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2-index chiral gauge theories

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Abstract: We undertake a systematic study of the 4-dimensional SU(N) 2-index chiral gauge theories and investigate their faithful global symmetries and dynamics. These are a finite set of theories with fermions in the 2-index symmetric and anti-symmetric representations, with no fundamentals, and they do not admit a large-N limit. We employ a combination of perturbative and nonperturbative methods, enabling us to constrain their infrared (IR) phases. Specifically, we leverage the 't Hooft anomalies associated with continuous and discrete groups to eliminate a few scenarios. In some cases, the anomalies rule out the possibility of fermion composites. In other cases, the interplay between the continuous and discrete anomalies leads to multiple higher-order condensates, which inevitably form to match the anomalies. Further, we pinpoint the most probable symmetry-breaking patterns by searching for condensates that match the full set of anomalies resulting in the smallest number of IR degrees of freedom. Higher-loop β-function analysis suggests that a few theories may flow to a conformal fixed point.

Keywords: Anomalies in Field and String Theories, Discrete Symmetries, Global Symmetries, Confinement

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Contents

1 Introduction

Chiral gauge theories form the fundamental framework of the Standard Model (SM) of particle physics. Within the SM, the electroweak sector undergoes Higgsing at weak coupling, allowing us to apply perturbative techniques. However, without a Higgs field, gauge theories generally flow towards a strongly-coupled regime, rendering their study considerably more challenging. A non-comprehensive list of some of the recent papers that studied chiral gauge theories is [1–10].

This paper focuses on a class of SU(N) chiral gauge theories that accommodate fermions in the 2-index symmetric and anti-symmetric representations. These theories, referred to as 2-index chiral gauge theories, can be characterized by the pair (N, k), where N represents the color and k serves as a common divisor of $N+4$ and $N-4$. Moreover, k is directly associated with the number of flavors in the 2-index symmetric and anti-symmetric representations. What makes these theories particularly intriguing is the absence of a requirement to introduce fundamental fermions to cancel the gauge anomaly. Additionally, they exhibit non-asymptotic freedom for $N > 44$. This class encompasses a collection of 14 distinct theories, occupying a distinct region within asymptotically-free chiral gauge theories. Consequently, a systematic approach to studying this class is justified. It is divided into two subclasses: bosonic and fermion theories. The latter can accommodate gauge-invariant massless fermions. In comparison, the gauge-invariant operators in bosonic theories cannot have a spinor index, as the fermion number is gauged.

We initiated this study in [8], utilizing 't Hooft anomalies to constrain the infrared dynamics of two theories. Namely, these are $(N = 8, k = 4)$ and $(N = 8, k = 2)$ theories. One important development was the identification of the faithful global symmetry acting on fermions. This enabled us to turn on the most general discrete fluxes, the color-flavor-U(1) (CFU) fluxes compatible with the theory and, thus, utilize the full power of 't Hooft anomaly matching conditions. These anomalies are dubbed CFU anomalies. A theory with 't Hooft anomalies cannot be trivially gapped; the infrared (IR) spectrum must contain massless particles or multi-vacua. In the case $(N = 8, k = 4)$, we found that the condensation of two operators can saturate the anomalies.

Continuing our explorations within this comprehensive framework, our current investigation exhausts all the 14 theories and introduces a few novel aspects.

1. We incorporate anomalies stemming from discrete symmetries, thereby imposing additional constraints on the infrared spectra. In the context of the 2-index chiral gauge theories we are examining, in addition to continuous non-abelian flavor symmetries, an axial $\mathrm{U}(1)_A$ symmetry comes into play. When a bosonic operator condenses, it generally breaks the $\mathrm{U}(1)_A$ symmetry down to a discrete subgroup, which typically is anomalous. Consequently, we face the challenge of identifying a set of condensates that not only matches the anomalies associated with non-abelian symmetries but also avoids the presence of any anomalous unbroken discrete subgroups. For example, the two candidate condensates we previously considered in the case $(N = 8, k = 4)$, [8], fail to match the anomaly of an unbroken discrete group. We revise the situation in light of the new understanding and propose that the set of anomalies can be matched by other condensates.

   Interestingly, in a few cases, matching the full set of anomalies, particularly anomalies of discrete symmetries, can be achieved only via the condensation of multiple higher-order operators. Given the strong dynamics, such formation is not a surprise. However, anomalies explain the kinematical reasons why such condensates have to form.

2. Another significant aspect of our work lies in our pursuit of the minimal scenario that satisfies the entire set of anomalies and yields the smallest number of massless particles

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in the infrared spectrum. Such a scenario holds particular appeal as it minimizes the free energy associated with the theory.

3. We adhered to a systematic algorithm during our quest to identify composite massless fermions capable of satisfying the anomalies within the fermionic class. Regrettably, we were unable to find such composites. Notably, in the case of $(N = 6, k = 2)$, we demonstrate that these composites cannot solely match the CFU anomaly. Consequently, we are left with two plausible explanations: either these composites do not exist altogether, or the formation of condensates alongside the composites is necessary to match the anomaly. However, the latter scenario is rather contrived in that it requires the formation of special condensates (that do not alter the anomalies matched by the composites) in addition to the composite fermions, prompting us to lean toward the likelihood that composites cannot form in this case.

4. To complete our study, we also examined the possibility that a theory flows to a conformal fixed point in the IR. Generally speaking, a theory can form a strongly-coupled IR fixed point, which is beyond the scope of any perturbative analysis. Such a fixed point automatically satisfies the full set of anomalies, albeit it remains an open question how this can be seen. Nevertheless, by scrutinizing the higher-order β-function, we successfully identified several instances where the analysis of the β-function offers indications suggestive of a perturbative nature inherent to a fixed point.

This paper is organized as follows. In section 2, we review the global symmetries and anomalies of the 2-index chiral theories. This includes the anomalies of continuous symmetries, the CFU anomalies, as well as anomalies of discrete symmetries. In section 3, we revise the matching of the CFU anomalies in the IR and introduce the novelty of matching the discrete subgroups of the axial $\mathrm{U}(1)_A$ that can be left unbroken by a condensate. Sections 4 and 5 are devoted to applying these ideas to both the fermionic and bosonic field theories, respectively. We summarize our findings in section 6, and in particular, the reader is referred to table 10, which, for all theories, it gives the global symmetries, the proposed IR condensates that yield the smallest number of Goldstones, and the fate of the symmetries in the IR.

2 Theory: symmetry structure and anomalies

We consider $\mathrm{SU}(N)$ gauge theory with n_ψ flavors of left-handed Weyl fermions ψ transforming in the 2-index representation along with n_χ flavors of left-handed Weyl fermions χ transforming in the complex conjugate 2-index anti-symmetric representation:

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}[f^c \wedge \star f^c] - i\bar{\psi}\bar{\sigma}^\mu D_\mu \psi - i\bar{\chi}\bar{\sigma}^\mu D_\mu \chi \,, \tag{2.1}$$

where $D_\mu \equiv \partial_\mu - i a^c_\mu$ is the covariant derivative, a^c is the color gauge field, and f^c is its field strength. In this work, we use the lower-case letters, a^c and $f^c = da^c$, to denote the

N	5	6	8	10	12	16	20	28	36	44
k	1	$1,2$	$2,4$	2	$4,8$	4	$4,8$	8	8	8

Table 1. A list of the 2-index chiral gauge theories.

dynamical (color) 1-form gauge field and its field strength, while we use upper-case letters, A and $F = dA$, for background fields. To keep track of the color indices, we choose $\psi_{(a_1 a_2)}$ to carry two down indices, while $\chi_{[a_1 a_2 \ldots a_{N-2}]}$ carries $N-2$ down indices. A round (square) bracket indicates symmetrizing (anti-symmetrizing) over the indices. The cubic anomaly coefficients of the 2-index symmetric and the conjugate of the 2-index anti-symmetric representations are $\mathcal{A}_\psi = N+4$, $\mathcal{A}_\chi = -(N-4)$, respectively. Cancellation of the gauge anomaly demands that n_ψ and n_χ are fixed as

$$n_\psi = \frac{N-4}{k}\,, \qquad n_\chi = \frac{N+4}{k}\,, \tag{2.2}$$

where k is a common divisor of $N-4$ and $N+4$. The theory is asymptotically free provided that $11N - \frac{2(N^2-8)}{k} > 0$. This leaves us with the finite set of theories in table 1. These theories do not possess a large-N limit, as they become infrared-free for $N > 44$. Also, except for $N = 5, 6, 10$, the other allowed colors are multiples of 4. These are bosonic theories because all their gauge invariant operators cannot carry a spinor index. In other words, the $(-1)^F$ fermion number in bosonic theories is gauged, and thus, they cannot have gauge-singlet fermionic operators.

One important aspect of this work is to systematically analyze these theories, paying particular attention to the faithful global symmetries, and exhausting the class of generalized 't Hooft anomalies that enable us to constrain the infrared phases.

2.1 Symmetries

The theory enjoys two global flavor groups $\mathrm{SU}(n_\psi) = \mathrm{SU}((N-4)/k)$ and $\mathrm{SU}(n_\chi) = \mathrm{SU}((N+4)/k)$ acting on ψ and χ, respectively. In addition, the theory is endowed with two U(1) global classical symmetries, $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$. Their action on ψ and χ is chosen as

$$\begin{aligned} \mathrm{U}(1)_1: &\quad \psi \longrightarrow e^{i\alpha_1}\psi\,, \quad \chi \longrightarrow e^{i\beta_1}\chi\,, \\ \mathrm{U}(1)_2: &\quad \psi \longrightarrow e^{i\alpha_2}\psi\,, \quad \chi \longrightarrow e^{i\beta_2}\chi\,. \end{aligned} \tag{2.3}$$

The two transformations $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$ come naturally with two parameters. Here, however, we introduce the 4 parameters $\alpha_{1,2}$ and $\beta_{1,2}$ to account for the fermions charges, in addition to the transformation parameters. The gauge sector instantons break most of the classical U(1) symmetries. The effective action in the instanton background acquires the terms

$$\Delta S = i\left(n_\psi \alpha_1 T_\psi + n_\chi \beta_1 T_\chi\right) \int_{\mathbb{M}^4} \lambda_0^{u_1} \frac{\mathrm{tr}\left[f^c \wedge f^c\right]}{8\pi^2} + i\left(n_\psi \alpha_2 T_\psi + n_\chi \beta_2 T_\chi\right) \int_{\mathbb{M}^4} \lambda_0^{u_2} \frac{\mathrm{tr}\left[f^c \wedge f^c\right]}{8\pi^2} \tag{2.4}$$

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upon performing simultaneous transformations of $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$, where

$$T_\psi = N+2\,, \qquad T_\chi = N-2 \tag{2.5}$$

are the Dynkin indices of the representations. Here, f^c is the 2-form field strength of the color group, while λ^{u_1} and λ^{u_2} are the gauge parameters of $\mathrm{U}(1)_1$ and $\mathrm{U}(2)_2$, respectively, i.e., $A^{u_{1,2}} \longrightarrow A^{u_{1,2}} + d\lambda^{u_{1,2}}$. We can find a combination of the parameters α_1 and β_1 that kills the first term in ΔS, leaving behind a genuine symmetry. We call this symmetry the axial $\mathrm{U}(1)_A$. It acts on ψ and χ with transformation parameter α:

$$\mathrm{U}(1)_A: \quad \psi \longrightarrow e^{i2\pi\alpha q_\psi}\psi\,, \quad \chi \longrightarrow e^{i2\pi\alpha q_\chi}\chi\,, \tag{2.6}$$

and we have defined the $\mathrm{U}(1)_A$ charges of ψ and χ as

$$q_\psi \equiv -\frac{N_\chi}{r}\,, \qquad q_\chi \equiv \frac{N_\psi}{r}\,, \tag{2.7}$$

where $r = \gcd(n_\chi T_\chi, n_\psi T_\psi)$, and

$$N_\chi \equiv n_\chi T_\chi\,, \qquad N_\psi \equiv n_\psi T_\psi\,. \tag{2.8}$$

Yet, we can find values of α_2 and β_2 that leave the discrete subgroup $\mathbb{Z}_{p_\psi N_\psi + p_\chi N_\chi} \subset \mathrm{U}(1)_2$ invariant in the color background, where p_ψ and p_χ are arbitrary integers. In appendix A, we show that most of the $\mathbb{Z}_{p_\psi N_\psi + p_\chi N_\chi}$ elements belong to $\mathrm{U}(1)_A$ and that only a subgroup $\mathbb{Z}_r \subset \mathbb{Z}_{p_\psi N_\psi + p_\chi N_\chi}$, which is p_χ and p_ψ-independent, can *potentially* act as a *genuine* symmetry on the fermions. Also, we can always choose $\mathbb{Z}_r$ to act solely on χ:

$$\mathbb{Z}_r: \quad (\psi,\chi) \longrightarrow \left(\psi, e^{i\frac{2\pi\ell}{r}}\chi\right)\,, \quad \ell = 0,1,2,\ldots,r-1\,. \tag{2.9}$$

Yet, one must check that $\mathbb{Z}_r$ or a subgroup of it cannot be absorbed in the centers of the color or flavor groups, which leaves a proper subgroup of $\mathbb{Z}_r$ as the genuine discrete symmetry. This will be checked on a case-by-case basis. In the following, we will use $\mathbb{Z}_p^{d\chi} \subseteq \mathbb{Z}_r$ to denote the genuine discrete chiral symmetry. For completeness, we remind the reader that the fermion number symmetry $\mathbb{Z}_2^F = (-1)^F$ operates on ψ and χ as $(\psi,\chi) \longrightarrow -(\psi,\chi)$.

Finally, we also note that when N is even, the theory is endowed with a $\mathbb{Z}_2^{(1)}$ 1-form center symmetry acting on the fundamental Wilson loops. In summary, the good global symmetry of the theory is

$$G^{\mathrm{g}} \sim \mathrm{SU}(n_\psi) \times \mathrm{SU}(n_\chi) \times \mathrm{U}(1)_A \times \mathbb{Z}_p^{d\chi} \times \mathbb{Z}^{(1)}_{\gcd(N,2)}\,, \tag{2.10}$$

where the tilde indicates that this is the correct group modulo a discrete group needed to fix the faithful global symmetry. Thus, the faithful global symmetry is a quotient group.

To determine the correct quotient group, we follow [8, 11]. Here, we keep our treatment short as the details can be found in [8]. We put the theory on a general compact 4-D spin manifold $\mathbb{M}^4$, define a principal bundle of the continuous part of the global symmetry G^{g} on $\mathbb{M}^4$, and take the transition functions of G^{g} to act on fibers by left multiplication. Spinors are sections of the bundle, and we use the notations ψ_i and χ_i for their values on a

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local patch $U_i \subset \mathbb{M}^4$. We denote the transition functions of the color SU(N), (non-abelian) flavor, and U(1)$_A$ group as g, f, and u, respectively, along with the proper superscript to distinguish those of ψ and χ. On the overlap $U_i \cap U_j$ we have

$$\psi_i = (g^\psi, f^\psi, u^\psi)_{ij}\,\psi_j\,, \qquad \chi_i = (g^\chi, f^\chi, u^\chi)_{ij}\,\chi_j\,. \tag{2.11}$$

The fermions are well defined on $\mathbb{M}^4$ provided they satisfy the cocycle condition (a necessary consistency condition) on the triple overlap $U_i \cap U_j \cap U_k$. Now, we turn on background gauge fields for centers of the gauge, flavor, and U(1)$_A$ groups and determine the most general combination compatible with the cocycle condition,[1] which reads

$$\left(g^\psi, f^\psi, u^\psi\right)_{ij}\circ\left(g^\psi, f^\psi, u^\psi\right)_{jk}\circ\left(g^\psi, f^\psi, u^\psi\right)_{ki} = (z_c, z_f, z_u) \quad \text{with} \quad z_c z_f z_u = 1\,, \tag{2.12}$$

where z's refer to the center elements: $z_c \in \mathbb{Z}_{N/\gcd(N,2)}$, $z_f \in \mathbb{Z}_{n_\psi}$, and $z_u \in \mathrm{U}(1)_A$. The condition $z_c z_f z_u = 1$ is required for the equivalence relation

$$(z_c, z_f, z_u) \sim (1,1,1)\,, \tag{2.13}$$

which is needed to obtain the correct compatibility condition. Similar expressions hold for the cocycle condition of χ. The following two equations give the consistency (compatibility) conditions

$$\begin{aligned}
\psi: &\quad \underbrace{e^{i2\pi\frac{2m}{N}}}_{z_c}\underbrace{e^{i2\pi\frac{pk}{N-4}}}_{z_\psi}\underbrace{e^{-i2\pi s\frac{(N+4)(N-2)}{kr}}}_{z_u} = 1\,,\\
\chi: &\quad \underbrace{e^{-i2\pi\frac{2m}{N}}}_{z_c}\underbrace{e^{-i2\pi\frac{p'k}{N+4}}}_{z_\chi}\underbrace{e^{i2\pi s\frac{(N-4)(N+2)}{kr}}}_{z_u} = 1\,.
\end{aligned} \tag{2.14}$$

Here, $m \in \mathbb{Z}_{N/\gcd(N,2)}$, $p \in \mathbb{Z}_{n_\psi}$, $p' \in \mathbb{Z}_{n_\chi}$ and s is a U(1)$_A$ parameter. The factor of 2 that appears in z_c accounts for the N-ality of ψ and χ, and the negative sign that appears in the z_c factor in the second line accounts for the fact that χ transforms in the complex conjugate of the 2-index anti-symmetric representation. We also take ψ to transform in the fundamental representation of SU(n_ψ) and χ to transform in the anti-fundamental representation of SU(n_χ). Following [8], we shall dub the discrete color-flavor-U(1)$_A$ fluxes as the CFU fluxes. The full set of solutions of (2.14) determines the quotient group in (2.10). These solutions will be found on a case-by-case basis. In general, we divide (2.10) by $\mathbb{Z}_{N/\gcd(N,2)} \times \mathbb{Z}_{(N-4)/k} \times \mathbb{Z}_{(N+4)/k}$ or a subgroup of it.

