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The emergence proposal and the emergent string

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Abstract: We explore the Emergence Proposal for the moduli metric and the gauge couplings in a concrete model with 7 saxionic and 7 axionic moduli fields, namely the compactification of the type IIA superstring on a 6-dimensional toroidal orbifold. We show that consistency requires integrating out precisely the 12 towers of light particle species arising from KK and string/brane winding modes and one asymptotically tensionless string up to the species scale. After pointing out an issue with the correct definition of the species scale in the presence of string towers, we carry out the emergence computation and find that the KK and winding modes indeed impose the classical moduli dependence on the one-loop corrections, while the emergent string induces moduli dependent logarithmic suppressions. The interpretation of these results for the Emergence Proposal are discussed revealing a couple of new and still not completely settled aspects.

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1 Introduction

Even though technically string theory is a well understood and applied mathematical framework of quantum gravity, the recently developed swampland program has revealed that, despite all its technical beauty, it conceals some of its underlying physical principles. It is the aim of the swampland program (see [1–4] for reviews) to bring them to the front and, in particular, to uncover its consequences for our low-energy effective field theory framework, which we are used to employ very successfully, for instance, in the description of particle physics in the form of the Standard Model.

One of the first formulated swampland conjectures is the Swampland Distance Conjecture [5] (see [6, 7] for a refined version), which was studied in the context of here relevant $N = 2$ supersymmetry in 4D in [8–17]. It has been generalized to more specialized set-ups, like the AdS Distance Conjecture [18] and the Gravitino Distance Conjecture [19, 20]. The Swampland Distance Conjecture says that in asymptotic limits in field space towers of exponentially light particle species appear. These are threatening the reliability of a Wilsonian low-energy effective field theory (EFT) that is keeping only the degrees of freedom below a certain cut-off. In some sense this is an awkward situation, as the stringy corrections to such a low-energy description are usually believed to be only under control in such asymptotic, i.e. weak coupling limits. And now another effect of quantum gravity is about to threaten its validity in precisely such regions or at least to restrict strongly its regime of validity? By moving over trans-Planckian distances in field space, new modes need to be integrated into the EFT which in principle could change certain couplings in the old EFT by inducing large corrections.

The so-called Emergence Proposal [8, 11, 21, 22] provides a new aspect to this situation. It is based on a field theoretical analysis of the one-loop effects the light towers of states have on certain quantities in the low-energy effective action. Initially, mostly the kinetic terms for scalars, fermions and gauge couplings in four dimensions had been considered, but more recently a more thorough analysis of emergence was carried out in [23, 24], which also considered arbitrary dimensions and also e.g. scalar potentials. In [25, 26] it was shown that the concept of emergence can also be applied to the effective field theory living in a strongly warped throat. This appears close to a conifold point in complex structure space and is at finite distance.

The important aspect of such a one-loop computation is that one integrates out those states whose masses are below the cut-off of quantum gravity, which naively is the four-dimensional Planck scale that, however, in the presence of many light species is known to be lowered to the species scale [27, 28]. Interestingly, there are two classes of arguments regarding the definition of the latter. In what we will call the QFT picture, it can be defined as the energy scale where the one-loop corrections to graviton scattering processes are of the same order as the tree-level ones. As such, it is defined by a one-loop effect leading to the intriguing result that the so-defined sum over loop amplitudes turns out to be independent of $\hbar$ and looks like a classical contribution. Alternatively, the species scale may be defined as the inverse radius of the smallest Black Hole that the EFT can describe. We will refer to this as the BH picture. The compatibility of those two definitions is a tricky issue and will be of relevance in the course of this work. Moreover, the species scale is moduli dependent which was analyzed more recently in [29–33].

Furthermore, it turned out that in simple examples the moduli dependent induced field metrics were of the same functional form as the usual tree-level ones. This gave rise to the idea of two possible meanings of emergence in quantum gravity. Following the formulation of [24], the Strong Emergence Proposal [1] states

> *Strong Emergence: in a theory of Quantum Gravity all light particles in a perturbative regime have no kinetic terms in the UV. The required kinetic terms appear as an IR effect due to loop corrections involving the sum over a tower of massless states.*

This is a very far-reaching and general proposal which would mean that quantum gravity in the UV is maybe a very simple theory, e.g. of purely topological nature, which only gives rise to geometric low-energy effective theories by integrating out the light towers of states up to the species scale. See [34] for an application of this principle to the Yukawa coupling in the Standard Model. Since the Strong Emergence Proposal is so general, it can in principle easily be falsified by finding a controllable model that violates its claim.

A much milder version is the Weak Emergence Proposal which does not assume vanishing kinetic terms in the UV and makes a statement about quantum corrections to the metrics that match the "tree level" behavior. To describe the objective and the results of this paper, it turns out to be useful to distinguish between two variants of Weak Emergence, called Variant A and Variant B. The more restrictive Variant A may be formulated as

> *Weak Emergence (Variant A): in a consistent theory of Quantum Gravity, for any singularity at infinite distance in the moduli space of the EFT, there are associated infinite towers of states becoming massless. These towers induce quantum corrections to the singular kinetic terms matching their tree-level behavior.*

Implicitly this also means that no new singular dependence is generated at one-loop. This formulation restricts to the dependence of the kinetic terms on the single modulus taken to infinity. This is what was mostly studied so far in the literature. However, the full moduli metric will also depend on the many directions orthogonal to the asymptotic one and we would like to check whether also this dependence is (at least partially) recovered. We therefore add a second variant of Weak Emergence, which reads

> *Weak Emergence (Variant B): at infinite distance in the EFT moduli space, infinite towers of light states appear which induce quantum corrections matching, beyond the singular ones from Variant A, some or even all the orthogonal kinetic terms.*

Since the second variant talks also about the non-singular terms, it is clearly stronger than the first and closer in spirit to the strong version of the Emergence Proposal. An even more restrictive assumption would be that there exists a notion of exact emergence in asymptotic limits, so that the full moduli dependence is emerging from integrating out the towers of light states.

In this paper, we will shed some new light on these proposals by generalizing the mostly single modulus computations to a model featuring multiple moduli fields. Of course, the main obstacle in performing a concrete computation is the precise knowledge of the light towers in the asymptotic region one is interested in. For this reason, we consider the simple, supersymmetric compactification of the type IIA superstring on the untwisted sector of a $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold of the 6-torus and keep track of 7 saxionic and 7 axionic moduli fields, i.e. the complex structure moduli, the complexified Kähler and the complexified 4D dilaton moduli. These are all moduli arising in the NS-NS sector of the type IIA superstring and for simplicity we will set all the R-R moduli to zero in our analysis. The advantage of this model is that in the weak string coupling limit, we know the precise moduli dependent spectrum so that we can carry out the full loop computation of the kinetic terms keeping the full moduli dependence. Building upon some of the computational techniques described in [24], we can eventually check which components of the full tree-level 14 dimensional field metric emerge in this asymptotic limit.

This paper is organized as follows: in section 2 we define our model of interest and make a first approach to compute the one-loop corrections by integrating out only the

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Kaluza-Klein (KK) towers of light states. We find that components of the field metric emerge whose classical counterparts are vanishing. Some of the basic notions and relations of the Emergence Proposal are relegated to appendix A.

In section 3 we first observe that in the asymptotic weak string coupling limit, a consistent computation requires to include not only the KK towers but also the winding modes and the fundamental string excitations. As we will discuss, the handling of the latter is not straightforward, as there is an issue already with the definition of the species scale. It turns out that a naive QFT approach does not give exactly the same result as an approach based on Black Holes. They differ by certain logarithmic factors of type $\log(M_{\rm pl}/M_{\rm s})$. Such factors will accompany our computations in the remainder of the paper.

In the second part of this section, we also consider two other limits of type IIA on a 6-torus, namely where one Kähler modulus goes to infinity and where one complex structure modulus becomes asymptotically large. Consistent with the Emergent String Conjecture [16, 35, 36] (see also [17, 37]), we will show that in these limits there are the same number of light towers of 4D particles and one low-tension string as were present in the weak coupling limit. However, now these states are mostly given by wrapped D-branes and NS-branes. In fact, for the large Kähler modulus regime, the relevant light 4D string is given by a wrapped NS5-brane, whereas in the large complex structure limit it becomes a wrapped KK-monopole. Despite the difficulty of knowing all bound states of these branes and their masses, in appendix C we provide some admittedly speculative arguments how such a mass formula could look like and unscrupulously use it in the body of the paper.

In section 4, we carry out a full emergence computation including all 13 towers of light states. As we will emphasize, only in the QFT approach there are techniques available for such computations. As a consequence, ubiquitous log-suppressions will appear, whose potential interpretation will be discussed at the end of this section. However, it will turn out that we recover almost the full tree-level metric and gauge couplings. To be more precise, no terms are generated that were absent at tree-level even though a few singular classical couplings are just not generated. This would disfavor both Variant A and Variant B of the Weak Emergence Proposal. However, the non-vanishing ones do have the same moduli dependence as their classical counterparts. Hence, ignoring for the moment the vanishing singular ones, we could say that our results go beyond Variant A of the Weak Emergence Proposal and be more in favor of Variant B. Moreover, we encounter a non-trivial numerical relation that combines contributions from the KK and winding towers as well as contributions related to the string tower. All this makes us confident that despite the unsettled question about the QFT versus the Black Hole picture, our computation captures already a large portion of the story.

2 Preliminaries

In this section, we perform a non-trivial test of the emergence proposal by confronting it with a field space metric of 14 moduli. First we define more concretely the toroidal orbifold background that we consider and attempt to compute the one-loop corrections to the field space metric taking only the KK-modes into account. This is meant for pedagogical

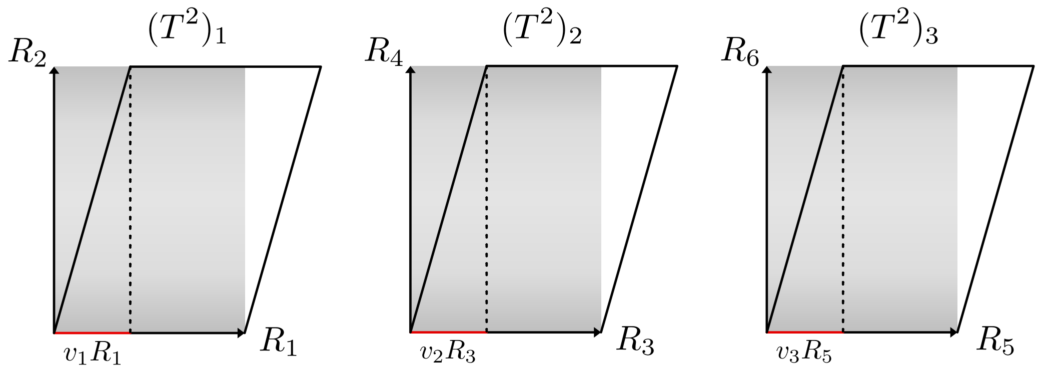

Figure 1. Conventions for the components of the factorized 6-torus. For instance, $(T^2)_1$ has a complex structure modulus with saxion $u_1 = R_2/R_1$ and the respective Kähler modulus $t_1 = R_1R_2/\alpha'$ is measured in string units.

purposes and for sharpening our computational tools. For completeness, we have collected the basic notions and a number of useful relations about the Emergence Conjecture in appendix A.

2.1 The orbifold background

We consider the type IIA superstring compactified on the usual $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold of a 6-dimensional torus. To allow for this action, the torus takes the factorized form $T^6 = T^2 \times T^2 \times T^2$ so that after introducing a complex coordinate Z_I on each T^2 the action takes the form

$$\Theta : \begin{cases} Z_1 \to -Z_1 \\ Z_2 \to -Z_2 \\ Z_3 \to Z_3 \,, \end{cases} \qquad \Theta' : \begin{cases} Z_1 \to Z_1 \\ Z_2 \to -Z_2 \\ Z_3 \to -Z_3 \,. \end{cases} \tag{2.1}$$

As we will see below, for our purpose we can restrict ourselves to the untwisted sector of this $N = 2$ supersymmetric background. As shown in figure 1, each T^2 comes with a complex structure modulus $\mathcal{U}_I = v_I + iu_I$ and a complexified Kähler modulus $\mathcal{T}_I = b_I + it_I$, where b_I is the Kalb-Ramond two-form field B_2 integrated over the 2-torus. Moreover, t_I is the string frame volume of the I-th T^2-factor measured in units of α'. The complex structure moduli are part of three $N = 2$ hypermultiplets and the complexified Kähler moduli reside in three $N = 2$ vector multiplets.

In order to apply the formulas reviewed in appendix A, we need to know the spectrum of particles in 4D whose mass is below the UV cut-off scale, i.e. the species scale. To identify all of these states is in general a non-trivial question, but in certain asymptotic limits one might get better control. In fact, so far the emergence proposal has only been tested in such infinite field distance limits. The best understood limit is, of course, the perturbative string limit, i.e. where the string coupling is very small $g_s = e^{\phi} \ll 1$. It is appropriate to merge the 4D dilaton and its axionic partner into a complex field

$$S = \rho + ie^{-\phi}\sqrt{t_1 t_2 t_3} = \rho + i\sigma\,. \tag{2.2}$$

Here the axion ρ is defined as the 4D magnetic dual of the Kalb-Ramond field B_2 with both legs along the 4D flat space-time. It can be thought of as the 10D magnetic dual 6-form B_6 with all legs along the T^6. The field S is the NS-NS part of an $N = 2$ hypermultiplet.

In the weak coupling limit, the usual CFT string partition function provides the lightest states, which are the KK, winding and string oscillator modes. The mass dependence of these states is known explicitly for arbitrary values of the complex structure, the complexified Kähler moduli and the 4D dilaton as [38, 39]

$$M^2 = \frac{M_{\rm pl}^2}{\sigma^2}\left\{\sum_{I=1}^{3}\left[\left(\frac{m_1^I - v_I m_2^I + b_I n_1^I + b_I v_I n_2^I}{u_I^{\frac{1}{2}}\, t_I^{\frac{1}{2}}}\right)^2 + \left(\frac{(m_2^I - b_I n_2^I)\, u_I^{\frac{1}{2}}}{t_I^{\frac{1}{2}}}\right)^2 \right.\right. \\ \left.\left. + \left(\frac{(n_1^I + v_I n_2^I)\, t_I^{\frac{1}{2}}}{u_I^{\frac{1}{2}}}\right)^2 + \left(n_2^I u_I^{\frac{1}{2}}\, t_I^{\frac{1}{2}}\right)^2\right] + \kappa^2 N\right\}. \tag{2.3}$$

Here, the $m_{1,2}^I$ denote the KK modes and the $n_{1,2}^I$ the winding modes[1] of the fundamental string and can be more compactly denoted as $\vec{m}^I = (m_1^I, m_2^I)$ and $\vec{n}^I = (n_1^I, n_2^I)$.[2] Moreover, N is the level of the tower of string oscillator modes, which comes with a degeneracy of states scaling at large level N as

$$\deg_N = \frac{\gamma}{N^{\frac{\nu}{2}}}\, e^{\beta\sqrt{N}}\,, \tag{2.4}$$

where for later purposes we left open the values of the parameters β, γ and ν. For the 10D type IIA superstring one has e.g. $\beta = 4\pi\sqrt{2}$ and $\nu = 11$.

