Double field theory at order α'

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ABSTRACT: We investigate α' corrections of bosonic strings in the framework of double field theory. The previously introduced "doubled α'-geometry" gives α'-deformed gauge transformations arising in the Green-Schwarz anomaly cancellation mechanism but does not apply to bosonic strings. These require a different deformation of the duality-covariantized Courant bracket which governs the gauge structure. This is revealed by examining the α' corrections in the gauge algebra of closed string field theory. We construct a four-derivative cubic double field theory action invariant under the deformed gauge transformations, giving a first glimpse of the gauge principle underlying bosonic string α' corrections. The usual metric and b-field are related to the duality covariant fields by non-covariant field redefinitions.

Contents

1 Introduction

At low energy string theory is well described by supergravity. Stringy corrections beyond supergravity are captured by higher-derivative α' corrections. While Einstein's gravity and supergravity are well understood in terms of Riemannian geometry, we have no good understanding of the geometry of string theory or even of *classical* string theory. Classical string theory includes α' corrections. Our goal in this paper is to better understand the geometry behind these corrections.

Concretely, we ask whether there is a symmetry explanation for higher-derivative α' corrections, i.e., a symmetry principle that *requires* α' corrections. We know of such symmetry principles in some cases; for instance, in heterotic string theory Green-Schwarz anomaly cancellation [1] requires an $\mathcal{O}(\alpha')$ deformation of the gauge transformations of the b-field, which in turn requires higher-derivative terms in the action. We have encountered this phenomenon as a special case of our geometrical formalism [2]. Building up on the work of [3] we will give extra evidence that there is indeed a gauge principle governing the α' corrections of classical string theory more generally.

Conventionally, α' corrections to the effective field theory of bosonic strings are written in terms of higher powers of curvature tensors, the three-form field strength H of the b-field, and their covariant derivatives. These actions are manifestly compatible with diffeomorphism invariance and the abelian b-field gauge invariance. Therefore, these corrections are not required by gauge symmetries. In this paper we will invoke T-duality covariance to study α' corrections, using closed string field theory [4, 5] and double field theory (DFT) [6–10]. While T-duality results in a global continuous symmetry of the effective theory after dimensional reduction, DFT features a T-duality covariance prior to any reduction. It also features duality-covariant generalized diffeomorphisms, and the duality symmetry that emerges after dimensional reduction is realized as gauge symmetries [9, 11, 12]. In a duality covariant formulation, gauge symmetries acquire α' corrections and in that sense 'explain' the origin of α' corrections to the effective action.

In closed string field theory on torus backgrounds T-duality covariance is built in by having coordinates dual to both momentum and winding modes, thereby realizing the T-duality group on this doubled space. More precisely, in closed string field theory we have a perturbative expansion in which the (fluctuating) field variables around T-dual backgrounds are related by simple transformations that make T-duality manifest [5]. String field theory enables one to read off gauge transformations and actions, including α' corrections. String field theory was the starting point for the construction of DFT in [7]. While T-duality is manifest in string field theory variables, the gauge symmetries do not have the form expected for 'Einstein' variables that originate from the the conventional metric tensor. To $\mathcal{O}(\alpha')$ the field redefinitions needed to connect Einstein variables to T-duality covariant fields are *not* generally covariant, leading to fields that transform in a non-standard way under gauge symmetries.

DFT is what follows from closed string field theory after restricting to the massless sector, performing duality-covariant field redefinitions, and implementing background independence. Moreover, one generally imposes a duality covariant "strong constraint" that means that effectively all fields depend only on half of the doubled coordinates.[1] To zeroth order in α', duality-covariant field and parameter redefinitions in closed string field theory (CSFT) simplify the gauge transformations, which then form the algebra governed by the C-bracket [8]. This bracket becomes the Courant bracket defined in [18] upon reduction to un-doubled coordinates.

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[1]While there is work on DFT without the strong constraint [13–17], our understanding of such theories is still preliminary.

In a DFT formulation of bosonic strings we have to describe the Riemann-squared term well known to appear to first order in α'. There is a duality-covariant generalized Riemann tensor, but it cannot be fully determined in terms of physical fields because the connection contains undetermined components [6, 21, 22]. Therefore, we cannot write directly an α' corrected action that preserves duality covariance. Additionally the Riemann-squared action cannot be written in terms of higher derivatives of the generalized metric [22]. To cubic order, however, the tensor structure in Riemann-squared that causes this difficulty can be removed by a *non-covariant* field redefinition of the metric. This leads to fields with non-standard gauge transformations, and a gauge algebra with α' corrections. This is in quantitative agreement with [25] that studied T-duality in reductions to one dimension. In that work α' corrections require field redefinitions of $\mathcal{O}(\alpha')$ that are quadratic in first derivatives of the metric, and thus cannot originate from covariant field redefinitions.

Doubled α' geometry [3] is also a formulation in which T-duality is unchanged but the gauge structure is changed. It features a field independent deformation of the C-bracket and an action that is exactly gauge invariant. This deformation, however, does not correspond to the $\mathcal{O}(\alpha')$ deformation of bosonic string theory; it does not give rise to Riemann-squared terms. The construction of [3] was based on a chiral CFT introduced in [6] and further studied in [3]. This CFT has one-loop worldsheet anomalies and captures some of the structure needed for heterotic string theory [2]. Indeed, in this theory the gauge transformations for the b-field make the field strength H with gravitational Chern-Simons modification gauge invariant. Although this geometry does not describe the full α' corrections of heterotic string theory, it contains important ingredients. A different approach to describe α' corrections to heterotic DFT has been discussed in [27], as we will discuss in the conclusions. See also [28–31] for Courant algebroids in 'generalized geometry' formulations of heterotic strings.

We use closed SFT to compute the gauge algebra to first nontrivial order in α'. After simplification, the result is a deformation of the C-bracket that differs from that of doubled α' geometry by a sign factor linked to the symmetry of bosonic closed strings under orientation reversal — a $\mathbb{Z}_2$ symmetry that is not part of the T-duality group. The four-derivative terms in [3] are in fact $\mathbb{Z}_2$ parity odd. We call this theory DFT$^-$ although it has no overall $\mathbb{Z}_2$ symmetry. We find that higher-derivative actions that respect the $\mathbb{Z}_2$ symmetry of bosonic strings exist. This theory, called DFT$^+$, is built to cubic order. The correction to the C-bracket in DFT$^+$ features the appearance of background fields, as opposed to the background independent deformation of DFT$^-$. In $O(D,D)$ covariant notation, with fundamental indices $M, N = 1, \ldots, 2D$, and gauge parameters ξ^M, the gauge algebra of DFT$^-$ reads

$$\text{DFT}^- : \quad \left[\xi_1, \xi_2\right]_-^M = \left[\xi_1, \xi_2\right]_C^M - \frac{1}{2}\eta^{KL}\eta^{PQ}\eta^{MN}K_{[1\,KP}\,\partial_N K_{2]\,LQ}. \qquad (1.1)$$

Here $K_{i\,MN} = 2\partial_{[M}\xi_{i\,N]} = \partial_M\xi_{i\,N} - \partial_N\xi_{i\,M}$, $i = 1, 2$, antisymmetrization of indices or labels is defined by $A_{[1}B_{2]} \equiv \frac{1}{2}(A_1B_2 - A_2B_1)$, and η denotes the $O(D,D)$ invariant metric. Moreover, $[\xi_1, \xi_2]_C$ denotes the C-bracket governing the gauge algebra of the two-derivative

DFT:

$$\left[\xi_1, \xi_2\right]_C^M = \xi_1^K \partial_K \xi_2^M - \xi_2^K \partial_K \xi_1^M - \frac{1}{2}(\xi_1^K \partial^M \xi_{2\,K} - \xi_2^K \partial^M \xi_{1\,K})$$
$$= 2\,\xi_{[1}^K \partial_K \xi_{2]}^M - \eta^{KL}\eta^{MN}\xi_{[1K}\partial_N \xi_{2]L}\,. \tag{1.2}$$

The second term in (1.1) is the higher-derivative correction. The factor of α' that multiplies it is left implicit. In contrast, the gauge algebra for DFT$^+$ reads

$$\text{DFT}^+: \quad \left[\xi_1, \xi_2\right]_+^M = \left[\xi_1, \xi_2\right]_C^M + \frac{1}{2}\,\bar{\mathcal{H}}^{KL}\eta^{PQ}\,\eta^{MN}K_{[1\,KP}\,\partial_N K_{2]\,LQ}\,, \tag{1.3}$$

where $\bar{\mathcal{H}}^{KL}$ denotes the background value of the generalized metric that encodes the background metric and b-field. It should be emphasized that this is not the complete algebra of DFT$^+$ which, given the appearance of $\bar{\mathcal{H}}$, is expected to be field dependent. At the present stage of our perturbative calculation only the background value of the fields appear. Starting from the DFT$^+$ gauge algebra we are able to write α'-deformed gauge transformations that realize the algebra, and we show that they are related to standard tensor gauge transformations by duality-violating redefinitions of precisely the expected form that absorbs the problematic structure of Riemann-squared.

Given this discrete freedom in the deformation of the gauge structure of DFT, it is natural to ask whether one can built an 'interpolating' theory with both $\mathbb{Z}_2$ even and $\mathbb{Z}_2$ odd contributions. Such a theory indeed exists at this cubic level, and it corresponds to having both the gravitational Chern-Simons modification of H and a Riemann-squared term. The gauge algebra for the interpolating theory reads

$$\left[\xi_1, \xi_2\right]_{\alpha'}^M = \left[\xi_1, \xi_2\right]_C^M + \frac{1}{2}\left(\gamma^+\bar{\mathcal{H}}^{KL} - \gamma^-\eta^{KL}\right)\eta^{PQ}\,K_{[1\,KP}\,\partial^M K_{2]\,LQ}\,, \tag{1.4}$$

with parameters $\gamma^\pm$ that at this level are unconstrained.

To confirm the consistency of our constructions we build the cubic action, both for DFT$^+$ and DFT$^-$, including all terms with four derivatives and show that it is consistent with gauge invariance. While the cubic DFT$^-$ action is simple, the DFT$^+$ action is quite involved, but we can show that it encodes Riemann-squared (or Gauss-Bonnet) at the cubic level.

The main conclusion suggested by the results in this paper can be summarized as follows: while it is always possible to write α' corrections in terms of standard 'Einstein variables' g and b, string theory strongly suggests that these are not the best variables when α' effects are turned on. Rather, making the duality symmetries of string theory manifest requires field variables that have non-covariant transformations of $\mathcal{O}(\alpha')$ under standard diffeomorphisms. This may seem a radical step since diffeomorphism invariance is the basic principle of Riemannian geometry, but in string theory this gauge principle is replaced by a duality covariant one, with a gauge algebra that extends the Lie bracket to the duality covariant bracket (1.4) with $\mathcal{O}(\alpha')$ contributions. It is to be expected that there will be a (generalized) geometric formulation of classical string theory that organizes the notoriously complicated α' corrections in an efficient way that is manifestly covariant under all symmetries.

This paper is organized as follows. In section 2 we review some of the basics of closed string field theory and then determine the gauge algebra including terms with one and three derivatives. This algebra is simplified by doing duality covariant field-dependent parameter redefinitions in section 3. We use the simple final form to write field transformations that realize this DFT$^+$ algebra. At this stage we also note that a simple variant gives the DFT$^-$ algebra. In section 4 we discuss the relation between the CSFT perturbative field variables and the 'Einstein' variables. We do this for DFT$^+$ showing that duality non-covariant field redefinitions relate the DFT variables to Einstein variables. For DFT$^-$ the relation is more subtle, as reviewed here. In section 5 we develop the perturbation theory of DFT$^-$, which is a useful step to develop the same perturbation theory for DFT$^+$. We discuss in detail the $\mathbb{Z}_2$ orientation reversal transformation and its action on the perturbative DFT fields. We also explain how to relate CSFT variables to the perturbative DFT variables and confirm our identification of DFT$^+$ with the theory that arises from CSFT. Finally, in section 6 we perform a very nontrivial check of the existence of DFT$^+$: we show that an invariant cubic action including four-derivative terms exists. The cubic terms show direct evidence of the Gauss-Bonnet terms in the effective action. We conclude with some additional discussion of our results in section 7.

2 The gauge algebra from string field theory

In this section we review the facts about string field theory necessary to extend the results of [7] to include α' corrections in the gauge algebra. We compute the algebra of gauge transformations directly from the string field theory, including the first nontrivial α' corrections. We also review the simplification of the gauge algebra to zeroth order in α'. This section prepares the ground for the next where we will perform redefinitions directly on the gauge algebra in order to obtain a simple form of the α' corrections to the algebra.

2.1 Generalities of closed string field theory

The string field theory action is non-polynomial and takes the form

$$(2\kappa^2)S = -\frac{4}{\alpha'}\left(\frac{1}{2}\langle\Psi, Q\Psi\rangle + \frac{1}{3!}\langle\Psi, [\Psi, \Psi]\rangle + \frac{1}{4!}\langle\Psi, [\Psi, \Psi, \Psi]\rangle + \cdots\right). \tag{2.1}$$

Here $|\Psi\rangle$ is the classical off-shell closed string field, a ghost-number two, Grassmann even state of the full matter and ghost conformal field theory that describes the closed string background. The off-shell string field must satisfy $(L_0 - \bar{L}_0)|\Psi\rangle = 0$ and $(b_0 - \bar{b}_0)|\Psi\rangle = 0$. The ghost-number one operator Q is the BRST operator of the conformal field theory and $\langle\cdot, \cdot\rangle$ denotes the (linear) inner product:

$$\langle A, B\rangle \equiv \langle A| c_0^-|B\rangle, \quad c_0^\pm \equiv \frac{1}{2}(c_0 \pm \bar{c}_0), \tag{2.2}$$

where $\langle A|$ is the BPZ conjugate of the string field $|A\rangle$. The inner product vanishes unless $\mathrm{gh}(A) + \mathrm{gh}(B) = 5$. The cubic interaction is defined in terms of a closed string bracket $[\cdot, \cdot]$ or product. This product, whose input is two string fields and its output is another string

field, is graded commutative: $[B_1, B_2] = (-1)^{B_1 B_2}[B_2, B_1]$ where the B_1 and B_2 in the sign factor denote the Grassmanality of the string fields B_1 and B_2, respectively. Moreover we have $\mathrm{gh}([B_1, B_2]) = \mathrm{gh}(B_1) + \mathrm{gh}(B_2) - 1$. Thus, for the Grassmann even classical field $[\Psi, \Psi]$ does not vanish and it has ghost number three, which is suitable for the cubic coupling of the theory not to vanish. The quartic term in the action is defined in terms of a three-product $[B_1, B_2, B_3]$ that is also graded commutative. This product parametrizes the failure of the bracket to be a Lie bracket, and is the next element in the L_∞ structure of the classical theory. The dots in the action denote terms quartic and higher order in the string field.

The field equations $\mathcal{F}(\Psi) = 0$ and gauge transformations $\delta_\Lambda \Psi$ of the theory take the form

$$\mathcal{F}(\Psi) \equiv Q\Psi + \frac{1}{2}[\Psi, \Psi] + \frac{1}{3!}[\Psi, \Psi, \Psi] + \cdots = 0$$
$$\delta_\Lambda \Psi = Q\Lambda + [\Psi, \Lambda] + \frac{1}{2}[\Psi, \Psi, \Lambda] + \cdots, \tag{2.3}$$

where Λ is a ghost-number one string field and the dots denote terms with higher powers of the string field Ψ. The gauge algebra of the theory takes the form

$$\left[\delta_{\Lambda_1}, \delta_{\Lambda_2}\right] = \delta_{\Lambda_{12}(\Psi)} + [\Lambda_1, \Lambda_2, \mathcal{F}] + \cdots. \tag{2.4}$$

The transformations only close on shell (using the three-product) and the dots represent higher terms that also vanish on-shell. The resulting gauge parameter takes the form

$$\Lambda_{12}(\Psi) = [\Lambda_2, \Lambda_1] + [\Lambda_2, \Lambda_1, \Psi] + \cdots \tag{2.5}$$

showing that the algebra has field-dependent structure constants. Since the gauge parameters are ghost number one string fields they are Grassmann odd and thus the bracket $[\Lambda_2, \Lambda_1]$ is properly antisymmetric under the exchange of the gauge parameters. We will compute the first term on the above right hand side.

The theory generically has gauge invariances of gauge invariances. A gauge parameter of the form $\widehat{\Lambda} = Q\chi$ will generate no leading order gauge transformations in (2.3) because $Q^2 = 0$. To all orders one only has on-shell gauge invariances of gauge invariances. Indeed, for

$$\widehat{\Lambda} = Q\chi + [\Psi, \chi] + \frac{1}{2}[\Psi, \Psi, \chi] + \cdots, \tag{2.6}$$

one finds a gauge transformation that vanishes on-shell: $\delta_{\widehat{\Lambda}} \Psi = -[\mathcal{F}, \chi] - [\mathcal{F}, \Psi, \chi] + \mathcal{O}(\Psi^3)$.

2.2 String field and gauge parameter

The closed string field for the massless sector takes the form

$$|\Psi\rangle = \int dp \left(-\frac{1}{2} e_{ij}(p) \alpha^i_{-1} \bar\alpha^j_{-1} c_1 \bar c_1 + e(p) c_1 c_{-1} + \bar e(p) \bar c_1 \bar c_{-1} \right.$$
$$\left. + i\sqrt{\frac{\alpha'}{2}} \left(f_i(p) c_0^+ c_1 \alpha^i_{-1} + \bar f_j(p) c_0^+ \bar c_1 \bar\alpha^j_{-1} \right) \right) |p\rangle. \tag{2.7}$$

This string field features five component fields: $e_{ij}, e, \bar{e}, f_i$, and $\bar{f}_i$. The field e_{ij} contains the gravity and b-field fluctuations as its symmetric and antisymmetric parts, respectively. One linear combination of the e and $\bar{e}$ fields (the difference) is the dilaton and the other linear combination (the sum) can be gauged away. The fields f_i and $\bar{f}_i$ are auxiliary fields and can be solved for algebraically. The gauge parameter $|\Lambda\rangle$ associated to the above string field takes the form

$$|\Lambda\rangle = \int [dp] \left(\frac{i}{\sqrt{2\alpha'}} \lambda_i(p)\, \alpha^i_{-1} c_1 - \frac{i}{\sqrt{2\alpha'}} \bar{\lambda}_i(p)\, \bar{\alpha}^i_{-1} \bar{c}_1 + \mu(p)\, c_0^+ \right) |p\rangle. \qquad (2.8)$$

The string field Λ has ghost number one and is annihilated by b_0^-. It contains two vectorial gauge parameters λ_i and $\bar{\lambda}_i$ that encode infinitesimal diffeomorphisms and infinitesimal b-field gauge symmetries in some suitable linear combinations. There is also one scalar gauge parameter μ that can be used to gauge away the field $e + \bar{e}$. The linearized gauge transformations are

$$\delta_\Lambda e_{ij} = D_i \bar{\lambda}_j + \bar{D}_j \lambda_i \,,$$

$$\delta_\Lambda f_i = -\frac{1}{2} \Box\, \lambda_i + D_i \mu \,,$$

$$\delta_\Lambda \bar{f}_i = \frac{1}{2} \Box\, \bar{\lambda}_i + \bar{D}_i \mu \,, \qquad (2.9)$$

$$\delta_\Lambda e = -\frac{1}{2} D^i \lambda_i + \mu \,,$$

$$\delta_\Lambda \bar{e} = \frac{1}{2} \bar{D}^i \bar{\lambda}_i + \mu \,.$$

All indices are raised and lowered with the background metric G^{ij}. The derivatives D and $\bar{D}$ are defined as

$$D_i = \frac{1}{\sqrt{\alpha'}} \left(\frac{\partial}{\partial x^i} - E_{ik} \frac{\partial}{\partial \tilde{x}_k} \right), \qquad \bar{D}_i = \frac{1}{\sqrt{\alpha'}} \left(\frac{\partial}{\partial x^i} + E_{ki} \frac{\partial}{\partial \tilde{x}_k} \right). \qquad (2.10)$$

The weak constraint means that the following equality holds acting on any field or gauge parameter

$$\Box \equiv D^2 = \bar{D}^2 \,. \qquad (2.11)$$

The strong constraint is $D^i A D_i B = \bar{D}^i A \bar{D}_i B$ for any A, B. We can now introduce fields d and χ by

$$d = \frac{1}{2}(e - \bar{e}), \qquad \text{and} \qquad \chi = \frac{1}{2}(e + \bar{e}). \qquad (2.12)$$

The gauge transformations of d and χ are

$$\delta_\Lambda d = -\frac{1}{4}(D^i \lambda_i + \bar{D}^i \bar{\lambda}_i), \qquad \delta_\Lambda \chi = -\frac{1}{4}(D^i \lambda_i - \bar{D}^i \bar{\lambda}_i) + \mu. \qquad (2.13)$$

It is clear that a choice of μ can be used to set $\chi = 0$. Since further $\lambda, \bar{\lambda}$ gauge transformations would then reintroduce χ, these gauge transformations must be accompanied by compensating μ gauge transformations with parameter $\mu(\lambda, \bar{\lambda})$

$$\mu(\lambda, \bar{\lambda}) = \frac{1}{4}(D \cdot \lambda - \bar{D} \cdot \bar{\lambda}) \,. \qquad (2.14)$$

Effectively, the new gauge transformations δ_Λ are $\delta_\lambda + \delta_{\bar\lambda} + \delta_{\mu(\lambda,\bar\lambda)}$. The extra term does not affect d nor e_{ij}, as neither transforms under μ gauge transformations. It changes the gauge transformations of f and $\bar f$, but this is of no concern as these are auxiliary fields to be eliminated. We denote by δ_Λ the gauge transformations generated by λ and $\bar\lambda$, and use δ_λ and $\delta_{\bar\lambda}$ for the separate transformations. We have

$$\delta_\Lambda e_{ij} \;=\; D_i \bar\lambda_j \;+\; \bar D_j \lambda_i \,, \qquad \delta_\Lambda d \;=\; -\frac{1}{4} D \cdot \lambda - \frac{1}{4} \bar D \cdot \bar\lambda \,, \qquad (2.15)$$

The theory is invariant under the $\mathbb{Z}_2$ symmetry

$$e_{ij} \;\to\; e_{ji} \,, \quad D_i \;\to\; \bar D_i \,, \quad \bar D_i \;\to\; D_i \,, \quad d \;\to\; d \;, \qquad (2.16)$$

related to the invariance of the closed string theory under orientation reversal. Note that this relates the transformations under λ to those under $\bar\lambda$. Invariance under one set of gauge transformations implies invariance under the other set. This holds both as we include field dependent terms and higher derivatives.

The component fields in the string field theory have simple transformations under T-duality. Since the formulation of the theory is not background independent the theory around some background E must be compared with the theory formulated around a T-dual background E'. The fluctuation fields of the two theories, as explained in section 4.2 of [7], are related by simple matrix transformations. Schematically $e_{ij} = M_i{}^k \bar M_j{}^l e'_{kl}$ and the dilaton d is duality invariant. Note that the first index of e transforms with the unbarred M and the second with the barred M. Every expression in which indices are contracted consistently, i.e., unbarred with unbarred and barred with barred indices, is therefore T-duality covariant. T-duality covariant redefinitions respect such structure in contractions of indices.