Once a non-trivial solution of (2.14) is found, we can calculate the topological charges associated with the center fluxes, which are fractional charges in general. Let $\mathbb{M}^4$ admit two independent 2-cycles and let two integers, e.g., m_1 and m_2, account for the number of quanta piercing through them. For example, we can take $\mathbb{M}^4 = \mathbb{T}^4$, a 4-torus with a period

[1] See [12–15] for applications of the anomalies resulting from turning on these fluxes.

length L, and turn on the gauge fields that are compatible with the cocycle condition:

$$\begin{aligned}
a_1^c &= \frac{2\pi m_1}{L^2}\boldsymbol{H}_c\cdot\boldsymbol{\nu}_c\,, & a_2^c &= 0\,, & a_3^c &= \frac{2\pi m_2}{L^2}\boldsymbol{H}_c\cdot\boldsymbol{\nu}_c\,, & a_4^c &= 0\,,\\
A_1^\psi &= \frac{2\pi p_1}{L^2}\boldsymbol{H}_\psi\cdot\boldsymbol{\nu}_\psi\,, & A_2^\psi &= 0\,, & A_3^\psi &= \frac{2\pi p_2}{L^2}\boldsymbol{H}_\psi\cdot\boldsymbol{\nu}_\psi\,, & A_4^\psi &= 0\,,\\
A_1^\chi &= \frac{2\pi p_1'}{L^2}\boldsymbol{H}_\chi\cdot\boldsymbol{\nu}_\chi\,, & A_2^\chi &= 0\,, & A_3^\chi &= \frac{2\pi p_2'}{L^2}\boldsymbol{H}_\chi\cdot\boldsymbol{\nu}_\chi\,, & A_4^\chi &= 0\,,\\
A_1^u &= \frac{2\pi s_1}{L^2}\,, & A_2^u &= 0\,, & A_3^u &= \frac{2\pi s_2}{L^2}\,, & A_4^u &= 0\,.
\end{aligned} \tag{2.15}$$

a^c_μ, A^ψ_μ, A^χ_μ, and A^u_μ are the background gauge fields of the center of the color, $SU(n_\psi)$ flavor, $SU(n_\chi)$, flavor, and $U(1)_A$, respectively. The bold-face letters $\boldsymbol{H}\equiv(H_1,\ldots,H_{N-1})$ are the Cartan generators of $SU(N)$ group, while $\boldsymbol{\nu}\equiv(\nu_1,\ldots,\nu_{N-1})$ is a weight in the defining representation of the group. Notice that the integers $m_{1,2}, p_{1,2}, p'_{1,2}, s_{1,2}$ are the same integers that solve the consistency conditions (2.14). Given the set of the background fields (2.15), one immediately obtains the topological charges defined as

$$Q_c=\int_{\mathbb{T}^4}\frac{\mathrm{tr}\left[f^c\wedge f^c\right]}{8\pi^2}\,,\quad Q_\psi=\int_{\mathbb{T}^4}\frac{\mathrm{tr}\left[F^\psi\wedge F^\psi\right]}{8\pi^2}\,,\quad Q_\chi=\int_{\mathbb{T}^4}\frac{\mathrm{tr}\left[F^\chi\wedge F^\chi\right]}{8\pi^2}\,,\quad Q_u=\int_{\mathbb{T}^4}\frac{F^u\wedge F^u}{8\pi^2}\,, \tag{2.16}$$

and f^c, $F^{\psi,\chi,u}$ are the field strengths of the corresponding background. Substituting (2.15) into (2.16), we obtain

$$\begin{aligned}
Q_c &= k_c-\frac{m_1m_2}{N}\,, & Q_\psi &= k_\psi-\frac{p_1p_2k}{N-4}\,,\\
Q_\chi &= k_\chi-\frac{p_1'p_2'k}{N+4}\,, & Q_u &= (s_1-k_1)(s_2-k_2)\,,
\end{aligned} \tag{2.17}$$

and $k_c, k_\psi, k_\chi, k_1, k_2\in\mathbb{Z}$ are arbitrary integers that are always allowed. These fluxes will support fermion zero modes, and the Dirac indices give their number:

$$\begin{aligned}
\mathcal{I}_\psi &= n_\psi T_\psi Q_c+\dim_\psi Q_\psi+\dim_\psi n_\psi q_\psi^2 Q_u\,,\\
\mathcal{I}_\chi &= n_\chi T_\chi Q_c+\dim_\chi Q_\chi+\dim_\chi n_\chi q_\chi^2 Q_u\,,
\end{aligned} \tag{2.18}$$

and $\dim_\psi=\frac{N(N+1)}{2}$, $\dim_\chi=\frac{N(N-1)}{2}$ are the dimensions of the representations. Dirac indices count the number of the Weyl zero modes in the background of center fluxes. The integrality of the indices can work as a check on the consistency of the fluxes on $\mathbb{M}^4$.

One may also turn on the CFU fluxes on nonspin $\mathbb{M}^4$. A nonspin manifold does not admit fermions in the sense that there is an obstruction in lifting the $SO(4)$ rotation group bundle to a $\mathrm{Spin}(4)$ bundle on $\mathbb{M}^4$. A diagnosis of a non-spin manifold is that the Dirac index of a Weyl fermion, $\mathcal{I}=\frac{1}{196\pi^2}\int_{\mathbb{M}^4}\mathrm{tr}R\wedge R$, where R is the curvature 2-form, is non-integer. An example of a nonspin manifold is $\mathbb{CP}^2$, which has $\frac{1}{196\pi^2}\int_{\mathbb{CP}^2}\mathrm{tr}R\wedge R=-\frac{1}{8}$. To put a Weyl fermion on $\mathbb{CP}^2$, we need to excite a $U(1)$ flux F through its 2-cycle $\mathbb{CP}^1\subset\mathbb{CP}^2$ and demand that $\int_{\mathbb{CP}^1}F\in\pi(2\mathbb{Z}+1)$, which implies $\frac{1}{8\pi^2}\int_{\mathbb{CP}^2}F\wedge F\in\frac{\mathbb{Z}}{8}$. Now, one can easily check the integrality of the Dirac index $\frac{1}{196\pi^2}\int_{\mathbb{CP}^2}\mathrm{tr}R\wedge R+\frac{1}{8\pi^2}\int_{\mathbb{CP}^2}F\wedge F\in\mathbb{Z}$, and thus, the fermions are well-defined on $\mathbb{CP}^2$ in the presence of such $U(1)$ fluxes. Here, although

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one cannot define a Spin(4) bundle on pure $\mathbb{CP}^2$, in the sense that the corresponding cocycle condition fails on a triple overlap, nonetheless, we can define the $\text{Spin}_c(4)$ structure $\text{Spin}(4) \times \text{U}(1)/\mathbb{Z}_2$ in the presence of the U(1) background.

This idea can be generalized in the presence of the CFU fluxes; see [16] for details. One just needs to replace the consistency conditions (2.14) with

$$\begin{aligned} \psi: \quad & \underbrace{e^{i2\pi\frac{2m}{N}}}_{z_c}\underbrace{e^{i2\pi\frac{pk}{N-4}}}_{z_\psi}\underbrace{e^{-i2\pi s\frac{(N+4)(N-2)}{kr}}}_{z_u} = -1\,, \\ \chi: \quad & \underbrace{e^{-i2\pi\frac{2m}{N}}}_{z_c}\underbrace{e^{-i2\pi\frac{p'k}{N+4}}}_{z_\chi}\underbrace{e^{i2\pi s\frac{(N-4)(N+2)}{kr}}}_{z_u} = -1\,. \end{aligned} \tag{2.19}$$

The minus sign on the right-hand side compensates for the minus sign arising from parallel transporting the spinor fields around appropriate closed paths in $\mathbb{CP}^2$; see the detailed discussion in [16]. Given that a solution, $m \in \mathbb{Z}_{N/\gcd(N,2)}$, $p \in \mathbb{Z}_{n_\psi}$, $p' \in \mathbb{Z}_{n_\chi}$ and s, to (2.19) can be found, the topological charges corresponding to the CFU fluxes and gravity are given by (see [16])

$$\begin{aligned} & Q_c = \frac{m^2}{2}\left(1-\frac{1}{N}\right), \qquad Q_\psi = \frac{p^2}{2}\left(1-\frac{k}{N-4}\right), \\ & Q_\chi = \frac{p'^2}{2}\left(1-\frac{k}{N+4}\right), \qquad Q_u = \frac{1}{2}s^2, \qquad Q_g = -\frac{1}{8}\,. \end{aligned} \tag{2.20}$$

The Dirac-indices of ψ and χ are

$$\begin{aligned} \mathcal{I}_\psi^{\mathbb{CP}^2} &= n_\psi T_\psi Q_c + \dim_\psi Q_\psi + \dim_\psi n_\psi \left(q_\psi^2 Q_u + Q_g\right), \\ \mathcal{I}_\chi^{\mathbb{CP}^2} &= n_\chi T_\chi Q_c + \dim_\chi Q_\chi + \dim_\chi n_\chi \left(q_\chi^2 Q_u + Q_g\right), \end{aligned} \tag{2.21}$$

which are always integers. Except for $(N = 6, k = 2)$ and $(N = 10, k = 2)$ in table 1, we can always find solutions to (2.19), and thus, we can put these theories on $\mathbb{CP}^2$.

2.2 Anomalies

The theory has a set of 't Hooft anomalies that can help constrain the possible IR phases. In the following, we list the 't Hooft anomalies we shall encounter in our study.

(I) $[\mathbf{SU}(n_\psi)]^3$ and $[\mathbf{SU}(n_\chi)]^3$ anomalies. These are perturbative (triangle) anomalies and their inflow from 5-D to 4-D is captured via 5-D Chern-Simons theories:

$$\begin{aligned} [\text{SU}(n_\psi)]^3: \quad & \exp\left[i\dim_\psi \int_{\mathbb{M}^5}\omega_5(A^\psi)\right], \\ [\text{SU}(n_\chi)]^3: \quad & \exp\left[i\dim_\chi \int_{\mathbb{M}^5}\omega_5(A^\chi)\right], \end{aligned} \tag{2.22}$$

where A^ψ and A^χ are the $\text{SU}(n_\psi)$ and $\text{SU}(n_\chi)$ 1-form background gauge fields, extended from 4-D to 5-D, and $\omega_5(A)$ is the 5-D Chern-Simons form defined via the descent equation:

$$d\omega_5(A) = \frac{1}{3!(2\pi)^2}\text{tr}_\square F^3\,, \tag{2.23}$$

and F is the 2-form field strength of A.

(II) $\mathrm{U}(1)_A$- and $\mathbb{Z}_p^{d\chi}$-gravitational anomalies. These anomalies are captured via the 5-D anomaly inflow actions:

$$\begin{aligned} \mathrm{U}(1)_A[\text{grav}]: &\quad \exp\left[i\left(q_\psi n_\psi \dim_\psi + q_\chi n_\chi \dim_\chi\right)\int_{\mathbb{M}^5} A^u \wedge \frac{p_1(\mathbb{M}^5)}{24}\right], \\ \mathbb{Z}_p^{d\chi}[\text{grav}]: &\quad \exp\left[i\left(n_\chi \dim_\chi\right)\int_{\mathbb{M}^5} A^{d\chi} \wedge \frac{p_1(\mathbb{M}^5)}{24}\right]. \end{aligned} \tag{2.24}$$

The 1-form gauge fields A^u and $A^{d\chi}$ are the backgrounds of $\mathrm{U}(1)_A$ and $\mathbb{Z}_p^{d\chi}$, respectively. $p_1(\mathbb{M}^5) = -\frac{1}{8\pi^2} R \wedge R$ is the first Pontryagin number and R is the curvature 2-form. On a spin manifold, we have $\int_{\mathbb{M}^4} p_1(\mathbb{M}^4) \in 48\,\mathbb{Z}$, and thus, there are 2 zero modes per Weyl fermion in a gravitational background. Under $\mathrm{U}(1)_A$ and $\mathbb{Z}_p^{d\chi}$ transformations we have $A^u \longrightarrow A^u + d\lambda^u$ with $\oint d\lambda^u = 2\pi\mathbb{Z}$ and $A^{d\chi} \longrightarrow A^{d\chi} + d\lambda^{d\chi}$, with $\oint d\lambda^{d\chi} = \frac{2\pi\mathbb{Z}}{p}$, and the anomaly inflow actions produce the 4-D anomalies.

The result (2.24) is "perturbative" as it can be seen from a triangle diagram with two vertices that couple the fermions to a gravitational background via the energy-momentum tensor, while the third vertex couples the fermions to an external $\mathrm{U}(1)_A$ or $\mathbb{Z}_p^{d\chi}$ sources.

(III) CFU anomalies. These anomalies were identified in [11]; however, see [17, 18] for earlier encounters. They are anomalies of $\mathrm{U}(1)_A$ and $\mathbb{Z}_p^{d\chi}$ symmetries in the background of the CFU fluxes that are supported on a general spin manifold. As was shown in [8], 5-D anomaly inflow actions can also capture them. However, we find it more convenient to express such anomalies in terms of the non-trivial phases that are acquired by the partition function $\mathcal{Z}$ under the action of $\mathrm{U}(1)_A$ and $\mathbb{Z}_p^{d\chi}$ symmetries in the background of the CFU fluxes:

$$\begin{aligned} \mathrm{U}(1)_A[\text{CFU}]: &\quad \mathcal{Z} \longrightarrow e^{i2\pi\alpha\left(q_\psi \mathcal{I}_\psi + q_\chi \mathcal{I}_\chi\right)}\mathcal{Z}, \\ \mathbb{Z}_p^{d\chi}[\text{CFU}]: &\quad \mathcal{Z} \longrightarrow e^{i\frac{2\pi}{p}\mathcal{I}_\chi}\mathcal{Z}, \end{aligned} \tag{2.25}$$

and the Dirac indices $\mathcal{I}_\psi$ and $\mathcal{I}_\chi$ are given in (2.18). The contribution from the color topological charge Q_c drops out in the computation of the $\mathrm{U}(1)_A[\text{CFU}]$ anomaly, as can be easily checked, since $\mathrm{U}(1)_A$ is a good symmetry in the background of the color flux. This is not the case with $\mathbb{Z}_p^{d\chi}[\text{CFU}]$ anomaly, where Q_c contributes to the anomaly. As we shall discuss, this observation has important consequences for anomaly matching in the IR.

It is also important to notice that the perturbative anomalies $\mathrm{U}(1)_A[\mathrm{SU}(n_\psi)]^2$, $\mathrm{U}(1)_A[\mathrm{SU}(n_\chi)]^2$, $\mathbb{Z}_p^{d\chi}[\mathrm{SU}(n_\chi)]^2$, and $[\mathrm{U}(1)_A]^3$ are a subset of the CFU anomalies, obtained by turning off the center fluxes and keeping only the integer topological charges in (2.17). Again, one can express them using anomaly inflow actions as

$$\begin{aligned} \mathrm{U}(1)_A[\mathrm{SU}(n_\psi)]^2: &\quad \exp\left[i q_\psi n_\psi \dim_\psi \int_{\mathbb{M}^5} A^u \wedge \frac{\operatorname{tr}\left[F^\psi \wedge F^\psi\right]}{8\pi^2}\right], \\ \mathrm{U}(1)_A[\mathrm{SU}(n_\chi)]^2: &\quad \exp\left[i q_\chi n_\chi \dim_\chi \int_{\mathbb{M}^5} A^u \wedge \frac{\operatorname{tr}\left[F^\chi \wedge F^\chi\right]}{8\pi^2}\right], \\ \mathbb{Z}_p^{d\chi}[\mathrm{SU}(n_\chi)]^2: &\quad \exp\left[i n_\chi \dim_\chi \int_{\mathbb{M}^5} A^{d\chi} \wedge \frac{\operatorname{tr}\left[F^\chi \wedge F^\chi\right]}{8\pi^2}\right], \\ [\mathrm{U}(1)_A]^3: &\quad \exp\left[i\left(q_\psi^3 n_\psi \dim_\psi + q_\chi^3 n_\chi \dim_\chi\right)\int_{\mathbb{M}^5} A^u \wedge \frac{F^u \wedge F^u}{24\pi^2}\right]. \end{aligned} \tag{2.26}$$

In addition, for N even, the CFU anomalies encompass the $\mathrm{U}(1)_A$ 0-form/ $\mathbb{Z}_2^{[1]}$ 1-form as well as the $\mathbb{Z}_r^{d\chi}$ 0-form/ $\mathbb{Z}_2^{[1]}$ 1-form mixed anomalies. These can be easily found by turning off the flavor and the $\mathrm{U}(1)_A$ fluxes. In practice, one uses (2.25), (2.17), and (2.18), after setting $p_{1,2} = p'_{1,2} = s_{1,2} = 0$ and $m_1 = m_2 = N/2$. This choice enforces the consistency conditions (2.14) and gives $Q_\psi = Q_\chi = Q_u = 0$ and $Q_c = \frac{N}{4}$.

