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Fusion of conformal defects in interacting theories

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Abstract: We study fusion of two scalar Wilson defects. We propose that fusion holds at a quantum level by showing that bare one-point functions are the same. This is an expected result as the path integral is invariant under fusion of the two defects. The difference instead lies in renormalization of local quantities on the defects. Those on the fused defect takes into account UV divergences in the fusion limit when the two defects approach eachother, in addition to UV divergences in the coincident limit of defect-local fields and in the near defect limits of bulk-local fields. At the fixed point of the corresponding RG flow the two conformal defects have fused into a single conformal defect identical to one of the original scalar Wilson defects.

Parts of this paper was first presented in my thesis [1].

Keywords: Effective Field Theories, Renormalization and Regularization, Scale and Conformal Symmetries, Wilson, 't Hooft and Polyakov loops

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1 Introduction

A defect is an extended object of dimension $p \geq 1$. E.g. a line or a surface. In this paper we study systems with two defects. The Poincaré or conformal symmetry in the bulk is broken in the same way as for one defect: each defect will be charged under an orthogonal group $\mathrm{SO}(d-p)$ (with p being the dimension of the defect), and a defect-local field with support on a defect is charged under $\mathrm{SO}(p-1,1) \times \mathrm{SO}(d-p)$ (assuming flat defects). Due to localization, each defect is only affected by the nearby bulk theory.

Local characteristics, such as anomalous dimensions, β-functions and *operator product expansion* (OPE) coefficients, of the bulk theory are not affected by the defects (since the *ultra-violet* (UV) divergences these quantities arise from are the coincident-limits of bulk fields). Likewise, the corresponding characteristics on each defect are not affected by other defects (since these defect quantities arise from their corresponding coincident-limits of defect-local fields and defect-limits of bulk-local fields).

There will be several new OPE's in play. In addition to the usual bulk-bulk OPE there is a defect-defect OPE (similar to the bulk-bulk OPE) on each defect and a *defect operator* *product expansion* (DOE) for each defect. The DOE allows us to expand bulk-local fields in terms of defect-local ones [2, 3].

If the defects intersect, there is also one defect-intersection DOE for each defect and an intersection-intersection OPE. In the conformal case (when the theories on the intersection, both of the defects and the bulk are all conformal), these give rise to a conformal bootstrap equation for bulk one- and bulk-intersection two-point functions [4].

We will consider two (parallel) *scalar Wilson defects* (or pinning defects) separated by a distance $2\,R$. These conformal defects are given by

$$D = \exp\left(-h\int_{\mathbb{R}^p} d^p x_\parallel \hat{\phi}\right)\,, \tag{1.1}$$

where h describes a magnetic field along the defect[1] and hatted operator denote those localized to the defect. From a technical point of view, h can be treated as a coupling constant of finite size localized on the defect [5]. See [6–14] for recent development on these defects. The dimension of the defect is $p = 1$ (a line) if $d = 4 - \epsilon$, and $p = 2$ (a surface) if $d = 6 - \epsilon$. Both of these two models have their $O(N)$-symmetry explicitly broken by the scalar Wilson defect. See [15] for a similar defect in a fermionic QFT.

In [16], *fusion* of two scalar Wilson defects was studied in the four dimensional free theory. In the limit $R \to 0$ it was found that the two defects can be described by a single defect which does not preserve the conformal symmetry

$$D_f = \exp\left(-2\,h\sum_{n\geq 0}\frac{R^{2\,n}}{(2\,n)!}\int_{\mathbb{R}} dx_\parallel \partial_R^{2\,n}\hat{\phi}(x_\parallel)\right)\,. \tag{1.2}$$

One way to understand this statement is that the distance, R, between the two defects is a scale of the theory, and thus has to be preserved after the fusion. This scale then enters in the interactions on D_f, making them dimensionfull. In turn, this makes the fused defect action non-conformal.

Using the language of fusion categories [17–19], this is an example when there is only one fused defect (with the OPE coefficient being one)

$$D(-R)D(+R) = D_f(0)\,. \tag{1.3}$$

Defects have also been fused in two-dimensional conformal field theories (CFT's) [20–22], wherein the fusion categories have been found for different examples. Unlike all of these works, defects were fused in [16] without using super-, topological- or Virasoro-symmetry. A non-perturbative Casimir energy is often factored out in front of each coefficient in the fusion category (due to the normalization used for the correlators). We will consider a higher dimensional CFT, and see that this Casimir energy is reproduced by the expectation value, $\langle D_f\rangle$, (or self-energy) of the fused defect itself (including the dimensionfull couplings).

In this paper we study fusion of two scalar Wilson defects (1.1) in interacting theories, and find the *renormalization group* (RG) flow of the interactions, $R^{2\,n}h$, on D_f (1.2). As expected, the dimensionfull couplings will not have any perturbative *fixed points* (f.p.'s).

[1] This is seen from the equation of motion, where h will act as a source term along the defect.

This means that after we have fused the defects, we can turn on interactions in the bulk and find a f.p. for D_f where we have restored the conformal symmetry. At this point in the RG, D_f is on the same form as one of the original defects (1.1).

We will mostly consider a model with cubic bulk-interactions in $d = 6 - \epsilon$. In section 2 we study the one-point function of bulk fields in the presence of the two defects and find the RG flow for the defect couplings. This is a slight generalization of the corresponding results in [8], and we use the more traditional way of calculating Feynman diagrams [5] assuming the bulk interactions are small w.r.t. those on the defects. In particular, we find that the couplings on the two defects are not affected by each other, which is what we expect since their corresponding β-functions measure UV divergences in their respective defect-limit of bulk-local fields as well as UV divergences in the coincident-limit of defect-local fields on the corresponding defect.

In section 3 we improve the results from [16], which concerns fusion of scalar Wilson defects (1.1) in $d = 4$ free theories. In the free theory we generalize this result to hold for any d, and show that the Casimir energy in between the two defects is reproduced by the fused defect as well. Specifying to the real-valued f.p.'s of the defects in $d = 6 - \epsilon$, we compare the bare one-point function in the presence of $D(+R)$ and $D(-R)$ with that near D_f (upto second order in the bulk couplings). We find that they are exactly the same, and there are no modifications needed to D_f. The underlying reason for this is that the path integral for $D(\pm R)$ is the same as that for D_f (meaning that the theory with $D(\pm R)$ is the same as that with D_f). We check that this is indeed the case for line defects in $d = 4 - \epsilon$ with a quartic bulk-interaction as well.

The difference between the theory with two defects and that with the fused defect lies in renormalization of the theory. Diagrams with bulk vertices connecting the two defects have logarithmic divergences in the *fusion-limit* (as the distance between the defects goes to zero). Such divergences are absorbed in the bare coupling constants on the fused defect, giving us different renormalized correlators. So in addition to UV divergences in the coincident-limit of defect-local fields and in the defect-limit of bulk-local fields, the β-functions on the fused defect also take into account UV divergences in the fusion-limit of the two defects. That said, the RG f.p.'s stay the same (as expected).

2 Renormalization group fixed points

Let us first introduce the main model we consider. In the bulk we have

$$S = \int_{\mathbb{R}^d} d^d x \left(\frac{(\partial_\mu \phi^i)^2}{2} + \frac{(\partial_\mu \sigma)^2}{2} + \frac{g_1}{2} \sigma (\phi^i)^2 + \frac{g_2}{3!} \sigma^3 \right) , \tag{2.1}$$

where $d = 6 - \epsilon$ and $i \in \{1, \ldots, N\}$. The scalars ϕ^i are invariant under $O(N)$. We consider two parallel surface defects, $D_\pm$, of dimension $p = 2$, spanned along $\hat{x}^a_\parallel$, $a \in \{1, 2\}$. They are separated by a distance $2\,R$, $R \equiv |R_i|$, in the orthogonal directions $\hat{x}^i_\perp$, $i \in \{1, \ldots, d-p\}$

$$D_\pm = \exp\left(- \int_{\mathbb{R}^p} d^p x \left[h^\phi_\pm \hat{\phi}^{i_\pm}(x_\pm) + h^\sigma_\pm \hat{\sigma}(x_\pm) \right] \right) . \tag{2.2}$$

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Here $x_\pm \equiv x_a \hat{x}^a_\| \pm R_i \hat{x}^i_\perp$ and $h^\phi_\pm$, $h^\sigma_\pm$ are couplings (or magnetic fields) of finite size localized on the respective defects. Due to their $\phi^{i_\pm}$-interaction, the $O(N)$-symmetry of the model is broken down to $O(N-2)$ by the defects (in the case when $i_+ = i_-$ the symmetry is broken down to $O(N-1)$). This is an explicit symmetry breaking caused by the defect interactions, and thus differs from e.g. the extraordinary p.t. near a boundary (which is a spontaneous symmetry breaking [23–25]).

The effective action is given by

$$S_{\text{eff}} = S + \sum_\pm \log D_\pm \,. \tag{2.3}$$

Since the β-functions for the bulk couplings arise from divergences in the coincident-limit of the bulk fields, they are not affected by the defect couplings. This means that we can borrow these results from the bulk theory (2.1) without the defects [26, 27]

$$\begin{aligned}
\beta_1 &= -\frac{\epsilon}{2} g_1 + \frac{(N-8)g_1^3 - 12 g_1^2 g_2 + g_1 g_2^2}{12(4\pi)^3} + \mathcal{O}(g^4)\,, \\
\beta_2 &= -\frac{\epsilon}{2} g_2 - \frac{4\,N\,g_1^3 - N\,g_1^2 g_2 + 3\,g_2^3}{4(4\pi)^3} + \mathcal{O}(g^4)\,.
\end{aligned} \tag{2.4}$$

In the case when $N = 0$ and the ϕ^i-fields are not present, there is a negative sign in front of the g_2^3-term in β_2. Due to this we find no real-valued RG f.p. in this case.

By including the $O(N)$-scalars we can expand in large $N \gg 1$

$$\begin{aligned}
\beta_1 &= -\frac{\epsilon}{2} g_1 + \frac{N\,g_1^3}{12(4\pi)^3} + \mathcal{O}(g^4)\,, \\
\beta_2 &= -\frac{\epsilon}{2} g_2 - \frac{4\,N\,g_1^3 - N\,g_1^2 g_2}{4(4\pi)^3} + \mathcal{O}(g^4)\,.
\end{aligned} \tag{2.5}$$

Setting these β-functions to zero yields in addition to a Gaussian f.p., two non-trivial, real-valued f.p.'s[2]

$$g_2^* = 6\,g_1^* + \mathcal{O}\left(\epsilon, \frac{1}{N}\right)\,, \quad g_1^* = \pm\sqrt{\frac{6(4\pi)^3\epsilon}{N}} + \mathcal{O}\left(\epsilon, \frac{1}{N}\right)\,. \tag{2.6}$$

Note that these f.p.'s go as $\sqrt{\epsilon}$, which differ from the *Wilson-Fisher* (WF) f.p. in $d = 4 - \epsilon$. The RG flow is depicted in figure 1.

We will proceed with finding the f.p.'s of the defect-interactions in (2.2). The corresponding β-functions measure divergence in the respective near distance limits. This means that e.g. the β-functions on D_+ does not depend on the interactions on D_-. In turn this tells us that the defect β-functions are the same on the two defects, and it can be found from the theory with only one defect. We will calculate Feynman diagrams in the theory with both defects (considering small bulk-interactions) to show that this is indeed the case.

[2] In particular, this result is valid for $N \geq 1039$ [27].

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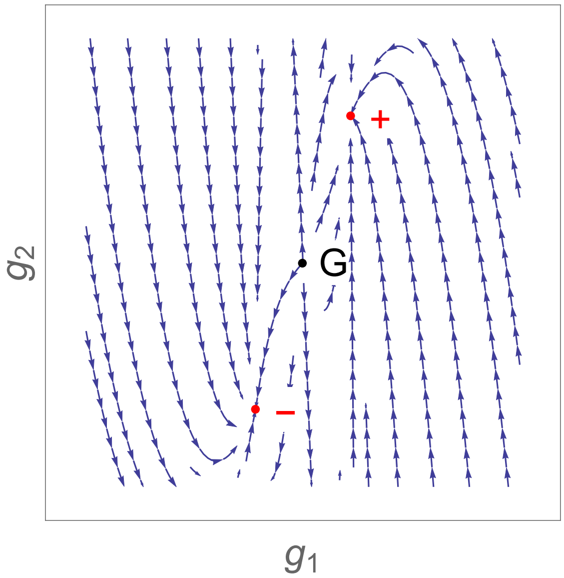

Figure 1. The RG flow of the cubic $O(N)$-model (2.1) at large N near six dimensions. The black dot (G) is the trivial Gaussian f.p. and the two red dots ($\pm$) are the attractive f.p.'s at (2.6).

2.1 Free theory

Correlators in the presence of the two defects (2.2) are found by expanding $D_\pm$ in its interactions and then applying Wick's theorem. This was done for a single insertion of a bulk field in [16]. In general, it gives us

$$\langle D_+ D_- \ldots\rangle = \langle D_+\rangle\langle D_-\rangle\langle D_+ D_-\rangle \left(\sum_{\pm}\langle D_\pm \ldots\rangle_N + \delta\langle D_+ D_- \ldots\rangle_N\right) . \tag{2.7}$$

Here the dots represent any combination of operators. $\langle D_\pm\rangle$ describes self-interactions on $D_\pm$ (their self-energies), and $\langle D_+ D_-\rangle$ is a non-perturbative (w.r.t. R) Casimir energy between the defects. See figure 2 for a diagrammatic representation of these correlators.

For the purposes of this section we are not interested in $\langle D_\pm\rangle$ and $\langle D_+ D_-\rangle$. Thus we normalize correlators in the following way[3]

$$\langle D_+ D_- \ldots\rangle_N \equiv \frac{\langle D_+ D_- \ldots\rangle}{\langle D_+\rangle\langle D_-\rangle\langle D_+ D_-\rangle} = \sum_{\pm}\langle D_\pm \ldots\rangle_N + \delta\langle D_+ D_- \ldots\rangle_N\,. \tag{2.8}$$

The remaining three correlators, $\langle D_\pm \ldots\rangle_N$ and $\delta\langle D_+ D_- \ldots\rangle_N$, can be found using standard Feynman diagrams techniques. $\langle D_\pm \ldots\rangle_N$ is the one-point function in the presence of the single defect $D_\pm$, and $\delta\langle D_+ D_- \ldots\rangle_N$ is the sum of Feynman diagrams connecting the two

[3]Likewise if we consider n defects, $\{D_i\}_{i=1}^n$, then we use the normalization

$$\langle D_1 \ldots D_n \ldots\rangle_N \equiv \frac{\langle D_1 \ldots D_n \ldots\rangle}{\langle D_1\rangle \ldots \langle D_n\rangle\langle D_1 \ldots D_n\rangle} = \sum_{i=1}^n \langle D_i \ldots\rangle_N + \delta\langle D_1 \ldots D_n \ldots\rangle_N\,,$$

where $\delta\langle D_1 \ldots D_n \ldots\rangle_N$ contains Feynman diagrams connecting two or more defects.