In addition, the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold has 3 twisted sectors, each of them comprised of 16 fixed points of codimension 4. Depending on whether one turns on discrete torsion or not, the twisted sector gives rise to 48 complex structure or Kähler moduli, respectively. Moreover, in each of them there are 2 KK and 2 winding modes and the excitation number of the corresponding twisted string. We will see in section 4.1 that these additive towers of light states give subdominant contributions to the number of light species.

Observe that the axion ρ does not appear in the mass formula (2.3). In the following, we will sometimes collectively denote all 14 real moduli as $\mathcal{M}_A$ or if we restrict to the set of 12 complex structure and complexified Kähler moduli as $\mathcal{M}_a$.

[1]Let us note that the presence of these KK and winding modes are special for this orbifold limit of a Calabi-Yau manifold, as they correspond to torsional one-cycles which are e.g. absent for a Calabi-Yau X with vanishing fundamental group $\pi_1(X) = 0$. Strictly speaking, only KK and winding modes are present which are invariant under the orbifold action. However, we have convinced ourselves that this issue only induces a change in some numerical factors.

[2]$\bar{N}$ has been eliminated by imposing the level matching condition $N - \bar{N} + \sum_I \vec{m}^I \cdot \vec{n}^I = 0$, which would at most change our final results by a numerical factor.

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The 4D dilaton relates the string to the 4D Planck scale

$$M_{\rm s} \simeq \frac{M_{\rm pl}}{\sigma}\,, \tag{2.5}$$

where we left open a numerical factor. The classical field space metric on this 14-dimensional moduli space is known to be of the form [40] (see also [17, 41, 42])

$$\begin{aligned} ds^2 &= g^{(0)}_{AB}\, d\mathcal{M}_A\, d\mathcal{M}_B \\ &= \frac{1}{\sigma^2}d\sigma^2 + \frac{1}{4\sigma^4}d\rho^2 + \sum_{I=1}^{3}\frac{1}{4u_I^2}(du_I^2 + dv_I^2) + \sum_{I=1}^{3}\frac{1}{4t_I^2}(dt_I^2 + db_I^2)\,, \end{aligned} \tag{2.6}$$

so that $G^{(0)}_{AB} = M_{\rm pl}^2\, g^{(0)}_{AB}$. Note that both this metric and the mass formula enjoy a (discrete) shift symmetry for v_I, b_I and ρ so that these moduli are (quasi)-axions.

In the type IIA case, the three complexified Kähler moduli are part of a full $N = 2$ vector multiplet, where the three gauge fields come from the dimensional reduction of the R-R three-form C_3 along the three 2-cycles. There is one more vector field, which is the graviphoton residing in the $N = 2$ gravity multiplet. This is just the R-R one-form C_1. These four 4D gauge fields are denoted as $\mathcal{A}^\Lambda$ (for $\Lambda = 0, 1, 2, 3$) with field strength $\mathcal{F}^\Lambda = d\mathcal{A}^\Lambda$.

The kinetic terms of the Kähler moduli and these vector fields are governed by special geometry, which we here only very briefly summarize. First, one defines homogeneous coordinates X^Λ which eventually are related to the inhomogeneous coordinates of the Kähler moduli space as $X^0 = 1$ and $\mathcal{T}_I = X^I/X^0$. Moreover, one introduces a prepotential F which is a homogeneous function of degree two of the coordinates X^Λ. Defining $F_\Lambda = \partial_\Lambda F$, the metric on the moduli space is given in terms of the Kähler potential

$$K = -\log\Big(i(X^\Lambda \overline{F}_\Lambda - \overline{X}^\Lambda F_\Lambda)\Big)\,. \tag{2.7}$$

The gauge kinetic terms are then expressed as [43]

$$S_{\rm gauge} = -\frac{1}{2}\int d^4x \Big(f_{\Lambda\Sigma}\, \mathcal{F}^\Lambda \wedge \star\mathcal{F}^\Sigma + \Theta_{\Lambda\Sigma}\, \mathcal{F}^\Lambda \wedge \mathcal{F}^\Sigma\Big)\,, \tag{2.8}$$

where at the classical level the gauge kinetic function $f_{\Lambda\Sigma}$ and the Theta-angles $\Theta_{\Lambda\Sigma}$ are given in terms of the imaginary and real parts of the period matrix of the underlying Calabi-Yau threefold. The latter can be expressed in terms of the prepotential as

$$\mathcal{N}_{\Lambda\Sigma} = \overline{F}_{\Lambda\Sigma} + 2i\frac{{\rm Im}(F_{\Lambda\Gamma}){\rm Im}(F_{\Sigma\Delta})X^\Gamma X^\Delta}{{\rm Im}(F_{\Gamma\Delta})X^\Gamma X^\Delta} \tag{2.9}$$

where $F_{\Lambda\Sigma} = \partial^2 F/\partial X^\Lambda \partial X^\Sigma$. In our toroidal orbifold case, we have the prepotential

$$F = \frac{X^1 X^2 X^3}{X^0} \tag{2.10}$$

leading to the gauge couplings (see e.g. [44])

$$f = \mathrm{Im}(\mathcal{N}) = \begin{pmatrix} t_1 t_2 t_3 q & -t_2 t_3 \frac{b_1}{t_1} & -t_1 t_3 \frac{b_2}{t_2} & -t_1 t_2 \frac{b_3}{t_3} \\ \dots & \frac{t_2 t_3}{t_1} & 0 & 0 \\ \dots & \dots & \frac{t_1 t_3}{t_2} & 0 \\ \dots & \dots & \dots & \frac{t_1 t_2}{t_3} \end{pmatrix} \tag{2.11}$$

with

$$q = 1 + \left(\frac{b_1}{t_1}\right)^2 + \left(\frac{b_2}{t_2}\right)^2 + \left(\frac{b_3}{t_3}\right)^2 . \tag{2.12}$$

Similarly, for the Θ-angles one gets

$$\Theta = \mathrm{Re}(\mathcal{N}) = \begin{pmatrix} 2b_1 b_2 b_3 & -b_2 b_3 & -b_1 b_3 & -b_1 b_2 \\ \dots & 0 & b_3 & b_2 \\ \dots & \dots & 0 & b_1 \\ \dots & \dots & \dots & 0 \end{pmatrix} . \tag{2.13}$$

To discriminate among the various versions of the Emergence Proposal, the question is which parts of the tree-level moduli metric and gauge kinetic terms can be recovered via integrating out (a subset of) the light perturbative string states at the one-loop level in the asymptotic field regimes.

2.2 Emergence for KK modes

One basic ingredient is the quantum gravity cut-off, which in the case of a large number of light species is not the Planck-scale but the species scale $\tilde{\Lambda}$ [27, 28]. In the QFT picture, one considers the quantum corrections to the graviton propagator due to the coupling of $N_{\rm sp}$ light species to gravity. Starting from the Einstein-Hilbert term of the action, the species scale is defined as the mass scale where the one-loop contributions become of the same order as the tree level ones. In particular, in 4D, which we will focus on in the following sections, one finds a propagator

$$\pi^{-1}(p^2) = p^2 \left(1 - \frac{N_{\rm sp}\, p^2}{120\pi M_{\rm pl}^2} \log\left(-\frac{p^2}{\mu^2}\right) + \gamma \sum_{n=1}^{N_{\rm sp}} \frac{p^2}{M_{\rm pl}^2} \frac{m_n}{\sqrt{-p^2}} \right) , \tag{2.14}$$

where we also included the form of the mass dependent terms [45] with γ denoting an order one parameter.

In this way, we get the scaling[3]

$$\tilde{\Lambda} \sim \frac{M_{\rm pl}}{\sqrt{N_{\rm sp}}} . \tag{2.15}$$

[3] We have checked that for KK-towers and string towers, the mass dependent term in (2.14) really gives $N_{\rm sp}\tilde{\Lambda}^2/M_{\rm pl}^2$ so that for the natural choice $\mu \sim \tilde{\Lambda}$ one indeed arrives at the relation (2.15) without any log-correction. We thank Niccolò Cribiori for bringing this to our attention.

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In practice one has two coupled relations, namely (2.15) and the definition of the number of light species

$$N_{\rm sp} = \#(m \leq \tilde{\Lambda})\,, \tag{2.16}$$

which can be solved for $N_{\rm sp}$ and $\tilde{\Lambda}$.

In the Black Hole picture, the species scale is defined via the radius $r_0 = 1/\tilde{\Lambda}$ of the minimal-sized Black Hole that can be described within the EFT. The mass and Bekenstein-Hawking entropy of such a Black Hole are

$$M_{\rm BH} = \frac{M_{\rm pl}^2}{\tilde{\Lambda}}\,, \qquad\qquad S_{\rm BH} = \frac{M_{\rm pl}^2}{\tilde{\Lambda}^2}\,. \tag{2.17}$$

The number of species is defined via the statistical entropy as

$$S_{\rm BH} = \log \Omega(M_{\rm BH}) =: N_{\rm sp}\,, \tag{2.18}$$

where $\Omega(M_{\rm BH})$ is the number of ways the macroscopic Black Hole of mass $M_{\rm BH}$ can be realized by the microstates. Note that this definition of the number of species also satisfies the relation (2.15). In practice, a second relation between $N_{\rm sp}$ and $\tilde{\Lambda}$ follows from the microcanonical relation (2.18) so that again both are determined.

Species scale for KK-modes. In appendix B, we apply these two definitions to a one-dimensional KK tower of states with spacing

$$\Delta m = \frac{M_{\rm s}}{r} = \frac{M_{\rm pl}}{\sigma r}\,. \tag{2.19}$$

Here r denotes the radius of the circle in units of the string length. In this case, the QFT approach is very simple whereas the BH approach turns out to be a bit more involved.[4] However, at the end of the day both approaches give the same result

$$\tilde{\Lambda} \sim \frac{M_{\rm pl}}{(\sigma r)^{\frac{1}{3}}}\,, \qquad\qquad N_{\rm sp} \sim (\sigma r)^{\frac{2}{3}}\,. \tag{2.20}$$

Note that the species scale is nothing else than the 5D Planck-scale.

Next, we generalize the computation in the QFT picture to the full toroidal orbifold model, where we consider the weak string coupling limit $\sigma \to \infty$ while keeping all the other moduli in a moderate regime. Since σ only appears as an overall factor in the mass formula, in the regime $t_I > 1$ the KK modes are the lightest modes. For this truncated mass spectrum, the number of light species reads

$$N_{\rm sp} = \underbrace{\sum_{\vec{m}^I}}_{M_{KK} \leq \tilde{\Lambda}^{\rm (KK)}} \approx \int \prod_{I=1}^{3} dm_1^I\, dm_2^I\,, \tag{2.21}$$

[4]We acknowledge the support of Niccolò Cribiori for carrying out this computation.

where we are summing over modes with non-zero excitations such that the total mass lies below the threshold $\tilde{\Lambda}^{(\mathrm{KK})}$. In what follows, sums over excitation numbers will always be approximated by integrals, requiring a sufficiently dense spectrum to be accurate. Since the mass (2.3) has an overall suppression by σ, the weak coupling limit indeed justifies taking the continuum limit in all directions. To implement the bound $M_{KK} \leq \tilde{\Lambda}^{(\mathrm{KK})}$, we introduce the 6 variables x^I, y^I via

$$M_{\mathrm{KK}}^2 = \frac{M_{\mathrm{pl}}^2}{\sigma^2} \sum_{I=1}^{3} \left[\underbrace{\left(\frac{m_1^I - v_I m_2^I}{u_I^{\frac{1}{2}}\, t_I^{\frac{1}{2}}} \right)}_{x^I}^2 + \underbrace{\left(\frac{m_2^I\, u_I^{\frac{1}{2}}}{t_I^{\frac{1}{2}}} \right)}_{y^I}^2 \right]. \tag{2.22}$$

The determinant of the Jacobian of this change is $\det(J) = t_1 t_2 t_3 \equiv \mathcal{V}_6$. In these coordinates we are integrating over a ball in 6 dimensions with radius $R = \frac{\tilde{\Lambda}^{(\mathrm{KK})}}{M_{\mathrm{pl}}}\sigma$. Hence, it is convenient to introduce the 6D spherical coordinates

$$\begin{aligned} x^1 &= r\cos\varphi_1 \\ y^1 &= r\sin\varphi_1\, \cos\varphi_2 \\ x^2 &= r\sin\varphi_1\, \sin\varphi_2\, \cos\varphi_3 \\ y^2 &= r\sin\varphi_1\, \sin\varphi_2\, \sin\varphi_3\, \cos\varphi_4 \\ x^3 &= r\sin\varphi_1\, \sin\varphi_2\, \sin\varphi_3\, \sin\varphi_4\, \cos\varphi_5 \\ y^3 &= r\sin\varphi_1\, \sin\varphi_2\, \sin\varphi_3\, \sin\varphi_4\, \sin\varphi_5 \end{aligned} \tag{2.23}$$

with $r \geq 0$, $\varphi_1, \ldots, \varphi_4 \in [0, \pi]$ and $\varphi_5 \in [0, 2\pi]$. The integration measure becomes

$$\mu = dr\, r^5\, d\varphi_1 \ldots d\varphi_5\ \sin^4\varphi_1\, \sin^3\varphi_2\, \sin^2\varphi_3\, \sin\varphi_4\,. \tag{2.24}$$

Now, we can compute N_{sp} in the new coordinates and invoke (2.15) (eliminating the $\log(N_{\mathrm{sp}})$ correction by setting $\tilde{\Lambda} = \mu$ in (2.14)) to determine the species scale

$$\tilde{\Lambda}^{(\mathrm{KK})} = \frac{M_{\mathrm{pl}}}{\sigma^{3/4}} \left(\frac{6}{\mathcal{V}_6\, \mathrm{vol}(S^5)} \right)^{\frac{1}{8}} \sim \frac{\sigma^{1/4}}{\mathcal{V}_6^{1/8}} M_{\mathrm{s}} \sim M_{\mathrm{pl},10}\,. \tag{2.25}$$

The quantum gravity cut-off turns out to be the ten-dimensional Planck scale, as expected given that only the KK modes have been taken under consideration in this calculation.

One-loop corrections to the moduli space metric. Next, let us determine the one-loop moduli metric from integrating out the light KK modes. In appendix A we recall that these emergent one-loop diagrams mimic a classical behavior in the sense that $\hbar$ drops out. However, in the following we nevertheless just call them one-loop diagrams. When we really mean the standard stringy loop diagrams, i.e. the higher genus diagrams coming with extra factors of the string coupling constant g_{s}, we call them "stringy one-loop corrections".