One may also use the CFU fluxes on $\mathbb{CP}^2$ to calculate the $\mathrm{U}(1)_A$ [CFU] and $\mathbb{Z}_p$ [CFU] anomalies, which sometimes are more restrictive than the corresponding anomalies on a spin manifold. We use the Dirac indices on $\mathbb{CP}^2$, as given by (2.21), to find

$$\begin{aligned} \mathrm{U}(1)_A\,[\mathrm{CFU}]_{\mathbb{CP}^2}: &\quad \mathcal{Z} \longrightarrow e^{i2\pi\alpha\left[q_\psi \mathcal{I}_\psi^{\mathbb{CP}^2} + q_\chi \mathcal{I}_\chi^{\mathbb{CP}^2}\right]} \mathcal{Z}\,, \\ \mathbb{Z}_p^{d\chi}\,[\mathrm{CFU}]_{\mathbb{CP}^2}: &\quad \mathcal{Z} \longrightarrow e^{i\frac{2\pi}{p}\mathcal{I}_\chi^{\mathbb{CP}^2}} \mathcal{Z}\,. \end{aligned} \tag{2.27}$$

From here on, we shall write all anomalies in terms of their phases to reduce clutter. For example, instead of (2.25), we write:

$$\mathrm{U}(1)_A[\mathrm{CFU}] : q_\psi \mathcal{I}_\psi + q_\chi \mathcal{I}_\chi\,, \quad \mathbb{Z}_p^{d\chi}[\mathrm{CFU}] : \mathcal{I}_\chi\,. \tag{2.28}$$

(IV) Anomalies of discrete groups. Here, we consider anomalies of a discrete symmetry $\mathbb{Z}_n$, where n is a general positive integer. An example of the discrete symmetry is the $\mathbb{Z}_p^{d\chi}$ chiral symmetry or a discrete subgroup of $\mathrm{U}(1)_A$ left unbroken in the IR. The proper way to detect anomalies of discrete symmetries is to use the Dai-Freed prescription [19, 20]. The idea stems from the fact that a chiral massless fermion defined on $\mathbb{M}^4$ can be realized as the chiral zero mode residing on the boundary $\mathbb{M}^4$ of a 5-dimensional manifold $\mathbb{M}^5$ that is endowed with massive fermions, with $\mathbb{Z}_n$ turned on in the 5-dimensional manifold. One can also consider a different 5-dimensional manifold $\mathbb{M}'^5$ with the same boundary $\mathbb{M}^4$. If the partition functions defined on $\mathbb{M}^5$ and $\mathbb{M}'^5$ have the same phase, then the theory on $\mathbb{M}^4$ is uniquely defined and anomaly-free; otherwise, it is anomalous. Applying the Dai-Freed prescription to study the IR phases of strongly-coupled theories is innovative. However, see [13, 21] for previous applications.[2]

If $\mathbb{M}^4$ is a spin manifold, the geometrical obstruction of uniquely extending a 4-D theory to a 5-D bulk can be inferred by computing the bordism group $\Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$, where $B\mathbb{Z}_n$ is the classifying space of $\mathbb{Z}_n$.[3] If $\Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$ is non-trivial, the theory might have a nonperturbative anomaly. To find the anomaly, one computes the η-invariant, a resolvent of the spectral asymmetry of the Dirac operator, on specific closed 5-dimensional spin manifolds that can detect the anomaly. For example, one puts the theory on Lens spaces to gauge $\mathbb{Z}_n$ and discover whether the theory exhibits a nonperturbative anomaly. For n even, $n = 2m$, one can take $\mathbb{M}^4$ to be nonspin by employing the twisted symmetry group $\mathrm{Spin}^{\mathbb{Z}_{2m}}(4) = (\mathrm{Spin}(4) \times \mathbb{Z}_{2m})/\mathbb{Z}_2$ instead of $\mathrm{Spin}(4)$. Here, one needs to compute the η-invariant that detects the bordism group $\Omega_5^{\mathrm{Spin}^{\mathbb{Z}_{2m}}}$.

[2] Also, see [22, 23] for applications of Dai-Freed anomalies in particle physics.

[3] A classifying space of symmetry G is an infinite dimensional space with the property that any principal G-bundle on a manifold $\mathbb{M}$ is the pullback via some map $f : \mathbb{M} \longrightarrow BG$. Then, the set of topologically distinct principal G-bundles over $\mathbb{M}$ is equivalent to the set of the homotopy classes of maps from $\mathbb{M}$ to BG.

The computations of the relevant η-invariants were carried out in [24] (see also [25] for an alternative perspective). For a theory of a left-handed Weyl fermion with a charge s under $\mathbb{Z}_n$ defined on a spin manifold, the anomaly is given by the pair of phases:

$$\left\{(n^2+3n+2)s^3 \bmod 6n\,,\, 2s \bmod n\right\}\,. \tag{2.29}$$

This pair can be thought of as contributions from $[\mathbb{Z}_n]^3$ and mixed $\mathbb{Z}_n$ [grav] anomalies. Indeed, the second entry in (2.29) is precisely the anomaly we computed in the second line of (2.24). The first entry can be obtained from pure $[\mathrm{U}(1)]^3$ anomaly by restricting U(1) to a $\mathbb{Z}_n$ discrete group; this is the Ibanez-Ross anomaly we comment on below. Also, a Weyl fermion defined on a twisted background and carrying a charge s under $\mathbb{Z}_{2m}$ has an anomaly given by the pair[4]

$$\left\{\Big((2m^2+m+1)s^3-(m+3)s\Big) \bmod 48m\,, \Big(ms^3+s\Big) \bmod 2m\right\}\,. \tag{2.30}$$

The charge s is assumed to be odd such that the fermion transforms under the $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_{2m}$. Generally, the anomaly (2.30) is more restrictive than (2.29). We shall use both (2.30) and (2.29) to constrain our theories. It is important to note that $\mathbb{Z}_2$ symmetry is anomaly-free, as can be easily seen from (2.29) and (2.30). This observation will play an essential role in the IR anomaly matching by condensates, as many of them break the global symmetries to $\mathbb{Z}_2$, as we discuss below.

We also comment on the Ibanez-Ross anomaly-matching conditions [26]. These are obtained from $[\mathrm{U}(1)]^3$ and U(1)[grav] anomalies by restricting U(1) to a $\mathbb{Z}_n$ subgroup. The Ibanez-Ross anomaly-cancellation conditions read:

$$s^3 = p'n + \frac{r'n^3}{8}\,, \qquad s = p''n + r''n\,, \tag{2.31}$$

where $p', r', p'', r'' \in \mathbb{Z}$, $p' \in 3\mathbb{Z}$ if $n \in 3\mathbb{Z}$, and $r', r'' = 0$ if n is odd. It can be shown that (2.31) and (2.29) are equivalent [24]. Hence, in what follows, we use either (2.29) or (2.30) to calculate discrete anomalies.

Finally, we also may have a discrete anomaly of the form $\mathbb{Z}_m[\mathbb{Z}_n]^2$. Such an anomaly can descend from $\mathrm{U}(1)_A$[CFU] anomaly after a given condensate breaks $\mathrm{U}(1)_A$ down to a discrete subgroup. Let s and s' be the charges of a left-handed Weyl fermion under $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively. Then, the anomaly cancelation condition, which follows from the Ibanez-Ross conditions, is given by [27]

$$s^2 s' = p' \gcd(m,n) + \frac{p''}{8} mn^2\,, \tag{2.32}$$

where $p', p'' \in \mathbb{Z}$ and p'' can be non-vanishing only if n and m are even. Notice that this anomaly is trivial when $\mathbb{Z}_m = \mathbb{Z}_2$.

[4]More generally, the bordism group $\Omega_5^{\mathrm{Spin}^{\mathbb{Z}_{2m}}} \cong \mathbb{Z}_a \times \mathbb{Z}_b$, where a, b, and the associated anomalies are given by eqs. (2.11)–(2.13) in [13]. In the present work, eq. (2.30) suffices to tackle the theories at hand.

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3 Anomaly matching and the IR phase

3.1 IR anomaly matching

't Hooft anomalies preclude a trivially gapped IR phase: a theory with 't Hooft anomaly must have gapless excitations, degenerate vacua, or symmetry-preserving topological quantum field theory (TQFT). In this work, we use the safe assumption that the gauge group does not break spontaneously under its strong dynamics. The breaking of a gauge group is under control in the presence of scalars at weak coupling, an ingredient absent from our theory from the get-go.[5] This leaves us with three possible IR scenarios.

(I) Conformal fixed point. In the first scenario, the theory flows to a conformal field theory (CFT). When the CFT is weakly coupled, the renormalization group flow from the UV to the IR is very slow. The UV matter content (fermions) can be considered the IR gapless excitations, and the anomalies are automatically matched. Anomaly matching by strongly interacting CFT is still an open problem; we have nothing to say here. The existence of a well-controlled Banks-Zaks fixed point implies a strictly weakly coupled CFT. However, such a reliable fixed point can only be obtained in the large-N limit. The 3-loop β-function reads

$$\beta(g) = -\beta_0 \frac{g^3}{(4\pi)^2} - \beta_1 \frac{g^5}{(4\pi)^4} - \beta_2 \frac{g^7}{(4\pi)^6} \,. \tag{3.1}$$

If $\beta_0 > 0$ and $\beta_1 < 0$, the theory flows to an IR fixed point, $g_*^2 = -\frac{(4\pi)^2\beta_0}{\beta_1}$, up to corrections from β_2. In the large-N limit and close to the boundary of the asymptotic-freedom region, the contribution from the third term is suppressed compared to the second term, and thus, this term and higher-order terms can be safely neglected. Our theories, however, do not admit large-N analysis. Yet, as we shall discuss, in a few cases, the numerical value of the third term is extremely small compared to the first two terms, so one can conclude that such a weakly-coupled CFT exists. In appendix B, we work out the fixed points using the 2 and 3-loop β-functions. We consider values of $\frac{g_*^2}{4\pi} < 0.1$, using both the 2-loop and 3-loop calculations, small enough to conclude that the theory has an IR fixed point. Also, the perturbative nature of the β-function calculations will be trusted when the third term in (3.1) is small compared to the first two terms. More stringent coupling constant values at the fixed point could also be assumed. This, however, will only mean doing more work to find out the fate of the IR phases of such theories.

(II) Composite massless fermions. In the second scenario, the theory becomes strongly coupled; it confines (for N even), preserves the global symmetries, and flows to a phase of composite massless fermions. This can happen in the non-bosonic theories $N = 5, 6, 10$. We sketch how one can systematically search for such composites. Let $\mathcal{F}_i$ be a gauge-invariant fermionic operator (a composite that transforms as a left-handed Weyl fermion under the Lorentz group) built of ψ and χ:

$$\mathcal{F}_i = \psi^{\kappa_i} \chi^{\rho_i} \,, \tag{3.2}$$

[5] Tumbling, [28], is a mechanism by which the breakdown of a gauge group occurs without the aid of fundamental scalar fields. We do not discuss tumbling in this work.

where $\kappa_i, \rho_i \in \{0\} \cup \mathbb{Z}^+$, and we suppressed the color and spinor indices to reduce notational clutter. Insertion of gluon fields can be used whenever fermi statistics cause $\mathcal{F}_i$ to vanish. Using the convention that ψ and χ carry 2 and $N-2$ indices, respectively, and demanding that $\mathcal{F}_i$ be a gauge invariant fermion yields the two conditions:

$$2\kappa_i + (N-2)\rho_i \in N\mathbb{Z}^+ \,, \quad \kappa_i + \rho_i \in (2\mathbb{Z}^+ - 1) \,. \tag{3.3}$$

The $\mathrm{U}(1)_A$ charge of $\mathcal{F}_i$ is

$$q_{\mathcal{F}_i} = \frac{-\kappa_i N_\chi + \rho_i N_\psi}{r} \,. \tag{3.4}$$

Generally, the composites $\mathcal{F}_i$ transform in higher representations of $\mathrm{SU}(n_\psi)$ and $\mathrm{SU}(n_\chi)$, making the process of matching anomalies containing flavor groups a daunting task. Thus, it is more convenient to start with matching $[\mathrm{U}(1)_A]^3$, $\mathrm{U}(1)_A[\text{grav}]$, and nonperturbative $\mathbb{Z}_p^{d\chi}$ anomalies. For generality, we assume there are $\mathcal{N}_i$ copies of composites $\mathcal{F}_i$. Then, matching these anomalies gives the conditions

$$\begin{aligned}
&\sum_i \mathcal{N}_i q_{\mathcal{F}_i}^3 = q_\psi^3 n_\psi \mathrm{dim}_\psi + q_\chi^3 n_\chi \mathrm{dim}_\chi \,, \\
&\sum_i \mathcal{N}_i q_{\mathcal{F}_i} = q_\psi n_\psi \mathrm{dim}_\psi + q_\chi n_\chi \mathrm{dim}_\chi \,, \\
&\quad 2\sum_i \mathcal{N}_i = 2 n_\chi \mathrm{dim}_\chi \,(\mathrm{mod}\, p) \,, \quad (p^2+3p+2)\sum_i \mathcal{N}_i = (p^2+3p+2) n_\chi \mathrm{dim}_\chi \,(\mathrm{mod}\, 6p) \,.
\end{aligned} \tag{3.5}$$

The number of the IR fermionic species $\mathcal{N} = \sum_i \mathcal{N}_i$ is bounded from above by the a-theorem:

$$\underbrace{\underbrace{2(N^2-1)}_{\text{gluons}} + \frac{7}{4}\left(n_\psi \mathrm{dim}_\psi + n_\chi \mathrm{dim}_\chi\right)}_{\text{UV degrees of freedom}} \geq \underbrace{\frac{7\mathcal{N}}{4}}_{\text{IR degrees of freedom}} \,. \tag{3.6}$$

In principle, one could systematically search for copies of composites $\{\mathcal{N}_1, \mathcal{N}_2, \ldots\}$ that satisfy (3.5). However, this would require finding the partitions of $\mathcal{N}$ (all integers that their sums give $\mathcal{N}$), a number that grows exponentially with $\sqrt{\mathcal{N}}$. The composites that satisfy (3.5) must also match the rest of the anomalies that involve the flavor groups. In all non-bosonic theories, $N = 5, 6, 10$, we could not find a set of composites that matched the full set of anomalies using the systematic approach sketched above. Simply, the algorithm takes an extremely long time, which makes such a systematic search impractical.

In fact, we can utilize the $\mathbb{Z}_p^{d\chi}[\text{CFU}]$ anomaly to show that in some cases, such candidates, if they exist, cannot solely match this anomaly. This approach was used in [11] in the case of vector-like theories, and we repeat it here for chiral theories. To this end, we assume that there exists a set of gauge-invariant composite fermions that match $[\mathrm{SU}(n_\psi)]^3$, $[\mathrm{SU}(n_\chi)]^3$, $[\mathrm{U}(1)_A]^3$, $\mathbb{Z}_p^{d\chi}[\mathrm{U}(1)_A]^2$, $\mathbb{Z}_p^{d\chi}[\mathrm{SU}(n_\psi)]^2$, $\mathbb{Z}_p^{d\chi}[\mathrm{SU}(n_\chi)]^2$, $\mathrm{U}(1)_A[\text{grav}]$, and $\mathbb{Z}_p^{d\chi}[\text{grav}]$ anomalies. Then, we turn the CFU fluxes on $\mathbb{M}^4$ and perform a $\mathbb{Z}_p^{d\chi}$ rotation. We denote the UV coefficients that multiply Q_c, Q_χ, and Q_u in (2.25), (2.18) by $D_c^{\mathrm{UV}} \equiv n_\chi T_\chi$,

$D_\chi^{\rm UV} \equiv \dim_\chi$, and $D_u^{\rm UV} \equiv q_\chi^2 n_\chi \dim_\chi$. Under a discrete chiral rotation, the UV partition function transforms as

$$\mathcal{Z}_{\rm UV} \longrightarrow e^{i\frac{2\pi}{p}\left(D_c^{\rm UV} Q_c + D_\chi^{\rm UV} Q_\chi + D_u^{\rm UV} Q_u\right)} \mathcal{Z}_{\rm UV}\,, \tag{3.7}$$

while the IR partition function transforms as[6]

$$\mathcal{Z}_{\rm IR} \longrightarrow e^{i\frac{2\pi}{p}\left(D_\chi^{\rm IR} Q_\chi + D_u^{\rm IR} Q_u\right)} \mathcal{Z}_{\rm IR}\,, \tag{3.8}$$

where $D_\chi^{\rm IR}, D_u^{\rm IR} \in \mathbb{Z}$ are group-theoretical coefficients that are chosen to match $\mathbb{Z}_p^{d\chi}[{\rm U}(1)_A]^2$ and $\mathbb{Z}_p^{d\chi}[{\rm SU}(n_\chi)]^2$ anomalies. Since the UV-IR anomaly matching is mod p, we must have[7]

$$D_c^{\rm UV} = p\ell_c\,, \quad D_\chi^{\rm UV} - D_\chi^{\rm IR} = p\ell_\chi\,, \quad D_u^{\rm UV} - D_u^{\rm IR} = p\ell_u\,, \tag{3.9}$$

for some $\ell_{c,\chi,u} \in \mathbb{Z}$. Thus, the ratio between the UV and IR partition functions reads

$$\frac{\mathcal{Z}_{\rm UV}}{\mathcal{Z}_{\rm IR}} = e^{i2\pi\left(\ell_c Q_c + \ell_\chi Q_\chi + \ell_u Q_u\right)}\,, \tag{3.10}$$

and the matching of the $\mathbb{Z}_p^{d\chi}[{\rm CFU}]$ anomaly requires

$$\ell_c Q_c + \ell_\psi Q_\chi + \ell_u Q_u \in \mathbb{Z}\,, \tag{3.11}$$

for all allowed topological charges. Suppose no integers $\ell_{c,\chi,u}$ exist that satisfy this condition for a given allowed fractional topological charges. In that case, the composites cannot solely match the $\mathbb{Z}_p^{d\chi}[{\rm CFU}]$ anomaly.

A minimal way out would be breaking $\mathbb{Z}_p^{d\chi} \longrightarrow \mathbb{Z}_{q<p}$ via condensate formation provided that the anomaly $\mathbb{Z}_q[{\rm CFU}]$ vanishes. Usually, a condensate would ordinarily break ${\rm SU}(n_\psi)$, ${\rm SU}(n_\chi)$, and ${\rm U}(1)_A$. Thus, one must postulate that all gauge-invariant condensates charged under these symmetries have zero vacuum expectation values. Otherwise, the condensation of such operators would oversaturate these anomalies, which are assumed to be matched by composites. In addition, one needs to build a neutral operator under the continuous symmetries, charged under $\mathbb{Z}_p^{d\chi}$, and has a non-zero vacuum expectation value. If it exists, such an operator would have a scaling dimension larger than the vanishing lower-order condensates. Although this scenario cannot be ruled out, we find it contrived in the examples of the 2-index chiral theories we discuss here.

This leaves us with the possibility that if condition (3.11) is violated, the $\mathbb{Z}_p^{d\chi}[{\rm CFU}]$ anomaly can be matched by a symmetry-preserving topological quantum field theory (TQFT). In [29, 30], it was shown that the matching of $\mathbb{Z}_p^{d\chi}$-gravitational anomalies by a unitary and symmetry-preserving TQFT is obstructed on a spin manifold. This obstruction can also be shown to hold in the case of $\mathbb{Z}_p^{d\chi}[{\rm CFU}]$ anomaly [8].

We conclude that if condition (3.11) is violated, the theory probably cannot flow to a phase with massless composites.

[6] Q_c does not contribute to the IR phase since the composites are color singlets.

[7] $\mathbb{Z}_p^{d\chi}$ is a good symmetry in the color background, and thus we must have $D_c^{\rm UV} = p\ell_c$.

(III) Spontaneous symmetry-breaking. In this scenario, the theory becomes strongly coupled; it confines (for N even) and breaks its global symmetries spontaneously. We say that the theory flows to a spontaneous symmetry-breaking (SSB) phase. An important aspect of this work involves identifying the minimal set of condensates that break global symmetries while matching the anomalies. These condensates break G^{g} down to $H \subset G^{\mathrm{g}}$, with the requirement that H remains anomaly-free. Without satisfying this condition, the symmetry breaking alone would not sufficiently match the UV anomaly. It is possible for composite fermions to match a non-vanishing anomaly in H, but it is crucial that these fermions do not undermine the matching of the G^{g} anomalies achieved by the condensates. Our focus did not involve searching for composites that could match the anomalous unbroken subgroups.

