A study of Feynman integrals with uniform transcendental weights and their symbology

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ABSTRACT: Multi-loop Feynman integrals are key objects for the high-order correction computations in high energy phenomenology. These integrals with multiple scales may have complicated symbol structures, and we show that twistor geometries of closely related dual conformal integrals shed light on their alphabet and symbol structures. In this paper, first, as a cutting-edge example, we derive the two-loop four-external-mass Feynman integrals with uniform transcendental (UT) weights, based on the latest developments on UT integrals. Then we find that all the symbol letters of these integrals can be explained nontrivially by studying the so-called Schubert problem of certain dual conformal integrals with a point at infinity. Certain properties of the symbol such as first two entries and extended Steinmann relations are also studied from analogous properties of dual conformal integrals.

Contents

1 Introduction

With the start of the run III stage of the Large Hadron Collider (LHC), demands for the high-order perturbative computations of the precision physics get higher. For the high-order perturbation computations in quantum field theory, multiloop Feynman integrals are the key objects which are difficult to reduce or evaluate.

Frequently multiloop Feynman integrals with multiple scales (Mandelstam variables or mass parameters) appear in high-order perturbative computations. These integrals, even if they can be expressed in term of polylogarithm functions, their symbol structure [1] would be very complicated: the alphabet (the list of symbol letters) may be very long and contains complicated square roots of the kinematics variables; the adjacency condition of letters in the symbols can be intricate. On the other hand, the information on the alphabet and symbol of multi-loop Feynman integrals are invaluable. It enables the simplification

algorithm of polylogarithm functions [1–3], the bootstrap of supersymmetric Yang-Mills amplitudes [4–8], the bootstrap of Feynman integrals [9], the finite-field interpolation of the canonical differential equation [10] and a lot of other important applications. Therefore, it is of great interests to study the alphabet and symbol structures of multi-loop Feynman integrals.

In the literature there are many interesting studies of the theoretical properties of alphabet and symbol structures, for example, the so-called first entry-condition [11], the relation between symbols and Landau Singularity [12, 13], the relation between the second entry condition and the Steinmann relation [7, 14], and the final entry condition [15]. The cluster algebra structure was applied for the computation of the symbol structure in $\mathcal{N} = 4$ SYM theory [16, 17]. Recently, the refs. [18–20] studied multi-loop Feynman integrals' alphabet from the cluster algebra structure. Besides these exciting developments, the multi-scale multi-loop Feynman integrals' alphabet and symbol structures still have a lot of mysteries. In this paper, we try to illustrate the alphabet and symbol structures for Feynman integrals with uniform transcendental (UT) weight [21, 22], from the viewpoint of dual conformal symmetry [23–26].

The method [21, 22] of applying UT Feynman integrals and the corresponding differential equation is currently a standard way of evaluating Feynman integrals analytically. In this paper, we derive a cutting-edge example of the UT Feynman integral basis for the two-loop double box diagrams with four different massive external legs, and then show that its alphabet and symbol structure can be well understood from the dual conformal properties. This family is particularly interesting because it is a sub-family of the two-loop eight-point massless diagrams, and also a sub-family of the two-loop five-point diagrams with three massive external legs. So, this family is important both for formal theories like $\mathcal{N} = 4$ SYM and also the phenomenology. It contains two Mandelstam variables as well as four mass parameters, thus the kinematics is very complicated. It is not an easy task to find the corresponding UT basis, so we turn to the latest UT basis determination methods [27, 28] with the leading singularity analysis in the Baikov representation. The complete UT basis for this family is found, and the canonical differential equation is calculated analytically via the finite-field method [10]. From the canonical differential equation, we obtained the full alphabet. Furthermore, the symbols of these UT integrals are calculated. As expected for other UT integrals with multiple scales, the alphabet is long and contains many square roots in the kinematic variables. The symbol structure for this family is also complicated.