2.3 Cubic terms and gauge transformations from CSFT

The algebra of gauge transformations is described by (2.5), and the field-independent part is given by

$$\Lambda_{12} \;\equiv\; [\,\Lambda_2\,, \Lambda_1\,] \,. \qquad (2.17)$$

We use uppercase gauge parameters to encode all the component gauge parameters:

$$\Lambda_1 = (\lambda_1, \bar\lambda_1, \mu_1), \qquad \Lambda_2 = (\lambda_2, \bar\lambda_2, \mu_2) \,, \qquad \Lambda_{12} = (\lambda_{12}, \bar\lambda_{12}, \mu_{12}). \qquad (2.18)$$

The computation of the gauge algebra is a straightforward but somewhat laborious matter in string field theory. There are contributions with various numbers of derivatives or powers of α'. We will be interested in the terms at zero order and first order in α'. We will write this as

$$\begin{aligned}
\lambda_{12}^i &= \lambda_{12}^{(0)i} + \alpha' \lambda_{12}^{(1)i} + \dots \\
\bar\lambda_{12}^i &= \bar\lambda_{12}^{(0)i} + \alpha' \bar\lambda_{12}^{(1)i} + \dots \\
\mu_{12}^i &= \mu_{12}^{(0)i} + \alpha' \mu_{12}^{(1)i} + \dots \,,
\end{aligned} \qquad (2.19)$$

where the superscripts in parenthesis denote the power of α'. The result to zeroth order in α' is

$$
\lambda_{12}^{(0)i} = \frac{1}{2}\left(\lambda_2 \cdot D\lambda_1^i - \lambda_1 \cdot D\lambda_2^i\right) - \frac{1}{4}(\lambda_2 \cdot D^i\lambda_1 - \lambda_1 \cdot D^i\lambda_2) - \frac{1}{4}(\lambda_2^i\, D \cdot \lambda_1 - \lambda_1^i\, D \cdot \lambda_2)
$$

$$
+ \frac{1}{4}\left(\bar{\lambda}_2 \cdot \bar{D}\lambda_1^i - \bar{\lambda}_1 \cdot \bar{D}\lambda_2^i\right) + \frac{1}{8}\left(\lambda_1^i \bar{D} \cdot \bar{\lambda}_2 - \lambda_2^i \bar{D} \cdot \bar{\lambda}_1\right) - \frac{1}{4}(\lambda_1^i\, \mu_2 - \lambda_2^i \mu_1)\,,
$$

$$
\bar{\lambda}_{12}^{(0)i} = \frac{1}{2}\left(\bar{\lambda}_2 \cdot \bar{D}\bar{\lambda}_1^i - \bar{\lambda}_1 \cdot \bar{D}\bar{\lambda}_2^i\right) - \frac{1}{4}(\bar{\lambda}_2 \cdot \bar{D}^i\bar{\lambda}_1 - \bar{\lambda}_1 \cdot \bar{D}^i\bar{\lambda}_2) - \frac{1}{4}(\bar{\lambda}_2^i\, \bar{D} \cdot \bar{\lambda}_1 - \bar{\lambda}_1^i\, \bar{D} \cdot \bar{\lambda}_2)
$$

$$
+ \frac{1}{4}\left(\lambda_2 \cdot D\bar{\lambda}_1^i - \lambda_1 \cdot D\bar{\lambda}_2^i\right) + \frac{1}{8}\left(\bar{\lambda}_1^i D \cdot \lambda_2 - \bar{\lambda}_2^i D \cdot \lambda_1\right) + \frac{1}{4}(\bar{\lambda}_1^i\, \mu_2 - \bar{\lambda}_2^i\mu_1)\,,
$$

$$
\mu_{12}^{(0)} = -\frac{1}{8}\left(\lambda_1 \cdot D + \bar{\lambda}_1 \cdot \bar{D}\right)\mu_2 - \frac{1}{16}\left(D \cdot \lambda_1 + \bar{D} \cdot \bar{\lambda}_1\right)\mu_2\,. \tag{2.20}
$$

A partial version of this result is given in equation (3.8) of [7]. In that reference we only determined the contribution to $\lambda_{12}^{(0)i}$ from λ_1 and λ_2. Such terms are in the first line of $\lambda_{12}^{(0)i}$. There are also contributions that involve $\bar{\lambda}_1$ and $\bar{\lambda}_2$ as well as μ_1 and μ_2. Such terms were not needed for [7], where the gauge algebra was recalculated after a number of parameter and field redefinitions. The order α' results will be given below in (2.28).

The gauge transformations of the fields, including terms linear in fields but only two derivatives, are somewhat complicated and were given in [7]. Their simplification took a few steps. One must substitute the leading values for the auxiliary fields f and $\bar{f}$. Again, one can gauge fix $e + \bar{e}$ to zero and work with just the dilaton d. This is followed by a redefinition of the gauge parameters:

$$
\lambda_i' = \lambda_i + \frac{3}{4}\lambda_i d - \frac{1}{4}\bar{\lambda}^k e_{ik}\,,
$$

$$
\bar{\lambda}_i' = \bar{\lambda}_i + \frac{3}{4}\bar{\lambda}_i d - \frac{1}{4}\lambda^k e_{ki}\,, \tag{2.21}
$$

and finally a duality-covariant redefinition of the fields:

$$
e_{ij}' = e_{ij} + e_{ij}d\,,
$$

$$
d' = d + \frac{1}{32}e_{ij}e^{ij} + \frac{9}{16}d^2\,. \tag{2.22}
$$

Dropping primes, the final form of the α'-independent gauge transformations is

$$
\delta_\Lambda e_{ij} = \bar{D}_j\,\lambda_i + \frac{1}{2}\big[\,(D_i\lambda^k - D^k\lambda_i)\,e_{kj} + \lambda_k D^k e_{ij}\,\big]
$$

$$
+ D_i\,\bar{\lambda}_j + \frac{1}{2}\big[\,(\bar{D}_j\bar{\lambda}^k - \bar{D}^k\bar{\lambda}_j)\,e_{ik} + \bar{\lambda}_k \bar{D}^k e_{ij}\,\big]\,, \tag{2.23}
$$

$$
\delta_\Lambda d = -\frac{1}{4}D \cdot \lambda + \frac{1}{2}(\lambda \cdot D)\,d - \frac{1}{4}\bar{D} \cdot \bar{\lambda} + \frac{1}{2}(\bar{\lambda} \cdot \bar{D})\,d\,.
$$

Trivial gauge parameters do not generate gauge transformations and take a simple form

$$
\lambda_i = D_i\chi\,, \quad \bar{\lambda}_i = -\bar{D}_i\chi\,, \tag{2.24}
$$

as can be checked using the strong constraint. These trivial gauge parameters have no field dependence and the resulting transformations of fields vanish without using equations of motion. This is simpler than what could have been expected from (2.6). We will see that such simplicity is preserved with α' corrections. The algebra of gauge transformations can be recalculated using (2.23) with the conventions

$$\left[\delta_{\Lambda_1}, \delta_{\Lambda_2}\right] = \delta_{\Lambda_{c,12}}, \qquad \Lambda_{c,12} = (\lambda_{c,12}, \bar{\lambda}_{c,12}), \tag{2.25}$$

with subscripts 'c' for C-bracket. We find

$$
\begin{aligned}
\lambda_{c,12}^i = {} & \frac{1}{2} \left[\, (\lambda_2 \cdot D + \bar{\lambda}_2 \cdot \bar{D}) \, \lambda_1^i - (\lambda_1 \cdot D + \bar{\lambda}_1 \cdot \bar{D}) \lambda_2^i \, \right] \\
& + \frac{1}{4} \left[\, \lambda_1 \cdot D^i \lambda_2 - \lambda_2 \cdot D^i \lambda_1 \right] - \frac{1}{4} \left[\, \bar{\lambda}_1 \cdot D^i \bar{\lambda}_2 - \bar{\lambda}_2 \cdot D^i \bar{\lambda}_1 \right], \\
\bar{\lambda}_{c,12}^i = {} & \frac{1}{2} \left[\, (\lambda_2 \cdot D + \bar{\lambda}_2 \cdot \bar{D}) \, \bar{\lambda}_1^i - (\lambda_1 \cdot D + \bar{\lambda}_1 \cdot \bar{D}) \, \bar{\lambda}_2^i \, \right] \\
& - \frac{1}{4} \left[\, \lambda_1 \cdot \bar{D}^i \lambda_2 - \lambda_2 \cdot \bar{D}^i \lambda_1 \right] + \frac{1}{4} \left[\, \bar{\lambda}_1 \cdot \bar{D}^i \bar{\lambda}_2 - \bar{\lambda}_2 \cdot \bar{D}^i \bar{\lambda}_1 \right].
\end{aligned}
\tag{2.26}
$$

This is the gauge algebra that written in background independent language gives the C-bracket [8]: $\Lambda_{c,12} = [\Lambda_2, \Lambda_1]_{\mathrm{C}}$. This algebra is different from the zeroth-order algebra in (2.20). It would have been convenient if (2.26) could have been derived from (2.20) without recourse to the gauge transformations of fields. We will do this in the next section as a warm-up, before extending the analysis to include the α' corrections. As a first step, we rewrite here $\lambda_{12}^{(0)i}$ in terms of $\lambda_{c,12}^i$. A short calculation gives

$$
\begin{aligned}
\lambda_{12}^{(0)i} = {} & \lambda_{12\,c}^i + \frac{1}{4} \left(\bar{\lambda}_1 \cdot D^i \bar{\lambda}_2 - \bar{\lambda}_2 \cdot D^i \bar{\lambda}_1 \right) - \frac{1}{4} \left(\lambda_2^i \, D \cdot \lambda_1 - \lambda_1^i \, D \cdot \lambda_2 \right) \\
& - \frac{1}{4} \left(\bar{\lambda}_2 \cdot \bar{D} \lambda_1^i - \bar{\lambda}_1 \cdot \bar{D} \lambda_2^i \right) + \frac{1}{8} \left(\lambda_1^i \bar{D} \cdot \bar{\lambda}_2 - \lambda_2^i \bar{D} \cdot \bar{\lambda}_1 \right) - \frac{1}{4} (\lambda_1^i \, \mu_2 - \lambda_2^i \mu_1).
\end{aligned}
\tag{2.27}
$$

The order α' terms in the gauge algebra, as defined in (2.19) are also calculated from the string field theory and the result is

$$
\begin{aligned}
\lambda_{12}^{(1)i} = {} & -\frac{1}{32} \big(D_1^i - D_2^i\big)\big(D_1^j + 2D_2^j\big)\big(2D_1^k + D_2^k\big)\lambda_{1j}\lambda_{2k} \\
& -\frac{1}{64} \big(D_1^i - D_2^i\big)\big(D_1^j + 2D_2^j\big)\big(2\bar{D}_1^k + \bar{D}_2^k\big)(\lambda_{1\,j}\bar{\lambda}_{2\,k} - \lambda_{2\,j}\bar{\lambda}_{1\,k}) \\
& +\frac{1}{32} \big(D_1^i - D_2^i\big)\big(D_1^j + 2D_2^j\big)(\lambda_{1\,j}\mu_2 - \lambda_{2\,j}\mu_1) \\
& +\frac{1}{32} \big(D_1^i - D_2^i\big)\big(\bar{D}_1^j + 2\bar{D}_2^j\big)(\bar{\lambda}_{1\,j}\,\mu_2 - \bar{\lambda}_{2\,j}\,\mu_1), \\
\bar{\lambda}_{12}^{(1)i} = {} & -\frac{1}{32} \big(\bar{D}_1^i - \bar{D}_2^i\big)\big(\bar{D}_1^j + 2\bar{D}_2^j\big)\big(2\bar{D}_1^k + \bar{D}_2^k\big)\bar{\lambda}_{1j}\bar{\lambda}_{2k} \\
& -\frac{1}{64} \big(\bar{D}_1^i - \bar{D}_2^i\big)\big(\bar{D}_1^j + 2\bar{D}_2^j\big)\big(2D_1^k + D_2^k\big)(\bar{\lambda}_{1\,j}\lambda_{2\,k} - \bar{\lambda}_{2\,j}\lambda_{1\,k}) \\
& -\frac{1}{32} \big(\bar{D}_1^i - \bar{D}_2^i\big)\big(\bar{D}_1^j + 2\bar{D}_2^j\big)(\bar{\lambda}_{1\,j}\mu_2 - \bar{\lambda}_{2\,j}\mu_1) \\
& -\frac{1}{32} \big(\bar{D}_1^i - \bar{D}_2^i\big)\big(D_1^j + 2D_2^j\big)(\lambda_{1\,j}\,\mu_2 - \lambda_{2\,j}\,\mu_1), \\
\mu_{12} = {} & \frac{1}{32} \left(D_1^j + 2D_2^j \right) \left(2\bar{D}_1^k + \bar{D}_2^k \right) \lambda_{1,j}\,\bar{\lambda}_{2,k} \; - \; (1 \leftrightarrow 2).
\end{aligned}
\tag{2.28}
$$

In here we have a collection of derivatives acting on products (or a sum of products) of gauge parameters. The convention is that D_1 acts on the first function and D_2 on the second function. Thus, for example, $D_1^i(f \cdot g) = D^i f \cdot g$, $D_2^i(f \cdot g) = f \cdot D^i g$, and $D_1^j D_2^i(f \cdot g) = D^j f \cdot D^i g$. Care must be exercised not to exchange the order of functions until all derivatives have been applied. Note that $\bar{\lambda}_{12}^i$ is obtained from λ_{12}^i by conjugating all objects and changing the sign of any term involving μ_1 or μ_2.

3   Simplifying the closed string theory gauge algebra

In this section we perform redefinitions of the gauge parameters in order to simplify the closed string field theory gauge algebra obtained in section 2. We begin by showing how to compute in general the change of a gauge algebra under a field-dependent parameter redefinition. Then we illustrate this technique by applying it to the CSFT gauge algebra to zeroth order in α', to recover the C-bracket result (2.26). We apply it next to the CSFT gauge algebra to first order in α', and after some steps we obtain the rather simple form given in (3.34). Based on this final form of the gauge algebra, we determine the associated gauge transformations to first order in α'. In the last subsection, we point out that consistency of such a higher-derivative deformation of bracket and gauge transformations does not uniquely determine these transformations. Rather, there is a $\mathbb{Z}_2$ freedom that leaves one sign undetermined.

3.1   General remarks on gauge parameter redefinitions

We start with a general discussion of (perturbative) gauge transformations and show how field-dependent redefinitions of the gauge parameters can change the gauge algebra. Note that, in contrast, field redefinitions leave the gauge algebra unchanged, even though they can change the form of gauge transformations. We consider gauge transformations of fields, collectively denoted by ϕ, with respect to a gauge parameter λ. They are perturbatively defined to first order in fields,

$$\delta_\lambda \phi = f(\lambda) + g(\lambda, \phi) + \mathcal{O}(\phi^2) \,, \tag{3.1}$$

where f is a linear function of λ, and g is a linear function of both λ and ϕ. We also write

$$f(\lambda) = \delta_\lambda^{[0]} \phi \,, \qquad g(\lambda, \phi) = \delta_\lambda^{[1]} \phi \,, \tag{3.2}$$

indicating by the superscript in brackets the power of fields. In general, closure of the gauge transformations requires

$$\left[\delta_{\lambda_1}, \delta_{\lambda_2} \right] \phi = \delta_{\lambda_{12}(\lambda_1, \lambda_2; \phi)} \phi \; + \; h(\lambda_1, \lambda_2, F(\phi)) \,, \tag{3.3}$$

where $\lambda_{12}(\lambda_1, \lambda_2; \phi)$ are field dependent structure constants, and $F(\phi)$ is a function of the fields such that $h = 0$ on-shell. To linear order in ϕ we can only determine the part of the gauge algebra (3.3) that is independent of ϕ, since terms $\mathcal{O}(\phi)$ are affected by

the $\delta^{[0]}$ variation of unknown terms $\mathcal{O}(\phi^2)$. Thus, h cannot be calculated from the field transformations to this order. Similarly, writing

$$\lambda_{12}(\lambda_1, \lambda_2; \phi) = \lambda_{12}(\lambda_1, \lambda_2) + \mathcal{O}(\phi), \qquad (3.4)$$

we can only determine the ϕ-independent part. Equation (3.3) then reduces to

$$\left[\delta_{\lambda_1}, \delta_{\lambda_2} \right] \phi = f(\lambda_{12}(\lambda_1, \lambda_2)) + \mathcal{O}(\phi) = \delta^{[0]}_{\lambda_{12}(\lambda_1, \lambda_2)} \phi + \mathcal{O}(\phi). \qquad (3.5)$$

On the other hand, computing the left-hand side directly from the transformations we find

$$\begin{aligned} \left[\delta_{\lambda_1}, \delta_{\lambda_2} \right] \phi &= \delta_{\lambda_1}\big(f(\lambda_2) + g(\lambda_2, \phi) + \mathcal{O}(\phi^2) \big) - (1 \leftrightarrow 2) \\ &= g(\lambda_2, f(\lambda_1)) - g(\lambda_1, f(\lambda_2)) + \mathcal{O}(\phi). \end{aligned} \qquad (3.6)$$

Comparing with (3.5) we learn that

$$f(\lambda_{12}(\lambda_1, \lambda_2)) = g(\lambda_2, f(\lambda_1)) - g(\lambda_1, f(\lambda_2)). \qquad (3.7)$$

Now we examine how a field-dependent parameter redefinition changes the gauge algebra. We consider

$$\lambda \;\to\; \lambda + \Lambda(\lambda, \phi), \qquad (3.8)$$

with $\Lambda(\lambda, \phi)$ linear in λ and ϕ. More precisely, we define new gauge transformations $\widetilde{\delta}_\lambda$ by

$$\widetilde{\delta}_\lambda \phi \;\equiv\; \delta_{\lambda + \Lambda(\lambda, \phi)} \phi = \delta_\lambda \phi + \delta_{\Lambda(\lambda, \phi)} \phi = f(\lambda) + g(\lambda, \phi) + f(\Lambda(\lambda, \phi)) + \mathcal{O}(\phi^2). \qquad (3.9)$$

From this we next compute the new gauge algebra, which is of the form

$$\left[\widetilde{\delta}_{\lambda_1}, \widetilde{\delta}_{\lambda_2} \right] \phi = \delta^{[0]}_{\widetilde{\lambda}_{12}(\lambda_1, \lambda_2)} \phi + \mathcal{O}(\phi) = f(\widetilde{\lambda}_{12}(\lambda_1, \lambda_2)) + \mathcal{O}(\phi). \qquad (3.10)$$

From the left-hand side we get

$$\begin{aligned} \left[\widetilde{\delta}_{\lambda_1}, \widetilde{\delta}_{\lambda_2} \right] \phi &= \widetilde{\delta}_{\lambda_1}\Big(f(\lambda_2) + g(\lambda_2, \phi) + f(\Lambda(\lambda_2, \phi)) + \mathcal{O}(\phi^2) \Big) - (1 \leftrightarrow 2) \\ &= g(\lambda_2, f(\lambda_1)) + f(\Lambda(\lambda_2, f(\lambda_1))) - (1 \leftrightarrow 2) + \mathcal{O}(\phi) \\ &= f(\lambda_{12}(\lambda_1, \lambda_2)) + f(\Lambda(\lambda_2, f(\lambda_1))) - f(\Lambda(\lambda_1, f(\lambda_2))) + \mathcal{O}(\phi), \end{aligned} \qquad (3.11)$$

using (3.7) in the last step. Recalling that $\delta^{[0]}_\lambda \phi = f(\lambda)$, we compare with (3.10) and infer that up to irrelevant trivial parameters

$$\widetilde{\lambda}_{12}(\lambda_1, \lambda_2) = \lambda_{12}(\lambda_1, \lambda_2) + \Lambda\big(\lambda_2, \delta^{[0]}_{\lambda_1}\phi\big) - \Lambda\big(\lambda_1, \delta^{[0]}_{\lambda_2}\phi\big). \qquad (3.12)$$

This relation allows us to compute the modification of the gauge algebra under field-dependent parameter redefinitions generated by $\Lambda(\lambda, \phi)$ knowing only the inhomogeneous transformations $\delta^{[0]}$ of the fields. We will apply this repeatedly below.

3.2 Simplifying the gauge algebra

We first illustrate the above method by simplifying the gauge algebra following from CSFT to zeroth order in α'. After using the gauge fixing condition (2.14) in (2.27) we find that λ_{12} to zeroth order in α' can be written as

$$
\lambda_{12}^{(0)i} = \lambda_{c,12}^i + \frac{1}{4}\big[\,\bar{\lambda}_1^k\,D^i\bar{\lambda}_{2,k} - \bar{\lambda}_2^k\,D^i\bar{\lambda}_{1,k}\,\big] - \frac{1}{4}\big[\,\bar{\lambda}_2^k\bar{D}_k\lambda_1^i - \bar{\lambda}_1^k\,\bar{D}_k\lambda_2^i\big]
$$
$$
+ \frac{3}{4}\left[-\lambda_1^i\left(-\frac{1}{4}(D\cdot\lambda_2 + \bar{D}\cdot\bar{\lambda}_2)\right) + \lambda_2^i\left(-\frac{1}{4}(D\cdot\lambda_1 + \bar{D}\cdot\bar{\lambda}_1)\right)\right]. \tag{3.13}
$$

Next let us combine terms in the first and second line to find

$$
\lambda_{12}^{(0)i} = \lambda_{c,12}^i + \frac{1}{4}\bar{\lambda}_1^k\,(D^i\bar{\lambda}_{2,k} + \bar{D}_k\lambda_2^i) + \frac{3}{4}\left[-\lambda_1^i\left(-\frac{1}{4}(D\cdot\lambda_2 + \bar{D}\cdot\bar{\lambda}_2)\right)\right] - (1 \leftrightarrow 2)\,, \tag{3.14}
$$

where the $(1 \leftrightarrow 2)$ antisymmetrization applies to all terms except $\lambda_{c,12}^i$. We wrote the terms such that they take the form of the $\delta^{[0]}$ transformations of e_{ij} and d as given in (2.15). Thus, using the $(1 \leftrightarrow 2)$ antisymmetry, we can write

$$
\lambda_{12}^{(0)i} = \lambda_{c,12}^i - \frac{1}{4}\,\bar{\lambda}_2^k\,\delta_{\lambda_1}^{[0]}e_k^i + \frac{3}{4}\,\lambda_2^i\,\delta_{\lambda_1}^{[0]}d - (1 \leftrightarrow 2)\,. \tag{3.15}
$$

Looking back at (3.12) we infer that the final two terms have precisely the structure needed to be removable by a parameter redefinition. More precisely, with

$$
\Lambda^i(\lambda,\bar{\lambda},e,d) = \frac{1}{4}\,\bar{\lambda}^k\,e_k^i - \frac{3}{4}\,\lambda^i\,d\,, \tag{3.16}
$$

we obtain with (3.12) and (3.15) for the redefined gauge algebra

$$
\widetilde{\lambda}_{12}^i(\lambda_1,\lambda_2) = \lambda_{12}^{(0)i}(\lambda_1,\lambda_2) + \frac{1}{4}\,\bar{\lambda}_2^k\,\delta_{\lambda_1}^{[0]}e_k^i - \frac{3}{4}\,\lambda_2^i\,\delta_{\lambda_1}^{[0]}d - (1 \leftrightarrow 2) = \lambda_{c,12}^i\,. \tag{3.17}
$$

The extra terms have cancelled and the gauge algebra reduces to the one defined by the C-bracket. The above parameter redefinitions are those in (2.21), and combined with the field redefinitions (2.22) lead to the simplified form (2.23) of the gauge transformations. Note that, more efficiently, terms in the gauge algebra of the form $\Lambda\big(\lambda_2, \delta_{\lambda_1}^{[0]}\phi\big) - \Lambda\big(\lambda_1, \delta_{\lambda_2}^{[0]}\phi\big)$ can be simply dropped.