Generally, H can be expressed as $H = H^c \times \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2}$, where H^c represents the continuous part of H, $\mathbb{Z}_{q_1}$ "collectively" denotes the unbroken discrete subgroups of $\mathrm{SU}(n_\psi) \times \mathrm{SU}(n_\chi) \times \mathrm{U}(1)_A$, and $\mathbb{Z}_{q_2}$ represents the unbroken subgroup of $\mathbb{Z}_p^{d\chi}$. If the condensates leave a discrete subgroup unbroken, we must examine its anomalies. In addition, if the theory possesses a 1-form/0-form mixed anomaly, there can be fractionalization classes, and hence, an ambiguity in calculating the cubic discrete anomalies [31]. One must ensure the condensates do not leave any discrete anomaly in any fractionalization class.[8] In a few examples, we observe that lower-order condensates (such as the 2-fermion condensates) lead to anomalous unbroken discrete subgroups. Consequently, the formation of other (higher-order) condensates becomes necessary to break the symmetries into non-anomalous subgroups.

In strongly-coupled theories, it is generally believed that higher-order bosonic operators undergo condensation. In this work, through anomaly matching conditions, we provide kinematical reasons behind this condensation.

Now we turn our attention to the matching of CFU anomalies. Given the better understanding of the nature of the unbroken discrete subgroups of $\mathrm{U}(1)_A$ and their anomalies, here, we provide a more in-depth discussion of the CFU anomalies than the earlier work [8]. As mentioned above, we encounter two types of such anomalies: $\mathrm{U}(1)_A[\mathrm{CFU}]$ and $\mathbb{Z}_p^{d\chi}[\mathrm{CFU}]$ anomalies. In all the examples we have examined, we consistently observe the trivialization of $\mathbb{Z}_p^{d\chi}[\mathrm{CFU}]$ by the condensates, which break the $\mathrm{U}(1)_A$ symmetry. On the other hand, the matching of the $\mathrm{U}(1)_A[\mathrm{CFU}]$ anomaly through condensates is a more intricate process that required closer examination.

As emphasized earlier, the color topological charge does not play a role in this particular anomaly. Consequently, we can view it as an anomaly of $\mathrm{U}(1)_A$ in the presence of the flavor center and $\mathrm{U}(1)_A$ fluxes. In the examples we have studied, the condensates break the flavor center, rendering this anomaly irrelevant. In simpler terms, the full breaking of the flavor center automatically matches the $\mathrm{U}(1)_A[\mathrm{CFU}]$ anomaly. This can be understood through the following principle: if a triangle (anomaly) diagram involves three abelian symmetries, namely G_1, G_2, and G_3 (in this case, G_1 through G_3 are the abelian discrete groups corresponding to turning on the CFU fluxes), the complete breaking of at least one of these symmetry groups will resolve the anomaly.

To extract more valuable insights from this anomaly, we can focus our attention solely on the color-$\mathrm{U}(1)_A$ fluxes by deactivating the flavor background. In doing so, the $\mathrm{U}(1)_A[\mathrm{CU}]$

[8] We would like to thank Erich Poppitz for illuminating discussions about this point.

anomaly becomes a mixed anomaly of $\mathrm{U}(1)_A$ in the presence of fractional $\mathrm{U}(1)_A$ flux (keeping in mind that the $\mathrm{U}(1)_A$ flux still needs to combine with the color flux to satisfy the cocycle conditions. Yet, the color topological charge remains uninvolved in the anomaly). Superficially, one might consider this to be equivalent to the $[\mathrm{U}(1)_A]^3$ anomaly. However, this is not the case since the latter anomaly only encompasses integer fluxes of $\mathrm{U}(1)_A$, whereas the $\mathrm{U}(1)_A[\mathrm{CU}]$ anomaly incorporates the minimal flux of $\mathrm{U}(1)_A$. Consequently, the latter is more restrictive in nature compared to the $[\mathrm{U}(1)_A]^3$ anomaly. Let the discrete flux of $\mathrm{U}(1)_A$ be a $\mathbb{Z}_n$ flux, and let a particular condensate break $\mathrm{U}(1)_A$ to $\mathbb{Z}_m \supseteq \mathbb{Z}_n$. Then the $\mathrm{U}(1)[\mathrm{CU}]$ anomaly can be thought of as a $\mathbb{Z}_m[\mathbb{Z}_n]^2$ anomaly, which can be checked via (2.32). If $\mathbb{Z}_m \subset \mathbb{Z}_n$ and the anomaly $[\mathbb{Z}_m]^3$ vanishes, the breaking of $\mathrm{U}(1)_A$ to $\mathbb{Z}_m$ automatically matches the $\mathrm{U}(1)[\mathrm{CU}]$ anomaly as the symmetry corresponding to the discrete flux of $\mathrm{U}(1)_A$ is broken.

Next, we discuss the condensates that cause the symmetries to break. A gauge-invariant condensate is a bosonic operator

$$\mathcal{C} = \psi^{\alpha_\psi}\chi^{\alpha_\chi}\,, \tag{3.12}$$

and α_ψ and α_χ satisfy the conditions

$$2\alpha_\psi + (N-2)\alpha_\chi \in N\mathbb{Z}^+\,, \qquad \alpha_\psi + \alpha_\chi \in 2\mathbb{Z}^+\,. \tag{3.13}$$

One needs as many condensates as necessary to break G^{g} to an anomaly-free subgroup. Distinct condensates will break G^{g} down to $H_1 \subset G^{\mathrm{g}}$, $H_2 \subset G^{\mathrm{g}}$, etc. If the subgroups $\{H_1, H_2, \ldots\}$ do not share common generators, G^{g} will break to unity. Finding the breaking patterns of a group G^{g} because of the condensation of single or many operators transforming in the defining or higher-dimensional representations of G^{g} is, in general, a complicated problem. Only a few cases have been discussed in the literature; see, e.g., [32–36] and references therein. The question then is, how many condensates does the theory develop in the IR? There is no known answer to this question. However, there must be at least as many condensates as needed to match all anomalies.

3.2 Minimizing the IR degrees of freedom

Beyond 't Hooft anomalies, are there additional sources of information that can be harnessed to make conjectures about the infrared (IR) phase of a strongly coupled theory? In [37–39], a constraint on the structure of strongly coupled asymptotically-free field theories was proposed. The constraint is an inequality favoring an IR phase with fewer degrees of freedom (DOF). It was also proposed to use the free energy to characterize DOF. The effective degrees of freedom $\mathcal{A}$ of free n_B massless real scalars and free n_f massless Weyl fermions are given in terms of the free energy density F as (T is an infinitesimal temperature)

$$\mathcal{A} \equiv \frac{90F}{\pi^2 T^4} = n_B + \frac{7}{4}n_f\,. \tag{3.14}$$

First, we may use (3.14) to favor between a phase of composite fermions or a phase with spontaneous symmetry breaking (SSB). As we pointed out above, we could not find composite fermions that matched the anomalies. Yet, one may be tempted to use (3.14) to

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predict whether the theory flows to an IR CFT. In a weakly-coupled CFT, the IR DOF are the same UV DOF. On the other hand, the DOF in a spontaneously broken phase, assuming the global symmetry $\mathrm{SU}(N_f)$ entirely breaks, are $N_f^2 - 1$ Goldstones.[9] Let us define $\Delta\mathcal{A}$ as the difference between the DOF in the two scenarios. Then, we have

$$\Delta\mathcal{A} = \underbrace{n_\psi^2 + n_\chi^2 - 2}_{\text{Goldstones}} - \left\{ \underbrace{2(N^2-1)}_{\text{gluons DOF}} + \frac{7}{4}\left(n_\psi \mathrm{dim}_\psi + n_\chi \mathrm{dim}_\chi\right) \right\}. \tag{3.15}$$

According to the conjecture, a theory with $\Delta\mathcal{A} > 0$ disfavors an SSB phase. It can be easily checked that all the theories in table 1 yield $\Delta\mathcal{A} < 0$, favoring a phase with broken symmetries. This is to be expected since $n_\psi, n_\chi \sim N$ and $\mathrm{dim}_\psi, \mathrm{dim}_\chi \sim N^2$. Thus, while the SSB phase has $\sim N^2$ DOF, a phase with CFT has $\sim N^3$ DOF. Then, one may naively conclude that all theories in table 1 will break their symmetries and flow to a Goldstone phase. This conclusion, however, totally ignores the dynamics of the theory on the way from UV to IR. A theory must enter a strongly-coupled regime to form condensates and break its continuous symmetries, i.e., breaking the symmetries has to happen dynamically since no elementary scalars exist. As we argued above, some of our theories have robust IR fixed points at weak coupling, indicating that it is most unlikely they can form condensates. Consequently, in the subsequent analysis, we avoid employing the aforementioned hypothesis to favor between an SSB or CFT phase. Instead, we use the β-function analysis to check whether a theory flows to an IR CFT.[10]

However, assuming the existence of multiple sets of condensates, each capable of accounting for all observed anomalies via SSB, we can employ the aforementioned line of reasoning to make a prediction. Presumably, the set of condensates that causes the flavor group to break into the largest subgroup will be preferred due to its associated reduction in the number of infrared degrees of freedom.

The following sections are devoted to systematically applying the above ideas to the concrete theories in table 1. We start our discussion by working out all the details. As we progress through the list of theories, we build on the previous experience and shorten our discussion.

4 Fermionic theories

This section systematically studies theories that admit fermionic operators in their spectrum. These are $(N = 5, k = 1)$, $(N = 6, k = 2)$, $(N = 6, k = 1)$, and $(N = 10, k = 2)$. Our analysis indicates that the first two theories form condensates and break their global symmetries, while the last two flow to a CFT.

[9] Notice that a theory that fully breaks its global symmetries will match its 't Hooft anomalies in the IR. We assume that enough condensates form to obey the matching conditions.

[10] This method was used in [13] to predict the IR phase of a theory with a single adjoint and N_f fundamental flavors of Weyl fermions. It was found that the $\Delta\mathcal{A}$ calculations are consistent with the prediction of perturbative β-function. The fact that this analysis does not hold for the 2-index chiral theories is attributed to the large number of degrees of freedom of a CFT, which always exceeds the number of degrees of freedom of an SSB phase.

Anomaly	Equation	Value
$[\mathrm{U}(1)_A]^3$	$\kappa_u^3 = n_\psi q_\psi^3 \dim\psi + n_\chi q_\chi^3 \dim\chi$	-264375
$\mathrm{U}(1)_A[\mathrm{SU}(9)_\chi]^2$	$q_\chi \dim\chi$	70
$[\mathrm{SU}(9)_\chi]^3$	$\dim\chi$	10
$\mathrm{U}(1)_A[\mathrm{grav}]$	$2(n_\psi q_\psi \dim\psi + n_\chi q_\chi \dim\chi)$	450
$\mathrm{U}(1)_A[\mathrm{CFU}]$	$q_\chi \dim\chi\, Q_\chi + \kappa_{u^3} Q_u$	$\frac{560}{9}p'^2 - 264375 s^2$

Table 2. Anomalies of $\mathrm{SU}(5), k=1$.

4.1 $\mathrm{SU}(5), k=1$

This theory admits a single Weyl fermion ψ and $n_\chi = 9$ flavors of χ Weyl fermions. In addition, we have $r = \gcd(n_\psi T_\psi, n_\chi T_\chi) = \gcd(7,27) = 1$, indicating that the theory does not possess a discrete chiral symmetry. The solutions to the cocycle conditions (2.14) give $\mathbb{Z}_5 \times \mathbb{Z}_9$ as the discrete division group. Thus, the faithful global symmetry is

$$G^{\mathrm{g}} = \frac{\mathrm{SU}(9)_\chi \times \mathrm{U}(1)_A}{\mathbb{Z}_5 \times \mathbb{Z}_9}\,, \tag{4.1}$$

and the $\mathrm{U}(1)_A$ charges of ψ and χ are

$$q_\psi = -27\,, \qquad q_\chi = 7\,. \tag{4.2}$$

Since both q_ψ and q_χ are odd, $(-1)^F \equiv \mathbb{Z}_2^F$ fermion-number symmetry, which acts on (ψ,χ) as $(\psi,\chi) \longrightarrow -(\psi,\chi)$, is a subgroup of $\mathrm{U}(1)_A$.

The topological charges of the CFU fluxes are given by:

$$Q_c = \frac{4m^2}{5}\,, m \in \mathbb{Z}_5\,, \qquad Q_\chi = \frac{8p'^2}{9}\,, p' \in \mathbb{Z}_9\,, \qquad Q_u = s^2\,, s \in \mathbb{Z}_{45}\,, \tag{4.3}$$

and (m, p', s) are chosen to satisfy (2.14). The theory admits a set of anomalies listed in table 2 (from here on, we give the phase of the corresponding anomaly).

Notice that, as pointed out above, the $\mathrm{U}(1)_A[\mathrm{CFU}]$ anomaly does not depend on the color topological charge. We can also put the theory on $\mathbb{CP}^2$ by employing fluxes in the centers of $\mathrm{SU}(5)$ and $\mathrm{SU}(9)_\chi$ accompanied by a $\mathrm{U}(1)_A$ flux, as can be easily checked from (2.19).

The 2-loop and 3-loop β-function analysis show that the theory has an IR fixed point at *somewhat large* coupling-constant: $\frac{g_*^2}{4\pi} \approx 0.64$ and $\frac{g_*^2}{4\pi} \approx 0.34$, respectively. Therefore, such a fixed point is not robust. We conclude that either the theory forms composite fermions or flows to an SSB phase.

Matching by composites. We used the systematic approach discussed in section 3 to search for a set of composite fermions. We found a pair of operators

$$\mathcal{F}_1 = \psi\chi^6\,, \quad \mathcal{F}_2 = \psi^7\chi^{22}\,, \tag{4.4}$$

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with $\mathcal{N}_1 = 36$ and $\mathcal{N}_2 = 9$ copies that matched the $[\mathrm{U}(1)_A]^3$ and $\mathrm{U}(1)_A[\mathrm{grav}]$ anomalies. Yet, this pair failed to match the $\mathrm{U}(1)_A\,[\mathrm{SU}(9)_\psi]^2$ anomaly. The upper bound on the number of the IR fermion species is $\mathcal{N} \sim 132$. The large number of partitions of $\mathcal{N}$ is $\mathcal{O}(10^7)$, which hindered the abilities of our search algorithm. We failed to find a set of composites that matches the full set of anomalies.

Matching by condensates. We now turn to the formation of condensates. The lowest-order condensate is

$$\mathcal{C}_1^i = \epsilon^{a_1a_2a_3a_4a_5}\epsilon_{\alpha_1\alpha_2}\psi^{\alpha_1}_{(a_1a_2)}\chi^{\alpha_2,\,i}_{[a_3a_4a_5]}\,, \tag{4.5}$$

where $a_1, \ldots, a_5$ are color, α_1, α_2 are spinor, and i is a $\mathrm{SU}(9)_\chi$ flavor indices. This condensate vanishes identically owing to the symmetrizing over a_1, a_2. Yet, one can evade this problem by inserting gauge-covariant gluonic fields $(f^c_{\mu\nu})^{a_j}_{a_i}\sigma^{\mu\nu}$:

$$\mathcal{C}_1^i \longrightarrow \tilde{\mathcal{C}}_1^i = \epsilon^{a_1a_2a_3a_4a_5}\epsilon_{\alpha_1\alpha_2}(f^c_{\mu\nu})^{a_6}_{a_2}\sigma^{\mu\nu}\psi^{\alpha_1}_{(a_1a_6)}\chi^{\alpha_2,\,i}_{[a_3a_4a_5]}\,. \tag{4.6}$$

This trick will always be followed whenever the statistics of indices cause some operator to vanish. $\mathcal{C}_1^i$ transforms in the defining representation of $\mathrm{SU}(9)_\chi$, and thus, it breaks it down to $\mathrm{SU}(8)$. However, the condensation of $\mathcal{C}_1^i$ leaves a $\mathrm{U}(1)$ generator of $\mathrm{SU}(9) \times \mathrm{U}(1)_A$ unbroken. To see that, we go to a basis where $\mathcal{C}_1^i \propto \delta_{i,9}$. In this basis, the unbroken $\mathrm{SU}(8)$ group acts on the 8×8 upper block matrices of the original 9×9 unitary matrices of $\mathrm{SU}(9)$. Now, it is easy to see that the $\mathrm{SU}(9)$ Cartan generator $H_8 = \mathrm{diag}\,(1, 1, \ldots, -8)$ combines with the $\mathrm{U}(1)_A$ generator to leave the vacuum $\delta_{i,9}$ invariant:

$$e^{i2\pi(-20\beta)}\begin{bmatrix} e^{i2\pi\alpha} & 0\ldots & 0 \\ ,\ldots & \ldots & \ldots \\ 0 & \ldots & e^{i2\pi(-8\alpha)} \end{bmatrix}\begin{bmatrix} 0 \\ \ldots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \ldots \\ 1 \end{bmatrix}, \tag{4.7}$$

where α and β are the Cartan and $\mathrm{U}(1)_A$ phases, respectively. The direction $2\alpha = -5\beta$ is the unbroken $\mathrm{U}(1)$ direction. The unbroken $\mathrm{SU}(8)$ has a non-vanishing cubic anomaly. In addition, the unbroken $\mathrm{U}(1)$ symmetry inherits the $\mathrm{U}(1)_A[\mathrm{grav}]$ anomaly, signaling that such breaking is incomplete or inconsistent with the anomaly-matching conditions.

Another condensate is (we suppress color and spinor indices to reduce clutter)

$$\mathcal{C}_2^{(ij)} = \psi^2\chi^{(i}\chi^{j)}\,, \tag{4.8}$$

which transforms in the 2-index symmetric representation of the flavor group.[11] The general form of the "Higgs" potential of the condensate is

$$V(\mathcal{C}_2) = -\frac{1}{2}\mu^2\mathcal{C}_2^{(ij)}\mathcal{C}_{2(ij)} + \frac{1}{4}\lambda_1(\mathcal{C}_2^{(ij)}\mathcal{C}_{2(ij)})^2 + \frac{1}{4}\lambda_2(\mathcal{C}_{2(ij)}\mathcal{C}_2^{(jk)}\mathcal{C}_{2(kl)}\mathcal{C}_2^{(li)})\,, \tag{4.9}$$

for some real parameters $\mu^2 > 0$, λ_1, and λ_2. In the case $\lambda_2 > 0$, the condensate has a non-zero vacuum expectation value and we can pick the form of the condensate to be $\mathcal{C}_2 \propto I_9$ [32]. This breaks $\mathrm{SU}(9)$ to the anomaly-free subgroup $\mathrm{SO}(9)$.