Therefore, in this paper, we propose to analyze the symbology of these Feynman integrals, namely the alphabet and properties of the symbols, from certain *dual conformal invariant* (DCI) Feynman integrals [29] with a dual point at infinity. Similar considerations have appeared in [18] for one-mass five-point process, whose symbology is closely related to that of eight-point DCI integrals. The symbology of DCI integrals, strictly in $D = 4$, is much better understood by explicit computations [30–35] as well as by Landau analysis [12, 13, 36, 37], considerations based on cluster algebras [18–20, 38] etc. Moreover, properties of the symbols such as conditions on first two entries and (extended) Steinmann relations have also been studied more extensively for DCI integrals (see [39] and references therein). By sending a generic dual point, which are not null separated from adjacent points, to infinity,

we obtain non-DCI integrals if the latter are finite. There is strong evidence [18] that such non-DCI limits are still relevant for IR divergent integrals in dimensional regularization, and we find it very rewarding to further study the implications of DCI results in $D = 4$ for more general Feynman integrals in $D = 4 - 2\epsilon$. The method of symbology in this paper, which was proposed very recently in [40, 41], is based on analysis of geometric configurations of DCI integrals in (momentum-)twistor space, known as Schubert problems. It has been successfully applied for predicting alphabets of DCI integrals and amplitudes up to $n = 9$ [40], as well as those of multi-loop Feynman integrals in dimensional regularization.

For our integrals with four external masses, it is necessary to consider fully massive DCI pentagon kinematics, which depends on five cross-ratios and reduces to four-mass (non-DCI) kinematics by sending any dual point to infinity. In such limits, the five cross-ratios reduce to ratios of kinematic invariants, $s, t, m_1^2, \cdots, m_4^2$. It is highly non-trivial that all the 68 letters can be explained from Schubert problems of such DCI integrals in $D = 4$. Out of all 12 non-trivial rational letters, 11 of them turn out to be (the square of) leading singularities associated with one- and two-loop DCI integrals, including four-mass boxes and double-boxes (the last one is a Gram determinant which can be obtained from following odd letters). The remaining 50 letters are parity odd with respect to the 11 square roots, which fall into two classes. There are 16 of them that has only one square root, which can all be identified with algebraic letters of four-mass boxes and two-loop generalizations; for the 34 "mixed" odd letters that depends on two square roots, we find their origin from twistor geometries of corresponding one- and two-loop DCI integrals. The appearance of these letters in the matrix of canonical DE also exhibit nice patterns. Moreover, we also find that, to all orders in ϵ, the first two entries of the symbols of such integrals can always be written as linear combinations of 5 four-mass box functions (and log log terms), which provides an explanation of the first-two entry conditions. Relatedly, we have also checked that extended Steinmann relations are satisfied by all the integrals, which follows from properties of canonical DE.

This paper is organized as: in section 2 we review the UT integrals, canonical differential equation and symbol. In section 3, we introduce the main example of two-loop double box diagram with four different external mass, derive the UT basis, canonical differential equation and the symbols. In section 4, we will first review basics of DCI integrals, focusing on one- and two-loop integrals which depend on fully-massive pentagon kinematics; we then explain that by taking any dual point to infinity, letters of DCI integrals nicely become those of double-box etc., including all square roots and odd letters; finally, we will comment on how certain properties of the symbols can be explained from DCI considerations. In section 5, we summarize this paper and provide an outlook for the future application of the DCI properties on the multi-loop UT integrals. More technical details are given in the appendices of this paper.

2 A review of differential equations, UT integrals and symbol

2.1 The canonical differential equation

Differential equation method is one of the most popular method of evaluating Feynman integrals [42–44]. The differential equation of a Feynman integral basis I, via the integration-

$f(x)$	$\mathcal{T}(f(x))$	$f(x)$	$\mathcal{T}(f(x))$
rational number	0	rational function	0
π	1	$\mathrm{Log}(x)$	1
ζ_n	n	$\mathrm{Li}_n(x)$	n
$H(a_1, \cdots, a_n; x)$	n	$G(a_1, \cdots, a_n; x)$	n

Table 1. Transcendental weight of some functions.

by-parts reduction, reads

$$\frac{\partial}{\partial x_i} I = A_i(\epsilon, x) I, \tag{2.1}$$

where x_i's are kinematic variables and A_i's are matrices for the differential equations.