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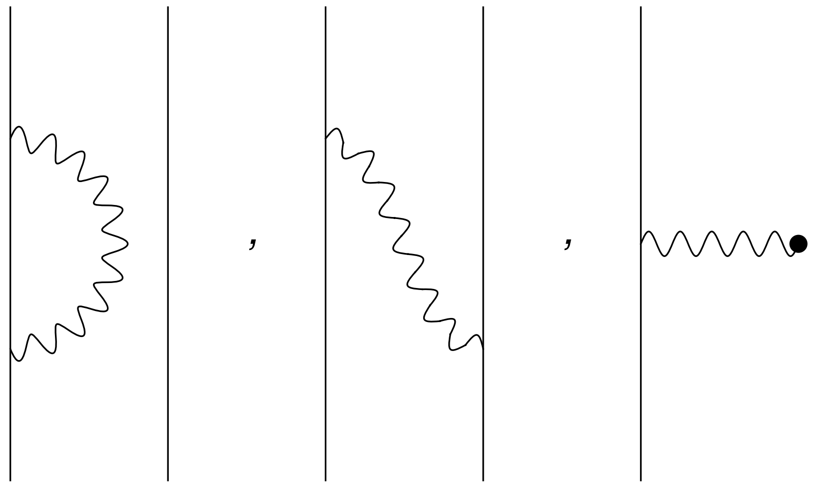

Figure 2. The three Feynman diagrams in the free theory. The one to the left correspond to the defect self-energy, $\log\langle D_-\rangle$, the middle to the Casimir energy, $\log\langle D_- D_+\rangle$, and the right to the normalized one-point function, $\langle D_-\phi^i(x)\rangle$. This figure is originally from [16].

defects. We will see examples of $\delta\langle D_+ D_- \dots\rangle_N$-diagrams later in when we take into account the bulk-interactions.

The one-point function of ϕ^i and σ in the presence of the two defects $D_\pm$ are given by the third Feynman diagram in figure 2

$$\begin{aligned} \langle D_+ D_-\phi^i(x)\rangle_N &= \sum_\pm \langle D_\pm \phi^i(x)\rangle_N\,, \qquad & \langle D_\pm\phi^i(x)\rangle_N &= -h^\phi_\pm \delta^{i i_\pm} K_\pm(x)\,, \\ \langle D_+ D_-\sigma(x)\rangle_N &= \sum_\pm \langle D_\pm \sigma(x)\rangle_N\,, \qquad & \langle D_\pm\sigma(x)\rangle_N &= -h^\sigma_\pm K_\pm(x)\,, \end{aligned} \tag{2.9}$$

where the integral $K_\pm$, which is the corresponding integral from [16], is given by

$$K_\pm(x) = \int_{\mathbb{R}^p} d^p z \langle \sigma(x)\sigma(z_\pm)\rangle\big|_{h^\phi_\pm, h^\sigma_\pm = 0}\,. \tag{2.10}$$

The integrand is the same as the connected part of $\langle D_+ D_- \sigma(x)\sigma(y)\rangle_N$. It is not affected by the defects interactions, and is thus the massless scalar correlator found from the Klein-Gordon equation

$$\langle \sigma(x)\sigma(y)\rangle\big|_{h^\phi_\pm, h^\sigma_\pm = 0} = \langle D_+ D_- \sigma(x)\sigma(y)\rangle^{\text{conn}}_N = \frac{A_d}{|x-y|^{2\Delta_\phi}}\,. \tag{2.11}$$

The constant A_d is given by

$$A_d = \frac{1}{(d-2)S_d}\,, \qquad S_d = \frac{2\,\Gamma_{\frac{d}{2}}}{\pi^{\frac{d}{2}}}\,. \tag{2.12}$$

Here S_d is the solid angle and $\Gamma_x \equiv \Gamma(x)$ is a shorthand notation for the Gamma function. The integrals $K_\pm$ are thus given by

$$K_\pm(x) = A_d I^p_{\Delta_\phi}(0, x_\parallel, x_\perp \mp R)\,, \tag{2.13}$$

which are written in terms of the master integral (given in terms of a modified Bessel function of the second kind)

$$
\begin{aligned}
I^n_\Delta(k,w,z^2) &= \int_{\mathbb{R}^n} d^n x \frac{e^{i\,k\,x}}{[(x-w)^2+z^2]^\Delta} \\
&= \frac{\pi^{\frac{n}{2}}}{2^{\Delta-\frac{n}{2}-1}\Gamma_\Delta} e^{i\,k\,w} \left(\frac{|k|}{|z|}\right)^{\Delta-\frac{n}{2}} K_{\Delta-\frac{n}{2}}(|k|\,z) \\
&= \begin{cases} \dfrac{\pi^{\frac{n}{2}}\Gamma_{\frac{n}{2}-\Delta}}{2^{2\Delta-n}\Gamma_\Delta} \dfrac{e^{i\,k\,w}}{|k|^{2\,\Delta-n}}\,, \text{ if } z=0\,, \\[2ex] \dfrac{\pi^{\frac{n}{2}}\Gamma_{\Delta-\frac{n}{2}}}{\Gamma_\Delta} \dfrac{1}{|z|^{2\,\Delta-n}}\,, \text{ if } k=0\,. \end{cases}
\end{aligned}
\tag{2.14}
$$

This yields

$$
K_\pm(x) = \frac{A_d \pi^{\frac{p}{2}} \Gamma_{\Delta_\phi-\frac{p}{2}}}{\Gamma_{\Delta_\phi}} \frac{1}{|x_\perp \mp R|^{2\,\Delta_\phi-p}}\,. \tag{2.15}
$$

In the interacting theory we will find it useful to Fourier transform w.r.t. the normal distances, $s^\pm_\perp \equiv x_\perp \mp R$, to the defects

$$
\begin{aligned}
\prod_{c=\pm} \int_{\mathbb{R}^{d-p}} ds^c_\perp e^{i\,k^c_\perp s^c_\perp} K_\pm(x) &= \frac{A_d \pi^{\frac{p}{2}} \Gamma_{\Delta_\phi-\frac{p}{2}}}{\Gamma_{\Delta_\phi}} \delta(k_\pm) I^{d-p}_{\Delta_\phi-\frac{p}{2}}(k^\mp_\perp,0,0) \\
&= \frac{\delta(k_\pm)}{k^2_\mp}\,, \quad \text{(exactly)}.
\end{aligned}
\tag{2.16}
$$

The momenta $k_\pm$ is that flowing between the bulk field and the defect $D_\pm$. It describes how momenta is being absorbed/emitted by the two defects. The Dirac δ-function tell us that the momenta is only affected by one of the defects in the free theory.

Note that the one-point functions (2.9) are the forms we expect a one-point function of a scalar to have from conformal symmetry [3]

$$
\mathcal{O}(x) = \frac{\mu^{\mathcal{O}}{}_{\mathbb{1}_\pm}}{|x_\perp|^\Delta}\,, \tag{2.17}
$$

from which we can read off the DOE coefficients[4]

$$
\mu^{\phi^i}{}_{\mathbb{1}_\pm} = -h^\phi_\pm \delta^{ii_\pm} \frac{\Gamma_{\Delta_\phi-1}}{4\,\pi^{\Delta_\phi}}\,, \qquad \mu^\sigma{}_{\mathbb{1}_\pm} = -h^\sigma_\pm \frac{\Gamma_{\Delta_\phi-1}}{4\,\pi^{\Delta_\phi}}\,, \tag{2.18}
$$

where the $\mathbb{1}_\pm$ subscript denotes the identity exchange on the respective defect.

2.2 Interacting theory

We will now proceed to the interacting theory, and find the β-functions of the defect couplings as well as the corresponding RG f.p.'s.

The one-point functions at $\mathcal{O}(g)$ are given by the two Feynman diagrams in figure 3. If a diagram contains n defect points of the same field, we have to divide the symmetry factor

[4]The defects are placed at the orthogonal coordinates $\pm R_i$, hence a shift in the denominator.

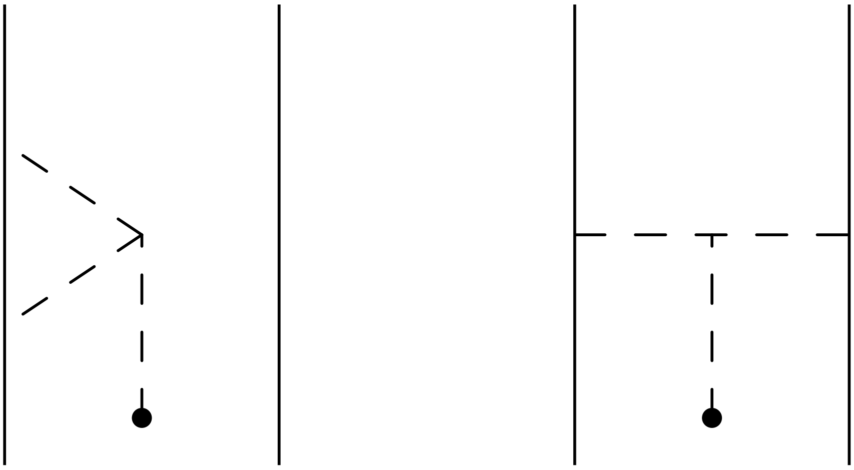

Figure 3. The two Feynman diagrams that contribute to the one-point functions of ϕ^i and σ. The dot is the external bulk point, the dotted lines are either $\phi - \phi$ or $\sigma - \sigma$ correlators and the solid lines are the two surface defects.

with a factor $n!$ to avoid overcounting (which is seen from the integration of the defect points). We leave the details of this calculation in appendix A.1, wherein we make use of the following master integral

$$
\begin{aligned}
J^n_{a,b}(z) &\equiv \int_{\mathbb{R}^n} \frac{d^n x}{|x|^{2\,a}|x-z|^{2\,b}} \\
&= \frac{\Gamma_{a+b}}{\Gamma_a \Gamma_b} \int_0^1 du (1-u)^{a-1} u^{b-1} \int_{\mathbb{R}^n} \frac{d^n x}{(x^2 + u(1-u) z^2)^{a+b}} \\
&= \frac{\pi^{\frac{n}{2}} \Gamma_{a+b-\frac{n}{2}} \Gamma_{\frac{n}{2}-a} \Gamma_{\frac{n}{2}-b}}{\Gamma_a \Gamma_b \Gamma_{n-a-b}} \frac{1}{|z|^{2(a+b)-n}} \,.
\end{aligned}
\tag{2.19}
$$

If we neglect finite constants, we find

$$
\begin{aligned}
\langle D_\pm \phi^i(k) \rangle^{(1)}_N &= -g_1 h^\phi_\pm h^\sigma_\pm \frac{\delta^{ii_\pm} \delta(k^\pm_\perp)}{8\,\pi^2 (k^\mp_\perp)^2} \left(\frac{1}{\epsilon} - \log |k^\mp_\perp| \right) , \\
\delta \langle D_+ D_- \phi^i(k) \rangle^{(1)}_N &= -g_1 \frac{h^\phi_+ h^\sigma_- \delta^{ii_+} + h^\phi_- h^\sigma_+ \delta^{ii_-}}{(k^+_\perp)^2 (k^-_\perp)^2 (k^+_\perp + k^-_\perp)^2} \,,
\end{aligned}
\tag{2.20}
$$

and

$$
\begin{aligned}
\langle D_\pm \sigma(x) \rangle^{(1)}_N &= -\frac{g_1 h^\phi_\pm h^\phi_\pm \delta + g_2 h^\sigma_\pm h^\sigma_\pm}{2} \frac{\delta(k^\pm_\perp)}{16\,\pi^2 (k^\mp_\perp)^2} \left(\frac{1}{\epsilon} - \log |k^\mp_\perp| \right) , \\
\delta \langle D_+ D_- \sigma(x) \rangle^{(1)}_N &= -\frac{g_1 h^\phi_+ h^\phi_- \delta^{i_+ i_-} + g_2 h^\sigma_+ h^\sigma_-}{(k^+_\perp)^2 (k^-_\perp)^2 (k^+_\perp + k^-_\perp)^2} \,.
\end{aligned}
\tag{2.21}
$$

Note that the Feynman diagram $\delta\langle D_+ D_- \phi^i(k_\pm)\rangle^{(1)}_N$, connecting the two defects is convergent. So only the diagrams $\langle D_\pm \phi^i(k_\pm)\rangle^{(1)}_N$, which are affected by one of the defects, are divergent. This means that in the renormalization procedure, the couplings on D_+ are not affected by those on D_- (and vice versa). This is the expected result since the bare couplings on D_+ should capture UV divergences in the coincident-limit of defect-local fields in addition to divergences in the limit as bulk-local fields approach D_+.[5]

The corresponding β-functions are also found in appendix A.1

$$\beta^{\phi}_{\pm} = -\frac{\epsilon}{2}\tilde{h}^{\phi}_{\pm} - \frac{\tilde{h}^{\phi}_{\pm}\tilde{h}^{\sigma}_{\pm}\tilde{g}_1}{8\,\pi^2}\,, \qquad \beta^{\sigma}_{\pm} \;= -\frac{\epsilon}{2}\tilde{h}^{\sigma}_{\pm} - \frac{(\tilde{h}^{\phi}_{\pm})^2\tilde{g}_1}{16\,\pi^2} - \frac{(\tilde{h}^{\sigma}_{\pm})^2\tilde{g}_2}{16\,\pi^2}\,. \tag{2.22}$$

Setting these to zero gives us a Gaussian f.p. where both defect couplings are zero.[6] We also find the following non-trivial ones

$$\left((h^{\phi}_{\pm})^*, (h^{\sigma}_{\pm})^*\right) \in \left\{\left(0, -\frac{8\,\pi^2\epsilon}{g_2^*}\right), \left(\pm 4\,\pi^2\epsilon\frac{\sqrt{2\,g_1^* - g_2^*}}{(g_1^*)^{\frac{3}{2}}}, -\frac{4\,\pi^2\epsilon}{g_1^*}\right)\right\}\,. \tag{2.23}$$

The first one is the same as that found in [8]. The bulk couplings are tuned to their respective f.p.'s (2.6), where we find four complex f.p.'s

$$\left((h^{\phi}_{\pm})^*, (h^{\sigma}_{\pm})^*\right) = \left(\pm i\sqrt{\frac{\pi\,N\,\epsilon}{6}}, \pm\frac{1}{2}\sqrt{\frac{\pi\,N\,\epsilon}{6}}\right)\,, \tag{2.24}$$

and two real-valued f.p.'s where only $h^{\sigma}_{\pm}$ is non-trivial

$$\left((h^{\phi}_{\pm})^*, (h^{\sigma}_{\pm})^*\right) = (0, h^*)\,, \qquad h^* = \mp\frac{1}{6}\sqrt{\frac{\pi\,N\,\epsilon}{6}}\,. \tag{2.25}$$

The sign of h^* is opposite to the bulk-couplings at their f.p. (2.6). If we restrict ourselves to real-valued f.p.'s then the $\phi^{i_\pm}$-term on the defects (2.2) vanish

$$D_{\pm} = \exp\left(-h^*\int_{\mathbb{R}^p} d^p x\,\hat{\sigma}(x_{\pm})\right)\,. \tag{2.26}$$

Note that since $N \gg 1$, none of the f.p.'s (2.24), (2.25) have to be small.

By studying the derivative of $\beta^{\sigma}_{\pm}$ we can check whether the real-valued f.p. is attractive or not

$$\partial_{\tilde{h}^{\sigma}_{\pm}}\beta^{\sigma}_{\pm}\Big|_{\tilde{h}^{\phi}_{\pm}=0,\ \tilde{h}^{\sigma}_{\pm}=h^*} = \frac{\epsilon}{2}\,, \tag{2.27}$$

which does not depend on the sign of h^* at (2.25). Since this is positive, the f.p.'s at (2.25) are minima of the defect σ-coupling and are thus attractive.