[11] We also insert gluons if the statistics cause the condensate to vanish.

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Is there a combination of $SU(9)_\chi \times U(1)_A$ that breaks to a remaining U(1) symmetry? As $U(1)_A$ is abelian, we need to consider the subgroup generated by the Cartan subalgebra of SU(9). The "unnormalized" generators of the Cartan subalgebra of SU(9) are:

$$[H_m]_{ij} = \sum_{k=1}^{m} \delta_{ik}\delta_{jk} - m\delta_{i,m+1}\delta_{j,m+1}\,, \quad m = 1, 2, \ldots, 8\,. \tag{4.10}$$

A general SU(9) element generated by the Cartan subalgebra has the form $\exp(2\pi i \alpha_m H_m)$, $m = 1, 2, \ldots, 8$ and $\alpha_m \in [0, 1)$. A combined $SU(9) \times U(1)_A$ transformation acts on $\mathcal{C}^{ij}$ via:

$$\mathcal{C}_2'^{(ij)} = e^{2\pi i(-40\beta)} \left(e^{2\pi i\alpha_m H_m}\right)^{ik} \left(e^{2\pi i\alpha_m H_m}\right)^{jl} \mathcal{C}_{2(kl)}\,, \tag{4.11}$$

and should leave the vacuum expectation value invariant. Thus, we need

$$e^{2\pi i(-40\beta)} \left(e^{2\pi i\alpha_m H_m}\right)^{ik} \left(e^{2\pi i\alpha_m H_m}\right)^{jl} I_9 = I_9\,. \tag{4.12}$$

It can be easily checked that there are no nontrivial solutions to the above equation, indicating that no U(1) direction is left unbroken.

Under the action of $U(1)_A$, the condensate transforms as $\mathcal{C}_2^{(ij)} = \psi^2\chi^{(i}\chi^{j)} \longrightarrow \mathcal{C}_2'^{(ij)} = e^{i2\pi(-40\beta)}\psi^2\chi^{(i}\chi^{j)}$, where $\beta \in [0, 1)$ is the $U(1)_A$ parameter. So it appears that the condensate is invariant under a discrete $\mathbb{Z}_{40}$ subgroup of $U(1)_A$. But recall that the global symmetry group includes a division by the $\mathbb{Z}_5$ center of the color group. $\mathbb{Z}_5$ is not a subgroup of SU(9), therefore it can only quotient $U(1)_A$, so the parameter β is in fact a $U(1)_A/\mathbb{Z}_5$ parameter and $\beta \in [0, 1/5)$. Therefore the condensate only exhibits an unbroken $\mathbb{Z}_8$ symmetry.

The discrete symmetry $\mathbb{Z}_8$ has non-perturbative anomalies, as can easily be checked using (2.29) and (2.30), meaning that the condensation of $\mathcal{C}_2^{(ij)}$ is insufficient to match the full set of anomalies. Notice that since both ψ and χ have odd charges under $U(1)_A$, any unbroken discrete subgroup of $U(1)_A$ necessarily contains $(-1)^F$, and thus, we can use the twisted group $\mathrm{Spin}^{\mathbb{Z}_{2m}}$ to detect the nonperturbative anomaly as given from (2.30). Moreover, since the theory does not possess a 1-form symmetry, there is no ambiguity in calculating the discrete-symmetry anomaly [31].

In searching for a condensate that does not leave behind a non-anomalous U(1) or discrete subgroup, we consider the most general bosonic operator:

$$\mathcal{C} = \psi^{\alpha_\psi}\chi^{\alpha_\chi}\,, \qquad 2\alpha_\psi + 3\alpha_\chi \in 5\mathbb{Z}^+\,, \alpha_\psi + \alpha_\chi \in 2\mathbb{Z}^+\,. \tag{4.13}$$

This condensate carries a charge of $-27\alpha_\psi + 7\alpha_\chi$ under $U(1)_A$, and thus, breaks $U(1)_A$ down to $\mathbb{Z}_{(-27\alpha_\psi+7\alpha_\chi)/5}$. We used both (2.29) and (2.30) to check the nonperturbative anomalies of $\mathbb{Z}_{(-27\alpha_\psi+7\alpha_\chi)/5}$ and found that both untwisted and twisted backgrounds yield the same results. The lowest-dimensional condensate that breaks $U(1)_A$ to a non-anomalous subgroup has $\alpha_\psi = 1$ and $\alpha_\chi = 11$. In this case, $\mathbb{Z}_{10}$ is the anomaly-free subgroup.

The condensate

$$\mathcal{C}_3 = \psi\chi^{11} \tag{4.14}$$

transforms in a higher representation of $SU(9)_\chi$. One can contract 9 out of the 11 flavor indices of $\mathcal{C}_3$ with the Levi-Civita tensor, leaving 2 free indices. Then, we can rearrange the free indices (possibly with insertions of gluons in case the statistics cause the condensate to vanish) such that $\mathcal{C}_3$ transforms in the 2-index symmetric representation of SU(9):

$$\mathcal{C}_3^{(ij)} = \psi\chi^9\chi^{(i}\chi^{j)}\,. \tag{4.15}$$

The condensing of $\mathcal{C}_3^{(ij)}$ breaks $\frac{SU(9)_\chi \times U(1)_A}{\mathbb{Z}_5\times\mathbb{Z}_9}$ down to the anomaly-free subgroup $SO(9)\times\mathbb{Z}_{10}$.

Alternatively, one can search for a companion condensate to $\mathcal{C}_2^{(ij)}$ that breaks $U(1)_A$ to a discrete subgroup $\mathbb{Z}_q$, such that $\gcd(q,8)=2$. This ensures that the formation of these two condensates breaks $U(1)_A$ down to the anomaly-free subgroup $\mathbb{Z}_2^F$, which is the fermion number. The companion condensate with the lowest dimension is $\mathcal{C}_4=\chi^{10}$, which, superficially, breaks $U(1)_A$ down to $\mathbb{Z}_q=\mathbb{Z}_{14}$. This, however, is an immature conclusion. One can contract 9 flavor indices of $\mathcal{C}_4$ with a Levi-Civita tensor leaving one free index. Then, $\mathcal{C}_4$ transforms in the fundamental representation of SU(9), and according to the discussion preceding (4.7), it breaks it down to $SU(8)\times U(1)$. Because of the unbroken U(1) generator, the condensation of $\mathcal{C}_2$ along with $\mathcal{C}_4$ break $\frac{SU(9)_\chi \times U(1)_A}{\mathbb{Z}_5\times\mathbb{Z}_9}$ down to a subgroup that contains the anomalous $\mathbb{Z}_8$. More than this is needed to match the full set of anomalies.

We might continue searching for suitable condensates that break G^{g} to an anomaly-free subgroup. However, the lesson from the above discussion is that it is generically a complex exercise.

Since SO(9) is the largest anomaly-free subgroup of SU(9), the condensation of $\mathcal{C}_3^{(ij)}$ leads to the smallest number of the IR Goldstones, and hence, we predict that the theory will flow to a phase with the global symmetry broken down to the anomaly-free $SO(9)\times\mathbb{Z}_{10}$. This is the minimal scenario. However, because of strong dynamics, nothing forbids the theory from forming all kinds of condensates, breaking G^{g} down to the anomaly-free $\mathbb{Z}_2^F$ fermion number symmetry.

In an equally alternative scenario, SU(9) could be broken down to the anomaly-free Sp(8) by a condensate transforming in the 2-index anti-symmetric representation. However, since the dimensions of Sp(8) and SO(9) are identical, anomalies and the argument of the number of Godstones cannot distinguish between the two possible symmetry-breaking scenarios. In general, a condensate transforming in the 2-index symmetric representation of $SU(2N+1)$ breaks this group down to $SO(2N+1)$, while a condensate in the 2-index anti-symmetric representation breaks it down to $Sp(2N)$. Both $SO(2N+1)$ and $Sp(2N)$ have dimension $N(2N+1)$.

Interestingly, the operator $\mathcal{C}_3^{(ij)}$ has a scaling dimension of at least 15 (it could have a higher dimension if gluon fields are needed to avoid the vanishing of the condensate because of fermi-statistics). That such condensate with a large-scaling dimension must condense in the IR to match the complete set of anomalies is remarkable. Generally, it is natural to expect that a strongly coupled theory forms higher-order condensates. In this example, however, this formation is not a question about the dynamics; rather, it is a necessary condition for the theory to obey the kinematical constraints imposed by anomalies.

Does our proposed condensate $\mathcal{C}_3^{(ij)}$ match the $U(1)_A$[CFU] anomaly? The answer is affirmative. $\mathcal{C}_3^{(ij)}$ breaks $SU(9)_\chi$ flavor group down to SO(9). The latter does not have a center symmetry, while the former group has a $\mathbb{Z}_9$ center. Thus, we conclude that the condensate breaks $\mathbb{Z}_9$ maximally, matching the $\mathrm{U}(1)_A$[CFU] anomaly. Next, we may turn off the flavor background, restricting ourselves to the color center and $\mathrm{U}(1)_A$ (CU) fluxes. In this case, we have $(m, p', s) = (1, 0, \frac{1}{5})$, and keeping in mind that the CU anomaly does not depend on the color topological charge, we find that this is an anomaly of the axial current in the background of a $\mathbb{Z}_5$ flux. The condensation of $\mathcal{C}_3$ breaks $\mathrm{U}(1)_A$ to $\mathbb{Z}_{10}$. Thus, the $\mathrm{U}(1)_A$[CU] anomaly becomes the $\mathbb{Z}_{10}[\mathbb{Z}_5]^2$ anomaly discussed around eq. (2.32). However, from the last line in table 2, the anomaly coefficient becomes, $\frac{-264375}{5^2} = -10575$, which is 0 modulo 5. Therefore, in this case, the anomaly becomes trivial, and the $\mathrm{U}(1)_A$[CU] anomaly is automatically matched.

4.2 $\mathrm{SU}(6), k = 2$

This theory has a single ψ Weyl fermion along with 5 flavors of χ fermions. Thus, the continuous global symmetry is $\mathrm{SU}(5)_\chi \times \mathrm{U}(1)_A$. The charges of ψ and χ under $\mathrm{U}(1)_A$ are

$$q_\psi = -5\,, \qquad q_\chi = 2\,. \tag{4.16}$$

Owing to the fact $r = \gcd(n_\psi T_\psi, n_\chi T_\chi) = \gcd(8, 20) = 4$, the theory is also endowed with a $\mathbb{Z}_4^{d\chi}$ chiral symmetry, which is taken to act on χ with a unit charge. It can be checked that this is a genuine symmetry since neither $\mathbb{Z}_4$ nor a subgroup of it can be absorbed in rotations in the centers of $\mathrm{SU}(6) \times \mathrm{SU}(5)_\chi$. To show that, we try to absorb the elements $e^{i\frac{2\pi\ell}{4}}$, $\ell = 1, 2, 3$, in the centers of $\mathrm{SU}(6) \times \mathrm{SU}(5)_\chi$:

$$\mathbb{Z}_4: \quad \psi \longrightarrow e^{2\pi i \frac{2m}{6}} \psi = \psi\,, \quad \chi \longrightarrow e^{-2\pi i \frac{2m}{6}} e^{-2\pi i \frac{p'}{5}} \chi = e^{2\pi i \frac{l}{4}} \chi\,. \tag{4.17}$$

No values of m and p' satisfy these equations for $\ell = 1, 2, 3$, and therefore, $\mathbb{Z}_4^{d\chi}$ is a genuine symmetry. An identical procedure is employed in the rest of the theories to ascertain the genuineness of discrete chiral symmetries.

To determine the faithful global symmetry, we must find the quotient group by solving the consistency conditions (2.14). This gives $\mathbb{Z}_3 \times \mathbb{Z}_5$ as the group we divide by. Putting everything together and remembering that the theory possesses a $\mathbb{Z}_2^{[1]}$ 1-form center symmetry, we write the faithful global group:

$$G^{\mathrm{g}} = \frac{\mathrm{SU}(5)_\chi \times \mathrm{U}(1)_A}{\mathbb{Z}_3 \times \mathbb{Z}_5} \times \mathbb{Z}_4^{d\chi} \times \mathbb{Z}_2^{(1)}\,. \tag{4.18}$$

The $\mathbb{Z}_2^F$ fermion number symmetry is contained in the generators of the product group $\mathrm{U}(1)_A \times \mathbb{Z}_4^{d\chi}$ (notice that the $\mathrm{U}(1)_A$ charges of ψ and χ are odd and even, respectively)

$$\begin{aligned} \mathbb{Z}_2^F \subset \mathrm{U}(1)_A: &\quad \psi \longrightarrow -\psi\,, \quad \chi \longrightarrow \chi\,, \\ \mathbb{Z}_2^F \subset \mathbb{Z}_4^{d\chi}: &\quad \psi \longrightarrow \psi\,, \quad\;\; \chi \longrightarrow -\chi\,. \end{aligned} \tag{4.19}$$

The topological charges of the CFU fluxes are given by:

$$Q_c = \frac{5m^2}{6}\,, m \in \mathbb{Z}_3\,, \qquad Q_\chi = \frac{4p'^2}{5}\,, p' \in \mathbb{Z}_5\,, \qquad Q_u = s^2\,, s \in \mathbb{Z}_{15}\,, \tag{4.20}$$

Anomaly	Equation	Value
$[\mathrm{U}(1)_A]^3$	$\kappa_{u^3} = n_\psi q_\psi^3 \dim\psi + n_\chi q_\chi^3 \dim\chi$	-2025
$\mathrm{U}(1)_A[\mathrm{SU}(5)_\chi]^2$	$q_\chi \dim\chi$	30
$[\mathrm{SU}(5)_\chi]^3$	$\dim\chi$	15
$\mathbb{Z}_4^{d\chi}\,[\mathrm{U}(1)_A]^2$	$\kappa_{zu^2} = n_\chi q_\chi^2 \dim\chi$	$300 \bmod 4$
$\mathbb{Z}_4^{d\chi}\,[\mathrm{SU}(5)_\chi]^2$	$\dim\chi$	$15 \bmod 4$
$\mathrm{U}(1)_A[\mathrm{grav}]$	$2(n_\psi q_\psi \dim\psi + n_\chi q_\chi \dim\chi)$	90
$\mathbb{Z}_4^{d\chi}[\mathrm{grav}]$	$2n_\chi \dim\chi$	$150 \bmod 8$
$[\mathbb{Z}_4^{d\chi}]^3$	(2.29)	$2250 \bmod 24$
$\mathrm{U}(1)_A[\mathrm{CFU}]$	$q_\chi \dim\chi\, Q_\chi + \kappa_{u^3} Q_u$	$24p'^2 - 2025s^2$
$\mathbb{Z}_4^{d\chi}[\mathrm{CFU}]$	$n_\chi T_\chi Q_c + \dim\chi\, Q_\chi + \kappa_{zu^2} Q_{u^2}$	$\frac{50}{3}m^2 + 12p'^2 + 300s^2$

Table 3. Anomalies of $\mathrm{SU}(6), k = 2$.

and (m, p', s) are chosen to satisfy (2.14). The anomalies of the theory are listed in table 3. It is worth noting that both $\mathbb{Z}_4^{d\chi}[\mathrm{grav}]$ and $\mathbb{Z}_4^{d\chi}[\mathrm{CFU}]$ anomalies give at most a $\mathbb{Z}_2$ phase. Also, this theory cannot be put on $\mathbb{CP}^2$, as there are no solutions to the conditions (2.19).

We first comment on the possibility that the theory flows to a Banks-Zaks fixed point in the IR. The 2-loop beta function of this theory gives $\frac{g_*^2}{4\pi} \approx 8.5 \gg 1$. This value of the coupling constant is too large for perturbation theory to hold. At 3-loops, we obtain $\frac{g_*^2}{4\pi} \approx 0.73$. Also, this coupling-constant value is large, so we cannot conclude that our theory flows to a conformal fixed point in the IR. In the following, we examine the possibilities of fermion composites and SSB.

Matching by composites. Here, we follow the argument in section 3 to show that composite fermions cannot solely match all the UV anomalies. The UV $\mathbb{Z}_4^{d\chi}[\mathrm{CFU}]$ anomaly of this theory is (unlike the $\mathrm{U}(1)_A[\mathrm{CFU}]$ anomaly, it is important to notice that the color flux contributes to the $\mathbb{Z}_4^{d\chi}[\mathrm{CFU}]$ anomaly)

$$n_\chi T_\chi Q_c + \dim\chi\, Q_\chi + \kappa_{zu^2} Q_{u^2} = \frac{50}{3} m^2 + 12p'^2 + 300s^2 \,. \tag{4.21}$$

In the IR, a set of gauge invariant composite fermions would generate the corresponding $\mathbb{Z}_4^{d\chi}[\mathrm{CFU}]$ anomaly:

$$D_\chi^{\mathrm{IR}} Q_\chi + D_u^{\mathrm{IR}} Q_u \tag{4.22}$$

for integers D_χ^{IR} and D_u^{IR}. The integers D_χ^{IR} and D_u^{IR} are group-theoretical coefficients that are assumed to be found by matching all anomalies of continuous symmetries. In the presence of a CFU background flux, the ratio between the UV and IR partition functions after undergoing a $\mathbb{Z}_4^{d\chi}$ transformation is given by:

$$\frac{\mathcal{Z}^{\mathrm{UV}}}{\mathcal{Z}^{\mathrm{IR}}} = e^{\frac{i2\pi}{4}\left(\frac{50}{3}m^2 + \left(12 - D_\chi^{\mathrm{IR}}\right)p^2 + \left(300 - D_u^{\mathrm{IR}}\right)s^2\right)} = e^{\frac{i2\pi}{4}\left(\frac{50}{3}m^2 + d_\chi p'^2 + d_u s^2\right)} \,, \tag{4.23}$$

where $d_\chi = 12 - D_\chi^{\rm IR} \in \mathbb{Z}$ and $d_u = 300 - D_u^{\rm IR} \in \mathbb{Z}$. If there exists a particular solution (m, p', s) of the consistency conditions (2.14) such that no integers d_χ, d_u exists such that

$$\frac{50}{3}m^2 + d_\chi p^2 + d_u s^2 \in 4\mathbb{Z}, \tag{4.24}$$

then we conclude that composite fermions cannot match the $\mathbb{Z}_4^{d\chi}$[CFU] anomaly.