Usually, solving the differential equation in (2.1) is difficult. A better choice of the integral basis will make it dramatically easier. One of the best choices of integral basis consists of integrals with uniformal transcendental (UT) weights [21]. These UT integrals have the following form in terms of ϵ expansion as,

$$I = \epsilon^k \sum_{i=0}^{\infty} I^{(n)} \epsilon^n, \tag{2.2}$$

where $I^{(n)}$ is a pure weight-n transcendental function, i.e. $\mathcal{T}(I^{(n)}) = n$ and $\mathcal{T}(\frac{\partial}{\partial x} I^{(n)}) = n - 1$, where $\mathcal{T}$ stands for the transcendental weight of a function. The transcendental weight of some common functions are shown in table 1, where $\mathrm{Li}_n(x)$, $H(\{a_1, \dots a_n\}, x)$ and $G(\{a_1, \dots a_n\}, x)$ are, respectively, the weight n classical, harmonic and Goncharov polylogarithm functions.

The differential equations for a basis formed by UT integrals (2.2) are called canonical differential equations. For the matrices of canonical differential equations, the ϵ parameter factorizes out as [21],

$$\frac{\partial}{\partial x_i} I = \epsilon A_i(x) I. \tag{2.3}$$

This is called the canonical differential equation. Consequently, we can solve the differential equations order by order in ϵ expansion as,

$$\frac{\partial}{\partial x_i} I^{(k)} = A_i I^{(k-1)}. \tag{2.4}$$

From the integrability condition, the canonical differential equation matrices for UT basis satisfies,

$$[A_i, A_j] = 0, \quad \frac{\partial}{\partial x_i} A_j - \frac{\partial}{\partial x_j} A_i = 0. \tag{2.5}$$

From the second property, we know that,

$$A_i = \frac{\partial}{\partial x_i} \tilde{A}, \tag{2.6}$$

where the matrix $\tilde{A}$ takes the form as

$$\tilde{A} = \sum_{i=1}^{N} a_i \log(W_i) \,, \tag{2.7}$$

where a_i's are matrices of rational numbers, and W_i's are algebraic functions of x_i's, called *symbol letters*. With the form of (2.7), the canonical equation (2.3) can be solved analytically with the information of the boundary condition. On the other hand, one can easily derive the *symbol* [1] of the solutions of the differential equation.

The symbol $\mathcal{S}$ of a function is defined as,

$$\mathcal{S}(\log R) \equiv S[R], \tag{2.8}$$

and

$$\mathcal{S}(F) \equiv \sum_{i} c_i \mathcal{S}(F_i) \otimes S[R_i], \tag{2.9}$$

if

$$dF = \sum_{i} c_i F_i d \log R_i, \tag{2.10}$$

where R and R_i are rational functions and $S[R_1, \cdots, R_n] \otimes S[R] \stackrel{\text{def}}{=} S[R_1, \cdots, R_n, R]$. With these, one can immediately derive the symbols of the solutions from the canonical differential equations as

$$\mathcal{S}(I^{(n)}) = \sum_{i_1, \cdots, i_n = 1}^{N} a_{i_n} \cdots a_{i_1} I^{(0)} S[W_{i_1}, \ldots, W_{i_n}] \,, \tag{2.11}$$

2.2 UT integral determination

For applying the method of canonical differential equation, finding a corresponding UT basis is not a trivial task. There are many methods and algorithms designed for determining a UT basis, including the leading singularity analysis [21, 22, 27], the Magnus and Dyson Series [45], the dlog integrand construction [46–49], the intersection theory [50–53], the initial algorithm [54], the Lee's algorithm [55–57] and so on. Based on these methods or algorithms, some public available packages were designed including Canonica [58, 59], Fuchsia [60], epsilon [61], initial [54] and libra [57]. In this paper, we are using the leading singularity analysis method in Baikov representation [27, 28, 48, 62] as well as other methods to determine a UT basis.