According to (A.15), up to numerical prefactors the one-loop metric is given by

$$G^{(1)}_{\mathcal{M}_A \mathcal{M}_B} \simeq \underbrace{\sum_{\vec{m}^I}}_{M_{\mathrm{KK}} \leq \tilde{\Lambda}^{(\mathrm{KK})}} (\partial_{\mathcal{M}_A} M_{\mathrm{KK}})\, (\partial_{\mathcal{M}_B} M_{\mathrm{KK}})\, \log\left(\frac{\tilde{\Lambda}^{(\mathrm{KK})}}{M_{\mathrm{KK}}} \right). \tag{2.26}$$

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To proceed, we need the derivatives of the mass formula, which can be expressed in the compact form

$$\begin{aligned}
\partial_{u_I} M_{\rm KK} &\simeq \frac{M_{\rm pl}}{\sigma}\frac{1}{2\,r\,u_I}\Big(-x_I^2+y_I^2\Big) = \frac{M_{\rm pl}}{\sigma}\frac{r}{2\,u_I}\,m_{u_I}(\underline{\varphi}) \\
\partial_{v_I} M_{\rm KK} &\simeq \frac{M_{\rm pl}}{\sigma}\frac{1}{2\,r\,u_I}\Big(-2\,x_I\,y_I\Big) = \frac{M_{\rm pl}}{\sigma}\frac{r}{2\,u_I}\,m_{v_I}(\underline{\varphi}) \\
\partial_{t_I} M_{\rm KK} &\simeq -\frac{M_{\rm pl}}{\sigma}\frac{1}{2\,r\,u_I}\Big(x_I^2+y_I^2\Big) = \frac{M_{\rm pl}}{\sigma}\frac{r}{2\,t_I}\,m_{t_I}(\underline{\varphi}) \\
\partial_{\sigma} M_{\rm KK} &\simeq -\frac{M_{\rm pl}}{\sigma^2} r\,, \qquad \partial_\rho M_{\rm KK} = 0\,,
\end{aligned} \tag{2.27}$$

where inserting the definition of the spherical coordinates (2.23) allows us to determine the functions $m_{u_I}(\underline{\varphi}),\dots,m_{t_I}(\underline{\varphi})$ which only depend on the angular variables $\varphi_1,\dots,\varphi_5$. From the last line in (2.27), we can also formally introduce $m_\sigma(\underline{\varphi})=1$. In the evaluation of (2.26) the following "angular metric" appears

$$\tilde{g}_{\mathcal{M}_A\mathcal{M}_B} := \int d\Omega_5\, m_{\mathcal{M}_A}(\underline{\varphi})\, m_{\mathcal{M}_B}(\underline{\varphi})\,. \tag{2.28}$$

The corresponding integrals over the 5 angular variables can be carried out explicitly, yielding

$$\tilde{g}_{u_I u_I} = \tilde{g}_{v_I v_I} = \tilde{g}_{t_I t_{J\neq I}} = \frac{\pi^3}{12}\,, \quad \tilde{g}_{t_I t_I} = \frac{\pi^3}{6}\,, \quad \tilde{g}_{\sigma t_I} = \frac{\pi^3}{3}\,, \quad \tilde{g}_{\sigma\sigma} = \pi^3\,. \tag{2.29}$$

The final radial integral can be evaluated explicitly using

$$\int_0^{r_0} r^7 \log\left(\frac{r_0}{r}\right) = \frac{r_0^8}{64}\,. \tag{2.30}$$

Collecting the remaining prefactors affecting the relative normalizations of the metric components and taking into account an overall not yet determined coefficient λ, we find

$$\begin{aligned}
G^{(1)}_{u_I u_J} = G^{(1)}_{v_I v_J} &= \frac{1}{128}\lambda G^{(0)}_{u_I u_J}\,, \qquad G^{(1)}_{t_I t_I} = \frac{1}{64}\lambda G^{(0)}_{t_I t_I}\,, \qquad G^{(1)}_{\sigma\sigma} = \frac{3}{32}\lambda G^{(0)}_{\sigma\sigma}\,, \\
G^{(1)}_{t_I\sigma} &= \frac{M_{\rm pl}^2}{64\,t_I\sigma}\,, \qquad G^{(1)}_{t_I t_{J\neq I}} = \frac{M_{\rm pl}^2}{512\,t_I t_J}\,, \qquad G^{(1)}_{\rho\mathcal{M}_A} = 0\,.
\end{aligned} \tag{2.31}$$

A comparison of this one-loop field metric with equation (2.6) reveals that the components involving the complex structure moduli u_I, v_I have the right relative normalization but that all other components are at odds. Since the mass formula does not contain the axion ρ, it is immediately clear that it completely decouples. Moreover, there are non-vanishing off-diagonal components in the second line of (2.31) whose tree-level counterparts are zero. In addition, the relative normalization of various diagonal components is not consistent with the tree-level one. Therefore, the one-loop induced metric misses some of the structure of the tree-level one and due to the singular off-diagonal components $G^{(1)}_{t_I\sigma}$ it is not even consistent with the most conservative version of Weak Emergence, namely Variant A.

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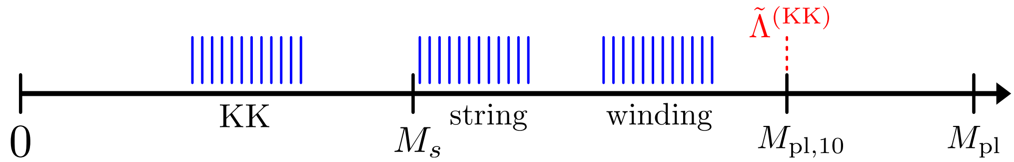

Figure 2. Tower mass scales on the energy axis. Compared to the string scale M_{s}, KK masses are suppressed by Kähler moduli, while winding modes are enhanced by them. The exponentially degenerate string excitations begin at M_{s}.

3 String towers

The result from the previous section might not be too surprising in view of a serious shortcoming of our computation, namely that in the large σ limit with all t_I moderately large one has the hierarchy

$$\tilde{\Lambda}^{(\mathrm{KK})} \gg M_{\mathrm{wind}} > M_{\mathrm{s}} > M_{\mathrm{KK}}\,. \tag{3.1}$$

As shown in figure 2, for sufficiently large σ, both the winding modes and the string excitations turn out to be lighter than the species scale and, therefore, should have been included from the very beginning. In this case, the mass formula treats the complex structure and Kähler moduli symmetrically and we would have a chance to recover the classical Kähler moduli metric.

3.1 The species scale for string towers

Being forced to include the highly degenerate tower of string excitations into our considerations, the first question is how to generalize the emergence computation to this case. Unfortunately, this is not as straightforward as one might think, as an issue already arises for the determination of the species scale and the number of light species.

To explain this, we have to recall the two ways to determine them, namely the QFT picture and the BH picture. Again let us discuss this for a simplified model where one only has a string tower with mass levels $M = M_{\mathrm{s}}\sqrt{N}$ and degeneracy $\deg_N$. For sufficiently large mass levels N, one can use the asymptotic expansion (2.4), which we recall here

$$\deg_N = \frac{\gamma}{N^{\frac{\nu}{2}}}\, e^{\beta\sqrt{N}}\,. \tag{3.2}$$

Let us look at the BH picture first [46]. Then, the excitation level required for the BH mass is

$$\sqrt{N_{\mathrm{BH}}} \sim \frac{M_{\mathrm{BH}}}{M_{\mathrm{s}}} \sim \frac{M_{\mathrm{pl}}^2}{M_{\mathrm{s}}\tilde{\Lambda}_{\mathrm{BH}}} \tag{3.3}$$

which for large σ is expected to be a very large number. Then, up to a β factor, the BH entropy is

$$S_{\mathrm{BH}} \sim \log\left(\deg_{N_{\mathrm{BH}}}\right) \sim \sqrt{N_{\mathrm{BH}}} - \frac{\nu}{\beta}\log\left(\sqrt{N_{\mathrm{BH}}}\right). \tag{3.4}$$

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QFT picture	BH picture
$\tilde{\Lambda}_{\rm QFT} \sim \frac{M_{\rm pl}}{\sigma} \log(\sigma)$	$\tilde{\Lambda}_{\rm BH} \sim \frac{M_{\rm pl}}{\sigma}$
$N_{\rm sp} = \frac{\sigma^2}{\log^2 \sigma}$	$N_{\rm sp} = \sigma^2$

Table 1. Species scale and $N_{\rm sp}$ in QFT and BH approach.

Setting this equal to the Bekenstein-Hawking entropy (2.17) and using (3.3) gives an implicit equation for $\tilde{\Lambda}$ which can be solved approximately by

$$\tilde{\Lambda}_{\rm BH} \sim M_{\rm s} + \frac{\nu}{\beta} M_{\rm s} \frac{\log \sigma}{\sigma^2} + \dots \,. \tag{3.5}$$

Hence, for $\sigma \to \infty$ the species scale approaches the string scale $M_{\rm s}$ from above. At leading order the number of species is then given by the entropy, i.e.

$$N_{\rm sp,BH} \sim \frac{M_{\rm pl}^2}{M_{\rm s}^2} \sim \sigma^2 \,. \tag{3.6}$$

Note that, as mentioned in [4, 31], this result is consistent with the proposal [29] that for the vector multiplet moduli space, the number of light species is given by the topological one-loop free-energy.[5]

That something is at odds with the corresponding QFT computation can already be expected from the observation that the number of species can still be big for large σ with at the same time the species scale being close to $M_{\rm s}$. Indeed, following the same strategy as [24, 47] and computing the number of species by integrating over all string states with energy smaller than the species scale and then solving the resulting implicit equation (2.15), one gets

$$\tilde{\Lambda}_{\rm QFT} \sim M_{\rm s} \log\left(\frac{M_{\rm pl}}{M_{\rm s}}\right) \sim M_{\rm s} \log(\sigma) \,. \tag{3.7}$$

In this case, the species scale comes out exponentially larger than $M_{\rm s}$ so that indeed a large number of string modes

$$N_{\rm sp,QFT} \sim \frac{\sigma^2}{\log^2 \sigma} \tag{3.8}$$

can be lighter, which is self-consistent with having used the asymptotic expression (2.4). Let us summarize the results in table 1.

Thus, even though the two computations seem to be self-consistent, they are not mutually consistent, at least as long as one is not willing to be agnostic about log-factors.

Unfortunately, this tension has not been resolved yet. However, as the authors of [4, 31], we tend to eventually rather trust the Black Hole picture. In the QFT approach we extrapolate the quantum field theory loop diagrams to energies that are much higher than

[5]We thank Max Wiesner for helpful discussions on this point.

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the string scale M_s. However, this high energy regime $M_\mathrm{s} < E < M_\mathrm{pl}$ is strongly believed to be very different from the sub-stringy one. There are indications that the string scattering amplitudes simplify in the high energy regime [48], and there is of course the issue with the Hagedorn temperature $T_H \sim M_\mathrm{s}$, where the canonical partition functions ceases to be well defined and a phase transition supposedly occurs [49].

However, the idea and the formalism of the Emergence Proposal are intimately linked to the QFT approach and so far there is no way to carry out any emergence computation using, for instance, the species scale and N_sp derived from the BH picture. Therefore, in section 4 we will take a pragmatic approach and explore what results we get by applying the QFT approach for the KK, winding and string states. Since the results for $\tilde{\Lambda}$ and N_sp differ just by some log-factors, we might still see a large portion of the emerging structure.

3.2 Asymptotic regime $t_1 \gg 1$

Before we move on to this computation, let us consider other asymptotic regions in the moduli space. We will see that here one gets the same number of light particle species, as well as an asymptotically tensionless string. First consider the $t_1 \gg 1$ regime.

Towers of light states. First, we determine the spectrum of light states below the quantum gravity cut-off $\tilde{\Lambda}$, where for simplicity, we set all axions to zero and consider only the dependence on the saxionic moduli (please see [14] for a related analysis). As we will see, the mass scale of the lightest modes is $M_\mathrm{pl}/\sqrt{t_1}$.

Perturbative string states. From the KK and string winding modes, the KK-modes on the first torus are the lightest ones with masses

$$M_\mathrm{KK}^{(1)} \sim \frac{M_\mathrm{pl}}{\sqrt{t_1}} \frac{1}{\sigma\sqrt{u_1}}\,, \qquad M_\mathrm{KK}^{(2)} \sim \frac{M_\mathrm{pl}}{\sqrt{t_1}} \frac{\sqrt{u_1}}{\sigma}\,. \tag{3.9}$$

All the other 4 KK- and 6 string winding modes are much heavier.

Wrapped D-brane states. As already pointed out in [8], in this limit one needs to take into account wrapped D-brane states, as well. As long as these D-branes do not wrap any cycle on the first T^2 factor, their masses also scale like $M_\mathrm{pl}/\sqrt{t_1}$. The resulting light modes are listed in table 2. Note that all D-brane masses do not contain any factor of σ.

Wrapped NS5-brane states. It turns out that the wrapped $NS5$-brane contributes two more light modes, which are listed in table 3. Hence, in total we have found 12 light modes which is precisely the same number as found in the perturbative string limit in section 2. Moreover, upon exchanging $\sigma \leftrightarrow \sqrt{t_1}$ and $u_2 \leftrightarrow t_3$ their masses follow the same pattern. This analogy suggests that there might also be a low-tension string in 4D, which in the former perturbative limit was just the fundamental string.

Light 4D string. First, we observe that e.g. a wrapped $D2$-brane yields a string tension that scales like

$$T_{D2} \sim \frac{M_\mathrm{pl}^2}{\sqrt{t_1}}\,, \tag{3.10}$$

D-branes	wrapping	mass scale
$D0$	$(--,--,--)$	$M^{(3)}_{D0} \sim \frac{M_{\rm pl}}{\sqrt{t_1}}\frac{1}{\sqrt{t_2 t_3}}$
$D2$	$(--,++,--)$	$M^{(4)}_{D2} \sim \frac{M_{\rm pl}}{\sqrt{t_1}}\sqrt{\frac{t_2}{t_3}}$
	$(--,--,++)$	$M^{(5)}_{D2} \sim \frac{M_{\rm pl}}{\sqrt{t_1}}\sqrt{\frac{t_3}{t_2}}$
	$(--,+-,+-)$	$M^{(6)}_{D2} \sim \frac{M_{\rm pl}}{\sqrt{t_1}}\frac{1}{\sqrt{u_2 u_3}}$
	$(--,-+,-+)$	$M^{(7)}_{D2} \sim \frac{M_{\rm pl}}{\sqrt{t_1}}\sqrt{u_2 u_3}$
	$(--,+-,-+)$	$M^{(8)}_{D2} \sim \frac{M_{\rm pl}}{\sqrt{t_1}}\sqrt{\frac{u_3}{u_2}}$
	$(--,-+,+-)$	$M^{(9)}_{D2} \sim \frac{M_{\rm pl}}{\sqrt{t_1}}\sqrt{\frac{u_2}{u_3}}$
$D4$	$(--,++,++)$	$M^{(10)}_{D4} \sim \frac{M_{\rm pl}}{\sqrt{t_1}}\sqrt{t_2 t_3}$

Table 2. Light wrapped D-brane states. In the second column we indicate by a $+$ which cycles on the internal $T^2 \times T^2 \times T^2$ are wrapped by the branes.