Consider $(m, p, s) = (1, 0, 2/3)$. This is a solution to the consistency conditions and therefore corresponds to a CFU flux. In the presence of this CFU background, the l.h.s. of (4.24) becomes

$$\frac{50}{3} + d_u\frac{4}{9} = \frac{150 + 4d_u}{9}\,. \tag{4.25}$$

However, $150 + 4d_u \equiv 2 \mod 4$ for any $d_u \in \mathbb{Z}$. Therefore we can conclude that for this theory, composite fermions cannot solely match the $\mathbb{Z}_4^{d\chi}$[CFU] anomaly in the IR.

Matching by a condensate. Without composites, the anomalies are matched by spontaneous symmetry breaking via condensates. First, the 2-fermion condensate cannot match the anomalies, as it breaks $\mathrm{SU}(5)_\chi \times \mathrm{U}(1)_A$ down to the anomalous subgroup $\mathrm{SU}(4) \times \mathrm{U}(1)$. Next, consider the operator

$$\mathcal{C}^{(ij)} = \psi^2\chi^{(i}\chi^{j)}\,, \tag{4.26}$$

where i, j are $\mathrm{SU}(5)_\chi$ flavor indices, and in particular, this condensate is in the two-index symmetric irrep of $\mathrm{SU}(5)_\chi$. Thus, the condensation of this operator breaks SU(5) to the anomaly-free subgroup SO(5).

Under the action of $\mathrm{U}(1)_A$, the condensate transforms as $\mathcal{C}^{ij} = \psi^2\chi^{(i}\chi^{j)} \longrightarrow \mathcal{C}'^{ij} = e^{2\pi(6\beta)}\psi^2\chi^{(i}\chi^{j)}$ where $\beta \in [0, 1)$ is the $\mathrm{U}(1)_A$ parameter. So it appears that the condensate is invariant under a discrete $\mathbb{Z}_6$ subgroup of $\mathrm{U}(1)_A$. But recall that the global symmetry group includes a division by the $\mathbb{Z}_3$ center of the color group. $\mathbb{Z}_3$ is not a subgroup of SU(5), therefore it can only quotient $\mathrm{U}(1)_A$, so the parameter β is in fact a $\mathrm{U}(1)_A/\mathbb{Z}_3$ parameter and $\beta \in [0, 1/3)$. Therefore the condensate only exhibits an unbroken $\mathbb{Z}_2$ symmetry, which has no global anomaly, and the $\mathrm{U}(1)_A$ breaks to a non-anomalous subgroup.

The condensate also breaks $\mathbb{Z}_4^{d\chi}$ down to $\mathbb{Z}_2$, leading to 2 vacua connected via a domain wall. Recalling that the $\mathbb{Z}_4^{d\chi}$[grav] anomaly is only a $\mathbb{Z}_2$ phase, the unbroken subgroup $\mathbb{Z}_2 \subset \mathbb{Z}_4^{d\chi}$ is anomaly free (remember that $\mathbb{Z}_2$ is also free from nonperturbative anomalies). In addition, the $\mathbb{Z}_4^{d\chi}$[CFU] anomaly is valued in $\mathbb{Z}_2$, meaning that the same condensate saturates it. The breaking of $\mathbb{Z}_4^{d\chi}$ down to $\mathbb{Z}_2$ will also automatically match the $[\mathbb{Z}_4^{d\chi}]^3$ anomaly, since $\mathbb{Z}_2$ is anomaly free.

Let us examine the fate of the $\mathrm{U}(1)_A$[CFU] anomaly. First, when we turn on the flavor center flux, the breaking of SU(5) into SO(5) matches the anomaly, as the breaking causes the center of SU(5) to break. Next, we solely turn on the color and $\mathrm{U}(1)_A$ fluxes. In this case, $s \in \mathbb{Z}_3$, and the breaking of $\mathrm{U}(1)_A$ down to $\mathbb{Z}_2$ implies that we are after $\mathbb{Z}_2[\mathbb{Z}_3]^2$ anomaly. The anomaly coefficient can be read from table 3, and according to (2.32), the anomaly is automatically matched since $\gcd(3, 2) = 1$.

We conclude that the global symmetry $G^{\rm g}$ breaks down to $\mathrm{SO}(5) \times \frac{(\mathbb{Z}_2 \subset \mathrm{U}(1)_A) \times (\mathbb{Z}_2 \subset \mathbb{Z}_4^{d\chi})}{\mathbb{Z}_2}$. The first $\mathbb{Z}_2$ symmetry acts only on ψ, while the second $\mathbb{Z}_2$ acts only on χ. Then, from (4.19),

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we see that the combination of these symmetries acts like the fermion number, which is left intact in the IR. The extra modding by $\mathbb{Z}_2$ is employed to avoid overcounting.

Since SO(5) is the largest anomaly-free subgroup of SU(5), this breaking pattern minimizes the number of Goldstones and is the most favorable scenario.

4.3 $\mathbf{SU(6)}, \boldsymbol{k=1}$

This theory has 2 flavors of ψ and 10 flavors of χ, and thus, the flavor symmetry is $\mathrm{SU}(2)_\psi \times \mathrm{SU}(10)_\chi$. The $\mathrm{U}(1)_A$ charges are

$$q_\psi = -5\,, \quad q_\chi = 2\,. \tag{4.27}$$

Because $r = \gcd(n_\chi T_\chi, n_\psi T_\psi) = \gcd(40, 16) = 8$, we may be tempted to conclude the theory has a $\mathbb{Z}_8$ chiral symmetry. However, one can show that a $\mathbb{Z}_2$ subgroup of the $\mathbb{Z}_{10}$ center of $\mathrm{SU}(10)_\chi$ can be used to identify elements of $\mathbb{Z}_8$:

$$\chi : e^{-2\pi i \frac{p'}{10}} e^{2\pi i \frac{l}{8}} = e^{2\pi i \frac{l'}{8}}\,, \tag{4.28}$$

for $l, l' = 1, 2, \ldots, 7$. For example, setting $p' = -5$ identifies $\ell = 1$ and $\ell = 5$, etc. In addition, the solutions to the consistency conditions (2.14) yield the division group $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_5$. Thus, the faithful global symmetry is

$$G^{\mathrm{g}} = \frac{\mathrm{SU}(2)_\psi \times \mathrm{SU}(10)_\chi \times \mathrm{U}(1)_A}{\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_5} \times \mathbb{Z}_4^{d\chi} \times \mathbb{Z}_2^{(1)}\,. \tag{4.29}$$

The β-function indicates that the theory flows to an IR fixed point. At 2 loops, the coupling constant at the fixed point is $\frac{g_*^2}{4\pi} \approx 0.094$. At 3 loops, we obtain $\frac{g_*^2}{4\pi} \approx 0.075$. Both values are much smaller than our threshold value of 0.1, and the 2- and 3-loop analysis is only 10% apart. Also, the 3-loop to the 2-loop ratio in (3.1) is ≈ 0.2. Thus, the fixed point is reliable. As we pointed out above, the lowest-order bosonic operator in this theory, $F_{\mu\nu}\sigma^{\mu\nu}\chi\psi$, necessitates the introduction of a color field to prevent its vanishing due to statistics. This is a dimension-5 operator, and due to the smallness of the coupling constant, we do not expect this operator to condense. Not to mention that this operator by itself is not enough to match the full set of anomalies, and higher-order condensates must also form to match them. We, thus, conclude that the most probable scenario is that the theory flows to a CFT.

4.4 $\mathbf{SU(10)}, \boldsymbol{k=2}$

The theory admits 3 flavors of ψ and 7 flavors of χ. The charges of the fermions under $\mathrm{U}(1)_A$ are

$$q_\psi = -14\,, \quad q_\chi = 9\,. \tag{4.30}$$

We also have $r = \gcd(N_\psi, N_\chi) = 4$, so that the theory is endowed with a $\mathbb{Z}_4^{d\chi}$ chiral symmetry, which cannot be absorbed in a combination of the centers of the color or flavor groups. After solving the consistency equations, we obtain the faithful global symmetry group

$$G^{\mathrm{g}} = \frac{\mathrm{SU}(3)_\psi \times \mathrm{SU}(7)_\chi \times \mathrm{U}(1)_A}{\mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_7} \times \mathbb{Z}_4^{d\chi} \times \mathbb{Z}_2^{(1)}\,. \tag{4.31}$$

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Theory	n_ψ	n_χ	$\mathbb{Z}_p^{d\chi}$	Γ	(q_ψ, q_χ)	2-loop	3-loop	$\frac{\alpha_*\beta_2}{4\pi\beta_1}$
$\mathrm{SU}(16), k=4$	3	5	2	$\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	$(-35, 27)$	0.09	0.064	0.62
$\mathrm{SU}(20), k=4$	4	6	2	$\mathbb{Z}_{10} \times \mathbb{Z}_4 \times \mathbb{Z}_3$	$(-27, 22)$	0.017	0.015	0.11
$\mathrm{SU}(28), k=8$	3	4	1	$\mathbb{Z}_7 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	$(-52, 45)$	0.086	0.051	1.12
$\mathrm{SU}(36), k=8$	4	5	1	$\mathbb{Z}_{18} \times \mathbb{Z}_4 \times \mathbb{Z}_5$	$(-85, 76)$	0.019	0.016	0.25
$\mathrm{SU}(44), k=8$	5	6	1	$\mathbb{Z}_{11} \times \mathbb{Z}_5 \times \mathbb{Z}_6$	$(-126, 115)$	0.0002	0.0002	0.003

Table 4. A list of conformal bosonic theories.

The theory develops a Banks-Zaks fixed point. The 2 and 3-loop values of the coupling constant at the fixed point are $\frac{g_*^2}{4\pi^2} \approx 0.059$ and $\frac{g_*^2}{4\pi^2} \approx 0.046$, respectively. Also, the 3-loop to the 2-loop ratio in (3.1) is ≈ 0.2. Thus, like $\mathrm{SU}(6), k=1$, this theory is expected to flow to a CFT.

5 Bosonic theories

All gauge-invariant operators in this class of theories are bosonic. In the following, we provide a systematic study of this class.

5.1 Conformal theories

We start by listing theories that flow to a conformal fixed point. These theories are displayed in table 4. In each case, the global symmetry is given by

$$G = \frac{\mathrm{SU}(n_\psi) \times \mathrm{SU}(n_\chi) \times \mathrm{U}(1)_A}{\Gamma} \times \mathbb{Z}_p^{d\chi} \times \mathbb{Z}_2^{(1)} \,. \tag{5.1}$$

We also display the coupling constant $\frac{g_*^2}{4\pi^2}$ at the 2- and 3-loop fixed points. The smallness of the coupling constant and its consistency between the 2- and 3-loop calculations is an indicator of the robustness of the fixed point. To quantify this robustness, we may truncate the β-function to the second term in (3.1) and find the fixed point is given by $\alpha_* = -\frac{4\pi\beta_0}{\beta_1}$. The existence of such a fixed point implies that the first and second terms possess comparable magnitudes. Consequently, the ratio between the third and second (or first) term $\frac{\alpha_*\beta_2}{4\pi\beta_1}$ represents the error incurred by neglecting the third term. A low ratio indicates the perturbative nature of the fixed point.

The two theories $(N=20, k=4)$ and $(N=44, k=8)$ have the most reliable fixed points. While the theory $(N=28, k=8)$ has $\frac{\alpha_*\beta_2}{4\pi\beta_1} = 1.12$, and its fixed point is under question.

5.2 Confining theories

5.2.1 $\mathrm{SU}(8), k=4$

This theory was studied in [8]. Here, we revisit it in light of the discrete anomalies not discussed in [8]. The theory admits $n_\psi = 1$ and $n_\chi = 3$ flavors of fermions. The fermion

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Anomaly	Equation	Value
$[\mathrm{SU}(3)_\chi]^3$	dim_χ	28
$\mathrm{U}(1)_A\,[\mathrm{SU}(3)_\chi]^2$	$q_\chi \mathrm{dim}_\chi$	140
$\mathbb{Z}_2^{d\chi}\,[\mathrm{SU}(3)_\chi]^2$	dim_χ	28
$\mathbb{Z}_2^{d\chi}\,[\mathrm{U}(1)_A]^2$	$\kappa_{zu^2} = q_\chi^2 \mathrm{dim}_\chi n_\chi$	2100
$\mathrm{U}(1)_A[\mathrm{grav}]$	$2(q_\psi \mathrm{dim}_\psi + q_\chi \mathrm{dim}_\chi n_\chi)$	192
$[\mathrm{U}(1)_A]^3$	$\kappa_{u^3} = q_\chi^3 \mathrm{dim}_\chi n_\chi + q_\psi^3 \mathrm{dim}_\psi$	-15744
$\mathbb{Z}_2^{d\chi}[\mathrm{grav}]$	$2\mathrm{dim}_\chi n_\chi$	168 (trivial)
$\mathrm{U}(1)_A[\mathrm{CFU}]$	$q_\chi \dim\chi\, Q_\chi + \kappa_{u^3} Q_u$	$\frac{280p'^2}{3} - 15744s^2$, $p' \in \mathbb{Z}_3\,, s \in \mathbb{Z}_{12}$
$\mathbb{Z}_2^{d\chi}[\mathrm{CFU}]$	$n_\chi T_\chi Q_c + \dim\chi\, Q_\chi + \kappa_{zu^2} Q_u$	$\frac{27m^2}{2} + \frac{56p'^2}{3} + 2100s^2\,, m \in \mathbb{Z}_4$

Table 5. Anomalies of $\mathrm{SU}(8), k = 4$.

charges under $\mathrm{U}(1)_A$ are

$$q_\psi = -9\,, \quad q_\chi = 5\,. \tag{5.2}$$

Also, the theory admits a $\mathbb{Z}_2^{d\chi}$ discrete chiral symmetry. Solving the consistency conditions (2.14) yield the faithful global symmetry

$$G^{\mathrm{g}} = \frac{\mathrm{SU}(3)_\chi \times \mathrm{U}(1)_A}{\mathbb{Z}_4 \times \mathbb{Z}_3} \times \mathbb{Z}_2^{d\chi} \times \mathbb{Z}_2^{(1)}\,. \tag{5.3}$$

The theory admits many anomalies in table 5. In addition, the theory admits a $\mathbb{Z}_2^{d\chi}[\mathrm{CFU}]$ anomaly, which yields a phase of π upon turning on a flux with, e.g., $(m, p', s) = (1, 0, \frac{1}{4})$, i.e., this is a $\mathbb{Z}_4 \subset \mathrm{U}(1)_A$ flux. We also find that there is an anomaly of $\mathbb{Z}_2^{d\chi}$ on a nonspin manifold, as the partition function acquires a phase of π by turning on a pure $\mathbb{Z}_2^{(1)}$ flux on $\mathbb{CP}^2$.

In [8], it was argued that all the anomalies could be matched by condensing two operators:

$$\mathcal{C}_1^i = \psi\chi^i\,, \quad \mathcal{C}_2^{i_4} = \epsilon_{i_1 i_2 i_3}\chi^{i_1}\chi^{i_2}\chi^{i_3}\chi^{i_4}\,. \tag{5.4}$$

Let us review the anomaly matching using these two operators and comment on why they cannot match the discrete anomalies.

Both operators $\mathcal{C}_1^i$ and $\mathcal{C}_2^i$ transform in the defining representation of SU(3) and break it down to the anomaly-free SU(2) (it has no Witten anomalies because the dimensions of the representations are even). Yet, the condensation of $\mathcal{C}_1^i$ or $\mathcal{C}_2^i$ leaves behind an unbroken SU(3) generator. We take $\mathcal{C}_1^i \propto \delta_{i,1}$ and $\mathcal{C}_2^i \propto \delta_{i,1}$ and parametrize the SU(3) matrix that corresponds to the unbroken Cartan generator of SU(3) as $\mathrm{diag}\left(e^{i4\pi\alpha}, e^{-i2\pi\alpha}, e^{-i2\pi\alpha}\right)$. Then, under $\mathrm{SU}(3)_\chi \times \mathrm{U}(1)_A \times \mathbb{Z}_2^{d\chi}$, the operators transform as

$$\mathcal{C}_1^i \longrightarrow e^{i4\pi\alpha - i8\pi\beta + in\pi}\mathcal{C}_1^i\,, \quad \mathcal{C}_2^i \longrightarrow e^{i40\pi\beta + i4\pi\alpha}\mathcal{C}_2^i\,, \tag{5.5}$$

where β corresponds to the $\mathrm{U}(1)_A$ transformation, whereas $n = 1$ corresponds to the $\mathbb{Z}_2^{d\chi}$ transformation. Taking $\alpha = -\frac{5}{24}$, $\beta = \frac{1}{48}$, and $n = 1$ leaves $\mathcal{C}_1^i$ and $\mathcal{C}_2^i$ invariant under the

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combined transformations of $\mathrm{SU}(3)_\chi \times \mathrm{U}(1)_A \times \mathbb{Z}_2^{d\chi}$. This superficially hints at an unbroken $\mathbb{Z}_{24}$ symmetry. However, owing to the modding by $\mathbb{Z}_4$ in (5.3), the genuine unbroken subgroup is $\mathbb{Z}_6$. This unbroken symmetry can be written as $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$, where $\mathbb{Z}_3$ is a genuine subgroup of $\mathrm{U}(1)_A$. This can be seen by setting $n = 0$, then we find that $\mathcal{C}_1^i$ and $\mathcal{C}_2^i$ are left invariant by taking $\alpha = -\frac{5}{12}$ and $\beta = \frac{1}{24}$. Remembering the modding by $\mathbb{Z}_4$ in (5.3), we conclude that there is a $\mathbb{Z}_3$ unbroken subgroup of $\mathrm{U}(1)_A$.

