young tableau with two rows). We shall use the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ weights (j_1, j_2) to label a representation of the Lorentz group (with spin $s = j_1+j_2$), i.e. a conformal group SO(2, 4) representation with scaling dimension Δ will be denoted as $(\Delta; j_1, j_2)$.

Our aim will be to determine the expressions for the S^3 Casimir (or vacuum) energy E_c and 4d conformal anomaly coefficients a and c corresponding to a generic AdS_5 field for the representation $(\Delta; j_1, j_2)$. Our results will generalize those for $j_1 = j_2 = \frac{s}{2}$ found for a in [1, 3, 5] and for E_c in [6, 7]. We shall also propose a general expression for the $\mathrm{c}(\Delta; j_1, j_2)$ coefficient which matches all known values in special cases and provides very non-trivial consistency checks of AdS/CFT.

We shall then discuss applications of our general relations to the adjoint and vectorial AdS/CFT dualities.

1.1 Structure of 4d conformal anomaly

Let us first recall the general expression for the stress tensor trace anomaly in a free 4d CFT defined on a curved space [15, 16][2]

$$\mathcal{A} = \sum b_4 = -\mathrm{a}\,\mathcal{E} + \mathrm{c}\,C^2 + \mathrm{g}\,D^2R\,. \tag{1.1}$$

Here b_4 is the Seeley coefficient (often called also a_2) for the corresponding kinetic operator. There may be several operators in the case of gauge symmetries and they may be of higher order than 2 in general. $\mathcal{E} = R^*R^*$ is the Euler density and C the Weyl tensor ($C^2 = \mathcal{E} + 2R^2_{\mu\nu} - \frac{2}{3}R^2$). The coefficient g of the total derivative term is a priori ambiguous (regularization-dependent) as it can be changed by adding a local R^2 counterterm.[3] It enters the expression for the Casimir energy on S^3 that can be found from the stress tensor [17]

$$E_c = \frac{3}{4}\left(\mathrm{a} + \frac{1}{2}\mathrm{g}\right). \tag{1.2}$$

Like g, the vacuum energy E_c also depends on a choice of regularization.[4] Computed in the standard heat kernel or ζ-function scheme the coefficient g happens to vanish in theories with large amount of supersymmetry[5] so that E_c and a-coefficient become directly

[2] Our choice of normalisation is such that for a real conformal scalar $\mathrm{a} = \frac{1}{360}$, $\mathrm{c} = \frac{1}{120}$, $\mathrm{g} = \frac{1}{180}$.

[3] If one uses dimensional regularisation [15] and defines Weyl tensor in d dimensions then $\mathrm{g} = \frac{3}{2}\mathrm{c}$. This implies $\mathcal{A} = (\mathrm{c}-\mathrm{a})\mathcal{E} - 4\,\mathrm{c}\,Q$ where $Q = \frac{1}{4}[\mathcal{E} - (C^2 + \frac{2}{3}\,D^2R)]$ is the "Q-curvature".

This relation is not true in the standard heat kernel (proper time cutoff) [16] or ζ-function regularization that we shall assume. For example, for standard spin ≤ 1 fields one then finds $\mathrm{a} = \frac{31}{180}n_1 + \frac{11}{720}n_{\frac{1}{2}} + \frac{1}{360}n_0$, $\mathrm{c} = \frac{1}{10}n_1 + \frac{1}{40}n_{\frac{1}{2}} + \frac{1}{120}n_0$, $\mathrm{g} = -\frac{1}{10}n_1 + \frac{1}{60}n_{\frac{1}{2}} + \frac{1}{180}n_0$, where n_i are the numbers of gauge vectors, Majorana fermions and real conformal scalars.

[4] E_c computed from the spectrum of the Hamiltonian is given by a formally divergent sum which may be defined using spectral ζ-function regularization.

[5] This was found [18] in $\mathcal{N} = 4$ SYM and also appears to be the case in $\mathcal{N} = 3, 4$ conformal supergravity as we shall see below.

proportional (see also [19, 20]). In UV finite theories with (extended) supersymmetry one also finds that c is equal to a, and the two conditions appear to hold at the same time if the number of global supersymmetries is $\mathcal{N} \geq 3$, i.e.

$$\mathcal{N} \geq 3 \ \ \text{susy}: \qquad E_c = \frac{3}{4}\mathrm{a}\,, \qquad \mathrm{a} = \mathrm{c}\,, \qquad \mathrm{g} = 0\,. \tag{1.3}$$

We would like to find the general expressions for the conformal anomaly coefficients a, c and also E_c (or g) as functions of the representation labels Δ, j_1, j_2 by starting with a dual 5d description of a given conformal 4d field.

Consider a 2nd-order operator $\mathcal{O} = -D^2 + X$ defined on a 5d field ϕ which corresponds to a representation $(\Delta; j_1, j_2)$. In the case when 5d space is AdS_5 X is a constant "mass" term (we shall make the definition of $\mathcal{O}$ precise below). More generally, we may consider a generalization of AdS_5 to an Einstein space $ds^2 = \frac{1}{z^2}\,[dz^2 + g_{\mu\nu}(x,z)\,dx^\mu\,dx^\nu]$ which asymptotes to a curved boundary metric $g_{\mu\nu}(x) \equiv g_{\mu\nu}(x,0)$.[6] The corresponding one-loop partition function (with Dirichlet-type "+" or Neumann-type "−" boundary conditions)

$$Z^{\pm} = (\det\,\mathcal{O})_{\pm}^{-1/2}\,, \tag{1.4}$$

will then be a functional of the boundary metric $g_{\mu\nu}$. One may define the associated boundary conformal anomaly $\mathcal{A}^{\pm}$ as the variation of $Z^{\pm}$ under the variation of the conformal factor of the boundary metric: $\delta \log Z^{\pm} = -\frac{1}{(4\pi)^2}\int d^4x\,\sqrt{g}\,\delta\sigma\,\mathcal{A}^{\pm}$, $\delta g_{\mu\nu} = 2\,\delta\sigma\,g_{\mu\nu}$ (generalizing the "tree-level" 5d derivation of 4d conformal anomaly [21]). It was argued in [22, 23] that one should find

$$\mathcal{A}^{+} = (\Delta - 2)\,\bar{\mathcal{A}}\,, \tag{1.5}$$

where according to [23] $\bar{\mathcal{A}} = -\frac{1}{2}b_4(\bar{\mathcal{O}})$ and $\bar{\mathcal{O}}$ is a 4d operator corresponding to a "restriction" of $\mathcal{O}$ to the boundary. In this case $\bar{\mathcal{A}}$ (which should have the same structure as (1.2)) can not depend on Δ.

As we shall see below, while (1.5) is indeed true, i.e. both a and c are proportional to $\Delta - 2$,

the coefficient $\bar{\mathcal{A}}$ should have a non-trivial dependence on Δ (in addition to its dependence on j_1, j_2).[7] Our expressions for a and c will thus be different from the ones proposed in [23] for spins $j_1 + j_2 \leq 2$. The individual field contributions to $\mathrm{c} - \mathrm{a}$ will also disagree with the general ansatz in [24–26] based on the prescription of [23], though the agreement (for $\mathrm{c} - \mathrm{a}$ but not for a in [23]) will be restored when fields are combined into for $\mathcal{N} = 1$ superconformal multiplets.

[6]In general, for a higher spin field ϕ in a 5d Einstein background the corresponding kinetic operator may contain non-minimal curvature couplings and its consistency may require an existence of a proper embedding into an interacting higher spin theory.

[7]To find the a-coefficient it is enough to consider the case of AdS_5 with conformally flat boundary, while to determine c one may specialize to the case of Ricci flat boundary metric. That a coefficient in $\bar{\mathcal{A}}$ should have 4-th order polynomial dependence on Δ follows already from the results for a general massive 5d scalar in euclidean AdS_5 with boundary S^4 [9, 10, 1].

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1.2 Relation between 5d and 4d partition functions

To understand the precise relation between the 5d determinants ([1.4](#)) and the conformal anomaly of the associated 4d operator let us start with a 5d action $S_5 = \int d^5x\, \phi \mathcal{O} \phi + \dots$ and evaluate it on a solution of the Dirichlet problem $\phi\big|_\partial = \varphi$, i.e. symbolically

$$S_5 = \int d^5x\ \phi \mathcal{O} \phi + \dots \quad \to \quad S_4 = \int d^4x\ \varphi \mathrm{K} \varphi\ \sim\ \log \varepsilon \int d^4x\ \varphi \tilde{\mathcal{O}} \varphi + \dots . \tag{1.6}$$

Here $\varepsilon = \mathrm{R}^{-1} \to 0$ is an IR cutoff in 5d. In the case of $\Delta = 2 + s$ when ϕ is a massless higher spin field the 4d field φ is the conformal higher spin field and $\tilde{\mathcal{O}} \sim D^{2s} + \dots$ is the corresponding Weyl-invariant 4d operator depending on $g_{\mu\nu}$.[8]

Let us now consider the following path integral[9]

$$\mathrm{Z}(\varphi) = \int_{\phi\big|_\partial = \varphi} d\phi\ e^{-S_5(\phi)} = \ Z^+ \, e^{-S_4(\varphi)} \, (1 + \dots) \ , \tag{1.7}$$

where in the r.h.s. we considered semiclassical expansion near the solution of the Dirichlet problem. Here Z^+ is the "free" one-loop 5d partition function in (1.4). Next, let us integrate (1.7) over the 4d field φ. As was argued in a similar context in [8], this results in path integral over ϕ with "free" Neumann boundary conditions, with the leading 1-loop term then being Z^- in (1.4)

$$\int d\varphi\ \mathrm{Z}(\varphi) = \int_- d\phi\ \ e^{-S_5(\phi)} = Z^- \, (1 + \dots) \ . \tag{1.8}$$

Combining this with (1.7) we find at the one-loop order

$$Z^- = Z^+ \ Z \, , \qquad\quad Z = (\det \mathrm{K})^{-1/2} \to (\det \tilde{\mathcal{O}})^{-1/2} \, . \tag{1.9}$$

Here we assume that Δ is such that K has leading local term $\tilde{\mathcal{O}}$ as in (1.6) and the subleading terms can be ignored in the limit. The overall singular constant will not contribute to observables like conformal anomaly. The case of an arbitrary Δ will be defined by an analytic continuation, which should give consistent results at least for the boundary conformal anomaly parts of the corresponding determinants.

We thus get a relation between the 5d and 4d determinants of local operators. In general, for a 5d field corresponding to a massive or massless representation $(\Delta; j_1, j_2)$ of $\mathrm{SO}(2,4)$ the associated boundary conformal field will have canonical dimension equal to $\Delta_- = 4 - \Delta$. Thus $\Delta \geq 4$ cases will correspond to 4d fields with higher $2(\Delta - 2) \geq 2(j_1 + j_2)$ derivative kinetic operators $\sim D^{2(\Delta-2)} + \dots$ which should give a Weyl-invariant action in curved 4d background. This implies, in particular, that the corresponding anomaly should

[8] The boundary operator becomes local only for special values of Δ (see, e.g., a discussion of the scalar case in [10]). In general, we shall assume analytic continuation in Δ.

[9] In the AdS/CFT context this should be equal to the generating functional for correlators of bilinear currents $J \sim \Phi^* \partial^s \Phi$ in the boundary CFT, $\mathrm{Z}(\varphi) = \int d\Phi\ \exp[-S_4(\Phi) + J \cdot \varphi]$. Integrating over N fields Φ gives induced action for φ starting with $N \int \varphi \mathrm{K} \varphi \sim N \log \varepsilon \int \varphi \tilde{\mathcal{O}} \varphi + \dots$ where ε is playing the role of a UV 4d cutoff.

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vanish at $\Delta = 2$ as in (1.5) as then the operator becomes algebraic. One simple case (cf. [10]) is when $\mathcal{O} = -D^2 + X$ is the 5d scalar operator with $X = \Delta(\Delta - 4) = 0$, i.e. corresponding to the representation $(4;0,0)$. Then $\tilde{\mathcal{O}}$ is the 4-derivative Weyl invariant scalar operator of [27, 28].[10]

As another example, we may consider $\mathcal{O}$ being a massless higher spin gauge field operator in (a generalization of) AdS$_5$ space. Then $\tilde{\mathcal{O}}$ will be the kinetic operator of the corresponding 4d CHS field and we will get the following 5d representation for its 1-loop partition

$$Z = \frac{Z^-}{Z^+} \,. \tag{1.10}$$

This relation was verified for the leading (log divergent) part of $Z_{\rm CHS}$ on S^4 and the corresponding IR divergent parts of $Z^\pm$ in the euclidean AdS$_5$ space, i.e. for the conformal anomaly a-coefficient [1, 3]. In the case of the "thermal" cover of AdS$_5$ with $S^1 \times S^3$ boundary eq. (1.10) was demonstrated explicitly (for any value of the length $\beta = -\ln q$ of S^1) in [7]. In particular, it then relates the Casimir energy E_c of a CHS field on S^3 to the vacuum energy of the corresponding massless higher spin field in AdS$_5$ space.

The above heuristic argument makes clear the simple kinematic origin of the relation (1.9) or (1.10) and suggests that it should also extend to the case when AdS$_5$ is deformed to an Einstein space asymptotic to a generic curved 4d boundary. Then the variation over the boundary metric should provide a 5d representation for the 4d conformal anomaly

$$\mathcal{A} = \mathcal{A}^- - \mathcal{A}^+ \,, \tag{1.11}$$

which should apply to all (a, c and g) coefficients in (1.1). It was noticed in the special case of the symmetric tensor representation $(2+s; \frac{s}{2}, \frac{s}{2})$ that the a-coefficients corresponding to $\mathcal{A}^\pm$ obey [5] $\mathrm{a}^+ = -\mathrm{a}^-$. Then (1.11) implies that $\mathrm{a} = -2\mathrm{a}^+$. Similar relation is true [7] for the Casimir energy and thus for the g coefficient in (1.1), (1.2).

We shall see below that the same applies also for the general representations $(\Delta; j_1, j_2)$. This is a consequence of the change of sign of the expressions for a^+ and E_c^+ under $\Delta_- \to \Delta_+$, i.e. under $\Delta - 2 \to -(\Delta - 2)$. It is then natural to assume that the same should be true also for the c-coefficient,[11] i.e. that in general[12]

$$\mathcal{A}^- = -\mathcal{A}^+ \,, \qquad \text{i.e.} \qquad \mathcal{A} = -2\mathcal{A}^+ \,. \tag{1.12}$$

[10] Weyl-invariant operators are not unique in general: for example, one can add a Weyl-invariant C^2 term to the $D^4 + \ldots$ Weyl-invariant operator with an arbitrary coefficient [27, 18] and the same is true for the 2nd-derivative Weyl-invariant operator defined on symmetric traceless tensor [29–31] corresponding to representation $(3;1,1)$ and on 4th rank tensor with symmetries of Weyl tensor [32] corresponding to representation $(3;2,0)+(3;0,2)$. The relation to a consistent 5d operator should fix this ambiguity. This ambiguity is absent in the case of D^4 operator defined on dimension zero tensor or $(4;1,1)$ coming out of the expansion of the C^2 Weyl action related [33] to the Einstein gravity action in 5d.

[11] A (not directly related) indication that local properties of variations of 5d determinants may have opposite signs for the Dirichlet and Neumann boundary conditions is that this is what happens for the coefficients of $\mathcal{E}$ and C^2 in the expression for the boundary b_5 Seeley coefficient in [34].

[12] Equivalently, in the notation of (1.5) that means $\bar{\mathcal{A}}(\Delta) = \bar{\mathcal{A}}(4-\Delta)$.

1.3 Higher spin operators in AdS_5

Let us now describe the structure of the 5d operators $\mathcal{O}$ we will be considering below. Let ϕ be a massive ($\Delta > 2 + j_1 + j_2$ for $j_1 j_2 \neq 0$ or $\Delta > 1 + j_1 + j_2$ for $j_1 j_2 = 0$) or massless ($\Delta = \Delta_0 \equiv 2 + j_1 + j_2,\ \ j_1 j_2 \neq 0$) field in AdS_5 corresponding the $SO(2,4)$ representation $(\Delta;\, j_1, j_2)$ (see (A.1)). One may also define the weights

$$h_1 = j_1 + j_2 \equiv s\,, \qquad h_2 = j_1 - j_2\,, \qquad h_1 \geq h_2\,, \tag{1.13}$$

which are integer for bosonic fields and half-integer for fermionic fields. In the bosonic case, h_1 and $|h_2|$ are the lengths of a two-row Young tableau. According to [35–37], the covariant equation of motion for such bosonic transverse field ϕ is (for $j_1 \geq j_2$)

$$\mathcal{O}\phi = 0\,, \quad \mathcal{O} = -D^2 + X\,, \qquad X = \Delta\,(\Delta - 4) - h_1 - |h_2| = (\Delta - 2)^2 - 2\,j_1\,, \tag{1.14}$$

where D^2 is the standard Laplacian in AdS_5. This equation is also valid not only for the bosonic, but also for the fermionic fields after squaring the 5d Dirac operator. For a generic fermion spinor-tensor field Ψ one has $(\not{D} + \Delta - 2)\,\Psi = 0$ [38]. After squaring, this turns out to be $\big[-D^2 + \frac{1}{4}R - h_1 - |h_2| + 1 + (\Delta - 2)^2\big]\ \Psi = 0$ (see [39] for details), where $R = -20$ is the scalar curvature of AdS_5 assumed to have unit scale. This gives the same X as in (1.14). A natural definition of mass of a bosonic field in AdS_5 is such that it vanishes for the massless representation with $\Delta = \Delta_0 = 2 + s$, i.e.[13]

$m^2 \equiv \Delta\,(\Delta - 4) - \Delta_0\,(\Delta_0 - 4) = (\Delta - 2)^2 - s^2$, so that $X = m^2 + (j_1 + j_2)^2 - 2j_1$.

The partition function of a massive higher spin field with standard (Dirichlet) boundary conditions corresponding to $\Delta = \Delta_+$ is then given by (1.4) with $\mathcal{O}$ defined on transverse fields in representation (j_1, j_2). We shall denote the massive case quantities with $\widehat{\ }$ in what follows, i.e.

$$Z^+_{\rm massive} \equiv \widehat{Z}^+(\Delta; j_1, j_2) = \big[\det(-D^2 + X)_\perp\big]^{-1/2}\,. \tag{1.15}$$

In the massless case of $\Delta = \Delta_0 = 2 + s$ we need to take into account the contribution of the corresponding ghosts that belong to representation $(\Delta_0 + 1; j_1 - \frac{1}{2}, j_2 - \frac{1}{2})$ (the gauge transformation parameters ξ in $\delta\phi \sim \partial\xi$ have one unit of spin and canonical dimension $4 - \Delta$ less):

$$Z^+_{\rm massless} \equiv Z^+(\Delta; j_1, j_2) = \frac{\widehat{Z}^+(\Delta + 1; j_1 - \frac{1}{2}, j_2 - \frac{1}{2})}{\widehat{Z}^+(\Delta; j_1, j_2)}\,, \qquad \Delta = 2 + j_1 + j_2\,. \tag{1.16}$$

For example, in the case of the totally symmetric massless higher spin field representation one finds [40, 41]

$$\mathcal{Z}^+\left(2 + s; \frac{s}{2}, \frac{s}{2}\right) \equiv Z^+_s = \left[\frac{\det\ \big(-D^2 + X'\big)_{s-1\,\perp}}{\det\ (-D^2 + X)_{s\,\perp}}\right]^{1/2}, \tag{1.17}$$
$$X(\Delta, s) = \Delta(\Delta - 2) - s = s^2 - s - 4\,, \qquad X' = X(\Delta + 1, s - 1) = s^2 + s - 2\,.$$

[13] In the fermionic case there is a possible alternative definition of mass as the parameter in the Dirac equation: $(\not{D} + m_{\rm D})\,\Psi = 0,\ \ m_{\rm D} = \Delta - 2$.

Below we will use (1.14), (1.15), (1.16) to compute the corresponding E_c and a coefficients. A direct 5d computation of c or c − a would require a generalization of $\mathcal{O}$ in (1.14) to an Einstein space which is asymptotically AdS_5 with Ricci flat boundary which is not known in general for $s > 2$ (cf. [42, 43]). However, the expresions for E_c and a and known results in special cases will allow us to suggest a unique expression for the c-coefficient which will then pass AdS/CFT consistency checks.

1.4 Summary

Let us summarize the content of the rest of this paper. In section 2 we consider the $S^1 \times S^3$ partition function Z in (1.10) and also find the corresponding S^3 Casimir energy for the case of generic representation $(\Delta; j_1, j_2)$. The resulting expression for E_c will follow the pattern in (1.12). The one-particle partition functions corresponding to Z^+ will be given directly by the SO(2, 4) characters but the case of Z^- will be more subtle, and we will determine it in few special cases.

In section 3 we find the general expression for the $\mathrm{a}(\Delta; j_1, j_2)$ conformal anomaly coefficient in (1.1), (1.12), generalizing the computation of [1, 5] done in the totally symmetric $(2 + s; \frac{s}{2}, \frac{s}{2})$ bosonic case. Combined together, the results for E_c and a determine also the form of the coefficient $\mathrm{g}(\Delta; j_1, j_2)$ in (1.1), (1.2).

In section 4 we determine a similar expression for the second conformal anomaly coefficient c. While we are presently unable to give its systematic derivation, we shall make a proposal for $\mathrm{c}(\Delta; j_1, j_2)$ that reproduces all known special cases and leads to non-trivial consistency checks and predictions in the context of AdS/CFT.

In section 5 we apply our general expressions for E_c (2.31), (2.32), a (3.3), (3.4) and c (4.10), (4.3) to compute the corresponding quantities for sets of fields forming long or short $\mathrm{SU}(2, 2|\mathcal{N})$ superconformal multiplets. We shall find that the total a and c vanish for long "massive" $\mathcal{N} = 1$ supermultiplets and observe that c − a for short $\mathcal{N} = 1$ supermultiplets agrees with the expressions in [24, 25] formally extended to all values of spins $j_1, j_2 \geq 1$. We will also rederive from the 5d approach the values of $K = (E_c, \mathrm{a}, \mathrm{c})$ for $\mathcal{N} \leq 4$ Maxwell and conformal supergravity supermultiplets, verifying the relation (1.3) for $\mathcal{N} = 3, 4$ cases. We will demonstrate that all the three quantities vanish when $\mathcal{N} = 4$ conformal supergravity is combined with exactly four $\mathcal{N} = 4$ Maxwell multiplets as in [27, 44]. The 5d approach provides a direct relation between the conformal anomaly of $\mathcal{N} = 4$ conformal supergravity and the one-loop contribution of fields of $\mathcal{N} = 8$, $d = 5$ gauged supergravity as the two theories are described by the equivalent short $P\mathrm{SU}(2, 2|4)$ supermultiplet (this generalizes to the one-loop level the known tree-level relation [33]). We will also show that $K = 0$ for a general long massless supermultiplet of $P\mathrm{SU}(2, 2|4)$.

In section 6 we turn to applications of our expressions for $K = (E_c, \mathrm{a}, \mathrm{c})$ to AdS/CFT dualities. We first consider in section 6.1 the "adjoint" duality between $\mathcal{N} = 4$ SU(N) SYM and string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$. We find that the values of K for the states at level p of Kaluza-Klein tower of 10d type IIB supergravity compactified on S^5 are equal to the values of p copies of $\mathcal{N} = 4$ Maxwell multiplets, in line with the fact that these states appear in the tensor product of p superdoubletons [45]. Under a particular regularization of the sum over p, this is consistent with the adjoint AdS/CFT duality with SU(N) SYM contribution to K being N^2-1 times that of one $\mathcal{N}=4$ Maxwell multiplet. As we explain on the example of the vacuum energy E_c, the required regularization of the sum over the KK states is, in fact, a spectral ζ-function one applied to 10d instead of 5d energy states.