In the following we apply this strategy to the $\mathcal{O}(\alpha')$ corrections of the gauge algebra. With the above simplification, the $\mathcal{O}((\alpha')^0)$ part of the algebra is that of the C-bracket and the $\mathcal{O}(\alpha')$ terms remain unchanged so that, deleting tilde's, we can write

$$
\lambda_{12}^i = \lambda_{c,12}^i + \alpha'\lambda_{12}^{(1)i}\,. \tag{3.18}
$$

Here $\lambda_{12}^{(1)i}$ represents the α' correction given in (2.28) that, grouping differential operators, can be written as:

$$
\lambda_{12}^{(1)i} = -\frac{1}{32}(D_1^i - D_2^i)(D_1^j + 2D_2^j)\Big[(2D_1^k + D_2^k)\lambda_{1j}\lambda_{2k}
$$
$$
+ \frac{1}{2}(2\bar{D}_1^k + \bar{D}_2^k)(\lambda_{1j}\bar{\lambda}_{2k} - \lambda_{2j}\bar{\lambda}_{1k}) - \lambda_{1j}\mu_2 + \lambda_{2j}\mu_1\Big] \tag{3.19}
$$
$$
+ \frac{1}{32}(D_1^i - D_2^i)(\bar{D}_1^j + 2\bar{D}_2^j)(\bar{\lambda}_{1,j}\mu_2 - \bar{\lambda}_{2,j}\mu_1)\,.
$$

Next we eliminate μ in favor of λ using (2.14) and expand the innermost differential operator

$$\lambda_{12}^{(1)i} = -\frac{1}{32}(D_1^i - D_2^i)(D_1^j + 2D_2^j)\left[2D^k\lambda_{1j}\,\lambda_{2k} + \lambda_{1j}D^k\lambda_{2k} + \bar{D}^k\lambda_{1j}\,\bar{\lambda}_{2k} - \bar{D}^k\lambda_{2j}\,\bar{\lambda}_{1k}\right.$$
$$\left. + \frac{1}{2}\lambda_{1j}\bar{D}^k\bar{\lambda}_{2k} - \frac{1}{2}\lambda_{2j}\bar{D}^k\bar{\lambda}_{1k} - \frac{1}{4}\lambda_{1j}(D\cdot\lambda_2 - \bar{D}\cdot\bar{\lambda}_2) + \frac{1}{4}\lambda_{2j}(D\cdot\lambda_1 - \bar{D}\cdot\bar{\lambda}_1)\right]$$
$$+ \frac{1}{32}(D_1^i - D_2^i)(\bar{D}_1^j + 2\bar{D}_2^j)\left(\frac{1}{4}\bar{\lambda}_{1,j}(D\cdot\lambda_2 - \bar{D}\cdot\bar{\lambda}_2) - \frac{1}{4}\bar{\lambda}_{2,j}(D\cdot\lambda_1 - \bar{D}\cdot\bar{\lambda}_1)\right). \tag{3.20}$$

Acting then with the second differential operator yields

$$\lambda_{12}^{(1)i} = -\frac{1}{32}(D_1^i - D_2^i)\left\{2D^jD^k\lambda_{1j}\,\lambda_{2k} + D^j\lambda_{1j}\,D^k\lambda_{2k} + D^j\bar{D}^k\lambda_{1j}\,\bar{\lambda}_{2k} - D^j\bar{D}^k\lambda_{2j}\,\bar{\lambda}_{1k}\right.$$
$$+ \frac{1}{2}D^j\lambda_{1j}\,\bar{D}^k\bar{\lambda}_{2k} - \frac{1}{2}D^j\lambda_{2j}\,\bar{D}^k\bar{\lambda}_{1k} + 4D^k\lambda_{1j}\,D^j\lambda_{2k} + 2\lambda_{1j}D^jD^k\lambda_{2k}$$
$$+ 2\bar{D}^k\lambda_{1j}\,D^j\bar{\lambda}_{2k} - 2\bar{D}^k\lambda_{2j}D^j\bar{\lambda}_{1k} + \lambda_{1j}D^j\bar{D}^k\bar{\lambda}_{2k} - \lambda_{2j}D^j\bar{D}^k\bar{\lambda}_{1k}$$
$$- \frac{1}{4}D^j\lambda_{1j}(D\cdot\lambda_2 - \bar{D}\cdot\bar{\lambda}_2) + \frac{1}{4}D^j\lambda_{2j}(D\cdot\lambda_1 - \bar{D}\cdot\bar{\lambda}_1)$$
$$- \frac{1}{2}\lambda_{1j}D^j(D\cdot\lambda_2 - \bar{D}\cdot\bar{\lambda}_2) + \frac{1}{2}\lambda_{2j}D^j(D\cdot\lambda_1 - \bar{D}\cdot\bar{\lambda}_1)$$
$$- \frac{1}{4}\bar{D}^j\bar{\lambda}_{1j}(D\cdot\lambda_2 - \bar{D}\cdot\bar{\lambda}_2) + \frac{1}{4}\bar{D}^j\bar{\lambda}_{2j}(D\cdot\lambda_1 - \bar{D}\cdot\bar{\lambda}_1)$$
$$\left. - \frac{1}{2}\bar{\lambda}_{1j}\bar{D}^j(D\cdot\lambda_2 - \bar{D}\cdot\bar{\lambda}_2) + \frac{1}{2}\bar{\lambda}_{2j}\bar{D}^j(D\cdot\lambda_1 - \bar{D}\cdot\bar{\lambda}_1)\right\}. \tag{3.21}$$

We can now combine and simplify various terms inside the parenthesis (i.e. before acting with the outer differential operator). We note that $(D_1 - D_2)$ imposes an antisymmetry: when exchanging the first and second factor of any term we get a sign. It is then an easy calculation to show, for instance, that the terms quadratic in $D\cdot\lambda$ and $\bar{D}\cdot\bar{\lambda}$ combine into

$$\frac{1}{2}\big(D\cdot\lambda_1 + \bar{D}\cdot\bar{\lambda}_1\big)\big(D\cdot\lambda_2 + \bar{D}\cdot\bar{\lambda}_2\big) = -\big(\delta_{\lambda_1}^{[0]}d\big)\big(D\cdot\lambda_2 + \bar{D}\cdot\bar{\lambda}_2\big) - \big(D\cdot\lambda_1 + \bar{D}\cdot\bar{\lambda}_1\big)\big(\delta_{\lambda_2}^{[0]}d\big). \tag{3.22}$$

Performing similar manipulations for the remaining terms in (3.21) we find in total

$$\lambda_{12}^{(1)i} = -\frac{1}{32}(D_1^i - D_2^i)\left\{-\big(D\cdot\lambda_1 + \bar{D}\cdot\bar{\lambda}_1\big)\delta_{\lambda_2}^{[0]}d + \frac{3}{2}\lambda_1^jD_j\big(D\cdot\lambda_2 + \bar{D}\cdot\bar{\lambda}_2\big)\right.$$
$$\left. + \frac{1}{2}\bar{\lambda}_1^j\bar{D}_j\big(D\cdot\lambda_2 + \bar{D}\cdot\bar{\lambda}_2\big) + 2D^k\lambda_{1j}\,D^j\lambda_{2k} + 2\bar{D}^k\lambda_{1j}\,D^j\bar{\lambda}_{2k} - (1\leftrightarrow 2)\right\},$$
$$= -\frac{1}{32}(D_1^i - D_2^i)\left\{-\big(D\cdot\lambda_1 + \bar{D}\cdot\bar{\lambda}_1\big)\big(\delta_{\lambda_2}^{[0]}d\big) - 6\lambda_1^jD_j\big(\delta_{\lambda_2}^{[0]}d\big) - 2\bar{\lambda}_1^j\bar{D}_j\big(\delta_{\lambda_2}^{[0]}d\big)\right.$$
$$\left. + 2D^k\lambda_{1j}\,D^j\lambda_{2k} + 2\bar{D}^k\lambda_{1j}\,D^j\bar{\lambda}_{2k} - (1\leftrightarrow 2)\right\}, \tag{3.23}$$

where $(1\leftrightarrow 2)$ means $(\lambda_1 \leftrightarrow \lambda_2)$. Consider the second term on the last line. We write it in terms of $\delta^{[0]}e$ and use the strong constraint to find

$$2\bar{D}^k\lambda_{1j}\big(\delta_{\lambda_2}^{[0]}e^j{}_k - \bar{D}_k\lambda_2^j\big) - (1\leftrightarrow 2) = 2\bar{D}^k\lambda_{1j}\,\delta_{\lambda_2}^{[0]}e^j{}_k - 2D^k\lambda_{1j}\,D_k\lambda_2^j - (1\leftrightarrow 2). \tag{3.24}$$

As a result, we have

$$\lambda_{12}^{(1)i} = -\frac{1}{16}\bigl(D_1^i - D_2^i\bigr)\Bigl\{ -\frac{1}{2}\bigl(D\cdot\lambda_1 + \bar{D}\cdot\bar{\lambda}_1\bigr)\bigl(\delta_{\lambda_2}^{[0]}d\bigr) - 3\lambda_1^j D_j\bigl(\delta_{\lambda_2}^{[0]}d\bigr) - \bar{\lambda}_1^j \bar{D}_j\bigl(\delta_{\lambda_2}^{[0]}d\bigr)$$
$$+ \bar{D}^k\lambda_{1j}\bigl(\delta_{\lambda_2}^{[0]}e^j{}_k\bigr) + D^k\lambda_{1j}\,D^j\lambda_{2k} - D^k\lambda_{1j}\,D_k\lambda_2^j - (1\leftrightarrow 2)\Bigr\}. \quad (3.25)$$

It is now clear that all terms that involve a $\delta^{[0]}$ have the structure that allows them to be removed by a suitable parameter redefinition. It thus follows that the new gauge algebra is

$$\tilde{\lambda}_{12}^{(1)i} = -\frac{1}{16}\bigl(D_1^i - D_2^i\bigr)\bigl\{\, 2D^k\lambda_{1j}\,D^j\lambda_{2k} - 2D^k\lambda_{1j}\,D_k\lambda_2^j \,\bigr\}, \quad (3.26)$$

where we noted that the antisymmetry in $\lambda_1 \leftrightarrow \lambda_2$ is automatic under the operator $(D_1^i - D_2^i)$. Dropping the tilde from now on, we have simplified the gauge algebra to

$$\lambda_{12}^{(1)i} = -\frac{1}{16}\bigl(D_1^i - D_2^i\bigr)\bigl(D^k\lambda_{1j} - D_j\lambda_1^k\bigr)\bigl(D^j\lambda_{2k} - D_k\lambda_2^j\bigr)$$
$$= \frac{1}{16}\bigl(D^k\lambda_{1j} - D_j\lambda_1^k\bigr)\overset{\leftrightarrow}{D}{}^i\bigl(D^j\lambda_{2k} - D_k\lambda_2^j\bigr), \quad (3.27)$$

with $A\overset{\leftrightarrow}{D}B \equiv ADB - DA\,B$. In the notation of (3.18) we have identified the α'-corrected gauge algebra or bracket as

$$\lambda_{12}^i \;\equiv\; \lambda_{c,12}^i \;-\; \frac{1}{16}\alpha'\bigl(D_j\lambda_1^k - D^k\lambda_{1j}\bigr)\overset{\leftrightarrow}{D}{}^i\bigl(D^j\lambda_{2k} - D_k\lambda_2^j\bigr). \quad (3.28)$$

Note that for trivial parameters $\lambda^i = D^i\chi$ the full α' corrected bracket vanishes. Moreover, this algebra is purely holomorphic. An exactly analogous treatment of the barred parameter would yield

$$\bar{\lambda}_{12}^i \;\equiv\; \bar{\lambda}_{c,12}^i \;-\; \frac{1}{16}\alpha'\bigl(\bar{D}_j\bar{\lambda}_1^k - \bar{D}^k\bar{\lambda}_{1j}\bigr)\overset{\leftrightarrow}{\bar{D}}{}^i\bigl(\bar{D}^j\bar{\lambda}_{2k} - \bar{D}_k\bar{\lambda}_2^j\bigr). \quad (3.29)$$

Employing the notation

$$K_{ij} \;\equiv\; 2D_{[i}\lambda_{j]}, \qquad \bar{K}_{ij} \;\equiv\; 2\bar{D}_{[i}\bar{\lambda}_{j]}, \quad (3.30)$$

the algebra takes the form

$$\lambda_{12}^i \;\equiv\; \lambda_{c,12}^i \;-\; \frac{1}{16}\alpha'\bigl(K_1^{kl}D^i K_{2kl} - (1\leftrightarrow 2)\bigr),$$
$$\bar{\lambda}_{12}^i \;\equiv\; \bar{\lambda}_{c,12}^i \;-\; \frac{1}{16}\alpha'\bigl(\bar{K}_1^{kl}\bar{D}^i\bar{K}_{2kl} - (1\leftrightarrow 2)\bigr). \quad (3.31)$$

Although this holomorphic/antiholomorphic presentation of the bracket is intriguing, it turns out to be useful to perform one more parameter redefinition that mixes holomorphic and antiholomorphic parts. In fact, the original string field theory gauge algebra mixes holomorphic and antiholomorphic parameters, and the C-bracket does as well. Such mixing leads to a simplified form of the gauge transformations, which we will discuss in the next

subsection. The parameter redefinition, in the form (3.12), uses parameters Λ^i and $\bar{\Lambda}^i$ given by

$$\Lambda^i = -\frac{1}{8}\alpha'\big(\bar{D}_k\bar{\lambda}_l - \bar{D}_l\bar{\lambda}_k\big)\bar{D}^k e^{il}, \qquad \bar{\Lambda}^i = -\frac{1}{8}\alpha'\big(D_k\lambda_l - D_l\lambda_k\big)D^k e^{li}. \tag{3.32}$$

This leads to the redefined gauge algebra

$$\begin{aligned}
\tilde{\lambda}_{12}^i &= \lambda_{12}^i - \frac{1}{8}\alpha'\big[\big(\bar{D}_k\bar{\lambda}_{2l} - \bar{D}_l\bar{\lambda}_{2k}\big)\bar{D}^k\big(D^i\bar{\lambda}_1^l + \bar{D}^l\lambda_1^i\big) - (1 \leftrightarrow 2)\big] \\
&= \lambda_{12}^i - \frac{1}{16}\alpha'\big[\big(\bar{D}_k\bar{\lambda}_{2l} - \bar{D}_l\bar{\lambda}_{2k}\big)D^i\big(\bar{D}^k\bar{\lambda}_1^l - \bar{D}^l\bar{\lambda}_1^k\big) - (1 \leftrightarrow 2)\big],
\end{aligned} \tag{3.33}$$

where we noted that the second derivative $\bar{D}^k\bar{D}^l$ is symmetric in k, l and so drops out of the antisymmetric contraction. Dropping the tilde, and combining with (3.31) the gauge algebra finally becomes

$$\begin{aligned}
\text{CSFT gauge algebra:} \quad \lambda_{12}^i &\equiv \lambda_{c,12}^i - \frac{1}{16}\alpha'\big(K_1^{kl}D^iK_{2kl} - \bar{K}_1^{kl}D^i\bar{K}_{2kl} - (1 \leftrightarrow 2)\big), \\
\bar{\lambda}_{12}^i &\equiv \bar{\lambda}_{c,12}^i - \frac{1}{16}\alpha'\big(\bar{K}_1^{kl}\bar{D}^i\bar{K}_{2kl} - K_1^{kl}\bar{D}^iK_{2kl} - (1 \leftrightarrow 2)\big),
\end{aligned} \tag{3.34}$$

where we have included the corresponding antiholomorphic part. This is the final form of the CSFT gauge algebra that we will use next to determine the α'-deformed gauge transformation. The $C_{\alpha'}$ bracket is read from the above results and the definitions:

$$\big[\delta_{\Lambda_1}, \delta_{\Lambda_2}\big] = \delta_{\Lambda_{12}}, \quad \Lambda_{12} \equiv \big[\Lambda_2, \Lambda_1\big]_{C_{\alpha'}} \quad \to \quad (\lambda_{12}, \bar{\lambda}_{12}) \equiv \big[(\lambda_2, \bar{\lambda}_2), (\lambda_1, \bar{\lambda}_1)\big]_{C_{\alpha'}}. \tag{3.35}$$

3.3 Gauge transformations

We now determine the corrected gauge transformations that close according to the gauge algebra (3.34). Rather than finding them from CSFT by performing a series of laborious field and parameter redefinitions it is easier to obtain them from the gauge algebra. To this end we consider the commutator of transformations on the field e_{ij}. If we only know the $\delta^{[0]}$ and $\delta^{[1]}$ transformations we find

$$\begin{aligned}
\Big[\delta_{\Lambda_1}, \delta_{\Lambda_2}\Big]e_{ij} &= \delta_{\Lambda_1}^{[0]}\big(\delta_{\Lambda_2}^{[1]}e_{ij}\big) - (1 \leftrightarrow 2) + \mathcal{O}(e) \\
&= \delta_{\Lambda_{12}}^{[0]}e_{ij} + \mathcal{O}(e) = D_i\bar{\lambda}_{12\,j} + \bar{D}_j\lambda_{12\,i} + \mathcal{O}(e),
\end{aligned} \tag{3.36}$$

which means that

$$\delta_{\Lambda_1}^{[0]}\big(\delta_{\Lambda_2}^{[1]}e_{ij}\big) - (1 \leftrightarrow 2) = D_i\bar{\lambda}_{12\,j} + \bar{D}_j\lambda_{12\,i}. \tag{3.37}$$

We can look at the first order in α' part of this equation. Noting that $\delta^{[0]}$ receives no α' correction but $\delta^{[1]}$ does, we will write

$$\delta^{[1]} = \delta^{[1](0)} + \alpha'\delta^{[1](1)} + \mathcal{O}(\alpha'^2), \tag{3.38}$$

and use this to evaluate the left-hand side. For the right-hand side we need the parts in λ_{12} and $\bar{\lambda}_{12}$ (3.34) linear in α'. A quick computation gives

$$\delta^{[0]}_{\Lambda_1}\big(\delta^{[1](1)}_{\Lambda_2}e_{ij}\big) - (1 \leftrightarrow 2) = -\frac{1}{8}\big(D_i\bar{K}_1^{kl}\,\bar{D}_j\bar{K}_{2kl} - D_iK_1^{kl}\,\bar{D}_jK_{2kl} - (1 \leftrightarrow 2)\big)\,. \tag{3.39}$$

We note that the terms of the form $KDDK$ cancelled under the $(1 \leftrightarrow 2)$ antisymmetrization, while the terms of the form $DKDK$ added up. We now have to rewrite the right-hand side as a total $\delta^{[0]}$ variation. To this end we write out one of the K factors in each term, using the manifest antisymmetry imposed by the other factor, and compute

$$
\begin{aligned}
\delta^{[0]}_{\Lambda_1}\big(\delta^{[1](1)}_{\Lambda_2}e_{ij}\big) - (1 \leftrightarrow 2) &= -\frac{1}{4}\big(\bar{D}^kD_i\bar{\lambda}_1^l\,\bar{D}_j\bar{K}_{2kl} - D_iK_1^{kl}\,D_k\bar{D}_j\lambda_{2l} - (1 \leftrightarrow 2)\big) \\
&= -\frac{1}{4}\big(\delta^{[0]}_{\lambda_1}\big(\bar{D}^ke_i{}^l\big)\,\bar{D}_j\bar{K}_{2kl} + D_iK_2^{kl}\,\delta^{[0]}_{\lambda_1}\big(D_ke_{lj}\big) - (1 \leftrightarrow 2)\big) \\
&= -\frac{1}{4}\delta^{[0]}_{\lambda_1}\big(\bar{D}^ke_i{}^l\,\bar{D}_j\bar{K}_{2kl} + D_iK_2^{kl}\,D_ke_{lj} - (1 \leftrightarrow 2)\big)\,, \tag{3.40}
\end{aligned}
$$

where we used the $(1 \leftrightarrow 2)$ antisymmetry in passing from the first to the second line. Note also that while $\delta^{[0]}e_{ij}$ has two terms, only one term survives due to the contraction with the antisymmetric K's. After a slight reordering of terms, we infer that closure of the gauge algebra holds for

$$\delta^{[1](1)}_{\Lambda}e_{ij} = -\frac{1}{4}\big[D_ke_{lj}\,D_iK^{kl} + \bar{D}_ke_{il}\,\bar{D}_j\bar{K}^{kl}\big]\,. \tag{3.41}$$

Writing out K, the result takes the form

$$\delta^{[1](1)}_{\Lambda}e_{ij} = -\frac{1}{4}\big[D^ke_{lj}\,D_i\big(D_k\lambda^l - D^l\lambda_k\big) + \bar{D}^ke_{il}\,\bar{D}_j\big(\bar{D}_k\bar{\lambda}^l - \bar{D}^l\bar{\lambda}_k\big)\big]\,, \tag{3.42}$$

which closes according to the α'-deformed gauge algebra (3.34) predicted by CSFT. Let us finally note that for this gauge algebra, to order α', we have $D \cdot \lambda_{12} + \bar{D} \cdot \bar{\lambda}_{12} = 0$. This implies that the dilaton gauge transformations need not be deformed in order to be compatible with the deformed gauge algebra. Indeed, we will see below that a gauge invariant action can be constructed without changing the dilaton gauge transformations.

3.4 A two-parameter freedom in the gauge algebra

We have used CSFT to determine a consistent deformation of the gauge algebra of the two-derivative theory and the associated deformations of the gauge transformations. One may have suspected that this would be the unique deformation (up to parameter and field redefinitions) of the gauge structure to first order in α'. We will see, however, that there is more freedom, given that the gauge algebra deformation of [3] does not coincide with the CSFT deformation above. There are two possibilities with definite $\mathbb{Z}_2$ properties under the transformation $b \to -b$ and a continuum of possibilities with indefinite $\mathbb{Z}_2$.

The more general gauge transformation can be obtained by using independent coefficients for the two terms in (3.41); the term involving λ and the term involving $\bar{\lambda}$. Introducing parameters $\gamma^\pm$ we write this more general transformation as

$$\delta^{[1](1)}_{\Lambda}e_{ij} = -\frac{1}{4}\big[(\gamma^+ + \gamma^-)D_ke_{lj}\,D_iK^{kl} + (\gamma^+ - \gamma^-)\bar{D}_ke_{il}\,\bar{D}_j\bar{K}^{kl}\big]\,. \tag{3.43}$$

A short computation shows that these close according to the deformed gauge algebra

$$\lambda_{12}^i \equiv \lambda_{c,12}^i - \frac{1}{16}\alpha'\big[\,(\gamma^+ + \gamma^-)K_1^{kl}D^iK_{2kl} - (\gamma^+ - \gamma^-)\bar{K}_1^{kl}D^i\bar{K}_{2kl} - (1 \leftrightarrow 2)\,\big]\,,$$
$$\bar{\lambda}_{12}^i \equiv \bar{\lambda}_{c,12}^i - \frac{1}{16}\alpha'\big[\,(\gamma^+ - \gamma^-)\bar{K}_1^{kl}\bar{D}^i\bar{K}_{2kl} - (\gamma^+ + \gamma^-)K_1^{kl}\bar{D}^iK_{2kl} - (1 \leftrightarrow 2)\,\big]\,. \tag{3.44}$$

For $\gamma^+ = 1$, $\gamma^- = 0$ this reduces to the CSFT transformations and gauge algebra, respectively. The second interesting case is $\gamma^+ = 0$, $\gamma^- = 1$, which introduces a relative sign between holomorphic and antiholomorphic parts and for which we obtain the gauge transformation

$$\delta_\Lambda^{[1](1)-}e_{ij} \equiv -\frac{1}{4}\big[D^k e_{lj}\,D_i\big(D_k\lambda^l - D^l\lambda_k\big) - \bar{D}^k e_{il}\,\bar{D}_j\big(\bar{D}_k\bar{\lambda}^l - \bar{D}^l\bar{\lambda}_k\big)\big]\,, \tag{3.45}$$

where we indicated the new transformation by adding the superscript $^-$. The corresponding gauge algebra reads

$$\lambda_{12}^{i-} = \lambda_{12,c}^i - \frac{1}{16}\alpha'\big(K_1^{kl}\,D^iK_{2kl} + \bar{K}_1^{kl}\,D^i\bar{K}_{2kl} - (1 \leftrightarrow 2)\big)\,,$$
$$\bar{\lambda}_{12}^{i-} = \bar{\lambda}_{12,c}^i + \frac{1}{16}\alpha'\big(K_1^{kl}\,\bar{D}^iK_{2kl} + \bar{K}_1^{kl}\,\bar{D}^i\bar{K}_{2kl} - (1 \leftrightarrow 2)\big)\,. \tag{3.46}$$

We note that for arbitrary γ^+ and γ^- we still have $D \cdot \lambda_{12} + \bar{D} \cdot \bar{\lambda}_{12} = 0$. Therefore, this deformation is also consistent with a dilaton gauge transformation that is not changed.