Brane	wrapping	mass scale
$NS5$	$(+-,++,++)$	$M^{(11)}_{NS5} \sim \frac{M_{\rm pl}}{\sqrt{t_1}}\frac{\sigma}{\sqrt{u_1}}$
	$(-+,++,++)$	$M^{(12)}_{NS5} \sim \frac{M_{\rm pl}}{\sqrt{t_1}}\sigma\sqrt{u_1}$

Table 3. Light wrapped $NS5$-brane states.

whose excitations provide 4D particles of mass $m \sim \sqrt{T_{D2}} \sim M_{\rm pl}/t_1^{1/4}$. Thus, these modes are parametrically heavier than the light modes listed so far. However, we can also wrap the $NS5$-brane on the last two T^2 factors yielding a string with tension and corresponding excitations

$$T_{NS5} = \frac{M_{\rm pl}^2}{t_1} \quad \Rightarrow \quad M^{(13)}_{NS5} = \frac{M_{\rm pl}}{\sqrt{t_1}} \,. \tag{3.11}$$

We think it is compelling that also in the asymptotic regime $t_1 \to \infty$, one finds the same pattern of 12 light particles and one low-tension string in 4D as we have seen in the much better understood perturbative limit $\sigma \to \infty$. Additionally, the appearance of an emergent string due to an $NS5$ brane in this limit is also in agreement with [16, 24, 47].

One might wonder whether, if there is an asymptotically tensionless string in 4D, maybe there also exists an asymptotically tensionless membrane. However, in accordance with the Emergent String Conjecture [16, 35, 36], wrapping D-branes and NS5-branes such that one gets a membrane M_2 in 4D, leads to its excitations of mass $M \sim T_{M_2}^{1/3}$ always being heavier than $M_{\rm pl}/\sqrt{t_1}$.

3.3 Asymptotic regime $u_1 \gg 1$

For completeness, let us now consider the type IIA superstring compactified on $T^6 = (T^2)^3$ in the limit $u_1 \to \infty$. Again, we set all axions to zero.

D-branes	wrapping	mass scale
$D2$	$(+-,+-,--)$	$M^{(3)}_{D2} \sim \frac{M_{\rm pl}}{\sqrt{u_1}} \frac{1}{\sqrt{u_2 t_3}}$
	$(+-,-+,--)$	$M^{(4)}_{D2} \sim \frac{M_{\rm pl}}{\sqrt{u_1}} \sqrt{\frac{u_2}{t_3}}$
	$(+-,--,+-)$	$M^{(5)}_{D2} \sim \frac{M_{\rm pl}}{\sqrt{u_1}} \frac{1}{\sqrt{t_2 u_3}}$
	$(+-,--,-+)$	$M^{(6)}_{D2} \sim \frac{M_{\rm pl}}{\sqrt{u_1}} \sqrt{\frac{u_3}{t_2}}$
$D4$	$(+-,-+,++)$	$M^{(7)}_{D4} \sim \frac{M_{\rm pl}}{\sqrt{u_1}} \sqrt{u_2 t_3}$
	$(+-,+-,++)$	$M^{(8)}_{D4} \sim \frac{M_{\rm pl}}{\sqrt{u_1}} \sqrt{\frac{t_3}{u_2}}$
	$(+-,++,-+)$	$M^{(9)}_{D4} \sim \frac{M_{\rm pl}}{\sqrt{u_1}} \sqrt{t_2 u_3}$
	$(+-,++,+-)$	$M^{(10)}_{D4} \sim \frac{M_{\rm pl}}{\sqrt{u_1}} \sqrt{\frac{t_2}{u_3}}$

Table 4. Light wrapped D-brane states.

Brane	wrapping	mass scale
$NS5$	$(+-,++,++)$	$M^{(11)}_{NS5} \sim \frac{M_{\rm pl}}{\sqrt{u_1}} \frac{\sigma}{\sqrt{t_1}}$
KK-monopole	$(-+,++,++)$	$M^{(12)}_{\rm KK-monop.} \sim \frac{M_{\rm pl}}{\sqrt{u_1}} \sigma \sqrt{t_1}$

Table 5. Light wrapped NS-brane states.

Perturbative string states. One KK and one winding mode on the first torus will be lighter than the others

$$M^{(1)}_{\rm KK} \sim \frac{M_{\rm pl}}{\sqrt{u_1}} \frac{1}{\sigma\sqrt{t_1}}\,, \qquad M^{(2)}_{\rm wind} \sim \frac{M_{\rm pl}}{\sqrt{u_1}} \frac{\sqrt{t_1}}{\sigma}\,. \tag{3.12}$$

Wrapped *D*-brane states. The light D-branes are the ones that wrap the x-cycle on the first T^2 and some other directions on the second and third T^2. As a consequence, only the $D2$ and $D4$ branes do lead to light modes scaling as $M_{\rm pl}/\sqrt{u_1}$. They are listed in table 4.

A closer look reveals that, so far, the spectrum of light states is related to the one in the $t_1 \to \infty$ limit by a T-duality in two directions, e.g. exchanging $u_1 \leftrightarrow t_1$ and $u_3 \leftrightarrow t_3$.

Wrapped *NS*5-brane states. It turns out that there is only one light state coming from a wrapped $NS5$-brane, namely the 5-brane wrapping the x-direction on the first T^2 and both directions in the remaining T^2's. This provides the eleventh light mode so that relative to the former asymptotic regimes, one state seems to be missing. However, by applying the just mentioned T-duality transformation, we realize that the former $NS5$-brane is mapped to a KK-monopole (see e.g. [50–52]). Hence, we find the two additional NS-branes listed in table 5.

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We notice that when expressed in string units, the contribution from the wrapped KK-monopole is

$$M = \frac{M_{\rm pl}}{\sqrt{u_1}}\sigma\sqrt{t_1} = \frac{M_{\rm s}}{g_{\rm s}^2} r_1^2\, r_2\, r_3\, r_4\, r_5\, r_6\,. \tag{3.13}$$

The, at first sight strange, factor r_1^2 comes from the energy f^2 of the non-trivial U(1) field strength $f = da$ supported on the KK-monopole.

Light 4D string. Now it is clear that the lightest 4D string also arises from the KK-monopole wrapped on the last two T^2 factors. This yields a tension and the corresponding masses are

$$T_{\text{KK-monop.}} = \frac{M_{\rm s}^2}{g_{\rm s}^2} r_1^2\, r_3\, r_4\, r_5\, r_6 = \frac{M_{\rm pl}^2}{u_1} \;\Rightarrow\; M^{(13)}_{\text{KK-monop.}} = \frac{M_{\rm pl}}{\sqrt{u_1}}\,. \tag{3.14}$$

Hence, again, in this asymptotic region of a large complex structure we have identified precisely 12 light particles and one low-tension string.

4 Emergence in asymptotic regions

Being aware of the potential limitations, in this section we carry out a complete emergence computation in the only currently accessible QFT approach. Beyond the KK modes, we will include the winding and string modes. First, we do this for the perturbative string limit ($\sigma \to \infty$) and afterwards also consider the generalization to the limits of a single large Kähler modulus and a single large complex structure modulus. Finally, we discuss the implications of our concrete computation for the Emergence Proposal.

4.1 Emergence in the weak string coupling limit

Employing similar computational methods as for the already presented KK example, let us now integrate out all light towers of states with a mass smaller than the species scale.

The species scale. For the latter, we first need to compute $N_{\rm sp}$ as given by (A.1)

$$N_{\rm sp} = \underbrace{\sum_{\vec{m}^I,\vec{n}^I,N}}_{M\le\tilde{\Lambda}} \deg_N \approx \int \prod_{I=1}^{3} dm_1^I\, dm_2^I\, dn_1^I\, dn_2^I\, dN\;\; \deg_N\,, \tag{4.1}$$

where in the $\sigma \gg 1$ regime we can again safely approximate the sum by an integral. This time, we need to define 13 new variables w^I, x^I, y^I, z^I and q via

$$M^2 = \frac{M_{\rm pl}^2}{\sigma^2}\left\{\sum_{I=1}^{3}\left[\left(\overbrace{\frac{m_1^I - v_I m_2^I + b_I n_1^I + b_I v_I n_2^I}{u_I^{\frac12}\, t_I^{\frac12}}}^{w^I}\right)^2 + \left(\overbrace{\frac{(m_2^I - b_I n_2^I)\, u_I^{\frac12}}{t_I^{\frac12}}}^{x^I}\right)^2 \right.\right.$$
$$\left.\left. + \left(\underbrace{\frac{(n_1^I + v_I n_2^I)\, t_I^{\frac12}}{u_I^{\frac12}}}_{y^I}\right)^2 + \left(\underbrace{n_2^I u_I^{\frac12}\, t_I^{\frac12}}_{z^I}\right)^2\right] + \underbrace{\kappa^2 N}_{q^2}\right\}. \tag{4.2}$$

The determinant of the Jacobian for this change of variables is $\det(J) = 2q/\kappa^2$. In these variables, we now integrate over a ball in 13 dimensions with radius $R = \frac{\tilde{\Lambda}}{M_{\rm pl}}\sigma$, which makes it convenient to introduce 13D spherical coordinates

$$
\begin{aligned}
q &= r\cos\varphi_0 \\
w^1 &= r\sin\varphi_0\,\cos\varphi_1 \\
x^1 &= r\sin\varphi_0\,\sin\varphi_1\,\cos\varphi_2 \\
y^1 &= r\sin\varphi_0\,\sin\varphi_1\,\sin\varphi_2\,\cos\varphi_3 \\
&\vdots \\
y^3 &= r\sin\varphi_0\,\sin\varphi_1\ldots\sin\varphi_{10}\,\cos\varphi_{11} \\
z^3 &= r\sin\varphi_0\,\sin\varphi_1\ldots\sin\varphi_{10}\,\sin\varphi_{11}
\end{aligned}
\tag{4.3}
$$

with $r \geq 0$, $\varphi_0, \ldots, \varphi_{10} \in [0,\pi]$ and $\varphi_{11} \in [0, 2\pi]$. The measure then becomes

$$
\mu = dr\, r^{12} d\varphi_0\, \sin^{11}\varphi_0\, d\varphi_1 \ldots d\varphi_{11}\, \sin^{10}\varphi_1\, \sin^9\varphi_2 \ldots \sin\varphi_{10}\,. \tag{4.4}
$$

Putting everything together, we arrive at the following integral for $N_{\rm sp}$

$$
N_{\rm sp} \simeq 2\gamma\kappa^{\nu-2}\,\mathrm{vol}(S^{11}) \int_0^{\tilde{\Lambda}/M_{\rm s}} dr\, r^{13-\nu} \int_0^{\pi/2} d\varphi_0\, \frac{\sin^{11}\varphi_0}{\cos^{\nu-1}\varphi_0} \exp\left(\frac{\beta}{\kappa} r\cos\varphi_0\right)\,, \tag{4.5}
$$

where we have carried out already the integral over the angular variables $\varphi_1, \ldots, \varphi_{11}$, which gives the volume $\mathrm{vol}(S^{11}) = \pi^6/60$ of the unit sphere S^{11}. The large $\tilde{\Lambda}/M_{\rm s}$ behavior of the r and φ_0 integral can be determined analytically and reads

$$
\int_0^{r_0} dr\, r^k \int_0^{\arccos(\kappa/r)} d\varphi_0\, \frac{\sin^{2n+1}\varphi_0}{\cos^m\varphi_0}\, e^{\alpha r\cos\varphi_0} = \frac{\delta_n}{\alpha^{n+2}}\, r_0^{k-n-1}\, e^{\alpha r_0} + \ldots\,, \tag{4.6}
$$

with the values of the coefficients δ_n listed in table 6 for the first values of n. For the convergence of the integral it was important that we imposed the physical constraint $N \geq 1$, which modifies the upper bound of the φ_0 integration accordingly. It is easy to see that the integral (4.6) is independent of the exponent m of the $\cos\varphi$-factor in the denominator. Applying this relation for $n = 5$, $k = 13 - \nu$, $r_0 = \tilde{\Lambda}/M_{\rm s}$ and $\alpha = \beta/\kappa$ leads to the number of light species

$$
N_{\rm sp} \simeq 7680 \frac{\gamma\kappa^{\nu+5}}{\beta^7} \mathrm{vol}(S^{11}) \left(\frac{M_{\rm s}}{\tilde{\Lambda}}\right)^{\nu-7} e^{\frac{\beta}{\kappa}\frac{\tilde{\Lambda}}{M_{\rm s}}}\,. \tag{4.7}
$$

Using (2.15), one gets a transcendental equation for the species scale $\tilde{\Lambda}$

$$
\frac{M_{\rm pl}^2}{M_{\rm s}^2} = \underbrace{7680 \frac{\gamma\kappa^{\nu+5}}{\beta^7} \mathrm{vol}(S^{11})}_{A} \left(\frac{\tilde{\Lambda}}{M_{\rm s}}\right)^{9-\nu} e^{\frac{\beta}{\kappa}\frac{\tilde{\Lambda}}{M_{\rm s}}}\,. \tag{4.8}
$$

n	0	1	2	3	4	5	6	7
δ_n	1	2	8	48	384	3840	46080	645120

Table 6. Values of δ_n coefficients.

This is solved by the Lambert function $W(y)$.[6] For $\nu \leq 9$ one chooses the branch W_0 and for $\nu > 9$ the branch W_{-1}

$$\frac{\tilde{\Lambda}}{M_{\rm s}} = (9-\nu)\frac{\kappa}{\beta}\, W\left(\frac{1}{(9-\nu)}\frac{\beta}{\kappa A^{\frac{1}{9-\nu}}}\left(\frac{M_{\rm pl}}{M_{\rm s}}\right)^{\frac{2}{9-\nu}}\right) . \tag{4.9}$$

In any case, in the asymptotic regime $\sigma = M_{\rm pl}/M_{\rm s} \gg 1$ one gets[7]

$$\frac{\tilde{\Lambda}}{M_{\rm s}} \sim \frac{2\kappa}{\beta}\log\left(\frac{M_{\rm pl}}{M_{\rm s}}\right) , \tag{4.10}$$

which is also independent of ν. Hence, as already mentioned in section 3.1, namely in equation (3.7), after including all the light modes in the perturbative string regime the actual UV cut-off is essentially the string scale however amplified by a logarithmic correction.