In section 6.2 we compute $(E_c, \mathrm{a}, \mathrm{c})$ on both sides of the vectorial AdS/CFT examples. We consider the earlier studied cases of type A and type B higher spin theories in AdS$_5$ corresponding to the scalar and spin $\frac{1}{2}$ fermion 4d boundary theories and also a novel example of "type C" theory dual to a singlet sector of N Maxwell fields at the boundary. We also discuss supersymmetric generalizations of vectorial AdS/CFT. In section 6.3 we point out that consistency conditions of vectorial AdS/CFT in non-minimal scalar and fermion theory cases implying cancellation of total a and c coefficients are equivalent to the consistency (cancellation of conformal anomalies or UV finiteness) of the corresponding 4d conformal higher spin theories. Some concluding remarks are made in section 7.

In appendix A we summarize basic representations of SO(2, 4), decompositions of products of two doubleton and superdoubleton representations and present useful relations for their characters that play important role in the discussion of one-particle partition functions in vectorial AdS/CFT examples. Appendix B contains the computation of $S^1 \times S^3$ partition functions of low-spin conformal 4d fields that appear in extended conformal supergravities and provide useful examples for the discussion in section 2. In appendix C we give details of the derivation of the spectral ζ-function for massive higher spin AdS$_5$ operator $\mathcal{O}$ in (1.14) which is used in section 3. In appendix D we complement the discussion in section 4 by presenting a more general ansatz for the c-coefficient that contains one free parameter. Appendix E summarizes the spectrum of 5d fields appearing in 10d type IIB supergravity compactified on S^5 which we use in sections 5 and 6.

2 Partition function on $S^1 \times S^3$ and Casimir energy

In this section we shall consider the expressions for one-particle partition function and S^3 Casimir energy. We shall start with the previously discussed case of totally symmetric $(2+s; \frac{s}{2}, \frac{s}{2})$ representation and then turn to the case of mixed representation $(\Delta; j_1, j_2)$.

2.1 Totally symmetric bosonic spin s conformal fields

The canonical partition function of a free CFT in $S^1 \times S^3$ can be computed by direct evaluation of the free QFT path-integral, i.e. by finding the eigenmodes of the quadratic kinetic operator. The same expression can be obtained by the operator counting method [46, 47, 7]. In radial quantisation, conformal operators in $\mathbb{R}^4$ with dimensions Δ_n are related to eigenstates of the Hamiltonian on $\mathbb{R}_t \times S^3$. From the spectrum of eigenvalues $\omega_n = \Delta_n$ and their degeneracies d_n one gets the *one-particle*, or canonical, partition function

$$\mathcal{Z}(q) = \mathrm{Tr}\, e^{-\beta H} = \sum_n \mathrm{d}_n\, e^{-\beta\, \omega_n} = \sum_n \mathrm{d}_n\, q^{\Delta_n}, \qquad q \equiv e^{-\beta}\,. \tag{2.1}$$

The multi-particle, or grand canonical, partition function is then given, in the bosonic and fermionic cases, by

$$B: \quad \log Z(q) = -\sum_n \mathrm{d}_n \log(1 - e^{-\beta\omega_n}) = \sum_{m=1}^{\infty} \frac{1}{m}\, \mathcal{Z}(q^m)\,, \tag{2.2}$$

$$F: \quad \log Z(q) = -\sum_n \mathrm{d}_n \log(1 + e^{-\beta\omega_n}) = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m} \mathcal{Z}(q^m)\,. \tag{2.3}$$

The analysis of the counting of states implies the following structure of $\mathcal{Z}(q)$ [7]

$$\mathcal{Z}(q) = \mathcal{Z}_-(q) - \mathcal{Z}_+(q), \qquad \mathcal{Z}_- = \mathcal{Z}^{\text{off}-\text{shell}}, \qquad \mathcal{Z}_+ = \mathcal{Z}^{\text{e.o.m.}}\,. \tag{2.4}$$

Here $\mathcal{Z}_-$ counts the off-shell components (and their derivative descendants) of a suitable gauge invariant field strength modulo non-trivial gauge identities while

$\mathcal{Z}_+$ counts the components of the equations of motion for the field strength (and their derivatives).

In the case of totally symmetric conformal higher spin gauge field with spin s, canonical dimension $2 - s$ and generalized s-derivative field strength of dimension $\Delta = 2$ (with $s = 1$ being Maxwell vector, $s = 2$ being Weyl graviton, etc.) one finds [7]

$$\mathcal{Z}_{+,s} = \frac{(s+1)^2 q^{s+2} - s^2 q^{s+3}}{(1-q)^4}\,, \qquad \mathcal{Z}_{-,s} = \frac{2(2s+1)q^2}{(1-q)^4} - \mathcal{Z}_{+,s}\,. \tag{2.5}$$

The form of $\mathcal{Z}_{-,s}$ reflects the fact that the counts of gauge identities and of equations of motion are isomorphic.

These expressions can be interpreted also from the AdS_5 perspective. In general, [7]

$$\mathcal{Z}(q) = \mathcal{Z}^-(q) - \mathcal{Z}^+(q)\,, \qquad \mathcal{Z}_+(q) = \mathcal{Z}^+(q)\,, \qquad \mathcal{Z}_-(q) = \mathcal{Z}^-(q)\,, \tag{2.6}$$

where $\mathcal{Z}^{\pm}(q)$ are the one-particle partition functions (2.2) for the one-loop partition function $Z^{\pm}$ of the corresponding massless higher spin gauge fields in thermal quotient of AdS_5 computed with teh standard ("Dirichlet") or alternative ("Neumann") boundary conditions. This is the special case of the general relation (1.10) with (1.17).

Explicitly, one finds from the AdS_5 heat kernel expression [41, 48] that $\mathcal{Z}_s^+$ is given by the same expression as $\mathcal{Z}_{+,s}$ in (2.5). The full singlet-sector partition function of the boundary CFT is then given by the sum of $\mathcal{Z}_{+,s} = \mathcal{Z}_s^+$ contributions over all spins.

Massless higher spin s field in AdS_5 with standard boundary condition is dual to the conserved spin s current operator of dimension $\Delta_+ = 2+s$ in the free complex scalar CFT_4. $\mathcal{Z}_s^+ = \mathcal{Z}_{+,s}$ has the interpretation of counting the bilinear current field J_s components (and its derivative descendants) modulo the on-shell conservation condition. This counting problem is isomorphic to that of counting the equations of motion for the 4d conformal higher spin field. Similarly, $\mathcal{Z}_s^- = \mathcal{Z}_{-,s}$ is counting the components of CHS fields ϕ_s modulo gauge identities and also counting the components of the shadow spin s conformal field of dimension $\Delta_- = 2 - s$ (conjugate to J_s) in the 4d scalar CFT modulo gauge identities. The negative term in $\mathcal{Z}_{+,s}$ in (2.5) corresponds [7] to the subtraction of the contribution of identities among equations of motion from the 4d CHS theory point of view, of the current conservation condition from the 4d scalar CFT point of view and of the ghost spin $s - 1$ field contribution from the AdS_5 bulk point of view.

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2.2 Mixed-symmetry conformal fields

Let us now consider the case of a conformal primary field in SO(2,4) representation $(\Delta; j_1, j_2)$. For generic Δ the character of this long representation of SO(2,4) should be equal to the one-particle partition function for the massive AdS$_5$ higher spin field partition function (1.15) which should just count all the components of (derivative descendants of) such field weighted with its dimension Δ (see (A.8) in appendix A)

$$\widehat{\mathcal{Z}}^+(\Delta;\, j_1, j_2) = d(j_1, j_2)\, \frac{q^\Delta}{(1-q)^4}\,, \qquad d(j_1, j_2) \equiv (2j_1+1)(2j_2+1)\,. \tag{2.7}$$

In the special case of $\Delta = 2 + j_1 + j_2$ such primary field should correspond to a conserved current in the boundary CFT or to its dual mixed-symmetry massless AdS$_5$ higher spin gauge field. In this case $\mathcal{Z}_+ = \mathcal{Z}^+$ should be given by the character of the associated short representation of SO(2,4) (A.9), i.e. should correspond to (1.16) where the ghost contribution is included. Taking into account the current conservation condition or, equivalently, subtracting the 5d ghost contribution gives the massless partition function [49, 50]

$$\mathcal{Z}^+(\Delta;\, j_1, j_2) = \widehat{\mathcal{Z}}^+(\Delta;\, j_1, j_2) - \widehat{\mathcal{Z}}^+\left(\Delta+1;\, j_1 - \frac{1}{2}, j_2 - \frac{1}{2}\right), \tag{2.8}$$

$$\mathcal{Z}^+(\Delta;\, j_1, j_2) = \mathcal{Z}_+(\Delta;\, j_1, j_2) = \frac{q^\Delta}{(1-q)^4}\left[(2j_1+1)(2j_2+1) - 4\,q\,j_1\,j_2\right]. \tag{2.9}$$

Note that eq. (2.9) reduces to (2.5) for $j_1 = j_2 = \frac{s}{2}$, $\Delta = 2+s$.

To find the partition function $\mathcal{Z}$ in (2.6) corresponding to (1.10) for a 4d conformal spin (j_1, j_2) field of canonical dimension $\Delta_- = 4 - \Delta$ it remains to determine the expression for the shadow partition function $\mathcal{Z}_-(\Delta;\, j_1, j_2)$. Let us start with the special case of "matter" conformal fields in SO(4) representation $(j, 0) + (0, j)$ corresponding to massive 5d fields (here the subtraction term in (2.9) is absent as $j_1 j_2 = 0$ so formally $\widehat{\mathcal{Z}}^+ = \mathcal{Z}^+$). In this non-degenerate case it is natural to expect $\mathcal{Z}_- = \mathcal{Z}^-$ to be related to $\mathcal{Z}_+ = \mathcal{Z}^+$ by the substitution

$$\Delta = \Delta_+ \quad \to \quad \Delta = \Delta_- = 4 - \Delta\,, \tag{2.10}$$

which, according to (2.7), is equivalent to

$$\mathcal{Z}^-(q) = \mathcal{Z}^+(q^{-1})\,. \tag{2.11}$$

Then using (2.7) we get for $\mathcal{Z}$ in (2.6)[14]

$$\mathcal{Z}(\Delta;\, 0, 0) = \frac{q^{4-\Delta} - q^\Delta}{(1-q)^4}\,, \qquad \mathcal{Z}(\Delta;\, j, 0) = \mathcal{Z}(\Delta;\, 0, j) = (2j+1)\,\frac{q^{4-\Delta} - q^\Delta}{(1-q)^4}\,. \tag{2.12}$$

Examples of such 4d conformal fields are provided by matter fields appearing in extended conformal supergravities [51, 44] (see table 2 below)

$$\phi \sim (3;\, 0, 0), \qquad \Phi \sim (4;\, 0, 0), \qquad T \sim (3;\, 1, 0) + (3;\, 0, 1),$$
$$\psi \sim \left(\frac{5}{2};\, \frac{1}{2}, 0\right) + \left(\frac{5}{2};\, 0, \frac{1}{2}\right), \qquad \Psi \sim \left(\frac{7}{2};\, \frac{1}{2}, 0\right) + \left(\frac{7}{2};\, 0, \frac{1}{2}\right). \tag{2.13}$$

[14] We split the two cases in (2.12) because $(j, 0) + (0, j)$ counts scalars as complex for $j = 0$. Instead, we shall always assume that scalars are real.

Here ϕ and ψ are the standard 4d massless scalar and spinor, Φ and Ψ are conformal fields with ∂^4 and $\partial\!\!\!/^3$ kinetic operators and T is conformal antisymmetric 2-tensor with ∂^2 kinetic term and no gauge invariance. Δ in $(\Delta; j_1, j_2)$ stands for Δ_+ dimension associated to the corresponding massive 5d field with standard boundary conditions while the canonical dimensions of these 4d fields are $\Delta_- = 4 - \Delta$ (i.e. ϕ has dimension 1, Φ has dimension 0, etc.). The partition functions $\mathcal{Z}$ for these fields are derived in appendix B by the explicit path-integral computation on $S^1 \times S^3$ and also by the operator counting method and the results are consistent with (2.12).

Turning to the massless gauge field case with $j_1 j_2 \neq 0$ let us recall the derivation [7] of the expression (2.6) for $\mathcal{Z}^- = \mathcal{Z}_-$ in the case of the bosonic totally symmetric field with $(\Delta;\, j_1, j_2) = (2+s;\, \frac{s}{2}, \frac{s}{2})$. The presence of gauge degeneracy or ghost contribution implies that in this case the simple relation (2.11) between $\mathcal{Z}^+$ and $\mathcal{Z}^-$ is no longer true. The shadow field with dimension $\Delta_- = 2 - s$ corresponds to a non-unitary SO(2, 4) representation which in general contains singular states with their associated submodules. The AdS$_5$ counterpart of this complication is that in the case of the alternative boundary condition one has additional gauge transformations allowed by non-normalizability [1]. These can be put in one-to-one correspondence with the conformal Killing tensors that may be associated to the finite dimensional SO(6) representation $(s-1, s-1, 0)$ labelled by the Young tableau with two rows with $s-1$ columns. Then (2.11) is replaced by [7] (same for lower $\pm$ labels)

$$\mathcal{Z}_s^-(q) = \mathcal{Z}_s^+(q^{-1}) + \sigma_s(q)\,, \tag{2.14}$$

where $\sigma_s(q)$ is the character of the representation for the conformal Killing tensors. Computing $\sigma_s(q)$ one then arrives at the expression in (2.5).

A similar derivation should be possible in the mixed representation case leading to

$$\begin{aligned} &\mathcal{Z}(q) = \mathcal{Z}^-(q) - \mathcal{Z}^+(q) = \big[\mathcal{Z}^+(q^{-1}) + \sigma(q)\big] - \mathcal{Z}^+(q) = \bar{\mathcal{Z}}(q) - 2\,\mathcal{Z}^+(q)\,,\\ &\bar{\mathcal{Z}}(q) \equiv \mathcal{Z}^+(q^{-1}) + \mathcal{Z}^+(q) + \sigma(q)\,. \end{aligned} \tag{2.15}$$

Below we will demonstrate this on the example of the fermionic conformal higher spin gauge fields described by totally symmetric spinor-tensor with one spinor index and $\mathrm{s} = 0, 1, 2, \ldots$ vector indices. Its total spin is $s = \mathrm{s} + \frac{1}{2}$ and it is represented by the sum of two mixed SO(4) representations:

$$\left[\left(\frac{1}{2}, 0\right) + \left(0, \frac{1}{2}\right)\right] \times \left(\frac{\mathrm{s}}{2}, \frac{\mathrm{s}}{2}\right) = \left(\frac{\mathrm{s}+1}{2}, \frac{\mathrm{s}}{2}\right) + \left(\frac{\mathrm{s}}{2}, \frac{\mathrm{s}+1}{2}\right)\,. \tag{2.16}$$

Here $\Delta = \Delta_+ = 2 + j_1 + j_2 = 2 + s$. The $\mathcal{Z}^+$ partition function is given by (2.9), i.e.

$$\begin{aligned} \mathcal{Z}^+_{\mathrm{s}+\frac{1}{2}}(q) &\equiv \mathcal{Z}^+\left(2+\mathrm{s}+\frac{1}{2};\, \frac{\mathrm{s}}{2}, \frac{\mathrm{s}+1}{2}\right) + \mathcal{Z}^+\left(2+\mathrm{s}+\frac{1}{2};\, \frac{\mathrm{s}+1}{2}, \frac{\mathrm{s}}{2}\right)\\ &= 2\,\frac{(\mathrm{s}+1)(\mathrm{s}+2)\, q^{\frac{5}{2}+\mathrm{s}} - \mathrm{s}\,(\mathrm{s}+1)\, q^{\frac{7}{2}+\mathrm{s}}}{(1-q)^4}. \end{aligned} \tag{2.17}$$

Then by analogy with the bosonic CHS case (2.14) we should find

$$\mathcal{Z}^-_{\mathrm{s}+\frac{1}{2}}(q) = \mathcal{Z}^+_{\mathrm{s}+\frac{1}{2}}(q^{-1}) + \sigma_{\mathrm{s}+\frac{1}{2}}(q)\,, \tag{2.18}$$

where $\sigma_{\mathrm{s}+\frac{1}{2}}(q)$ is the character for the conformal algebra representation corresponding to the conformal Killing spinor-tensors. The latter may be associated to the SO(6) representation $\left(\mathrm{s}-\frac{1}{2},\mathrm{s}-\frac{1}{2},\pm\frac{1}{2}\right)$ with dimension[15]

$$\dim\left(\mathrm{s}-\frac{1}{2},\mathrm{s}-\frac{1}{2},\pm\frac{1}{2}\right)=\frac{1}{3}\,\mathrm{s}\,(\mathrm{s}+1)^3\,(\mathrm{s}+2)\,. \tag{2.19}$$

The relevant character can be found by a specialization of the discussion in [7]

$$\sigma_{\mathrm{s}+\frac{1}{2}}(q)=\lim_{x\to 1}\chi_{(\mathrm{s}-\frac{1}{2},\mathrm{s}-\frac{1}{2},\pm\frac{1}{2})}(q,x,1)=2\lim_{x\to 1}\frac{\det M(\mathrm{s}-\frac{1}{2};x,q)}{\det N(x,q)}\,, \tag{2.20}$$

$$M(\mathrm{s}-\frac{1}{2};x,q)=\begin{pmatrix} 2 & 2 & 2\\ x^{-\mathrm{s}-\frac{3}{2}}+x^{\mathrm{s}+\frac{3}{2}} & x^{-\mathrm{s}-\frac{1}{2}}+x^{\mathrm{s}+\frac{1}{2}} & \sqrt{x}+\frac{1}{\sqrt{x}}\\ q^{-\mathrm{s}-\frac{3}{2}}+q^{\mathrm{s}+\frac{3}{2}} & q^{-\mathrm{s}-\frac{1}{2}}+q^{\mathrm{s}+\frac{1}{2}} & \sqrt{q}+\frac{1}{\sqrt{q}}\end{pmatrix}\,, \tag{2.21}$$

$$N(x,q)=\begin{pmatrix} 2 & 2 & 2\\ x^2+\frac{1}{x^2} & x+\frac{1}{x} & 2\\ q^2+\frac{1}{q^2} & q+\frac{1}{q} & 2\end{pmatrix}\,. \tag{2.22}$$

This gives $\sigma_{\mathrm{s}+\frac{1}{2}}(q)$ as a finite sum[16]

$$\sigma_{\mathrm{s}+\frac{1}{2}}(q)=\frac{\mathrm{s}+1}{3}\sum_{p=1}^{\mathrm{s}}(p-\mathrm{s}-2)(p-\mathrm{s}-1)(2p+\mathrm{s})\,(q^{p-\frac{1}{2}}+q^{\frac{1}{2}-p}), \tag{2.23}$$

obeying the important property

$$\sigma_{\mathrm{s}+\frac{1}{2}}(q)=\sigma_{\mathrm{s}+\frac{1}{2}}(q^{-1})\,, \tag{2.24}$$

which was also true for the bosonic σ_s in (2.14). Doing the sum over p in (2.23) gives

$$\sigma_{\mathrm{s}+\frac{1}{2}}(q)=\frac{2(\mathrm{s}+1)q^{\frac{1}{2}-\mathrm{s}}\left(q^{\mathrm{s}+1}-1\right)\left[\mathrm{s}\,q^{\mathrm{s}+2}-(\mathrm{s}+2)q^{\mathrm{s}+1}+(\mathrm{s}+2)q-\mathrm{s}\right]}{(1-q)^4}\,. \tag{2.25}$$

Then using this in (2.14), (2.15) leads to the final result for $\mathcal{Z}_{\mathrm{s}+\frac{1}{2}}=\mathcal{Z}^-_{\mathrm{s}+\frac{1}{2}}(q)-\mathcal{Z}^+_{\mathrm{s}+\frac{1}{2}}(q)$

$$\mathcal{Z}_{\mathrm{s}+\frac{1}{2}}=4\,\frac{(\mathrm{s}+1)\,q^{\frac{3}{2}}+(\mathrm{s}+1)\,q^{\frac{5}{2}}-(\mathrm{s}+1)(\mathrm{s}+2)\,q^{\frac{5}{2}+\mathrm{s}}+\mathrm{s}(\mathrm{s}+1)\,q^{\frac{7}{2}+\mathrm{s}}}{(1-q)^4}\,. \tag{2.26}$$

As a check, for the standard massless spin $\frac{1}{2}$ fermion ($\mathrm{s}=0$) this agrees with (2.12) with $j=\frac{1}{2}$ and $\Delta=\frac{5}{2}$. Also, for the conformal gravitino ($\mathrm{s}=1$) this leads to

$$\mathcal{Z}_{\frac{3}{2}}=8\,\frac{q^{\frac{3}{2}}+q^{\frac{5}{2}}-3\,q^{\frac{7}{2}}+q^{\frac{9}{2}}}{(1-q)^4}\,, \tag{2.27}$$

which is the same expression (B.17) as derived in appendix (B) by directly computing the conformal gravitino partition function on $S^1\times S^3$.

[15] See also footnote 24 of [1].

[16] Some explicit values are $\sigma_{\frac{1}{2}}(q)=0,\quad \sigma_{\frac{3}{2}}(q)=4\sqrt{q}+\frac{4}{\sqrt{q}},\quad \sigma_{\frac{5}{2}}(q)=12q^{3/2}+\frac{12}{q^{3/2}}+24\sqrt{q}+\frac{24}{\sqrt{q}}$.

2.3 General expression for the Casimir energy on S^3

The Casimir energy on S^3 can be extracted from the one-particle partition function $\mathcal{Z}(q)$ in (2.1) using the standard relations (see, e.g., [50])[17]

$$E_c = \frac{1}{2}(-1)^F \sum_n \mathrm{d}_n\, \omega_n = \frac{1}{2}(-1)^F\, \zeta_E(-1)\,, \tag{2.28}$$

$$\zeta_E(z) = \sum_n \frac{\mathrm{d}_n}{\omega_n^z} = \frac{1}{\Gamma(z)}\int_0^\infty d\beta\, \beta^{z-1}\, \mathcal{Z}(e^{-\beta})\,. \tag{2.29}$$

The representation in terms of $\zeta_E(-1)$ has the advantage that it allows one to show that the Casimir energy *vanishes* if the partition function obeys $\mathcal{Z}(q) = \mathcal{Z}(q^{-1})$ [6] (see also [52]).

If we start with $\mathcal{Z}_+$ corresponding to a primary field $(\Delta;\, j_1, j_2)$, the associated Casimir energy $E_c^+(\Delta;\, j_1, j_2)$ is then the same as the vacuum energy of a single massless higher spin field in AdS$_5$ with standard boundary conditions. If we consider a 4d conformal higher spin field, its Casimir energy on S^3 can be found from the corresponding one-particle partition function in (2.15). The Killing tensor character should in general obey the property (2.24), implying that the same should be true for $\bar{\mathcal{Z}}(q)$ in (2.15), and if $\bar{\mathcal{Z}}(q) = \bar{\mathcal{Z}}(q^{-1})$ then it does not contribute to E_c. As a result, we conclude that the Casimir energy of a 4d conformal field in representation $(\Delta;\, j_1, j_2)$ is given by -2 of the AdS$_5$ vacuum energy of the corresponding 5d field with the standard boundary condition

$$E_c(\Delta;\, j_1, j_2) = E_c^-(\Delta;\, j_1, j_2) - E_c^+(\Delta;\, j_1, j_2) = -2\, E_c^+(\Delta;\, j_1, j_2)\,. \tag{2.30}$$

In the non-gauge 4d field case (corresponding to a massive 5d field) we thus get from $\widehat{\mathcal{Z}}^+$ in (2.7) that $E_c = \widehat{E}_c$, where

$$\begin{aligned}\widehat{E}_c(\Delta;\, j_1, j_2) &= -2\widehat{E}_c^+(\Delta;\, j_1, j_2)\\ &= -\frac{1}{720}(-1)^{2j_1+2j_2}\,(2j_1+1)(2j_2+1)(\Delta-2)\left[6\,(\Delta-2)^4 - 20\,(\Delta-2)^2 + 11\right]\,.\end{aligned} \tag{2.31}$$

For a gauge conformal field (or a massless 5d field) with $\Delta = 2 + j_1 + j_2$ we get according to (1.16)

$$E_c(\Delta;\, j_1, j_2) = \widehat{E}_c(\Delta;\, j_1, j_2) - \widehat{E}_c\left(\Delta+1;\, j_1 - \frac{1}{2}, j_2 - \frac{1}{2}\right)\,. \tag{2.32}$$

As in (2.9), the second term here vanishes if $j_1 j_2 = 0$.