Here we briefly introduce the leading singularity method in Baikov representation [63–65]. For a Feynman integral defined as

$$G_{\alpha_1, \cdots, \alpha_n} \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D l_i}{i \pi^{D/2}} \frac{1}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}}, \tag{2.12}$$

in the Baikov representation, it is

$$G_{\alpha_1, \cdots, \alpha_n} = C_E^L U^{\frac{E-D+1}{2}} \int_{\Omega} \mathrm{d}z_1 \cdots \mathrm{d}z_n P(z)^{\frac{D-L-E-1}{2}} \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}. \tag{2.13}$$

In this representation, E and L are the number of independent external momenta and loop momenta respectively, C_E^L is a constant irrelevant to kinematic variables, and U and P are Gram diterminants as

$$U = \det G \begin{pmatrix} p_1, \ \dots \ p_E \\ p_1, \ \dots \ p_E \end{pmatrix}, \tag{2.14}$$

$$P = \det G \begin{pmatrix} l_1, \ \dots \ l_L, \ p_1, \ \dots \ p_E \\ l_1, \ \dots \ l_L, \ p_1, \ \dots \ p_E \end{pmatrix}. \tag{2.15}$$

The idea of using Baikov leading singularity method to determine UT integrals is based on an conjecture [51] that an integral with constant leading singularity should be a UT integral. In the Baikov representation, the leading singularity of an integral can be derived using the cut [66–68], which is to replace the integration by taking residues at $z_i \to 0$, schematically as

$$\text{LS} = \oint_{z_i \to 0} \mathrm{d}z_1 \cdots \mathrm{d}z_n P(z)^{\frac{D-L-E-1}{2}} \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}. \tag{2.16}$$

For multi-loop integrals, a more practical way to derive the leading singularities is the loop-by-loop Baikov representations [68, 69]. Take the double box with 4 external masses diagram (to be introduced in section 3.2 in detail) as an example. The propagator and kinematic information are given in (3.1) and (3.9). Consider the scalar integral in the top sector in (3.10), which is

$$J_1 = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \frac{\mathrm{d}^D l_2}{i\pi^{D/2}} \frac{1}{D_1 \cdots D_7}, \tag{2.17}$$

one can compute the leading singularity from full Baikov representation introduced in (2.13). In comparison, a more computationally economical way is to derive the Baikov representation loop by loop: first we treat l_1 as loop momentum and l_2, p_1, p_2, and p_4 as external momenta, so that

$$\begin{aligned} J_1 &= \int \frac{\mathrm{d}^D l_2}{i\pi^{D/2}} \frac{1}{D_4 D_5 D_6} \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \frac{1}{D_1 D_2 D_3 D_7} \\ &= C \int \frac{\mathrm{d}^D l_2}{i\pi^{D/2}} \frac{1}{D_4 D_5 D_6} U_1^{\frac{E-D+1}{2}} \int_\Omega \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \mathrm{d}z_7 P_1(z)^{\frac{D-L-E-1}{2}} \frac{1}{z_1 z_2 z_3 z_7}. \end{aligned} \tag{2.18}$$

Here, $E = 3$, $L = 1$, $D = 4$, C is some constant irrelevant to kinematics and

$$P_1 = \det G \begin{pmatrix} l_1, \ p_1, \ p_2, \ l_2 \\ l_1, \ p_1, \ p_2, \ l_2 \end{pmatrix}. \tag{2.19}$$

Then the leading singularity for the "left" loop integral is

$$\oint \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \mathrm{d}z_7 P_1(z)^{\frac{D-L-E-1}{2}} \frac{1}{z_1 z_2 z_3 z_7} = \frac{4}{\Delta}, \tag{2.20}$$

where

$$\Delta^2 = \frac{1}{16} \left(m_1^4 D_4^2 + \left(m_2^2 D_6 - s D_9 \right)^2 - 2 m_1^2 D_4 \left(m_2^2 D_6 + s D_9 \right) \right). \tag{2.21}$$