The one-point functions of ϕ^i are trivial at this f.p. (restoring $O(N)$-symmetry), while those for σ can be resummed in ϵ

$$\langle D_+ D_- \sigma(k_\perp^{\pm})\rangle_N = \sum_{a=\pm}\langle D_a\sigma(k_\perp^{\pm})\rangle_N + \delta\langle D_+ D_- \sigma(k_\perp^{\pm})\rangle_N + \mathcal{O}(g^2)\,, \tag{2.28}$$

[5]At higher orders in the bulk couplings, there are divergent diagrams in $\delta\langle D_+D_-\phi^i\rangle$. However, the divergences in these diagrams are taken care of by renormalization of local quantities at lower orders in the bulk couplings. One such example is the first diagram in figure 7.

[6]The defect couplings can be zero while those in the bulk (2.6) are not.

$$
\begin{aligned}
\langle D_\pm \sigma(k_\perp^\pm)\rangle_N &= -h^* \frac{\delta(k_\perp^\pm)}{(k_\perp^\mp)^{2-\frac{\epsilon}{2}}}\,,\\
\delta\langle D_+ D_- \sigma(k_\perp^\pm)\rangle_N &= -\frac{(h^*)^2 g_2^*}{(k_\perp^+)^2(k_\perp^-)^2(k_\perp^+ + k_\perp^-)^2}\,.
\end{aligned}
\tag{2.29}
$$

Note that the RG scale has completely vanished at the f.p. We have

$$
(h^*)^2 g_2^* = \pm \frac{\pi^{\frac{5}{2}}}{\sqrt{N}}\left(\frac{2\,\epsilon}{3}\right)^{\frac{3}{2}}\,,
\tag{2.30}
$$

which is at a subleading order in N. This means that $\delta\langle D_+ D_- \sigma(k_\perp^\pm)\rangle_N$ is small compared to $\langle D_\pm \sigma(k_\perp^\pm)\rangle_N$.

In Euclidean space we find

$$
\begin{aligned}
\langle D_\pm \sigma(x)\rangle_N &= \prod_{c=\pm}\int_{\mathbb{R}^{d-p}} \frac{d^{d-p}k_\perp^c}{(2\,\pi)^{d-p}} e^{-i\,k_\perp^c s_\perp^c}\langle D_\pm \sigma(k_\perp^\pm)\rangle_N\\
&= -\frac{h^*}{(2\pi)^{d-p}} I^{d-p}_{1-\frac{\epsilon}{4}}(-s_\perp^\pm,0,0) + \mathcal{O}(\epsilon^{\frac{3}{2}}) = -\frac{h^*}{(2\pi)^{2-\frac{\epsilon}{2}}|s_\perp^\pm|^{2-\frac{\epsilon}{2}}}\,,
\end{aligned}
\tag{2.31}
$$

which agrees with the free theory result at $\mathcal{O}(\sqrt{\epsilon})$, and has the correct scaling dimension of σ. From this we can also read off the DOE coefficients

$$
\mu^{\sigma}{}_{\mathbb{1}_\pm} = -\frac{h^*}{(2\pi)^{2-\frac{\epsilon}{2}}} + \mathcal{O}(g^2)\,.
\tag{2.32}
$$

2.3 Order g^2

Before we study fusion, let us calculate the Feynman diagrams at $\mathcal{O}(g^2)$, and find the defect f.p. upto $\mathcal{O}(\epsilon^{\frac{3}{2}})$. To do this we study the one-point function of σ in the presence of one defect (2.26) with only a σ-interaction. The Feynman diagrams at $\mathcal{O}(g^2)$ is in figure 4, and are calculated in appendix A.2. We find the full one-point function upto $\mathcal{O}(g^2)$ to be given by

$$
\begin{aligned}
\langle \sigma(k)\rangle = -\frac{h}{k_\perp^2}\Bigg(1 &+ \frac{g_2 h \csc\left(\frac{\pi\,\epsilon}{2}\right)}{4^{3-\epsilon}\pi^{\frac{1-\epsilon}{2}}\Gamma_{\frac{3-\epsilon}{2}}|k_\perp|^\epsilon} - \frac{(N g_1^2 + g_2^2)\csc\left(\frac{\pi\,\epsilon}{2}\right)}{4^{5-\epsilon}\pi^{\frac{3-\epsilon}{2}}\Gamma_{\frac{5-\epsilon}{2}}|k_\perp|^\epsilon} +\\
&+ \frac{g_2^2 h^2 \epsilon\csc(\pi\,\epsilon)\Gamma^2_{-\frac{\epsilon}{2}}}{2^{11-3\,\epsilon}\pi^{\frac{5}{2}-\epsilon}\Gamma_{2-\frac{3\,\epsilon}{2}}\Gamma_{\frac{3-\epsilon}{2}}|k_\perp|^\epsilon} + \mathcal{O}(g^3)\Bigg)\,.
\end{aligned}
\tag{2.33}
$$

At $\mathcal{O}(g^2)$ the field σ receives an anomalous dimension [27]. Thus we also have to introduce a Z-factor for this field when we find the bare defect coupling. Details on this calculation are again in appendix A.2

$$
h = \mu^{\frac{\epsilon}{2}}\tilde{h}\left(1 - \frac{\tilde{g}_2\tilde{h}}{16\,\pi^2\epsilon} + \tilde{g}_2^2\tilde{h}^2\left(\frac{1}{256\,\pi^4\epsilon^2} + \frac{1}{512\,\pi^4\epsilon}\right) + \mathcal{O}(g^3)\right)\,,
\tag{2.34}
$$

and the one-point function (neglecting finite constants)

$$
\begin{aligned}
\langle \tilde{\sigma}(k)\rangle = -\frac{\tilde{h}}{k_\perp^2}\Bigg(1 &- \frac{\tilde{g}_2\tilde{h}}{32\,\pi^2}\log\left(\frac{k_\perp^2}{\mu^2}\right) - \frac{N\,\tilde{g}_1^2 + \tilde{g}_2^2}{768,\pi^3}\log\left(\frac{k_\perp^2}{\mu^2}\right) +\\
&+ \frac{\tilde{g}_2^2\tilde{h}^2}{1024\,\pi^4}\log\left(\frac{k_\perp^2}{\mu^2}\right)^2 + \mathcal{O}(g^3)\Bigg)\,.
\end{aligned}
\tag{2.35}
$$

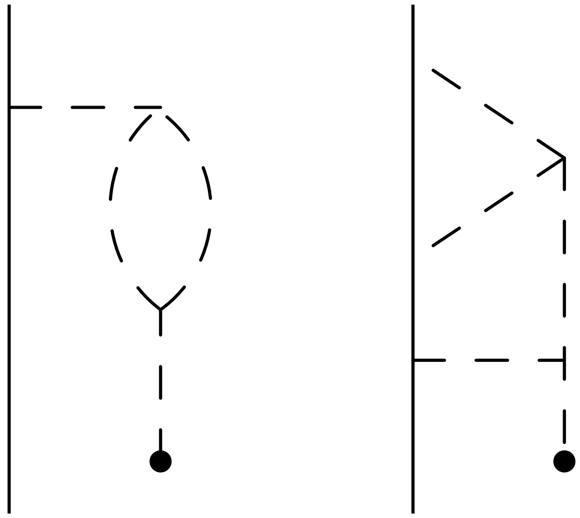

Figure 4. The two Feynman diagrams at $\mathcal{O}(g^2)$ in $\langle D_+\sigma\rangle$. In the bulk-loop of the first diagram there are either ϕ- or σ-internal fields.

As another sanity check we find that all logarithms are dimensionless.

Finally, from (2.34) we find the β-function for the defect coupling

$$\beta_h = -\frac{\tilde{h}\,\epsilon}{2} - \frac{\tilde{g}_2\tilde{h}^2}{16\,\pi^2} - \frac{\tilde{g}_2^2\tilde{h}^3}{256\,\pi^4} + \mathcal{O}(g^3)\,, \tag{2.36}$$

which has the perturbative f.p.

$$h^* = -8\,\pi^2\left(\frac{1}{\tilde{g}_2} \mp \sqrt{\frac{1-2\,\epsilon}{\tilde{g}_2^2}}\right)\,. \tag{2.37}$$

The sign in front of the square root is opposite to that in the bulk f.p. (2.6), which we write out here again to higher orders of ϵ and N^{-1} [27]

$$\begin{aligned} g_1^* = \pm\sqrt{\frac{6(4\,\pi)^3\epsilon}{N}}\Bigg[&1+\frac{22}{N}+\frac{726}{N^2}-\frac{326,180}{N^3}-\frac{349,658,330}{N^4}+\mathcal{O}\left(\frac{1}{N^5}\right)+\\ &-\frac{\epsilon}{N}\left[\frac{155}{6}+\frac{1,705}{N}-\frac{912,545}{N^2}-\frac{3,590,574,890}{N^3}+\mathcal{O}\left(\frac{1}{N^4}\right)\right]+\mathcal{O}\left(\frac{\epsilon^3}{N^3}\right)\Bigg]\,, \end{aligned}$$

$$\begin{aligned} g_2^* = \pm\sqrt{\frac{6(4\,\pi)^3\epsilon}{N}}\Bigg[&6\left[1+\frac{162}{N}+\frac{68,760}{N^2}+\frac{41,224,420}{N^3}+\frac{28,762,554,870}{N^4}+\mathcal{O}\left(\frac{1}{N^5}\right)\right]+\\ &-\frac{\epsilon}{N}\left[\frac{215}{2}+\frac{86,335}{N}-\frac{75,722,265}{N^2}-\frac{69,633,402,510}{N^3}+\mathcal{O}\left(\frac{1}{N^4}\right)\right]+\mathcal{O}\left(\frac{\epsilon^3}{N^3}\right)\Bigg]\,. \end{aligned}$$

If we now expand the defect f.p. (2.37) in small ϵ and large N we find the defect f.p.

$$\begin{aligned} h^* = \mp\frac{1}{6}\sqrt{\frac{\pi\,N\,\epsilon}{6}}\Bigg(&1-\frac{162}{N}-\frac{45,522}{N^2}-\frac{23,195,764}{N^3}-\frac{15,402,417,210}{N^4}+\mathcal{O}\left(\frac{1}{N^5}\right)+\\ &+\epsilon\left[\frac{1}{2}-\frac{757}{2\,N}-\frac{76,061}{N^2}-\frac{9,386,189}{2\,N^3}-\frac{4,845,204,490}{N^4}+\mathcal{O}\left(\frac{1}{N^5}\right)\right]+\mathcal{O}(\epsilon^3)\Bigg)\,. \end{aligned}$$

3 Fusion

Let us now fuse the two defects (2.26). This can be done by Taylor expanding D_+ and D_- w.r.t. each component of R_i (remember that the two defects are placed at $\pm R_i$ along the orthogonal coordinates)

$$D_\pm = \exp\left(-h \prod_{i=1}^{d-p} \sum_{n_i \geq 0} \frac{(\pm 1)^{n_i} R_i^{n_i}}{n_i!} \lim_{R_i \to 0} \partial_i^{n_i} \int_{\mathbb{R}^p} d^p x \, \sigma(x_+)\right) . \tag{3.1}$$

Adding the exponents gives us the fused defect

$$D_f = D_+ D_- = \exp\left(-2\, h \prod_{i=1}^{d-p} \sum_{n_i \geq 0} \frac{R_i^{2\, n_i}}{(2\, n_i)!} \int_{\mathbb{R}^p} d^p x \, \partial_i^{2\, n_i} \hat{\sigma}(x_+)\right) . \tag{3.2}$$

This is the multivariate version of the result in [16]. Since this is just a Taylor expansion, we find the path integral, which generates all of the correlators, to be the same for D_+D_- as for D_f (see section 3 of [16] for a proof on this). Note that the entire tower of terms w.r.t. R_i has to be kept to find the same path integral. R_i should be treated as a distance scale of the theory, and thus we keep it even after fusion of the two defects.

Let us also mention that two straight parallel lines are conformally equivalent to two concentric circles. This means that above fusion is also true for two concentric circular Wilson lines. Although not commented upon, this was seen in [16]. I.e. its eq.'s (2.21) and (5.4) are the same.

Since the path integral is the same for D_+D_- and D_f, we expect the fusion (3.2) to hold even in an interacting theory. In the rest of this paper we will perform several consistency checks to see that this indeed the case. Firstly, we will show that the defect correlators without any field insertions (the normalization factors in (2.8)) are the same upto $\mathcal{O}(g)$: $\langle D_+\rangle\langle D_-\rangle\langle D_+D_-\rangle = \langle D_f\rangle$. Then we will show that the expansion of $\langle D_+D_-\sigma\rangle_N$ in R is the same as $\langle D_f\sigma\rangle_N$ upto $\mathcal{O}(g^2)$ (before renormalization of the couplings).

To simplify the calculations, we will choose a coordinate system s.t. R is one-dimensional. In addition, we let the normal coordinate of the external field (when we study one-point functions) be one-dimensional as well

$$R^i = R\, \delta^{i1} , \quad R > 0 , \quad x_\perp^i = x_\perp \, \delta^{i1} . \tag{3.3}$$

3.1 Normalization factor

We will start by calculating the normalization factors. Note that in [16] (where we divide $\langle D_+D_- \ldots\rangle$ with only $\langle D_-\rangle\langle D_+\rangle$), we used a normalization different from (2.8). Thus in [16] it looks like the fusion only holds upto the Casimir energy, $\langle D_+D_-\rangle$. In this section we show that the normalization factors in (2.8) are in fact exactly reproduced by the self-energy, $\langle D_f\rangle$, of the fused defect if we include the entire tower of dimensionfull couplings (upto first order in the bulk coupling, g).

In the free theory, the logarithm of the correlators of interest are given by the two first Feynman diagrams in figure 2

$$\log\langle D_\pm\rangle = \frac{h^2}{2} \int_{\mathbb{R}^p} d^p x \int_{\mathbb{R}^p} d^p y \, \langle \sigma(x_\pm)\sigma(y_\pm)\rangle|_{h=0} = \frac{A_d h^2}{2} \mathrm{vol}(\mathbb{R}^p) \int_{\mathbb{R}^p} \frac{d^p x}{(x^2)^{\Delta_\phi}} , \tag{3.4}$$

$$
\begin{aligned}
\log\langle D_+ D_-\rangle &= h^2 \int_{\mathbb{R}^p} d^p x \int_{\mathbb{R}^p} d^p y \, \langle \sigma(x_+)\sigma(y_-)\rangle|_{h=0} \\
&= A_d h^2 \mathrm{vol}(\mathbb{R}^p) \int_{\mathbb{R}^p} \frac{d^p x}{(x^2 + 4\,R^2)^{\Delta_\phi}} \\
&= A_d h^2 \mathrm{vol}(\mathbb{R}^p) \sum_{n\geq 0} \binom{-\Delta_\phi}{n} (2\,R)^{2\,n} \int_{\mathbb{R}^p} \frac{d^p x}{(x^2)^{\Delta_\phi + n}} \,,
\end{aligned}
\tag{3.5}
$$

where we do not perform the last (divergent) integral over x. Together they give

$$
\begin{aligned}
\log(\langle D_+\rangle\langle D_-\rangle\langle D_+ D_-\rangle) = A_d h^2 \mathrm{vol}(\mathbb{R}^p) \Bigg(2 \int_{\mathbb{R}^p} \frac{d^p x}{(x^2)^{\Delta_\phi}} + \\
+ \sum_{n\geq 1} \binom{-\Delta_\phi}{n} (2\,R)^{2\,n} \int_{\mathbb{R}^p} \frac{d^p x}{(x^2)^{\Delta_\phi + n}} \Bigg) \,.
\end{aligned}
\tag{3.6}
$$

For D_f we have a single diagram similar to the first one in figure 2 (without the other defect)

$$
\begin{aligned}
\log\langle D_f\rangle &= \frac{(2\,h)^2}{2!} \sum_{m_1,m_2\geq 0} \frac{R^{2(m_1+m_2)}}{(2\,m_1)!(2\,m_2)!} \int_{\mathbb{R}^p} d^p x_\parallel \int_{\mathbb{R}^p} d^p y_\parallel \times \\
&\quad \times \lim_{R',R''\to 0} \partial_{R'}^{2\,m_1} \partial_{R''}^{2\,m_1} \, \langle \sigma(x\hat{x}_\parallel + R'\hat{x}_\perp^1)\sigma(y\hat{x}_\parallel + R''\hat{x}_\perp^1)\rangle\Big|_{h=0} \\
&= 2\,A_d h^2 \mathrm{vol}(\mathbb{R}^p) \sum_{n\geq 0} \sum_{m=0}^{n} \frac{R^{2n}}{(2\,m)!(2\,n-2\,m)!} \int_{\mathbb{R}^p} d^p x_\parallel \times \\
&\quad \times \lim_{R',R''\to 0} \partial_{R'}^{2\,m} \partial_{R''}^{2(n-m)} \frac{1}{(x^2 + (R'-R'')^2)^{\Delta_\phi + n}} \,.
\end{aligned}
\tag{3.7}
$$

This is exactly $\log(\langle D_+\rangle\langle D_-\rangle\langle D_+ D_-\rangle)$.