Special cases include fields of extended conformal supergravity with values of E_c listed in table 2. For the general spin s totally symmetric bosonic $(2+s;\, \frac{s}{2}, \frac{s}{2})$ [7] and fermionic

[17] Given the data (d_n, ω_n) the formal sum over n is usually divergent and requires a regularization. A natural regularization is a spectral ζ-function one as above which is also equivalent to computing E_c as the finite part of the $\epsilon \to 0$ expansion of the following regularized expression (see, e.g., [6])

$$E_c = \frac{1}{2}(-1)^F \sum_n \mathrm{d}_n\, \omega_n\, e^{-\epsilon\, \omega_n}\Big|_{\epsilon\to 0,\ \text{finite}}\,.$$

$(2+s;\frac{s+\frac{1}{2}}{2},\frac{s-\frac{1}{2}}{2})+(2+s;\frac{s-\frac{1}{2}}{2},\frac{s+\frac{1}{2}}{2})$ conformal 4d fields we obtain from (2.31), (2.32) (or directly from (2.5) and (2.17)) the following expressions for the Casimir energies

$$E_{c,s}=E_c\Big(2+s;\frac{s}{2},\frac{s}{2}\Big)=\frac{1}{720}\,\nu_b\,(18\,\nu_b^2-14\,\nu_b-11)\,,\qquad s=1,2,\dots \tag{2.33}$$

$$E_{c,s}=2E_c\left(2+s;\frac{s+\frac{1}{2}}{2},\frac{s-\frac{1}{2}}{2}\right)=\frac{1}{5760}\,\nu_f\,(36\,\nu_f^2+140\,\nu_f+85),\quad s=\frac{1}{2},\frac{3}{2},\dots \tag{2.34}$$

$$\nu_b\equiv s(s+1),\qquad\qquad \nu_f\equiv -2\left(s+\frac{1}{2}\right)^2=-2\nu_b-\frac{1}{2}\,. \tag{2.35}$$

Here ν_b and ν_f are the numbers of dynamical degrees of freedom of the bosonic and fermionic CHS fields [3]. The coefficient 2 in the fermionic case accounts for the equal contributions of the two $j_1\leftrightarrow j_2$ representations.

3 Conformal anomaly a-coefficient

Next, let us turn to the computation of the conformal anomaly a-coefficient of 4d conformal field with canonical dimension $4-\Delta$ and SO(4) spins (j_1,j_2) corresponding to a generic representation $(\Delta;\,j_1,j_2)$.

As follows from (1.1), to find the a-coefficient it is sufficient to consider the case of conformally flat S^4 background (for unit-radius sphere $\mathcal{A}^+=-24\mathrm{a}^+$). We shall use (1.10), (1.12) to give the AdS_5 derivation of the a-anomaly generalizing the computation of [1, 5] in the totally symmetric $(2+s;\,\frac{s}{2},\frac{s}{2})$ bosonic case. The expressions for a for both bosonic and fermionic totally symmetric conformal higher spin fields were found directly in 4d in [3].

$\mathcal{A}^+$ in (1.12) is associated with the variation of the one-loop partition function of 5d field corresponding to the representation $(\Delta;\,j_1,j_2)$ under a local conformal variation of the boundary metric. In the case of the Euclidean AdS_5 with boundary S^4 (i.e. hyperboloid $\mathbb{H}^5$) the conformal anomaly is proportional to the logarithmic IR singular part of the one-loop partition function (see, e.g., [9, 1])

$$\log Z^+=-\frac{1}{2}\log\det{}_+\,\mathcal{O}=\frac{1}{2}\,\zeta'(0)=-4\mathrm{a}^+\log\mathrm{R}+\dots\,. \tag{3.1}$$

Here $\zeta(z)$ is the spectral zeta function defined by evaluating the trace of the $\mathbb{H}^5$ heat kernel associated with the "massive" 5d operator $\mathcal{O}$ in (1.14) (see [53, 1]).

The trace is proportional to the regularised volume of $\mathbb{H}^5$ that has a factor $\log\mathrm{R}$ depending on IR cutoff.

The explicit derivation of $\zeta(z)$ for the operator $\mathcal{O}$ acting on a transverse field in a general representation $(\Delta;\,j_1,j_2)$ is given in appendix C. Using (C.14) the a-coefficient for the 4d conformal field associated to "massive" $(\Delta;\,j_1,j_2)$ representation can be represented as (-2 factor is as in (1.12))

$$\begin{aligned}\widehat{\mathrm{a}}(\Delta;\,j_1,j_2)&=-2\widehat{\mathrm{a}}^+(\Delta;\,j_1,j_2)=\frac{1}{4\,\log\mathrm{R}}\,\zeta'(0)=\frac{1}{48\,\pi}(-1)^{2(j_1+j_2)}(2j_1+1)(2j_2+1)\\&\times\lim_{z\to0}\frac{\partial}{\partial z}\int_0^\infty d\lambda\,\frac{\left[\lambda^2+(j_1-j_2)^2\right]\left[\lambda^2+(j_1+j_2+1)^2\right]}{[\lambda^2+(\Delta-2)^2]^z}\,.\end{aligned} \tag{3.2}$$

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A straightforward computation gives

$$\widehat{\mathrm{a}}(\Delta;\, j_1, j_2) = \frac{1}{720}(-1)^{2(j_1+j_2)}(2j_1+1)(2j_2+1)(\Delta-2)$$
$$\times \left[-3(\Delta-2)^4 + 10\left(j_1^2+j_2^2+j_1+j_2+\frac{1}{2}\right)(\Delta-2)^2 - 15(j_1-j_2)^2(j_1+j_2+1)^2\right]. \tag{3.3}$$

This expression is odd under $\Delta \to 4-\Delta$, i.e. under (2.10). This implies that the anomaly corresponding to Z^- computed with the alternative boundary condition has the opposite sign, i.e. we have $\widehat{\mathrm{a}} = \widehat{\mathrm{a}}^- - \widehat{\mathrm{a}}^+ = -2\widehat{\mathrm{a}}^+$. This is also the same pattern that was found for the Casimir energy (2.30).[18]

In the massless field case one is to subtract the ghost contribution in (1.16), i.e.

$$\mathrm{a}(\Delta;\, j_1, j_2) = \widehat{\mathrm{a}}(\Delta;\, j_1, j_2) - \widehat{\mathrm{a}}\left(\Delta+1;\, j_1-\frac{1}{2}, j_2-\frac{1}{2}\right). \tag{3.4}$$

As in the case of E_c in (2.31), (2.32), the second term in (3.4) vanishes for $j_1 j_2 = 0$.

It is easy to check that in the special cases of conformal fields appearing in extended conformal supergravity the expressions (3.3), (3.4) reproduce the known values [27, 44] of the corresponding a-coefficients (see table 2). Also, for the totally symmetric bosonic and fermionic conformal higher spin gauge fields we find as in (2.33), (2.34)

$$\mathrm{a}_s = \mathrm{a}\left(s+2;\, \frac{s}{2}, \frac{s}{2}\right) = \frac{1}{720}\,\nu_b\,(14\,\nu_b^2 + 3\,\nu_b)\,, \qquad s = 1, 2, \ldots \tag{3.5}$$

$$\mathrm{a}_s = 2\,\mathrm{a}\left(s+2;\, \frac{s+\frac{1}{2}}{2}, \frac{s-\frac{1}{2}}{2}\right) = \frac{1}{2880}\,\nu_f\,\left(14\nu_f^2 + 45\nu_f + 12\right), \qquad s = \frac{1}{2}, \frac{3}{2}, \ldots \tag{3.6}$$

Eq. (3.5) was first found in the 5d approach in [1]; both expressions were also obtained by direct computation in 4d [3]. derived there for the $(\Delta;\, \frac{s}{2}, \frac{s}{2})$ fields

$$\widehat{\mathrm{a}}(\Delta;\, \frac{s}{2}, \frac{s}{2}) = -\frac{(s+1)^2}{48\,\pi}\int_2^\Delta dx\,(x-2)(x+s-1)(x-s-3)\,\Gamma(x-1)\Gamma(x-3)\sin(\pi x)$$
$$= \frac{1}{720}\,(s+1)^2\,(\Delta-2)^3\,\left(-3\,\Delta^2 + 12\,\Delta + 5s^2 + 10s - 7\right). \tag{3.7}$$

Also, a special case of a massive scalar field with $m^2 = \Delta(\Delta-d)$ in AdS_{d+1} with even d corresponding to a conformal field $(\Delta;\, 0, 0)$ at the boundary was considered in [9, 10], where it was found ($(\ldots)_n$ is Pochhammer symbol)

$$\frac{\partial}{\partial\Delta}\widehat{\mathrm{a}}(\Delta;\, 0, 0) = -\frac{1}{2}\,\frac{(-1)^{d/2}}{\Gamma(d+1)}\,(\Delta-2)_2\,(2-\Delta)_2\,. \tag{3.8}$$

[18] As was discussed in section 2, in the case of $S^1 \times S^3$ boundary the Z^- partition function is not simply given by Z^+ with $\Delta_+ \to \Delta_- = 4 - \Delta_+$ (eq. (2.14) contains non-trivial σ term) but this relation still holds for E_c in (2.30). Same may be true in the case of S^4 boundary: while the IR divergent parts of $\log Z^-$ and $\log Z^+$ proportional to a^- and a^+ are the same up to sign, the relation between the finite (non-universal) parts of the partition functions may be more involved.

In $d = 4$ this gives

$$\frac{\partial}{\partial\Delta}\widehat{\mathrm{a}}(\Delta;\,0,0) = -\frac{1}{48}\,(\Delta-3)(\Delta-2)^2(\Delta-1)\,, \tag{3.9}$$

in agreement with (3.7). For general $(\Delta;\,j_1,j_2)$, it follows from (3.3) that the Δ derivative of $\widehat{\mathrm{a}}$ has a simple factorized structure

$$\begin{aligned}\frac{\partial}{\partial\Delta}\widehat{\mathrm{a}}(\Delta;\,j_1,j_2) &= -\frac{1}{48}\,(-1)^{2\,(j_1+j_2)}\,(2j_1+1)\,(2j_1+1)\\ &\times(\Delta-j_1-j_2-3)\,(\Delta-j_1+j_2-2)\,(\Delta+j_1-j_2-2)\,(\Delta+j_1+j_2-1)\,.\end{aligned} \tag{3.10}$$

Let us note in passing that since this expression is an obvious generalization of (3.8), (3.9) (obtained by $\Delta\to\Delta-j_1-j_2$ in the Pochhammer symbols, etc.), this suggests that the general field bulk-to-bulk propagator can be obtained from the scalar one by a similar replacement (with the prefactor coming from the trace over spin). This is indeed consistent with the known expressions in the case of totally symmetric tensors considered in [54].

4 Conformal anomaly c-coefficient

In this section we shall propose the general expression for the $\mathrm{c}(\Delta; j_1, j_2)$ coefficient in the 4d conformal anomaly (1.1) which will be the counterpart of the expression for $\mathrm{a}(\Delta; j_1, j_2)$ in (3.3), (3.4). We shall motivate it by imposing various consistency conditions and agreement with known special cases.

4.1 Expression for c in low spin cases

Once the value of a is known, to find c it is sufficient to compute $\mathrm{c}-\mathrm{a}$ by considering the case of Ricci flat 4d space when the conformal anomaly (1.1) becomes $\mathcal{A} = (\mathrm{c}-\mathrm{a})\mathcal{E}$.

In the case of "massive" low spin 5d fields appearing in supergravity (e.g., in the KK spectrum of 10d type IIB supergravity compactified on S^5) ref. [24, 26] suggested, following the proposal in [22], a general parametrization of $\mathrm{c}-\mathrm{a}$ coefficient in the boundary conformal anomaly[19]

$$\begin{aligned}&\widehat{\mathrm{c}}^+ - \widehat{\mathrm{a}}^+ = -\frac{1}{2}(\Delta-2)b_4(\bar{\mathcal{O}}_{j_1,j_2}) = -\frac{1}{360}\,(-1)^{2\,(j_1+j_2)}(\Delta-2)\,d(j_1,j_2)\,\left[1+f(j_1)+f(j_2)\right],\\ &d(j_1,j_2) = (2j_1+1)(2j_2+1), \qquad\qquad f(j) \equiv j\,(j+1)\,\left[6j\,(j+1)-7\right]\,.\end{aligned} \tag{4.1}$$

This expression follows from the ansatz (1.5) with $\bar{\mathcal{A}} = -\frac{1}{2}b_4(\bar{\mathcal{O}})$ assuming that $\bar{\mathcal{O}}$, i.e. the 4d boundary restriction of the 5d massive kinetic operator defined on an Einstein space which is a generalization of AdS_5 space asymptotic to the Ricci-flat boundary, is the standard $\bar{\mathcal{O}} = -D^2 + U$ operator defined on 4d field in Lorentz representation (j_1, j_2)

[19] To recall, we use $\widehat{\ }$ to indicate massive representation and $^+$ indicates the one-loop 5d field contribution computed with standard (Dirichlet) boundary conditions. The normalization of $\mathrm{c}-\mathrm{a}$ in [24, 26] is such that that it corresponds to 1-loop contributions of 5d fields dual to composite 4d operators in the AdS/CFT picture; summing over all such contributions should reproduce the conformal anomaly of the boundary CFT. Thus $\widehat{\mathrm{c}}^+ - \widehat{\mathrm{a}}^+$ for, e.g., a scalar field corresponding the $(3; 0, 0)$ representation is $-\frac{1}{2}$ of the standard value $\frac{1}{180}$.

with "minimal" curvature coupling. Then applying the standard algorithm to compute its Seeley coefficient b_4 [16] gives (4.1).[20]

Applying (4.1) together with our result (3.3) for the value of a-coefficient to compute the corresponding 4d conformal field anomaly c-coefficient according to (1.12), we find in the non-gauge 5d massive $\Delta > 2 + j_1 + j_2$ case

$$\begin{aligned}\widehat{\mathrm{c}}(\Delta;\, j_1, j_2) = -2\,\widehat{\mathrm{c}}^{+}(\Delta;\, j_1, j_2) = \frac{1}{720}(-1)^{2(j_1+j_2)}(2j_1+1)(2j_2+1)(\Delta-2) \\ \times\Big[-3(\Delta-2)^4+10\left(j_1^2+j_2^2+j_1+j_2+\frac{1}{2}\right)(\Delta-2)^2+9(j_1^4+j_2^4)+30j_1^2j_2^2 \\ +18(j_1^3+j_2^3)+30j_1j_2(j_1+j_2+1)-19(j_1^2+j_2^2)-28(j_1+j_2)+4\Big]\,.\end{aligned} \tag{4.2}$$

To get c for CHS gauge fields corresponding to massless 5d fields with $\Delta = 2 + j_1 + j_2$ we are to subtract the 5d ghost contribution as in (1.16), (3.4):

$$\mathrm{c}(\Delta;\, j_1, j_2) = \widehat{\mathrm{c}}(\Delta;\, j_1, j_2) - \widehat{\mathrm{c}}\left(\Delta+1;\, j_1-\frac{1}{2}, j_2-\frac{1}{2}\right)\,. \tag{4.3}$$

This expression reproduces the known values of c for spin ≤ 2 $\mathcal{N}=4$ conformal supergravity fields [27] in table 2 (which are dual to fields of 5d $\mathcal{N}=8$ gauged supergravity).

If we formally assume (4.2), (4.3) to be valid also for all totally symmetric higher spin fields with $j_1 = j_2 = \frac{s}{2}$ then we find as in (3.5), (3.6)[21]

$$\mathrm{c}_s = \mathrm{c}\left(s+2;\, \frac{s}{2}, \frac{s}{2}\right) = \frac{1}{1080}\,\nu_b\,\left[\nu_b\,(43\,\nu_b-59)+r_b(\nu_b-2)(\nu_b-6)\right], \tag{4.4}$$

$$\mathrm{c}_s = 2\,\mathrm{c}\left(s+2;\, \frac{s+\frac{1}{2}}{2}, \frac{s-\frac{1}{2}}{2}\right) = \frac{1}{23040}\,\nu_f\,\left[\nu_f\,(173\,\nu_f+490)+r_f(\nu_f+2)(\nu_f+8)\right], \tag{4.5}$$

with ν_b, ν_f defined in (2.35) and

$$r_b = \frac{1}{2}\,, \qquad\qquad r_f = 59\,. \tag{4.6}$$

These are the same expressions as obtained in [3] by the direct computation in 4 dimensions. The key assumption there was that the factorization of the higher-derivative CHS kinetic operator on Ricci-flat background into a product of standard 2nd derivative operators known to apply for $s \leq 2$ continues to be valid also for $s > 2$.

It is useful to understand the reason for this agreement. Let us consider, for example, the bosonic CHS field on a curved Ricci-flat background. Assuming factorization of the conformal $D^{2s} + \ldots$ kinetic operator into a product of s 2nd-derivative massless spin s

[20] Here the 4d operator obtained by restricting the 5d operator defined on transverse fields to the boundary acts on unconstrained 4d fields.

[21] This parametrization of c_s in terms of two a priori arbitrary constants r_b, r_f was introduced in [3] to ensure the agreement with known values for low spins $s = \frac{1}{2}, 1, \frac{3}{2}, 2$.

operators with minimal coupling to curvature the corresponding CHS partition function can be written as [3]

$$Z_s = \left[\frac{(\det \bar{\mathcal{O}}_{s-1})^{s+1}}{(\det \bar{\mathcal{O}}_s)^s}\right]^{1/2}, \tag{4.7}$$

where $\bar{\mathcal{O}}_k = (-D^2+U)_k$, $U = -R^{ab}{}_{mn}\Sigma^{mn}\Sigma_{ab}$ are covariant 2nd-order differential operators defined on traceless rank k tensors and having the standard massless higher spin form that was assumed also in [16]. Then the conformal anomaly $\beta_1 \equiv \mathrm{c} - \mathrm{a}$ coefficient for spin s CHS field can be expressed in terms of β_1 coefficients for the operators $\bar{\mathcal{O}}_s$

$$\beta_{1,s} = s\,\beta_1(\bar{\mathcal{O}}_s) - (s+1)\,\beta_1(\bar{\mathcal{O}}_{s-1})\,, \qquad\qquad \beta_1 \equiv \mathrm{c} - \mathrm{a}\,. \tag{4.8}$$

Here the scaling dimension is $\Delta = 2 + s$ so that (4.8) has exactly the same structure $(\Delta-2)\beta_1(\bar{\mathcal{O}}_s) - (\Delta-1)\,\beta_1(\bar{\mathcal{O}}_{s-1})$ as required for a massless 5d field anomaly (cf. (4.2), (4.3)). Since $\beta_1(\bar{\mathcal{O}}_s)$ was computed in [3] from the same expression for $b_4(\bar{\mathcal{O}}_s)$ in [16] as used in (4.1) we conclude that the expressions for $\mathrm{c} - \mathrm{a}$ should indeed match. As the a coefficients are already known to agree, this implies the agreement of the c coefficients found from the 5d approach based on (4.1) and from the 4d approach based on (4.7).

However, there are good reasons to believe that both (4.1) and (4.7) are to be modified for spins $j_1, j_2 > 1$. First, the expression for the Seeley coefficient of 4d operator on (j_1, j_2) field used in (4.1) was taken from [16] which formally applies only for spins ≤ 2: for higher spins the consistency of "minimal coupling" operators considered in [16] requires extra constraints on the curvature (in addition to Ricci flatness) invalidating the derivation of $\mathrm{c}-\mathrm{a}$. Indeed, kinetic operators of higher spin 4d fields should in general contain terms with non-minimal (e.g., $R_{...}D_.D_.$) coupling to the curvature [42, 43] which does not allow the application of the standard algorithm for computing the b_4 Seeley coefficient used in [16].

Second, the assumption of factorization of the CHS operator on Ricci flat background made in [3] was questioned in [55]. It is likely that $\mathrm{c}-\mathrm{a}$ for CHS fields may still be computed by assuming that factorization formally applies (extra terms obstructing factorization appear to involve derivatives of the curvature that can not produce non-trivial contribution to conformal anomaly in 4d) but the corresponding 2nd-derivative factor-operators should then also have non-minimal structure rather than being minimal operators as assumed in [3].

While the form of such 2nd-derivative higher spin operators that may appear in factorization of CHS operator on a Ricci-flat background remains to be understood, below we shall present a conjecture for what should be the correct generalization of c in (4.2) to higher spins $j_1, j_2 > 1$. Our expression will lead to unique consistency properties when applied in the context of AdS/CFT.

4.2 Proposal for general expression for $\mathrm{c}(\Delta;\, j_1, j_2)$

Our proposal for c that replaces (4.2) in the massive representation case $(\Delta;\, j_1, j_2)$ is

$$\begin{aligned}\widehat{\mathrm{c}}(\Delta;\, j_1, j_2) = -2\,\widehat{\mathrm{c}}^+(\Delta;\, j_1, j_2) &= \frac{1}{720}\,(-1)^{2(j_1+j_2)}(2j_1+1)(2j_2+1)\,(\Delta-2) \\ &\times \big[-6\,(\Delta-2)^4 + 20\,(\Delta-2)^2 + 6\,(j_1^4+j_2^4) + 20\,j_1^2 j_2^2 + 12\,(j_1^3+j_2^3) \\ &\quad +20\,(j_1^2 j_2 + j_1 j_2^2) - 6\,(j_1^2+j_2^2) + 20\,j_1 j_2 - 12\,(j_1+j_2) - 8\big]\,. \end{aligned} \tag{4.9}$$

The corresponding expression for $\mathrm{c}-\mathrm{a}$ following from (3.3) and (4.9) is then

$$\begin{aligned}\widehat{\mathrm{c}}(\Delta;j_1,j_2)-\widehat{\mathrm{a}}(\Delta;j_1,j_2) = \frac{1}{720}&(-1)^{2(j_1+j_2)}(2j_1+1)(2j_2+1)\,(\Delta-2)\\ \times\big[&-3\,(\Delta-2)^4-5(2j_1^2+2j_2^2+2j_1+2j_2-3)(\Delta-2)^2\\ &+21\,(j_1^4+j_2^4)-10\,j_1^2j_2^2+42\,(j_1^3+j_2^3)-10\,(j_1^2j_2+j_1j_2^2)\\ &+9\,(j_1^2+j_2^2)-10\,j_1j_2-12\,(j_1+j_2)-8\big]\,. \end{aligned} \tag{4.10}$$

This is different from (4.1) as the dependence on Δ is not just via the overall $\Delta-2$ factor.