It is straightforward to calculate the $\mathbb{Z}_3$ anomaly using (2.29) to find that it is non-vanishing, meaning that the condensation of $\mathcal{C}_1^i$ and $\mathcal{C}_2^i$ is not enough to match the complete set of anomalies. The way out is to consider the condensation of the operator

$$\mathcal{C}_3^{(ij)} = \psi^2 \chi^{(i}\chi^{j)}\,, \tag{5.6}$$

which transforms in the 2-index symmetric representation of $\mathrm{SU}(3)_\chi$ and breaks it down to $\mathrm{SO}(3)$. $\mathrm{U}(1)_A$ is broken to $\mathbb{Z}_2$, after taking into account the modding by $\mathbb{Z}_4$ in (5.3). The $\mathbb{Z}_2^{d\chi}[\mathrm{CFU}]$ anomaly is automatically matched as $\mathrm{U}(1)_A$ is broken down to $\mathbb{Z}_2$ (remember, however, that this $\mathbb{Z}_2$ is the fermion number since both fermions carry odd charges under $\mathrm{U}(1)_A$, and the fermion number is gauged). Recalling that we had to turn on a $\mathbb{Z}_4 \subset \mathrm{U}(1)_A$ flux in the first place to see this anomaly (a π phase), the breaking of $\mathrm{U}(1)_A$ to a smaller subgroup than $\mathbb{Z}_4$ (in this case $\mathbb{Z}_2 \subset \mathbb{Z}_4$) trivializes the anomaly. Thus, at this level, one does not need to break $\mathbb{Z}_2^{d\chi}$. This differs from the findings in [8], where it was argued that the CFU anomaly is not trivial. Here, we arrive at a different IR condensate by scrutinizing the discrete subgroups of $\mathrm{U}(1)_A$.

What about matching the anomaly of $\mathbb{Z}_2^{d\chi}$ on $\mathbb{CP}^2$? Since this anomaly is valued in $\mathbb{Z}_2$, it can be matched by a TQFT on a nonspin manifold, as was argued in [29]. Yet, another scenario is to form the condensate $\mathcal{C}_1^i$, which breaks $\mathbb{Z}_2^{d\chi}$ to unity (remember that the $\mathbb{Z}_2^{(1)}$ 1-form symmetry is unbroken assuming confinement). Thus, the condensation of both $\mathcal{C}_1^i$ and $\mathcal{C}_3^{(ij)}$ match all anomalies and break the global group down to $\mathrm{SO}(3)$, resulting in 2 vacua (because of the breaking of $\mathbb{Z}_2^{d\chi}$) connected via domain walls.

5.2.2 $\mathrm{SU}(8), k = 2$

This case was also considered briefly in [8]. The theory admits 2 flavors of ψ fermions and 6 flavors of χ fermions. The $\mathrm{U}(1)_A$ charges of the fermions are

$$q_\psi = -9\,, \quad q_\chi = 5\,. \tag{5.7}$$

Since $\gcd(N_\psi, N_\chi) = 4$, one may naively conclude that the discrete symmetry is $\mathbb{Z}_4$. Yet, two elements of $\mathbb{Z}_4$ are identified with elements in $\mathbb{Z}_2 \subset \mathbb{Z}_6$, where $\mathbb{Z}_6$ is the center of $\mathrm{SU}(6)_\chi$. This leaves us with $\mathbb{Z}_2^{d\chi}$ as the genuine discrete group, which we take to act solely on χ. The faithful global symmetry is

$$G^{\mathrm{g}} = \frac{\mathrm{SU}(2)_\psi \times \mathrm{SU}(6)_\chi \times \mathrm{U}(1)_A}{\mathbb{Z}_4 \times \mathbb{Z}_6} \times \mathbb{Z}_2^{d\chi} \times \mathbb{Z}_2^{(1)}\,. \tag{5.8}$$

The UV theory has the 't Hooft anomalies in table 6. The $\mathbb{Z}_2^{d\chi}[\mathrm{CFU}]$ anomaly does not provide new information. However, there is a non-trivial $\mathbb{Z}_2^{d\chi}[\mathrm{CFU}]_{\mathbb{CP}^2}$ anomaly (a π phase)

Anomaly	Equation	Value
$[\mathrm{SU}(6)_\chi]^3$	dim_χ	28
$\mathrm{U}(1)_A[\mathrm{grav}]$	$2(q_\psi \mathrm{dim}_\psi + q_\chi \mathrm{dim}_\chi n_\chi)$	384
$\mathbb{Z}_2^{d\chi}[\mathrm{grav}]$	$\mathrm{dim}_\chi n_\chi$	336 (trivial)
$\mathrm{U}(1)_A[\mathrm{SU}(6)_\chi]^2$	$q_\chi \mathrm{dim}_\chi$	140
$\mathrm{U}(1)_A[\mathrm{SU}(2)_\psi]^2$	$q_\psi \mathrm{dim}_\psi$	-324
$[\mathrm{U}(1)_A]^3$	$q_\psi^3 \mathrm{dim}_\psi + q_\chi^3 \mathrm{dim}_\chi n_\chi$	-31488
$\mathbb{Z}_2^{d\chi}[\mathrm{SU}(6)_\chi]^2$	dim_χ	28 (trivial)
$\mathbb{Z}_2^{d\chi}[\mathrm{U}(1)_A]^2$	$q_\psi^2 \mathrm{dim}_\psi n_\psi + q_\chi^2 \mathrm{dim}_\chi n_\chi$	4200 (trivial)

Table 6. Anomalies of $\mathrm{SU}(8), k = 2$.

in the background of a CFU configuration with all fluxes turned on, e.g., $(m, p, p', s) = (1, 1, 1, -5/12)$.

The condensation of the operator

$$\mathcal{C}^i_{1\,j} = \psi_j \chi^i \tag{5.9}$$

break $\mathrm{SU}(2)_\psi \times \mathrm{SU}(6)_\chi$ down to $\mathrm{SU}(2) \times \mathrm{SU}(4)$. The unbroken $\mathrm{SU}(4)$ is anomalous.

Another operator is

$$\mathcal{C}_2^{[i_1 i_2]} = \psi^2 \chi^{[i_1} \chi^{i_2]}\,, \tag{5.10}$$

which is neutral under $\mathrm{SU}(2)_\psi \times \mathbb{Z}_2^{d\chi}$ but transforms in the 2-index anti-symmetric representation of $\mathrm{SU}(6)$ and breaks it down to the anomaly-free $\mathrm{Sp}(6)$.[12] In addition, the condensation of $\mathcal{C}_2^{[i_1 i_2]}$ breaks $\mathrm{U}(1)_A$ to the anomaly-free $\mathbb{Z}_2$, after taking into account the modding by $\mathbb{Z}_4$ in (5.8). What about the $\mathbb{Z}_2^{d\chi}[\mathrm{CFU}]_{\mathbb{CP}^2}$ anomaly? Remember that one needs to turn on a configuration with $\mathrm{U}(1)_A$ flux in $\mathbb{Z}_{12}$. Since $\mathrm{U}(1)_A$ breaks down to $\mathbb{Z}_2$, the anomaly trivializes. Recall that this $\mathbb{Z}_2$ is the fermion number since both fermions have odd charges under $\mathrm{U}(1)_A$, and that the fermion number is gauged. Thus, the condensation of $\mathcal{C}_2^{[i_1 i_2]}$ leaves behind the unbroken $\frac{\mathrm{SU}(2)_\psi \times \mathrm{Sp}(6)}{\mathbb{Z}_2} \times \mathbb{Z}_2^{d\chi}$ subgroup and matches all anomalies.[13]

5.2.3 $\mathrm{SU}(12), k = 4$

The number of flavors in this case is $n_\psi = 2$ and $n_\chi = 4$, and the $\mathrm{U}(1)_A$ charges are:

$$q_\psi = -10\,, \quad q_\chi = 7\,. \tag{5.11}$$

Since $\gcd(n_\chi T_\chi, n_\psi T_\psi) = \gcd(40, 28) = 4$, one may conclude that the theory is endowed with a $\mathbb{Z}_4$ chiral symmetry that acts on χ. However, this $\mathbb{Z}_4$ is the center of the $\mathrm{SU}(4)_\chi$

[12] Alternatively, one could propose the formation of a condensate transforming in the 2-index symmetric representation of $\mathrm{SU}(6)$. This condensate, however, would break $\mathrm{SU}(6)$ down to $\mathrm{SO}(6)$, resulting in a larger number of Goldstones.

[13] The symplectic group $\mathrm{Sp}(2N)$ has a $\mathbb{Z}_2$ center symmetry, see, e.g., [40]. This is why we needed to mod by $\mathbb{Z}_2$ that is common between $\mathrm{Sp}(6)$ and $\mathrm{SU}(2)_\psi$.

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Anomaly	Equation	Value
$[\mathrm{U}(1)_A]^3$	$q_\chi^3 \mathrm{dim}_\chi n_\chi + q_\psi^3 n_\psi \mathrm{dim}_\psi$	-65448
$\mathrm{U}(1)_A[\mathrm{SU}(2)_\psi]^2$	$q_\psi \mathrm{dim}_\psi$	-780
$\mathrm{U}(1)_A[\mathrm{SU}(4)_\chi]^2$	$q_\chi \mathrm{dim}_\chi$	462
$[\mathrm{SU}(4)_\chi]^3$	dim_χ	66
$\mathrm{U}(1)_A[\mathrm{grav}]$	$\kappa_{u^3} = q_\chi \mathrm{dim}_\chi n_\chi + q_\psi n_\psi \mathrm{dim}_\psi$	576
$\mathrm{U}(1)_A[\mathrm{CFU}]$	$q_\psi\, \mathrm{dim}_\psi\, Q_\psi + q_\chi\, \mathrm{dim}_\chi\, Q_\chi + \kappa_{u^3} Q_u$	$390p^2 + \frac{693}{2}p'^2 - 65448s^2,\ p, p' \in \mathbb{Z}_2$

Table 7. Anomalies of $\mathrm{SU}(12), k = 4$.

flavor symmetry. Therefore, the theory does not possess a discrete chiral symmetry. Solving the consistency conditions (2.14), we find that the faithful global symmetry group is:

$$G^{\mathrm{g}} = \frac{\mathrm{SU}(2)_\psi \times \mathrm{SU}(4)_\chi \times \mathrm{U}(1)_A}{\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2} \times \mathbb{Z}_2^{(1)} . \tag{5.12}$$

This theory is endowed with the anomalies in table 7. The theory does not possess a Witten anomaly of $\mathrm{SU}(2)_\psi$ since $\mathrm{dim}_\psi = 66$ is an even number.

The 2-loop and the 3-loop β-functions predict fixed points at $\frac{g_*^2}{4\pi} = 0.514$ and 0.202, respectively. Both values are large for the fixed points to be robust.

In searching for candidates that break the symmetries spontaneously, let us study the bilinear condensate:

$$\mathcal{C}_j^i = \epsilon^{a_1 \ldots a_{12}} \left(f^c_{\mu\nu}\right)_{a_2}^{a_{13}} \sigma^{\mu\nu} \epsilon_{\alpha_1\alpha_2} \psi^{\alpha_1}_{j,\,(a_1 a_{13})} \chi^{\alpha_2,\,i}_{[a_3 \ldots a_{12}]} , \qquad j = 1, 2\,, i = 1, 2, 3, 4, \tag{5.13}$$

where, as usual, $a_1, a_2, ..$ are color indices, α_1, α_2 are spinor indices, while j and i are respectively $\mathrm{SU}(2)_\psi$ and $\mathrm{SU}(4)_\chi$ indices. The transformation of $\mathcal{C}_j^i$ is noteworthy as it occurs in the fundamental representation of $\mathrm{SU}(2)\psi$ and the anti-fundamental representation of $\mathrm{SU}(4)\chi$. Consequently, upon condensation, it has the potential to break down $\mathrm{SU}(2)_\psi \times \mathrm{SU}(4)_\chi$ to $\mathrm{SU}(2)_V \times \mathrm{SU}(2)$. This symmetry-breaking pattern can be explained as follows [32].

To create an invariant potential for the 4×2 matrix $\mathcal{C}_j^i$, we define the 4×4 matrix $\Phi^{i,i'} \equiv \sum_{j=1}^2 \mathcal{C}_j^i \mathcal{C}_j^{i'}$. By considering the effective potential as a trace over quadratic and quartic terms of $\Phi^{i,i'}$, we might initially assume that we can transform to a basis where $\Phi^{i,i'}$ becomes a non-degenerate diagonal matrix. However, this assumption leads to a contradiction because the 4×1 column vectors in $\Phi^{i,i'}$ are dependent due to the construction of $\Phi^{i,i'}$ from a 4×2 matrix. In other words, $\Phi^{i,i'}$ possesses two zero eigenvalues. Hence, we conclude that we can only transform to a basis that diagonalizes $\mathrm{SU}(2)\psi \times (\mathrm{SU}(2) \subset \mathrm{SU}(4)\chi)$. This results in the diagonal (vector-like) matrix $\mathrm{SU}(2)_V$, while $\mathrm{SU}(4-2) = \mathrm{SU}(2) \subset \mathrm{SU}(4)\chi$ remains unbroken. Both $\mathrm{SU}(2)_V$ and $\mathrm{SU}(2) \subset \mathrm{SU}(4)\chi$ are subgroups devoid of anomalies. The potential anomaly, namely the Witten anomaly, does not afflict any of these subgroups. The UV particle content ensures that the number of fermions transforming under $\mathrm{SU}(2)\psi$ and $\mathrm{SU}(2) \subset \mathrm{SU}(4)_\chi$ is $\mathrm{dim}\psi = 78$ and $\mathrm{dim}\chi = 66$, respectively, both of which are even numbers. Therefore, none of these groups can exhibit Witten anomalies.

Anomaly	Equation	Value
$[\mathrm{U}(1)_A]^3$	$\kappa_{u^3} = q_\chi^3 \mathrm{dim}_\chi n_\chi + q_\psi^3 \mathrm{dim}_\psi$	-32724
$\mathrm{U}(1)_A[\mathrm{SU}(2)_\chi]^2$	$q_\chi \mathrm{dim}_\chi$	462
$\mathrm{U}(1)_A[\mathrm{grav}]$	$q_\chi \mathrm{dim}_\chi n_\chi + q_\psi \mathrm{dim}_\psi$	288
$\mathrm{U}(1)_A[\mathrm{CFU}]$	$q_\chi\, \mathrm{dim}_\chi\, Q_\chi + \kappa_{u^3} Q_u$	$231 p'^2 - 32724 s^2$, $p' \in \mathbb{Z}_2$

Table 8. Anomalies of $\mathrm{SU}(12), k = 8$.

Moreover, due to the $\mathbb{Z}_3$ modding in (5.12), the axial symmetry $\mathrm{U}(1)_A/\mathbb{Z}_3$ identifies a transformation phase α with $\alpha + \frac{2\pi}{3}$. The charge of the condensate $\mathcal{C}^i_j$ under $\mathrm{U}(1)_A$ is -3, leading to the breaking of $\mathrm{U}(1)_A$ to unity. Hence, we conclude that the 2-fermion condensate $\mathcal{C}^i_j$ successfully saturates all the anomalies and breaks the global symmetry down to $\frac{\mathrm{SU}(2)_V \times (\mathrm{SU}(2) \subset \mathrm{SU}(4)_\chi)}{\mathbb{Z}_2}$, where we mod by the $\mathbb{Z}_2$ common center of both groups.

5.2.4 $\mathrm{SU}(12), k = 8$

The number of flavors in this case is $n_\psi = 1$ and $n_\chi = 2$ and the $\mathrm{U}(1)_A$ charges are:

$$q_\psi = -10\,, \quad q_\chi = 7\,. \tag{5.14}$$

Given that $r = \gcd(n_\psi T_\psi, n_\chi T_\chi) = \gcd(14, 20) = 2$, we may conclude that the theory has a $\mathbb{Z}_2$ discrete chiral symmetry. Yet, one can absorb this $\mathbb{Z}_2$ in the center of $\mathrm{SU}(2)_\chi$, leaving behind no genuine discrete symmetry. After solving the consistency conditions, we find that the faithful global symmetry group is:

$$G^{\mathrm{g}} = \frac{\mathrm{SU}(2)_\chi \times \mathrm{U}(1)_A}{\mathbb{Z}_3 \times \mathbb{Z}_2} \times \mathbb{Z}_2^{(1)}\,. \tag{5.15}$$

The theory possesses the anomalies in table 8. The potential Witten anomaly of $\mathrm{SU}(2)_\chi$ is absent because $\mathrm{dim}_\chi = 66$ is an even number.

The 2-loop and 3-loop β-functions do not predict fixed points, and the theory needs to break its symmetries spontaneously by forming condensates. The operator

$$\mathcal{C}^i_1 = \psi \chi^i\,, \tag{5.16}$$

where the index i is the $\mathrm{SU}(2)_\chi$ flavor, breaks the global symmetry down to $\mathrm{U}(1)$. To see that, let us fix the vacuum to be $[1 \quad 0]^T$. Then, if a $\mathrm{U}(1)$ generator is left unbroken by the vacuum, one should find a nontrivial solution to

$$\exp\left[i2\pi\beta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right] e^{-i6\pi\alpha I_{2\times 2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{5.17}$$

It is easy to check that the solution $\beta = 3\alpha$ satisfies the above equation, which is the unbroken $\mathrm{U}(1)$ direction. The unbroken $\mathrm{U}(1)$ symmetry inherits the UV mixed $\mathrm{U}(1)_A[\mathrm{grav}]$ anomaly, and thus, condensing $\mathcal{C}^i_1$ is not enough to match the anomalies.

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Anomaly	Equation	Value
$[\mathrm{U}(1)_A]^3$	$\kappa_{u^3} = q_\chi^3 \dim_\chi n_\chi + q_\psi^3 \dim_\psi$	-2197500
$\mathrm{U}(1)_A\,[\mathrm{SU}(2)_\psi]^2$	$q_\psi n_\psi \dim_\psi$	-5670
$\mathrm{U}(1)_A\,[\mathrm{SU}(3)_\chi]^2$	$q_\chi n_\chi \dim_\chi$	4180
$[\mathrm{SU}(3)_\chi]^3$	$\dim_\chi$	190
$\mathrm{U}(1)_A\,[\text{grav}]$	$2(q_\chi \dim_\chi n_\chi + q_\psi \dim_\psi)$	2400
$\mathrm{U}(1)_A[\text{CFU}]$	$q_\psi \dim_\psi\, Q_\psi + q_\chi \dim_\chi\, Q_\chi + \kappa_{u^3} Q_u$	$-2835p^2 + \frac{8360}{3}p'^2 - 2197500s^2$

Table 9. Anomalies of $\mathrm{SU}(20), k = 8$.