Let us now turn on the interactions and study these correlators at $\mathcal{O}(g)$. For $D_\pm$ we have the two Feynman diagrams in figure 5

$$
\begin{aligned}
\log\langle D_\pm\rangle^{(1)} &= \frac{3!}{3!} \left(-\frac{g_2}{3!}\right) (-h)^3 A_\pm^\pm \,, \\
\log\langle D_+ D_-\rangle^{(1)} &= \frac{3!}{2!} \left(-\frac{g_2}{3!}\right) (-h)^3 \left(A_-^+ + A_+^-\right) \,,
\end{aligned}
\tag{3.8}
$$

which gives us the full normalization factor

$$
\log(\langle D_+\rangle\langle D_-\rangle\langle D_+ D_-\rangle)^{(1)} = \frac{g_2 h^3}{2} \left(\frac{A_+^+ + A_-^-}{3!} + A_-^+ + A_+^- \right) \,.
\tag{3.9}
$$

This is given in terms of the following integral

$$
\begin{aligned}
A_b^a &= \int_{\mathbb{R}^d} d^d z K_a(z)^2 K_b(z) \\
&= \frac{A_d^3 \pi^{\frac{3\,p}{2}} \mathrm{vol}(\mathbb{R}^p) \Gamma^3_{\Delta_\phi - \frac{p}{2}}}{\Gamma^3_{\Delta_\phi}} \int_{\mathbb{R}^{d-p}} \frac{d^{d-p} z_\perp}{|z_\perp - a\,R|^{2\,\Delta_\phi - p} |z_\perp - b\,R|^{\Delta_\phi - \frac{p}{2}}} \,.
\end{aligned}
\tag{3.10}
$$

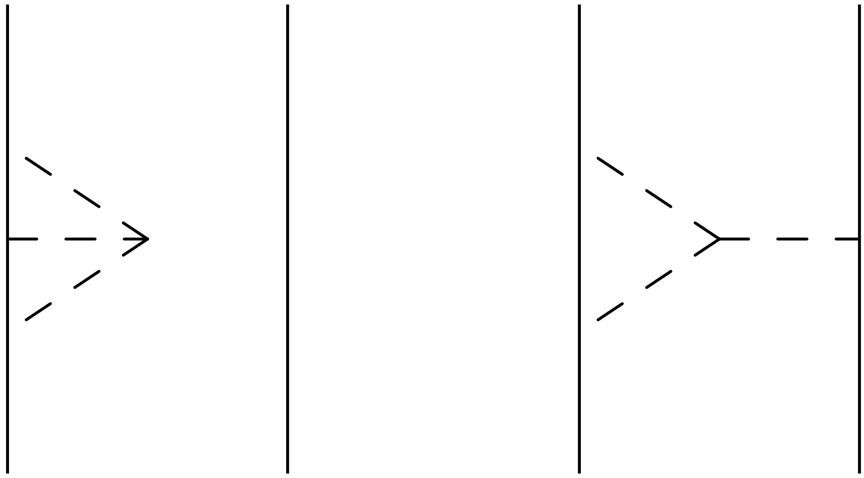

Figure 5. The two Feynman diagrams that contribute to the normalization factor at $\mathcal{O}(g)$.

Here we only integrated over the parallel part of the vertex ($z_{\|} \in \mathbb{R}^p$). The normalization of D_f is given by a diagram similar to the first one in figure 5

$$\begin{aligned}\log\langle D_f\rangle^{(1)} = \frac{3!}{3!}\left(-\frac{g_2}{3!}\right)(-2\,h)^3 \int_{\mathbb{R}^d} d^d z \prod_{i=1}^{3} \sum_{m_i \geq 0} \frac{R^{2\,m_i}}{(2\,m_i)!} \int_{\mathbb{R}^p} d^p x_i \times \\ \times \lim_{R_i \to 0} \partial_{R_i}^{2\,m_i} \left.\langle \sigma(z)\sigma(x_i \hat{x}_{\|} + R_i \hat{x}_{\perp}^1)\rangle\right|_{h=0} .\end{aligned} \tag{3.11}$$

Performing the integration over the parallel coordinates, and differentiating

$$\lim_{R' \to 0} \partial_{R'}^{2\,m}[(a-R')^2]^{-\Delta_\phi + \frac{p}{2}} = (2\,m)!\, b_m |a|^{-2\,\Delta_\phi + p - 2\,m}\,, \quad b_m \equiv \frac{(2\,\Delta_\phi + p)_{2\,m}}{(2\,m)!}\,, \tag{3.12}$$

gives us

$$\log\langle D_f\rangle^{(1)} = 4\,A_d^3 \pi^{\frac{3\,p}{2}} g_2 h^3 \mathrm{vol}(\mathbb{R}^p) \frac{\Gamma^3_{\Delta_\phi - \frac{p}{2}}}{\Gamma^3_{\Delta_\phi}} \sum_{n \geq 0} c_n R^{2\,n} \int_{\mathbb{R}^{d-p}} \frac{d^{d-p} z_\perp}{|z_\perp|^{3(2\,\Delta_\phi - p) + 2\,n}}\,. \tag{3.13}$$

This is expressed in terms of the constant

$$\begin{aligned} c_n &= \sum_{m=0}^{n} \sum_{m'=0}^{m} b_{n-m} b_{m-m'} b_{m'} \\ &= \sum_{m=0}^{n} \frac{\Gamma_{d-4+2\,m}\Gamma_{d-4+2(n-m)}}{\Gamma^2_{d-4}\Gamma_{2\,m+1}\Gamma_{2(n-m)+1}} \times \\ &\quad \times {}_4F_3\left(\frac{d-2}{2}, \frac{d-3}{2}, \frac{1}{2} - m, -m; \frac{1}{2}, \frac{5-d}{2} - m, 3 - \frac{d}{2} - m; 1\right) . \end{aligned} \tag{3.14}$$

By expanding (3.9) in R we find perfect agreement with $\log\langle D_f\rangle^{(1)}$. Thus we have shown that the normalization factor, including the self and Casimir energy, is exactly the same upto $\mathcal{O}(g)$ (as expected)

$$\langle D_f\rangle = \langle D_+\rangle\langle D_-\rangle\langle D_+ D_-\rangle\,. \tag{3.15}$$

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The Casimir energy (3.8), (3.10) between the two defects $D_\pm$ can be calculated using the master integral (2.19). It contains a pole in ϵ, which is taken care of by the bare coupling constants in section 2.3. At leading order, the renormalized Casimir energy is given by

$$\log\langle D_+ D_-\rangle = \frac{\mathrm{vol}(\mathbb{R}^2)}{(4\,\pi\, R)^2}\tilde{h}^2 + \mathcal{O}(\epsilon)\,, \tag{3.16}$$

which is proportional to R^{-2} as expected from conformal symmetry.

3.2 One-point function

Let us now check that fusion also holds for the one-point function of σ. In the free theory we have

$$\begin{aligned} \langle D_f \sigma(x)\rangle_N^{(0)} &= -2\, A_d \pi^{\frac{p}{2}} h^2 g_2 \frac{\Gamma_{\Delta_\phi - \frac{p}{2}}}{\Gamma_{\Delta_\phi}} \sum_{n\geq 0} \frac{(2\,\Delta_\phi - p)_{2\,n}}{(2\,n)!} \frac{R^{2\,n}}{x_\perp^{2\,\Delta_\phi - p + 2\,n}} \\ &= -\frac{h^2 g_2}{2\,\pi^2} \sum_{n\geq 1} (2\,n+3) \frac{R^{2\,n}}{x_\perp^{2(n+1)}} + \mathcal{O}(\epsilon) = \langle D_+ D_- \sigma(x)\rangle_N^{(0)}\,, \end{aligned} \tag{3.17}$$

which is in perfect agreement with $\langle D_+ D_- \sigma(x)\rangle_N^{(0)}$ in (2.9).

At $\mathcal{O}(g)$ we need the full $\langle D_+ D_- \sigma\rangle_N^{(1)}$ in Euclidean space (A.2). We already have $L_\pm^\pm$ (A.4), and are thus left to find

$$\delta\langle D_+ D_- \sigma(x)\rangle_N^{(1)} = -(h^*)^2 g_2^* L_-^+\,. \tag{3.18}$$

We know from its Fourier transform (A.7) that L_-^+ is free of UV divergences. So we are free to set $\epsilon = 0$ before integration over $z_\perp$ in (A.3)

$$L_-^+ = \int_{\mathbb{R}^4} \frac{d^4 z_\perp}{64\,\pi^6} \frac{1}{z_\perp^2 (z_\perp + s_\perp^+)^2 (z_\perp + s_\perp^-)^2}\,. \tag{3.19}$$

This integral has been done in the amplitude literature [28]. Its a rather lengthy expression for general R, but by specifying to one dimensional $x_\perp$ and R (3.3) it simplifies to[7]

$$L_-^+ = \frac{1}{64\,\pi^4 R}\left(\frac{1}{s_+}\log\left(\frac{|s_-|}{2\,R}\right) + \frac{1}{s_-}\log\left(\frac{|s_+|}{2\,R}\right)\right)\,. \tag{3.20}$$

The full $\langle D_+ D_- \sigma\rangle_N^{(1)}$ is thus

$$\langle D_+ D_- \sigma(k_\perp^\pm)\rangle_N = -\frac{h^*}{4\,\pi^2}\sum_{a=\pm}\left[\frac{1}{(s_\perp^a)^2} + \frac{h^* g_2^*}{8\,\pi^2}\left(\frac{\log|s_\perp^a|}{(s_\perp^a)^2} + \frac{1}{2\,R|s_\perp^{-a}|}\log\left|\frac{s_\perp^a}{2\,R}\right|\right)\right]\,.$$

In the expansion of L_-^+ in R we find a logarithmic divergence

$$\delta\langle D_+ D_- \sigma(x)\rangle_N^{(1)} \ni -\frac{(h^*)^2 g_2^*}{32\,\pi^4}\log(R)\sum_{n\geq 0}\frac{R^{2\,n}}{x_\perp^{2(n+1)}}\,. \tag{3.21}$$

[7]This integral can also be done using Feynman parametrization. Then the integrals over the Feynman parameters simplify greatly in the case of (3.3).

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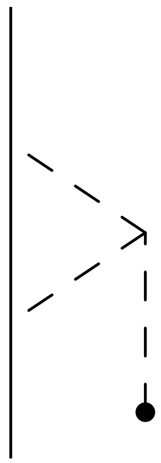

Figure 6. The single Feynman diagram at $\mathcal{O}(g_2)$ in $\langle D_f\sigma(x)\rangle$.

Note that this is not an IR divergence since R is a distance scale. Still it should not be absorbed in the bare couplings on $D_\pm$.

To avoid this logarithmic divergence we instead expand the integrands (A.3) of L^a_b in R before we integrate over $z_\perp$. In this way we capture the logarithmic divergence in R as a pole in ϵ

$$\langle D_+D_-\sigma(x)\rangle^{(1)}_N = -h^2 g_2 A_d^3 \pi^{\frac{3p}{2}} \frac{\Gamma^3_{\Delta_\phi-\frac{p}{2}}}{\Gamma^3_{\Delta_\phi}} \sum_{n\geq 0} a_n R^{2n} J^{d-p}_{\Delta_\phi-\frac{p}{2},2\Delta_\phi-p+n}(-x_\perp,0)\,, \tag{3.22}$$

where $J^n_{a,b}(z,w^2)$ is the master integral (2.19), and a_n is the constant

$$a_n = \frac{(2\Delta_\phi-p)_n}{n!} + \frac{(4\Delta_\phi-2p)_{2n}}{(2n)!} = \frac{2(n+1)(2n^2+4n+3)}{3} + \mathcal{O}(\epsilon)\,. \tag{3.23}$$

We will now compute $\langle D_f\sigma(x)\rangle^{(1)}_N$ and see that it exactly equals (3.22). For one-dimensional R (3.3), it is given by

$$D_f = \exp\left(-2h\sum_{n\geq 0}\frac{R^{2n}}{(2n)!}\int_{\mathbb{R}^p} d^p x \partial_R^{2n}\hat{\sigma}(x_+)\right)\,. \tag{3.24}$$

$\langle D_f\sigma(x)\rangle^{(1)}_N$ is found from the single Feynman diagram in figure 6

$$\begin{aligned}\langle D_f\sigma(x)\rangle^{(1)}_N = -\frac{(2h)^2 g_2}{2}\int_{\mathbb{R}^d} d^d z\, \langle\sigma(x)\sigma(z)\rangle|_{h=0}\times \\ \times\prod_{i=1}^{2}\sum_{m_i\geq 0}\frac{R^{2m_i}}{(2m_i)!}\int_{\mathbb{R}^p} d^p y_i \lim_{R_i\to 0}\partial^{2m_i}_{R_i}\,\langle\sigma(z)\sigma(y_i\hat{x}_\parallel + R_i\hat{x}^1_\perp)\rangle\Big|_{h=0}\,.\end{aligned} \tag{3.25}$$

Performing the integration over the parallel coordinates, and differentiating (3.12) gives us

$$\begin{aligned}\langle D_f\sigma(x)\rangle^{(1)}_N &= -2h^2 g_2 A_d^3\pi^{\frac{3p}{2}}\frac{\Gamma^3_{\Delta_\phi-\frac{p}{2}}}{\Gamma^3_{\Delta_\phi}}\sum_{n\geq 0} c_n R^{2n} J^{d-p}_{\Delta_\phi-\frac{p}{2},2\Delta_\phi-p+n}(-x_\perp,0)\,,\\ c_n &= \sum_{m=0}^{n} b_m b_{n-m} = \frac{a_n}{2}\,.\end{aligned} \tag{3.26}$$

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This is exactly the same as (3.22)

$$\langle D_f \sigma(x)\rangle_N^{(1)} = \langle D_+ D_- \sigma(x)\rangle_N^{(1)} \,. \tag{3.27}$$

Thus the fusion (3.24) seems to hold even in the interacting theory. Note that using D_f, instead of $D_\pm$, simplified the Feynman diagram calculation as we did not need to calculate L_-^+ in (3.19).