Eq. (4.9) and its massless representation counterpart (4.3) is consistent with all low-spin data, giving, e.g., the correct values for all the fields of extended conformal supergravity (see table 2): scalars with $\Delta=3,4$, spin $\frac{1}{2}$ fermions with $\Delta=\frac{5}{2},\frac{7}{2}$, non-gauge antisymmetric tensor, conformal gravitino and conformal graviton. Applying (4.3), (4.9) to the cases of totally symmetric bosonic and fermionic CHS fields we find again the expressions in (4.4), (4.5) but now with

$$r_b=-1\,,\qquad\qquad r_f=51\,, \tag{4.11}$$

instead of (4.6). These values of the parameters are precisely the ones that lead to the vanishing of the sum $\sum_s \mathrm{c}_s$ over all totally symmetric CHS fields [3, 5], assuming the same regularization that implies the vanishing of $\sum_s \mathrm{a}_s$ [1, 3] and $\sum_s E_{c,s}$ [7].

The crucial feature of (4.9) is that it leads to important consistency checks of vectorial AdS/CFT duality which are direct analogs of the earlier checks based on the expressions for a-coefficient and E_c.[22] These checks will be discussed in detail in section 6. Here we just mention two non-trivial relations in the case of a particular mixed representation satisfied by c (4.3) defined by (4.9) but not by (4.2):

$$\sum_{s=1,2,\dots}^{\infty}\mathrm{c}\Big(2+s;\,\frac{s+1}{2},\frac{s-1}{2}\Big)=-\frac{1}{120}\,,\qquad\qquad \sum_{s=2,4,\dots}^{\infty}\mathrm{c}\Big(2+s;\,\frac{s+1}{2},\frac{s-1}{2}\Big)=-\frac{1}{30}\,. \tag{4.12}$$

5 E_c, a, c for superconformal $SU(2,2|\mathcal{N})$ multiplets

In this section we shall compute E_c, a, c for collections of primary fields of $SO(2,4)$ representations $(\Delta;j_1,j_2)$ forming superconformal multiplets. It turns out that the difference between $\mathrm{c}-\mathrm{a}$ in (4.1) and our proposal (4.10) disappears once one sums over all fields in the supermultiplet, implying that the resulting $\mathrm{c}-\mathrm{a}$ is linear in Δ as in (4.1) (but separate values of the coefficients a and c are still different from the ones implied by the prescription of [23]).

[22]We present a more general ansatz for c that reduces to (4.9) after imposing this consistency constraint in appendix D.

5.1 Summary of contributions of a single conformal $(\Delta;\, j_1, j_2)$ field

It is useful first to summarize the expressions for $E_c, \mathrm{a}, \mathrm{c}$ and $\mathrm{c}-\mathrm{a}$ in (2.31), (3.3), (4.9), (4.10) for a non-gauge (massive 5d) field in a compact form using the variables $d_1 = 2j_1 + 1$, $d_2 = 2j_2 + 1$:

$$\widehat{E}_c(\Delta;\, j_1, j_2) = -\frac{1}{720}(-1)^{d_1+d_2}\, d_1\, d_2\, (\Delta-2)\, \left[6\,(\Delta-2)^4 - 20\,(\Delta-2)^2 + 11\right], \tag{5.1}$$

$$\begin{aligned}\widehat{\mathrm{a}}(\Delta;\, j_1, j_2) &= \frac{1}{11520}(-1)^{d_1+d_2}\, d_1\, d_2\, (\Delta-2) \\ &\quad \times \left[-48\,(\Delta-2)^4 + 40\,(d_1^2+d_2^2)\,(\Delta-2)^2 - 15\,(d_1^2-d_2^2)^2\right],\end{aligned} \tag{5.2}$$

$$\begin{aligned}\widehat{\mathrm{c}}(\Delta;\, j_1, j_2) &= \frac{1}{5760}(-1)^{d_1+d_2}\, d_1\, d_2\, (\Delta-2) \\ &\times \left[-48\,(\Delta-2)^4 + 160\,(\Delta-2)^2 + 3\,(d_1^4+d_2^4) + 10\, d_1^2\, d_2^2 - 40\,(d_1^2+d_2^2)\right],\end{aligned} \tag{5.3}$$

$$\begin{aligned}\widehat{\mathrm{c}}(\Delta;\, j_1, j_2) - \widehat{\mathrm{a}}(\Delta;\, j_1, j_2) &= \frac{1}{11520}\,(-1)^{d_1+d_2} d_1\, d_2\, (\Delta-2) \\ &\times \left[-48\,(\Delta-2)^4 - 40\,(d_1^2+d_2^2-8)\,(\Delta-2)^2 + 21\,(d_1^4+d_2^4) - 10\, d_1^2\, d_2^2 - 80\,(d_1^2+d_2^2)\right].\end{aligned} \tag{5.4}$$

Note that these expressions are odd under $\Delta \to 4-\Delta$, cf. (2.10). The values in the gauge (massless 5d) field case with $\Delta = 2 + j_1 + j_2$ follow from (2.32), (3.4), (4.3). Written in terms of the variables

$$s = h_1 = j_1 + j_2\,, \qquad h_2 = j_1 - j_2\,, \qquad \nu = s(s+1)\,, \qquad \Delta = 2 + s\,, \tag{5.5}$$

they read

$$E_c(j_1, j_2) = \frac{1}{720}(-1)^{2s}\left[\nu(18\nu^2 - 14\nu - 11) - 3h_2^2(10\nu^2 - 10\nu - 1)\right], \tag{5.6}$$

$$\mathrm{a}(j_1, j_2) = \frac{1}{720}(-1)^{2s}\left[\nu(14\nu^2 + 3\nu) - 3h_2^2(20\nu^2 + 10\nu + 1) + 5h_2^4(6\nu + 1)\right], \tag{5.7}$$

$$\mathrm{c}(j_1, j_2) = \frac{1}{360}(-1)^{2s}\left[\nu(14\nu^2 - 17\nu - 4) - h_2^2(15\nu^2 - 15\nu - 7) - 5h_2^4 + h_2^6\right], \tag{5.8}$$

generalizing (2.33), (2.34), (3.5), (3.6), (4.4), (4.5).[23] These expressions are symmetric under $j_1 \leftrightarrow j_2$ so that in the case of $j_1 \neq j_2$ when the physical combination is $(j_1, j_2)_c = (j_1, j_2) + (j_2, j_1)$ an extra factor of 2 is to be added (in our notation bosonic $j_1 = j_2$ fields are real).

5.2 $\mathcal{N} = 1$ superconformal multiplets

Let us now find the total contributions of $\mathcal{N} = 1$ superconformal multiplets containing $(\Delta;\, j_1, j_2)$ field as the lowest dimension member. The structure of relevant multiplets was given, e.g., in [56]. In addition to long massive multiplets there are shortened ones: chiral and right-handed semi-long (SLII), as well as their CP conjugates — anti-chiral and left-handed semi-long (SLI). There are also CP self-conjugate ("conserved") multiplets that are the sums of one SLI and one SLII multiplet (thus they need not be considered separately).

[23] Here we wrote the fermionic contribution in terms of $\nu = s(s+1)$ rather than ν_f.

SO(2,4) representation content of massive long $\mathcal{N}=1$ superconformal multiplet is[24]

$$
\begin{aligned}
[\Delta;\, j_1, j_2]_{\text{long}} = (\Delta;\, j_1, j_2) &+ \left(\Delta+\frac{1}{2};\, j_1+\frac{1}{2}, j_2\right) + \left(\Delta+\frac{1}{2};\, j_1-\frac{1}{2}, j_2\right) \\
&+ \left(\Delta+\frac{1}{2};\, j_1, j_2+\frac{1}{2}\right) + \left(\Delta+\frac{1}{2};\, j_1, j_2-\frac{1}{2}\right) + 2\,(\Delta+1;\, j_1, j_2) \\
&+ \left(\Delta+1;\, j_1+\frac{1}{2}, j_2+\frac{1}{2}\right) + \left(\Delta+1;\, j_1+\frac{1}{2}, j_2-\frac{1}{2}\right) \\
&+ \left(\Delta+1;\, j_1-\frac{1}{2}, j_2+\frac{1}{2}\right) + \left(\Delta+1;\, j_1-\frac{1}{2}, j_2-\frac{1}{2}\right) \\
&+ \left(\Delta+\frac{3}{2};\, j_1, j_2+\frac{1}{2}\right) + \left(\Delta+\frac{3}{2};\, j_1, j_2-\frac{1}{2}\right) \\
&+ \left(\Delta+\frac{3}{2};\, j_1-\frac{1}{2}, j_2\right) + \left(\Delta+\frac{3}{2};\, j_1+\frac{1}{2}, j_2+\frac{1}{2}\right) + (\Delta+2;\, j_1, j_2).
\end{aligned} \tag{5.9}
$$

Using the above expressions we find that the total a and c anomalies of a long massive multiplet vanish but the Casimir energy does not:

$$
\mathrm{a}_{\text{long}} = \mathrm{c}_{\text{long}} = 0, \qquad\qquad E_{c\ \text{long}} = -\frac{1}{16}\,(-1)^{2\,(j_1+j_2)}\,(2j_1+1)(2j_2+1)\,(\Delta-1)\,. \tag{5.10}
$$

The vanishing of $\mathrm{c}-\mathrm{a}$ for long multiplets follows also from (4.1) [24, 26]. The fact that E_c is not proportional to a-coefficient as in (1.3) means that the coefficient g of the D^2R term in the trace anomaly (1.1) does not vanish (in the heat kernel scheme we are using to define E_c); indeed, g is expected to cancel only in $\mathcal{N}>2$ extended supersymmetric cases (cf. (1.3)).

The content of the chiral short multiplet is

$$
[\Delta;\, j, 0]_{\text{chiral}} = (\Delta;\, j, 0) + \left(\Delta+\frac{1}{2};\, j+\frac{1}{2}, 0\right) + \left(\Delta+\frac{1}{2};\, j-\frac{1}{2}, 0\right) + (\Delta+1;\, j, 0)\,, \tag{5.11}
$$

and thus we find

$$
\begin{aligned}
\mathrm{a}_{\text{chiral}} &= \frac{1}{96}\,(-1)^{2j}\,(2j+1)\,(2\,\Delta-3)\,(-2\,\Delta^2+6\,\Delta+6j^2+6j-3), \\
\mathrm{c}_{\text{chiral}} &= -\frac{1}{48}\,(-1)^{2j}\,(2j+1)\,(2\,\Delta-3)\,(\Delta^2-3\,\Delta+j^2+j+1), \\
E_{c\ \text{chiral}} &= -\frac{1}{384}\,(-1)^{2j}\,(2j+1)\,(16\Delta^3-72\Delta^2+94\Delta-33)\,, \\
(\mathrm{c}-\mathrm{a})_{\text{chiral}} &= -\frac{1}{96}\,(-1)^{2j}\,(2j+1)\,(8j^2+8j-1)\,(2\,\Delta-3)\,.
\end{aligned} \tag{5.12}
$$

[24] The term $2\,(\Delta+1;\, j_1, j_2)$ comes from two representations with the same SO(2,4) labels but different R-charge. For the computation of $E_c, \mathrm{a}, \mathrm{c}$ we do not need to keep track of the R charge (in general, it is constrained by the shortening conditions).

$\mathcal{N}$	ϕ	ψ	V_μ	E_c	a	c
1	–	1	1	$\frac{7}{64}$	$\frac{3}{16}$	$\frac{1}{8}$
2	2	2	1	$\frac{13}{96}$	$\frac{5}{24}$	$\frac{1}{6}$
3, 4	6	4	1	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{1}{4}$

Table 1. Values of $E_c, \mathrm{a}, \mathrm{c}$ for $\mathcal{N} \leq 4$ supersymmetric Maxwell multiplets.

The SLII short multiplet has the content

$$\begin{aligned}[\Delta;\, j_1, j_2]_{\rm SLII} = (\Delta;\, j_1, j_2) + \left(\Delta + \frac{1}{2};\, j_1, j_2 + \frac{1}{2}\right) + \left(\Delta + \frac{1}{2};\, j_1 + \frac{1}{2}, j_2\right) \\ + \left(\Delta + \frac{1}{2};\, j_1 - \frac{1}{2}, j_2\right) + \left(\Delta + 1;\, j_1 + \frac{1}{2}, j_2 + \frac{1}{2}\right) + \\ + \left(\Delta + 1;\, j_1 - \frac{1}{2}, j_2 + \frac{1}{2}\right) + (\Delta + 1;\, j_1, j_2) + \left(\Delta + \frac{3}{2};\, j_1, j_2 + \frac{1}{2}\right),\end{aligned} \tag{5.13}$$

and we obtain

$$\begin{aligned}\mathrm{a}_{\rm SLII} &= \frac{(-1)^{2\,(j_1+j_2)}}{96}\,(2j_1+1)\,(2\,\Delta+2\,j_2-1)\,[2\,(\Delta+j_2-1)(\Delta+j_2)-6j_1(j_1+1)-1]\,, \\ \mathrm{c}_{\rm SLII} &= \frac{(-1)^{2\,(j_1+j_2)}}{48}\,(2j_1+1)\,(2\,\Delta+2\,j_2-1)\,[(\Delta+j_2-1)(\Delta+j_2)+j_1(j_1+1)-1]\,, \\ E_{c\ \rm SLII} &= \frac{(-1)^{2\,(j_1+j_2)}}{384}\,(2j_1+1)\,\left[16\,\Delta^3-24\,\Delta^2-26\,\Delta+29+2\,(24\,\Delta^2-60\,\Delta+31)\,j_2\right], \\ (\mathrm{c}-\mathrm{a})_{\rm SLII} &= \frac{1}{96}\,(-1)^{2\,(j_1+j_2)}\,(2j_1+1)\,(8j_1^2+8j_1-1)\,(2\Delta+2j_2-1)\,.\end{aligned} \tag{5.14}$$

The same expressions (5.12) and (5.14) for c − a follow [26] if we use (4.1) instead of our (4.10), i.e. the chiral and SLII multiplet expressions for c are not sensitive to the difference between (4.2) and (4.9).[25]

5.3 $\mathcal{N} > 1$ superconformal multiplets

Next, let us present the expressions for $E_c, \mathrm{a}, \mathrm{c}$ in the case of some $\mathcal{N} > 1$ superconformal multiplets.

5.3.1 Maxwell supermultiplets

Considering massless 4d multiplets with the highest spin 1 we get the values in table 1.

We notice that for $\mathcal{N} = 3, 4$ eq. (1.3) is satisfied, i.e.

$$E_c = \frac{3}{4}\mathrm{a}\,, \qquad \mathrm{a} = \mathrm{c}\,, \tag{5.15}$$

[25]This equality of c − a for $\mathcal{N} = 1$ multiplets computed using c from (4.2) or from (4.9) is non-trivial. Consider the difference between (4.2) and (4.9) for the basic combination of representations $\langle\Delta; j\rangle \equiv (\Delta;\, j, 0) + (\Delta + \frac{1}{2};\, j + \frac{1}{2}, 0)$. This turns out to be a function of $\Delta + j$ multiplied by $(-1)^{2j}$ and the contribution of a chiral multiplet happens to be the same as of $\langle\Delta; j\rangle + \langle\Delta + \frac{1}{2}; j - \frac{1}{2}\rangle$. It is then possible to see that the contribution of this sum vanishes.

Field	$(\Delta; j_1, j_2)$	E_c	a	c
ϕ ($\Box$)	$(3; 0, 0)$	$\frac{1}{240}$	$\frac{1}{360}$	$\frac{1}{120}$
Φ ($\Box^2$)	$(4; 0, 0)$	$-\frac{3}{40}$	$-\frac{7}{90}$	$-\frac{1}{15}$
ψ ($\not\partial$)	$(\frac{5}{2}; \frac{1}{2}, 0) + (\frac{5}{2}; 0, \frac{1}{2})$	$\frac{17}{960}$	$\frac{11}{720}$	$\frac{1}{40}$
Ψ ($\not\partial^3$)	$(\frac{7}{2}; \frac{1}{2}, 0) + (\frac{7}{2}; 0, \frac{1}{2})$	$-\frac{29}{960}$	$-\frac{3}{80}$	$-\frac{1}{120}$
$T_{\mu\nu}$ ($\Box$)	$(3; 1, 0) + (3; 0, 1)$	$\frac{1}{40}$	$-\frac{19}{60}$	$\frac{1}{20}$
V_μ ($\Box$)	$(3; \frac{1}{2}, \frac{1}{2})$	$\frac{11}{120}$	$\frac{31}{180}$	$\frac{1}{10}$
ψ_μ ($\not\partial^3$)	$(\frac{7}{2}; 1, \frac{1}{2}) + (\frac{7}{2}; \frac{1}{2}, 1)$	$-\frac{141}{80}$	$-\frac{137}{90}$	$-\frac{149}{60}$
$g_{\mu\nu}$ ($\Box^2$)	$(4; 1, 1)$	$\frac{553}{120}$	$\frac{87}{20}$	$\frac{199}{30}$

Table 2. Values of E_c, a, c for fields of extended conformal supergravities.

$\mathcal{N}$	ϕ	Φ	ψ	Ψ	$T_{\mu\nu}$	V_μ	ψ_μ	$g_{\mu\nu}$	E_c	a	c
1	–	–	–	–	–	1	1	1	$\frac{47}{16}$	3	$\frac{17}{4}$
2	–	–	2	–	1	4	2	1	$\frac{145}{96}$	$\frac{41}{24}$	$\frac{13}{6}$
3	6	–	9	1	3	9	3	1	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{2}$
4	20	2	20	4	6	15	4	1	$-\frac{3}{4}$	-1	-1

Table 3. Values of E_c, a, c for $\mathcal{N} \leq 4$ extended conformal supergravity.

and thus the coefficient g of the derivative term in (1.1) vanishes [18].

Let us mention that $\mathcal{N} = 4$ Maxwell multiplet is isomorphic to the $\mathcal{N} = 4$ superdoubleton multiplet $\{\mathcal{N} = 4\} = \{\ 1, 0\}_c + 4\{\ \frac{1}{2}, 0\}_c + 6\{0, 0\}$ of $P\mathrm{SU}(2, 2|4)$ [57] and thus their quantum characteristics should be the same,

$$K(\{\mathcal{N} = 4\}) = K(\mathcal{N} = 4 \text{ Maxwell})\,, \qquad K \equiv (E_c, \mathrm{a}, \mathrm{c})\,. \tag{5.16}$$

Also, the one-particle partition functions match, see (A.31).

5.3.2 Conformal supergravity multiplets

The case of short multiplets with highest spin value is 2 is that of 4d extended conformal supergravity (CSG) multiplets. The relevant fields are listed in table 2 together with their individual E_c, a, c values. The total values for $\mathcal{N} \leq 4$ conformal conformal supergravity multiplets are given in table 3 (the numbers in the central square are multiplicities of the fields, i.e. dimensions of their $\mathrm{U}(\mathcal{N})$ or $\mathrm{SU}(4)$ representations).

As in the case of Maxwell supermultiplets, for $\mathcal{N} = 3, 4$ we find the relation (1.3) or (5.15) satisfied, implying $\mathrm{g} = 0$ (cf. (1.2)). The values of E_c and g for conformal supergravities were not computed previously.

As was found in [27, 44], the conformal anomalies of the combined system of $\mathcal{N} = 4$ conformal supergravity and *four* $\mathcal{N} = 4$ Maxwell multiplets cancel, i.e. this is a UV finite theory. This is readily seen from the values in tables 1 and 2:

$$K(\mathcal{N}=4\ \text{CSG}) + 4\,K(\mathcal{N}=4\ \text{Maxwell}) = 0\,, \qquad\qquad K = (E_c, \text{a}, \text{c})\,. \tag{5.17}$$

The vanishing of the total E_c is a new result (implied by (5.15) which is valid for each of the $\mathcal{N}=4$ multiplets).

The $\mathcal{N}=4$ conformal supergravity multiplet[26] is isomorphic to the supercurrent multiplet of $\mathcal{N}=4$ Maxwell theory [58] and also to the short massless multiplet of fields of gauged $\mathcal{N}=8$ supergravity in 5 dimensions whose AdS_5 vacuum isometry is $P\text{SU}(2,2|4)$ [57, 59, 60, 33].[27] The field content of the latter is given in $p=2$ entry in table 6 in appendix E. Indeed, the 5d expression for the conformal anomaly and the Casimir energy for $\mathcal{N}=4$ CSG is directly given by the one-loop contributions of fields of $\mathcal{N}=8$ 5d supergravity, i.e.

$$K(\mathcal{N}=4\ \text{CSG}) = -2\,K^{+}(\mathcal{N}=8\ \text{5d SG})\,. \tag{5.18}$$

This one-loop relation between the two theories generalizes the tree-level one in [33].

In view of (5.17) this also implies that one-loop contribution of $\mathcal{N}=8$ 5d supergravity is the same as of two $\mathcal{N}=4$ Maxwell multiplets,

$$K^{+}(\mathcal{N}=8\ \text{5d SG}) = 2\,K(\mathcal{N}=4\ \text{Maxwell})\,. \tag{5.19}$$

Remarkably, this non-trivial relation may be interpreted as expressing the fact that the states of $\mathcal{N}=8$ 5d supergravity appear in the product of two $\mathcal{N}=4$ superdoubletons [45]. We shall return to this observation in section 6.1 below.

5.3.3 General long higher spin massless $P\text{SU}(2,2|4)$ supermultiplet

The general long massless multiplet of $P\text{SU}(2,2|4)$ [61, 45] has spin range 4 (8 supercharges). Its conformal representation content is that of $[j_1,j_2]\oplus[j_2,j_1]$ where $[j_1,j_2]$ is summarized in table 4. There $j_1,j_2\geq 1$ are the labels of the supermultiplet and all states have $\Delta = 2+j_1+j_2$. The members of this multiplets may be viewed as representing massless higher spin AdS_5 fields or the corresponding 4d conformal higher spin gauge fields.

Using (5.6), (5.7), (5.8) we find that for all choices of the j_1, j_2 labels of the supermultiplet

$$E_c = \text{a} = \text{c} = 0\,. \tag{5.20}$$

Thus in contrast to the case the massive $\mathcal{N}=1$ long multiplet in (5.10) here the total Casimir energy vanishes along with a and c. This is another manifestation of the relation (1.3), (5.15) valid for $\mathcal{N}\geq 3$.

[26] In addition to fields listed in table 2 this $P\text{SU}(2,2|4)$ short multiplet contains also 20 auxiliary scalars with $\Delta=2$ which do not contribute to physical quantities (the total number of helicity $2j_1+2j_2+1$ states is 256).

[27] $\mathcal{N}=8$ supersymmetry of 5d supergravity corresponds to 4 Poincare and 4 conformal supersymmetries of $\mathcal{N}=4$ conformal supergravity in 4d.

spin (j_L, j_R)	SU(4)
$(j_1{+}1, j_2{+}1)$	1
$(j_1{+}1, j_2{+}\frac{1}{2}){+}(j_1{+}\frac{1}{2}, j_2{+}1)$	$4{+}4^*$
$(j_1{+}\frac{1}{2}, j_2{+}\frac{1}{2})$	$1{+}15$
$(j_1{+}1, j_2){+}(j_1, j_2{+}1)$	$6{+}6$
$(j_1{+}\frac{1}{2}, j_2){+}(j_1, j_2{+}\frac{1}{2})$	$4{+}4^*{+}20{+}20^*$
$(j_1{+}1, j_2{-}\frac{1}{2}){+}(j_1{-}\frac{1}{2}, j_2{+}1)$	$4{+}4^*$
(j_1, j_2)	$1{+}15{+}20'$
$(j_1{+}\frac{1}{2}, j_2{-}\frac{1}{2}){+}(j_1{-}\frac{1}{2}, j_2{+}\frac{1}{2})$	$6{+}6{+}10{+}10^*$
$(j_1{+}1, j_2{-}1){+}(j_1{-}1, j_2{+}1)$	$1{+}1$

spin (j_L, j_R)	SU(4)
$(j_1, j_2{-}\frac{1}{2}){+}(j_1{-}\frac{1}{2}, j_2)$	$4{+}4^*{+}20{+}20^*$
$(j_1{+}\frac{1}{2}, j_2{-}1){+}(j_1{-}1, j_2{+}\frac{1}{2})$	$4{+}4^*$
$(j_1{-}\frac{1}{2}, j_2{-}\frac{1}{2})$	$1{+}15$
$(j_1, j_2{-}1){+}(j_1{-}1, j_2)$	$6{+}6$
$(j_1{-}\frac{1}{2}, j_2{-}1){+}(j_1{-}1, j_2{-}\frac{1}{2})$	$4{+}4^*$
$(j_1{-}1, j_2{-}1)$	1

Table 4. Spin and SU(4) content of general long massless supermultiplet $[j_1, j_2]$ of PSU(2, 2|4).