As we will show in more detail below, the δ^- gauge transformation is an inequivalent deformation of the two-derivative gauge structure of DFT and is the one that arises in [3]. In fact, while the α' deformation implied by CSFT preserves the $\mathbb{Z}_2$ symmetry of the two-derivative DFT, the deformation δ^- violates $\mathbb{Z}_2$ maximally. In the following these two different theories are referred to as DFT$^+$ and DFT$^-$, respectively. We will discuss in the next section their relation to higher-derivative deformations of Einstein gravity with conventional gauge transformations.

4 α' corrections in Einstein variables

In this section we discuss the relation of the CSFT field variable e_{ij}, that has α'-deformed gauge transformations, to the usual variables h_{ij} in Einstein gravity, that transform under conventional diffeomorphisms. We first show that in order to write the Riemann-squared term appearing in the α' expansion of string theory in a T-duality covariant way, we have to perform a redefinition that is not diffeomorphism covariant. This redefinition induces an α' deformed gauge transformation that in turn can be matched with that of CSFT. Finally, we discuss the $\mathbb{Z}_2$ odd gauge transformations of DFT$^-$. We find that on the b-field the deformed gauge transformation cannot be related to that of a conventional 2-form. It has an anomalous term that, however, is exactly as required by the familiar Green-Schwarz anomaly cancellation.

4.1 Riemann-squared and T-duality

We start with the low-energy effective action of closed bosonic string theory to first order in α' [24, 25]. For simplicity we set for now the dilaton and the b-field to zero. The action is then given by

$$S = \int dx \sqrt{g} \left(R + \frac{1}{4} \alpha' R_{ijkl} R^{ijkl} \right) , \tag{4.1}$$

where R_{ijkl} denotes the Riemann tensor. We recall that the Riemann-squared term gives a tensor structure in g_{ij} that cannot be written in a $O(D,D)$ covariant way [22]. In a perturbative expansion $g_{ij} = \eta_{ij} + h_{ij}$ around a constant background and to cubic order in fluctuations one finds

$$S = \int dx \sqrt{g}\, R + \frac{1}{4} \alpha' \int dx\, \partial^k h^{lp}\, \partial^i h_{pq}\, \partial_i \partial_k h^q{}_l + \cdots , \tag{4.2}$$

where to order α' we indicated only the cubic structure that is problematic. This term can be read off from eq. (4.42) in [22], upon expanding to cubic order in h. The claim is that all other cubic terms, indicated by dots, can be written in $O(D,D)$ covariant form.

Before proceeding, let us briefly explain why this term is problematic for $O(D,D)$ covariance. We claim that there is no $O(D,D)$ covariant term that reduces to this structure upon setting $\tilde{\partial} = 0$ and $b = 0$. Such a term would have to be written in terms of e_{ij} and derivatives D_i and $\bar{D}_i$. It is easy to convince oneself, however, that such a term cannot be written, for a natural candidate like

$$D^k e^{lp}\, D^i e_{pq}\, D_i D_k e_l{}^q , \tag{4.3}$$

violates the rules for consistent index contractions reviewed in section 2. Indeed, the summation index p in the first factor has to be considered barred, but in the second factor unbarred, therefore violating $O(D,D)$ covariance. There is no other index assignment that would be consistent. Thus, Riemann-squared expanded to cubic order cannot be written in a T-duality covariant way in terms of e_{ij}.

In order to proceed we now perform a field redefinition that removes the problematic term. We first note that the term can be written as

$$\partial^k h^{lp}\, \partial^i h_{pq}\, \partial_i \partial_k h^q{}_l = \frac{1}{2} \partial_i \big(\partial^k h^{lp}\, \partial^i h_{pq}\, \partial_k h^q{}_l \big) - \frac{1}{2} \partial^2 h_{pq}\, \partial^k h^{lp}\, \partial_k h_l{}^q . \tag{4.4}$$

Ignoring the boundary term, the action (4.2) becomes

$$S[g] = \int dx \sqrt{g}\, R - \frac{1}{8}\, \alpha' \int dx\, \partial^2 h_{pq}\, \partial^k h^{lp}\, \partial_k h_l{}^q + \cdots . \tag{4.5}$$

Consider now a field redefinition of the metric fluctuation,

$$g'_{ij} = \eta_{ij} + h'_{ij} = \eta_{ij} + h_{ij} + \delta h_{ij} , \tag{4.6}$$

where we view δh_{ij} to be of first order in α'. Under such a redefinition, the Einstein-Hilbert term is shifted by

$$\delta(\sqrt{g}R) = \sqrt{g}\, \delta g^{ij} \left(R_{ij} - \frac{1}{2} g_{ij} R \right) = -\sqrt{g}\, \delta h_{ij} \left(R^{ij} - \frac{1}{2} g^{ij} R \right) . \tag{4.7}$$

We thus get for the action (4.5) expressed in terms of the redefined fields, to first order in α',

$$
\begin{aligned}
S[g] = S[g' - \delta g] &= S[g'] + \int dx \sqrt{g} \, \delta h_{ij} \left(R^{ij} - \frac{1}{2} g^{ij} R \right) + \mathcal{O}(\alpha'^2) \\
&= \int dx \, \sqrt{g'} \, R(g') + \int dx \sqrt{g} \, \delta h_{ij} \left(R^{ij} - \frac{1}{2} g^{ij} R \right) \\
&\quad - \frac{1}{8} \alpha' \int dx \, \partial^2 h_{pq} \partial^k h^{lp} \partial_k h_l{}^q + \mathcal{O}(\alpha'^2).
\end{aligned}
\tag{4.8}
$$

As this is valid up to cubic terms in h, we can employ the linearized Ricci tensor and Ricci scalar in the second term,

$$
\begin{aligned}
S[g] = \int dx \, \sqrt{g'} \, R(g') \\
- \int dx \, \delta h_{ij} \left(\frac{1}{2} \partial^2 h^{ij} - \partial^{(i} \partial_k h^{j)k} + \frac{1}{2} \partial^i \partial^j h + \frac{1}{2} \eta^{ij} (-\partial^2 h + \partial^p \partial^q h_{pq}) \right) \\
- \frac{1}{8} \alpha' \int dx \, \partial^2 h_{pq} \, \partial^k h^{lp} \, \partial_k h_l{}^q + \mathcal{O}(\alpha'^2) \, .
\end{aligned}
\tag{4.9}
$$

We now specialize the field redefinition to be of the form

$$
\delta h_{ij} = -\frac{1}{4} \alpha' \, \partial_k h_i{}^l \, \partial^k h_{jl} \, .
\tag{4.10}
$$

This cancels precisely the undesired term in the last line of (4.9). It is easy to see that the remaining terms in (4.9) can be written in $O(D, D)$ covariant form.

To summarize, performing the following redefinitions of the metric fluctuation

$$
h'_{ij} = h_{ij} - \frac{1}{4} \alpha' \, \partial_k h_i{}^p \, \partial^k h_{jp} + \cdots \, ,
\tag{4.11}
$$

we removed the problematic structure in Riemann-squared, which is necessary in order to make T-duality manifest. This result is compatible with a similar conclusion of Meissner [25], that analyzed reductions to $D = 1$ of the low-energy action to first order in α' and found that field redefinitions are necessary in order to make T-duality manifest. Specifically, he found the need for a redefinition of the external components g_{ij} of the metric by terms quadratic in the first derivatives of g_{ij}. This redefinition precisely reduces to (4.11) when expanded in fluctuations and for zero b-field. Being first order in derivatives, such redefinitions are not diffeomorphism covariant and lead to modified metric gauge transformations, as expected from the CSFT results. In the next subsection we determine the full field redefinition including terms involving the b-field.

4.2 Relation to Einstein variables for $\mathbb{Z}_2$ even transformations

We now aim to connect the full closed SFT field e_{ij} to the (perturbative) Einstein variable $\check{e}_{ij}$ defined as the fluctuation of the field $\mathcal{E}_{ij}$ formed by adding the metric to the Kalb-Ramond field

$$
\mathcal{E}_{ij} = G_{ij} + h_{ij} + B_{ij} + b_{ij} = E_{ij} + \check{e}_{ij} \, .
\tag{4.12}
$$

Here $E_{ij} = G_{ij} + B_{ij}$ is the sum of the background metric and Kalb-Ramond field and $\check{e}_{ij} = h_{ij} + b_{ij}$ is the sum of their fluctuations. In the two-derivative DFT this field redefinition is given by [8, 19]

$$\check{e}_{ij} = e_{ij} + \frac{1}{2}e_i{}^k e_{kj} + \cdots , \tag{4.13}$$

where we omitted terms of higher order in fields (that are known in closed form). The form of the field redefinition can be fixed from the standard gauge transformation of $\check{e}_{ij}$ under diffeomorphisms and b-field gauge transformations for $\tilde{\partial} = 0$ [8]. The conventional diffeomorphism and b-field gauge transformations are given by

$$\delta\check{e}_{ij} = \partial_i\epsilon_j + \partial_j\epsilon_i + \partial_i\tilde{\epsilon}_j - \partial_j\tilde{\epsilon}_i + \epsilon^k\partial_k e_{ij} + \partial_i\epsilon^k e_{kj} + \partial_j\epsilon^k e_{ik} , \tag{4.14}$$

where ϵ^i is the diffeomorphism parameter and $\tilde{\epsilon}_i$ the one-form parameter. The relation to the DFT gauge parameter $\xi^M = (\tilde{\xi}_i, \xi^i)$ is given by

$$\epsilon^i = \xi^i , \qquad \tilde{\epsilon}_i = \tilde{\xi}_i + B_{ij}\xi^j . \tag{4.15}$$

The parameters ϵ^i and $\tilde{\epsilon}_i$ are related to the CSFT parameters by

$$\lambda_i = \epsilon_i - \tilde{\epsilon}_i , \qquad \bar{\lambda}_j = \epsilon_j + \tilde{\epsilon}_j . \tag{4.16}$$

The form of the quadratic term in the field redefinition (4.13) is such that the gauge transformation of $\check{e}_{ij}$ on the left-hand side follows as required by (4.14), with the right-hand side transforming according to the CSFT gauge transformations to zeroth order in α', as shown in detail in [8].

Let us now investigate how (4.13) generalizes when including the first α' correction. Since in this case δe_{ij} receives a higher-derivative correction, there must be higher-derivative terms in the field redefinition (4.13) so that the extra variations cancel and the Einstein variable still transforms as in (4.14). In general, the relation (4.16) between the gauge parameters may also receive α' corrections. Making a general ansatz one finds that the field redefinition takes the form

$$\begin{aligned}
\check{e}_{ij} &= e_{ij} + \frac{1}{2}e_i{}^k e_{kj} + \cdots \\
&+ \frac{1}{4}\alpha'\bigg[\partial^k e_i{}^l\,\partial_k e_{lj} - \partial^l e_i{}^k\,\partial_k e_{lj} - \partial^k e^l{}_j\,\partial_i(e_{kl} - e_{lk}) \\
&\qquad + \partial^k e_i{}^l\,\partial_j(e_{kl} - e_{lk}) - \frac{1}{2}\partial_i e^{kl}\partial_j(e_{kl} - e_{lk}) \bigg] + \cdots ,
\end{aligned} \tag{4.17}$$

where the dots represent terms higher order in fields and higher order in α'. Moreover, the relation between gauge parameters indeed gets α' corrected,

$$\begin{aligned}
\lambda_i &= \epsilon_i - \tilde{\epsilon}_i - \frac{1}{4}\alpha'\partial_i(e_{kl} - e_{lk})\partial^k\epsilon^l + \mathcal{O}(\alpha'^2) , \\
\bar{\lambda}_j &= \epsilon_j + \tilde{\epsilon}_j + \frac{1}{4}\alpha'\partial_j(e_{kl} - e_{lk})\partial^k\epsilon^l + \mathcal{O}(\alpha'^2) ,
\end{aligned} \tag{4.18}$$

or for the inverse

$$
\begin{aligned}
\tilde{\epsilon}_i &= -\frac{1}{2}\big(\lambda_i - \bar{\lambda}_i\big) - \frac{1}{8}\alpha'\,\partial_i\big(e_{kl} - e_{lk}\big)\partial^l\big(\lambda^k + \bar{\lambda}^k\big) + \mathcal{O}(\alpha'^2)\,, \\
\epsilon_i &= \frac{1}{2}\big(\lambda_i + \bar{\lambda}_i\big) + \mathcal{O}(\alpha'^2)\,.
\end{aligned}
\tag{4.19}
$$

Note that these redefinitions are T-duality violating, as it should be. In order to verify the claim that the above redefinitions are the right ones one has to compute the gauge transformation of the right-hand side of (4.17) by means of the α'-deformed gauge transformation (3.42) and the inhomogeneous transformation $\delta^{[0]}e_{ij}$ in the $\mathcal{O}(\alpha')$ terms, setting $\tilde{\partial} = 0$. A straightforward computation yields

$$
\begin{aligned}
\delta^{[1](1)}\check{e}_{ij} &= -\frac{1}{4}\partial_i e^{[kl]}\,\partial_j\partial_k\big(\lambda_l + \bar{\lambda}_l\big) + \frac{1}{4}\partial_j e^{[kl]}\,\partial_i\partial_k\big(\lambda_l + \bar{\lambda}_l\big) \\
&= \partial_i\left(+\frac{1}{4}\partial_j e_{[kl]}\,\partial^k\big(\lambda^l + \bar{\lambda}^l\big)\right) - \partial_j\left(\frac{1}{4}\partial_i e_{[kl]}\,\partial^k\big(\lambda^l + \bar{\lambda}^l\big)\right)\,,
\end{aligned}
\tag{4.20}
$$

where, as indicated by the notation on δ on the left-hand side, we included only the terms $\mathcal{O}(\alpha')$ and linear in fields. This is precisely of the form of the $\mathcal{O}(\alpha')$ terms originating in $\delta\check{e}_{ij} = \partial_i\tilde{\epsilon}_j - \partial_j\tilde{\epsilon}_i$ through the deformation of the parameter redefinition in (4.19). Thus, we trivialized the higher-derivative deformation. Together with the analysis in [8] it follows that the gauge transformations reduce to the conventional diffeomorphism and b-field ransformations (4.14) for $\check{e}_{ij}$. This proves that the field and parameter redefinitions (4.17), (4.18) connect to conventional Einstein variables and symmetries. From the leading term in the second line of (4.17) one may verify that this field redefinition indeed contains the minimal redefinition (4.11) needed in order to describe Riemann-squared (note here that e_{ij} has to be identified with h'_{ij} and $\check{e}_{ij}$ with h_{ij}).

4.3 Relation to Einstein variables for $\mathbb{Z}_2$ odd transformations

Let us now turn to the $\mathbb{Z}_2$ violating gauge transformations of DFT$^-$ defined in (3.45),

$$
\delta^{[1](1)-}_{\Lambda}e_{ij} = -\frac{1}{4}\left[D^k e_{lj}\,D_i\big(D_k\lambda^l - D^l\lambda_k\big) - \bar{D}^k e_{il}\,\bar{D}_j\big(\bar{D}_k\bar{\lambda}^l - \bar{D}^l\bar{\lambda}_k\big)\right]\,.
\tag{4.21}
$$

We will show that in contrast to the DFT$^+$ transformations discussed above, these transformations cannot be related to those of conventional metric and b-field fluctuations upon field and parameter redefinitions. More precisely, the deformed gauge transformation (4.21) leads a gauge transformation for the antisymmetric b-field part of the fluctuation that has a non-removable higher-derivative deformation of the diffeomorphism transformation.

To analyze the relation of (4.21) to standard gauge transformations of Einstein-type variables we have to set $\tilde{\partial} = 0$. Useful relations between the different gauge parameters then follow from (4.16)

$$
\begin{aligned}
\partial_k\lambda_l - \partial_l\lambda_k &= 2\partial_{[k}\epsilon_{l]} - 2\partial_{[k}\tilde{\epsilon}_{l]}\,, \\
\partial_k\bar{\lambda}_l - \partial_l\bar{\lambda}_k &= 2\partial_{[k}\epsilon_{l]} + 2\partial_{[k}\tilde{\epsilon}_{l]}\,.
\end{aligned}
\tag{4.22}
$$

Here ϵ_i and $\tilde{\epsilon}_i$ are the diffeomorphism and b-field gauge parameter, respectively. Thus, the linearized gauge transformations for the symmetric and antisymmetric part of $e_{ij} \equiv h_{ij} + b_{ij}$ read to lowest order in fields

$$\delta h_{ij} = \partial_i \epsilon_j + \partial_j \epsilon_i , \qquad \delta b_{ij} = \partial_i \tilde{\epsilon}_j - \partial_j \tilde{\epsilon}_i . \tag{4.23}$$

Next, we evaluate the deformed gauge transformation (4.21) for h and b by using (4.22) and decomposing into the symmetric and antisymmetric parts,

$$
\begin{aligned}
\delta^{[1](1)-} h_{ij} &= \frac{1}{2} \partial^k h^l{}_j \, \partial_i \partial_{[k} \tilde{\epsilon}_{l]} + \frac{1}{2} \partial^k b_i{}^l \, \partial_j \partial_{[k} \epsilon_{l]} + (i \leftrightarrow j) , \\
\delta^{[1](1)-} b_{ij} &= -\frac{1}{2} \partial^k h^l{}_j \, \partial_i \partial_{[k} \epsilon_{l]} + \frac{1}{2} \partial^k b^l{}_j \, \partial_i \partial_{[k} \tilde{\epsilon}_{l]} - (i \leftrightarrow j) .
\end{aligned}
\tag{4.24}
$$

These higher-derivative deformations, which are not present for standard Einstein variables, were the starting point for the analysis in [2]. There we showed that these gauge transformations can be brought to the form of those needed for Green-Schwarz anomaly cancellation. Specifically, we showed that through a combined parameter and field redefinition the gauge transformation of h_{ij} can be trivialized, so that, to this order, it reduces to (4.23), while the gauge transformation of b_{ij} can be brought to the form

$$\delta b_{ij} = \partial_i \tilde{\epsilon}_j - \partial_j \tilde{\epsilon}_i + \partial_{[i} \partial^k \epsilon^l \, \omega^{(1)}_{j]kl} , \tag{4.25}$$

with the linearized spin connection $\omega^{(1)}_{j,kl} \equiv -\partial_{[k} h_{l]j}$. To this order, this is the gauge transformation of the Green-Schwarz mechanism, viewed as a deformation of diffeomorphisms (as opposed to local Lorentz transformations). We also showed in [2] that the non-linear form of these deformed diffeomorphisms provides an exact realization of the deformed C-bracket of DFT$^-$.

5 Perturbation theory of DFT$^-$ and DFT$^+$

In this section we compare the gauge structure discussed so far to that of the theory developed in the context of a 'doubled α'-geometry' in [3]. We will show that this theory corresponds, in the above terminology, to DFT$^-$, i.e., to the $\mathbb{Z}_2$ violating case. To this end we first develop the perturbation theory for the fundamental 'double metric' field $\mathcal{M}$ introduced in [3] and discuss the $\mathbb{Z}_2$ action on these fields. We finally show how to relate these perturbative variables to those appearing in CSFT.

5.1 Perturbative expansion of double metric in DFT$^-$

The theory constructed in [3] features as fundamental fields the 'double metric' $\mathcal{M}_{MN}$, with $O(D,D)$ indices $M, N = 1, \ldots, 2D$, and the dilaton density ϕ (which is related to the CSFT dilaton used above by $\phi = -2d$). In contrast to the generalized metric formulation of double field theory in [10], the field $\mathcal{M}_{MN}$ is not constrained by assuming that it takes values in $O(D,D)$. Rather, it is an unconstrained field that does not even need to be invertible off-shell. In [3] an exactly gauge invariant action with up to six derivatives

was constructed. Although $\mathcal{M}$ is unconstrained, its field equations read $\mathcal{M}_M{}^K \mathcal{M}_{KN} = \eta_{MN} + \cdots$, where the dots represent higher-derivative corrections. To lowest order this equation implies $\mathcal{M} \in O(D,D)$, from which invertibility follows, but since this equation receives higher-derivative corrections its relation to the usual generalized metric and thus to the conventional metric and b-field is subtle.

In the following we discuss the perturbative expansion of this theory around a constant background $\langle \mathcal{M} \rangle$. Being constant, the higher-derivative terms in the background field equations vanish and so the field equations are solved for any $\langle \mathcal{M} \rangle \in O(D,D)$. Thus, the background double metric can be identified with a background *generalized* metric,

$$\langle \mathcal{M}_{MN} \rangle \; \equiv \; \bar{\mathcal{H}}_{MN} \; = \; \begin{pmatrix} G^{ij} & -G^{ik}B_{kj} \\ B_{ik}G^{kj} & G_{ij} - B_{ik}G^{kl}B_{lj} \end{pmatrix} \, , \tag{5.1}$$

where G and B are the (constant) background metric and B-field. In the following it will be convenient to use a notation introduced in [22]. To explain this notation note that due to $\mathcal{H}_{MN}\mathcal{H}^N{}_P = \eta_{MP}$ we may introduce the two background projectors [10]

$$P = \frac{1}{2}\big(\eta - \bar{\mathcal{H}}\big) \, , \qquad \bar{P} = \frac{1}{2}\big(\eta + \bar{\mathcal{H}}\big) \, , \tag{5.2}$$

satisfying $P^2 = P$, $\bar{P}^2 = \bar{P}$ and $P\bar{P} = 0$. Then we define projected $O(D,D)$ indices by

$$W_{\underline{M}} \; \equiv \; P_M{}^N \, W_N \, , \qquad W_{\bar{M}} \; \equiv \; \bar{P}_M{}^N \, W_N \, , \tag{5.3}$$

and similarly for arbitrary $O(D,D)$ tensors. Note that due to the projector identity $P + \bar{P} = \mathbf{1}$ we can decompose any tensor into components with projected indices, e.g., for a vector $W_M = W_{\underline{M}} + W_{\bar{M}}$. We also use this notation for the partial derivatives, so that the strong constraint implies

$$\partial^M \partial_M = 0 \quad \Rightarrow \quad \partial^{\underline{M}} \partial_{\underline{M}} = -\partial^{\bar{M}} \partial_{\bar{M}} \, . \tag{5.4}$$

We are now ready to set up the perturbative expansion of $\mathcal{M}$ around the background $\bar{\mathcal{H}}$. Since $\mathcal{M}$ is unconstrained off-shell, the expansion is simply

$$\mathcal{M}_{MN} = \bar{\mathcal{H}}_{MN} + m_{MN} = \bar{\mathcal{H}}_{MN} + m_{\bar{M}\bar{N}} + m_{\bar{M}\underline{N}} + m_{\underline{M}\bar{N}} + m_{\underline{M}\underline{N}} \, , \tag{5.5}$$

with unconstrained symmetric fluctuations $m_{MN} = m_{NM}$ that we decomposed into projected indices as explained above. Being unconstrained, the perturbation fields m_{MN} has more than the $D \times D$ components needed to encode the metric and b-field fluctuations, but we will show that the projections $m_{\bar{M}\bar{N}}$ and $m_{\underline{M}\underline{N}}$ are auxiliary fields, while the physical part is encoded in $m_{\underline{M}\bar{N}} = m_{\bar{N}\underline{M}}$ (symmetry properties of tensors imply the same properties for the projected components).