At this order $\langle D_f \sigma(x)\rangle_N^{(1)}$ has a single pole in ϵ (neglecting constants of $\mathcal{O}(\epsilon^0)$)

$$\begin{aligned} \langle D_+ D_- \sigma(x)\rangle_N^{(1)} &= -\frac{h^2 g_2}{16\,\pi^4 x_\perp^2}\left(\frac{1}{\epsilon} + \log(x_\perp^2)\right) + \\ &\quad + \frac{h^2 g_2}{96\,\pi^4}\sum_{n\geq 1}\frac{2\,n^2+4\,n+3}{n}\frac{R^{2\,n}}{x_\perp^{2(n+1)}} + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{3.28}$$

The sum over n was done in the thesis [1].

3.3 Order g^2

We will now proceed to the next order in the bulk couplings, and see that fusion still holds. At this order, we will have $\mathcal{O}(h\,g^2)$- and $\mathcal{O}(h^3 g^2)$-terms. We leave calculational details for this entire section in appendix B.

Terms of $\mathcal{O}(h\,g^2)$ are affected by only one of the defects, and are given by the first diagram in figure 4. Its expansion in R is given by

$$\langle D_+ D_- \sigma\rangle_N^{(2)}\Big|_{h\,g^2} = -\frac{32\,h(N\,g_1^2+g_2^2)\pi^{d+1}A_d^4}{(d-4)^2(d^2-9\,d+18)\Gamma_{2\,d-9}\Gamma^2_{\frac{d}{2}-2}}\sum_{n\geq 0}\frac{\Gamma_{\frac{d+5-5}{2}}}{(2\,n)!}\frac{R^{2\,n}}{x_\perp^{2(d-5+n)}}\,. \tag{3.29}$$

The corresponding part of $\langle D_f\sigma\rangle$ is found to be in perfect agreement (since $\Delta_\phi = \frac{d-2}{2}$, $p=2$ in the Feynman diagrams)

$$\begin{aligned} \langle D_f \sigma(x)\rangle_N^{(2)}\Big|_{h\,g^2} &= -h(N\,g_1^2+g_2^2)\pi^{d+\frac{p}{2}}A_d^4\frac{\Gamma_{\frac{d}{2}-2\,\Delta_\phi}\Gamma_{4\,\Delta_\phi-d-\frac{p}{2}}\Gamma^2_{\frac{d}{2}-\Delta_\phi}}{\Gamma_{\frac{3\,d}{2}-4\,\Delta_\phi}\Gamma_{2\,\Delta_\phi}\Gamma^2_{\Delta_\phi}}\times \\ &\quad\times\sum_{n\geq 0}\frac{(8\,\Delta_\phi-2\,d-p)_n}{(2\,n)!}\frac{R^{2\,n}}{|x_\perp|^{8\,\Delta_\phi-2\,d-p+2\,n}} \\ &= \langle D_+ D_- \sigma(x)\rangle_N^{(2)}\Big|_{h\,g^2}\,. \end{aligned} \tag{3.30}$$

It has a single pole in ϵ (neglecting constants at $\mathcal{O}(\epsilon^0)$)

$$\langle D_f \sigma(x)\rangle_N^{(2)}\Big|_{h\,g^2} = \frac{(N\,g_1^2+g_2^2)h}{768\,\pi^5}\sum_{n\geq 0}(2\,n+1)\frac{R^{2\,n}}{|x_\perp|^{2(n+1)}}\left(\frac{1}{\epsilon}+\log(x_\perp^2)+\mathcal{O}(\epsilon)\right)\,. \tag{3.31}$$

At $\mathcal{O}(h^3 g^2)$ we have in addition to the second diagram in figure 4 also the two connecting diagrams in figure 7. Its expansion in R is given by (using Mathematica we can write out higher orders)

$$\begin{aligned} \langle D_+ D_- \sigma\rangle_N^{(2)}\Big|_{h^3 g^2} &= -4\,\pi^{\frac{d}{2}2\,p}A_d^5 h^3 g_2^2\frac{\Gamma_{\frac{d+p}{2}-2\,\Delta_\phi}\Gamma_{\frac{d}{2}-\Delta_\phi}\Gamma_{3\,\Delta_\phi-\frac{d}{2}-p}\Gamma^4_{\Delta_\phi-\frac{p}{2}}}{\Gamma_{d+\frac{p}{2}-3\,\Delta_\phi}\Gamma_{2\,\Delta_\phi-p}\Gamma^5_{\Delta_\phi}}\times \\ &\quad\times J^{d-p}_{4\,\Delta_\phi-\frac{d+3\,p}{2},\Delta_\phi-\frac{p}{2}}(x_\perp)+\mathcal{O}(R^2)\,. \end{aligned} \tag{3.32}$$

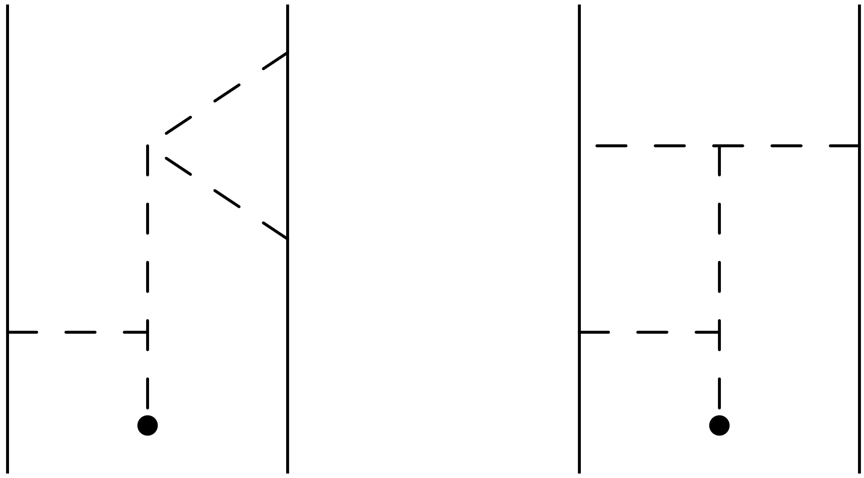

Figure 7. The two Feynman diagrams that contribute to the one-point function of σ (in the presence of the two defects $D_\pm$) at $\mathcal{O}(g^2)$.

By comparing order by order in R, we find again perfect agreement for the fused defect

$$\langle D_f \sigma\rangle_N^{(2)}\Big|_{h^3 g^2} = -\frac{128\,\pi^{d+3} A_d^5 h^3 g_2^2}{\Gamma^2_{\frac{d-2}{2}}\Gamma^3_{d-3}} \sum_{n\geq 0} c_n \frac{R^{2\,n}}{|x_\perp|^{3\,d-16+2\,n}} = \langle D_+ D_- \sigma\rangle_N^{(2)}\Big|_{h^3 g^2}\,. \tag{3.33}$$

The constant c_n is given by a finite sum

$$\begin{aligned} c_n &= \sum_{m=0}^{n}\sum_{m'=0}^{m} a_{n-m,m-m',m'} \\ &= \frac{4}{(3\,d+2\,n-16)(d+n-6)\Gamma^2_{d-4}} \sum_{m=0}^{n} \frac{\Gamma_{d+2(n-m)-4}\Gamma_{d+2\,m-4}}{(d+m-5)(d+2\,m-6)\Gamma_{2(n+m)+1}\Gamma_{2\,m+1}} \times \\ &\quad \times {}_4F_3\left(\frac{d-2}{2},\frac{d-3}{2},\frac{1}{2}-m,-m;\frac{1}{2},\frac{5-d}{2}-m,3-\frac{d}{2}-m;1\right)\,. \end{aligned}$$

Here a_{m_1,m_2,m_3} is

$$\begin{aligned} a_{m_1,m_2,m_3} = b_{m_1} b_{m_2} b_{m_3} &\frac{\Gamma_{d+p-4\,\Delta_\phi-m_1-m_2-m_3}\Gamma_{3\,\Delta_\phi-\frac{d}{2}-p+m_2+m_3}}{\Gamma_{\frac{3\,d}{2}+p-5\,\Delta_\phi-m_1-m_2-m_3}\Gamma_{2\,\Delta_\phi-p+m_2+m_3}} \times \\ \times &\frac{\Gamma_{5\,\Delta_\phi-d-\frac{3\,p}{2}+m_1+m_2+m_3}\Gamma_{\frac{d+p}{2}-2\,\Delta_\phi-m_2-m_3}}{\Gamma_{4\,\Delta_\phi-\frac{d+3\,p}{2}+m_1+m_2+m_3}\Gamma_{d+\frac{p}{2}-3\,\Delta_\phi-m_2-m_3}}\,. \end{aligned} \tag{3.34}$$

$\langle D_f \sigma\rangle_N^{(2)}\Big|_{h^3 g^2}$ has the ϵ-expansion (we do not care about constants at $\mathcal{O}(\epsilon^0)$)

$$\begin{aligned} \langle D_f \sigma\rangle_N^{(2)}\Big|_{h^3 g^2} = -\frac{h^3 g_2^2}{128\,\pi^6 x_\perp^2} &\left(\frac{1}{\epsilon^2} + 3\frac{\mathcal{A}+\log(x_\perp^2)}{2\,\epsilon} + \frac{9}{2}\left(\frac{\log(x_\perp^2)}{4}\right)^2 + \right. \\ &\left. + \frac{9\,\mathcal{A}\,\log(x_\perp^2)}{4} + \sum_{n\geq 1}\frac{2\,n+1}{n(n+1)} R^{2\,n}\left(\frac{1}{\epsilon} + 3\frac{\log(x_\perp^2)}{2}\right) + \mathcal{O}(\epsilon)\right)\,, \\ \mathcal{A} = \log\left(\pi\, e^{\gamma_E+\frac{5}{3}}\right)\,. \end{aligned} \tag{3.35}$$

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To summarize this section, we found (by studying terms of the same order in h) in (3.30), (3.33) that fusion holds at $\mathcal{O}(g^2)$

$$\langle D_f\sigma\rangle_N^{(2)} = \langle D_+D_-\sigma\rangle_N^{(2)}\,. \tag{3.36}$$

3.4 Renormalization

In this section we will renormalize the one-point function (3.26), (3.30), (3.33) of σ in the presence of the fused defect, D_f. To do this we treat each order in R on D_f (3.24) as an independent coupling

$$D_f = \exp\left(-2\sum_{n\geq 0} h_n \frac{R^{2\,n}}{(2\,n)!}\int_{\mathbb{R}^p} d^px\partial_R^{2\,n}\hat{\sigma}(x_+)\right)\,, \tag{3.37}$$

where h_n is the set of bare couplings on D_f. For $n\geq 1$ the couplings are dimensionfull ($R^{2\,n}$), and thus they will have no non-trivial conformal f.p.'s.[8] Thus we will only focus on the renormalization of the dimensionless coupling ($n=0$) in this section. We expect it to flow to the same RG f.p. (2.37) as on $D_\pm$ (which in the notation above would differ by a factor of two). In this section we will see that this is indeed the case, which serves as a good sanity check on our fusion. This will in turn mean that after we have fused the two (conformal) defects, $D_\pm$, we can turn on the bulk-interactions and flow to a conformal f.p. in the RG where D_f is also conformal.

The following bare defect coupling cancels the poles in ϵ (details on this renormalization as well as several consistency checks are in appendix B)

$$h_0 = \mu^{\frac{\epsilon}{2}}\tilde{h}_0\left(1-\frac{\tilde{h}_0\tilde{g}_2}{8\,\pi^2\epsilon}+\frac{\tilde{h}_0^2\tilde{g}_2^2}{64\,\pi^4\epsilon^2}-\frac{\tilde{h}_0^2\tilde{g}_2^2}{128\,\pi^4\epsilon}+\mathcal{O}(g^3)\right)\,, \tag{3.38}$$

from which we find the β-function

$$\beta_0 = -\epsilon\,\tilde{h}_0 - \frac{\tilde{h}_0^2\tilde{g}_2}{8\,\pi^2} - \frac{\tilde{h}_0^3\tilde{g}_2^2}{64\,\pi^4}\,. \tag{3.39}$$

From which we find a trivial f.p. and the following non-trivial f.p.

$$h_0^* = -4\,\pi^2\left(\frac{1}{\tilde{g}_2}\mp\sqrt{\frac{1-2\,\epsilon}{\tilde{g}_2^2}}\right)\,. \tag{3.40}$$

As expected, this if half of that (2.37) on $D_\pm$

$$h_0^* = \frac{h^*}{2}\,. \tag{3.41}$$

Specifying to this non-trivial f.p., we find the renormalized one-point function

$$\begin{aligned}\langle D_f\tilde{\sigma}(x)\rangle = \frac{\mu^{\sigma_{\mathbb{1}}}}{|x_\perp|^{2-\epsilon}}\Bigg(1&+\frac{g_2^*h_0^*}{(4\,\pi)^2}\log(x_\perp^2\mu^2)-\frac{N(g_1^*)^2+(g_2^*)^2}{768\,\pi^3}\log(x_\perp^2\mu^2)+\\ &+\frac{(g_2^*)^2(h_0^*)^2}{(4\,\pi)^4}\log(x_\perp^2\mu^2)^2+\mathcal{O}(g^3)\Bigg)\,.\end{aligned} \tag{3.42}$$

[8] For details on this, see appendix B.

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As expected, it only contain dimensionless logarithms. This is expressed in terms of the DOE coefficient

$$\mu^{\sigma}{}_{\mathbb{1}} = -\frac{h_0^*\Gamma_{1-\frac{\epsilon}{2}}}{2\,\pi^{2-\frac{\epsilon}{2}}}\,. \tag{3.43}$$

From (3.42) we find the correct bulk anomalous dimension of σ [27]

$$\Delta_\sigma = \frac{d-2}{2} + (N(g_1^*)^2 + (g_2^*)^2)\gamma_\sigma + \mathcal{O}(g^3)\,, \qquad \gamma_\sigma = \frac{1}{768\,\pi^3}\,. \tag{3.44}$$

3.5 Line defects near four dimensions

We will end this section by showing that fusion also seems to hold at a quantum level in $d = 4-\epsilon$. This calculation is similar to that in section 3.2. Near four dimensions we can consider a quartic bulk-interaction invariant under $O(N)$

$$S = \int_{\mathbb{R}^d} d^dx \left(\frac{(\partial_\mu\phi^i)^2}{2} + \frac{\lambda}{8}\phi^4\right)\,, \tag{3.45}$$

where $\phi^4 \equiv [(\phi^i)^2]^2$ and $i \in \{1,\ldots,N\}$. In addition we consider two $p=1$ dimensional line defects on the form (2.2) (without the σ-term). The bulk interactions has the non-trivial WF f.p.

$$\lambda^* = \frac{(4\,\pi)^2\epsilon}{N+8} + \mathcal{O}(\epsilon^2)\,, \tag{3.46}$$

and on the defects we have the following attractive f.p.'s [5]

$$h_\pm^* = \pm\sqrt{N+8} \pm \frac{4\,N^2 + 45\,N + 170}{4(N+8)^{\frac{3}{2}}}\epsilon + \mathcal{O}(\epsilon^2)\,, \tag{3.47}$$

which are of finite size.

Fusing the defects (2.2) with a multivariate Taylor expansion yields

$$D_f = \exp\left(-h\prod_{i=1}^{d-p}\sum_{n_i\geq 0}\frac{\delta^{ji_+} + (-1)^n\delta^{ji_-}}{n_i!}R_i^{n_i}\int_{\mathbb{R}^p} d^px\,\partial_i^{n_i}\hat{\phi}^j(x_+)\right)\,. \tag{3.48}$$

This reduces to the form (3.2) when $i_\pm = N$ (under the exchange $\sigma \to \phi^N$).