6 Applications to AdS/CFT

Let us now apply the general expressions for $(E_c, \mathrm{a}, \mathrm{c})$ to specific examples of AdS/CFT duality. This will require summation of contributions of infinite collections of 5d fields (in the above discussion of supermultiplets the sets of fields were finite), and thus a choice of a regularization that should be consistent with symmetries of the underlying theory.

6.1 Adjoint $\mathrm{AdS}_5/\mathrm{CFT}_4$

Let us start with the canonical example of the duality between type IIB superstring on $\mathrm{AdS}_5{\times}\mathrm{S}^5$ and $\mathcal{N} = 4$ SU(N) SYM theory [62–64]. The partition function of SYM theory defined on a curved 4d background M^4 should match the one of the superstring defined on a generalization of AdS_5 asymptotic to M^4. This implies, in particular, the matching of conformal anomalies and Casimir energies computed on the two sides of the duality. The direct perturbative comparison is possible due to the expected non-renormalization of these quantities, with the SYM side giving

$$K(\mathcal{N} = 4 \text{ SU(N) SYM}) = (N^2 - 1)\,\mathrm{k}\,, \qquad\qquad K \equiv (E_c, a, c)\,, \tag{6.1}$$

where $\mathrm{k} = (\frac{3}{16}, \frac{1}{4}, \frac{1}{4})$ are the single $\mathcal{N} = 4$ Maxwell multiplet entries in table 1.

At the leading N^2 order (string tree level or classical type IIB supergravity) this matching was demonstrated in [21] (for the conformal anomalies) and in [65] (for the vacuum energy). To consider the next — string one-loop order it is natural to assume that the contributions of loops of all massive string modes should vanish.

Indeed, string modes form long massive PSU(2, 2|4) multiplets[28] and thus should give zero contribution (cf. section 5). Equivalently, string mode masses depend on 't Hooft coupling ($m^2 \sim \alpha'^{-1} \sim \sqrt{\lambda}$) and thus a non-trivial contribution from them would contradict the expectred non-renormalization of (6.1).

[28] KK descendants of massive string excitations sit in long multiplets given by tensoring string primaries with the long Konishi multiplet [66].

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Assuming this, the subleading $O(N^0)$ term in (6.1) should be reproduced just by the loop of massless string modes, i.e. by the one-loop correction in 10d type IIB supergravity compactified on S^5. The latter is given by the sum of the contributions of the massless $\mathcal{N} = 8$ $5d$ supergravity multiplet and an infinite tower of massive KK multiplets [67]. Thus for consistency with (6.1) one should find that

$$\text{one-loop 10d IIB supergravity on } S^5: \qquad E_c^+ = -\frac{3}{16}, \quad \mathrm{a}^+ = -\frac{1}{4}, \quad \mathrm{c}^+ = -\frac{1}{4}. \tag{6.2}$$

Here we put superscript + as we are interested in direct contributions of 5d fields with standard ("Dirichlet") boundary conditions given by

$$K^+ = (E_c^+, \mathrm{a}^+, \mathrm{c}^+) = -\frac{1}{2}(E_c, \mathrm{a}, \mathrm{c}) \,, \tag{6.3}$$

in terms of the corresponding elementary 4d conformal field values quoted in section 5.1. Eq. (6.2) may be written also as (cf. (5.19))

$$K^+(\text{10d IIB SG on } \mathrm{S}^5) = -K(\mathcal{N} = 4 \text{ Maxwell}) \,. \tag{6.4}$$

This matching of both a and c coefficients at the one-loop supergravity level was earlier claimed in [22, 23]. In particular, using (4.1) motivated by the prescription of [23], the vanishing of the type IIB supergravity contribution to $\mathrm{c} - \mathrm{a}$ implied by (6.2) was interpreted in [24, 25] as a consequence of the vanishing of the contributions of each of the long KK multiplet and the separate cancellation of the $\mathrm{c} - \mathrm{a}$ contributions from states in the massless multiplet.[29] Reproducing the explicit value of a in (6.2) is much more non-trivial, requiring a specific choice of a regularization of the sum over the infinite number of KK modes. While our final conclusion is the same as in [23] the intermediate steps of the derivation disagree.

Starting with our general expressions for $E_c, \mathrm{a}, \mathrm{c}$ given in section 5.1 we shall explicitly demonstrate the validity of (6.2) or (6.4). The proportionality (1.3) of E_c and a-coefficient is expected due to the maximal supersymmetry, implying, in particular, that E_c (i.e. the AdS_5 vacuum energy) does not vanish in the one-loop type IIB supergravity compactified on S^5. This is different from the vanishing of the vacuum energy in $\mathcal{N} > 4$ gauged supergravities in 4 dimensions [69] and in also in 11d supergravity compactified on S^7 [70–72].[30] The non-vanishing of the vacuum energy in the pure $\mathcal{N} = 8$ 5d supergravity was already noted in [50] but the inclusion of the contribution of the KK multiplets leading to the value of E_c in (6.2) is a new result.

The $P\mathrm{SU}(2,2|4)$ multiplet content of 10d supergravity compactified on S^5 is recalled in table 6 in appendix E (where p is KK level). The degeneracies, i.e. the dimensions of the corresponding SU(4) representations can be found using (E.1). Summing up the

[29] Similar pattern applies to matching of axial anomalies [68].

[30] A possible way to reconcile these different conclusions from the AdS/CFT point of view is to note that Casimir energy should automatically vanish in the case of 3d boundary theory, but need not in the 4d case (see also below).

elementary 5d field contributions using (5.6)–(5.8) and (6.3) we find for the massless $p = 2$ supermultiplet in table 6[31]

$$p = 2: \qquad E_c = \frac{3}{8}, \qquad \mathrm{a} = \frac{1}{2}, \qquad \mathrm{c} = \frac{1}{2}. \tag{6.5}$$

The $p = 2$ multiplet corresponding to the states of pure $\mathcal{N} = 8$ 5d gauged supergravity is isomorphic to the $\mathcal{N} = 4$ 4d conformal supergravity multiplet.[32] The corresponding values for the conformal anomaly and E_c should thus be related as in (6.3): indeed, -2 times the values in (6.5) gives the values in the last line of table 3, i.e. we get the expression given above in (5.18). The equivalent form of (6.5) was given in (5.19).

For both $p = 3$ and $p \geq 4$ massive KK multiplets in table 6 we obtain

$$p \geq 3: \qquad E_c = \frac{3p}{16}, \qquad \mathrm{a} = \frac{p}{4}, \qquad \mathrm{c} = \frac{p}{4}, \tag{6.6}$$

Remarkably, despite the different structure of the $p = 2$, $p = 3$ and $p \geq 4$ multiplets in table 6, their contributions to $K = (E_c, \mathrm{a}, \mathrm{c})$ are thus universally described by[33]

$$K^+(\text{KK level p of 10d IIB SG on } \mathrm{S}^5) = p\, K(\mathcal{N} = 4 \text{ Maxwell}), \qquad p = 2, 3, 4, \ldots \tag{6.7}$$

As the $p = 1$ level may be interpreted as the $\mathcal{N} = 4$ superdoubleton multiplet, this relation formally applies also for $p = 1$, becoming (5.16). For $p = 2$ eq. (6.7) is equivalent to (5.19), while for $p > 2$ to (6.6). A natural interpretation of this non-trivial identity (which relies on the particular values of $E_c, \mathrm{a}, \mathrm{c}$ we used)[34] is that it expresses the fact that the 5d states at the KK level p appear in the tensor product of p copies of $\mathcal{N} = 4$ superdoubleton [45].

It remains to sum up the supermultiplet contributions (6.7) over the KK level p, i.e. to assign a consistent value to the divergent sum $\sum_{p=2}^{\infty} p$. The prescription that is required to reproduce (6.4) is

$$\sum_{p=1}^{\infty} p = 0\,, \qquad \text{i.e.} \qquad \sum_{p=2}^{\infty} p = -1\,. \tag{6.8}$$

This can be interpreted as follows. As was noted above, the $p = 1$ case of (6.7) is the same as the contribution of one $\mathcal{N} = 4$ Maxwell multiplet (5.15) or superdoubleton. The

[31]The value of E_c is the same as found in [50].

[32]The full set of states of 10d supergravity compactified on S^5 will then correspond in 4d to $\mathcal{N} = 4$ conformal supergravity coupled to infinite collection of conformal fields with canonical dimensions $\Delta_- = 4 - \Delta$ corresponding to massive $p \geq 3$ states in 5d spectrum in table 6.

[33]Note, in particular, that the relation (1.3) or (5.15) applies level by level, i.e. for each $\mathcal{N} = 4$ supermultiplet.

[34]For example, this relation would not be true for the c-coefficient had we used (4.2) instead of (4.9). The expressions for the contributions of each p level to a and c coefficients found in [23] were very different: they were not linear in p but polynomials of order 5. The reason for this was that the expressions for the individual 5d field contributions to a and c used there (cf. (1.5)) were linear in $\Delta - 2$ and thus linear in p (cf. table 6), while the higher powers in p were coming from the multiplicities given by the dimensions (E.1) of the corresponding SU(4) representations. The correct expressions for E_c, a and c found here are instead 5th order polynomials in $\Delta - 2$ (and thus in p, for the states in table 6), but, remarkably, the non-linearity in p cancels out after multiplying by the dimensions of SU(4) representations and summing over the members of each supermultiplet.

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contribution of the $p=1$ superdoubleton should not to be included [57] in the list of physical multiplets in table 6 as it is gauged away [67] but if we would formally include it then under (6.8) the total 10d supergravity contribution would vanish.[35] The condition (6.8) is satisfied if one defines the sum over KK level p with a sharp cutoff and then drops all cutoff-dependent terms.[36]

While the prescription (6.8) may look artificial (e.g., it is not the ubiquitous Riemann ζ-function rule) it is possible, in fact, to justify it by starting with the standard spectral ζ-function regularization. The key point is that a regularization consistent with symmetries of the theory should be applied directly at the 10d rather than 5d level, i.e. it should be based on the spectrum of the original 10d differential operators defined on $\mathrm{AdS}_5\times \mathrm{S}^5$ or its generalization.

Let us demonstrate this on the example of the sum of E_c contributions. The expression for the contribution of a massive $(\Delta;\, j_1,j_2)$ 5d field to the vacuum energy E_c can be obtained from the partition function (2.7) which may be written as

$$\widehat{\mathcal{Z}}(\Delta;\, j_1,j_2) = d(j_1,j_2)\sum_{k=0}^{\infty}\binom{k+3}{3}\, q^{\Delta+k}\,. \tag{6.9}$$

Then (2.28) implies that a formal (divergent) expression for E_c is given by

$$\widehat{E}_c(\Delta;\ j_1,j_2) = \sum_{k=0}^{\infty} e_k(\Delta;\ j_1,j_2)\,, \tag{6.10}$$

$$e_k(\Delta;\ j_1,j_2) = \frac{1}{2}\,(-1)^{2\,(j_1+j_2)}\,d(j_1,j_2)\binom{k+3}{3}(\Delta+k)\,. \tag{6.11}$$

This sum can be computed using the ζ-function prescription (2.28) applied to the full effective energy eigenvalue $\Delta+k$, or, equivalently, by introducing an exponential cutoff

$$e_k \quad\to\quad e_k\, e^{-\epsilon(\Delta+k)}\,, \tag{6.12}$$

doing the sum, expanding in $\epsilon\to 0$ and finally dropping all singular terms. Keeping ϵ finite we may find the contribution to the sum (6.10) from all states of the $p\geq 4$ massive KK multiplet in table 6. This gives the total summand $e_k(p;\epsilon)$. Summing over both k and p

[35] Adding the $p=1$ superdoubleton contribution would be equivalent to adding the decoupled U(1) D3-brane contribution, i.e. the same as replacing SU(N) by U(N) group on the dual SYM side and thus dropping -1 term in (6.1). An alternative interpretation might be in terms of an effective bulk+boundary anomaly cancellation (conformal anomaly analog of "anomaly inflow"). That would also formally imply the cancellation of the total $\mathrm{AdS}_5\times \mathrm{S}^5$ vacuum energy as in in the $\mathrm{AdS}_4\times \mathrm{S}^7$ case.

[36] Explicitly, one has $\sum_{p=1}^{P} p = \frac{1}{2}P^2+\frac{1}{2}P \to 0$. The same sharp cutoff regularization of the sum over KK level was assumed in [23]. In such a regularization all sums $\sum_{p=1}^{\infty} p^n$ with positive integer n are just set to zero. This formally explains why a different expression for the summand in [23] still led to the same correct expression for the result in (6.2).

we obtain

$$\sum_{k=0}^{\infty}\sum_{p=4}^{\infty} e_k(p;\epsilon) = \frac{e^{-2\epsilon}\left(95e^{\epsilon}+120e^{3\epsilon/2}-220e^{2\epsilon}-420e^{5\epsilon/2}+50e^{3\epsilon}+420e^{7\epsilon/2}+210e^{4\epsilon}-6\right)}{\left(e^{\epsilon/2}-1\right)^2\left(e^{\epsilon/2}+1\right)^{10}}$$
$$= \frac{249}{256\,\epsilon^2} - \frac{9}{8} + \mathcal{O}(\epsilon^2)\,. \tag{6.13}$$

Keeping only the finite part and adding the contributions of the $p = 2$ and $p = 3$ multiplets in (6.5), (6.6) gives finally for the total 10d supergravity contribution

$$E_c^{+} = \frac{3}{8} + \frac{9}{16} - \frac{9}{8} = -\frac{3}{16}\,. \tag{6.14}$$

This is in agreement with (6.2) and thus confirms the prescription in (6.8).

6.2 Vectorial $\mathrm{AdS}_5/\mathrm{CFT}_4$

In the case of vectorial $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence one considers N free fields transforming in a vector (fundamental) representation of $\mathrm{U}(N)$ or $O(N)$. The restriction to the singlet sector of bilinear conserved higher spin current operators implies duality to massless higher spin fields in AdS_{d+1} described by Vasiliev-type theories (see, e.g., [73–75]). The coefficient in front of the classical action in AdS_{d+1} is proportional to N, with the cubic and higher amplitudes supposed to match free-theory correlators of conserved currents at the boundary in $1/N$ expansion.

The original examples were for $d = 3$ [76–79] while generalizations to $d > 3$ were studied in [80, 81, 1, 2, 5, 6] (see also [82, 84–86] for related work). In $d = 3$ one may build conserved higher spin currents as bilinears of free scalars or spin $\frac{1}{2}$ fermions and then get the spectrum of dual massless higher spin theories in AdS_4 containing totally symmetric tensors (these are the only options to get a consistent 3d theory with higher-spin symmetry under natural assumptions [87]).

In $d = 4$ case we will be interested in here in the free fermion case there is a new feature: the corresponding conserved currents belong to particular mixed-symmetry representations of SO(4) [88, 49, 6]. Another novelty of the $d = 4$ case is that here one can also use spin 1 fields[37] as building blocks for higher spin conserved currents (free spin $0, \frac{1}{2}, 1$ are he only options to get a 4d theory with a higher spin symmetry if one assumes unitarity [89–92]).[38] as in [93, 94, 61, 95, 96, 88, 49, 97] are also in specific mixed-symmetry representations of SO(2, 4). The singlet sector of a theory of N real Maxwell vectors should then be dual to

[37] In $d = 3$ Maxwell vector is dual to a scalar.

[38] In principle, one can also explore the possibility of defining the boundary theory in terms of higher spin singletons which are unitary and conformal when described in terms of field strengths. This possibility was noticed in [83] where the corresponding higher spin algebras were studied.

AdS_5	CFT_4 (singlet sector)
non-minimal type A theory $(2;0,0) + \bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})$	N complex scalars : $\mathrm{U}(N)$
minimal type A theory $(2;0,0) + \bigoplus_{s=2,4,\dots}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})$	N real scalars : $O(N)$
non-minimal type B theory $2\,(3;0,0)+$ $2\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2}) + \bigoplus_{s=1}^{\infty}(2+s;\frac{s+1}{2},\frac{s-1}{2})_c$	N Dirac fermions : $\mathrm{U}(N)$
minimal type B theory $2\,(3;0,0)+$ $\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2}) + \bigoplus_{s=2,4,\dots}^{\infty}(2+s;\frac{s+1}{2},\frac{s-1}{2})_c$	N Majorana fermions : $O(N)$
non-minimal type C theory $2\,(4;0,0) + (4;1,0)_c$ $2\bigoplus_{s=2}^{\infty}(2+s;\frac{s}{2},\frac{s}{2}) + \bigoplus_{s=2}^{\infty}(2+s;\frac{s+2}{2},\frac{s-2}{2})_c$	N complex Maxwell vectors : $\mathrm{U}(N)$
minimal type C theory $2\,(4;0,0)+$ $\bigoplus_{s=2}^{\infty}(2+s;\frac{s}{2},\frac{s}{2}) + \bigoplus_{s=2,4,\dots}^{\infty}(2+s;\frac{s+2}{2},\frac{s-2}{2})_c$	N real Maxwell vectors : $O(N)$

Table 5. Field content of vectorial $\text{AdS}_5/\text{CFT}_4$ dualities.

a particular version of higher spin theory in AdS_5 involving mixed-symmetry fields which should exist but was not studied detail so far (we shall call it "type C" theory).[39]

The field content of the corresponding dual pairs is summarized in table 5 where we use the notation $(\Delta;\,j_1,j_2)_c \equiv (\Delta;\,j_1,j_2) + (\Delta;\,j_2,j_1)$.

Higher spin theory content matches the list of bilinear conserved currents in the boundary theory. It can be obtained by taking the product of two doubleton representations corresponding to the boundary fields (see appendix A). In addition to conserved currents there are also scalar bilinears dual to $(2;0,0)$ AdS_5 scalars in type A theory (see (A.5)) and fermion bilinears dual to $(3;0,0)$ AdS_5 scalar and pseudoscalar in type B theory (see (A.6)).[40] Type A theories contain symmetric tensors while type B and type C theories include also particular mixed-symmetry representations of massless higher spin

[39] Interacting higher spin theory for totally symmetric fields in AdS_5 was considered in [98, 99]. Mixed-symmetry fields in AdS_5 and the associated currents were discussed in [36, 37, 100–103, 39]. Cubic interactions of mixed-symmetry higher spin fields in flat space were studied in [104] and in AdS_5 they were considered in [105, 97, 106, 89, 107, 108]. The question of consistency of an interacting AdS_5 theory involving mixed-symmetry fields goes beyond the cubic order and requires, in particular, the closure of the symmetry algebra [89]. Unitarity imposes additional constraints, excluding, e.g., partially massless fields.

[40] The massless $(2;0,0)$ scalars having $\Delta - 2 = 0$ will not contribute to the quantities $K = (E_c, \mathrm{a}, \mathrm{c})$ discussed below.

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fields in AdS$_5$. The second series of massless $(2+s;\frac{s}{2},\frac{s}{2})$ fields in U(N) type B and type C theories are parity-odd. Restriction to real fields at the boundary implies projecting out some (odd-spin parity even and even-spin parity odd) fields in the bulk that either vanish or become total derivatives (see [6] for discussion of the minimal type B theory case).

Since the content of type C theory dual to (complex or real) 4d Maxwell fields was not explicitly studied in the literature let us comment on it in some detail. It can be obtained by taking the product of two spin 1 doubletons as in (A.7).[41] In the complex Maxwell field case the tower of relevant operators starts with dimension 4 operators appearing in the decomposition of $F^*_{\mu\nu}F_{\kappa\rho}$ into SO(4) irreps:[42]

(i) scalar $F^*_{\mu\nu}F^{\mu\nu}$ and pseudoscalar $F^*_{\mu\nu}\tilde{F}^{\mu\nu}$ in massive representation $(4;0,0)$;

(ii) antisymmetric tensor $F^*_{\mu[\nu}F_{\kappa]\mu}$ which is not conserved on shell and corresponds to massive selfdual + anti-selfdual rank 2 tensor, i.e. representation $(4;1,0)_c=(4;1,0)+(4;0,1)$;[43]

(iii) spin 2 conserved stress tensor $(4;1,1)$ and its parity-odd counterpart with one $F_{\mu\nu}$ replaced by $\tilde{F}_{\mu\nu}$;

(iv) conserved current with symmetries of Weyl tensor, i.e. the massless state $(4;2,0)_c$ described by the Young tableu with 2 rows and 2 columns.

In addition, the product (A.7) of two spin 1 doubletons $(\{1,0\}+\{0,1\})\otimes(\{1,0\}+\{0,1\})$ (where $\{1,0\}$ and $\{0,1\}$ correspond to selfdual and antiselfdual parts of $F_{\mu\nu}$) contains also higher spin conserved currents dual to massless AdS$_5$ fields. The real vector case (minimal type C theory) is found by a projection similar to the one in type B theory case: removing one set (parity-odd) of symmetric tensor states and odd-spin mixed-symmetry states. This results in the spectrum given in table 5.

The AdS/CFT duality implies the equality of the corresponding partition functions. For example, the singlet-sector partition function $Z_{\rm CFT}$ of U(N) conformal scalar defined on a curved space M^4 should be equal to the quantum partition function $Z_{\rm HS}$ of the corresponding higher spin theory with coupling constant N^{-1} defined on an AdS$_5$ type Einstein space which is asymptotic to M^4 boundary. If M^4 has no non-trivial holonomies $\log Z_{\rm CFT}$ should be given just by the free-theory one-loop contribution.[44] It should match the leading classical term in $\log Z_{\rm HS}$ that should thus scale as N.

[41]Related discussions appeared in [109, 94, 110]; see also [111] for a general construction of higher spin currents as bilinears in higher spin fields in flat space.

[42]Here * is complex conjugation, tilde denotes dual tensor and we suppress U(N) vector index.

[43]The corresponding antisymmetric tensor field in AdS$_5$ appears, e.g., in S^5 compactification of type IIB supergravity and was discussed in [67, 112]. Its AdS$_5$ Lagrangian has first-derivative topological kinetic term plus the standard mass term.

[44]The singlet constraint may be imposed by integrating over an auxiliary pure-gauge vector field gauging the U(N) or $O(N)$ global symmetry. This constraint does not change the leading order N term in the partition function, i.e. is not relevant for computing vacuum energy and conformal anomaly coefficients, but in presence of non-trivial holonomy like in $S^1\times S^3$ case it leads to an additional $O(N^0)$ contribution to the non-trivial β-dependent part of the partition function (see [113–115, 6] and refs. there). Note that the case of adjoint-representation vector fields (cf. [116] and also [117]) is different from the vector-representation one we consider here.

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As the full non-linear classical actions for higher spin theories in AdS$_5$ are presently unknown, one is not able to compare the leading large N terms in the corresponding observables like $(E_c, \mathrm{a}, \mathrm{c})$. Remarkably, it is still possible [2] to perform non-trivial next-order checks: as $O(N^0)$ term in $\log Z_{\rm CFT}$ is absent in the free theory case, the one-loop contribution to $\ln Z_{\rm HS}$ should vanish too. This was explicitly demonstrated for the a-coefficient of type A theories in [5], and for the Casimir energy of type A and B theories in [6].