With the definition $D_9 = (l_2 - p_1)^2$, we can similarly derive the leading singularity for the "right" loop as

$$
\begin{aligned}
\mathrm{LS}(J_1) &= \mathrm{LS}\left(\int \frac{\mathrm{d}^D l_2}{i\pi^{D/2}} \frac{4}{D_4 D_5 D_6 \Delta} \right) \\
&= \oint_0 \mathrm{d}z_4 \mathrm{d}z_5 \mathrm{d}z_6 \mathrm{d}z_9 P_2(z)^{\frac{D-L-E-1}{2}} \frac{1}{z_4 z_5 z_6 \Delta(z)} = \frac{16}{sr_1},
\end{aligned}
\tag{2.22}
$$

where

$$
r_1^2 = s^2 t^2 - 2st m_1^2 m_3^2 + m_1^4 m_3^4 - 2st m_2^2 m_4^2 - 2 m_1^2 m_2^2 m_3^2 m_4^2 + m_2^4 m_4^4.
\tag{2.23}
$$

Requiring the leading singularity of this integral to be constant, we guess that $sr_1 J_1$ is a UT integral candidate. Later, from the differential equations, we verify that it indeed has the uniform transcendental weights.

3 Feynman integrals with four different external masses

In this paper, we study the one and two-loop Feynman integrals with four different external masses. These integrals are interesting because they appear in the eight-point massless scattering processes which is the focus of the amplitude study of formal theories, and also in the five-point scattering processes with three external masses. These integrals have rich structures related to dual conformal invariance.

The kinematic condition for these integrals is

$$
p_1^2 = m_1^2, \quad p_2^2 = m_2^2, \quad p_3^2 = m_3^2, \quad p_4^2 = m_4^2, \quad (p_1 + p_2)^2 = s, \quad (p_2 + p_3)^2 = t,
\tag{3.1}
$$

with $p_1 + p_2 + p_3 + p_4 = 0$.

3.1 One loop UT integral basis

The propagators of one-loop Feynman integral family with four different external masses are:

$$
D_1 = l_1^2, \quad D_2 = (l_1 - p_1)^2, \quad D_3 = (l_1 - p_1 - p_2)^2, \quad D_4 = (l_1 + p_4)^2.
\tag{3.2}
$$

There are 11 master integrals after the IBP reduction.

$$
\begin{aligned}
&G_{1,1,1,1}, \qquad G_{1,1,1,0}, \qquad G_{1,1,0,1}, \qquad G_{1,0,1,1}, \qquad G_{0,1,1,1}, \\
&G_{1,2,0,0}, \qquad G_{1,0,2,0}, \qquad G_{1,0,0,2}, \qquad G_{0,1,2,0}, \qquad G_{0,1,0,2}, \qquad G_{0,0,1,2}.
\end{aligned}
\tag{3.3}
$$

These master integrals are almost UT integrals up to some rational function factors in kinematic variables. It is not difficult to determine these factor by looking at the differential
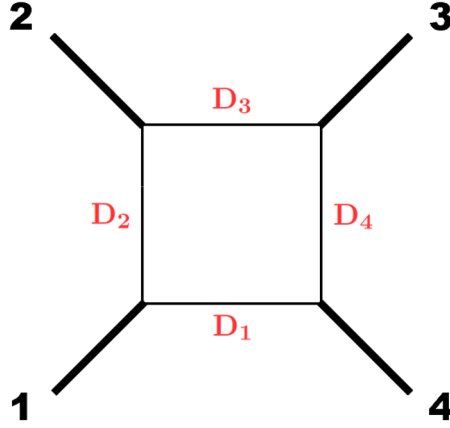
Figure 1. 1-loop diagram with 4 external massive legs.