At $\mathcal{O}(\lambda)$ we find $\langle D_+D_-\phi^i(x)\rangle_N^{(1)}$ from the Feynman diagrams in figure 8

$$\langle D_+D_-\phi^i(x)\rangle_N^{(1)} = (-h)^3\left(-\frac{\lambda}{8}\right)\sum_{a=\pm}\left(\frac{4!}{3!}\delta^{i\,i_a}\tilde{L}_a^a + \frac{8!}{2}(2\,\delta^{i_+i_-}\delta^{i\,i_{+a}} + \delta^{i\,i_{-a}})\tilde{L}_{-a}^{+a}\right)\,.$$

This is expressed in terms of the following integral

$$\tilde{L}_b^a = \int_{\mathbb{R}^d} d^dz\,\langle\phi(x)\phi(z)\rangle|_{h=0}\,K_a(z)^2K_b(z)\,. \tag{3.49}$$

We will now perform the following steps:

1. Integrate over the parallel coordinate, $z_\parallel \in \mathbb{R}$, in $\tilde{L}_b^a$.

2. Expand in R.

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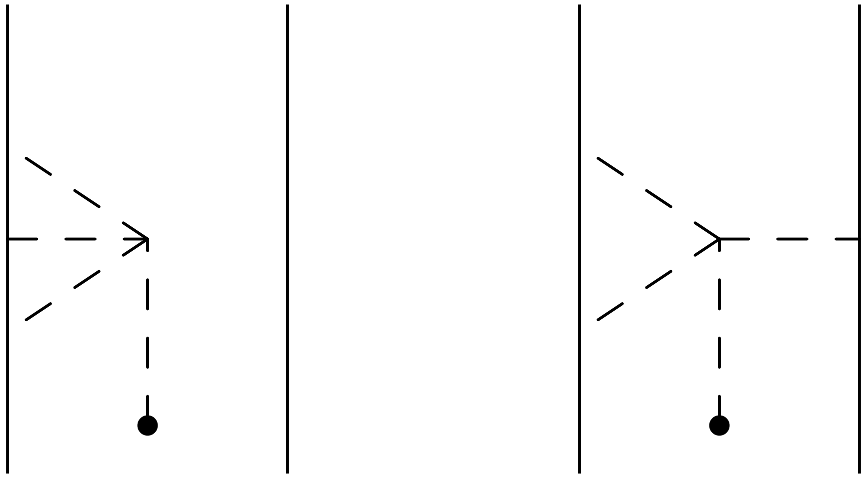

Figure 8. The diagrams that contribute to the one-point function of ϕ^i in $d = 4 - \epsilon$.

3. Integrate over $z_\perp \in \mathbb{R}^{d-1}$.

Doing this yields

$$\tilde{L}^a_b = \pi^{2\,p} A_d^4 \frac{\Gamma^4_{\Delta_\phi - \frac{p}{2}}}{\Gamma^4_{\Delta_\phi}} \left[J^{d-p}_{\Delta_\phi - \frac{p}{2}, 3\,\Delta_\phi - \frac{3\,p}{2}}(-x_\perp, 0) + \right. \\ \left. + (2\,a + b)(p - 2\,\Delta_\phi) R\, J^{d-p}_{\Delta_\phi - \frac{p}{2}, 3\,\Delta_\phi - \frac{3\,p}{2} + 1}(-x_\perp, 0) + \mathcal{O}(R^2) \right] . \tag{3.50}$$

On the other hand, for D_f we have

$$\begin{aligned} \langle D_f \phi^i(x) \rangle^{(1)}_N &= \frac{4!}{3!} (-h)^3 \left(-\frac{\lambda}{8} \right) \int_{\mathbb{R}^d} d^d z \, \langle \phi(x) \phi(z) \rangle |_{h=0} \prod_{i=1}^{3} \int_{\mathbb{R}^p} d^p w_i \times \\ &\quad \times \sum_{n_i \geq 0} \sigma_{n_1,n_2,n_3} \frac{R^{2\,n_i}}{(2\,n_i)!} \lim_{R_i \to 0} \partial_i^{2\,n_i} \, \langle \phi(z) \phi(w_i) \rangle |_{h=0} \\ &= \frac{4\,\pi^{2\,p} A_d^4 h^3 \lambda\, \Gamma^4_{\Delta_\phi - \frac{p}{2}} \delta^{i\,N}}{\Gamma^4_{\Delta_\phi}} \sum_{n \geq 0} d_n R^{2\,n} J^{d-p}_{\Delta_\phi - \frac{p}{2}, 3\,\Delta_\phi - \frac{3\,p}{2} + n}(-x_\perp, 0) , \end{aligned} \tag{3.51}$$

with the constant

$$\begin{aligned} d_n &= \sum_{m=0}^{n} \sum_{m'=0}^{m} b_{\frac{n-m}{2}} b_{\frac{m-m'}{2}} b_{\frac{m'}{2}} \sigma_{n-m, m-m', m'} \\ &= 154 \frac{\delta^{ii_+} + (-1)^n \delta^{ii_-}}{\Gamma^2_{2\,\Delta_\phi - p}} \sum_{\substack{m \geq 0 \\ \text{even } m}} \frac{1}{(n - 2\,m)!} \left(\frac{\Gamma_{2\,\Delta_\phi - p + m} \Gamma_{2\,\Delta_\phi - p + n - 2\,m}}{m!} + \right. \\ &\quad \left. - \frac{\pi \, \csc \left[\pi (p - 2\,\Delta_\phi) \right]}{(2\,m)!} {}_2F_1 \left(-2\,m, 2\,\Delta_\phi - p; p - 2\,\Delta_\phi - 2\,m + 1; 1 \right) \right) . \end{aligned} \tag{3.52}$$

Here b_m is the constant in (3.12) and σ_{n_1,n_2,n_3} is a factor from applying Wick's theorem to the integrand

$$\sigma_{n_1,n_2,n_3} = \left(1 + (-1)^{n_2 + n_3} + \left[(-1)^{n_2} + (-1)^{n_3} \right] \delta^{i_+ i_-} \right) \left(\delta^{ii_+} + (-1)^{n_1} \delta^{ii_-} \right) . \tag{3.53}$$

$\langle D_f \phi^i(x)\rangle_N^{(1)}$ at (3.51) is in perfect agreement with $\langle D_+ D_- \phi^i(x)\rangle_N^{(1)}$ at (3.50) (seen order by order in R). This suggests that the fusion (3.24) is valid in $d = 4 - \epsilon$ as well

$$\langle D_f \phi^i(x)\rangle_N^{(1)} = \langle D_+ D_- \phi^i(x)\rangle_N^{(1)} \,. \tag{3.54}$$

If we expand $\langle D_f \phi^i(x)\rangle_N^{(1)}$ in ϵ we find (neglecting constants of $\mathcal{O}(\epsilon^0)$)

$$\begin{aligned} \langle D_+ D_- \phi^j(x)\rangle_N^{(1)} = \frac{h^3 \lambda}{128\,\pi^3 |x_\perp|} \Bigg[&(1+\delta^{i_+ i_-})(\delta^{j i_+} + \delta^{j i_-}) \left(\frac{1}{\epsilon} + 3\,\log|x_\perp|\right) + \\ &- \sum_{n\geq 1} \Bigg(\frac{(N+\delta^{i_+ i_-})(\delta^{j i_+} - \delta^{j i_-})}{2\,N-1} \frac{R^{2\,n-1}}{|x_\perp|^{2\,n-1}} + \\ &+ \frac{(N+1)(N+1+\delta^{i_+ i_-})(\delta^{j i_+} + \delta^{j i_-})}{N(2\,N+1)} \frac{R^{2\,n}}{|x_\perp|^{2\,n}} \Bigg) \Bigg] \,. \end{aligned} \tag{3.55}$$

Writing D_f as (3.37), we find that $\langle D_f \phi^i(x)\rangle_N^{(1)}$ is renormalized by the following bare couplings constants[9]

$$h_0 = \mu^{\frac{\epsilon}{2}} \tilde{h}_0 \left(1 + \tilde{h}_0^2 \tilde{\lambda} \frac{1+\delta^{i_+ i_-}}{32\,\pi^2\,\epsilon} + \mathcal{O}(\tilde{\lambda}^2)\right) \,, \qquad h_n = \mu^{\frac{\epsilon}{2}} \tilde{h}_n + \mathcal{O}(\tilde{\lambda}^2) \,. \tag{3.56}$$

which gives us the β-functions

$$\beta_0 = -\frac{\epsilon}{2} \tilde{h}_0 + \tilde{h}_0^3 \tilde{\lambda} \frac{1+\delta^{i_+ i_-}}{(4\,\pi)^2} + \mathcal{O}(\tilde{\lambda}^2) \,, \quad \beta_n = -\frac{\epsilon}{2} \tilde{h}_n + \mathcal{O}(\tilde{\lambda}^2) \,. \tag{3.57}$$

This β-function has the non-trivial f.p.

$$h_0^* = \pm 2\,\pi \sqrt{\frac{2\,\epsilon}{(1+\delta^{i_+ i_-})\lambda^*}} = \pm \sqrt{\frac{N+8}{2(1+\delta^{i_+ i_-})}} + \mathcal{O}(\epsilon) \,, \quad h_n^* = 0 \,. \tag{3.58}$$

If $i_+ = i_-$, then the factor of two in the denominator of h_0^* is canceled by adding $\phi^{i_+} + \phi^{i_-}$ in D_f (3.48). It yields the same RG f.p. (3.47) as that on $D_\pm$. This is also the case when $i_+ \neq i_-$, in which case the scalar on D_f is tilted in the $O(N)$-space[10]

$$\Phi = \frac{\phi^{i_+} + \phi^{i_-}}{\sqrt{2}} \,. \tag{3.59}$$

Note that this vector is normalized due to the factor of $\sqrt{2}$ from h_0^* (3.58). We thus recover the RG f.p. (3.47) as expected.

The renormalized one-point function at the conformal f.p. is

$$\langle D_f \phi^j(x)\rangle_N = -\frac{\delta^{j i_+} + \delta^{j i_-}}{4\,\pi^{2-\frac{\epsilon}{2}} |x_\perp|^{2-\epsilon}} \Gamma_{1-\frac{\epsilon}{2}} h_0^* \left(1 - (h_0^*)^2 \lambda^* \frac{1+\delta^{i_+ i_-}}{16\,\pi^2} \log|\mu\, x_\perp| + \mathcal{O}(\tilde{\lambda}^2)\right) \,,$$

[9] At $\mathcal{O}(\lambda)$ the bulk-field, ϕ^i, receives no anomalous dimension and thus we do not have to bother with a Z-factor [29].

[10] We are grateful to Himanshu Khanchandani for a discussion on this.

where

$$(h_0^*)^2\lambda^* = \frac{8\,\pi^2\epsilon}{1+\delta^{i_+i_-}} + \mathcal{O}(\epsilon^2)\,. \tag{3.60}$$

By resumming in ϵ we find the expected behavior from conformal symmetry

$$\langle D_f\phi^j(x)\rangle_N = \frac{\mu_{\mathbb{1}}^{\phi^j}}{|x_\perp|^{\Delta_\phi}}\,, \quad \Delta_\phi = \frac{d-2}{2} + \mathcal{O}(\epsilon^2)\,, \tag{3.61}$$

given in terms of the DOE coefficient

$$\mu_{\mathbb{1}}^{\phi^j} = -\frac{\delta^{ji_+}+\delta^{ji_-}}{4\,\pi^{2-\frac{\epsilon}{2}}\mu^{\frac{\epsilon}{2}}}\Gamma_{1-\frac{\epsilon}{2}}h_0^*\,. \tag{3.62}$$

4 Conclusion

In this paper we have fused two scalar Wilson defects (1.1) in $d = 4-\epsilon$ and $d = 6-\epsilon$ dimensions, and presented results which indicate that this fusion (1.2) also holds in the interacting theory. In particular, we showed that bulk one-point functions stay invariant (before renormalization). This is an expected result since the path integral is the same before and after fusion. In addition we also found the Casimir energy (3.16) between the two defects.

From our results we see the power of fusion. Firstly it gives rise to an infinite tower of interactions (1.2). However, as we have shown in this paper, the dimensionfull couplings does not have non-trivial f.p.'s and can thus be tuned to zero in conformal field theories. Assuming this to start with would greatly simplify the calculation of Feynman diagrams.

We found that the coupling constants on the fused defect, D_f, also takes into account divergences in the fusion-limit of the two defects, giving us the β-functions (3.39), (3.57). Without these additional divergences we would not be able to reproduce the same conformal f.p.'s (3.41), (3.58) as those on $D_\pm$. It would be interesting to study whether this fusion can be understood using only symmetry arguments, e.g. by studying the OPE of the defect-local operators, $\sigma_\pm$, on the two defects.

Our results indicate that the two conformal defects have fused into one conformal defect of the same type. This means that if we are given a scalar Wilson line at the RG f.p., then we cannot determine whether it is actually a product of fusion of two such defects or not. This might sound exotic, but it makes sense from an OPE point of view

$$D(-R)D(+R) = D(0)\,. \tag{4.1}$$

This serves as a motivation to study fusion of more scalar Wilson defects. A starting point could be to consider three scalar Wilson defects (1.1), and study whether first fusing two of them and then fuse the resulting fused defect with the last one gives the same result as fusing all three defects at once.

Note that we have not used the conformal symmetry in any way when fusing the defects. Meaning the methods we have used should be applicable to several other kinds of defects and theories as well, assuming we have a Lagrangian description for the defects.

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E.g. it would be interesting to study fusion of scalar Wilson defects (1.1) in theories with other bulk interactions. Or for that matter push our calculations to higher orders in the bulk couplings. Due to how the Z-factor for the external field is introduced [13] (see the discussion above (A.16)), it could be that the double scaling limit results (upto $\mathcal{O}(g^4)$, where all bulk-loops are suppressed) of [12] are fully valid.

Another interesting direction to pursue would be to find a more non-trivial fusion (using the methods of this paper), where two defects are fused into a sum of several fused defects

$$D_1 D_2 = \sum_n C^{12}{}_n D_n \,, \tag{4.2}$$

where $C^{12}{}_n$ are the OPE coefficients in the corresponding fusion category [17–19]. A possible candidate could be the *monodromy twist defects* (or symmetry defects) [30–34]. These are of codimension two (exactly), and carry a monodromy constraint for the bulk fields which breaks the global symmetry of the theory. It might be worthwhile to study whether this symmetry breaking can be seen from fusion.

A generalization to monodromy twist defects are *replica twist defects* (or rényi defects) [35]. These are used in quantum information to find entanglement entropies [36, 37]. We believe the c-function (monotonous under the RG flow) in quantum information can be found from fusion of two replica twist defects [38]. It would be interesting whether we can apply the techniques from this paper to get more insight into this problem. A drawback with these codimension two defects (both monodromy and replica twist defects) is that we are not aware of any Lagrangian descriptions for them.