In the non-minimal type A and B theories where one sums over all spins one finds the vanishing results for the one-loop corrections to a-coefficient (from $Z_{\rm HS}$ on AdS$_5$ with S^4 boundary) and to E_c (from $Z_{\rm HS}$ on AdS$_5$ with $\mathbb{R}\times S^3$ boundary). In the minimal theories the one-loop HS correction turns out to be non-zero and equal to that of one real 4d scalar (in the minimal type A case) and one Majorana fermion (in the minimal type B case). The proposed interpretation [2] of this fact is that the bulk coupling constant in the minimal HS theory is not N^{-1} but $(N-1)^{-1}$, so that there is an extra $O(N^0)$ contribution that comes from the corresponding $N-1$ coefficient of the tree-level term that cancels the non-zero one-loop HS correction.

As for the c-coefficient, its matching was not attempted so far (apart from a remark in [5] that similar conclusions as for a-coefficient may apply in type A theory if one uses the expression (4.4) with the special "finite" choice of $r_b = -1$ [3]). Neither a- nor c- coefficients were discussed previously in type B theories containing mixed-symmetry 5d fields.

The expressions for a and c coefficients corresponding to one-loop corrections of general $(\Delta; j_1, j_2)$ fields in AdS$_5$ presented in section 5.1 allow us to complete the picture and explicitly demonstrate that the above matching pattern applies universally not only to E_c [6] but also to a and c in all type A and type B cases. The matching of both conformal anomaly coefficients provides further non-trivial test of the consistency of the vectorial AdS/CFT duality. Note that here there is no supersymmetry, so there is no a priori reason to expect a correlation between the values of a and c or a and E_c as in (1.3). As we shall see below, the novel case of type C theory appears to require a different matching pattern.

Since HS theories contain infinite number of fields, one needs a prescription of how to regularize the infinite sum of individual contributions. In the computations of the a-coefficient and E_c (from the partition functions in AdS$_5$ with S^d and $\mathbb{R}\times S^{d-1}$ boundaries where the heat kernel is explicitly known) there is a preferred regularization equivalent to the use of the spectral ζ-function [5, 6]. Its use should be required by the preservation of symmetries of the theory at the quantum level. This regularization amounts to first doing the sum over spins of individual-field $\zeta(z)$-functions for an arbitrary z and then analytically continuing the result (or its derivative) to the required value of z. As was found in [5], in the case of d-dimensional boundary this regularization is equivalent to introducing a specific exponential cutoff factor $\exp\left[-\epsilon\,(s+\frac{d-3}{2})\right]$ into the sum over spins s, doing the sum and then dropping all singular terms in the $\epsilon\to 0$ limit.

Below we shall apply the same prescription also for the summation of the contributions to the c-coefficient where a direct spectral ζ-function regularization is not available. In the present $d=4$ case this prescription amounts to

$$\sum_s K(s) \equiv \sum_s e^{-\epsilon\,(s+\frac{1}{2})}\, K(s)\,\Big|_{\epsilon\to 0,\ \text{finite part}}\,, \qquad\qquad K=(E_c, \mathrm{a}, \mathrm{c})\,. \tag{6.15}$$

Here $s = j_1 + j_2$ is the total spin and the sum includes summation over all states. Let us denote by $K^+(\Delta; j_1, j_2)$ any of the three quantities $E_c^+, \mathrm{a}^+, \mathrm{c}^+$ corresponding to the one-loop contribution of a 5d field in the representation $(\Delta; j_1, j_2)$. Then, as in (6.3), $K = -2K^+$ will give the quantities for the associated elementary 4d conformal field with the canonical dimension equal to $\Delta_- = 4 - \Delta$.

Starting with the non-minimal type A theory and using the expressions in (2.33), (3.5) and (4.4), (4.11) together with the regularization (6.15) one finds that the total one-loop HS contribution to each of the three quantities is indeed zero

$$\sum_{s=1}^{\infty} K^+\left(2+s;\, \frac{s}{2}, \frac{s}{2}\right) = 0\,. \tag{6.16}$$

In the minimal type A theory we get instead

$$\sum_{s=2,4,\ldots}^{\infty} K^+\left(2+s;\, \frac{s}{2}, \frac{s}{2}\right) = K(3;\, 0, 0), \tag{6.17}$$

i.e. the total AdS_5 HS theory one-loop correction is equal exactly to the one-loop contribution of a single real massless 4d scalar.[45] As the contribution of N such scalars should match the classical plus one loop minimal type A higher spin theory result, this is consistent with the AdS/CFT duality provided the coefficient in front of the classical minimal HS theory action is not N but $N-1$.

Similarly, in the non-minimal type B theory we get from (2.34), (3.6) and (4.5), (4.11)

$$2\, K^+(3;\, 0, 0) + 2\sum_{s=1}^{\infty} K^+\left(2+s;\, \frac{s+1}{2}, \frac{s-1}{2}\right) = 0\,. \tag{6.18}$$

Here the first term $2\, K^+(3;\, 0, 0) = -K(3; 0, 0)$ stands for the contribution of the two 5d scalars appearing in the type B spectrum in table 5. The contribution of the totally symmetric higher spin fields vanishes separately due to (6.16). The contributions of $(\Delta; j_1, j_2)$ and $(\Delta; j_2, j_1)$ states are equal so the mixed-symmetry term doubles. For c^+ this is equivalent to the first relation in (4.12) (where $\mathrm{c} = -2\mathrm{c}^+$).

In the minimal type B theory we find

$$2\, K^+(3;\, 0, 0) + 2\sum_{s=2,4,\ldots}^{\infty} K^+\left(2+s;\, \frac{s+1}{2}, \frac{s-1}{2}\right) = K\left(\frac{5}{2};\, \frac{1}{2}, 0\right)_c\,, \tag{6.19}$$

where the r.h.s. is the same as the contribution of a single 4d Majorana fermion (again equivalent to (4.12) in the case of c^+).[46]

Repeating the same computations for the spectrum of the non-minimal type C theory in table 5 we find (cf. (A.27) and the discussion of Casimir energy in appendix A)

$$\begin{aligned} &2\, K^+(4;\, 0, 0) + K^+(4;\, 1, 0)_c + 2\sum_{s=2}^{\infty} K^+\left(2+s;\, \frac{s}{2}, \frac{s}{2}\right) + \sum_{s=2}^{\infty} K^+\left(2+s;\, \frac{s+2}{2}, \frac{s-2}{2}\right)_c \\ &\quad = 2\, K\left(3;\, \frac{1}{2}, \frac{1}{2}\right) = -4\, K^+\left(3;\, \frac{1}{2}, \frac{1}{2}\right)\,. \end{aligned} \tag{6.20}$$

[45] Explicitly, $K(3;\, 0, 0) = (\frac{1}{240}, \frac{1}{360}, \frac{1}{120})$, see table 2.

[46] Here $K(\frac{5}{2};\, \frac{1}{2}, 0)_c = 2K(\frac{5}{2};\, \frac{1}{2}, 0) = (\frac{17}{960}, \frac{11}{720}, \frac{1}{40})$, see table 2.

Here the sum of all AdS_5 one-loop contributions is no longer zero but is twice $K(3; \frac{1}{2}, \frac{1}{2}) = (\frac{11}{120}, \frac{31}{180}, \frac{1}{10})$, i.e. is the same as the contribution of one complex 4d Maxwell field. This suggests that already in the non-minimal type C theory case one needs to assume that the coefficient in front of the corresponding HS classical action in AdS_5 is not N but $N-1$.[47]

In the minimal type C theory we get a relation similar to (6.20)

$$\begin{aligned} 2\,K^+(4;0,0) + \sum_{s=2}^{\infty} K^+\left(2+s; \frac{s}{2}, \frac{s}{2}\right) + \sum_{s=2,4,\ldots}^{\infty} K^+\left(2+s; \frac{s+2}{2}, \frac{s-2}{2}\right)_c \\ = 2\,K\left(3; \frac{1}{2}, \frac{1}{2}\right) = -4\,K^+\left(3; \frac{1}{2}, \frac{1}{2}\right). \end{aligned} \tag{6.21}$$

Since here the boundary vector field is real, this non-vanishing result could be accommodated by the shift $N \to N-2$ in the coefficient of the classical HS action. This is analogous to what happened in the type A and B theories where one required an extra -1 shift of the coefficient of the HS action when going from non-minimal to minimal case. The reason for the $N \to N-1$ shift required already in the non-minimal type C case remains to be understood.

Let us mention also that as discussed in appendix A, the one-particle partition functions on $S^1 \times S^3$ in the non-minimal and minimal type C theories satisfy the relations (A.22) and (A.23) which are the direct analogs of the relations (A.16), (A.17) and (A.18), (A.19) in the type A and type B theories [6]. It is straightforward to derive these relations from the large N limit of the singlet-sector partition function for the boundary spin 1 theory just like that was done in the spin 0 and spin $\frac{1}{2}$ cases in [115, 6].[48]

Finally, let us note that while supersymmetry is not a necessary ingredient in vectorial AdS/CFT duality, it is possible to consider also supersymmetric $\mathrm{AdS}_5/\mathrm{CFT}_4$ dual pairs.[49] An example of $\mathcal{N}=1$ supersymmetric higher spin theory in AdS_5 was constructed in [99]. The 4d boundary theory should be represented by N free spin $(0, \frac{1}{2})$ $\mathcal{N}=1$ supermultiplets having bosonic integer spin and fermionic half integer spin conserved currents. Equivalently, in addition to the bosonic HS 5d fields there will be the fermionic ones coming from the product of spin 0 and spin $\frac{1}{2}$ doubleton representations (cf. (A.29) for $n_1 = 0,\ n_0 = n_{1/2}$). The analog of (6.16) for the non-minimal theory should then be given by the sum of the bosonic and fermionic 5d field contributions. The bosonic part vanishes separately due to (6.16) while the fermionic part can be verified to satisfy the required identity (here we use $\mathrm{s} = s - \frac{1}{2}$ as in (2.16) which takes integer values for the fermions)

$$2\,K^+\left(\frac{5}{2}; \frac{1}{2}, 0\right) + \sum_{\mathrm{s}=1}^{\infty} K^+\left(2+\mathrm{s}+\frac{1}{2}; \frac{\mathrm{s}}{2}, \frac{\mathrm{s}+1}{2}\right)_c = 0\,. \tag{6.22}$$

[47] An alternative possibility may be to add 4 real massless 5d vectors to the bulk theory, i.e. to put the r.h.s. term in (6.20) to the l.h.s. as in (A.27), but it is unclear why that would lead to a consistent HS theory (and also which should be the corresponding conserved spin 1 currents in the boundary theory).

[48] We shall present details of this derivation elsewhere.

[49] Supersymmetric $\mathrm{AdS}_4/\mathrm{CFT}_3$ cases were discussed, e.g., in [77, 78, 118].

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There is also a minimal-theory analog of this relation

$$\sum_{\mathrm{s}=1,3,5,\dots}^{\infty} K^{+}\left(2+\mathrm{s}+\frac{1}{2};\,\frac{\mathrm{s}}{2},\frac{\mathrm{s}+1}{2}\right)_c = 0\,. \tag{6.23}$$

It should be possible also to consider the case of supersymmetric boundary theory containing spin 1 fields. This will generalize the type A, B and C theory examples considered above.

The most supersymmetric case of the free unitary boundary CFT will be a collection of N free $\mathcal{N}=4$ Maxwell supermultiplets. The spectrum of the dual AdS_5 HS theory will then be given by the product of two $\mathcal{N}=4$ superdoubletons [45, 61, 119, 96] with the low-spin ≤ 2 part [59] being the same as the set of fields of type IIB supergravity compactified on S^5 given in table 6. This HS theory with AdS_5 vacuum should correspond to the "leading Regge trajectory" part of the zero tension limit of $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring (cf. [66, 120]). This may suggest a way to consider a particular maximally supersymmetric case of the vectorial AdS/CFT duality as a truncation of zero gauge coupling limit of the adjoint AdS/CFT. As we have seen in sections 5.1 and 6.1, when 5d fields are combined into supermultiplets many cancellations happen, and this should especially be true in the maximally supersymmetric case.

We postpone detailed discussion of the supersymmetric case for the future, presenting here only the result of the computation of $K^+=(E_c^+,\mathrm{a}^+,\mathrm{c}^+)$ corresponding to the infinite set of higher spin 5d fields appearing in the product of two superdoubletons $\{\mathcal{N}\}$ representing $\mathcal{N}$-supersymmetric Maxwell theory (see appendix A). In general, if $\{\mathcal{N}\}$ contains n_1 vector, $n_{\frac{1}{2}}$ fermion and n_0 scalar doubletons (A.28) then we find from (A.30)[50]

$$\begin{aligned} K^+(\{\mathcal{N}\}\otimes\{\mathcal{N}\}) &= n_1\left(\frac{4\,n_0+17\,n_{\frac{1}{2}}+88\,n_1}{480},\,\frac{2\,n_0+11\,n_{\frac{1}{2}}+124\,n_1}{360},\,\frac{n_0+3\,n_{\frac{1}{2}}+12\,n_1}{60}\right)\\ &= 2\,n_0\,n_1\,K(3;\,0,0)+2\,n_{\frac{1}{2}}\,n_1\,K\left(\frac{5}{2};\,\frac{1}{2},0\right)_c+2\,n_1^2\,K\left(3;\,\frac{1}{2},\frac{1}{2}\right). \end{aligned} \tag{6.24}$$

This generalizes the above results (6.16) ($n_0=1,\,n_{\frac{1}{2}}=n_1=0$), (6.18) ($n_{\frac{1}{2}}=1, n_0=n_1=0$) and (6.20) ($n_1=1,\,n_{\frac{1}{2}}=n_0=0$) in non-minimal type A, B, and C theories: the r.h.s. of (6.24) contains no n_0^2 or $n_{\frac{1}{2}}^2$ terms, but there is n_1^2 term. For the particular choices of n_i corresponding to $\mathcal{N}\leq 4$ supersymmetric Maxwell theory, i.e. $(n_1,n_{\frac{1}{2}},n_0)=(1,1,0),(1,2,2),(1,4,6)$ we thus get a remarkable relation

$$K^+(\{\mathcal{N}\}\otimes\{\mathcal{N}\}) = 2\,K(\{\mathcal{N}\}) = 2\,K(\mathcal{N}\text{-Maxwell})\,. \tag{6.25}$$

[50]Here we again use the regularization (6.15) with $s=j_1+j_2$. It turns out that in $\mathcal{N}=4$ supersymmetric case the total result has no poles in $\epsilon\to 0$. This is due to supersymmetry and can be understood as follows. Here we are summing the contributions of bosonic and fermionic fields, $\sum_s K_b(s)+\sum_{\mathrm{s}} K_f(\mathrm{s})$, where in the fermionic case $\mathrm{s}=s-\frac{1}{2}$ is an integer. Ignoring regularization and separating finite number of low-spin terms, the remaining sum can be rewritten as $\sum_s\left[K_b(s)+K_f(s-\frac{1}{2})\right]$ and happens to vanish, implying finiteness of the total result.

Here the r.h.s. is twice the contribution of the $\mathcal{N}$-supersymmetric Maxwell theory, or, which is the same, the contribution of the $\mathcal{N}$-superdoubleton (cf. (A.14), see (5.16) for $\mathcal{N}=4$). This is the direct super-generalization of the relation (6.21) in type C theory.

Eq. (6.25) (i.e. "anomaly of a product is twice anomaly of a factor") may be viewed as the analog of the relation for the characters or partition functions $\mathcal{Z}(\{\mathcal{N}\}\otimes\{\mathcal{N}\}) = [\mathcal{Z}(\{\mathcal{N}\})]^2$ and also admits the following interpretation. As was observed above in (5.19), the one-loop contribution of the states of $\mathcal{N}=8$ 5d supergravity is already equal to the contribution of two $\mathcal{N}=4$ Maxwell multiplets. Thus all other states appearing in the product $\{\mathcal{N}\}\otimes\{\mathcal{N}\}$ (i.e. in (A.30) with $n_1=1, n_{\frac{1}{2}}=4, n_0=6$) should give zero contribution to (6.25). As they should form massless supermultiplets of $P\mathrm{SU}(2,2|4)$, this is indeed consistent with what was found in (5.18).

KK states of type IIB supergravity on S^5 are contained in tensor products of more than two $\mathcal{N}=4$ superdoubletons [45]. Their contribution (6.7) was computed in section 6.1 above. We leave the discussion of the contributions of their partner higher spin states for the future.

6.3 Conformal higher spin theories

The relation (6.16) written in terms of $K=-2K^+=(E_c,\mathrm{a},\mathrm{c})$ has also another interpretation: it expresses the vanishing of the total Casimir energy and the total conformal anomaly coefficients in the 4d conformal higher spin (CHS) theory of all symmetric bosonic gauge fields. The vanishing of the total a-coefficient was first observed in the 5d context [1] and then understood also directly from the 4d perspective [3]. The cancellation of E_c was demonstrated in [7]. The vanishing of the total c-coefficient requires the use of our proposed expression (4.9) leading to (4.4) with the specific choice of the parameter $r_b=-1$ [3] in (4.11). Similar conclusion applies also to the fermionic CHS theory with the individual field contributions given in (2.34), (3.6), (4.5), (4.11), generalizing earlier demonstration of the vanishing of its total a-coefficient [3].

The consistency of the vectorial AdS/CFT is thus tightly related with the consistency (cancellation of anomalies) of the associated CHS theories. This is not completely surprising in view of the direct connection of the CHS theory (viewed as induced by the boundary CFT [121–123, 1]) to the CFT conserved currents (CHS fields are shadow fields for the CFT currents) and then, via AdS/CFT, to the 5d higher spins (CHS fields are effectively boundary values for the 5d fields).

While it still remains to prove our conjecture for the c coefficient in (4.9) (implying the values in (4.4), (4.5), (4.11)) this is a strong indication that in addition to the $\mathcal{N}=4$ supersymmetric theory of conformal supergravity coupled to 4 Maxwell multiplets containing finite number of fields, the theory of an infinite collection of conformal higher spins is also a consistent quantum conformal theory with no Weyl anomalies (both theories are of course perturbatively non-unitary). The same should be true also for the $\mathrm{SU}(2,2|\mathcal{N})$ supersymmetric conformal higher spin theories like the one constructed in [124, 125] and its truncations [99].

7 Concluding remarks

There are many open questions. One interesting question is to understand better the vectorial AdS/CFT duality in the spin 1 boundary theory case, clarifying the structure of the dual type C theory in table 5 and providing the interpretation for the equation (6.20).

It remains to explore further the relation between vectorial AdS/CFT duality setup for $\mathcal{N}=4$ superdoubleton as boundary theory and a tensionless limit of the $\mathrm{AdS}_5\times\ \mathrm{S}^5$ string theory, computing, in particular, the quantities $(E_c, \mathrm{a}, \mathrm{c})$ and also the twisted and thermodynamic one-particle partition functions for the string spectrum of 5d fields.

Another direction is to attempt to build an example of vectorial AdS/CFT duality by starting with spin >1 conformal fields at the boundary and considering the set of (in general, non-unitary) 5d higher spin fields corresponding to their conserved currents.

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A SO(2, 4) representations, characters and generalised Flato-Fronsdal relations

Below we shall summarize some relations for relevant representations of the $d=4$ conformal group and their characters using some results of [49]. We shall then consider the relations between characters that have the interpretation in terms of one-particle partition functions in the context of vectorial AdS/CFT discussed in section 6.2. We shall also discuss the case of supersymmetric combination of representations.

We shall adopt the following short-hand notation for the unitary irreducible representations of SO(2, 4)

$$\begin{array}{rll} \text{“massive”}: & (\Delta;\, j_1, j_2), & \Delta > 2 + j_1 + j_2 \\ \text{“massless”}: & (2+j_1+j_2;\, j_1, j_2) & \Delta = 2 + j_1 + j_2 \\ \text{“doubleton”}: & \{j, 0\},\ \{0, j\} & \Delta = 1 + j\,, \end{array} \tag{A.1}$$

where j can take integer or half-integer values and the names refer to AdS_5 interpretation of the corresponding fields.[51] We shall also use $(\Delta; j_1, j_2)_c \equiv (\Delta; j_1, j_2) + (\Delta; j_2, j_1)$.

[51] As we consider the AdS_5 case we use name doubleton [93] instead of singleton. The massive case with $j_1 j_2 = 0$ was called *massive self-dual* in [126] where it is shown that, contrary to the doubleton case, this representation admits a realisation in terms of local fields in AdS_5. Examples of such fields are $(3; 1, 0)$ in table 2 and $(4; 1, 0)$ in non-minimal type C theory in table 5.