In order to verify this claim we have to inspect the Lagrangian in a derivative expansion around the background. The relevant action can be straightforwardly computed from the two-derivative truncation, see eq. (7.13) in [3], which reads

$$S = \int e^{\phi} \left[\frac{1}{2}\eta^{MN}(\mathcal{M} - \frac{1}{3}\mathcal{M}^3)_{MN} + \frac{1}{2}(\mathcal{M}^2 - 1)^{MP}\mathcal{M}_P{}^N \partial_M \partial_N \phi \right.$$
$$\left. + \frac{1}{8}\mathcal{M}^{MN}\partial_M \mathcal{M}^{PQ}\partial_N \mathcal{M}_{PQ} - \frac{1}{2}\mathcal{M}^{MN}\partial_N \mathcal{M}^{KL}\partial_L \mathcal{M}_{KM} - \mathcal{M}^{MN}\partial_M \partial_N \phi \right] \, . \tag{5.6}$$

Note that this action contains terms without derivatives. Inserting the expansion (5.5) and keeping all terms with no derivatives and quadratic terms with two derivatives we find the Lagrangian

$$
\begin{aligned}
\mathcal{L} = {} & \frac{1}{2}\, m^{\underline{MM}}\, m_{\underline{MN}} \; - \frac{1}{2}\, m^{\underline{MN}}\, m_{\underline{M}}{}^{\bar{P}}\, m_{\underline{N}\bar{P}} \; - \frac{1}{6}\, m^{\underline{MN}}\, m_{\underline{N}}{}^{\underline{P}}\, m_{\underline{NP}} \\
& - \frac{1}{2} m^{\bar{M}\bar{N}} m_{\bar{M}\bar{N}} \; - \frac{1}{2}\, m^{\bar{M}\bar{N}}\, m^{\underline{P}}{}_{\bar{M}}\, m_{\underline{P}\bar{N}} \; - \frac{1}{6}\, m^{\bar{M}\bar{N}}\, m_{\bar{N}}{}^{\bar{P}}\, m_{\bar{N}\bar{P}} \\
& + \frac{1}{2}\, \partial^{\bar{M}} m^{\underline{P}\bar{Q}}\, \partial_{\bar{M}} m_{\underline{P}\bar{Q}} \; + \frac{1}{2}\, \partial^{\underline{M}} m^{\underline{P}\bar{Q}}\, \partial_{\underline{P}} m_{\underline{M}\bar{Q}} \; - \frac{1}{2}\, \partial^{\bar{M}} m^{\underline{P}\bar{Q}}\, \partial_{\bar{Q}} m_{\underline{P}\bar{M}} \\
& - 2\, m^{\underline{M}\bar{N}}\, \partial_{\underline{M}} \partial_{\bar{N}} \phi - 2\phi\, \partial^{\bar{M}} \partial_{\bar{M}} \phi \\
& + \frac{1}{4}\, \partial^{\bar{M}} m^{\bar{P}\bar{Q}}\, \partial_{\bar{M}} m_{\bar{P}\bar{Q}} \; + \frac{1}{4}\, \partial^{\bar{M}} m^{\underline{PQ}}\, \partial_{\bar{M}} m_{\underline{PQ}} \\
& + \frac{1}{2} \partial^{\underline{M}} m^{\underline{PQ}}\, \partial_{Q} m_{\underline{PM}} \; - \frac{1}{2}\, \partial^{\bar{M}} m^{\bar{P}\bar{Q}}\, \partial_{\bar{Q}} m_{\bar{P}\bar{M}} \,.
\end{aligned}
\tag{5.7}
$$

The first two lines are the terms with no derivatives, the next two lines contain the physical fields, and the last two lines contain derivatives of the auxiliary fields. Solving for the auxiliary fields to lowest order in fields and without derivatives, the first two terms in the first and second lines give

$$
\begin{aligned}
m_{\underline{MN}} &= \frac{1}{2}\, m_{\underline{M}}{}^{\bar{P}} m_{\underline{N}\bar{P}} + \dots \,, \\
m_{\bar{M}\bar{N}} &= -\frac{1}{2}\, m^{\underline{P}}{}_{\bar{M}} m_{\underline{P}\bar{N}} + \dots \,,
\end{aligned}
\tag{5.8}
$$

where dots indicate terms with more fields or derivatives. Next we eliminate the auxiliary fields, which does not affect the two-derivative quadratic action for the physical fields. This action is then

$$
\begin{aligned}
\mathcal{L}^{(2)} = {} & \frac{1}{2}\, \partial^{\bar{M}} m^{\underline{P}\bar{Q}}\, \partial_{\bar{M}} m_{\underline{P}\bar{Q}} \; + \frac{1}{2}\, \partial^{\underline{M}} m^{\underline{P}\bar{Q}}\, \partial_{\underline{P}} m_{\underline{M}\bar{Q}} \; - \frac{1}{2}\, \partial^{\bar{M}} m^{\underline{P}\bar{Q}}\, \partial_{\bar{Q}} m_{\underline{P}\bar{M}} \\
& - 2\, m^{\underline{M}\bar{N}}\, \partial_{\underline{M}} \partial_{\bar{N}} \phi - 2\phi\, \partial^{\bar{M}} \partial_{\bar{M}} \phi \,.
\end{aligned}
\tag{5.9}
$$

This is the quadratic approximation to the two-derivative standard DFT action [6, 7]. Beyond this approximation the auxiliary fields will be determined non-trivially in terms of the physical fields.

Let us now turn to the gauge symmetries for the fluctuations m_{MN}. These can be obtained from the gauge transformations in [3], eq. (6.39), which are[2]

$$
\begin{aligned}
\delta_\xi \mathcal{M}^{MN} = {} & \xi^P \partial_P \mathcal{M}^{MN} + (\partial^M \xi_P - \partial_P \xi^M) \mathcal{M}^{PN} + (\partial^N \xi_P - \partial_P \xi^N) \mathcal{M}^{MP} \\
& - \frac{1}{2} \big[\partial^M \mathcal{M}^{PQ}\, \partial_P (\partial_Q \xi^N - \partial^N \xi_Q) + 2\, \partial_Q \mathcal{M}^{KM}\, \partial^N \partial_K \xi^Q + (M \leftrightarrow N) \big] \\
& - \frac{1}{2}\, \partial_K \partial^{(M} \mathcal{M}^{PQ}\, \partial^{N)} \partial_P \partial_Q \xi^K \,.
\end{aligned}
\tag{5.10}
$$

---

[2]The different coefficient on the final term arises because here we use a symmetrization convention with unit weight.

Upon insertion of (5.5), and including up to three derivatives in the transformation rules one obtains

$$
\begin{aligned}
\delta_\xi^- m_{MN} = {}& \xi^P \partial_P m_{MN} + \left(\partial_M \xi^P - \partial^P \xi_M\right)\bar{\mathcal{H}}_{PN} + \left(\partial_N \xi^P - \partial^P \xi_N\right)\bar{\mathcal{H}}_{PM} \\
& + \left(\partial_M \xi^P - \partial^P \xi_M\right)m_{PN} + \left(\partial_N \xi^P - \partial^P \xi_N\right)m_{PM} \\
& - \frac{1}{2}\partial_M m^{PQ}\, \partial_P\left(\partial_Q \xi_N - \partial_N \xi_Q\right) - \frac{1}{2}\partial_N m^{PQ}\, \partial_P\left(\partial_Q \xi_M - \partial_M \xi_Q\right) \\
& - \partial_Q m_{MK}\, \partial_N \partial^K \xi^Q - \partial_Q m_{NK}\, \partial_M \partial^K \xi^Q\,.
\end{aligned}
\tag{5.11}
$$

We added the minus superscript to δ_ξ to emphasize that these are the gauge transformations for DFT$^-$. Next we decompose the indices into their projected parts according to (5.3). Using $\bar{\mathcal{H}} = \bar{P} - P$, which follows from (5.2), we compute

$$
\begin{aligned}
\delta_\xi^- m_{MN} = {}& \left(\partial_M \xi_{\bar{N}} - \partial_{\bar{N}} \xi_M\right) + \left(\partial_N \xi_{\bar{M}} - \partial_{\bar{M}} \xi_N\right) - \left(\partial_M \xi_{\underline{N}} - \partial_{\underline{N}} \xi_M\right) - \left(\partial_N \xi_{\underline{M}} - \partial_{\underline{M}} \xi_N\right) \\
& + \xi^P \partial_P m_{MN} + \left(\partial_M \xi^P - \partial^P \xi_M\right)m_{PN} + \left(\partial_N \xi^P - \partial^P \xi_N\right)m_{PM} \\
& - \frac{1}{2}\partial_M m^{PQ}\, \partial_P\left(\partial_Q \xi_N - \partial_N \xi_Q\right) - \frac{1}{2}\partial_N m^{PQ}\, \partial_P\left(\partial_Q \xi_M - \partial_M \xi_Q\right) \\
& - \partial_Q m_{MK}\, \partial_N \partial^K \xi^Q - \partial_Q m_{NK}\, \partial_M \partial^K \xi^Q\,.
\end{aligned}
\tag{5.12}
$$

We now specialize this to the external projection corresponding to the physical fluctuation $m_{M\bar{N}}$ and eliminate the auxiliary fields by use of the lowest-order result (5.8). This yields for the gauge transformation of the physical field

$$
\begin{aligned}
\delta_\xi^- m_{\underline{M}\bar{N}} = {}& 2\left(\partial_{\underline{M}} \xi_{\bar{N}} - \partial_{\bar{N}} \xi_{\underline{M}}\right) \\
& + \xi^P \partial_P m_{\underline{M}\bar{N}} + \left(\partial_{\underline{M}} \xi^{\underline{P}} - \partial^{\underline{P}} \xi_{\underline{M}}\right)m_{\underline{P}\bar{N}} + \left(\partial_{\bar{N}} \xi^{\bar{P}} - \partial^{\bar{P}} \xi_{\bar{N}}\right)m_{\underline{M}\bar{P}} \\
& - \frac{1}{2}\partial_{\underline{M}} m^{\underline{P}\bar{Q}}\, \partial_{\underline{P}}\left(\partial_{\bar{Q}} \xi_{\bar{N}} - \partial_{\bar{N}} \xi_{\bar{Q}}\right) - \frac{1}{2}\partial_{\bar{N}} m^{\underline{Q}\bar{P}}\, \partial_{\bar{P}}\left(\partial_{\underline{Q}} \xi_{\underline{M}} - \partial_{\underline{M}} \xi_{\underline{Q}}\right) \\
& - \frac{1}{2}\partial_{\bar{N}} m^{\underline{P}\bar{Q}}\, \partial_{\underline{P}}\left(\partial_{\bar{Q}} \xi_{\underline{M}} - \partial_{\underline{M}} \xi_{\bar{Q}}\right) - \frac{1}{2}\partial_{\underline{M}} m^{\underline{Q}\bar{P}}\, \partial_{\bar{P}}\left(\partial_{\underline{Q}} \xi_{\bar{N}} - \partial_{\bar{N}} \xi_{\underline{Q}}\right) \\
& - \partial_Q m_{\underline{M}\bar{K}}\, \partial_{\bar{N}} \partial^{\bar{K}} \xi^Q\ -\ \partial_{\bar{Q}} m_{\underline{K}\bar{N}}\, \partial_{\underline{M}} \partial^{\underline{K}} \xi^{\bar{Q}} \\
& - \partial_Q m_{\underline{K}\bar{N}}\, \partial_{\underline{M}} \partial^{\underline{K}} \xi^Q\ -\ \partial_{\bar{Q}} m_{\underline{M}\bar{K}}\, \partial_{\bar{N}} \partial^{\bar{K}} \xi^{\bar{Q}}\,.
\end{aligned}
\tag{5.13}
$$

Similarly, we can compute from (5.12) the gauge transformation of the auxiliary fields, using again the lowest-order result (5.8). We find for $m_{\underline{M}\underline{N}}$

$$
\begin{aligned}
\delta_\xi^- m_{\underline{M}\underline{N}} = {}& \left(\partial_{\underline{M}} \xi^{\bar{P}} - \partial^{\bar{P}} \xi_{\underline{M}}\right)m_{\underline{N}\bar{P}} + \left(\partial_{\underline{N}} \xi^{\bar{P}} - \partial^{\bar{P}} \xi_{\underline{N}}\right)m_{\underline{M}\bar{P}} \\
& - \frac{1}{2}\partial_{\underline{M}} m^{\underline{P}\bar{Q}}\, \partial_{\underline{P}}\left(\partial_{\bar{Q}} \xi_{\underline{N}} - \partial_{\underline{N}} \xi_{\bar{Q}}\right) - \frac{1}{2}\partial_{\underline{N}} m^{\underline{P}\bar{Q}}\, \partial_{\underline{P}}\left(\partial_{\bar{Q}} \xi_{\underline{M}} - \partial_{\underline{M}} \xi_{\bar{Q}}\right) \\
& - \frac{1}{2}\partial_{\underline{M}} m^{\bar{P}Q}\, \partial_{\bar{P}}\left(\partial_Q \xi_{\underline{N}} - \partial_{\underline{N}} \xi_Q\right) - \frac{1}{2}\partial_{\underline{N}} m^{\bar{P}Q}\, \partial_{\bar{P}}\left(\partial_Q \xi_{\underline{M}} - \partial_{\underline{M}} \xi_Q\right) \\
& - \partial_Q m_{\underline{M}\bar{K}}\, \partial_{\underline{N}} \partial^{\bar{K}} \xi^Q - \partial_{\bar{Q}} m_{\underline{M}\bar{K}}\, \partial_{\underline{N}} \partial^{\bar{K}} \xi^{\bar{Q}} - \partial_Q m_{\underline{N}\bar{K}}\, \partial_{\underline{M}} \partial^{\bar{K}} \xi^Q \\
& - \partial_{\bar{Q}} m_{\underline{N}\bar{K}}\, \partial_{\underline{M}} \partial^{\bar{K}} \xi^{\bar{Q}}\,,
\end{aligned}
\tag{5.14}
$$

where we made the $\underline{M}\underline{N}$ symmetrization manifest in each line. We observe that there is no inhomogenous term, as required for (5.8) to be consistent with the gauge symmetries. The gauge transformations determine the form of the auxiliary field to next order, which we give here for completeness,

$$m_{\underline{M}\underline{N}} = \frac{1}{2}\, m_{\underline{M}}{}^{\bar{P}} m_{\underline{N}\bar{P}} \; - \frac{1}{4}\, \partial_{(\underline{M}} m^{\underline{P}\bar{Q}}\, \partial_{\underline{N})} m_{\underline{P}\bar{Q}} + \partial_{(\underline{M}} m^{\underline{P}\bar{Q}}\, \partial_{\underline{P}} m_{\underline{N})\bar{Q}}$$
$$-\frac{1}{2}\partial^{\bar{P}} m_{\underline{M}\bar{Q}}\, \partial^{\bar{Q}} m_{\underline{N}\bar{P}} + \frac{1}{2}\partial^{\bar{P}} m_{\underline{M}}{}^{\bar{Q}}\, \partial_{\bar{P}} m_{\underline{N}\bar{Q}} + \cdots . \tag{5.15}$$

It is straightforward to verify, using the gauge transformations of the physical field, that this expression gives rise to the required transformations (5.14). Analogous relations hold for the auxiliary field $m_{\bar{M}\bar{N}}$.

In order to relate the perturbative field variable here to that of CSFT we first simplify the gauge transformations (5.13) by field and parameter redefinitions. Consider the following field redefinition

$$m'_{\underline{M}\bar{N}} = m_{\underline{M}\bar{N}} + \frac{1}{2}\partial^{\bar{P}} m_{\underline{Q}\bar{N}}\, \partial_{\underline{M}} m^{\underline{Q}}{}_{\bar{P}} - \frac{1}{2}\partial^{\underline{P}} m_{\underline{M}\bar{Q}}\, \partial_{\bar{N}} m_{\underline{P}}{}^{\bar{Q}} . \tag{5.16}$$

With (5.13) we can compute the gauge transformation of m', after which we drop the prime,

$$\delta_\xi^- m_{\underline{M}\bar{N}} = 2\left(\partial_{\underline{M}}\xi_{\bar{N}} - \partial_{\bar{N}}\xi_{\underline{M}}\right)$$
$$+ \xi^P \partial_P m_{\underline{M}\bar{N}} + \left(\partial_{\underline{M}}\xi^{\underline{P}} - \partial^{\underline{P}}\xi_{\underline{M}}\right)m_{\underline{P}\bar{N}} + \left(\partial_{\bar{N}}\xi^{\bar{P}} - \partial^{\bar{P}}\xi_{\bar{N}}\right)m_{\underline{M}\bar{P}}$$
$$- \frac{1}{2}\partial_{\underline{M}} m^{\underline{P}\bar{Q}}\, \partial_{\underline{P}}\left(\partial_{\bar{Q}}\xi_{\bar{N}} - \partial_{\bar{N}}\xi_{\bar{Q}}\right) - \frac{1}{2}\partial_{\bar{N}} m^{Q\bar{P}}\, \partial_{\bar{P}}\left(\partial_{Q}\xi_{\underline{M}} - \partial_{\underline{M}}\xi_{Q}\right)$$
$$+ \frac{1}{2}\partial_{\bar{N}} m^{\underline{P}\bar{Q}}\, \partial_{\underline{P}}\left(\partial_{\bar{Q}}\xi_{\underline{M}} - \partial_{\underline{M}}\xi_{\bar{Q}}\right) + \frac{1}{2}\partial_{\underline{M}} m^{Q\bar{P}}\, \partial_{\bar{P}}\left(\partial_{Q}\xi_{\bar{N}} - \partial_{\bar{N}}\xi_{Q}\right) \tag{5.17}$$
$$+ \partial_{\bar{P}} m_{Q\bar{N}}\, \partial_{\underline{M}}\left(\partial^{Q}\xi^{\bar{P}} - \partial^{\bar{P}}\xi^{Q}\right) - \partial_{\underline{P}} m^{\underline{M}\bar{Q}}\, \partial_{\bar{N}}\left(\partial^{\underline{P}}\xi^{\bar{Q}} - \partial^{\bar{Q}}\xi^{\underline{P}}\right)$$
$$- \partial_{Q} m_{\underline{M}\bar{K}}\, \partial_{\bar{N}}\partial^{\bar{K}}\xi^{Q} \; - \; \partial_{\bar{Q}} m_{\underline{K}\bar{N}}\, \partial_{\underline{M}}\partial^{\underline{K}}\xi^{\bar{Q}}$$
$$- \partial_{Q} m_{\underline{K}\bar{N}}\, \partial_{\underline{M}}\partial^{\underline{K}}\xi^{Q} \; - \; \partial_{\bar{Q}} m_{\underline{M}\bar{K}}\, \partial_{\bar{N}}\partial^{\bar{K}}\xi^{\bar{Q}} .$$

The third and fourth lines combine and so do the fifth and sixth, giving

$$\delta_\xi^- m_{\underline{M}\bar{N}} = 2\left(\partial_{\underline{M}}\xi_{\bar{N}} - \partial_{\bar{N}}\xi_{\underline{M}}\right)$$
$$+ \xi^P \partial_P m_{\underline{M}\bar{N}} + \left(\partial_{\underline{M}}\xi^{\underline{P}} - \partial^{\underline{P}}\xi_{\underline{M}}\right)m_{\underline{P}\bar{N}} + \left(\partial_{\bar{N}}\xi^{\bar{P}} - \partial^{\bar{P}}\xi_{\bar{N}}\right)m_{\underline{M}\bar{P}}$$
$$+ \frac{1}{2}\partial_{\underline{M}} m^{\underline{P}\bar{Q}}\, \partial_{\bar{N}}\left(\partial_{\underline{P}}\xi_{\bar{Q}} - \partial_{\bar{Q}}\xi_{\underline{P}}\right) + \frac{1}{2}\partial_{\bar{N}} m^{\underline{P}\bar{Q}}\, \partial_{\underline{M}}\left(\partial_{\bar{Q}}\xi_{\underline{P}} - \partial_{\underline{P}}\xi_{\bar{Q}}\right) \tag{5.18}$$
$$- \partial^{\bar{Q}} m_{\underline{K}\bar{N}}\, \partial_{\underline{M}}\partial_{\bar{Q}}\xi^{\underline{K}} \; - \; \partial^{Q} m_{\underline{M}\bar{K}}\, \partial_{\bar{N}}\partial_{Q}\xi^{\bar{K}}$$
$$- \partial_{Q} m_{\underline{K}\bar{N}}\, \partial_{\underline{M}}\partial^{\underline{K}}\xi^{Q} \; - \; \partial_{\bar{Q}} m_{\underline{M}\bar{K}}\, \partial_{\bar{N}}\partial^{\bar{K}}\xi^{\bar{Q}} .$$

Next we use the strong constraint in the line before last and relabel both there and in the line below to obtain

$$
\begin{aligned}
\delta_\xi^- m_{\underline{M}\bar{N}} = {} & 2\left(\partial_{\underline{M}}\xi_{\bar{N}} - \partial_{\bar{N}}\xi_{\underline{M}}\right) \\
& + \xi^P \partial_P m_{\underline{M}\bar{N}} + \left(\partial_{\underline{M}}\xi^{\underline{P}} - \partial^{\underline{P}}\xi_{\underline{M}}\right)m_{\underline{P}\bar{N}} + \left(\partial_{\bar{N}}\xi^{\bar{P}} - \partial^{\bar{P}}\xi_{\bar{N}}\right)m_{\underline{M}\bar{P}} \\
& + \frac{1}{2}\partial_{\underline{M}}m^{\underline{P}\bar{Q}}\,\partial_{\bar{N}}\left(\partial_{\underline{P}}\xi_{\bar{Q}} - \partial_{\bar{Q}}\xi_{\underline{P}}\right) - \frac{1}{2}\partial_{\bar{N}}m^{\underline{P}\bar{Q}}\,\partial_{\underline{M}}\left(\partial_{\underline{P}}\xi_{\bar{Q}} - \partial_{\bar{Q}}\xi_{\underline{P}}\right) \\
& + \partial_{\underline{P}}m_{\underline{Q}\bar{N}}\,\partial_{\underline{M}}\left(\partial^{\underline{P}}\xi^{\underline{Q}} - \partial^{\underline{Q}}\xi^{\underline{P}}\right) + \partial_{\bar{P}}m_{\underline{M}\bar{Q}}\,\partial_{\bar{N}}\left(\partial^{\bar{P}}\xi^{\bar{Q}} - \partial^{\bar{Q}}\xi^{\bar{P}}\right) .
\end{aligned}
\tag{5.19}
$$

It is convenient to rewrite this in terms of

$$
K_{MN} \equiv \partial_M \xi_N - \partial_N \xi_M ,
\tag{5.20}
$$

which yields

$$
\begin{aligned}
\delta_\xi^- m_{\underline{M}\bar{N}} = {} & 2\,K_{\underline{M}\bar{N}} + \xi^P \partial_P m_{\underline{M}\bar{N}} + K_{\underline{M}}{}^{\underline{P}}\,m_{\underline{P}\bar{N}} + K_{\bar{N}}{}^{\bar{P}}\,m_{\underline{M}\bar{P}} \\
& + \frac{1}{2}\partial_{\underline{M}}m^{\underline{P}\bar{Q}}\,\partial_{\bar{N}}K_{\underline{P}\bar{Q}} - \frac{1}{2}\partial_{\bar{N}}m^{\underline{P}\bar{Q}}\,\partial_{\underline{M}}K_{\underline{P}\bar{Q}} \\
& + \partial_{\underline{P}}m_{\underline{Q}\bar{N}}\,\partial_{\underline{M}}K^{\underline{P}\underline{Q}} + \partial_{\bar{P}}m_{\underline{M}\bar{Q}}\,\partial_{\bar{N}}K^{\bar{P}\bar{Q}} .
\end{aligned}
\tag{5.21}
$$

The final form of the gauge transformations is obtained by performing a parameter redefinition, which eliminates the terms in the second line. We take

$$
\xi'_M = \xi_M - \frac{1}{4}\partial_M K_{\underline{P}\bar{Q}}\,m^{\underline{P}\bar{Q}} ,
\tag{5.22}
$$

or, more explicitly, for the different projections,

$$
\begin{aligned}
\xi'_{\underline{M}} &= \xi_{\underline{M}} - \frac{1}{4}\partial_{\underline{M}}K_{\underline{P}\bar{Q}}\,m^{\underline{P}\bar{Q}} \\
\xi'_{\bar{N}} &= \xi_{\bar{N}} - \frac{1}{4}\partial_{\bar{N}}K_{\underline{P}\bar{Q}}\,m^{\underline{P}\bar{Q}} .
\end{aligned}
\tag{5.23}
$$