Let us comment on the contribution to $N_{\rm sp}$ from the 48 twisted sectors of the orbifold. As mentioned, these provide additive towers of light states with in each case $\Delta_{\rm t} = 4$ KK and winding modes and the fundamental twisted string. Going through the same steps as in the untwisted case, one realizes that their number of light states is suppressed relative to the contribution from the untwisted sector with its $\Delta_{\rm u} = 12$ KK and winding modes

$$\frac{N_{\rm sp,u}}{N_{\rm sp,t}} \sim \left(\frac{\tilde{\Lambda}}{M_{\rm s}}\right)^{\frac{(\Delta_{\rm u}-\Delta_{\rm t})}{2}} \gg 1 . \tag{4.11}$$

Thus, they can be safely neglected in the asymptotic limit.

One-loop corrections to the moduli space metric. First, we compute the one-loop correction to $G^{(0)}_{\sigma\sigma}$. Again, we only keep the numerical factors affecting the relative normalization of the metric components. Since σ appears in the mass formula (4.2) only as a prefactor, the relation (A.16) becomes

$$G^{(1)}_{\sigma\sigma} \simeq \underbrace{\sum_{\vec{m}^I,\vec{n}^I,N}}_{M\leq\tilde{\Lambda}} \deg_N \left(\frac{M_{\rm pl}\, r}{\sigma^2}\right)^2 . \tag{4.12}$$

[6]The Lambert function $W(y)$ is defined as the solution to the equation $xe^x = y$, for $y \geq -e^{-1}$ and is a multivalued function when $-e^{-1} < y < 0$. The branch satisfying $-1 \leq W(y)$ is called the principle branch and is denoted as $W_0(x)$, while the one satisfying $W(y) \leq -1$ is denoted by $W_{-1}(x)$. If $y \geq 0$, then $W(y) = W_0(y)$.

[7]For us, the following expansions are relevant [53]:

$$\begin{aligned} W_0(x\to\infty) &= \log(x) - \log(\log(x)) + \ldots \\ W_{-1}(x\to 0) &= \log(-x) - \log(-\log(-x)) + \ldots \end{aligned}$$

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We can proceed in the same manner as for the computation of $N_{\rm sp}$ and arrive at[8]

$$G^{(1)}_{\sigma\sigma} \simeq \frac{M_{\rm pl}^2}{\sigma^4} \underbrace{7680\frac{\gamma\kappa^{\nu+5}}{\beta^7}{\rm vol}(S^{11})}_{A} \left(\frac{\tilde\Lambda}{M_{\rm s}}\right)^{9-\nu} e^{\frac{\beta}{\kappa}\frac{\tilde\Lambda}{M_{\rm s}}} . \tag{4.13}$$

Using the relation (4.8), we bring this to the simple form

$$G^{(1)}_{\sigma\sigma} \simeq \frac{M_{\rm pl}^2}{\sigma^2} . \tag{4.14}$$

This is precisely the tree-level metric $G^{(0)}_{\sigma\sigma}$ from (2.6). In particular, we realize that the still left open coefficients β, γ, κ, ν completely drop out in the final result. Before getting too excited, we have to keep in mind that this result is only true up to some overall numerical prefactor.

Next, we determine the one-loop corrections to all the other metric components, again using (A.16)

$$G^{(1)}_{\mathcal{M}_a\mathcal{M}_b} \simeq \underbrace{\sum_{\vec m^I,\vec n^I,N}}_{M\le\tilde\Lambda} {\rm deg}_N\, (\partial_{\mathcal{M}_a} M)\, (\partial_{\mathcal{M}_b} M) . \tag{4.15}$$

Proceeding analogously to our previous calculation for the KK spectrum, the derivatives now take the more symmetric form

$$\begin{aligned}
\partial_{u_I} M &\simeq \frac{M_{\rm pl}}{\sigma}\frac{1}{2\,r\,u_I}\Big(-w_I^2+x_I^2-y_I^2+z_I^2\Big) = \frac{M_{\rm pl}}{\sigma}\frac{r\sin\varphi_0}{2\,u_I}\,m_{u_I}(\underline{\varphi})\\
\partial_{v_I} M &\simeq \frac{M_{\rm pl}}{\sigma}\frac{1}{2\,r\,u_I}\Big(-2\,w_I\,x_I+2\,y_I\,z_I\Big) = \frac{M_{\rm pl}}{\sigma}\frac{r\sin\varphi_0}{2\,u_I}\,m_{v_I}(\underline{\varphi})\\
\partial_{t_I} M &\simeq \frac{M_{\rm pl}}{\sigma}\frac{1}{2\,r\,u_I}\Big(-w_I^2-x_I^2+y_I^2+z_I^2\Big) = \frac{M_{\rm pl}}{\sigma}\frac{r\sin\varphi_0}{2\,t_I}\,m_{t_I}(\underline{\varphi})\\
\partial_{b_I} M &\simeq \frac{M_{\rm pl}}{\sigma}\frac{1}{2\,r\,u_I}\Big(2\,w_I\,y_I-2\,x_I\,z_I\Big) = \frac{M_{\rm pl}}{\sigma}\frac{r\sin\varphi_0}{2\,t_I}\,m_{b_I}(\underline{\varphi})\\
\partial_\sigma M &\simeq -\frac{M_{\rm pl}}{\sigma^2}r\,,
\end{aligned} \tag{4.16}$$

where inserting the definition of the spherical coordinates (4.3) allows us to determine the functions $m_{u_I}(\underline{\varphi}),\ldots,m_{b_I}(\underline{\varphi})$ which only depend on the angular variables $\varphi_1,\ldots,\varphi_{11}$. Again, we can also formally introduce $m_\sigma(\underline{\varphi})=1$ and in the evaluation of (4.15) an angular metric appears

$$\tilde g_{\mathcal{M}_A\mathcal{M}_B} := \int d\Omega_{11}\, m_{\mathcal{M}_A}(\underline{\varphi})\, m_{\mathcal{M}_B}(\underline{\varphi}) . \tag{4.17}$$

The corresponding integrals over the 11 angular variables can be carried out explicitly which yields the only non-vanishing components

$$\tilde g_{u_Iu_I} = \tilde g_{v_Iv_I} = \frac{\pi^6}{1260}\,, \qquad \tilde g_{t_It_I} = \tilde g_{b_Ib_I} = \frac{\pi^6}{1260}\,, \tag{4.18}$$

[8] We have checked that using the exact form of the integral (A.13) gives the same result in the asymptotic limit $\tilde\Lambda/M_{\rm s}\gg 1$.

so that

$$\tilde{g}_{\mathcal{M}_a\mathcal{M}_b} = \frac{\pi^6}{1260}\delta_{\mathcal{M}_a\mathcal{M}_b}\,. \tag{4.19}$$

All off-diagonal components including $\tilde{g}_{u_I\sigma}, \ldots, \tilde{g}_{b_I\sigma}$ vanish. Hence, from these integrals we already get a large portion of the structure of the tree-level metric $G^{(0)}_{AB}$. It remains to carry out the full integrals, i.e. also the ones over r and φ_0

$$\begin{aligned} G^{(1)}_{\mathcal{M}_a\mathcal{M}_a} &\simeq M_{\rm pl}^2\,\frac{2\gamma\kappa^{\nu-2}}{4\mathcal{M}_a^2\sigma^2}\,\tilde{g}_{\mathcal{M}_a\mathcal{M}_a} \\ &\quad\times\int_0^{\tilde{\Lambda}/M_{\rm s}} dr\, r^{15-\nu}\int_0^{\arccos(\kappa/r)} d\varphi_0\,\frac{\sin^{15}\varphi_0}{\cos^{\nu-1}\varphi_0}\exp\left(\frac{\beta}{\kappa}r\cos\varphi_0\right)\,. \end{aligned} \tag{4.20}$$

Next, employing (4.6) for $n = 7$, $k = 15 - \nu$, $r_0 = \tilde{\Lambda}/M_{\rm s}$ and $\alpha = \beta/\kappa$ and using the relation (4.8) one arrives at

$$G^{(1)}_{\mathcal{M}_a\mathcal{M}_a} \simeq M_{\rm pl}^2\,\frac{\kappa^2}{\beta^2}\,\frac{168\,\tilde{g}_{\mathcal{M}_a\mathcal{M}_a}}{{\rm vol}(S^{11})}\,\frac{1}{4\mathcal{M}_a^2}\left(\frac{M_{\rm s}}{\tilde{\Lambda}}\right)^2\,, \tag{4.21}$$

where we used $\delta_7/\delta_5 = 168$. First, we observe the intriguing numerical relation

$$\frac{168}{8\,{\rm vol}(S^{11})}\,\tilde{g}_{\mathcal{M}_a\mathcal{M}_a} = \frac{168\cdot 60}{8\,\pi^6}\,\frac{\pi^6}{1260} = 1\,, \tag{4.22}$$

where the factors ${\rm vol}(S^{11})$ and $\tilde{g}_{\mathcal{M}_a\mathcal{M}_a}$ were coming from integration over KK and winding modes and the factor 168 from the integration over the string oscillators, namely the integral (4.6). Hence, it seems that something highly non-trivial between extra dimensions and strings is happening here, in fact on a quantitative level.

Then, invoking in addition the species scale (4.10), we finally get the one-loop moduli metric

$$G^{(1)}_{\mathcal{M}_a\mathcal{M}_b} \simeq \frac{M_{\rm pl}^2}{2\,\mathcal{M}_a^2}\,\frac{1}{\log^2\left(\frac{M_{\rm pl}}{M_{\rm s}}\right)}\,\delta_{\mathcal{M}_a\mathcal{M}_b}\,. \tag{4.23}$$

At this point it is tempting to speculate that working with the species scale found in the BH picture, namely $\tilde{\Lambda} \sim M_{\rm s}$, the $\log^2$-factor would be absent. However, the previous relation (4.21) was of course derived in the QFT picture.

Taking into account an overall not yet determined coefficient λ and that the axion ρ completely decoupled, we can summarize the results for the one-loop field metric as

$$\begin{aligned} &G^{(1)}_{\sigma\sigma} = \lambda\,G^{(0)}_{\sigma\sigma}\,, \qquad && G^{(1)}_{\rho\rho} = 0\,, \qquad G^{(1)}_{\rho\sigma} = 0\,, \\ &G^{(1)}_{\sigma\mathcal{M}_a} = 0\,, && G^{(1)}_{\rho\mathcal{M}_a} = 0\,, \\ &G^{(1)}_{\mathcal{M}_a\mathcal{M}_b} = \frac{2\lambda}{\log^2(\sigma)}\,G^{(0)}_{\mathcal{M}_a\mathcal{M}_b}\,. \end{aligned} \tag{4.24}$$

Thus, a large portion of the structure of the tree-level metric is there, in particular all off-diagonal components of the metric are vanishing. However, the classical singular behavior of $G_{\rho\rho}$ is not reproduced, which if taken seriously threatens both Variant A and Variant B of the Weak Emergence Proposal. On the positive side, up to the $\log(\sigma)$ factor, the dependence on all orthogonal moduli is reproduced correctly, in agreement with Variant B of the Weak Emergence Proposal. However, the log-factors that we have seen already in the result of the species scale also make their appearance for the one-loop field metric.

One-loop corrections to the gauge kinetic terms. We can also compute the one-loop corrections to the gauge kinetic terms. However, since all perturbative string states are neutral with respect to the R-R gauge fields, it is immediately clear that integrating them out does not lead to any contribution so that

$$f^{(1)}_{IJ} = 0\,. \tag{4.25}$$

The charged states are given by $D0$ and wrapped $D2$ branes, which in the asymptotic region $\sigma \gg 1$ are heavier than the species scale. Hence, the gauge kinetic terms cannot emerge in this limit. Note that the tree-level ones are singular in the Kähler moduli and not in σ, which is taken to infinity here. Therefore, we can state that although due to Variant B of the Weak Emergence Proposal one would hope to get a non-trivial correction, we are still meeting the requirements of Variant A and no direct contradiction to either of them is observed.

4.2 Emergence in the asymptotic regime $t_1 \gg 1$

Before we discuss potential consequences of our result, we consider other asymptotic regions in the moduli space. In this section we focus on the $t_1 \to \infty$ limit.

One loop corrections to the moduli space metric. It is a difficult question to decide what kinds of bound states exist for these wrapped branes and how the final mass formula for them reads. In appendix C, essentially by analogy we propose an analogous mass formula as for the perturbative string states (2.3). There, we also include the axions and realize the appearance of the axion ρ in the mass formula. Using the relations from appendix C, the computation for the species scale and the one-loop kinetic terms proceeds as before and we arrive at the analogous result

$$\begin{aligned}
G^{(1)}_{t_1 t_1} &= \lambda\, G^{(0)}_{t_1 t_1}\,, \qquad && G^{(1)}_{b_1 b_1} = 0\,, \qquad G^{(1)}_{t_1 b_1} = 0\,,\\
G^{(1)}_{t_1 \mathcal{M}_a} &= 0\,, && G^{(1)}_{b_1 \mathcal{M}_a} = 0\,,\\
G^{(1)}_{\mathcal{M}_a \mathcal{M}_b} &= \frac{2\lambda}{\log^2\left(\frac{M_{\rm pl}}{M_{\rm s}}\right)}\, G^{(0)}_{\mathcal{M}_a \mathcal{M}_b}\,,
\end{aligned} \tag{4.26}$$

with $M_{\rm pl}/M_{\rm s} = \sqrt{t_1}$ and a, b labelling all modes except t_1 and b_1.[9]

[9] Note that the KK-modes, the NS5-branes and the last four $D2$-branes wrap torsion cycles of the orbifold, which would be absent for a CY with $\pi_1(X) = 0$. As a consequence, one would not get any non-trivial one-loop metric for the hypermultiplets in the latter case.

One loop corrections to the gauge kinetic terms. In contrast to the weak string coupling limit, now some of the light states are wrapped D-branes and we can get a non-trivial one-loop contribution to the gauge kinetic terms. As in the previous paragraph, we assume that the suggestive mass formulas from appendix C are correct so that the computation can proceed similarly to the one from section 4.1. Let us only mention some of the main points here.

To evaluate the one-loop correction (A.20), we first need to identify those light states that are charged under the four gauge symmetries. Looking at table 2 we realize that the $D0$-brane is electrically charged under $\mathcal{A}^0$ and the first two wrapped $D2$-branes from that table are electrically charged under $\mathcal{A}^2$ and $\mathcal{A}^3$, respectively. There is no wrapped $D2$-brane that is electrically charged under $\mathcal{A}^1$, but there is the wrapped $D4$-brane, which is magnetically charged under $\mathcal{A}^1$. This means that this brane is electrically charged under the magnetic dual gauge field $\tilde{\mathcal{A}}^1$. For the following, we have to keep in mind that the perturbative one-loop correction (A.20) to the gauge coupling has been derived for electrically charged particles running in the loop and we will only apply it to such cases.