equation, then the UT integral basis of this four masses box diagram family is:

$$I_1 = r_1 G_{1,1,1,1}, \qquad I_2 = r_2 G_{1,1,1,0}, \qquad I_3 = r_3 G_{1,1,0,1}, \qquad (3.4)$$

$$I_4 = r_4 G_{1,0,1,1}, \qquad I_5 = r_5 G_{0,1,1,1}, \qquad I_6 = \frac{m_1^2 G_{1,2,0,0}}{\epsilon}, \qquad (3.5)$$

$$I_7 = \frac{s G_{1,0,2,0}}{\epsilon}, \qquad I_8 = \frac{m_4^2 G_{1,0,0,2}}{\epsilon}, \qquad I_9 = \frac{m_2^2 G_{0,1,2,0}}{\epsilon}, \qquad (3.6)$$

$$I_{10} = \frac{t G_{0,1,0,2}}{\epsilon}, \qquad I_{11} = \frac{m_3^2 G_{0,0,1,2}}{\epsilon}. \qquad (3.7)$$

Where we used the following roots, $r_1 \sim r_5$,

$$
\begin{aligned}
r_1^2 &= s^2 t^2 - 2stm_1^2 m_3^2 + m_1^4 m_3^4 - 2stm_2^2 m_4^2 - 2m_1^2 m_2^2 m_3^2 m_4^2 + m_2^4 m_4^4, \\
r_2^2 &= s^2 - 2sm_1^2 + m_1^4 - 2sm_2^2 - 2m_1^2 m_2^2 + m_2^4, \\
r_3^2 &= t^2 - 2tm_1^2 + m_1^4 - 2tm_4^2 - 2m_1^2 m_4^2 + m_4^4, \\
r_4^2 &= s^2 - 2sm_3^2 + m_3^4 - 2sm_4^2 - 2m_3^2 m_4^2 + m_4^4, \\
r_5^2 &= t^2 - 2tm_2^2 + m_2^4 - 2tm_3^2 - 2m_2^2 m_3^2 + m_3^4.
\end{aligned}
\qquad (3.8)
$$

3.2 Two loop UT integral basis

The propagators of two-loop Feynman integral family with four different external masses are:

$$
\begin{aligned}
D_1 &= l_1^2, & D_2 &= (l_1 - p_1)^2, & D_3 &= (l_1 - p_1 - p_2)^2, \\
D_4 &= (l_2 + p_1 + p_2)^2, & D_5 &= (l_2 + p_1 + p_2 + p_3)^2, & D_6 &= l_2^2, \\
D_7 &= (l_1 + l_2)^2, & D_8 &= (l_1 - p_1 - p_2 - p_3)^2, & D_9 &= (l_2 + p_1)^2.
\end{aligned}
\qquad (3.9)
$$

The top sector, $(1, 1, 1, 1, 1, 1, 1, 0, 0)$ for this family, is a double box diagram shown in figure 2. Here D_8 and D_9 are irreducible scalar products (ISPs). From the stand IBP reduction procedure, we see that the number of master integrals for this family is 74. The
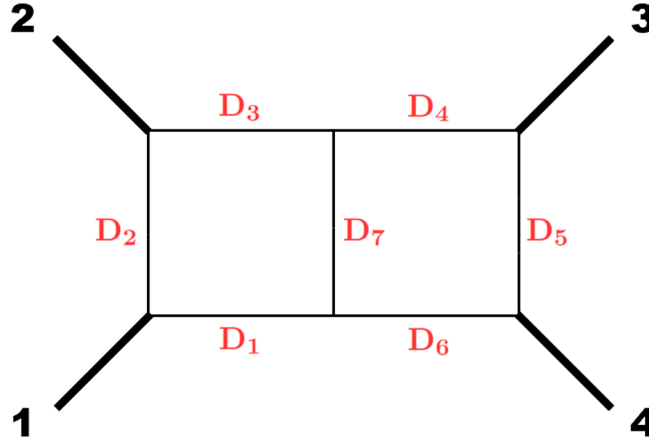
Figure 2. 2-loop diagram with 4 external massive legs.

diagrams for all master integrals are given in the appendix A. Some of the master integrals in the family were calculated in [70].