Another operator that can condense is

$$\mathcal{C}_2 = \epsilon_{ij} \psi\psi\chi^i\chi^j \,, \tag{5.18}$$

with possible insertions of gluon fields. The operator $\mathcal{C}_2$ is singlet under SU(2), but it has a charge -6 under $\mathrm{U}(1)_A$. Because of the modding by $\mathbb{Z}_3$ in (5.15), the condensation of $\mathcal{C}_2$ breaks $\mathrm{U}(1)_A$ down to $\mathbb{Z}_2$, an anomaly-free subgroup. We conclude that the condensation of $\mathcal{C}_2$ is enough to match the anomalies, a scenario with the minimum number of Goldstones.

5.2.5 $\mathrm{SU}(20), k = 8$

The number of flavors is $n_\psi = 2$ and $n_\chi = 3$, while the $\mathrm{U}(1)_A$ charges are:

$$q_\psi = -27 \,, \quad q_\chi = 22 \,. \tag{5.19}$$

Since $r = \gcd(n_\psi T_\psi, n_\chi T_\chi) = (36, 66) = 2$, we might conclude that the theory possesses a $\mathbb{Z}_2$ chiral symmetry. However, this symmetry can be rotated away in the following way. First, according to our choice, the would-be chiral symmetry acts only on χ. Thus, $(\psi, \chi) \longrightarrow (\psi, -\chi)$ under this $\mathbb{Z}_2$. Next, we apply a transformation by $(-1)^F$, which sends $(\psi, -\chi) \longrightarrow (-\psi, \chi)$. Finally, we apply another transformation by the center of $\mathrm{SU}(2)_\psi$, which sends $(-\psi, \chi) \longrightarrow (\psi, \chi)$. This shows that the theory does not possess a discrete chiral symmetry. Finding the solutions to the consistency conditions, the faithful global symmetry group is:

$$G^{\mathrm{g}} = \frac{\mathrm{SU}(2)_\psi \times \mathrm{SU}(3)_\chi \times \mathrm{U}(1)_A}{\mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_3} \times \mathbb{Z}_2^{(1)} \,. \tag{5.20}$$

The anomalies of this theory are given in table 9.

Since $\mathrm{SU}(2)_\psi$ is an anomaly-free group, it does not need to break. The scenario that gives the lowest number of Goldstones amounts to breaking $\mathrm{SU}(3)\chi \times \mathrm{U}(1)_A$ to an anomaly-free subgroup. This can be achieved by condensing

$$\mathcal{C}^{(ij)} = \psi^2 \chi^{(i}\chi^{j)} \,, \tag{5.21}$$

which is singlet under $\mathrm{SU}(2)_\psi$ and transforms in the 2-index symmetric representation of SU(3) breaking it to SO(3). As before, this condensate also breaks $\mathrm{U}(1)_A$ to the anomaly-free subgroup $\mathbb{Z}_2$. Thus, the IR unbroken 0-form symmetry is $\frac{\mathrm{SU}(2)\psi \times (\mathbb{Z}_2 \subset \mathrm{U}(1)_A)}{\mathbb{Z}_2} \times \mathrm{SO}(3)$.

Theory	Global Symmetries	Condensate(s)	IR Symmetries
$\mathrm{SU}(5), k=1$	$\frac{\mathrm{SU}(9)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_5\times\mathbb{Z}_9}$	$\psi\chi^9\chi^{(i}\chi^{j)}$	$\mathrm{SO}(9)\times(\mathbb{Z}_{10}\subset\mathrm{U}(1)_A)$
$\mathrm{SU}(6), k=1$	$\frac{\mathrm{SU}(2)_\psi\times\mathrm{SU}(10)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_3\times\mathbb{Z}_2\times\mathbb{Z}_5}\times\mathbb{Z}_4^{d\chi}$	—	CFT
$\mathrm{SU}(6), k=2$	$\frac{\mathrm{SU}(5)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_3\times\mathbb{Z}_5}\times\mathbb{Z}_4^{d\chi}$	$\psi^2\chi^{(i}\chi^{j)}$	$\mathrm{SO}(5)\times\frac{(\mathbb{Z}_2\subset\mathrm{U}(1)_A)\times(\mathbb{Z}_2\subset\mathbb{Z}_4^{d\chi})}{\mathbb{Z}_2}$
$\mathrm{SU}(10), k=2$	$\frac{\mathrm{SU}(3)_\psi\times\mathrm{SU}(7)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_5\times\mathbb{Z}_3\times\mathbb{Z}_7}\times\mathbb{Z}_4^{d\chi}$	—	CFT
$\mathrm{SU}(8), k=2$	$\frac{\mathrm{SU}(2)_\psi\times\mathrm{SU}(6)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_4\times\mathbb{Z}_6}\times\mathbb{Z}_2^{d\chi}$	$\psi^2\chi^{[i}\chi^{j]}$	$\frac{\mathrm{SU}(2)_\psi\times\mathrm{Sp}(6)}{\mathbb{Z}_2}\times\mathbb{Z}_2^{d\chi}$
$\mathrm{SU}(8), k=4$	$\frac{\mathrm{SU}(3)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_4\times\mathbb{Z}_3}\times\mathbb{Z}_2^{d\chi}$	$\psi\chi^i, \psi^2\chi^{(i}\chi^{j)}$	$\mathrm{SO}(3)$
$\mathrm{SU}(12), k=4$	$\frac{\mathrm{SU}(2)_\psi\times\mathrm{SU}(4)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_6\times\mathbb{Z}_2\times\mathbb{Z}_2}$	$\psi_i\chi^j$	$\frac{\mathrm{SU}(2)_V\times(\mathrm{SU}(2)\subset\mathrm{SU}(4)_\chi)}{\mathbb{Z}_2}$
$\mathrm{SU}(12), k=8$	$\frac{\mathrm{SU}(2)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_3\times\mathbb{Z}_2}$	$\epsilon_{ij}\psi^2\chi^i\chi^j$	$\frac{\mathrm{SU}(2)_\chi\times(\mathbb{Z}_2\subset\mathrm{U}(1)_A)}{\mathbb{Z}_2}$
$\mathrm{SU}(16), k=4$	$\frac{\mathrm{SU}(3)_\psi\times\mathrm{SU}(5)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_8\times\mathbb{Z}_3\times\mathbb{Z}_5}\times\mathbb{Z}_2^{d\chi}$	—	CFT
$\mathrm{SU}(20), k=8$	$\frac{\mathrm{SU}(2)_\psi\times\mathrm{SU}(3)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_5\times\mathbb{Z}_2\times\mathbb{Z}_3}$	$\psi^2\chi^{(i}\chi^{j)}$	$\frac{\mathrm{SU}(2)_\psi\times(\mathbb{Z}_2\subset\mathrm{U}(1)_A)}{\mathbb{Z}_2}\times\mathrm{SO}(3)$
$\mathrm{SU}(20), k=4$	$\frac{\mathrm{SU}(4)_\psi\times\mathrm{SU}(6)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_{10}\times\mathbb{Z}_4\times\mathbb{Z}_3}\times\mathbb{Z}_2^{d\chi}$	—	CFT
$\mathrm{SU}(28), k=8$	$\frac{\mathrm{SU}(3)_\psi\times\mathrm{SU}(4)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_7\times\mathbb{Z}_3\times\mathbb{Z}_4}$	—	CFT
$\mathrm{SU}(36), k=8$	$\frac{\mathrm{SU}(4)_\psi\times\mathrm{SU}(5)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_{18}}$	—	CFT
$\mathrm{SU}(44), k=8$	$\frac{\mathrm{SU}(5)_\psi\times\mathrm{SU}(6)_\chi\times\mathrm{U}(1)_A}{\mathbb{Z}_{11}\times\mathbb{Z}_5\times\mathbb{Z}_6}$	—	CFT

Table 10. A summary of the 2-index chiral theories, their global symmetries, and their IR realizations. Theories with N even also enjoy a $\mathbb{Z}_2^{(1)}$ 1-form symmetry acting on the Wilson lines. This symmetry is assumed to be unbroken in theories that confine.

6 Summary

In this paper, we exhaustively scrutinized the 2-index chiral gauge theories. By studying the 2-loop and 3-loop β-functions, we could pinpoint a few theories that may flow to an IR CFT. Theories that do not admit a fixed point break its global symmetries. We considered scenarios that give the minimal number of IR Goldstones, as this lowers the free energy of the theory. We paid particular attention to the anomaly-matching conditions and ensured that the condensates match any discrete subgroup of $\mathrm{U}(1)_A$. Our theories, their global symmetries, the proposed IR phase condensates, and the unbroken IR symmetries are shown in table 10. The first 4 theories are fermionic, while the rest are bosonic.

Our investigation included a closer examination of the CFU anomalies one of the authors studied in the previous work [8], giving a better interpretation of this class of anomalies in

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the light of the discrete-anomaly matching conditions. The general finding is that matching the full set of anomalies and, in particular, the anomalies of the discrete subgroups of the axial $\mathrm{U}(1)_A$ symmetry necessitates the formation of multiple higher-order condensates. One expects such higher-order condensates to form in strongly-coupled theories. Here, their formation is explained via the constraints of the anomaly-matching conditions. We also employed a systematic approach to search for massless composite fermions that could match the anomalies in the case of fermionic theories. We were not able to find such composites. In one case, we used the CFU anomaly to show that a set of composites cannot solely match this anomaly, hinting at a deeper reason why the composites could not be found.

Our work provides a systematic approach that can be applied to study other classes of strongly-coupled phenomena, including different chiral gauge theories.

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A Obtaining the discrete chiral symmetry

In this appendix, we show that there is a discrete symmetry $\mathbb{Z}_r$, where $r = \gcd(N_\psi, N_\chi)$, that acts on χ. To this end, we conisder the groups $\mathbb{Z}_{N_\psi p_\psi + N_\chi p_\chi}$ and $\mathrm{U}(1)_A$ we discussed in the text. Under $\mathrm{U}(1)_A \times \mathbb{Z}_{N_\psi p_\psi + N_\chi p_\chi}$, ψ transforms as

$$\psi \longrightarrow e^{2\pi i a q_\psi \alpha} e^{2\pi i p_\psi \frac{l}{N_\psi p_\psi + N_\chi p_\chi}} \psi \,, \tag{A.1}$$

where $l \in \mathbb{Z}_{N_\psi p_\psi + N_\chi p_\chi}$, a is a charge factor, and $\alpha \in [0,1)$. This transformation leaves ψ invariant if

$$a q_\psi \alpha + p_\psi \frac{l}{N_\psi p_\psi + N_\chi p_\chi} = k_1 \in \mathbb{Z} \implies \alpha = \frac{k_1}{a q_\psi} - \frac{p_\psi l}{a q_\psi (N_\psi p_\psi + N_\chi p_\chi)} \,. \tag{A.2}$$

Note that k_1 can be freely chosen. Then, χ transforms under $\mathrm{U}(1)_A \times \mathbb{Z}_{N_\psi p_\psi + N_\chi p_\chi}$ as:

$$\begin{aligned}
\chi \longrightarrow e^{2\pi i a q_\chi \alpha} e^{2\pi i p_\chi \frac{l}{N_\psi p_\psi + N_\chi p_\chi}} \chi &= e^{2\pi i \left(a q_\chi \left(\frac{k_1}{a q_\psi} - \frac{p_\psi l}{a q_\psi (N_\psi p_\psi + N_\chi p_\chi)} \right) + p_\chi \frac{l}{N_\psi p_\psi + N_\chi p_\chi} \right)} \chi \\
= e^{2\pi i \left(\frac{q_\chi}{q_\psi} k_1 + \frac{l}{N_\psi p_\psi + N_\chi p_\chi} \left(p_\chi - p_\psi \frac{q_\chi}{q_\psi} \right) \right)} \chi &= e^{2\pi i \left(\frac{q_\chi}{q_\psi} k_1 + \frac{l}{(N_\psi p_\psi + N_\chi p_\chi) q_\psi} \left(-p_\chi \frac{N_\chi}{r} - p_\psi \frac{N_\psi}{r} \right) \right)} \chi \,, \\
= e^{2\pi i \left(\frac{q_\chi}{q_\psi} k_1 - \frac{l}{r q_\psi} \right)} \chi \,,
\end{aligned} \tag{A.3}$$

where $r = \gcd(N_\psi, N_\chi)$ and we used $q_\psi = -\frac{N_\chi}{r}$ and $q_\chi = \frac{N_\psi}{r}$. We can rewrite $l = m_1 + m_2 r$, where $m_1 = 0, 1, \ldots, r-1$ and $m_2 \in \mathbb{Z}$. Bezout's theorem also tells us that since $r =$

$\gcd(N_\psi, N_\chi)$, there are integers k_1, k_2 such that $m_2 r = k_1 N_\psi + k_2 N_\chi$. Applying this to the transformation of χ gives us:

$$\begin{aligned}\chi \longrightarrow e^{2\pi i\left(\frac{q_\chi}{q_\psi}k_1 - \frac{l}{rq_\psi}\right)}\chi &= e^{2\pi i\left(\frac{q_\chi}{q_\psi}k_1 - \frac{m_1+m_2 r}{rq_\psi}\right)}\chi = e^{2\pi i\left(\frac{m_1+m_2 r}{N_\chi} - \frac{N_\psi}{N_\chi}k_1\right)}\chi \\ &= e^{2\pi i\frac{m_1}{N_\chi}}e^{2\pi i\frac{m_2 r - N_\psi k_1}{N_\chi}}\chi = e^{2\pi i\frac{m_1}{N_\chi}}e^{2\pi i k_2}\chi = e^{2\pi i\frac{m_1}{N_\chi}}\chi\,. \end{aligned} \tag{A.4}$$

Since $m_1 = 0, 1, \ldots, r-1$, there are only r distinct transformations generated by $\mathrm{U}(1)_A \times \mathbb{Z}_{N_\psi p_\psi + N_\chi p_\chi}$, and the symmetry group that acts on χ is $\mathbb{Z}_r$. For our purposes, we will assume that under $\mathbb{Z}_r$, χ transforms with charge 1 (in principle, we could fix any charge). Finally, one needs to check whether this $\mathbb{Z}_r$ is a genuine symmetry in the sense that it cannot be absorbed in the center of color or flavor groups. This will be done on a case-by-case basis.

B The 3-loop β-function and the IR fixed points

The 3-loop β function is given by (see [41–43])

$$\begin{aligned}\beta(g) &= -\beta_0\frac{g^3}{(4\pi)^2} - \beta_1\frac{g^5}{(4\pi)^4} - \beta_2\frac{g^7}{(4\pi)^6}\,, \\ \beta_0 &= \frac{11}{6}C_2(G) - \sum_{\mathcal{R}}\frac{1}{3}T_{\mathcal{R}}n_{\mathcal{R}}\,, \\ \beta_1 &= \frac{34}{12}C_2^2(G) - \sum_{\mathcal{R}}\left\{\frac{5}{6}n_{\mathcal{R}}C_2(G)T_{\mathcal{R}} + \frac{n_{\mathcal{R}}}{2}C_2(\mathcal{R})T_{\mathcal{R}}\right\}\,, \\ \beta_2 &= \frac{2857}{432}C_2^3(G) - \sum_{\mathcal{R}}\frac{n_{\mathcal{R}}T_{\mathcal{R}}}{4}\left[-\frac{C_2^2(\mathcal{R})}{2} + \frac{205C_2(G)C_2(\mathcal{R})}{36} + \frac{1415C_2^2(G)}{108}\right] \\ &\quad + \sum_{\mathcal{R},\mathcal{R}'}\frac{n_{\mathcal{R}}n'_{\mathcal{R}}T_{\mathcal{R}}T_{\mathcal{R}'}}{16}\left[\frac{44C_2(\mathcal{R})}{18} + \frac{158C_2(G)}{54}\right]\,. \end{aligned} \tag{B.1}$$

Here, G denotes the adjoint representation, and $n_{\mathcal{R}}$ is the number of the Weyl flavors in representation $\mathcal{R}$. Also, $C_2(\mathcal{R})$ is the quadratic Casimir operator of representation $\mathcal{R}$, defined as

$$t^a_{\mathcal{R}}t^a_{\mathcal{R}} = C_2(\mathcal{R})\mathbf{1}_{\mathcal{R}}\,. \tag{B.2}$$

We reserve $C_2(G)$ for the quadratic Casimir of the adjoint representation. $T_{\mathcal{R}}$ is the Dynkin index of $\mathcal{R}$, which is defined by

$$\mathrm{tr}\left[t^a_{\mathcal{R}}t^b_{\mathcal{R}}\right] = T_{\mathcal{R}}\delta^{ab}\,. \tag{B.3}$$

From eqs. (B.2) and (B.3), we easily obtain the useful relation

$$T_{\mathcal{R}}\mathrm{dim}_G = C_2(\mathcal{R})\mathrm{dim}_{\mathcal{R}}\,, \tag{B.4}$$

where $\mathrm{dim}_{\mathcal{R}}$ is the dimension of $\mathcal{R}$.

In particular, we have $C_2(G) = 2N$, $\dim_G = N^2 - 1$, $T_\psi = N + 2$, $\dim_\psi = \frac{N(N+1)}{2}$, $C_2(\psi) = \frac{2(N+2)(N-1)}{N}$, $T_\chi = N - 2$, $\dim_\chi = \frac{N(N-1)}{2}$, $C_2(\chi) = \frac{2(N-2)(N+1)}{N}$. Then, the values of β_0 to β_2 are

$$\begin{aligned}\beta_0 &= \frac{1}{3}\left[11N - \frac{2}{k}(N^2 - 8)\right] ,\\ \beta_1 &= \frac{2\left(-48 + 76N^2 + 17kN^3 - 8N^4\right)}{3kN} ,\\ \beta_2 &= \frac{1}{54k^2N^2}\Big[2857k^2N^5 + N(-8448 + 12448N^2 - 2584N^4 + 145N^6)\\ &\quad - 2k(864 + 3948N^2 - 8945N^4 + 988N^6)\Big] .\end{aligned} \tag{B.5}$$

Assuming that $\beta_0 > 0$ and $\beta_1 < 0$, the theory develops an IR fixed point to 2-loops. The value of the coupling constant at the fixed point is

$$\alpha_* \equiv \frac{g_*^2}{4\pi} = -\frac{4\pi\beta_0}{\beta_1} = \frac{2\pi N\left(16 + 11kN - 2N^2\right)}{48 - 76N^2 - 17kN^3 + 8N^4} . \tag{B.6}$$

To assess the stability of this fixed point, we can examine the roots of the β-function when the 3-loop term is taken into account.

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