The relevant piece from the mass formula for these four types of branes is precisely (C.1) which we repeat here for convenience

$$M^2 = \frac{M_{\rm pl}^2}{t_1}\Bigg[\Bigg(\overbrace{\frac{n_1+b_3n_2+b_2n_3+b_2b_3n_4}{t_2^{\frac12}\,t_3^{\frac12}}}^{w_1}\Bigg)^2+\Bigg(\overbrace{\frac{(n_2+b_2n_4)\,t_3^{\frac12}}{t_2^{\frac12}}}^{x_1}\Bigg)^2 \\ +\Bigg(\underbrace{\frac{(n_3+b_3n_4)\,t_2^{\frac12}}{t_3^{\frac12}}}_{y_1}\Bigg)^2+\Big(\underbrace{n_4t_2^{\frac12}\,t_3^{\frac12}}_{z_1}\Big)^2+\ldots\Bigg]\,. \tag{4.27}$$

Say one wants to compute $f_{00}^{(1)}$. As already observed in [24], in the presence of a non-trivial Kalb-Ramond field one has to recall (see e.g. [41]) that the 4D gauge fields $\mathcal{A}^\Lambda$ are defined via the exact pieces in $\hat{F}_2 = dC_1$ and $\hat{F}_4 = dC_3 - H_3\wedge C_1$. For the latter one expands $C_3=\sum_{I=1}^3 \mathcal{A}^I\wedge\omega_I$ so that the charges do not change and are still integer valued. Hence, we have $q_0 = n_1$ and the starting point of the computation is (A.20)

$$f_{00}^{(1)} \simeq \underbrace{\sum_{\vec{m}^I,\,\vec{n}^I,\,\vec{p}^I,\,N}}_{M\le\tilde{\Lambda}} \deg_N\,(n_1)^2\,, \tag{4.28}$$

where the sum is over all states. As explained, we have set all one-loop beta-coefficients to one. Let us stress that the independence of the one-loop corrections from the coefficients β, γ and ν implies that even if higher spin fields were inducing an extra polynomial or even exponential $\exp(\beta'\sqrt{N})$ factor, they would remain unchanged.

To avoid couplings to the wrapped $D4$-branes, we set $b_2b_3=0$ but allow either of them to be non-zero. After approximating the sum by an integral and expressing the $\mathcal{A}^0$ charge n_1 as

$$n_1 = t_2^{\frac12}t_3^{\frac12}\left(w_1-\frac{b_3}{t_3}x_1-\frac{b_2}{t_2}y_1\right)\,, \tag{4.29}$$

one can proceed as in section 4.1. Then one realizes that via the integral over the 11 angular directions $\varphi_1, \ldots, \varphi_{11}$, all off-diagonal contributions from n_1^2 vanish so that only the diagonal ones survive. Following the same steps as in the previous metric calculation, we finally arrive at

$$f_{00}^{(1)} \simeq t_1 t_2 t_3 \left(1 + \left(\frac{b_2}{t_2}\right)^2 + \left(\frac{b_3}{t_3}\right)^2\right) \frac{1}{2\log\left(\frac{M_{\rm pl}}{M_{\rm s}}\right)} . \tag{4.30}$$

This is proportional to the classical gauge coupling from (2.11), with only the term involving b_1 missing. But that is expected, as b_1 does not appear in the mass formula for the light states.

We can proceed similarly for the other electric components of the gauge coupling, i.e. $\Lambda, \Sigma = 0, 2, 3$. For $\Lambda = \Sigma = 1$ one can only determine the magnetically dual coupling $\tilde{f}_{11}^{(1)}$ in terms of $\tilde{f}_{11}^{(0)} = (f_{11}^{(0)})^{-1}$. Now, the wrapped $D4$ branes (for $b_2 = b_3 = 0$) are electrically charged and we can still apply the formalism. In this manner, the final result for the non-vanishing one-loop corrections to the gauge kinetic functions can be compactly written as

$$\begin{aligned} f_{\Lambda\Sigma}^{(1)} &= \frac{\xi}{2\log\left(\frac{M_{\rm pl}}{M_{\rm s}}\right)} f_{\Lambda\Sigma}^{(0)}\Big|_{b_1=0} \quad (\Lambda, \Sigma = 0, 2, 3)\,, \\ \tilde{f}_{11}^{(1)} &= \frac{\xi}{2\log\left(\frac{M_{\rm pl}}{M_{\rm s}}\right)} \tilde{f}_{11}^{(0)}\,, \end{aligned} \tag{4.31}$$

where ξ is a common numerical prefactor.

By electric-magnetic duality one might also extract some information on the one-loop corrections to the theta-angles, but we stop here and state that in the large Kähler modulus limit $t_1 \to \infty$, the loop corrections to the gauge couplings are essentially consistent with Variant B of the Weak Emergence Proposal, but again miss some dependence (on b_1) and receive a suppression by $\log(M_{\rm pl}/M_{\rm s}) \sim \log(t_1)$.

4.3 Emergence in the asymptotic regime $u_1 \gg 1$

For completeness, let us now discuss emergence in the limit $u_1 \to \infty$. If we again assume that we get an analogous mass formula as for the perturbative string states (2.3), then the computation for the species scale and the one-loop kinetic terms will proceed as before and we will arrive at the analogous result[10]

$$\begin{aligned} G^{(1)}_{u_1 u_1} &= \lambda\, G^{(0)}_{u_1 u_1}\,, & G^{(1)}_{v_1 v_1} &= 0\,, & G^{(1)}_{u_1 v_1} &= 0\,, \\ G^{(1)}_{u_1 \mathcal{M}_a} &= 0\,, & G^{(1)}_{v_1 \mathcal{M}_a} &= 0\,, & & \\ G^{(1)}_{\mathcal{M}_a \mathcal{M}_b} &= \frac{2\lambda}{\log^2\left(\frac{M_{\rm pl}}{M_{\rm s}}\right)}\, G^{(0)}_{\mathcal{M}_a \mathcal{M}_a}\,, & & & & \end{aligned} \tag{4.32}$$

[10] Note that the two perturbative as well as all the wrapped $D2$, $D4$ and NS-branes wrap torsion cycles of the orbifold, which would be absent for a CY with $\pi_1(X) = 0$. According to [13], in this case the classical infinite distance point in the complex structure moduli space could be obstructed by moving it to finite distance by the presence of Euclidean $D2$-brane instantons. As described below table 4, for this toroidal setting the large complex structure limit is related to the unobstructed large Kähler modulus limit by two T-dualities. This suggests that the $D2$-brane instantons do not carry the right fermionic zero mode structure to contribute to the hypermultiplet metric.

with $M_{\rm pl}/M_{\rm s} = \sqrt{u_1}$ and a, b labelling all modes except u_1 and v_1. Since all states in the light towers are neutral with respect to the graviphoton and the other three U(1) gauge symmetries, the one-loop corrections to their gauge couplings do vanish.

4.4 Consequences for emergence

In view of the discrepancy of the BH and the QFT approach in determining the species scale for string towers, let us now discuss what the consequences of these results for the emergence proposal could be. All the one-loop corrections have been derived in the QFT picture and as such in particular the log-factors are under scrutiny. There is the possibility that the extrapolation of the QFT techniques to the stringy regime with energies larger than $M_{\rm s}$ is at least questionable.

However, in the course of the computation, we have seen a certain universality in the final results for the emerging field metric and gauge couplings, which first of all show the expected classical moduli dependence in some of the orthogonal components. The latter arose from the inclusion of both the KK and the winding modes, which in the regime of interest are already heavier than the string scale. In addition, we observed this highly non-trivial conspiracy of coefficients in the relation (4.22), which we think is not just coincidental.

Thus, in the remainder of this section we approach our results with a positive attitude and discuss two possibilities to interpret the appearing log-factors:

(A) The log-factors are unphysical. Motivated by the BH picture we just ignore the log-suppressions. Going through the computation, one realizes that they really entered the expression for the one-loop field metric and gauge coupling by inserting $\tilde{\Lambda}/M_{\rm s} \sim \log(\sigma)$ in the final step (see e.g. the discussion around equation (4.23)).

In the weak string coupling limit, we would say that the field metric is almost fully emerging with only the $G_{\rho\rho}$ term missing. This is due to the non-appearance of the axion ρ in the mass formula for the light towers of states and seems to be at odds with supersymmetry. For the non-vanishing metric components, the relative normalizations are also fine, except for a factor of 2 between $G_{\sigma\sigma}$ and the other components.

Since the one-loop gauge couplings were all trivially vanishing, clearly these do not emerge. As mentioned, on a genuine CY with vanishing fundamental group, also the 1-loop metric components for the Kähler moduli would be vanishing. Essentially, this is just a reflection of the non-mixing of hyper and vector multiplets in 4D $N = 2$ supergravity.

Consistent with $N = 2$ supersymmetry, in the limit of a single large Kähler modulus, also some components of the gauge couplings emerged with the expected moduli dependence.

(B) The log-factors are physical. The second possibility is that the log-factors are physical and should be taken seriously. For concreteness, let us only consider the example of the weak string coupling case. As in the weak version of the Emergence

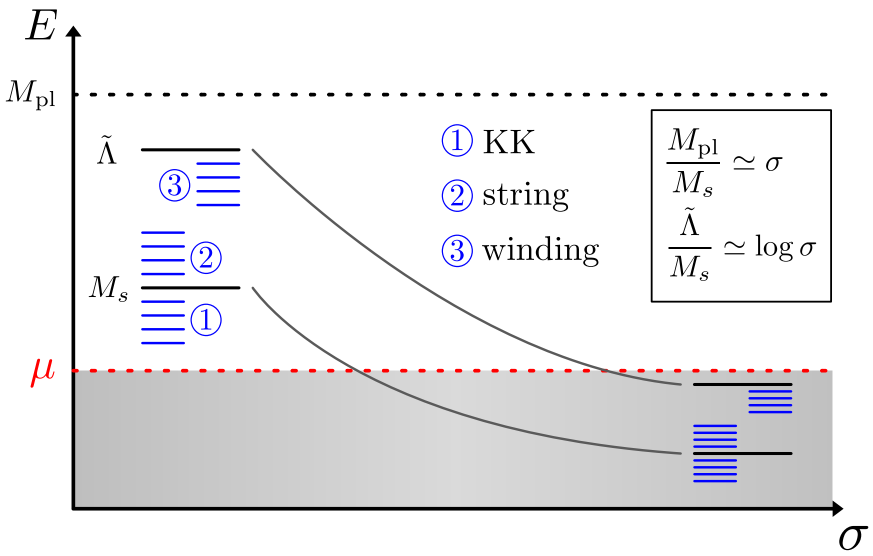

Figure 3. Dependence of energy scales on σ. The Swampland Distance Conjecture predicts a breakdown of the EFT at large (Planck scale) field distances when many states fall below its cut-off μ.

Proposal, including the classical contributions, we have up to the one-loop level

$$\begin{aligned}\mathcal{L}_{\rm kin} = M_{\rm pl}^2\left(\frac{1}{\sigma^2}+\frac{\lambda}{\sigma^2}+\dots\right)(\partial\sigma)^2+\frac{M_{\rm pl}^2}{4\sigma^4}(\partial\rho)^2\\ +M_{\rm pl}^2\sum_I\left(\frac{1}{4t_I^2}+\frac{\lambda}{2t_I^2\log^2(\sigma)}+\dots\right)\Big((\partial t_I)^2+(\partial b_I)^2\Big)\end{aligned} \tag{4.33}$$

and a similar term for the complex structure moduli $\mathcal{U}_I = v_I + iu_I$. The dots indicate that there will be higher order corrections, as in our computation we were just extracting the asymptotic form of the loop corrections.

We first notice that (4.33) indicates that the induced corrections due to the light towers of states appearing in asymptotic limits in field space are much milder than initially advocated. Against the intuition from the Swampland Distance Conjecture that the EFT becomes worse and worse with more and more states dropping below the UV cut-off (see also figure 3), in the infinite field distance limit the one-loop corrections are either proportional to the classical values or are subleading. However, this does not happen polynomially in $\sigma^{-1} = \exp(-\phi_4)$ like in perturbation theory, but logarithmically with a suppression $\log^{-2}(\sigma)$. Thus, in this sense the (kinetic) couplings of the classical EFT, in which one is only keeping the generically lightest states, are still valid and under control.

However, the correction is moduli dependent and one might wonder about the initial $N = 2$ supersymmetry. As we have already mentioned, in a 4D EFT with $N = 2$ supersymmetry, the moduli spaces of the vector- and the hyper multiplets are not allowed to mix. But since the σ-field resides in the type IIA hyper multiplet and σ-dependent corrections appear in the one-loop metric of the Kähler moduli (belonging to the vector multiplet), it seems that the two sectors do not decouple in our case.

A natural way to resolve this issue is to interpret the problematic log-factor in $G^{(1)}_{t_I t_I} = M^2_{\rm pl}\, g^{(1)}_{t_I t_I}$ not as a correction to the field metric, but as a one-loop correction to $M^2_{\rm pl}$. Recall that the latter is nothing else than the kinetic term for the graviton, which should/could also emerge. Indeed, the stringy one-loop correction for the type II string on 4D orbifold models with $N = 2$ supersymmetry was found to be non-vanishing in [54, 55] and in fact proportional to the string scale, i.e. $\delta(M^2_{\rm pl}) \sim M^2_{\rm s}$. With this modification, the expansions of the relevant quantities read

$$\begin{aligned} M^2_{\rm pl} &= M^{2\,(0)}_{\rm pl} + M^{2\,(1)}_{\rm pl} + \ldots \\ g_{AB} &= g^{(0)}_{AB} + g^{(1)}_{AB} + \ldots\,, \end{aligned} \tag{4.34}$$

where $g_{AB} = G_{AB}/M^2_{\rm pl}$. The previously computed one-loop coefficients therefore consist of two different components, namely

$$G^{(1)}_{AB} = M^{2\,(0)}_{\rm pl}\, g^{(1)}_{AB} + M^{2\,(1)}_{\rm pl}\, g^{(0)}_{AB}\,. \tag{4.35}$$

Let us now apply (4.35) to the log-corrected metric components of the Kähler moduli in (4.24). Here, the σ-dependent factor corrects the Planck mass according to

$$M^{2\,(1)}_{\rm pl} = \frac{2\lambda\, M^{2\,(0)}_{\rm pl}}{\log^2(\sigma)} \simeq N_{\rm sp}\, M^2_{\rm s}\,, \tag{4.36}$$

which for a small number of light species is consistent with [54, 55]. Now the one-loop corrections to the field metrics of the orthogonal directions are all vanishing

$$g^{(1)}_{ab} = 0\,, \qquad \text{with} \quad a, b = t_I, b_I, u_I, v_I\,. \tag{4.37}$$

As a consequence, the found emergence-like relation $G^{(1)}_{ab} \sim G^{(0)}_{ab} / \log^2(\sigma)$ is just a trivial consequence of (4.35). Next, determining the corrections to the remaining metric components we find

$$g^{(1)}_{\sigma\sigma} = \frac{\lambda}{\sigma^2} - \frac{2\lambda}{\log^2(\sigma)\sigma^2}\,, \qquad g^{(1)}_{\rho\rho} = -\frac{\lambda}{2\log^2(\sigma)\sigma^4}\,, \tag{4.38}$$

so that only the hyper multiplet metric receives logarithmic hyper multiplet depending corrections. However, the aforementioned asymmetry with respect to the saxion σ and its axionic partner ρ persists.