Products of two doubleton representations decompose as follows [88, 49]

$$\{j,0\}\otimes\{j',0\}=\bigoplus_{k=|j-j'|}^{j+j'}(2+j+j';\,k,0)+\bigoplus_{k=1}^{\infty}\left(2+j+j'+k;j+j'+\frac{k}{2},\frac{k}{2}\right)\,, \qquad (A.2)$$

$$\{0,j\}\otimes\{0,j'\}=\bigoplus_{k=|j-j'|}^{j+j'}(2+j+j';\,0,k)+\bigoplus_{k=1}^{\infty}\left(2+j+j'+k;\frac{k}{2},j+j'+\frac{k}{2}\right)\,, \qquad (A.3)$$

$$\{j,0\}\otimes\{0,j'\}=\bigoplus_{k=0}^{\infty}\left(2+j+j'+k;j+\frac{k}{2},j'+\frac{k}{2}\right)\,, \qquad (A.4)$$

where the first term in (A.2), (A.3) is the finite sum over representations corresponding to states appearing in the product $j\otimes j'=(j+j')\oplus(j+j'-1)\oplus\cdots\oplus|j-j'|$. For example, the product of two spin 0 doubletons gives the Flato-Fronsdal type relation [127, 88]

$$\{0,0)\otimes\{0,0\}=(2;\,0,0)+\bigoplus_{s=1}^{\infty}\left(2+s;\frac{s}{2},\frac{s}{2}\right)\,. \qquad (A.5)$$

For the product of two spin $\frac{1}{2}$ doubletons we get

$$\begin{aligned}&\left(\left\{\frac{1}{2},0\right\}+\left\{0,\frac{1}{2}\right\}\right)\otimes\left(\left\{\frac{1}{2},0\right\}+\left\{0,\frac{1}{2}\right\}\right)\\ &\quad=2\,(3;\,0,0)+(3;\,1,0)_c+2\bigoplus_{k=0}^{\infty}\left(3+k;\frac{k+1}{2},\frac{k+1}{2}\right)+\bigoplus_{k=1}^{\infty}\left(3+k;1+\frac{k}{2},\frac{k}{2}\right)_c\\ &\quad=2\,(3;\,0,0)+2\bigoplus_{s=1}^{\infty}\left(2+s;\frac{s}{2},\frac{s}{2}\right)+\bigoplus_{s=1}^{\infty}\left(2+s;\frac{s+1}{2},\frac{s-1}{2}\right)_c\,.\end{aligned} \qquad (A.6)$$

For two spin 1 doubletons one finds

$$\begin{aligned}&(\{1,0\}+\{0,1\})\otimes(\{1,0\}+\{0,1\})\\ &\quad=2\,(4;\,0,0)+(4;\,1,0)_c+(4;\,2,0)_c+2\bigoplus_{k=0}^{\infty}\left(4+k;\frac{k+2}{2},\frac{k+2}{2}\right)+\bigoplus_{k=1}^{\infty}\left(4+k;2+\frac{k}{2},\frac{k}{2}\right)_c\\ &\quad=2\,(4;\,0,0)+(4;\,1,0)_c+2\bigoplus_{s=2}^{\infty}\left(2+s;\frac{s}{2},\frac{s}{2}\right)+\bigoplus_{s=2}^{\infty}\left(2+s;\frac{s+2}{2},\frac{s-2}{2}\right)_c\,.\end{aligned} \qquad (A.7)$$

A.1 Characters of products of doubletons

Above relations have immediate counterparts in terms of ("blind") characters for the basic representations in (A.1)[52]

$$\text{"massive"}:\quad \mathrm{Z}(\Delta;j_1,j_2)=\frac{q^{\Delta}}{(1-q)^4}(2j_1+1)\,(2j_2+1)\,, \qquad (A.8)$$

[52] Note that the expression for the character of the massless representation (A.9) formally applies also for $j_1j_2=0$ when it gives the character of the corresponding massive self-dual representation, cf. (A.1).

$$\text{“massless”}: \quad \mathrm{Z}(2+j_1+j_2;j_1,j_2) = \frac{q^{j_1+j_2+2}}{(1-q)^4}\left[(2j_1+1)(2j_2+1)-4\,q\,j_1\,j_2\right], \tag{A.9}$$

$$\text{“doubleton”}: \quad \mathrm{Z}(\{j,0\}) = \mathrm{Z}(\{0,j\}) = \frac{q^{j+1}}{(1-q)^3}\left[2j+1-q\,(2j-1)\right]. \tag{A.10}$$

The character (A.8) has the interpretation of one-particle partition function $\widehat{\mathcal{Z}}^+$ in (2.7) corresponding to a massive 5d field while the one in (A.9) is the one-particle partition function $\mathcal{Z}^+$ in (2.7) corresponding to a massless 5d field (2.8) with $\Delta_0 = 2+j_1+j_2$. For the doubleton partition function we shall also use the notation $\mathcal{Z}(\{j,0\})$, i.e.

$$\mathrm{Z}(\Delta;j_1,j_2) = \widehat{\mathcal{Z}}^+(\Delta;j_1,j_2), \quad \mathrm{Z}(\Delta_0;j_1,j_2) = \mathcal{Z}^+(\Delta_0;j_1,j_2), \quad \mathrm{Z}(\{j,0\}) \equiv \mathcal{Z}(\{j,0\}) \tag{A.11}$$

Note that the massless and doubleton characters satisfy the following identity

$$\mathrm{Z}(2+2j;j,j) = [\mathrm{Z}(\{j,0\})]^2 - \left[\mathrm{Z}\left(\left\{j+\frac{1}{2},0\right\}\right)\right]^2, \tag{A.12}$$

$$\text{i.e.} \quad \mathrm{Z}\left(3;\frac{1}{2},\frac{1}{2}\right) = \left[\mathrm{Z}\left(\left\{\frac{1}{2},0\right\}\right)\right]^2 - [\mathrm{Z}(\{1,0\})]^2, \ldots. \tag{A.13}$$

There are also the following relations implying that doubletons can be identified with the corresponding boundary conformal fields (cf. (1.10), (2.4), (2.6), (2.26)):

$$\mathrm{Z}(\{0,0\}) = \mathcal{Z}(3;0,0), \quad \mathrm{Z}\left(\left\{\frac{1}{2},0\right\}\right) = \mathcal{Z}\left(\frac{5}{2};\frac{1}{2},0\right), \quad \mathrm{Z}(\{1,0\}_c) = \mathcal{Z}\left(3;\frac{1}{2},\frac{1}{2}\right), \tag{A.14}$$

$$\mathcal{Z}(\Delta;j_1,j_2) \equiv \mathcal{Z}^-(\Delta;j_1,j_2) - \mathcal{Z}^+(\Delta;j_1,j_2). \tag{A.15}$$

The relations for one-particle partition functions of non-minimal type A and type B theories in table 5 are direct character counterparts of (A.5) and (A.6):

$$[\mathrm{Z}(\{0,0\})]^2 = \mathrm{Z}(2;0,0) + \sum_{s=1}^{\infty}\mathrm{Z}\left(2+s;\frac{s}{2},\frac{s}{2}\right), \tag{A.16}$$

$$\left[2\mathrm{Z}\left(\left\{\frac{1}{2},0\right\}\right)\right]^2 = 2\,\mathrm{Z}(3;0,0) + 2\sum_{s=1}^{\infty}\mathrm{Z}\left(2+s;\frac{s}{2},\frac{s}{2}\right) + \sum_{s=1}^{\infty}\mathrm{Z}\left(2+s;\frac{s+1}{2},\frac{s-1}{2}\right)_c. \tag{A.17}$$

We also get the following character identities that express the relations between one-particle partition functions in minimal type A and type B theories [6][53]

$$\frac{1}{2}[\mathrm{Z}(\{0,0\})]^2 + \frac{1}{2}[\mathrm{Z}(\{0,0\})]_{q\to q^2} = \mathrm{Z}(2;0,0) + \sum_{s=2,4,\ldots}^{\infty}\mathrm{Z}\left(2+s;\frac{s}{2},\frac{s}{2}\right), \tag{A.18}$$

$$\frac{1}{2}\left[2\mathrm{Z}\left(\left\{\frac{1}{2},0\right\}\right)\right]^2 - \frac{1}{2}\left[2\mathrm{Z}(\{\tfrac{1}{2},0\})\right]_{q\to q^2}$$
$$= 2\,\mathrm{Z}(3;0,0) + \sum_{s=1}^{\infty}\mathrm{Z}\left(2+s;\frac{s}{2},\frac{s}{2}\right) + \sum_{s=2,4,6,\ldots}^{\infty}\mathrm{Z}\left(2+s;\frac{s+1}{2},\frac{s-1}{2}\right)_c. \tag{A.19}$$

[53] Here the notation $[\mathrm{Z}(\{0,0\})]_{q\to q^2}$ stands for $\frac{q^2}{(1-q^2)^3}(1+q^2)$, etc.

For spin 1 doubleton characters we find the following identities

$$[\mathrm{Z}(\{1,0\})]^2 = \sum_{s=2}^{\infty} \mathrm{Z}\left(2+s; \frac{s}{2}, \frac{s}{2}\right) = 4\,\mathrm{Z}(4;0,0) + \frac{1}{2}\sum_{s=2}^{\infty} \mathrm{Z}\left(2+s; \frac{s+2}{2}, \frac{s-2}{2}\right)_c, \quad \text{(A.20)}$$

$$[\mathrm{Z}(\{1,0\})]^2 + [\mathrm{Z}(\{1,0\})]_{q\to q^2} = 2\,\mathrm{Z}(4;0,0) + \sum_{s=2,4,\ldots}^{\infty} \mathrm{Z}\left(2+s; \frac{s+2}{2}, \frac{s-2}{2}\right)_c. \quad \text{(A.21)}$$

Since (A.8) implies that $\mathrm{Z}(4;1,0) = \mathrm{Z}(4;0,1) = 3\mathrm{Z}(4;0,0)$ we get the relation which is the counterpart of (A.7) at the character level:

$$\begin{aligned} [2\mathrm{Z}(\{1,0\})]^2 &= 2\,\mathrm{Z}(4;0,0) + \mathrm{Z}(4;1,0)_c \\ &\quad + 2\sum_{s=2}^{\infty} \mathrm{Z}\left(2+s; \frac{s}{2}, \frac{s}{2}\right) + \sum_{s=2}^{\infty} \mathrm{Z}\left(2+s; \frac{s+2}{2}, \frac{s-2}{2}\right)_c. \end{aligned} \quad \text{(A.22)}$$

It has the direct interpretation as the relation of one-particle partition functions in non-minimal type C theory in table 5. Similarly, from (A.20) and (A.21) we get the minimal type C theory counterpart of the relations (A.18) and (A.19) in the minimal type A and type B theories

$$\begin{aligned} &\frac{1}{2}[2\mathrm{Z}(\{1,0\})]^2 + \frac{1}{2}[2\mathrm{Z}(\{1,0\})]_{q\to q^2} \\ &= 2\,\mathrm{Z}(4;0,0) + \sum_{s=2}^{\infty} \mathrm{Z}\left(2+s; \frac{s}{2}, \frac{s}{2}\right) + \sum_{s=2,4,\ldots}^{\infty} \mathrm{Z}\left(2+s; \frac{s+2}{2}, \frac{s-2}{2}\right)_c. \end{aligned} \quad \text{(A.23)}$$

It would be interesting to know a group theoretic interpretation of this relation. It is possible to show, just like this was done in the scalar case in [6], that the l.h.s. of (A.23) corresponds to the leading large N term in the singlet-sector partition function of N real 4d Maxwell vectors.

Let us now comment on the corresponding (2.28), (2.29) Casimir energy. Note that the expressions

$$[\mathrm{Z}(\{0,0\})]^2 = \frac{q^2(1+q)^2}{(1-q)^6}, \qquad \left[2\mathrm{Z}\left(\left\{\frac{1}{2},0\right\}\right)\right]^2 = \frac{16\,q^3}{(1-q)^6}, \quad \text{(A.24)}$$

are invariant under $q\to q^{-1}$. This implies that the total Casimir energy of the 5d fields appearing in the r.h.s. of (A.5), (A.6) or (A.16), (A.17).

The presence of the additional $\mathrm{Z}_{q\to q^2}$ terms in the r.h.s. of (A.18), (A.19) which change sign under $q\to q^{-1}$ implies that the Casimir energy for the representations in the r.h.s. is no longer vanishing in minimal type A and type B theories suggesting the $N\to N-1$ shift in the 5d classical action of the dual HS theory for a consistent AdS/CFT interpretation [6] (see section 6.2).

In contrast, for spin 1 doubleton product (A.7) we get $q\to q^{-1}$ non-invariant expression

$$[2\mathrm{Z}(\{1,0\})]^2 = \frac{4q^4\,(3-q)^2}{(1-q)^6} \quad \text{(A.25)}$$

already in the r.h.s. (A.22). This implies that the Casimir energy in type C theory does not vanish even in the non-minimal case. Observing that one can form a $q \to q^{-1}$ invariant combination as

$$[2\mathrm{Z}(\{1,0\})]^2 + 4\mathrm{Z}\left(3; \frac{1}{2}, \frac{1}{2}\right) = \frac{16q^3}{(1-q)^6}\,, \tag{A.26}$$

we conclude that one can make the Casimir energy vanish by adding four $(3; \frac{1}{2}, \frac{1}{2})$ to the representations in (A.7), getting a theory with field content

$$4\left(3; \frac{1}{2}, \frac{1}{2}\right) + 2\,(4;\,0,0) + (4;\,1,0)_c + 2\bigoplus_{s=2}^{\infty}\left(2+s; \frac{s}{2}, \frac{s}{2}\right) + \bigoplus_{s=2}^{\infty}\left(2+s; \frac{s+2}{2}, \frac{s-2}{2}\right)_c\,. \tag{A.27}$$

In the case of the minimal type C theory the l.h.s. of (A.23) contains half of the same term plus an extra $q \to q^{-1}$ non-invariant term $\mathrm{Z}(\{1,0\})_{q\to q^2}$, and the two combined together give the same Casimir energy as in the non-minimal theory (see section 6.2).

A.2 Product of two $\mathcal{N} \le 4$ superdoubletons

A natural extension of the above discussion is to consider a supersymmetric combination of the 0, $\frac{1}{2}$, 1 doubletons forming a superdoubleton $\{\mathcal{N}\}$ representing $\mathcal{N}$-supersymmetric Maxwell theory [45, 61, 119]. One can then study the SO(2, 4) representation content of the tensor product of two superdoubletons $\{\mathcal{N}\}$. More generally, let us define

$$\{\mathcal{N}\} = n_0\,\{0,0\} + n_{\frac{1}{2}}\left[\left\{\frac{1}{2}, 0\right\} + \left\{0, \frac{1}{2}\right\}\right] + n_1\,[\{1,0\} + \{0,1\}]\,, \tag{A.28}$$

where $(n_1, n_{\frac{1}{2}}, n_0) = (1,1,0), (1,2,2), (1,4,6)$ for one vector multiplet with $\mathcal{N} = 1,2,4$ supersymmetries. The representations appearing in the tensor product $\{\mathcal{N}\} \otimes \{\mathcal{N}\}$ are easily found by using the above expressions for the doubletons[54]

$$\begin{aligned}
\{\mathcal{N}\} \otimes \{\mathcal{N}\} = {} & n_0^2\,(2\;0,0) + 2\,n_{\frac{1}{2}}^2\,(3;\,0,0) + 2\,n_1^2\,(4;\,0,0) + 2\,n_0\,n_{\frac{1}{2}}\left(\frac{5}{2}; 0, \frac{1}{2}\right)_c \\
& + 2\,n_{\frac{1}{2}}\,n_1\left(\frac{7}{2}; 0, \frac{1}{2}\right)_c + (2\,n_0\,n_1 + n_{\frac{1}{2}}^2)\,(3;\,0,1)_c + n_1^2\,(4;\,0,1)_c \\
& + 2\,n_{\frac{1}{2}}\,n_1\left(\frac{7}{2}; 0, \frac{3}{2}\right)_c + n_1^2\,(4;\,0,2)_c + 2\,n_{\frac{1}{2}}^2\sum_{k=0}^{\infty}\left(3+k; \frac{k+1}{2}, \frac{k+1}{2}\right) \\
& + n_0^2\sum_{k=1}^{\infty}\left(2+k; \frac{k}{2}, \frac{k}{2}\right) + 2\,n_1^2\sum_{k=0}^{\infty}\left(4+k; \frac{k+2}{2}, \frac{k+2}{2}\right) \\
& + 2\,n_{\frac{1}{2}}\,n_1\sum_{k=0}^{\infty}\left(\frac{7}{2}+k; \frac{k+1}{2}, \frac{k+2}{2}\right)_c + 2\,n_0\,n_{\frac{1}{2}}\sum_{k=1}^{\infty}\left(\frac{5}{2}+k; \frac{k}{2}, \frac{k+1}{2}\right)_c \\
& + (2\,n_0\,n_1 + n_{\frac{1}{2}}^2)\sum_{k=1}^{\infty}\left(3+k; \frac{k}{2}, \frac{k+2}{2}\right)_c + 2\,n_{\frac{1}{2}}\,n_1\sum_{k=1}^{\infty}\left(\frac{7}{2}+k; \frac{k}{2}, \frac{k+3}{2}\right)_c \\
& + n_1^2\sum_{k=1}^{\infty}\left(4+k; \frac{k}{2}, \frac{k+4}{2}\right)_c
\end{aligned} \tag{A.29}$$

[54] Here we ignore details of SU($\mathcal{N}$) index structure, i.e. just count different representations of SO(2, 4). We shall also not discuss in detail the organization into representations of the superconformal group SU(2, 2|$\mathcal{N}$), see [45].

Grouping terms together, this can be also written as[55]

$$
\begin{aligned}
\{\mathcal{N}\} \otimes \{\mathcal{N}\} = &\, n_0^2\,(2\,0,0) + 2\,n_{\frac{1}{2}}^2\,(3;\,0,0) + 2\,n_1^2\,(4;\,0,0) \\
&+ 2\,n_0\,n_{\frac{1}{2}}\left(\frac{5}{2};\,0,\frac{1}{2}\right)_c + 2\,n_{\frac{1}{2}}\,n_1\left(\frac{7}{2};\,0,\frac{1}{2}\right)_c + (2\,n_0\,n_1 + n_{\frac{1}{2}}^2)\,(3;\,0,1)_c + n_1^2\,(4;\,0,1)_c \\
&+ 2\,n_{\frac{1}{2}}\,n_1\left(\frac{7}{2};\,0,\frac{3}{2}\right)_c + n_1^2\,(4;\,0,2)_c - 2\,n_1^2\left(3;\,\frac{1}{2},\frac{1}{2}\right) \\
&+ (n_0^2 + 2\,n_{\frac{1}{2}}^2 + 2\,n_1^2)\sum_{s=1}^{\infty}\left(2+s;\,\frac{s}{2},\frac{s}{2}\right) + 2\,n_{\frac{1}{2}}\,(n_0+n_1)\sum_{\mathrm{s}=1}^{\infty}\left(\frac{5}{2}+\mathrm{s};\,\frac{\mathrm{s}}{2},\frac{\mathrm{s}+1}{2}\right)_c \\
&+ (2\,n_0\,n_1 + n_{\frac{1}{2}}^2)\sum_{s=2}^{\infty}\left(2+s;\,\frac{s-1}{2},\frac{s+1}{2}\right)_c + 2\,n_{\frac{1}{2}}\,n_1\sum_{\mathrm{s}=2}^{\infty}\left(\frac{5}{2}+\mathrm{s};\,\frac{\mathrm{s}-1}{2},\frac{\mathrm{s}+2}{2}\right)_c \\
&+ n_1^2\sum_{s=3}^{\infty}\left(2+s;\,\frac{s-2}{2},\frac{s+2}{2}\right)_c . \qquad \text{(A.30)}
\end{aligned}
$$

The previous expressions (A.5), (A.6), (A.7) for products of doubletons with the same spin are obtained as special cases — as the coefficients of n_0^2, $n_{\frac{1}{2}}^2$ and n_1^2 terms.

The r.h.s. of (A.30) could be reorganised in order to make manifest the supersymmetry, i.e. rewritten in terms of multiplets of the superconformal group $\mathrm{SU}(2,2|\mathcal{N})$. Doing so, for example, for $\mathcal{N}=4$ one would get an infinite sum of massless finite-dimensional $P\mathrm{SU}(2,2|4)$ multiplets. Each of them is fully characterised by its lowest weight state as discussed in [61, 45]; further details are illustrated in appendix A of [119] (for superconformal characters see [128, 129]).

Let us note that, as follows from (A.14), the partition functions of superdoubletons are the same as of the corresponding super Maxwell theories. For example, for the $\mathcal{N}=4$ case with $\{\mathcal{N}=4\} = \{1,0\}_c + 4\{\frac{1}{2},0\}_c + 6\{0,0\}$ we get (see (A.14))

$$
\mathcal{Z}(\{\mathcal{N}=4\}) = \mathcal{Z}(\mathcal{N}=4\ \text{Maxwell}) = \mathcal{Z}(3;\frac{1}{2},\frac{1}{2}) + 4\,\mathcal{Z}(\frac{5}{2};\frac{1}{2},0)_c + 6\,\mathcal{Z}(3;0,0)\,. \qquad \text{(A.31)}
$$

B Partition functions of free conformal supergravity fields on $S^1 \times S^3$

Here we shall explicitly compute the one-loop partition functions for low-spin fields that appear in $\mathcal{N} \le 4$ conformal supergravities (see tables 2 and 3). The resulting expressions for the one-particle partition functions will be the same that follow from the operator counting method. The cases of the standard scalar, vector and Weyl graviton were already discussed in [7]. For example, for the Maxwell vector (cf. (2.5), (2.6), (2.9), (2.14), (A.25))

$$
\mathcal{Z}_1 = \mathcal{Z}(\{1,0\}_c) = \mathcal{Z}\left(3;\frac{1}{2},\frac{1}{2}\right) = \mathcal{Z}^-\left(3;\frac{1}{2},\frac{1}{2}\right) - \mathcal{Z}^+\left(3;\frac{1}{2},\frac{1}{2}\right) = \frac{2(3-q)q^2}{(1-q)^3}\,. \qquad \text{(B.1)}
$$

[55] The term $-2\,n_1^2\,(3;\,\frac{1}{2},\frac{1}{2})$ appears as a consequence of $\sum_{s=1}^{\infty}(2+s;\,\frac{s}{2},\frac{s}{2}) + \sum_{s=1}^{\infty}(3+s;\,\frac{s+1}{2},\frac{s+1}{2}) = -(3;\,\frac{1}{2},\frac{1}{2}) + 2\sum_{s=1}^{\infty}(2+s;\,\frac{s}{2},\frac{s}{2})$. Also, terms labelled by s are bosonic while those labeled by $\mathrm{s} = s - \frac{1}{2}$ are fermionic.

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Let us start with the familiar case of spin $\frac{1}{2}$ Majorana fermion, i.e. $\mathscr{L}_{\frac{1}{2}} = \overline{\psi}\, e^{\mu}_{a}\, \gamma^{a}\, \mathcal{D}_{\mu}\psi$, $\mathcal{D}_{\mu} = \partial_{\mu} + \frac{1}{2}\,\sigma_{ab}\,\omega^{ab}_{\mu}(e)$, $\sigma_{ab} = \frac{1}{2}\gamma_{[a}\gamma_{b]}$. The corresponding partition function is

$$Z_{\frac{1}{2}} = (\det \not{\mathcal{D}})^{1/2} = (\det \not{\mathcal{D}}^{2})^{1/4}\,, \qquad \not{\mathcal{D}}^{2} = \mathcal{D}^{2} - \frac{1}{4}R = \partial_{0}^{2} + \boldsymbol{\mathcal{D}}^{2} - \frac{1}{4}R\,, \tag{B.2}$$

where $R = 6$ is the scalar curvature of the unit-radius S^{3} and ∂_{0} is derivative along the Euclidean time with period $\beta = -\ln q$. In general, the spectrum of the square of the Dirac operator on unit-radius S^{d-1} with odd $d-1$ is [130]

$$-\boldsymbol{\mathcal{D}}^{2} + \frac{1}{4}R \ \to\ \left(n + \frac{d-1}{2}\right)^{2}, \qquad \mathrm{d}_{n} = 2^{d/2}\,\frac{(n+d-2)!}{n!\,(d-2)!}\,, \qquad n = 0,1,2,\dots\,. \tag{B.3}$$

Then by the standard arguments the corresponding one-particle partition function in (2.2) is given by (see, e.g., [7])

$$\mathcal{Z}_{\frac{1}{2}} = \sum_{n=0}^{\infty} \mathrm{d}_{n}\, q^{n+\frac{d-1}{2}} = 2^{d/2}\,\frac{q^{\frac{d-1}{2}}}{(1-q)^{d-1}} = 2^{d/2}\,\frac{q^{\frac{d-1}{2}} - q^{\frac{d+1}{2}}}{(1-q)^{d}}\,. \tag{B.4}$$

This has direct operator-counting interpretation in $\mathbb{R}^{d}$: counting components of ψ (and their derivative descendants) minus equations of motion $\not{\partial}\psi = 0$. For $d = 4$ this gives as in [47]

$$\mathcal{Z}_{\frac{1}{2}} = \mathcal{Z}(\frac{5}{2};\frac{1}{2},0)_{c} = \frac{4\,q^{3/2}}{(1-q)^{3}}\,. \tag{B.5}$$

Next, let us consider the conformal gravitino [131] with the following quadratic Lagrangian in curved background (we omit $\mathcal{D}R\overline{\psi}\psi$ terms)

$$\begin{aligned}\mathscr{L}_{\frac{3}{2}} = {}& -4\, e^{-1}\,\epsilon^{\mu\nu\rho\sigma}\,\overline{\phi}_{\rho}\,\gamma_{5}\,\gamma_{\sigma}\,\mathcal{D}_{\mu}\phi_{\nu} \\ & - R^{\mu\nu}\left[2\,\overline{\psi}^{\lambda}\sigma_{\lambda\nu}\phi_{\mu} - 2\,\overline{\psi}_{\mu}\sigma_{\lambda\nu}\phi^{\lambda} + 2\,\overline{\psi}^{\lambda}\gamma_{\nu}\left(\mathcal{D}_{[\mu}\psi_{\lambda]} - \gamma_{[\mu}\phi_{\lambda]}\right)\right] + \frac{4}{3}\,R\,\overline{\psi}^{\lambda}\sigma_{\lambda\nu}\phi_{\nu},\end{aligned} \tag{B.6}$$

$$\phi_{\mu} \equiv \frac{1}{3}\gamma^{\nu}\left(\mathcal{D}_{\nu}\psi_{\mu} - \mathcal{D}_{\mu}\psi_{\nu} + \frac{1}{2}\gamma_{5}\epsilon_{\nu\mu\alpha\beta}\,\mathcal{D}^{\alpha}\psi^{\beta}\right), \qquad \mathcal{D}_{\mu}\psi_{\nu} = \left(\partial_{\mu} + \frac{1}{2}\sigma_{ab}\omega^{ab}_{\mu}(e)\right)\,\psi_{\nu}. \tag{B.7}$$