Acknowledgments

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A Renormalization group

A.1 Order g

In this appendix we calculate the Feynman diagrams in figure 3. The corresponding one-point functions are given by

$$\begin{aligned} \langle D_\pm \phi^i(x)\rangle_N^{(1)} &= 2\left(-\frac{g_1}{2}\right)(-h_\pm^\phi)(-h_\pm^\sigma)\delta^{ii_\pm} L_\pm^\pm(x)\,, \\ \delta\langle D_+ D_- \phi^i(x)\rangle_N^{(1)} &= 2\left(-\frac{g_1}{2}\right)\sum_{a=\pm}(-h_a^\phi)(-h_{-a}^\sigma)\delta^{ii_a} L_-^+(x)\,, \end{aligned} \tag{A.1}$$

and

$$\begin{aligned}
\langle D_\pm \sigma(x)\rangle_N^{(1)} &= \left(\frac{2}{2}\left(-\frac{g_1}{2}\right)(-h_\pm^\phi)(-h_\pm^\phi)\delta^{i_\pm i_\pm}+\right.\\
&\quad \left.+\frac{3!}{2}\left(-\frac{g_2}{3!}\right)(-h_\pm^\sigma)(-h_\pm^\sigma)\right)L_\pm^\pm(x)\,,\\
\delta\langle D_+ D_- \sigma(x)\rangle_N^{(1)} &= \left(2\left(-\frac{g_1}{2}\right)(-h_+^\phi)(-h_-^\phi)\delta^{i_+ i_-}+\right.\\
&\quad \left.+3!\left(-\frac{g_2}{3!}\right)(-h_+^\sigma)(-h_-^\sigma)\right)L_-^+(x)\,.
\end{aligned} \tag{A.2}$$

Here L_b^a, with $a,b=\pm$, is the following integral

$$\begin{aligned}
L_b^a(x) &\equiv \int_{\mathbb{R}^d} d^d z\,\langle\phi(x)\phi(z)\rangle|_{h=0}\,K_a(z)K_b(z)\\
&= A_d\int_{\mathbb{R}^{d-p}} d^{d-p}z_\perp K_a(z)K_b(z)I^p_{\Delta_\phi}(0,x_\parallel,(z_\perp-x_\perp)^2)\\
&= \frac{A_d^3\pi^{\frac{3p}{2}}\Gamma^3_{\Delta_\phi-\frac{p}{2}}}{\Gamma^3_{\Delta_\phi}}\int_{\mathbb{R}^{d-p}}\frac{d^{d-p}z_\perp}{(|z_\perp|\,|z_\perp+x_\perp-a\,R|\,|z_\perp+x_\perp-b\,R|)^{2\Delta_\phi-p}}\,,
\end{aligned} \tag{A.3}$$

where in the last step we shifted $z_\perp\to z_\perp+x_\perp$. When $a=b$ this integral can be solved using the master integral (2.19). It gives us

$$\begin{aligned}
L_\pm^\pm(x) &= A_d^3\pi^{\frac{3p}{2}}\frac{\Gamma^3_{\Delta_\phi-\frac{p}{2}}}{\Gamma^3_{\Delta_\phi}}J^{d-p}_{\Delta_\phi-\frac{p}{2},2\Delta_\phi-p}(x_\perp\mp R)\\
&= \frac{\Gamma_{\frac{d+p}{2}-2\Delta_\phi}\Gamma_{\frac{d}{2}-\Delta_\phi}\Gamma^2_{\Delta_\phi-\frac{p}{2}}\Gamma_{3\Delta_\phi-\frac{d}{2}+p}}{\Gamma_{d+\frac{p}{2}-3\Delta_\phi}\Gamma^3_{\Delta_\phi}\Gamma_{2\Delta_\phi-p}}\frac{A_d^3\pi^{\frac{3p}{2}}}{|x_\perp\mp R|^{6\Delta_\phi-d-2p}}\,.
\end{aligned} \tag{A.4}$$

We find it easier to study the UV divergences of the integral L_-^+ in momentum space, where we Fourier transform w.r.t. $s_\perp^\pm$

$$M_b^a(k_\perp^\pm)\equiv\prod_{c=\pm}\int_{\mathbb{R}^{d-p}} d^{d-p}s_\perp^c\,e^{i\,k_\perp^c s_\perp^c}L_b^a(x)\,. \tag{A.5}$$

This integral can then be performed using only the master integral (2.14).

$$\begin{aligned}
M_\pm^\pm(k_\perp^\pm) &= \frac{A_d^3\pi^{\frac{3p}{2}}\Gamma^3_{\Delta_\phi-\frac{p}{2}}}{\Gamma^3_{\Delta_\phi}}\delta(k_\perp^\pm)\int_{\mathbb{R}^{d-p}}\frac{d^{d-p}z_\perp}{|z_\perp|^{2\Delta_\phi-p}}I^p_{2\Delta_\phi-p}(k_\perp^\mp,-z_\perp,0)\\
&= \frac{A_d^3 2^{d+p-4\Delta_\phi}\pi^{\frac{d}{2}+p}\Gamma_{\frac{d+p}{2}-2\Delta_\phi}\Gamma^3_{\Delta_\phi-\frac{p}{2}}}{\Gamma^3_{\Delta_\phi}\Gamma_{2\Delta_\phi-p}|k_\perp^\mp|^{\frac{d+p}{2}-2\Delta_\phi}}\delta(k_\perp^\pm)I^{d-p}_{\Delta_\phi-\frac{p}{2}}(-k_\perp^\mp,0,0)\\
&= \frac{\delta(k_\perp^\pm)}{8\,\pi^2(k_\perp^\mp)^2}\left(\frac{1}{\epsilon}-\log|k_\perp^\mp|+\mathcal{A}\right)\,,
\end{aligned} \tag{A.6}$$

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$$\begin{aligned}
M_-^+(k_\perp^\pm) &= \frac{A_d^3 \pi^{\frac{3p}{2}} \Gamma^3_{\Delta_\phi - \frac{p}{2}}}{\Gamma^3_{\Delta_\phi}} \int_{\mathbb{R}^{d-p}} \frac{d^{d-p} z_\perp}{|z_\perp|^{2\Delta_\phi - p}} \prod_{a=\pm} I^p_{\Delta_\phi - \frac{p}{2}}(k_\perp^a, -z_\perp, 0) \\
&= \frac{A_d^3 4^{d-2\Delta_\phi} \pi^{d+\frac{p}{2}} \Gamma^2_{\frac{d}{2}-\Delta_\phi} \Gamma_{\Delta_\phi - \frac{p}{2}}}{\Gamma^3_{\Delta_\phi} |k_\perp^+|^{d-2\Delta_\phi} |k_\perp^-|^{d-2\Delta_\phi}} I^{d-p}_{\Delta_\phi - \frac{p}{2}}(-k_\perp^+ - k_\perp^-, 0, 0) \\
&= \frac{1}{(k_\perp^+)^2 (k_\perp^-)^2 (k_\perp^+ + k_\perp^-)^2}\,, \qquad \text{(exactly)}.
\end{aligned} \tag{A.7}$$

Here $\mathcal{A}$ is the following constant

$$\mathcal{A} = \log\left(\frac{2\sqrt{\pi}}{e^{\frac{\gamma_E}{2}-1}}\right), \tag{A.8}$$

which can be absorbed in the coupling constants (by defining *minimal subtraction* (MS) scheme couplings) without affecting the RG flow. Thus we will not care about it.

To find the bare defect couplings we add the free theory correlators (2.9) to those at first order in the coupling constants (A.1). We then make the following ansatz for the bare coupling constants

$$h_\pm^\phi = \mu^{\frac{\epsilon}{2}} \tilde{h}_\pm^\phi \left(1 + a_\pm^\phi \frac{\tilde{h}_\pm^\sigma \tilde{g}_1}{\epsilon}\right), \qquad h_\pm^\sigma = \mu^{\frac{\epsilon}{2}} \left(\tilde{h}_\pm^\sigma + b_\pm^\phi \frac{(\tilde{h}_\pm^\phi)^2 \tilde{g}_1}{\epsilon} + b_\pm^\sigma \frac{(\tilde{h}_\pm^\sigma)^2 \tilde{g}_2}{\epsilon}\right), \tag{A.9}$$

where the constants $a_\pm^\phi, b_\pm^\phi$ and $b_\pm^\sigma$ are tuned s.t. that the ϵ-poles in the correlators vanish. Coupling constants with a tilde are renormalized ones (dimensionless), and μ is the RG scale. By expanding the correlators in the bulk couplings and in ϵ we find (by matching powers of $k_\pm$)

$$b_\pm^\phi = b_\pm^\sigma = \frac{a_\pm^\phi}{2}, \quad a_\pm^\phi = -\frac{1}{8\pi^2}. \tag{A.10}$$

From which we find the β-functions (2.22) (by differentiating $\log h_\pm^\phi$, $\log h_\pm^\sigma$ w.r.t. $\log \mu$).[11]

A.2 Order g^2

At $\mathcal{O}(g^2)$ the one-point function of σ in the presence of one defect (2.26) with only a σ-interaction is given by

$$\langle D_+ \sigma\rangle^{(2)} = \left(2N\left(-\frac{g_1}{2}\right)^2 + 3 * 3!\left(-\frac{g_2}{2}\right)^2\right)(-h)A + \frac{(3!)^2}{2!}\left(-\frac{g_2}{2}\right)^2 (-h)^3 B\,, \tag{A.11}$$

where A is the first diagram in figure 4, and B the second

$$\begin{aligned}
A &= \prod_{i=1}^2 \int_{\mathbb{R}^d} d^d z_i \, \langle\sigma(x)\sigma(z_1)\rangle|_{h=0} \, \langle\sigma(z)\sigma(z_2)\rangle|^2_{h=0} \, K_+(z_2)\,, \\
B &= \prod_{i=1}^2 \int_{\mathbb{R}^d} d^d z_i \, \langle\sigma(x)\sigma(z_1)\rangle|_{h=0} \, \langle\sigma(z)\sigma(z_2)\rangle|_{h=0} \, K_+(z_1) K_+(z_2)^2\,.
\end{aligned} \tag{A.12}$$

These integrals can be calculated using the master integrals (2.14), (2.19). In order, we perform the following steps:

[11] Here we used that the bulk β-functions are given by (2.5). See e.g. appendix B of [39] for details on this.

1. Integrate over the parallel vertex coordinates: $z^i_{\parallel} \in \mathbb{R}^p$.
2. Integrate over $z_\perp \in \mathbb{R}^{d-p}$.
3. Integrate over $w_\perp \in \mathbb{R}^{d-p}$.
4. For simplicity, Fourier transform w.r.t. $s_\perp \equiv x_\perp - R \in \mathbb{R}^{d-p}$ and express the diagram in terms of the orthogonal momenta $k_\perp$. We denote the Fourier transform of A, B with $\tilde{A}$, $\tilde{B}$ respectively.
5. Neglect constant-terms at $\mathcal{O}(\epsilon^0)$ which can be absorbed in the coupling constants in the MS scheme (for simplicity).

This gives us

$$
\begin{aligned}
\tilde{A} &= \pi^{\frac{3\,d}{2}} A_d^4 \frac{\Gamma_{\frac{d}{2}-2\,\Delta_\phi}\Gamma_{\frac{d}{2}-\Delta_\phi}}{\Gamma_{2\,\Delta_\phi}\Gamma^2_{\Delta_\phi}} \left(\frac{2}{k_\perp^-}\right)^{3\,d-8\,\Delta_\phi} = -\frac{1}{192\,\pi^3 k_\perp^2}\left(\frac{1}{\epsilon} - \frac{\log(k_\perp^2)}{2} + \mathcal{O}(\epsilon)\right)\,, \\
\tilde{B} &= \pi^{\frac{3\,d}{2}+p} A_d^5 \frac{\Gamma_{\frac{d+p}{2}-4\,\Delta_\phi}\Gamma_{d+p-4\,\Delta_\phi}\Gamma_{3\,\Delta_\phi-\frac{d}{2}-p}\Gamma^2_{\frac{d}{2}-\Delta_\phi}\Gamma^2_{\frac{d}{2}-\Delta_\phi}}{\Gamma_{d+\frac{p}{2}-3\,\Delta_\phi}\Gamma_{2\,\Delta_\phi-p}\Gamma_{4\,\Delta_\phi-\frac{d+3\,p}{2}}\Gamma^5_{\Delta_\phi}} \left(\frac{2}{k_\perp^-}\right)^{3\,d+2(p-5\,\Delta_\phi)} \\
&= \frac{1}{128\,\pi^4 k_\perp^2}\left(\frac{1}{\epsilon^2} + \frac{\mathcal{A}^{(2)} - \log(k_\perp^2)}{\epsilon} + \left(\frac{\log(k_\perp^2)}{2}\right)^2 - 2\,\mathcal{A}^{(2)}\log(k_\perp^2)\right)\,.
\end{aligned}
\tag{A.13}
$$

Here $\tilde{A}$ captures the contribution to the bulk anomalous dimension of σ, and $\mathcal{A}^{(2)}$ is the following constant

$$
\mathcal{A}^{(2)} = \log\left(\frac{4\,\pi}{e^{\gamma_E - \frac{5}{2}}}\right)\,. \tag{A.14}
$$

The $\frac{\mathcal{A}^{(2)}}{\epsilon}$-term in B (A.13) will cancel due to the $\mathcal{O}(\epsilon)$-term from M_+^+ (A.6) when we renormalize the full one-point function of σ. This serves as a good sanity check on our result. Upto $\mathcal{O}(g^2)$, $\langle\sigma\rangle$ is given by (2.33).

To find the bare defect coupling we need to introduce a Z-factor for the bulk field [27]

$$
\sigma = \sqrt{Z}\tilde{\sigma} \quad \Rightarrow \quad \tilde{\sigma} = \frac{\sigma}{\sqrt{Z}}\,, \qquad \sqrt{Z} = 1 + z\frac{N\,g_1^2 + g_2^2}{\epsilon} + \mathcal{O}(g^3)\,. \tag{A.15}
$$

In this Z-factor we have a coefficient z which we can find from the bulk theory without a defect. Note that when we compute $\langle\sigma\rangle$ at (2.10) in the free theory we integrate over the two-point function (2.11) of σ in the presence of no defect ($h = 0$). This means that we need to include an extra factor of $\sqrt{Z}$ every time the integral K_+ appear in the Feynman diagrams in (A.3), (A.12) [13].[12] Technically, this means that we should divide every bare defect coupling, h, with $\sqrt{Z}$ in the one-point function of the renormalized field $\tilde{\sigma}$

$$
\begin{aligned}
\langle\tilde{\sigma}(k)\rangle = -\frac{h}{Z\,k_\perp^2}\Bigg(1 &+ \frac{g_2 h\csc\left(\frac{\pi\,\epsilon}{2}\right)}{4^{3-\epsilon}\pi^{\frac{1-\epsilon}{2}}\Gamma_{\frac{3-\epsilon}{2}}\sqrt{Z}\,|k_\perp|^\epsilon} - \frac{(N g_1^2 + g_2^2)\csc\left(\frac{\pi\,\epsilon}{2}\right)}{4^{5-\epsilon}\pi^{\frac{3-\epsilon}{2}}\Gamma_{\frac{5-\epsilon}{2}}|k_\perp|^\epsilon} + \\
&+ \frac{g_2^2 h^2\epsilon\csc(\pi\,\epsilon)\Gamma^2_{-\frac{\epsilon}{2}}}{2^{11-3\,\epsilon}\pi^{\frac{5}{2}-\epsilon}\Gamma_{2-\frac{3\,\epsilon}{2}}\Gamma_{\frac{3-\epsilon}{2}}Z\,|k_\perp|^\epsilon} + \mathcal{O}(g^3)\Bigg)\,.
\end{aligned}
\tag{A.16}
$$

At this order in the bulk couplings, only the Z-factor at $\mathcal{O}(g^0)$ will contribute.

[12] We are grateful to Diego Rodriguez-Gomez for a discussion on this.