5 Conclusions

In this paper we were exploring the Emergence Proposal with a concrete $N = 2$ supersymmetric toroidal orbifold model where, in particular in the weak string coupling limit,

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we had full control over the light particle and string towers of states and their detailed mass formula. We were able to carry out the computation keeping track of all 14 moduli fields from the NS sector. We also considered two other asymptotic regimes, namely the large Kähler and large complex structure limits, where the same number and mass pattern appeared once one included all light modes. Indeed, we identified 12 light particles mostly arising from wrapped D-branes and wrapped NS-branes and one low-tension 4D string arising from a wrapped NS-brane.

Concerning the string tower, we pointed out an issue with the definition of the species scale, namely the QFT and the BH picture were giving mutually non-consistent results. From the QFT point of view, this might be rooted in the fact that with an emergent string present in the asymptotic field limit, one inevitably probes the string at energies larger than $M_{\rm s}$. The most important question clearly is to completely resolve this issue and in the course develop a well-founded approach to treat the Emergence Proposal in the presence of asymptotically tensionless strings.

As the only available approach, carrying out the self-consistent computation in the QFT picture, we found that the one-loop corrections to the moduli field metric and the gauge couplings follow a very similar pattern as their classical results. However, the details turned out to be more intricate and we discussed essentially two different ways to interpret the results differing in how we treated the ubiquitously appearing log-suppressions. Just ignoring them, a large portion of the classical field metric and gauge couplings emerged, i.e. they in particular had the expected moduli dependence. No components were generated at one-loop that were absent classically.

The other option was to consider the log-factors as being physical. Avoiding any conflict with the decoupling of hyper- and vector multiplets in the $N = 2$ supersymmetric EFT led us to the inclusion of a moduli dependent one-loop correction to the 4D Planck mass. However, taken at face value, we have seen that neither of the two variants of the Weak Emergence Proposal is really fully satisfied.

As another new aspect of emergence, this interpretation suggested that the potentially induced corrections due to the light towers of states appearing in asymptotic limits in field space are much milder than initially advocated.

To get further confirmation, it would be interesting to further generalize our multiple moduli computation. Of course, one could consider many other infinite distance limits in the saxionic moduli space, like e.g. the overall large volume limit, where one scales all Kähler moduli like $t_I = \lambda \hat{t}_I$ with the 4D dilaton kept constant. This is nothing else than the decompactification limit of the dual M-theory on $CY \times S^1$ compactification. Such a large volume limit has been one of the prime examples of emergence discussed in [8, 24], where the lightest states were the $D0$-branes with the species scale being the 5D Planck-scale $\tilde{\Lambda} \sim M_{\rm pl}/\lambda^{\frac{1}{2}}$. In addition, one could consider backgrounds in other space-time dimensions or with less supersymmetry.

To resolve this issue about the correct value of the species scale one might wonder whether one could carry out the full emergence computation directly in string theory, i.e. without employing the field theory diagrams. For instance, in an asymptotic direction in the $N = 2$ vector multiplet moduli space one starts with the proposal [29] that the species scale is related to the topological one-loop free energy as $\tilde{\Lambda} \sim M_{\rm pl}/\sqrt{F_1}$ and then involves also the one-loop gauge thresholds corrections.

Acknowledgments

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A Generalities on emergence

To be self-contained, we collect here the basics of the Emergence Proposal, providing in particular the background material and formulas that are needed in the body of the paper.

As already mentioned, in the weaker version of the Emergence Proposal it is claimed that any infinite field distance singularity can be associated to a tower of light states becoming massless in that limit. The tree level singular behavior of the metric can then be matched by integrating out these states up to the quantum gravity cut-off, namely the species scale $\tilde{\Lambda}$. In general, we have multiple towers of states with moduli dependent mass scales and a further degeneracy in the mass spectrum. Therefore, the number of states $N_{\rm sp}$ is given by

$$N_{\rm sp} = \sum_{\vec{n}} \deg_{\vec{n}}\,, \tag{A.1}$$

where we are summing over all quantum numbers n_i, collectively denoted as $\vec{n}$, such that the corresponding states have masses below the species scale. Note that (A.1) is valid in the case of multiplicative towers, which we focus on, where states of mixed quantum numbers are allowed to appear. If that is not the case, then we have what is called additive towers, where $N_{\rm sp} = \sum_i N_{\rm sp,i}$ [24]. If the species scale is much bigger than the mass scales of all towers, one can employ an integral approximation and can often carry out these higher dimensional integrals either analytically or at least extract their asymptotic form.

For emergence, we are for instance interested in the kinetic terms for the moduli fields, whose classical action takes the form

$$S_{\rm kin} = -\frac{1}{2}\int d^4x\sqrt{-g}\,\underbrace{G^{(0)}_{ab}\,\partial_\mu\phi^a\,\partial^\mu\phi^b}_{\mathcal{L}_{\rm kin}}\,. \tag{A.2}$$

The above conventions correspond to dimensionless fields with $c = \hbar = 1$ and in 4D Minkowski space we have of course $g_{\mu\nu} = \eta_{\mu\nu}$. Note that in our conventions the classical field metric $G^{(0)}_{ab}$ contains a factor of $M_{\rm pl}^2$. Qualitatively speaking, emergence means that the one-loop contribution $G^{(1)}_{ab}$ to this moduli field metric arising from integrating out the aforementioned light species is proportional to the tree-level metric. Closely following [8, 24], consider a tower of massive real scalars $\varphi_{\vec{n}}$ or Dirac fermions $\psi_{\vec{n}}$, whose mass is

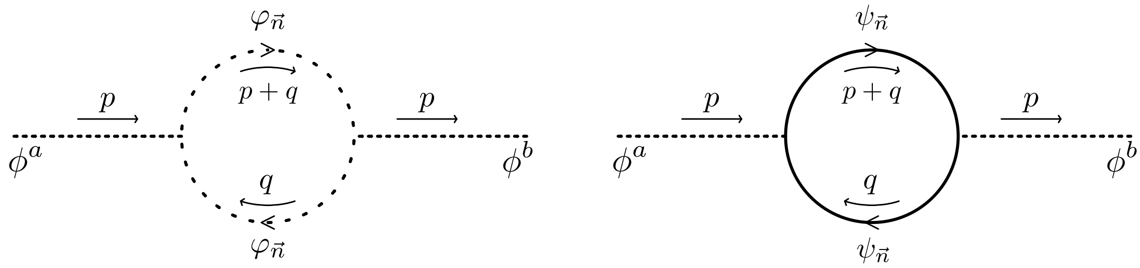

Figure 4. Feynman diagrams for the processes leading to the one-loop corrections to the field metric, with massive scalars $\varphi_{\vec{n}}$ (left) or massive fermions $\psi_{\vec{n}}$ (right) running in the loop.

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parametrized by the moduli fields ϕ^a and some quantum numbers n_i. This is captured by the mass term of the Lagrangian

$$\mathcal{L}^B_{\text{mass}} = M_{\text{pl}}^2 \sum_{\vec{n}} \frac{1}{2} m_{\vec{n}}^2(\phi^a)\, \varphi_{\vec{n}}^2 \qquad \text{or} \qquad \mathcal{L}^F_{\text{mass}} = M_{\text{pl}}^3 \sum_{\vec{n}} m_{\vec{n}}(\phi^a)\, \overline{\psi}_{\vec{n}}\, \psi_{\vec{n}} \,. \tag{A.3}$$

The expansion of each mass term to linear order in the perturbations around the vacuum expectation values of the fields ϕ^a results in a trilinear interaction vertex with coupling strengths

$$\lambda_{a,\vec{n}} = 2 m_{\vec{n}} \partial_a m_{\vec{n}} \text{ (scalars)} \qquad \text{or} \qquad \mu_{a,\vec{n}} = \partial_a m_{\vec{n}} \text{ (fermions)} \tag{A.4}$$

with $\partial_a = \partial/\partial\phi^a$. These vertices lead to the Feynman diagrams shown in figure 4.

Upon integrating out the states from the tower, the propagator matrix $D_{ab}(p^2)$ of the moduli receives a one-loop correction given by

$$D_{ab}(p^2) = \frac{1}{p^2 - \Pi_{ab}(p^2)} \,, \qquad \Pi_{ab}(p^2) = \sum_{\vec{n}} \Pi_{ab,\vec{n}}(p^2) \,. \tag{A.5}$$

Here, $\Pi_{ab,\vec{n}}(p^2)$ is the contribution of a single amputated one-loop Feynman diagram containing the boson or fermion of the tower characterized by $\vec{n}$ and the index structure is due to (A.4). The resulting wave-function renormalization of the moduli is equivalent to the one-loop metric we are looking for. Since it is given by the part of $\Pi_{ab}(p^2)$ proportional to p^2, we simply need to take the derivative of each $\Pi_{ab,\vec{n}}(p^2)$ with respect to p^2, evaluate at $p = 0$ and sum over the whole spectrum. The one-loop metric comes out as

$$G^{(1)}_{ab} = \sum_{\vec{n}} \left. \frac{\partial \Pi_{ab,\vec{n}}(p^2)}{\partial p^2} \right|_{p^2=0} \,. \tag{A.6}$$

With the conventions of figure 4, the contribution from a scalar loop reads

$$\Pi_{ab,\vec{n}}(p^2) = \frac{\lambda_{a,\vec{n}}\, \lambda_{b,\vec{n}}}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m_{\vec{n}}^2} \frac{1}{(p+q)^2 + m_{\vec{n}}^2} \,, \tag{A.7}$$

where the factor of 1/2 accounts for the symmetry of the bosonic diagram. For the contribution to the one-loop metric one obtains

$$\begin{aligned} \left.\frac{\partial \Pi_{ab,\vec{n}}(p^2)}{\partial p^2}\right|_{p^2=0} &= -\frac{\lambda_{a,\vec{n}}\,\lambda_{b,\vec{n}}}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+m_{\vec{n}}^2)^3} \\ &= -\frac{\lambda_{a,\vec{n}}\,\lambda_{b,\vec{n}}}{16\,(2\pi)^2}\frac{\tilde{\Lambda}^4}{m_{\vec{n}}^2(\tilde{\Lambda}^2+m_{\vec{n}}^2)^2}\,. \end{aligned} \tag{A.8}$$

The momentum integral was performed up to the species scale $\tilde{\Lambda}$, since only those light modes will be included in the emergence calculation. Now, one can distinguish two asymptotic limits: either $\tilde{\Lambda} \gg m_{\vec{n}}$, which is typically fulfilled by KK towers, or $\tilde{\Lambda} \simeq m_{\vec{n}}$, which holds for most states in a tower of string excitations. Apparently, both limits give the same functional behavior, namely

$$\left.\frac{\partial \Pi_{ab,\vec{n}}(p^2)}{\partial p^2}\right|_{p^2=0} \simeq \frac{\lambda_{a,\vec{n}}\,\lambda_{b,\vec{n}}}{m_{\vec{n}}^2} \tag{A.9}$$

and only the overall numerical coefficient is different. As mentioned, our computation is indifferent to such overall factors so that the form (A.9) is sufficient for our purposes.

Fermionic loop integrals can be computed in a similar way. The Feynman diagram on the right hand side of figure 4 gives

$$\Pi_{ab,\vec{n}}(p^2) = -\mu_{a,\vec{n}}\,\mu_{b,\vec{n}}\int \frac{d^4q}{(2\pi)^4}\mathrm{tr}\left(\frac{(-i\not{q}+m_{\vec{n}})(-i(\not{p}+\not{q})+m_{\vec{n}})}{(q^2+m_{\vec{n}}^2)((p+q)^2+m_{\vec{n}}^2)}\right)\,. \tag{A.10}$$

With the trace in the above integral explicitly performed, the part linear in p^2 splits into the two pieces

$$\left.\frac{\partial \Pi_{ab,\vec{n}}(p^2)}{\partial p^2}\right|_{p^2=0} = \underbrace{-\,4\,\mu_{a,\vec{n}}\,\mu_{b,\vec{n}}\int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+m_{\vec{n}}^2)^2}}_{\equiv (I)} \tag{A.11}$$

$$\underbrace{+\,2\cdot 4\,\mu_{a,\vec{n}}\,\mu_{b,\vec{n}}\int \frac{d^4q}{(2\pi)^4}\frac{m_{\vec{n}}^2}{(q^2+m_{\vec{n}}^2)^3}}_{\equiv (II)}\,, \tag{A.12}$$

where the factor 4 comes from the trace of gamma matrices and counts the number of fermionic degrees of freedom. For (II), one finds the exact same behavior of the loop-integral as in the bosonic case, only with opposite sign. We focus on supersymmetric setups, where the number of on-shell bosonic and fermionic degrees of freedom match, so these terms precisely cancel out. Carrying out the q-integration, for the remaining contribution (I) in (A.11) one obtains

$$\left.\frac{\partial \Pi_{ab,\vec{n}}(p^2)}{\partial p^2}\right|_{p^2=0} = -\frac{\mu_{a,\vec{n}}\,\mu_{b,\vec{n}}}{(2\pi)^2}\left(\log\left(\frac{\tilde{\Lambda}^2+m_{\vec{n}}^2}{m_{\vec{n}}^2}\right)-\frac{\tilde{\Lambda}^2}{\tilde{\Lambda}^2+m_{\vec{n}}^2}\right)\,. \tag{A.13}$$

In the KK-like limit $\tilde{\Lambda} \gg m_{\vec{n}}$, this becomes

$$\left.\frac{\partial \Pi_{ab,\vec{n}}(p^2)}{\partial p^2}\right|_{p^2=0} \simeq -\mu_{a,\vec{n}}\,\mu_{b,\vec{n}}\,\log\left(\frac{\tilde{\Lambda}^2}{m_{\vec{n}}^2}\right). \tag{A.14}$$

The one-loop metric is given by the sum of these contributions. Inserting (A.4) and (A.14) in (A.6) and taking into account possible mass degeneracies, we arrive at

$$G^{(1)}_{ab} \simeq \sum_{\vec{n}} \deg_{\vec{n}}\; \partial_a m_{\vec{n}}\,\partial_b m_{\vec{n}}\,\log\left(\frac{\tilde{\Lambda}}{m_{\vec{n}}}\right). \tag{A.15}$$

For the string-like limit $\tilde{\Lambda} \simeq m_{\vec{n}}$, the contribution (A.13) reduces to the already familiar expression (A.9) and leads us to the one-loop metric

$$G^{(1)}_{ab} \simeq \sum_{\vec{n}} \deg_{\vec{n}}\; \partial_a m_{\vec{n}}\,\partial_b m_{\vec{n}}\,. \tag{A.16}$$