Considering a Bach (e.g., an Einstein) space background we may fix the gauge symmetries by $\gamma^{\mu}\psi_{\mu} = 0$ and $\mathcal{D}_{\mu}\psi^{\mu} = 0$, i.e. restrict to transverse γ-traceless field. Then we get

$$\mathscr{L}_{\frac{3}{2}} = \overline{\psi}^{\lambda}\mathcal{O}_{\frac{3}{2}}\,\psi_{\lambda}, \qquad \mathcal{O}_{\frac{3}{2}} = -\not{\mathcal{D}}^{3} - R^{\mu\nu}\gamma_{\nu}\mathcal{D}_{\mu} + \frac{1}{6}R\not{\mathcal{D}}. \tag{B.8}$$

This operator factorizes [132, 133, 29] on an Einstein space background ($R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}$) as

$$\mathcal{O}_{\frac{3}{2}} = -\not{\mathcal{D}}\left(\not{\mathcal{D}}^{2} + \frac{1}{12}R\right), \qquad \not{\mathcal{D}}^{2} = \mathcal{D}^{2} + \frac{1}{2}[\mathcal{D}_{\mu},\mathcal{D}_{\nu}]\,\gamma^{\mu\nu} = \mathcal{D}^{2} - \frac{1}{4}R - \frac{1}{12}R\,. \tag{B.9}$$

Specializing to the $S^{1}\times S^{3}$ case, we have $R_{\mu\nu} \to R_{ij} = \frac{R}{3}\,g_{ij} = 2g_{ij}$ i.e.

$$\mathcal{O}_{\frac{3}{2}} = -\not{\mathcal{D}}^{3} - 2\vec{\not{\mathcal{D}}} + \not{\mathcal{D}} = -(\gamma^{0}\partial_{0} + \vec{\not{\mathcal{D}}})^{3} + \gamma^{0}\partial_{0} - \vec{\not{\mathcal{D}}}\,, \qquad \vec{\not{\mathcal{D}}} \equiv \gamma_{i}\,\mathcal{D}^{i}\,. \tag{B.10}$$

Taking into account that $\{\vec{\not{D}},\gamma^0\}=0$ the determinant of this operator can written as

$$\det\mathcal{O}_{\frac{3}{2}}=\left(\det(\partial_0^2+\vec{\not{D}}^2)\det\left[(\partial_0+1)^2+\vec{\not{D}}^2\right]\det\left[(\partial_0-1)^2+\vec{\not{D}}^2\right]\right)^{1/2}. \tag{B.11}$$

From (B.9) we get $\vec{\not{D}}^2\psi_i=\left(\boldsymbol{\mathcal{D}}^2-\frac{R}{4}\right)\psi_i+\frac{1}{2}(\gamma_{ij}R^j_k-\gamma_{kj}R^j_i-\frac{R}{3}\gamma_{ik})\psi^k$ so that for $\gamma_i\psi^i=0$

$$\vec{\not{D}}^2=\boldsymbol{\mathcal{D}}^2-\frac{R}{4}-\frac{R}{6}=\boldsymbol{\mathcal{D}}^2-\frac{5}{2}. \tag{B.12}$$

The spectrum of $\boldsymbol{\mathcal{D}}^2$ for a general spin s field on S^3 is (see, e.g., [40])

$$-\boldsymbol{\mathcal{D}}^2\ \to\ (n+s)(n+s+2)-s,\qquad\qquad \mathrm{d}_n=2(n+1)(n+2s+1), \tag{B.13}$$

so that for $s=\frac{3}{2}$ we get

$$\vec{\not{D}}^2\to-\left(n+\frac{5}{2}\right)^2,\qquad\qquad \mathrm{d}_n=2(n+1)(n+4). \tag{B.14}$$

Thus the contribution of the spatially transverse and traceless gravitino ψ_i to the one-particle partition function is

$$\mathcal{Z}^{\rm TT}_{\frac{3}{2}}(q)=\sum_{n=0}^{\infty}2\,(n+1)(n+4)(q^{n+\frac{3}{2}}+q^{n+\frac{5}{2}}+q^{n+\frac{7}{2}})=\frac{4\,q^{\frac{3}{2}}\,(2+q+q^2-q^3)}{(1-q)^3}. \tag{B.15}$$

To get the full partition function we still need to add the contribution of one Majorana spinor degree of freedom.[56] On $\mathbb{R}\times S^3$, we may further split TT ψ_μ into TT ψ_i and a spinor. This gives

$$Z=\left[\det\mathcal{O}^{\rm TT}_{\frac{3}{2}}\ \det{}'\mathcal{O}_{\frac{1}{2}}\right]^{1/4}, \tag{B.16}$$

where $\mathcal{O}^{\rm TT}_{\frac{3}{2}}$ is now defined on transverse γ_i-traceless ψ_i field. Adding together (B.15) and the contribution of a Majorana fermion (B.4) without the $n=0$ zero mode term[57] we arrive at the following conformal gravitino one-particle partition function

$$\mathcal{Z}_{\frac{3}{2}}(q)=\frac{4\,q^{\frac{3}{2}}\,(2+2\,q-6q^2+2q^3)}{(1-q)^4}. \tag{B.17}$$

This expression admits the following operator counting interpretation in flat space. The natural gravitino analog of the covariant Weyl tensor field strength for the conformal graviton is its superpartner [134] (tilde denotes the dual field)

$$\Phi_{\mu\nu}=\frac{1}{3}\left(\psi_{\mu\nu}-\gamma_5\tilde{\psi}_{\mu\nu}+2\,\gamma_{[\nu}{}^{\lambda}\psi_{\lambda\mu]}\right),\qquad\qquad \psi_{\mu\nu}=\partial_\mu\psi_\nu-\partial_\nu\psi_\mu, \tag{B.18}$$

[56] To recall, in covariant gauge the conformal gravitino partition function may be written as $Z=\left[(\det\mathcal{O}_{\frac{1}{2}})^2/\det\mathcal{O}_{\frac{3}{2}}\right]^{-1/4}$, where $\mathcal{O}_{\frac{3}{2}}$ is defined on transverse γ_μ-traceless field ψ_μ (see, e.g., [3]). This correctly accounts for -8 degrees dynamical degrees of freedom: transverse traceless (TT) field ψ_μ contributes 2×4 (with extra factor of 3 being due to the degree of the kinetic operator) and the fermion contributes 2×4.

[57] This mode must be dropped for the same reason as discussed in appendix D of [7].

obeying $\Phi_{\mu\nu} = \gamma_5 \tilde{\Phi}_{\mu\nu}$, $\gamma^\mu \Phi_{\mu\nu} = 0$. These conditions imply that $\Phi_{\mu\nu}$ has 4×2 components (the γ_5 self-duality reduces 6 to 3 and the γ-tracelessness adds one additional constraint). This explains the first $4 \times 2q^{\frac{3}{2}}$ term in the numerator of (B.17). The next term $4 \times 2q^{\frac{5}{2}}$ is associated with $\partial\!\!\!/ \Phi_{\mu\nu}$. The equations of motion and the Bianchi identities remove the term $4 \times (3+3)q^{\frac{7}{2}}$; the term $4 \times 2q^{\frac{9}{2}}$ compensates for overcounting in this subtraction (cf. [7]).

The Lagrangian of the conformal fermion Ψ with $\partial\!\!\!/^3$ kinetic term is [51, 27]

$$\mathscr{L}_\Psi = \overline{\Psi}\,\mathcal{O}_\Psi\,\Psi, \qquad \mathcal{O}_\Psi = \not{\mathcal{D}}^3 + \left(R_{\mu\nu} - \frac{1}{6}Rg_{\mu\nu}\right)\gamma^\mu\mathcal{D}^\nu. \tag{B.19}$$

On $S^1 \times S^3$ the kinetic operator takes the form $\mathcal{O}_\Psi = \not{\mathcal{D}}^3 - \not{\mathcal{D}} + 2\vec{\not{\mathcal{D}}}$, i.e. is the same as the one in (B.10) but now defined on a Majorana spinor. Using (B.13) for $s = \frac{1}{2}$ we get

$$\mathcal{Z}_\Psi(q) = \sum_{n=0}^{\infty} 2(n+1)(n+2)(q^{n+\frac{1}{2}} + q^{n+\frac{3}{2}} + q^{n+\frac{5}{2}}) = \frac{4\,q^{\frac{1}{2}}\,(1-q^3)}{(1-q)^4}\,, \tag{B.20}$$

which admits the same counting interpretation as in the case of the $\partial\!\!\!/$ spinor (B.4).

The Lagrangian for the conformal scalar Φ with ∂^4 kinetic term is [27]

$$\mathscr{L}_\Phi = D^2\Phi D^2\Phi - 2\left(R_{\mu\nu} - \frac{1}{3}Rg_{\mu\nu}\right) D^\mu\Phi D^\nu\Phi. \tag{B.21}$$

On $S^1 \times S^3$ the kinetic operator becomes

$$\mathcal{O}_\Phi = D^4 - 4D^2 + 4\mathbf{D}^2 \;\rightarrow\; (\partial_0^2 - n^2)\left[\partial_0^2 - (n+2)^2\right]\,, \tag{B.22}$$

where $D^2 = \partial_0^2 + \mathbf{D}^2$, and we used that $\mathbf{D}^2$ has the spectrum $-n(n+2)$ with multiplicity $(n+1)^2$. As a result,

$$\mathcal{Z}_\Phi(q) = \sum_{n=0}^{\infty}(n+1)^2(q^n + q^{n+2}) = \frac{1-q^4}{(1-q)^4}\,. \tag{B.23}$$

Similar computation can be done in the case of the non-gauge conformal antisymmetric tensor field $T_{\mu\nu}$ [51, 27, 44] with the Lagrangian (corresponding to the Weyl-invariant action)

$$\mathscr{L}_T = (D^\mu T_{\mu\nu})^2 - \frac{1}{4}(D_\mu T_{\rho\sigma})^2 - R_{\mu\nu}T^{\mu\lambda}T^\nu{}_\lambda + \frac{1}{8}RT_{\mu\nu}^2 + \frac{1}{2}R_{\mu\alpha\nu\beta}T^{\mu\nu}T^{\alpha\beta}\,. \tag{B.24}$$

Here we shall just quote the result for the corresponding partition function which is much easier to find by the counting method in flat space. $T_{\mu\nu}$ has 6 components with dimension 1. The equations of motion

$$E_{\mu\nu} \equiv \partial_\mu\partial_\lambda T^\lambda{}_\nu - \partial_\nu\partial_\lambda T^\lambda{}_\mu - \frac{1}{2}\partial^2 T_{\mu\nu} = 0 \tag{B.25}$$

reprsent 6 conditions with dimension 3. Thus

$$\mathcal{Z}_T(q) = \frac{6q - 6q^3}{(1-q)^4}\,. \tag{B.26}$$

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C Spectral ζ-function for 2nd-order operator on $(\Delta;\, j_1, j_2)$ fields in AdS$_5$

The computation of a-coefficient requires consideration of (in general, massive) higher spin field partition function in Euclidean AdS$_5$ with boundary S^4. The relevant kinetic operator $\mathcal{O}$ given in (1.14) is defined on transverse fields.

In general, for the operator $\mathcal{O}$ on a space $\mathcal{M}$ one can express the corresponding ζ-function in terms of heat kernel as

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty dt\, t^{z-1}\, \mathrm{Tr}\, K\,, \qquad\qquad K(x,y;t) = \langle x|e^{-t\,\mathcal{O}}|y\rangle\,. \tag{C.1}$$

For a homogeneous manifold $\mathcal{M}$ the trace over the position x gives a factor of (regularized) volume, i.e.

$$\zeta(z) = \mathrm{Vol}(\mathcal{M})\, \zeta(z;x)\,, \qquad \zeta(z;x) \equiv \frac{1}{\Gamma(z)} \int_0^\infty dt\, t^{z-1}\, \mathrm{tr}\, K(x,x;t)\,. \tag{C.2}$$

Here tr is the trace over the Lorentz indices of the operator and $\zeta(z;x)$ does not actually depend on x.

To determine $\zeta(z)$ in our case we shall use the results for the heat kernel of the Laplacian in AdS$_{2n+1}$ with even n in [53, 48] (see also [117]) specialising to the case of $n = 2$. Following [53, 48], we shall start with heat-kernel for the sphere S^5 and then analytically continue to AdS$_5$. Let us consider a field on S^5 transforming under the tangent space rotations in a representation H of SO(5). Since the sphere is a homogeneous space S^5 = SO(6)/SO(5) the heat kernel receives contributions from each representation $\mathcal{R}$ of SO(6) that contains $\mathcal{H}$ when restricted to SO(5). Let us denote $\mathcal{R}$ and $\mathcal{H}$ by the corresponding weights as

$$\mathcal{R} = (r_1, r_2, r_3), \qquad r_1 \geq r_2 \geq |r_3|, \qquad\qquad \mathcal{H} = (h_1, h_2), \qquad h_1 \geq h_2 \geq 0, \tag{C.3}$$

were all labels are integer or half integer. The branching condition on the representation $\mathcal{R}$ is

$$r_1 \geq h_1 \geq r_2 \geq h_2 \geq |r_3| \tag{C.4}$$

with the additional requirement that $r_i - h_i \in \mathbb{Z}$. The heat kernel at the coincident points, traced over representation indices, can be written as

$$\mathrm{tr}\, K(x,x;t) = \frac{1}{\pi^3} \sum_{r_i} d_{\mathrm{R}}\, e^{-t\, E_{\mathcal{R}}^{(\mathcal{H})}}\,, \tag{C.5}$$

where $E_{\mathcal{R}}^{(\mathcal{H})}$ are the eigenvalues of the Laplacian $-D^2$ on S^5 expressed in terms of the second Casimir values for the two representations and $d_{\mathcal{R}}$ is the dimension of $\mathcal{R}$

$$-D^2\big|_{\mathrm{S}^5} \;\to\; E_{\mathrm{R}}^{(\mathcal{H})} = C_2(\mathcal{R}) - C_2(\mathcal{H})\,, \tag{C.6}$$

$$C_2(\mathcal{R}) = r_1\,(r_1+4) + r_2\,(r_2+2) + r_3^2, \qquad C_2(\mathcal{H}) = h_1\,(h_1+3) + h_2(h_2+1), \tag{C.7}$$

$$d_{\mathcal{R}} = \frac{1}{12}\left[(r_1+2)^{\,2} - (r_2+1)^{\,2}\right]\left[(r_1+2)^{\,2} - r_3^2\right]\left[(r_2+1)^{\,2} - r_3^2\right]. \tag{C.8}$$

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The analytic continuation from S^5 to AdS_5 amounts to the replacement [53, 48]

$$r_1 \quad \to \quad i\,\lambda - 2\,, \tag{C.9}$$

with the sum over r_1 becoming an integral over the positive real λ. For the $(\Delta; j_1, j_2)$ representation of $SO(2,4)$ we have $h_1 = j_1 + j_2$ and $h_2 = j_1 - j_2$. The analytically continued $E^{(\mathcal{H})}_{\mathcal{R}}$ is then [48]

$$-D^2\big|_{\mathrm{AdS}_5} \;\to\; E^{(\mathcal{H})}_{\mathcal{R}} = \lambda^2 - r_2(r_2+2) - r_3^2 + 2\,j_1(j_1+2) + 2j_2(j_2+1) + 4. \tag{C.10}$$

For the general mixed-symmetry fields which are traceless and transverse (on which our operator $\mathcal{O}$ is defined) the branching condition (C.4) imposes the following restriction[58]

$$r_2 = h_1 = j_1 + j_2, \qquad |r_3| = h_2 = j_1 - j_2\,. \tag{C.11}$$

Then (C.10) becomes

$$-D^2\big|_{\mathrm{AdS}_5} \quad \to \quad \lambda^2 + 2\,j_1 + 4\,. \tag{C.12}$$

Thus finally for the full operator $\mathcal{O}$ in (1.14) with $X = \Delta(\Delta-4) - 2j_1$ we get the following eigenvalue

$$(-D^2 + X)\big|_{\mathrm{AdS}_5} \quad \to \quad \lambda^2 + (\Delta-2)^2\,. \tag{C.13}$$

The regularised volume of the Euclidean AdS_5 or hyperboloid $\mathbb{H}^5$ may be written as $\mathrm{Vol}(\mathbb{H}^5) = \pi^2 \log \mathrm{R} + \dots$ where R is an IR cutoff (the radius of S^4 measured in 5d metric $d\rho^2 + \sinh^2\rho\, d\Omega^2_{S^4}$ at large ρ). Doing the analytic continuation (C.9) in the dimension $d_{\mathcal{R}}$ in (C.8) we then finally obtain from (C.2), (C.5) and (C.13)

$$\begin{aligned}\zeta(z) &= \mathrm{Vol}(\mathbb{H}^5)\,\zeta(z;x)\\ &\to -\log \mathrm{R}\,\frac{(2j_1+1)(2j_2+1)}{12\pi^2}\int_0^\infty d\lambda\,\frac{\left[\lambda^2+(j_1-j_2)^2\right]\left[\lambda^2+(j_1+j_2+1)^2\right]}{\left[\lambda^2+(\Delta-2)^2\right]^z}.\end{aligned} \tag{C.14}$$

D One-parameter ansatz for c-coefficient

Here we present a generalization of our proposal for the c-coefficient (4.9) that preserves correspondence with all known results in special cases. It turns out that this leaves just one-parameter freedom. The remaining free parameter is fixed once we assume in addition the consistency conditions required for vectorial AdS/CFT.

Let us start with the following ansatz

$$\begin{aligned}\widehat{\mathrm{c}}(\Delta; j_1, j_2) =& \frac{1}{720}(-1)^{2(j_1+j_2)}(2j_1+1)(2j_2+1)(\Delta-2)\,[\,k_1\,(\Delta-2)^4\\ &+\left[k_2\,(j_1^2+j_2^2) + k_3\,j_1\,j_2 + k_4\,(j_1+j_2) + k_5\right]\,(\Delta-2)^2\\ &+k_6\,(j_1^4+j_2^4) + k_7\,(j_1^3 j_2 + j_1 j_2^3) + k_8\,j_1^2 j_2^2 + k_9\,(j_1^3+j_2^3) + k_{10}\,(j_1^2 j_2 + j_1 j_2^2)\\ &+k_{11}\,(j_1^2+j_2^2) + k_{12}\,j_1 j_2 + k_{13}\,(j_1+j_2) + k_{14}\,]\,,\end{aligned} \tag{D.1}$$

[58] For the special case of totally symmetric fields see the discussion after (2.17) in [48].

where k_n are some constants to be determined. We shall then require that this expression should reproduce (i) the values of c for the conformal supergravity fields in table 2; (ii) the representation (4.4) and (4.5) for c of totally symmetric fields (with any r_b, r_f); (iii) the value of c − a for all long and short SU(2, 2|1) supermultiplets as obtained in section 5.

Remarkably, these conditions fix all constants in (D.1) apart from one constant that can be identified with the parameter r_b in (4.4), i.e. we get

$$\begin{aligned}\widehat{\mathrm{c}}(\Delta; j_1, j_2) =& \frac{1}{720}(-1)^{2(j_1+j_2)}(2j_1+1)(2j_2+1)(\Delta-2)\Bigg[2\,(r_b-2)\,(\Delta-2)^4 \\ &+ \left[\frac{20}{3}\,(r_b+1)\,(j_1^2+j_2^2)+\frac{20}{3}\,(r_b+1)\,(j_1+j_2)-10\,(r_b-1)\right]\,(\Delta-2)^2 \\ &+2\,(r_b+4)\,(j_1^4+j_2^4)+\frac{20}{3}\,(r_b+4)\,j_1^2j_2^2+4\,(r_b+4)\,(j_1^3+j_2^3) \\ &+\frac{20}{3}\,(r_b+4)\,(j_1^2j_2+j_1j_2^2)+\frac{20}{3}\,(r_b+4)\,j_1j_2 \\ &-\frac{2}{3}(13r_b+22)\,(j_1^2+j_2^2)-\frac{4}{3}\,(8r_b+17)\,(j_1+j_2)+8\,r_b\Bigg]\,. \end{aligned} \tag{D.2}$$

The expression (4.5) for the totally symmetric fermionic fields then has

$$r_f = \frac{16}{3}\,r_b + \frac{169}{3}\,, \tag{D.3}$$

which is a generalization of both (4.6) and (4.11). Our proposal (4.9) corresponds to the choice of r_b in (4.11), i.e.

$$r_b = -1\,, \tag{D.4}$$

while (4.2) is reproduced if $r_b = \frac{1}{2}$ as in (4.6). Our choice (D.4) ensures, in particular, that the consistency conditions for vectorial AdS/CFT discussed in section 6.2 that hold for a-coefficient and E_c are valid also for the c-coefficient.

E AdS$_5$ field content of type IIB 10d supergravity compactified on S^5

In table 6 we summarize the field content of S^5 compactification of IIB supergravity [57, 67]. For each KK level p we list the corresponding SO(2, 4) and SU(4) representations.

The dimension of SU(4) representation (a, b, c) (where a, b, c are Dynkin labels)

$$\mathrm{d}(a,b,c) = \frac{1}{12}(a+1)(b+1)(c+1)(a+b+2)(b+c+2)(a+b+c+3)\,. \tag{E.1}$$

We recall that the level $p = 1$ states (doubleton multiplet) are decoupled from the physical spectrum. The level $p = 2$ is the massless multiplet of gauged $\mathcal{N} = 8$ 5d supergravity; it is isomorphic to the multiplet of states of $\mathcal{N} = 4$ conformal supergravity in tables 2 and 3.[59] The states with $p \geq 3$ form shortened massive multiplets with spin ≤ 2.

[59]There we ignored the auxiliary scalar in the SU(4) representation (0, 2, 0) of dimension 20.

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	$(\Delta;\, j_1, j_2)$	SU(4)		$(\Delta;\, j_1, j_2)$	SU(4)
				$(p+\frac{3}{2};\, \frac{1}{2}, 0)$	$(2, p-3, 1)_c$
	$(p;\, 0, 0)$	$(0, p, 0)$		$(p+\frac{5}{2};\, \frac{1}{2}, 0)$	$(0, p-3, 1)_c$
	$(p+\frac{1}{2};\, \frac{1}{2}, 0)$	$(0, p-1, 1)_c$	$p \geq 3$	$(p+2;\, \frac{1}{2}, \frac{1}{2})$	$(1, p-3, 1)_c$
	$(p+1;\, 1, 0)$	$(0, p-1, 0)_c$		$(p+2;\, 1, 0)$	$(2, p-3, 0)_c$
$p \geq 2$	$(p+1;\, 0, 0)$	$(0, p-2, 2)_c$		$(p+3;\, 1, 0)$	$(0, p-3, 0)_c$
	$(p+2;\, 0, 0)$	$(0, p-2, 0)_c$		$(p+\frac{5}{2};\, 1, \frac{1}{2})$	$(1, p-3, 0)_c$
	$(p+\frac{3}{2};\, \frac{1}{2}, 0)$	$(0, p-2, 1)_c$		$(p+2;\, 0, 0)$	$(2, p-4, 2)$
	$(p+1;\, \frac{1}{2}, \frac{1}{2})$	$(1, p-2, 1)$		$(p+3;\, 0, 0)$	$(0, p-4, 2)_c$
	$(p+\frac{3}{2};\, 1, \frac{1}{2})$	$(1, p-2, 0)_c$	$p \geq 4$	$(p+4;\, 0, 0)$	$(0, p-4, 0)$
	$(p+2;\, 1, 1)$	$(0, p-2, 0)$		$(p+\frac{5}{2};\, \frac{1}{2}, 0)$	$(2, p-4, 1)_c$
				$(p+\frac{7}{2};\, \frac{1}{2}, 0)$	$(0, p-4, 1)_c$
				$(p+3;\, \frac{1}{2}, \frac{1}{2})$	$(1, p-4, 1)$

Table 6. Field content of compactification of type IIB supergravity on S^5.

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