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To renormalize $\langle\tilde{\sigma}(k)\rangle$ we make the following ansatz for the bare couplings

$$
\begin{aligned}
g_i &= \mu^{\frac{\epsilon}{2}}\tilde{g}_i + \mathcal{O}(g^2)\,, \quad i\in\{1,2\}\,,\\
h &= \mu^{\frac{\epsilon}{2}}\tilde{h}\left(1-\frac{\tilde{g}_2\tilde{h}}{16\,\pi^2\epsilon}+a\frac{N\,g_1^2+g_2^2}{\epsilon}+\tilde{g}_2^2\tilde{h}^2\left(\frac{b_2}{\epsilon^2}+\frac{b_1}{\epsilon}\right)+\mathcal{O}(g^3)\right)\,,
\end{aligned}
\tag{A.17}
$$

where a, b_1 and b_2 are three coefficients to be fixed by cancelling the poles in ϵ. By expanding in the bulk couplings and then ϵ, we are able to cancel the poles with

$$
a=\frac{1}{384\,\pi^3}+z\,,\quad b_2=\frac{1}{256\,\pi^4}=\left(-\frac{1}{16\,\pi^2}\right)^2\,,\quad b_1=-\frac{1}{512\,\pi^4}\,. \tag{A.18}
$$

Note that b_2-term in the bare coupling (A.17) is exactly twice the coefficient in front of the $\mathcal{O}(g_2h^2)$-term. This serves as a consistency check on our result as it will cancel an ϵ-pole in our β-function (which we will soon calculate).

As input from the bulk theory, the Z-factor coefficient, z, will precisely tune a to zero

$$
z=-\frac{1}{384\,\pi^3}\quad\Rightarrow\quad a=0\,. \tag{A.19}
$$

This is another consistency check of our result, since if a were to be non-zero, the f.p. (2.25) from $\mathcal{O}(g)$ would get further corrections at $\mathcal{O}(\sqrt{N\,\epsilon})$ (due to the $\mathcal{O}(Ng_1^2)$-term in $\langle\tilde{\sigma}\rangle$).

All and all, we find the bare defect coupling (2.34).

B Fusion at order g^2

In this appendix we calculate and compare the Feynman diagrams for $D_\pm$ as well as D_f at $\mathcal{O}(g^2)$, where both $\mathcal{O}(h)$- and $\mathcal{O}(h^3)$-terms are present. Let us start with the former ones, which are only affected by one of the defects (A.11)

$$
\langle D_+D_-\sigma\rangle_N^{(2)}\Big|_{h\,g^2}=-\frac{(N\,g_1^2+g_2^2)h}{2}\left(A+A|_{R\to-R}\right)\,, \tag{B.1}
$$

where the Fourier transform of A is given by (A.13). If we take the inverse we find

$$
A=\int_{\mathbb{R}^{d-p}}\frac{d^{d-p}k_\perp}{(2\,\pi)^{d-p}}e^{-i\,k_\perp(x_\perp-R)}\tilde{A}=\frac{\Gamma_{2-\frac{d}{2}}\Gamma_{d-5}}{\Gamma_{4-\frac{d}{2}}\Gamma_{d-2}\Gamma^2_{\frac{d-2}{2}}}\frac{\pi^{d+1}A_d^4}{|x_\perp-R|^{2(d-5)}}\,. \tag{B.2}
$$

With this at hand we can expand the one-point function at (B.1) in R to find (3.29). We wish to point out that if we were to expand in R before doing the integrals (A.12) over $z_\perp^i$ in A we find divergences at $\mathcal{O}(R^{2\,n})$ that go as H_{-n} (the harmonic number) which we cannot regulate using dimensional regularization.

The $\mathcal{O}(h\,g^2)$-part of $\langle D_f\sigma\rangle$ can be found from the first diagram in figure 4 (neglecting the defect not connected to the vertex)

$$
\begin{aligned}
\langle D_f\sigma(x)\rangle_N^{(2)}\Big|_{h\,g^2}=(-2\,h)\frac{N\,g_1^2+g_2^2}{2}\int_{\mathbb{R}^d}d^dz\int_{\mathbb{R}^d}d^dw\,\langle\sigma(x)\sigma(z)\rangle|_{h=0}\,\langle\sigma(z)\sigma(w)\rangle^2\Big|_{h=0}\times\\
\times\sum_{n\geq0}\frac{R^{2\,n}}{(2\,n)!}\int_{\mathbb{R}^p}d^py\lim_{R'\to0}\partial_{R'}^{2\,n}\,\langle\sigma(w)\sigma(y\hat{x}_\parallel+R'\hat{x}_\perp^1)\rangle\Big|_{h=0}\,.
\end{aligned}
$$

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We can integrate over the parallel coordinates using the master integral (2.14), and over the normal coordinates using (2.19). After this we can differentiate w.r.t. R' using (3.12). This yields (3.30).

The other part of $\langle D_+D_-\sigma(x)\rangle_N^{(2)}$ are those of $\mathcal{O}(h^3g^2)$, which contain the B-part (here we denote $B = C_{+,+,+}$) of the one-point function (A.11) in the presence of only one defect as well as the two connecting diagrams in figure 7

$$\begin{aligned}\langle D_+D_-\sigma\rangle_N^{(2)}\Big|_{h^3g^2} = (3!)^2(-h)^3\left(-\frac{g_2}{3!}\right)^2\bigg(&\frac{C_{+,+,+}+C_{-,-,-}+C_{+,-,-}+C_{-,+,+}}{2!}+\\ &+C_{+,+,-}+C_{-,-,+}\bigg)\,.\end{aligned}\tag{B.3}$$

This is written in terms of the integral

$$\begin{aligned}C_{\alpha,\beta,\gamma} &= \int_{\mathbb{R}^d} d^dz\int_{\mathbb{R}^d} d^dw\ \langle\sigma(x)\sigma(z)\rangle|_{h=0}\ \langle\sigma(z)\sigma(w)\rangle|_{h=0}\,K_\alpha(z)K_\beta(z)K_\gamma(z)\\ &=\left(\pi^{\frac{p}{2}}A_d\frac{\Gamma_{\Delta_\phi-\frac{p}{2}}}{\Gamma_{\Delta_\phi}}\right)^5\int_{\mathbb{R}^{d-p}} d^{d-p}z_\perp\int_{\mathbb{R}^{d-p}} d^{d-p}w_\perp\times\\ &\quad\times\frac{1}{(|x_\perp-z_\perp|\,|z_\perp-w_\perp|\,|x_\perp-\alpha R|\,|x_\perp-\beta R|\,|x_\perp-\gamma R|)^{2\Delta_\phi-p}}\,.\end{aligned}\tag{B.4}$$

With this at hand we can expand (B.3) in R and integrate over the normal coordinates to find (3.32).

On the other hand, for D_f we have

$$\begin{aligned}\langle D_f\sigma\rangle_N^{(2)}\Big|_{h^3g^2} = &\frac{(3!)^2}{2!}(-2\,h)^3\left(-\frac{g_2}{3!}\right)^2\int_{\mathbb{R}^d} d^dz\int_{\mathbb{R}^d} d^dw\ \langle\sigma(x)\sigma(z)\rangle|_{h=0}\ \langle\sigma(z)\sigma(w)\rangle|_{h=0}\times\\ &\times\prod_{i=1}^{3}\sum_{m_i\geq 0}\frac{R^{2\,m_i}}{(2\,m_i)!}\int_{\mathbb{R}^p} d^py_i\lim_{R_1\to 0}\langle\sigma(z)\sigma(y_i\hat{x}_\parallel+R_1\hat{x}_\perp^1)\rangle\Big|_{h=0}\times\\ &\times\prod_{j=2}^{2}\lim_{R_j\to 0}\langle\sigma(w)\sigma(y_j\hat{x}_\parallel+R_j\hat{x}_\perp^1)\rangle\Big|_{h=0}\,.\end{aligned}$$

To solve this integral we perform the following steps:

1. Integrate over the parallel coordinates.
2. Differentiate w.r.t. R_i.
3. Integrate over $w_\perp$.
4. Integrate over $z_\perp$.

Doing this gives us (3.33).

Let us now renormalize $\langle D_f\sigma\rangle$ to find the bare couplings in (3.37). The one-point function near D_f is given by (3.26), (3.30), (3.33). To avoid $\frac{\gamma_E}{\epsilon}$-terms (of $\mathcal{O}(h^3g_2^2)$) after renormalization we have to factor out the free theory contribution. Let us write $\langle D_f\sigma\rangle_N$ in the following way

$$\langle D_f\sigma\rangle_N = D_0+\sum_{n\geq 0}R^{2\,n}D_n+\mathcal{O}(g^3)\,.\tag{B.5}$$

The $\mathcal{O}(R^0)$-terms are given by

$$D_0 = -\frac{h_0\Gamma_{1-\frac{\epsilon}{2}}}{2\,\pi^{2-\frac{\epsilon}{2}}|x_\perp|^{2-\epsilon}}\left(1+\frac{g_2h_0\Gamma_{-\frac{\epsilon}{2}}|x_\perp|^\epsilon}{16\,\pi^{2-\frac{\epsilon}{2}}(\epsilon-1)}+\frac{(N\,g_1^2+g_2^2)\Gamma_{-\frac{\epsilon}{2}}|x_\perp|^\epsilon}{256\,\pi^{3-\frac{\epsilon}{2}}(\epsilon^2-4\,\epsilon+3)}+ \right.$$
$$\left. +\frac{g_2^2h_0^2\Gamma^2_{-\frac{\epsilon}{2}}|x_\perp|^{2\,\epsilon}}{128(\epsilon^2-5\,\epsilon+2)}\right)\,, \tag{B.6}$$

and the $\mathcal{O}(R^{2\,n})$-terms are

$$\begin{aligned} D_n = &-\frac{h_n\Gamma_{1-\frac{\epsilon}{2}}(2-\epsilon)_n}{2\,\pi^{2-\frac{\epsilon}{2}}(2\,n)!|x_\perp|^{2(n+1)-\epsilon}}\Bigg[\,1+\\ &-\frac{4^\epsilon\pi\Gamma_{2(n+2-\epsilon)}+2^{3+2\,n}\Gamma_{n+\frac{1}{2}}\Gamma_{\frac{5}{2}-\epsilon}\Gamma_{n+2-\epsilon}}{\Gamma_{\frac{5}{2}-\epsilon}\Gamma_{n+2-\epsilon}}\frac{g_2h_n\Gamma_{1-\frac{\epsilon}{2}}|x_\perp|^\epsilon}{128\,\pi^{\frac{5-\epsilon}{2}}(2\,n-\epsilon)(n+1-\epsilon)}+\\ &-\frac{\Gamma_{2(n+1-\epsilon)}\Gamma_{-\frac{\epsilon}{2}}}{\Gamma_{\frac{3}{2}-\epsilon}\Gamma_{n+2-\epsilon}}\frac{(N\,g_1^2+g_2^2)|x_\perp|^\epsilon}{2^{9-2\,\epsilon}\pi^{5-\frac{\epsilon}{2}}(\epsilon-3)}+\\ &+\frac{\Gamma_{2\,n}\Gamma^2_{1-\frac{\epsilon}{2}}}{\Gamma_{2-\epsilon}\Gamma_{n+2-\epsilon}}\frac{g_2^2h_n^2n|x_\perp|^{2\,\epsilon}}{16\,\pi^{4-\epsilon}(3\,\epsilon+2\,n+2)(\epsilon-n)}\sum_{m=0}^{n}\frac{\Gamma_{2(n-m+1)-\epsilon}\Gamma_{2(m+1)-\epsilon}}{\Gamma_{2(n-m)+1}\Gamma_{2\,m+1}}\times\\ &\times\frac{1}{(\epsilon-2\,m)(\epsilon-m-1)}{}_4F_3\left(1-\frac{\epsilon}{2},\frac{3-\epsilon}{2},\frac{1}{2}-m,-m;\frac{1}{2},\frac{\epsilon-1}{2}-m,\frac{\epsilon}{2}-m;1\right)\Bigg]\,. \end{aligned}$$

Here $(x)_n$ is the Pochhammer symbol. $\langle D_f\sigma\rangle_N$ admit the ϵ-expansions at (3.28), (3.31), (3.35). To renormalize D_0, D_n we need the bare bulk couplings, g_i, at (A.17) and the Z-factor (A.15), (A.19) for σ. For the same reason as in section 2.3 (see the discussion above (A.16)) we divide each bare defect coupling in the one-point function of the renormalized field $\tilde{\sigma}$ with $\sqrt{Z}$: $h_n \to \frac{h_n}{\sqrt{Z}}$. We make the following ansatz for the bare defect couplings

$$h_n = \mu^{\frac{\epsilon}{2}}\tilde{h}_n\left(1+a_n\frac{\tilde{h}_n\tilde{g}_2}{\epsilon}+b_n\frac{N\,\tilde{g}_1^2+\tilde{g}_2^2}{\epsilon}+c_n^2\frac{\tilde{h}_n^2\tilde{g}_2^2}{\epsilon^2}+c_n^1\frac{\tilde{h}_n^2\tilde{g}_2^2}{\epsilon}+\mathcal{O}(g^3)\right)\,, \tag{B.7}$$

where we do not sum over n. By cancelling the poles in ϵ we are able to fix the constants a_n, b_n, c_n^2, c_n^1 (note that some of these are zero). This yields the dimensionless bare coupling constant (3.38), from which a couple of consistency checks can be made. Firstly, there is no $N\,g_1^2$-term in h_0 (it was exactly canceled by the Z-factor (A.19) of σ). This is expected as otherwise it would affect its f.p. found at the $\mathcal{O}(g)$. Secondly, $c_0^2 = a_0^2$ which is required to cancel a pole of ϵ in the β-function of h_0.

In addition, we also find the following dimensionfull bare coupling constant

$$h_{n\geq 1} = \mu^{\frac{\epsilon}{2}}\tilde{h}_n\left(1+\left(\frac{2^{2\,n-7}(\frac{3}{2})_n}{3(n+1)\pi^3}-\frac{1}{384\,\pi^3}\right)\frac{N\,\tilde{g}_1^2+\tilde{g}_2^2}{\epsilon}+\frac{2^{2\,n-5}\Gamma_{n+\frac{3}{2}}}{\pi^{\frac{9}{2}}n(n+1)^2}\frac{\tilde{h}_n^2\tilde{g}_2^2}{\epsilon}+\mathcal{O}(g^3)\right)\,.$$

The corresponding β-functions is[13]

$$\beta_{n\geq 1} = -\epsilon\,\tilde{h}_n + \left(\frac{2^{2\,n-7}(\frac{3}{2})_n}{3(n+1)\pi^3} - \frac{1}{384\,\pi^3}\right)(N\,\tilde{g}_1^2 + \tilde{g}_2^2)\tilde{h}_n + \frac{4^{n-2}\Gamma_{n+\frac{3}{2}}}{\pi^{\frac{9}{2}}n(n+1)^2}\tilde{h}_n^3\tilde{g}_2^2\,. \tag{B.8}$$

From which we can find in addition to a trivial f.p., the following non-trivial one

$$h^*_{n\geq 1} = \pm i\frac{\pi^{\frac{3}{4}}}{2^{n+3}3^{\frac{3}{2}}}\sqrt{n(n+1)\frac{2^{2\,n+1}(\frac{3}{2})_n - 3(n+1)}{\Gamma_{n+\frac{3}{2}}}}\sqrt{N} + \mathcal{O}\left(\frac{1}{N^2}\right)\,. \tag{B.9}$$

However, this f.p. is not perturbative, and thus only its trivial f.p. is valid

$$h^*_{n\geq 1} = 0\,. \tag{B.10}$$

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