Dropping primes, the final form of the gauge transformations is

$$
\begin{aligned}
\delta_\xi^- m_{\underline{M}\bar{N}} = {} & 2\,K_{\underline{M}\bar{N}} + \xi^P \partial_P m_{\underline{M}\bar{N}} + K_{\underline{M}}{}^{\underline{P}}\,m_{\underline{P}\bar{N}} + K_{\bar{N}}{}^{\bar{P}}\,m_{\underline{M}\bar{P}} \\
& + \partial_{\underline{P}}m_{\underline{Q}\bar{N}}\,\partial_{\underline{M}}K^{\underline{P}\underline{Q}} + \partial_{\bar{P}}m_{\underline{M}\bar{Q}}\,\partial_{\bar{N}}K^{\bar{P}\bar{Q}} .
\end{aligned}
\tag{5.24}
$$

Summarizing, the $\mathcal{O}(\alpha')$ correction to the gauge transformation is the second line above and is linear in the fields:

$$
\delta_\xi^{[1](1)-} m_{\underline{M}\bar{N}} = \partial_{\underline{P}}m_{\underline{Q}\bar{N}}\,\partial_{\underline{M}}K^{\underline{P}\underline{Q}} + \partial_{\bar{P}}m_{\underline{M}\bar{Q}}\,\partial_{\bar{N}}K^{\bar{P}\bar{Q}} .
\tag{5.25}
$$

5.2 $\mathbb{Z}_2$ action on fields

We will now show that the deformations of gauge transformations determined in the previous subsection are $\mathbb{Z}_2$ odd and thus belong to DFT$^-$. To this end we first have to determine the action of $\mathbb{Z}_2$ on the field variables $m_{\underline{M}\bar{N}}$, on derivatives and on gauge parameters. In the generalized metric formalism, the action of $\mathbb{Z}_2$ has been discussed in section 4.1 of [10]. This symmetry acts on the background fields as $B_{ij} \to -B_{ij}$, so it is easy to see that on the (background) generalized metric (5.1) it is implemented by the $2D \times 2D$ matrix

$$Z_M{}^N \equiv \begin{pmatrix} Z^i{}_j & Z^{ij} \\ Z_{ij} & Z_i{}^j \end{pmatrix} = \begin{pmatrix} -\delta^i{}_j & 0 \\ 0 & \delta_i{}^j \end{pmatrix} , \quad Z^2 = \mathbf{1} , \tag{5.26}$$

satisfying $Z^2 = \mathbf{1}$. More precisely, $\mathbb{Z}_2$ acts on $O(D,D)$ indices via

$$\begin{aligned} \partial_M &\to Z_M{}^N \partial_N , \\ \bar{\mathcal{H}}_{MN} &\to Z_M{}^P Z_N{}^Q \bar{\mathcal{H}}_{PQ} , \\ \xi^M &\to \xi^N Z_N{}^M . \end{aligned} \tag{5.27}$$

On the D-dimensional components this indeed reduces to the expected $\mathbb{Z}_2$ action, e.g.,

$$B_{ij} \to -B_{ij} , \qquad \tilde{\partial}^i \to -\tilde{\partial}^i , \qquad \tilde{\xi}_i \to -\tilde{\xi}_i , \tag{5.28}$$

leaving all objects without tilde unchanged. It is important to recall that $\mathbb{Z}_2$ is not part of $O(D,D)$. Indeed, the $\mathbb{Z}_2$ transformation does not leave the $O(D,D)$ metric invariant,

$$Z_M{}^P Z_N{}^Q \eta_{PQ} = -\eta_{MN} \quad \Longleftrightarrow \quad Z_N{}^K \eta_{MK} = -Z_M{}^K \eta_{NK} , \tag{5.29}$$

with the analogous relation for η^{MN} with upper indices. This has important consequences for the $\mathbb{Z}_2$ action on $O(D,D)$ tensors for which indices have been raised or lowered with η. Specifically, taking the $O(D,D)$ tensors in (5.27) as fundamental, the corresponding ones with raised and lowered indices transform as

$$\begin{aligned} \partial^M &\to -\partial^N Z_N{}^M , \\ \bar{\mathcal{H}}_M{}^N &\to -Z_M{}^P Z_Q{}^N \bar{\mathcal{H}}_P{}^Q , \\ \xi_M &\to -Z_M{}^N \xi_N , \end{aligned} \tag{5.30}$$

as a direct consequence of (5.29).

Let us now determine the $\mathbb{Z}_2$ action on the various objects of the perturbative formalism introduced above, starting with the background projectors (5.2). If we view them as having index structure $P_M{}^N$ and $\bar{P}_M{}^N$ the $\mathbb{Z}_2$ action changes the sign of the $\bar{\mathcal{H}}_M{}^N$ term according to (5.30), thereby exchanging P and $\bar{P}$. We thus find

$$P_M{}^N \to Z_M{}^P \bar{P}_P{}^Q Z_Q{}^N , \qquad \bar{P}_M{}^N \to Z_M{}^P P_P{}^Q Z_Q{}^N . \tag{5.31}$$

If we view P and $\bar{P}$ as tensors with lower indices, the leading η_{MN} term changes sign according to (5.29), leading to an exchange of P and $\bar{P}$ up to a global sign,

$$P_{MN} \to -Z_M{}^P Z_N{}^Q \bar{P}_{PQ} , \qquad \bar{P}_{MN} \to -Z_M{}^P Z_N{}^Q P_{PQ} . \tag{5.32}$$

From these results we can immediately determine the transformation of the projected derivatives,

$$
\begin{aligned}
\partial_{\underline{M}} &\rightarrow Z_M{}^P \partial_{\bar{P}} , & \partial_{\bar{M}} &\rightarrow Z_M{}^P \partial_{\underline{P}} , \\
\partial^{\underline{M}} &\rightarrow -\partial^{\bar{P}} Z_P{}^M , & \partial^{\bar{M}} &\rightarrow -\partial^{\underline{P}} Z_P{}^M .
\end{aligned}
\tag{5.33}
$$

This implies for the differential operator

$$
\partial^{\bar{M}} \partial_{\bar{M}} \rightarrow -\partial^{\underline{M}} \partial_{\underline{M}} = \partial^{\bar{M}} \partial_{\bar{M}} ,
\tag{5.34}
$$

using the strong constraint in the last step. Thus, the operator $\partial^{\bar{M}} \partial_{\bar{M}}$, which reduces to the usual Laplace operator for $\tilde{\partial} = 0$, is $\mathbb{Z}_2$ invariant. The same conclusion follows for $\partial^{\underline{M}} \partial_{\underline{M}}$. Note that this result is consistent with the fact that $\partial^M \partial_M = \partial^{\underline{M}} \partial_{\underline{M}} + \partial^{\bar{M}} \partial_{\bar{M}}$, containing one η, is odd under $\mathbb{Z}_2$, because by the strong constraint, which we used above, it is actually zero.

Next, we discuss the $\mathbb{Z}_2$ action on the fluctuation fields m. They are defined via $\mathcal{M} = \mathcal{H} + m$ and so according to the rules for the $\mathbb{Z}_2$ action on $O(D,D)$ indices we have

$$
m_{MN} \rightarrow Z_M{}^P Z_N{}^Q m_{PQ} .
\tag{5.35}
$$

The analogous relations follow for any of the projections with (5.31),

$$
\begin{aligned}
m_{\underline{M}\bar{N}} &\rightarrow Z_M{}^P Z_N{}^Q m_{Q\bar{P}} , \\
m_{\underline{M}\underline{N}} &\rightarrow Z_M{}^P Z_N{}^Q m_{\bar{P}\bar{Q}} , \\
m_{\bar{M}\bar{N}} &\rightarrow Z_M{}^P Z_N{}^Q m_{\underline{P}\underline{Q}} .
\end{aligned}
\tag{5.36}
$$

Similarly, the projected gauge parameters transforms as

$$
\xi^{\underline{M}} \rightarrow \xi^{\bar{P}} Z_P{}^M , \qquad \xi_{\underline{M}} \rightarrow -Z_M{}^P \xi_{\bar{P}} ,
\tag{5.37}
$$

and completely analogously for $\xi^{\bar{M}}$.

We are now in the position to test the $\mathbb{Z}_2$ properties of the gauge transformations for $m_{\underline{M}\bar{N}}$. On account of (5.36), for $\mathbb{Z}_2$ even transformations we should have

$$
\delta_\xi m_{\underline{M}\bar{N}} \rightarrow Z_M{}^P Z_N{}^Q \delta_\xi m_{Q\bar{P}} .
\tag{5.38}
$$

In order to verify the $\mathbb{Z}_2$ parity in tensors with several (free or contracted) $O(D,D)$ indices it can be a bit laborious to insert every single Z matrix, most of which drop out by $Z^2 = \mathbf{1}$. Rather, one may just apply the following simple rule which summarizes the above results:

Rule for $\mathbb{Z}_2$ parity: *An expression with free indices $\underline{M}$ and $\bar{N}$ is $\mathbb{Z}_2$ even/odd if the following action gives back the expression with the same/opposite sign. First exchange $\underline{M} \leftrightarrow \bar{N}$. Second, exchange bars and under-bars in all other indices, keeping the same letter as index label. Third, include a minus sign factor for each index that is not in its canonical position. For an expression without free indices, steps two and three must leave it invariant.*

The canonical positions for fluctuations, derivatives and gauge parameters are m_{MN}, ∂_M and ξ^M, respectively. On the m field the index substitution is implemented as $m_{\underline{P}\bar{Q}} \to m_{Q\bar{P}}$ since, by convention, we always put the under barred index first. Moreover, $\bar{\xi}^P \partial_P$, for example, is $\mathbb{Z}_2$ even.

We can verify now that in the gauge transformation (5.24) the part with one derivative is $\mathbb{Z}_2$ even but the higher derivative correction is $\mathbb{Z}_2$ odd. Applying the above rule to the inhomogeneous term we find that it is left invariant

$$2\big(\partial_{\underline{M}}\xi_{\bar{N}} - \partial_{\bar{N}}\xi_{\underline{M}}\big) \ \to \ -2\big(\partial_{\bar{N}}\xi_{\underline{M}} - \partial_{\underline{M}}\xi_{\bar{N}}\big) = 2\big(\partial_{\underline{M}}\xi_{\bar{N}} - \partial_{\bar{N}}\xi_{\underline{M}}\big)\,, \qquad (5.39)$$

where the sign originated because the gauge parameters have their index in the non-canonical position. Similarly, the terms homogeneous in fields and with one derivative are, as a whole, $\mathbb{Z}_2$ even:

$$\xi^P \partial_P m_{\underline{M}\bar{N}} \quad \to \quad \xi^P \partial_P m_{\underline{M}\bar{N}}\,,$$

$$K_{\underline{M}}{}^P m_{\underline{P}\bar{N}} = (\partial_{\underline{M}}\xi^P - \partial^P \xi_{\underline{M}})\, m_{\underline{P}\bar{N}} \quad \to \quad (\partial_{\bar{N}}\xi^{\bar{P}} - \partial^{\bar{P}}\xi_{\bar{N}})\, m_{\underline{M}\bar{P}} = K_{\bar{N}}{}^{\bar{P}}\, m_{\underline{M}\bar{P}}\,, \ \ (5.40)$$

$$K_{\bar{N}}{}^{\bar{P}} m_{\underline{M}\bar{P}} = (\partial_{\bar{N}}\xi^{\bar{P}} - \partial^{\bar{P}}\xi_{\bar{N}})\, m_{\underline{M}\bar{P}} \quad \to \quad (\partial_{\underline{M}}\xi^P - \partial^P \xi_{\underline{M}})\, m_{\underline{P}\bar{N}} = K_{\underline{M}}{}^P m_{\underline{P}\bar{N}}\,.$$

Note that the second and third terms were exchanged under the transformation. Consider now the higher-derivative terms in the gauge transformation of $m_{\underline{M}\bar{N}}$. For the first term

$$\partial_{\underline{P}}\, m_{Q\bar{N}}\, \partial_{\underline{M}} K^{\underline{PQ}} = \partial_{\underline{P}}\, m_{Q\bar{N}}\, \partial_{\underline{M}}(\partial^{\underline{P}}\xi^{\underline{Q}} - \partial^{\underline{Q}}\xi^{\underline{P}}) \ \to \ -\partial_{\bar{P}}\, m_{\underline{M}\bar{Q}}\, \partial_{\bar{N}}(\partial^{\bar{P}}\xi^{\bar{Q}} - \partial^{\bar{Q}}\xi^{\bar{P}})$$
$$= -\, \partial_{\bar{P}}\, m_{\underline{M}\bar{Q}}\, \partial_{\bar{N}} K^{\bar{P}\bar{Q}}\,, \qquad (5.41)$$

which is minus the second term. Similarly, the transformation of the second term is minus the first. Thus the α' terms in (5.24) are $\mathbb{Z}_2$ odd.

Let us finally point out that also the field redefinition (5.16) was $\mathbb{Z}_2$ violating. This is as it should be because it eliminates $\mathbb{Z}_2$ odd terms through variations of inhomogeneous $\mathbb{Z}_2$ even terms in $\delta_\xi m$. Similarly, the parameter redefinitions (5.23) are $\mathbb{Z}_2$ odd. Summarizing, the gauge transformations of order α', determined for the theory constructed in [3], are $\mathbb{Z}_2$ odd, and so this theory actually corresponds to DFT$^-$. In the next subsection we relate the field variables here to those in the CSFT language, confirming explicitly this conclusion.

5.3 Relating CSFT and DFT frameworks

We now relate in detail the gravitational field variable e_{ij} of CSFT to the double metric fluctuation $m_{\underline{M}\bar{N}}$. On the face of it they appear to be rather different: the former carries D-dimensional indices as in standard gravity, and the latter carries doubled $O(D, D)$ indices. Since the $O(D, D)$ indices are projected, however, they are effectively D-dimensional. The two formalisms are essentially equivalent, as we will show in the following.

The most efficient way to establish this relation is in terms of a frame or vielbein formalism [6, 21], see [26]. More specifically, here we employ a frame formalism for the constant background fields. The 'tangent space' symmetry in this case reduces to a global $GL(D) \times GL(D)$ symmetry, indicated by flat frame indices $A = (a, \bar{a})$, so that a tangent

space tensor is decomposed as $U_A = (U_a, U_{\bar{a}})$. Next we define the background vielbein for a particular 'gauge choice',

$$\mathcal{E}_A{}^M = \begin{pmatrix} \mathcal{E}_{ai} & \mathcal{E}_a{}^i \\ \mathcal{E}_{\bar{a}i} & \mathcal{E}_{\bar{a}}{}^i \end{pmatrix} = \begin{pmatrix} -E_{ai} & \delta_a{}^i \\ E_{i\bar{a}} & \delta_{\bar{a}}{}^i \end{pmatrix} . \tag{5.42}$$

Some components have been fixed to be Kronecker deltas, which in turn allows us to identify i, j indices with a, b indices. The matrix E describes, as in section 2, the sum of background metric and B-field. For completeness we also give the inverse frames $\mathcal{E}_M{}^A$, satisfying $\mathcal{E}_M{}^A \mathcal{E}_A{}^N = \delta_M{}^N$ as well as $\mathcal{E}_A{}^M \mathcal{E}_M{}^B = \delta_A{}^B$:

$$\mathcal{E}_M{}^A = \begin{pmatrix} \mathcal{E}^{ia} & \mathcal{E}^{i\bar{a}} \\ \mathcal{E}_i{}^a & \mathcal{E}_i{}^{\bar{a}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}G^{ia} & \frac{1}{2}G^{i\bar{a}} \\ \frac{1}{2}E_{bi}G^{ab} & \frac{1}{2}E_{\bar{i}\bar{b}}G^{\bar{a}\bar{b}} \end{pmatrix} . \tag{5.43}$$

Next, we inspect the tangent space metric, defined from the metric η_{MN} by

$$\mathcal{G}_{AB} \equiv \begin{pmatrix} \mathcal{G}_{ab} & \mathcal{G}_{a\bar{b}} \\ \mathcal{G}_{\bar{a}b} & \mathcal{G}_{\bar{a}\bar{b}} \end{pmatrix} \equiv \mathcal{E}_A{}^M \mathcal{E}_B{}^N \eta_{MN} = \mathcal{E}_A{}^i \mathcal{E}_{Bi} + \mathcal{E}_{Ai}\mathcal{E}_B{}^i = \begin{pmatrix} -2G_{ab} & 0 \\ 0 & 2G_{\bar{a}\bar{b}} \end{pmatrix} , \tag{5.44}$$

where the last equality follows by a direct calculation from (5.42). Consequently, the inverse metric $\mathcal{G}^{AB}$ is given by

$$\mathcal{G}^{AB} \equiv \begin{pmatrix} \mathcal{G}^{ab} & \mathcal{G}^{a\bar{b}} \\ \mathcal{G}^{\bar{a}b} & \mathcal{G}^{\bar{a}\bar{b}} \end{pmatrix} \equiv \mathcal{E}_M{}^A \mathcal{E}_N{}^B \eta^{MN} = \begin{pmatrix} -\frac{1}{2}G^{ab} & 0 \\ 0 & \frac{1}{2}G^{\bar{a}\bar{b}} \end{pmatrix} . \tag{5.45}$$

These tangent space metrics are used to raise and lower frame indices A, B. Due to the factors of ± 2 and $\pm \frac{1}{2}$ appearing in the metric $\mathcal{G}$ and its inverse, respectively, there is an ambiguity regarding which metric is used when D-dimensional indices are contracted. Here we follow the conventions in which

- the tangent space metric $\mathcal{G}$ (and its inverse) is used to contract indices whenever they are written with latin letters from the beginning of the alphabet, i.e., a, b or $\bar{a}$, $\bar{b}$, but

- the metric G (and its inverse) is used to contract indices whenever they are written with latin letters from the middle of the alphabet, i.e., i, j, etc.

The background projectors (5.2) are defined in terms of the frame fields as

$$P_M{}^N = \mathcal{E}_M{}^a \mathcal{E}_a{}^N , \qquad \bar{P}_M{}^N = \mathcal{E}_M{}^{\bar{a}} \mathcal{E}_{\bar{a}}{}^N . \tag{5.46}$$

Alternatively, we have

$$P^{MN} = \mathcal{G}^{ab} \mathcal{E}_a{}^M \mathcal{E}_b{}^N , \qquad \bar{P}_M{}^N = \mathcal{G}^{\bar{a}\bar{b}} \mathcal{E}_{\bar{a}}{}^M \mathcal{E}_{\bar{b}}{}^N . \tag{5.47}$$

Using the frame field and its inverse we now can introduce various 'flattened' objects. The partial derivatives in flat indices,

$$D_A \equiv \mathcal{E}_A{}^M \partial_M = (D_a, D_{\bar{a}}) , \tag{5.48}$$

take the following explicit form for the choice (5.42),

$$D_a = \partial_a - E_{ai}\tilde{\partial}^i , \qquad D_{\bar{a}} = \partial_{\bar{a}} + E_{ia}\tilde{\partial}^i . \qquad (5.49)$$

Looking back at section 2, we infer that these operators coincide with the differential operators introduced there under the same name (recalling that for (5.42) we can identify flat and curved indices). Similarly, for the flattened gauge parameters we identify

$$\Lambda_A \equiv \mathcal{E}_A{}^M \xi_M = (\Lambda_a, \Lambda_{\bar{a}}) = (-\lambda_a, \bar{\lambda}_{\bar{a}}) , \qquad (5.50)$$

so that we find with (5.42)

$$\lambda_a = -\tilde{\xi}_a + E_{ai}\xi^i , \qquad \bar{\lambda}_{\bar{a}} = \tilde{\xi}_{\bar{a}} + E_{ia}\xi^i . \qquad (5.51)$$

This coincides with the gauge parameters λ_i, $\bar{\lambda}_i$ of CSFT as discussed in section 3 of [8]. Note that we introduced a relative sign in (5.50) in order to comply with the conventions of CSFT. Note that in CSFT we view the parameters λ_i, $\bar{\lambda}_i$ with lower indices as fundamental, while in the $O(D,D)$ covariant language ξ^M with upper indices is fundamental. This requires some care when translating expressions from a frame-like basis to the CSFT basis. For instance, the contraction of two $O(D,D)$ vectors U and V, whose fundamental indices are lower, barred indices, reads

$$U^{\bar{P}}V_{\bar{P}} = \bar{P}^{PQ}U_P V_Q = \mathcal{G}^{\bar{a}\bar{b}}\mathcal{E}_{\bar{a}}{}^P \mathcal{E}_{\bar{b}}{}^Q U_P V_Q = \mathcal{G}^{\bar{a}\bar{b}}U_{\bar{a}}V_{\bar{b}} = \frac{1}{2}G^{ij}\bar{U}_i\bar{V}_j = \frac{1}{2}\bar{U}_i\bar{V}^i , \qquad (5.52)$$

using (5.47) and $\mathcal{G}^{\bar{a}\bar{b}} = \frac{1}{2}G^{\bar{a}\bar{b}}$. Here we indicated the barred nature of the indices on U and V by barring the objects, as it is customary in DFT. We also have, in completely analogous fashion

$$U^{\underline{P}}V_{\underline{P}} = P^{PQ}U_P V_Q = \mathcal{G}^{ab} \mathcal{E}_a{}^P \mathcal{E}_b{}^Q U_P V_Q = \mathcal{G}^{ab}U_a V_b = -\frac{1}{2}G^{ij}U_i V_j = -\frac{1}{2}U_i V^i . \qquad (5.53)$$

As a general translation tool, these are most useful in the form

$$U^{\bar{P}}V_{\bar{P}} = U^{\bar{a}}V_{\bar{a}} , \qquad U^{\underline{P}}V_{\underline{P}} = U^a V_a , \qquad (5.54)$$

where the flat indices are raised with the appropriate $\mathcal{G}$. It is a simple matter to verify that flattening of projected indices works according to the association $\underline{M} \leftrightarrow a$ of under-barred indices with normal latin indices and $\bar{M} \leftrightarrow \bar{a}$ of barred indices:

$$\mathcal{E}_a{}^M B_{\underline{M}} = B_a , \quad \mathcal{E}_a{}^M B_{\bar{M}} = 0 , \qquad \mathcal{E}_{\bar{a}}{}^M B_{\underline{M}} = 0 , \quad \mathcal{E}_{\bar{a}}{}^M B_{\bar{M}} = B_{\bar{a}} . \qquad (5.55)$$

The transport operator $\xi^M \partial_M$ has a simple translation into frame objects:

$$\xi^M \partial_M = \Lambda^A D_A = \mathcal{G}^{ab}\Lambda_a D_b + \mathcal{G}^{\bar{a}\bar{b}}\Lambda_{\bar{a}}D_{\bar{b}} = -\mathcal{G}^{ab}\lambda_a D_b + \mathcal{G}^{\bar{a}\bar{b}}\bar{\lambda}_{\bar{a}}\bar{D}_b = \frac{1}{2}\big(\lambda^i D_i + \bar{\lambda}^i \bar{D}_i\big) . \qquad (5.56)$$

In the above we converted curved into flat indices and decomposed into λ and $\bar{\lambda}$ components according to (5.50). In the last step we used the metric components G according to (5.45). This introduced a factor of $\frac{1}{2}$ and cancelled the minus sign from the frame definition of λ_i.