Strictly speaking, these relations are derived for scalars (spin-0) and spin-1/2 fermions. Since we can also include towers of string excitations, there can be contributions from higher spin bosons and fermions, as well. We assume that their contribution per degree of freedom will not essentially deviate from the lowest spin cases discussed. Hence, we apply the one-loop metrics (A.15) and (A.16) also to these higher spin states.[11]

Gauge kinetic terms at one-loop. The emergence idea can be similarly utilized to study the behavior of gauge couplings in the infrared. Let us briefly sketch the logic for a set of U(1) gauge fields A^a_μ with field strengths $F^a_{\mu\nu} = 2\partial_{[\mu}A^a_{\nu]}$ in 4D, allowing for classical gauge-kinetic mixings according to

$$S_{\rm kin} = \int d^4x\sqrt{-g}\left(-\frac{1}{4}\sum_{a,b} f^{(0)}_{ab}\,F^{\mu\nu,a}F^b_{\mu\nu}\right) \tag{A.17}$$

with the gauge kinetic function given in terms of the gauge couplings as $f^{(0)}_{ab} = g^{-2}_{ab}$. Once again, we incorporate towers of scalars $\varphi_{\vec{n}}$ and fermions $\psi_{\vec{n}}$ with mass $m_{\vec{n}}$. They are minimally coupled to the gauge fields via the covariant derivative

$$D_\mu\varphi_{\vec{n}} = (\partial_\mu - iq_{a,\vec{n}}\,A^a_\mu)\,\varphi_{\vec{n}}\,, \qquad D_\mu\psi_{\vec{n}} = (\partial_\mu - iq_{a,\vec{n}}\,A^a_\mu)\,\psi_{\vec{n}} \tag{A.18}$$

where $q_{a,\vec{n}}$ are the electric charges. The propagator of the U(1) fields receives corrections from analogous loop diagrams depicted in figure 5, so the corrected expression reads

$$D^{\mu\nu}_{ab}(p^2) = \left(\frac{p^2}{g^2_{ab}}\delta^{\mu\nu}_{ab} - \Pi^{\mu\nu}_{ab}(p^2)\right)^{-1}, \quad \Pi^{\mu\nu}_{ab}(p^2) = \Pi_{ab}(p^2)\delta^{\mu\nu}\,, \tag{A.19}$$

where we are assuming Lorentz gauge for all vector fields, $\partial^\mu A^a_\mu = 0$, and a flat background with Euclidean metric $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$. As before, the kinetic terms are given by

[11] After all, the contribution from the string excitations was very robust and independent of a couple of parameters. Therefore, one can probably even weaken this assumption.

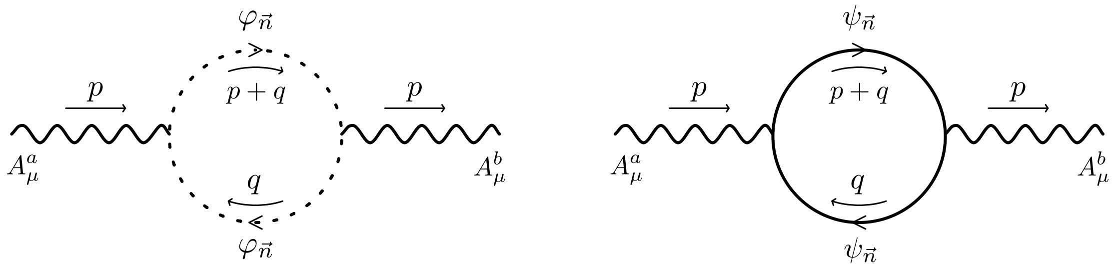

Figure 5. Feynman diagrams for the processes leading to the one-loop corrections to the gauge kinetic terms, with massive scalars $\varphi_{\vec{n}}$ (left) or massive fermions $\psi_{\vec{n}}$ (right) running in the loop.

wave-function renormalizations, so one has to sum over all contributions from amputated one-loop diagrams.

Similar to the kinetic terms for the moduli, one part from the fermionic loop integrals always cancels the scalar contribution if supersymmetry is unbroken. Likewise, both two interesting limits $\tilde{\Lambda} \gg m_{\vec{n}}$ and $\tilde{\Lambda} \simeq m_{\vec{n}}$ lead to the same asymptotic behavior as before. In view of our application, we only focus on the latter case, for which the total one-loop gauge couplings are given by

$$f^{(1)}_{ab} \simeq \sum_{\vec{n}} \deg_{\vec{n}}\ q_{a,\vec{n}}\, q_{b,\vec{n}}\,. \tag{A.20}$$

Analogous to the moduli metric, we will assume that the contribution per degree of freedom from higher spin bosons and fermions will not essentially deviate. Hence, we set all the usually appearing one-loop beta-function coefficients to one.

Cancellation of $\hbar$-factors. Following [24], let us comment on an important conceptual point. Although our starting point was accounting for quantum corrections, our results do remain classical. This subtlety is partially concealed because in natural units $M^2_{\rm pl} = \kappa_4^{-2}$, where κ_4 is Einstein's gravitational coupling constant. Restoring $\hbar$ in the graviton self energy expression (2.14) we have[12]

$$\pi^{-1}(p) = p^2\left(1 - \frac{N_{\rm sp}\, p^2 \kappa_4^2}{120\pi\hbar}\log\left(-\frac{p^2}{\mu^2}\right) + \dots\right)\,, \tag{A.21}$$

where for convenience we left out the mass dependent corrections. Noticing that the classical factor κ_4^2 comes from the graviton vertex, we realize that this one-loop diagram comes with an extra factor $\hbar^{-1}$. Now to illustrate our point and take advantage of (A.1), let us consider the example of a tower of states with mass $n\,\Delta m$ with polynomial degeneracy n^K, corresponding to $N_{\rm sp} = n^{K+1}_{\rm max}$, where $\tilde{\Lambda} = n_{\rm max}\Delta m$. Using (A.16), we get

$$G^{(1)}_{ab} \simeq \frac{1}{\hbar} n^{K+3}_{\rm max}\,(\partial_a \Delta m)\,(\partial_b \Delta m)\,. \tag{A.22}$$

[12]We will still set $c = 1$. Recall that we can express the d-dimensional Planck mass as $M^{d-2}_{\rm pl,d} = \hbar^{d-3}c^{5-d}/8\pi G_N$, where G_N is Newton's gravitational constant and $G_N = \kappa_d^2 c^4/8\pi$.

As for the one-loop amplitude (A.21), we have introduced an extra loop-factor of $\hbar^{-1}$. Then the correction to the metric can be expressed as

$$G^{(1)}_{ab} \simeq \frac{1}{\hbar} N_{\rm sp}\, \tilde{\Lambda}^2 \frac{\partial_a \Delta m\, \partial_b \Delta m}{\Delta m^2} \simeq \frac{M^2_{\rm pl}}{\hbar} \frac{\partial_a \Delta m\, \partial_b \Delta m}{\Delta m^2}\,, \tag{A.23}$$

which due to the relation $M^2_{\rm pl}/\hbar = \kappa_4^{-2}$ gives a classical result, where $\hbar$ has canceled out. This is possible, as the cut-off is also defined via a one-loop diagram.

B Species scale for KK modes

Assume for simplicity that we have a circle compactification leading to a non-degenerate one-dimensional KK tower with a mass spacing

$$\Delta m = \frac{M_{\rm s}}{r} = \frac{M_{\rm pl}}{\sigma r}\,, \tag{B.1}$$

where r is the radius of the circle in string length units and σ the inverse of the 4D string coupling. The mass of each KK mode will be given by

$$m_n = n\, \Delta m \tag{B.2}$$

so that the heaviest KK mode in our theory will be for $k = \tilde{\Lambda}/\Delta m$. In the QFT approach, the total number of states up to level k will be equal to $N_{\rm sp} = k$ so that invoking also the definition of the species scale we get

$$\tilde{\Lambda} = \frac{M_{\rm pl}}{(\sigma r)^{\frac{1}{3}}}\,, \qquad\qquad N_{\rm sp} = (\sigma r)^{\frac{2}{3}}\,. \tag{B.3}$$

Note that the species scale is nothing else than the 5D Planck-scale.

Now let us consider the same model using the BH approach,[13] where the calculation is more involved. In particular, while $k = \tilde{\Lambda}/\Delta m$ still holds, we now need to count the number of multiparticle states whose total mass is equal to the one of the BH. This will be described by a (large) number N, given by

$$N = \frac{M_{\rm BH}}{\Delta m} = S_{\rm BH}\, k = N_{\rm sp}\, k\,. \tag{B.4}$$

where we have used the formula for the Bekenstein-Hawking entropy. Hence, their total number will be given by the number of possible partitions of N into numbers smaller than k. Since $k \ll N$, we can approximate this number by

$$\Omega_{N,k} \sim \frac{N^{k-1}}{(k-1)!\, k!}\,. \tag{B.5}$$

One realizes that for $N_{\rm sp} \sim k$ this takes the asymptotic form

$$\Omega_{N,k} \sim \left(\frac{k^k}{k!}\right)^2 \sim e^{2k} \tag{B.6}$$

[13] We are indebted to Niccolò Cribiori for contributing in an essential way to this computation.

where we have used the leading exponential term in Stirling's formula $k! \sim (k/e)^k$. Taking now the logarithm of this expression we find

$$S = \log \Omega_{N,k} \sim k\,. \tag{B.7}$$

In the BH picture this is supposed to be equal to $N_{\rm sp}$, which is indeed consistent with our assumption. In conclusion, for KK modes both the QFT and the BH picture give the same values for the species scale and the number of light species.

C Axions in the large Kähler modulus limit

Relating to the discussion in section 4.2, in this appendix we speculate about the full mass formula in the $t_1 \to \infty$ limit. So far, we were not explicitly including the axions in the discussion. To do so, let us first consider the wrapped D-brane states from table 2. Up to the overall factor $M_{\rm pl}/\sqrt{t_1}$, these states involve only the Kähler moduli t_2, t_3 and the complex structure moduli u_2, u_3 on the second and the third T^2. Starting with the $D4$-brane, turning on b_2 (or b_3) via the Born-Infeld action one generates also a contribution to the tension that scales precisely like the first $D2$-branes from that table. Turning on both b_2 and b_3 one gets a contribution like the $D0$-brane. Hence, these four branes are related via turning on non-trivial axion values b_2 and b_3.

Note that such chains of four states were also present in the original perturbative mass formula (2.3). It is a yet not resolved issue which bound states of all these (relatively non-supersymmetric) wrapped branes can form and what their mass is, so we just speculate that the final form will be very similar to the one for the perturbative states. The shift symmetry of the axions provides some constraints. Hence, it is suggestive that the mass for the bound states of these four types of wrapped D-branes takes the form

$$\begin{aligned} M_1^2 = \frac{M_{\rm pl}^2}{t_1}\Bigg[\left(\frac{n_1+b_3n_2+b_2n_3+b_2b_3n_4}{t_2^{\frac12}\,t_3^{\frac12}}\right)^2+\left(\frac{(n_2+b_2n_4)\,t_3^{\frac12}}{t_2^{\frac12}}\right)^2 \\ +\left(\frac{(n_3+b_3n_4)\,t_2^{\frac12}}{t_3^{\frac12}}\right)^2+\left(n_4t_2^{\frac12}\,t_3^{\frac12}\right)^2\Bigg]\,. \end{aligned} \tag{C.1}$$

The n_i denote the four wrapping numbers of the wrapped $D0$, $D2$ and $D4$ branes.

Similarly, the other four wrapped $D2$-branes from table 2 are also related via turning on the (quasi-)axions v_2 and v_3 so that we propose

$$\begin{aligned} M_2^2 = \frac{M_{\rm pl}^2}{t_1}\Bigg[\left(\frac{p_1+v_3p_2+v_2p_3+v_2v_3p_4}{u_2^{\frac12}\,u_3^{\frac12}}\right)^2+\left(\frac{(p_2+v_2p_4)\,u_3^{\frac12}}{u_2^{\frac12}}\right)^2 \\ +\left(\frac{(p_3+v_3p_4)\,u_2^{\frac12}}{u_3^{\frac12}}\right)^2+\left(p_4u_2^{\frac12}\,u_3^{\frac12}\right)^2\Bigg]\,. \end{aligned} \tag{C.2}$$

So far, the arguments behind the mass formulas (C.1) and (C.2) were still fairly standard. Next, we are dealing with the two KK-modes (3.9) along the two one-cycles of the first T^2 factor and the corresponding two wrapped NS5-brane states listed in table 3. For these final four states we apply just the analogy to the fundamental string. For the fundamental string, a wrapped string along the $x(y)$-cycle in the presence of a Kalb-Ramond field B_2 changes the definition of the canonical momentum and provides a correction to the mass of the KK-mode along the $y(x)$-direction. Analogously, we now propose that a wrapped NS5-brane along the $x(y)$-cycle in the presence of the magnetic dual B_6 field (with all legs along T^6) changes the definition of the canonical momentum and provides a correction to the mass of the KK-mode along the $y(x)$-direction. Applying this logic we obtain the mass formula

$$M_3^2 = \frac{M_{\rm pl}^2}{t_1}\left[\left(\frac{m_1 + v_1 m_2 + \rho\, m_3 + v_1 \rho\, m_4}{\sigma\, u_1^{\frac{1}{2}}}\right)^2 + \left(\frac{(m_2 + \rho\, m_4)\, u_1^{\frac{1}{2}}}{\sigma}\right)^2 \right.$$
$$\left. + \left(\frac{(m_3 + v_1 m_4)\,\sigma}{u_1^{\frac{1}{2}}}\right)^2 + \left(m_4 u_1^{\frac{1}{2}}\,\sigma\right)^2\right]. \tag{C.3}$$

Recall that ρ is the magnetic dual of the Kalb-Ramond field B_2 with both legs along the 4D space-time. Therefore, it is B_6 with all legs along the 6 toroidal directions. Finally, we put all these the contributions together and also add the contribution from the oscillators of the low-tension 4D string arising from the wrapped NS5-brane

$$M^2 = M_1^2 + M_2^2 + M_3^2 + \frac{M_{\rm pl}^2}{t_1}\kappa^2 N\,. \tag{C.4}$$

Thus, we claim that all two KK-modes and the many wrapped brane states can form bound states whose mass will be given by this relation. This is certainly a strong claim, but the analogy to the weak coupling limit is striking.

Analogous to the mass expression (4.2), introducing now continuous variables, for the derivative with respect to the axion ρ we obtain

$$\partial_\rho M \simeq \frac{M_{\rm pl}}{\sqrt{t_1}}\frac{1}{2\, r\, \sigma^2}\Big(2\, w_3\, y_3 + 2\, x_3\, z_3\Big)\,. \tag{C.5}$$

Hence, compared to the results in the perturbative limit (4.16), we notice the factor σ^2 (instead of σ). This will correctly reproduce the term $d\rho^2/(4\sigma^4)$ in the tree-level metric (2.6).

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