Our main goal in this formalism is to translate the gauge variation of the double metric fluctuation $m_{\underline{M}\bar{N}}$ to that in terms of the CSFT fluctuation e_{ij} in order to compare results. We claim that these fluctuations are related by

$$e_{a\bar{b}} = \frac{1}{2}\,\mathcal{E}_a{}^M \mathcal{E}_{\bar{b}}{}^N m_{\underline{M}\bar{N}}\,, \qquad \text{or} \qquad m_{\underline{M}\bar{N}} = 2\,\mathcal{E}_M{}^a \mathcal{E}_N{}^{\bar{b}} e_{a\bar{b}}\,, \qquad (5.57)$$

as we will show that it relates the gauge transformations to leading order in derivatives

$$\delta_\Lambda^- e_{a\bar{b}} = \frac{1}{2}\,\mathcal{E}_a{}^M \mathcal{E}_{\bar{b}}{}^N \delta_\xi^- m_{\underline{M}\bar{N}}\,. \qquad (5.58)$$

We evaluate the right-hand side using (5.24):

$$
\begin{aligned}
\delta_\Lambda^- e_{a\bar{b}} =&\frac{1}{2}\,\mathcal{E}_a{}^M \mathcal{E}_{\bar{b}}{}^N \Big(2(\partial_{\underline{M}}\xi_{\bar{N}} - \partial_{\bar{N}}\xi_{\underline{M}}) + \xi^P \partial_P m_{\underline{M}\bar{N}} \\
&\qquad + (\partial_{\underline{M}}\xi^{\underline{P}} - \partial^{\underline{P}}\xi_{\underline{M}})\,m_{\underline{P}\bar{N}} + (\partial_{\bar{N}}\xi^{\bar{P}} - \partial^{\bar{P}}\xi_{\bar{N}})\,m_{\underline{M}\bar{P}} \\
&\qquad + \partial_{\underline{M}}(\partial_{\underline{P}}\xi^{\underline{Q}} - \partial^{\underline{Q}}\xi_{\underline{P}})\partial^{\underline{P}} m_{\underline{Q}\bar{N}} + \partial_{\bar{N}}(\partial_{\bar{P}}\xi^{\bar{Q}} - \partial^{\bar{Q}}\xi_{\bar{P}})\partial^{\bar{P}} m_{\underline{M}\bar{Q}} \Big) \\
=&\, D_a\Lambda_{\bar{b}} - D_{\bar{b}}\Lambda_a + \xi^P \partial_P e_{a\bar{b}} + (D_a\Lambda^c - D^c\Lambda_a)e_{c\bar{b}} + (D_{\bar{b}}\Lambda^{\bar{c}} - D^{\bar{c}}\Lambda_{\bar{b}})e_{a\bar{c}} \\
&\, + D_a(D_c\Lambda^d - D^d\Lambda_c)D^c e_{d\bar{b}} + D_{\bar{b}}(D_{\bar{c}}\Lambda^{\bar{d}} - D^{\bar{d}}\Lambda_{\bar{c}})D^{\bar{c}}e_{a\bar{d}}\,,
\end{aligned}
$$

where we used repeatedly (5.54). We pass to D-dimensional curved indices letting $a \to i$ and $\bar{b} \to j$. Note that then $\Lambda_a \to -\lambda_i$ and $\Lambda_{\bar{b}} \to \bar{\lambda}_j$. A short calculation then gives

$$
\begin{aligned}
\delta_\Lambda^- e_{ij} =&\, D_i\bar{\lambda}_j + \bar{D}_j\lambda_i \\
&+ \frac{1}{2}\big(\lambda^i D_i + \bar{\lambda}^i \bar{D}_i\big)e_{ij} + \frac{1}{2}\big(D_i\lambda^k - D^k\lambda_i\big)e_{kj} + \frac{1}{2}\big(\bar{D}_j\bar{\lambda}^k - \bar{D}^k\bar{\lambda}_j\big)e_{ik} \qquad (5.59) \\
&- \frac{1}{4}D_i(D_k\lambda^l - D^l\lambda_k)D^k e_{lj} + \frac{1}{4}\bar{D}_j(\bar{D}_k\bar{\lambda}^l - \bar{D}^l\bar{\lambda}_k)\bar{D}^k e_{il}\,,
\end{aligned}
$$

The factors of $\frac{1}{4}$ on the last line originate from the two inverse metrics $\mathcal{G}^{-1}$ required by the two index contractions. Since the same type metric is used twice on each term the sign difference between $\mathcal{G}^{ab}$ and $\mathcal{G}^{\bar{a}\bar{b}}$ is immaterial. The first two lines on the above equation are the familiar CSFT gauge transformations of e_{ij} [7]. This confirms the correctness of the identification (5.57) of e with m.

The α' correction of the gauge transformation is on the last line. It differs from the CSFT result (3.42) in the sign of the second term, but agrees precisely with the DFT$^-$ transformation (3.45). Thus, in agreement with the previous section, the theory studied so far in this section is DFT$^-$.

We close this subsection by verifying the above conclusion at the level of the gauge algebra. We first recall the gauge algebra for the background-independent DFT constructed in [3]

$$-\xi_{12}^M = \big[\xi_1, \xi_2\big]^M = \xi_1^N \partial_N \xi_2^M - \xi_2^N \partial_N \xi_1^M - \frac{1}{2}\xi_1^K \overset{\leftrightarrow}{\partial}{}^M \xi_{2K} + \frac{1}{2}\partial_K \xi_1^L \overset{\leftrightarrow}{\partial}{}^M \partial_L \xi_2^K\,, \quad (5.60)$$

where the last term encodes the $\mathcal{O}(\alpha')$ correction. We relate this algebra to the CSFT one by converting to flat indices. One finds for the flattened parameter (5.50)

$$\begin{aligned}
\Lambda_{12\,A} = &\ \frac{1}{2}(\bar{\lambda}_1 \cdot D + \bar{\lambda}_1 \cdot \bar{D})\Lambda_{2A} \ - \ (1 \leftrightarrow 2) \\
&+ \frac{1}{4}\big(\lambda_1 \cdot \overleftrightarrow{D}_A \lambda_2 - \bar{\lambda}_1 \cdot \overleftrightarrow{D}_A \bar{\lambda}_2\big) \\
&- \frac{1}{16}\Big(K_{1\,kl}\overleftrightarrow{D}_A K_2{}^{kl} + \bar{K}_{1\,kl}\overleftrightarrow{D}_A \bar{K}_2{}^{kl} - 2L_{1\,kl}\overleftrightarrow{D}_A L_2{}^{kl}\Big)\,,
\end{aligned} \tag{5.61}$$

where as before: $K_{kl} = D_k\lambda_l - D_l\lambda_k$, $\bar{K}_{kl} = \bar{D}_k\bar{\lambda}_l - \bar{D}_l\bar{\lambda}_k$ and we defined $L_{kl} \equiv D_k\bar{\lambda}_l + \bar{D}_l\lambda_k = \delta_\lambda e_{kl}$. As in various previous examples, the last term in (5.61) can thus be removed by a parameter redefinition. Doing this and converting the external flat index the gauge algebra reads

$$\begin{aligned}
\lambda_{12}^i &= \lambda_{12,c}^i - \frac{1}{16}\,\alpha'\,\big(K_1^{kl}\overleftrightarrow{D}^i K_{2\,kl} \ + \ \bar{K}_1^{kl}\overleftrightarrow{D}^i \bar{K}_{2\,kl}\big)\,, \\
\bar{\lambda}_{12}^i &= \bar{\lambda}_{12,c}^i + \frac{1}{16}\,\alpha'\,\big(K_1^{kl}\overleftrightarrow{D}^i K_{2\,kl} \ + \ \bar{K}_1^{kl}\overleftrightarrow{D}^i \bar{K}_{2\,kl}\big)\,.
\end{aligned} \tag{5.62}$$

The sign difference between the $\mathcal{O}(\alpha')$ contributions is due to the relative sign in the frame definition of λ and $\bar{\lambda}$ in (5.50). This agrees with the DFT$^-$ gauge algebra anticipated in (3.46).

5.4 Direct comparison of gauge algebras

We have seen that the background-independent gauge algebra (5.60) introduced in [3] corresponds to DFT$^-$. Since this is the unique field-independent deformation of the C-bracket there is no analogous background-independent form for the DFT$^+$ algebra. It is illuminating, however, to give a form in which every tensor is written with un-projected $O(D, D)$ indices.

To this end it is convenient to rewrite the DFT$^+$ gauge transformations and algebra in terms of objects with (doubled) $O(D, D)$ indices, using $m_{\underline{M}\bar{N}}$ as field variable, which is straightforward using the map (5.57) between the two formalisms. We should start from the form of the DFT$^-$ gauge transformations that gave the background independent gauge algebra (5.60) directly, without further parameter redefinitions, which is given in (5.21). Changing the relative signs in the last two lines of (5.21) in order to make it $\mathbb{Z}_2$ invariant, one finds

$$\begin{aligned}
\delta_\xi^+ m_{\underline{M}\bar{N}} = &\ 2\,K_{\underline{M}\bar{N}} \ + \xi^P\partial_P m_{\underline{M}\bar{N}} + K_{\underline{M}}{}^{\underline{P}} m_{\underline{P}\bar{N}} + K_{\bar{N}}{}^{\bar{P}} m_{\underline{M}\bar{P}} \\
&+ \frac{1}{2}\partial_{\underline{M}}m^{\underline{P}\bar{Q}}\,\partial_{\bar{N}}K_{\underline{P}\bar{Q}} \ + \ \frac{1}{2}\partial_{\bar{N}}m^{\underline{P}\bar{Q}}\,\partial_{\underline{M}}K_{\underline{P}\bar{Q}} \\
&+ \partial_{\underline{P}}m_{\underline{Q}\bar{N}}\,\partial_{\underline{M}}K^{\underline{P}\underline{Q}} \ - \ \partial_{\bar{P}}m_{\underline{M}\bar{Q}}\,\partial_{\bar{N}}K^{\bar{P}\bar{Q}}\,,
\end{aligned} \tag{5.63}$$

where we indicated by a super-script $^+$ that this describes the DFT$^+$ transformations. In the DFT$^-$ case the corresponding terms in the second line could be removed by the parameter redefinition (5.23), but this introduces a background dependence in the gauge algebra; in contrast, the terms here are removable by a field redefinition, which does not

change the algebra and so does not affect the background dependence. Specifically, the terms in the second line of (5.63) equal a total variation,

$$\frac{1}{4}\partial_{\underline{M}}m^{\underline{P}\bar{Q}}\,\partial_{\bar{N}}\big(\delta_\xi m_{\underline{P}\bar{Q}}\big) + \frac{1}{4}\partial_{\bar{N}}m^{\underline{P}\bar{Q}}\,\partial_{\underline{M}}\big(\delta_\xi m_{\underline{P}\bar{Q}}\big) = \frac{1}{4}\delta_\xi^{[0]}\big(\partial_{\underline{M}}m^{\underline{P}\bar{Q}}\,\partial_{\bar{N}}m_{\underline{P}\bar{Q}}\big)\,, \qquad (5.64)$$

and are thus removable by a field redefinition. Computing the gauge algebra directly from (5.63) one finds

$$\xi_{12}^M = -\frac{1}{4}K_2^{\underline{P}Q}\partial^M K_{1\underline{P}Q} + \frac{1}{4}K_2^{\bar{P}\bar{Q}}\partial^M K_{1\bar{P}\bar{Q}} - (1\leftrightarrow 2)\,. \qquad (5.65)$$

Next, we eliminate the background projectors by (5.2) in order to find the $O(D,D)$ covariant form without projected indices. A straightforward computation yields

$$\mathrm{DFT}^+: \qquad \xi_{12}^M = \xi_{12\,C}^M - \frac{1}{2}\bar{\mathcal{H}}^{KL}\eta^{PQ}\,K_{[1\,KP}\partial^M K_{2]\,LQ}\,. \qquad (5.66)$$

This is to be contrasted with the DFT$^-$ algebra, which in the same notation reads

$$\mathrm{DFT}^-: \qquad \xi_{12}^M = \xi_{12\,C}^M + \frac{1}{2}\eta^{KL}\eta^{PQ}\,K_{[1\,KP}\partial^M K_{2]\,LQ}\,. \qquad (5.67)$$

Note the background field dependence $\bar{\mathcal{H}}$ in the DFT$^+$ algebra. This strongly suggests that in a manifestly background independent formulation of DFT$^+$ the gauge algebra will be field dependent.

We close this section by discussing the general gauge algebra for arbitrary γ^+, γ^-. To this end it is convenient to start from the gauge transformations that follow from (5.21) and (5.63)

$$\begin{aligned}
\delta_\xi^\gamma m_{\underline{M}\bar{N}} = {}&2\,K_{\underline{M}\bar{N}} + \xi^P\partial_P m_{\underline{M}\bar{N}} + K_{\underline{M}}{}^{\underline{P}}\,m_{\underline{P}\bar{N}} + K_{\bar{N}}{}^{\bar{P}}\,m_{\underline{M}\bar{P}} \\
&+ \frac{1}{2}(\gamma^+ + \gamma^-)\partial_{\underline{M}}m^{\underline{P}\bar{Q}}\,\partial_{\bar{N}}K_{\underline{P}\bar{Q}} + \frac{1}{2}(\gamma^+ - \gamma^-)\partial_{\bar{N}}m^{\underline{P}\bar{Q}}\,\partial_{\underline{M}}K_{\underline{P}\bar{Q}} \qquad (5.68) \\
&+ (\gamma^+ + \gamma^-)\partial_{\underline{P}}m_{Q\bar{N}}\,\partial_{\underline{M}}K^{\underline{P}Q} - (\gamma^+ - \gamma^-)\partial_{\bar{P}}m_{\underline{M}\bar{Q}}\,\partial_{\bar{N}}K^{\bar{P}\bar{Q}}\,.
\end{aligned}$$

The terms in the second line proportional to γ^+ are removable by a field redefinition (and can thus be ignored for the sake of computing the gauge algebra); the terms in the second line proportional to γ^- are removable by a parameter redefinition (and thus should be kept as in (5.21)). A direct computation then shows closure with the effective parameter

$$\begin{aligned}
\xi_{12}^M = {}&\xi_{12\,C}^M - \frac{1}{4}(\gamma^+ + \gamma^-)K_2^{\underline{P}Q}\partial^M K_{1\underline{P}Q} + \frac{1}{4}(\gamma^+ - \gamma^-)K_2^{\bar{P}\bar{Q}}\partial^M K_{1\bar{P}\bar{Q}} \\
&- \frac{1}{2}\gamma^- K_2^{\underline{P}\bar{Q}}\partial^M K_{1\underline{P}\bar{Q}} - (1\leftrightarrow 2)\,.
\end{aligned} \qquad (5.69)$$

Eliminating now the projectors by (5.2) we find the gauge algebra

$$\xi_{12}^M = \xi_{12\,C}^M - \frac{1}{2}\big(\gamma^+\mathcal{H}^{KL} - \gamma^-\eta^{KL}\big)\,K_{[1\,K}{}^P\partial^M K_{2]\,LP}\,, \qquad (5.70)$$

which interpolates between the background-dependent (5.66) and the background-independent (5.67).

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6 Cubic actions for DFT$^-$ and DFT$^+$

We now explicitly construct cubic actions of order α', i.e., with three fields and four derivatives, for both DFT$^-$ and DFT$^+$ and thereby also for the interpolating theories. It is convenient to cast the cubic action into a semi-geometric form, partially written in terms of linearized connections and curvatures that have simple transformation rules under the lowest-order gauge symmetries. In the first subsection we introduce these objects and use them to define the $O(D,D)$ covariant form of the Gauss-Bonnet term (to quadratic order in fields). Then we define the cubic DFT$^-$ and DFT$^+$ actions and discuss their respective differences as well as the interpolating case related to the heterotic string.

6.1 Linearized connections, curvatures and Gauss-Bonnet

The two-derivative DFT can be cast into a geometric form, with generalized connections and curvatures, but an important difference to standard geometry is that not all connection components can be determined in terms of the physical fields [6, 21–23, 34]. (This is the very reason that α' corrections are non-trivial and require an extension of the framework, cf. the discussion in [22, 23].) In the following, however, it is sufficient to work with the linearized version of the determined connections, which are given by

$$
\begin{aligned}
\Gamma_{\bar{M}\underline{N}\underline{K}} &\equiv \partial_{\underline{N}} m_{\underline{K}\bar{M}} - \partial_{\underline{K}} m_{\underline{N}\bar{M}} \,, \\
\Gamma_{\underline{M}\bar{N}\bar{K}} &\equiv \partial_{\bar{N}} m_{\underline{M}\bar{K}} - \partial_{\bar{K}} m_{\underline{M}\bar{N}} \,, \\
\Gamma_{\underline{M}} &\equiv \partial^{\bar{N}} m_{\underline{M}\bar{N}} - 2\partial_{\underline{M}}\phi \,, \\
\Gamma_{\bar{M}} &\equiv \partial^{\underline{N}} m_{\underline{N}\bar{M}} + 2\partial_{\bar{M}}\phi \,.
\end{aligned}
\tag{6.1}
$$

It is convenient to record the $\mathbb{Z}_2$ properties of the connections. These are easily found applying the rules spelled out in section 5.2 (recalling that the dilaton is $\mathbb{Z}_2$ invariant):

$$
\begin{aligned}
\Gamma_{\bar{M}\underline{N}\underline{K}} &\xrightarrow{\mathbb{Z}_2} \Gamma_{\underline{M}\bar{N}\bar{K}} \,, \\
\Gamma_{\bar{M}} &\xrightarrow{\mathbb{Z}_2} -\Gamma_{\underline{M}} \,, \\
K_{\bar{M}\bar{N}} &\xrightarrow{\mathbb{Z}_2} -K_{\underline{M}\underline{N}} \,.
\end{aligned}
\tag{6.2}
$$

The gauge variations of these connections under the lowest-order gauge transformation, cf. (5.21),

$$
\delta^{[0]}_\xi m_{\underline{M}\bar{N}} = 2(\partial_{\underline{M}}\xi_{\bar{N}} - \partial_{\bar{N}}\xi_{\underline{M}}) \,, \qquad \delta^{[0]}_\xi \phi = \partial_{\underline{M}}\xi^{\underline{M}} + \partial_{\bar{M}}\xi^{\bar{M}} \,,
\tag{6.3}
$$

can be conveniently written in terms of the gauge parameters:

$$
K_{\bar{M}\bar{N}} \equiv \partial_{\bar{M}}\xi_{\bar{N}} - \partial_{\bar{N}}\xi_{\bar{M}} \,, \qquad K_{\underline{M}\underline{N}} \equiv \partial_{\underline{M}}\xi_{\underline{N}} - \partial_{\underline{N}}\xi_{\underline{M}} \,,
\tag{6.4}
$$

and read

$$
\delta^{[0]}_\xi \Gamma_{\bar{M}\underline{N}\underline{K}} = -2\partial_{\bar{M}} K_{\underline{N}\underline{K}} \,, \qquad\qquad \delta^{[0]}_\xi \Gamma_{\underline{M}\bar{N}\bar{K}} = 2\partial_{\underline{M}} K_{\bar{N}\bar{K}} \,,
\tag{6.5}
$$

$$
\delta^{[0]}_\xi \Gamma_{\underline{M}} = 2\partial^{\underline{N}} K_{\underline{N}\underline{M}} \,, \qquad\qquad \delta^{[0]}_\xi \Gamma_{\bar{M}} = -2\partial^{\bar{N}} K_{\bar{N}\bar{M}} \,.
\tag{6.6}
$$

Note, in particular, that $\Gamma_{\bar{M}\underline{N}\underline{K}}$ and $\Gamma_{\underline{M}}$ are gauge invariant under $\xi^{\bar{M}}$ transformations. Similarly, $\Gamma_{\underline{M}\bar{N}\bar{K}}$ and $\Gamma_{\bar{M}}$ are gauge invariant under $\xi^{\underline{M}}$ transformations. This fact simplifies the construction of gauge invariant actions below.

Next, we can define the linearized Ricci tensor and scalar curvature:

$$\begin{aligned}
\mathcal{R}_{\underline{M}\bar{N}} &\equiv \partial^{\underline{K}}\Gamma_{\bar{N}\underline{K}\underline{M}} + \partial_{\bar{N}}\Gamma_{\underline{M}} = -\partial^{\bar{K}}\Gamma_{\underline{M}\bar{K}\bar{N}} - \partial_{\underline{M}}\Gamma_{\bar{N}} \ , \\
\mathcal{R} &\equiv \partial^{\underline{M}}\Gamma_{\underline{M}} = \partial^{\bar{M}}\Gamma_{\bar{M}} \ .
\end{aligned}$$

(6.7)

The equivalence of the two definitions in each case can be verified with the explicit form of the connections (6.1). These tensors are gauge invariant as can be easily verified with (6.5). Inserting (6.1) the explicit form of the linearized curvatures is given by

$$\begin{aligned}
\mathcal{R}_{\underline{M}\bar{N}} &= \Box\, m_{\underline{M}\bar{N}} - \partial_{\underline{M}}\partial^{\underline{K}}m_{\underline{K}\bar{N}} + \partial_{\bar{N}}\partial^{\bar{K}}m_{\underline{M}\bar{K}} - 2\partial_{\underline{M}}\partial_{\bar{N}}\phi \ , \\
\mathcal{R} &= \partial^{\underline{M}}\partial^{\bar{K}}m_{\underline{M}\bar{K}} - 2\Box\phi \ ,
\end{aligned}$$

(6.8)

where $\Box = \partial^{\underline{M}}\partial_{\underline{M}}$. These curvatures appear in the general variation of the quadratic two-derivative action (5.9),

$$\delta S^{(2)} = \int \delta m^{\underline{M}\bar{N}}\, \mathcal{R}_{\underline{M}\bar{N}} - 2\,\delta\phi\,\mathcal{R} \ .$$

(6.9)

It is also interesting to note that, up to boundary terms, the two-derivative action can be written in terms of connections,

$$\mathcal{L}^{(2)} = \frac{1}{4}\,\Gamma^{\underline{M}\bar{P}\bar{Q}}\Gamma_{\underline{M}\bar{P}\bar{Q}} + \frac{1}{2}\,\Gamma^{\bar{M}}\Gamma_{\bar{M}} \ .$$

(6.10)

There is no $O(D,D)$ covariant Riemann tensor that is fully determined in terms of the physical fields [6, 22] or that even encodes the physical Riemann tensor among undetermined components [23]. However, there is a linearized gauge invariant Riemann tensor (that encodes the linearized physical Riemann tensor for vanishing b-field), as noted in [6].[3] It is defined by

$$\mathcal{R}_{\underline{M}\underline{N}\bar{K}\bar{L}} = 2\partial_{[\underline{M}}\Gamma_{\underline{N}]\bar{K}\bar{L}} \equiv 2\partial_{[\bar{K}}\Gamma_{\bar{L}]\underline{M}\underline{N}} \equiv \mathcal{R}_{\bar{K}\bar{L}\underline{M}\underline{N}} \ .$$

(6.11)

Its explicit form is given by

$$\mathcal{R}_{\underline{M}\underline{N}\bar{K}\bar{L}} = \partial_{\underline{M}}\partial_{\bar{K}}\,m_{\underline{N}\bar{L}} - \partial_{\underline{N}}\partial_{\bar{K}}\,m_{\underline{M}\bar{L}} - \partial_{\underline{M}}\partial_{\bar{L}}\,m_{\underline{N}\bar{K}} + \partial_{\underline{N}}\partial_{\bar{L}}\,m_{\underline{M}\bar{K}} \ .$$

(6.12)

It is easily seen with (6.5) or (6.3) that this tensor is indeed gauge invariant. The linearized Riemann and Ricci tensor and the curvature scalar satisfy differential Bianchi identities,

$$\begin{aligned}
\partial_{\underline{M}}\mathcal{R}^{\underline{M}\underline{N}\bar{K}\bar{L}} &= 2\partial^{[\bar{K}}\mathcal{R}^{|\underline{N}|\bar{L}]} \ , & \partial_{\bar{M}}\mathcal{R}^{\bar{M}\bar{N}\underline{K}\underline{L}} &= -2\partial^{[\underline{K}}\mathcal{R}^{\underline{L}]\bar{N}} \ , \\
\partial_{\underline{M}}\mathcal{R}^{\underline{M}\bar{N}} &= \partial^{\bar{N}}\mathcal{R} \ , & \partial_{\bar{M}}\mathcal{R}^{\underline{N}\bar{M}} &= -\partial^{\underline{N}}\mathcal{R} \ .
\end{aligned}$$

(6.13)

These are easily verified using the definition of these curvatures in terms of connections.

---

[3]This tensor does not have a non-linear completion: there is no tensor that is covariant under the non-linear (un-deformed) gauge transformations of DFT and reduces to it upon expansion around a background.

We close this subsection by giving an $O(D, D)$ covariant form of the Gauss-Bonnet combination (to quadratic order in fields), because this will be important below when relating to the usual $\mathcal{O}(\alpha')$ actions of string theory that are conveniently written in terms of Gauss-Bonnet [32]. Using the linearized $O(D, D)$ covariant curvatures above, the Gauss-Bonnet combination is defined by

$$\text{GB} \equiv \mathcal{R}_{\underline{M}\,\underline{N}\,\bar{K}\,\bar{L}}\,\mathcal{R}^{\underline{M}\,\underline{N}\,\bar{K}\,\bar{L}} + 4\,\mathcal{R}_{\underline{M}\,\bar{N}}\,\mathcal{R}^{\underline{M}\,\bar{N}} + 4\,\mathcal{R}^2 \,. \tag{6.14}$$

This combination is a total derivative (as is the conventional Gauss-Bonnet combination at the quadratic level). Indeed, we can write

$$\text{GB} = \partial_{\underline{M}}B^{\underline{M}} + \partial_{\bar{M}}B^{\bar{M}} \,, \tag{6.15}$$

where

$$\begin{aligned}
B^{\underline{M}} &= \Gamma_{\underline{N}\bar{K}\bar{L}}\,\mathcal{R}^{\underline{M}\,\underline{N}\,\bar{K}\,\bar{L}} + 2\Gamma^{\bar{K}\underline{M}\underline{N}}\,\mathcal{R}_{\underline{N}\bar{K}} - 2\Gamma_{\bar{N}}\,\mathcal{R}^{\underline{M}\bar{N}} + 2\Gamma^{\underline{M}}\,\mathcal{R} \,, \\
B^{\bar{M}} &= \Gamma_{\bar{N}\underline{K}\underline{L}}\,\mathcal{R}^{\bar{M}\bar{N}\underline{K}\underline{L}} - 2\Gamma^{\underline{K}\bar{M}\bar{N}}\,\mathcal{R}_{\underline{K}\bar{N}} + 2\Gamma_{\underline{N}}\,\mathcal{R}^{\underline{N}\bar{M}} + 2\Gamma^{\bar{M}}\,\mathcal{R} \,.
\end{aligned} \tag{6.16}$$

In order to check that the divergence of these vectors leads to the Gauss-Bonnet combination (thereby proving that the latter is a total derivative) one has to use repeatedly the Bianchi identities (6.13).

It is instructive to investigate the gauge transformations of $B^{\underline{M}}$ and $B^{\bar{M}}$, because they play a role analogous to the Chern-Simons three-forms whose exterior derivatives define the conventional Gauss-Bonnet term $\text{tr}(R \wedge R)$. These Chern-Simons forms are not gauge invariant but transform into exterior derivatives, and it is interesting to find the DFT analogue of this fact. Using again the Bianchi identities, one finds that the gauge variations of the B^M can be written as

$$\delta_\xi B^{\underline{M}} = \partial_{\bar{N}}C^{\underline{M}\bar{N}} + \partial_{\underline{N}}C^{\underline{M}\underline{N}} \,, \qquad \delta_\xi B^{\bar{M}} = -\partial_{\underline{N}}C^{\underline{N}\bar{M}} - \partial_{\bar{N}}C^{\bar{M}\bar{N}} \,, \tag{6.17}$$

where

$$\begin{aligned}
C^{\underline{M}\bar{N}} &= -4K^{\underline{M}\underline{K}}\,\mathcal{R}_{\underline{K}}{}^{\bar{N}} + 4K^{\bar{N}\bar{K}}\,\mathcal{R}^{\underline{M}}{}_{\bar{K}} \,, \\
C^{\underline{M}\underline{N}} &= 2K_{\bar{K}\bar{L}}\,\mathcal{R}^{\underline{M}\,\underline{N}\,\bar{K}\,\bar{L}} - 4K^{\underline{M}\underline{N}}\,\mathcal{R} \,, \\
C^{\bar{M}\bar{N}} &= 2K_{\underline{K}\underline{L}}\,\mathcal{R}^{\bar{M}\bar{N}\underline{K}\underline{L}} - 4K^{\bar{M}\bar{N}}\,\mathcal{R} \,.
\end{aligned} \tag{6.18}$$

As $C^{\underline{M}\underline{N}}$ and $C^{\bar{M}\bar{N}}$ are by definition antisymmetric, this makes it manifest that the total divergence of the B^M is gauge invariant.

6.2 Cubic action for DFT at order α'

We now turn to the construction of the cubic action to first order in α', i.e., with four derivatives. We will denote this action by $S^{(3,4)}$ where the first superscript denotes the number of fields and the second the number of derivatives. For DFT$^+$ we will call this action $S_+^{(3,4)}$ and for DFT$^-$ we will call it $S_-^{(3,4)}$. The quadratic action, which is known, is

written as $S^{(2,2)}$; it has two fields and two derivatives. The cubic action $S^{(3,4)}$ is determined by gauge invariance, which to this order in fields requires

$$\delta^{[1](1)} S^{(2,2)} + \delta^{[0]} S^{(3,4)} = 0 \,, \tag{6.19}$$

where we recall that the superscripts on δ indicate the number of fields in brackets and the power of α' in parenthesis. Here we assumed that the action does not contain terms quadratic in fields with four derivatives. This assumption is justified, because one can always choose a field basis in which the curvature-squared invariants enter in the Gauss-Bonnet combination, which reduces to a total derivative at the quadratic level. In fact, in CSFT there are no such terms. Sometimes it may be more convenient to work with another field basis, and we will return to this case below.

Let us now discuss the invariance condition in a little more detail. It turns out to be convenient to write the variation of order α' in terms of linearized connections. In fact, (5.25) can be written as

$$\delta_\xi^{[1](1)\sigma} m_{\underline{M}\bar{N}} = \frac{1}{2}\big(\partial_{\underline{M}} K^{\underline{K}\underline{L}} \Gamma_{\bar{N}\underline{K}\underline{L}} - \sigma\, \partial_{\bar{N}} K^{\bar{K}\bar{L}} \Gamma_{\underline{M}\bar{K}\bar{L}}\big) \,, \tag{6.20}$$

with $\sigma = +1$ for DFT$^+$ and $\sigma = -1$ for DFT$^-$. In constructing the cubic action it is sufficient to focus on one projection of the gauge parameter, provided the action has a definite $\mathbb{Z}_2$ parity. Indeed, for DFT$^+$ gauge invariance under $\underline{\xi}$ implies gauge invariance under $\bar{\xi}$:

$$\delta_{\underline{\xi}}^{[1](1)+} S^{(2,2)} + \delta_{\underline{\xi}}^{[0]} S_+^{(3,4)} = 0 \quad \xrightarrow{\mathbb{Z}_2} \quad \delta_{\bar{\xi}}^{[1](1)+} S^{(2,2)} + \delta_{\bar{\xi}}^{[0]} S_+^{(3,4)} = 0 \,. \tag{6.21}$$

Similarly, for DFT$^-$ we have

$$\delta_{\underline{\xi}}^{[1](1)-} S^{(2,2)} + \delta_{\underline{\xi}}^{[0]} S_-^{(3,4)} = 0 \quad \xrightarrow{\mathbb{Z}_2} \quad -\delta_{\bar{\xi}}^{[1](1)-} S^{(2,2)} - \delta_{\bar{\xi}}^{[0]} S_-^{(3,4)} = 0 \,. \tag{6.22}$$

As before, $\bar{\xi}$ invariance follows from $\underline{\xi}$ invariance. More generally, given cubic, four-derivative actions $S_-^{(3,4)}$ and $S_+^{(3,4)}$ we can construct an invariant action for linear combinations. In fact, the gauge transformations with parameters γ^+ and γ^- in (5.68) are equivalent to

$$\delta_\xi^{[1](1)\gamma} = \gamma^+\, \delta_\xi^{[1](1)+} + \gamma^-\, \delta_\xi^{[1](1)-} \,. \tag{6.23}$$

Then the cubic action

$$S_\gamma^{(3,4)} \equiv \gamma^+\, S_+^{(3,4)} + \gamma^-\, S_-^{(3,4)} \,, \tag{6.24}$$

leads to a gauge invariant action as a direct consequence of (6.21) and (6.22).

We now discuss the specific construction for DFT$^-$. As explained above, it is sufficient to focus on, say, the $\underline{\xi}$ variation, which is given by

$$\delta_{\underline{\xi}}^{[1](1)} m_{\underline{M}\bar{N}} = \frac{1}{2} \partial_{\underline{M}} K^{\underline{K}\underline{L}} \Gamma_{\bar{N}\underline{K}\underline{L}}. \tag{6.25}$$

(Note that, as long as we ignore $\bar{\xi}$, δ^+ and δ^- coincide.) Inserting this variation into the general form (6.9) we compute

$$\delta_{\underline{\xi}}^{[1](1)} S^2 = -\frac{1}{2}\, \partial^{\underline{M}} K^{\underline{P}\underline{Q}} \Gamma^{\bar{N}}{}_{\underline{P}\underline{Q}}\big(\partial^{\bar{K}} \Gamma_{\underline{M}\bar{K}\bar{N}} + \partial_{\underline{M}}\Gamma_{\bar{N}}\big) \,, \tag{6.26}$$

using the (second) definition of the linearized Ricci tensor in (6.7). In order to determine the cubic action we have to find cubic coupling whose $\delta^{[0]}$ variations cancel these terms. It turns out that these terms can be naturally written in terms of the connections (6.1). After some manipulations, discarding total derivatives and using the strong constraint, one can show that the cubic couplings are

$$
S_{-}^{(3,4)} = -\frac{1}{8}\Big(\Gamma^{\underline{P}\bar{M}\bar{N}} \, \Gamma_{\bar{M}}{}^{\underline{K}\underline{L}} \, \partial_{\underline{P}}\Gamma_{\bar{N}\underline{K}\underline{L}} \; + \; \Gamma^{\bar{P}\underline{M}\underline{N}} \, \Gamma_{\underline{M}}{}^{\bar{K}\bar{L}} \, \partial_{\bar{P}}\Gamma_{\underline{N}\bar{K}\bar{L}}
$$
$$
- \; \Gamma^{\bar{M}}{}_{\underline{K}\underline{L}} \Gamma^{\bar{N}\underline{K}\underline{L}} \partial_{\bar{M}}\Gamma_{\bar{N}} \; - \; \Gamma^{\underline{M}}{}_{\bar{K}\bar{L}} \, \Gamma^{\underline{N}\bar{K}\bar{L}} \, \partial_{\underline{M}}\Gamma_{\underline{N}} \Big) \, .
\tag{6.27}
$$

With the $\mathbb{Z}_2$ action (6.2) on the connections and the rules explained in section 5.2 it follows that this action is $\mathbb{Z}_2$ odd. Indeed, in the first line there are five η implicit, leading to a sign change under $\mathbb{Z}_2$; in the second line there are four η implicit, but $\mathbb{Z}_2$ acts as $\Gamma_{\bar{M}} \to -\Gamma_{\underline{M}}$, which also leads to a sign change.

6.3 Cubic action for DFT$^+$

We now turn to the cubic action for DFT$^+$. It can be written in various different forms, all related by total derivatives or covariant field redefinitions. Here we give two forms, one for a field-basis with Riemann-squared, one for the Gauss-Bonnet combination.

The Riemann-squared case turns out to be a little simpler, so we start with this one. We now have to include an $S^{(2,4)}$ term quadratic in fields and with four-derivatives, namely the square of the Riemann tensor (6.11). The full gauge invariance requires

$$
\delta_\xi^{[1](1)+} S^{(2,2)} + \delta_\xi^{[1](0)} S^{(2,4)} + \delta_\xi^{[0]} S^{(3,4)} = 0 \, .
\tag{6.28}
$$

A gauge invariant action to this order is then given by

$$
\begin{aligned}
S = \; & S^{(2,2)} + S^{(3,2)} \\
& + \frac{1}{4}\mathcal{R}^{\underline{M}\underline{N}\bar{K}\bar{L}}\mathcal{R}_{\underline{M}\underline{N}\bar{K}\bar{L}} + \frac{1}{4}\phi\,\mathcal{R}^{\underline{M}\underline{N}\bar{K}\bar{L}}\mathcal{R}_{\underline{M}\underline{N}\bar{K}\bar{L}} \\
& - \frac{1}{8}\Big(\Gamma^{\underline{P}\bar{M}\bar{N}} \, \Gamma_{\bar{M}}{}^{\underline{K}\underline{L}} \, \partial_{\underline{P}}\Gamma_{\bar{N}\underline{K}\underline{L}} \; - \; \Gamma^{\bar{P}\underline{M}\underline{N}} \, \Gamma_{\underline{M}}{}^{\bar{K}\bar{L}} \, \partial_{\bar{P}}\Gamma_{\underline{N}\bar{K}\bar{L}} \\
& \qquad - \Gamma^{\bar{M}}{}_{\underline{K}\underline{L}} \Gamma^{\bar{N}\underline{K}\underline{L}} \partial_{\bar{M}}\Gamma_{\bar{N}} \; + \; \Gamma^{\underline{M}}{}_{\bar{K}\bar{L}} \, \Gamma^{\underline{N}\bar{K}\bar{L}} \, \partial_{\underline{M}}\Gamma_{\underline{N}} \Big) \\
& - \frac{1}{2}\mathcal{R}_{\underline{M}\underline{N}\bar{K}\bar{L}} \Gamma^{\bar{K}\underline{M}\underline{P}} \Gamma^{\bar{L}\underline{N}}{}_{\underline{P}} + \frac{1}{2}\mathcal{R}_{\underline{K}\underline{L}\bar{M}\bar{N}} \Gamma^{\underline{K}\bar{M}\bar{P}} \Gamma^{\underline{L}\bar{N}}{}_{\bar{P}} \\
& - \frac{1}{2}m_{\underline{M}\bar{N}} \mathcal{R}^{\underline{M}\underline{K}\bar{P}\bar{Q}} \partial^{\bar{N}}\Gamma_{\underline{K}\bar{P}\bar{Q}} + \frac{1}{2}m_{\underline{M}\bar{N}} \mathcal{R}^{\underline{P}\underline{Q}\bar{N}\bar{K}} \partial^{\underline{M}}\Gamma_{\bar{K}\underline{P}\underline{Q}} \\
& + \frac{1}{2}\mathcal{R}_{\underline{M}\underline{N}\bar{K}\bar{L}} \partial^{\underline{P}}m^{\underline{M}\bar{K}} \partial_{\underline{P}}m^{\underline{N}\bar{L}} \, .
\end{aligned}
\tag{6.29}
$$

Here $S^{(2,2)}$ and $S^{(3,2)}$ are the quadratic and cubic couplings of the two-derivative theory, $S^{(2,4)}$ is the term quadratic in the Riemann tensor, and all remaining terms belong to $S^{(3,4)}$, the cubic couplings with four derivatives. Note that the explicit form of $S^{(3,2)}$ is not needed for the $\mathcal{O}(\alpha')$ proof of gauge invariance. The gauge invariance can be verified systematically by computing the variation in (6.28) and integrating by parts so that all terms appear with

an undifferentiated gauge parameter ξ. These terms have to cancel, without any total derivative ambiguities. We have verified this (and in fact constructed (6.29)) with the help of a `Mathematica` code.

Next we turn to the field basis with Gauss-Bonnet combination (6.14). In this case the quadratic terms with four derivatives contribute only a boundary term and can thus be ignored. A gauge invariant action to this order is then given by

$$
\begin{aligned}
S = {} & S^{(2,2)} + S^{(3,2)} \\
& + \frac{1}{4}\phi\left(\mathcal{R}^{\underline{M}\,\underline{N}\bar{K}\bar{L}}\,\mathcal{R}_{\underline{M}\,\underline{N}\bar{K}\bar{L}} + 4\,\mathcal{R}_{\underline{M}\bar{N}}\,\mathcal{R}^{\underline{M}\bar{N}} + 4\,\mathcal{R}^2\right) \\
& - \frac{1}{8}\Big(\Gamma^{\underline{P}\bar{M}\bar{N}}\,\Gamma_{\bar{M}}{}^{\underline{K}\underline{L}}\,\partial_{\underline{P}}\Gamma_{\bar{N}\underline{K}\underline{L}} \;-\; \Gamma^{\bar{P}\underline{M}\underline{N}}\,\Gamma_{\underline{M}}{}^{\bar{K}\bar{L}}\,\partial_{\bar{P}}\Gamma_{\underline{N}\bar{K}\bar{L}} \\
& \qquad\quad - \Gamma^{\bar{M}}{}_{\underline{K}\underline{L}}\,\Gamma^{\bar{N}\underline{K}\underline{L}}\partial_{\bar{M}}\Gamma_{\bar{N}} \;+\; \Gamma^{\underline{M}}{}_{\bar{K}\bar{L}}\,\Gamma^{\underline{N}\bar{K}\bar{L}}\partial_{\underline{M}}\Gamma_{\underline{N}}\Big) \\
& + 4\,m_{\underline{M}\bar{N}}\,\partial^{\underline{M}}\partial^{\bar{N}}\phi\,\Box\phi - 4\,m_{\underline{M}\bar{N}}\,\partial^{\underline{M}}\partial_{\underline{K}}\phi\,\partial^{\bar{N}}\partial^{\underline{K}}\phi + 4\,\Box\phi\,\partial^{\underline{K}}\phi\,\partial_{\underline{K}}\phi \\
& + \partial_{\underline{P}}\partial_{\underline{Q}}m_{\underline{M}\bar{N}}\,\partial^{\underline{P}}m^{\underline{M}\bar{K}}\,\partial_{\bar{K}}m^{\underline{Q}\bar{N}} + \partial_{\underline{P}}\partial_{\bar{Q}}m_{\underline{M}\bar{N}}\,\partial^{\underline{P}}m^{\underline{K}\bar{N}}\,\partial_{\underline{K}}m^{\underline{M}\bar{Q}} \\
& + \partial_{\underline{M}}\partial^{\underline{L}}m_{\underline{L}\bar{N}}\,\partial^{\bar{N}}m^{\underline{K}\bar{P}}\,\partial_{\underline{K}}m^{\underline{M}}{}_{\bar{P}} - \partial_{\bar{M}}\partial^{\bar{L}}m_{\underline{N}\bar{L}}\,\partial^{\underline{N}}m^{\underline{P}\bar{K}}\,\partial_{\bar{K}}m_{\underline{P}}{}^{\bar{M}} \\
& - \frac{1}{2}\partial^{\underline{M}}\partial^{\bar{N}}m_{\underline{M}\bar{N}}\,\partial^{\underline{K}}m_{\underline{K}\bar{P}}\,\partial_{\underline{L}}m^{\underline{L}\bar{P}} + \frac{1}{2}\partial^{\underline{M}}\partial^{\bar{N}}m_{\underline{M}\bar{N}}\,\partial^{\bar{K}}m_{\underline{P}\bar{K}}\,\partial_{\bar{L}}m^{\underline{P}\bar{L}} \\
& + \frac{1}{2}\partial^{\underline{M}}\partial^{\bar{N}}m_{\underline{M}\bar{N}}\,\partial_{\underline{K}}m_{\underline{L}\bar{P}}\,\partial^{\underline{K}}m^{\underline{L}\bar{P}} \\
& + \frac{1}{2}\mathcal{R}_{\underline{M}\,\underline{N}\bar{K}\bar{L}}\,\partial^{\underline{P}}m^{\underline{M}\bar{K}}\,\partial_{\underline{P}}m^{\underline{N}\bar{L}}\,.
\end{aligned}
\tag{6.30}
$$

Note that we obtained cubic couplings of the form dilaton times Gauss-Bonnet. This is consistent with the conventional spacetime action of $\mathcal{O}(\alpha')$ in string frame where such terms arise. Again, we proved the gauge invariance condition (6.19) by computing the variation and integrating by parts to show that all terms cancel.

Let us stress that there is a large field-redefinition ambiguity and total derivative ambiguitiy, so the forms given in (6.29) and (6.30) are not unique. The Riemann-squared completion in (6.29) takes a 'semi-geometric' form, written in terms of (linearized) connections and curvatures. We did not manage to find a similarly geometric form of (6.30). It would be interesting, however, to further elucidate the geometrical content of this action, thus arriving at a DFT-extended form of the Gauss-Bonnet action discussed at the linearized level in section 6.1.

We close this section by briefly mentioning the 'interpolating' heterotic case (for vanishing gauge vectors). The corresponding action is given by (6.24) with both γ^+ and γ^- switched on, thus containing a linear combination of the cubic action (6.27) and, depending on the field basis, (6.29) or (6.30). Dropping $\tilde{\partial}$ derivatives and writing the action in terms of conventional perturbative variables, it encodes both a Riemann-squared term and the gravitational Chern-Simons modification of the b-field field strength.

7 Conclusions and outlook

In this paper we have developed DFT$^+$, the double field theory for bosonic string theory to first order in α' and compared it to the 'doubled α' geometry' in [3]. As reviewed here and discussed in more detail in [2], the latter theory, DFT$^-$, has elements of heterotic string theory. Indeed, the gauge algebra for DFT$^+$ differs from that for DFT$^-$. We computed the gauge algebra for the cubic DFT$^+$ from closed string field theory to first order in α'. Then we computed the gauge transformations that close according to this gauge algebra and determined the cubic action. While the cubic action for DFT$^-$ describes (part of) the Chern-Simons modifications of the three-form curvature needed for Green-Schwarz anomaly cancellation, the cubic action for DFT$^+$ describes the T-duality invariant extension of the Riemann-squared term that is known to appear in bosonic string theory. The claim that DFT$^+$ encodes Riemann-squared requires a justification.[4] Therefore we summarize in the following three independent arguments that imply this result:

(1) As explained in section 4.1, writing the cubic terms of Riemann-squared in a T-duality invariant way requires a non-covariant field redefinition [22]. This leads to modified diffeomorphism transformations that agree with the gauge transformations of DFT$^+$.

(2) The results in [25] imply that, upon reduction to one dimension, writing the $\mathcal{O}(\alpha')$ terms in bosonic string theory in an $O(d,d)$ covariant way requires field redefinitions that are in quantitative agreement with those discussed under (1).

(3) The gauge algebra (and therefore, indirectly, the gauge transformations) of DFT$^+$ have been determined from bosonic closed string field theory and thus must lead to a theory that encodes the known Riemann-squared correction. In fact, taken together with the results under (1) our analysis determined the coefficient of the Riemann-squared term as predicted from string field theory and agrees perfectly with the literature.

A final observation supporting the conclusion that DFT$^+$ describes Riemann-squared is that the cubic actions (6.29) or (6.30) contain cubic couplings involving the dilaton times Riemann-squared or Gauss-Bonnet, exactly as expected for the cubic couplings of the $\mathcal{O}(\alpha')$ terms in the string frame.

So far we constructed only the cubic action for DFT$^+$ or, more generally, for the interpolating theory relevant for heterotic string theory, describing both gravitational Chern-Simons modifications and Riemann-squared. It is clearly desirable to construct the background-independent theory, i.e., to all orders in fluctuations. As the DFT$^+$ gauge algebra (5.66) is background-dependent this requires a further extension to a *field-dependent* gauge algebra. Most likely, this extension goes beyond replacing the background generalized metric by the full generalized metric.

---

[4]The fact that the cubic action (6.29) includes the square of the linearized Riemann tensor does not suffice: up to field redefinitions this term may be replaced by the Gauss-Bonnet combination (6.15), which is a total derivative [32].

Very recently an interesting proposal appeared [27] that aims to describe the complete $\mathcal{O}(\alpha')$ corrections of heterotic string theory in DFT. It starts from the heterotic DFT [6, 33, 35, 36] that incorporates n gauge vectors in an enlarged generalized metric taking values in $O(D, D+n)$. The theory is defined on a further extended space with n new coordinates and subject to additional constraints. By declaring part of the connections to be (torsionful) Lorentz connections one obtains the desired $\mathcal{O}(\alpha')$ corrections of heterotic string theory as in [37]. It is asserted in [27] that this procedure is compatible with $O(D, D)$, which however is modified in terms of the conventional fields. This would suggest an alternative picture in which the gauge transformations are unchanged, but instead the action of $O(D, D)$ on fields is α' corrected. It may be of interest to search for a relation to the constructions in [38–40] that also extend the coordinates to encode Lorentz algebra directions.

Acknowledgments

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