Using the Baikov leading singularity analysis techniques described in the previous section, as well as other methods that will be explained later, we have derived the UT basis for this diagram, as follows:

$$I_1 = sr_1 G_{1,1,1,1,1,1,1,1,0,0}, \tag{3.10}$$

$$I_2 = r_4 \left(-m_1^2 G_{1,1,1,0,1,1,1,0,0} - m_2^2 G_{1,1,1,1,1,0,1,0,0} + s G_{1,1,1,1,1,1,1,0,-1} \right), \tag{3.11}$$

$$I_3 = r_2 \left(-m_3^2 G_{0,1,1,1,1,1,1,0,0} - m_4^2 G_{1,1,0,1,1,1,1,0,0} + s G_{1,1,1,1,1,1,1,-1,0} \right), \tag{3.12}$$

$$I_4 = s G_{1,1,1,1,1,1,1,1,-1,-1} + \frac{1}{2} s \left(s - m_1^2 - m_2^2 \right) G_{1,1,1,1,1,1,1,1,-1,0}$$

$$+ \frac{1}{2} s \left(s - m_3^2 - m_4^2 \right) G_{1,1,1,1,1,1,1,1,0,-1} + \cdots \tag{3.13}$$

$$I_5 = r_9 G_{0,1,1,1,1,1,1,1,0,0}, I_6 = r_8 G_{1,1,0,1,1,1,1,0,0}, I_7 = r_{11} G_{1,1,1,0,1,1,1,0,0}, \tag{3.14}$$

$$I_8 = r_{10} G_{1,1,1,1,1,0,1,0,0}, I_9 = r_2 r_4 G_{1,1,1,1,1,1,1,0,0,0}, I_{10} = \frac{r_1 G_{0,1,0,1,1,1,2,0,0}}{\epsilon}, \tag{3.15}$$

$$I_{11} = \frac{r_4 G_{0,2,0,1,1,1,1,0,-1}}{\epsilon}, I_{12} = \frac{m_2^2 r_4 G_{0,1,2,0,1,1,1,0,0}}{\epsilon}, I_{13} = \frac{m_2^2 r_3 G_{0,2,1,0,1,1,1,0,0}}{\epsilon}, \tag{3.16}$$

$$I_{14} = \frac{m_4^2 r_2 G_{0,1,1,0,1,2,1,0,0}}{\epsilon}, I_{15} = \frac{m_4^2 r_5 G_{0,1,1,0,2,1,1,0,0}}{\epsilon}, I_{16} = \frac{r_1 G_{0,1,1,0,1,1,2,0,0}}{\epsilon}, \tag{3.17}$$

$$I_{17} = r_6 G_{0,1,1,0,1,1,1,0,0}, \tag{3.18}$$

$$I_{18} = -\frac{\left(st - m_1^2 m_3^2 + m_2^2 m_4^2\right) G_{0,1,1,0,1,1,2,0,0}}{2\epsilon} - \frac{\left(s - m_1^2 + m_2^2\right) m_4^2 G_{0,1,1,0,1,2,1,0,0}}{\epsilon}$$

$$- \frac{m_2^2 \left(s - m_3^2 + m_4^2\right) G_{0,1,2,0,1,1,1,0,0}}{\epsilon} + \frac{s m_2^2 m_4^2 G_{0,1,2,0,1,2,1,0,0}}{\epsilon^2}, \tag{3.19}$$

$$I_{19} = r_2 G_{0,1,1,1,0,1,1,0,0}, I_{20} = r_5 G_{0,1,1,1,1,0,1,0,0}, I_{21} = \frac{m_2^2 r_4 G_{0,1,2,1,1,1,0,0,0}}{\epsilon}, \tag{3.20}$$

$$I_{22} = r_4 G_{1,0,1,0,1,1,1,0,0}, I_{23} = r_4 G_{1,0,1,1,1,0,1,0,0}, I_{24} = s r_4 G_{1,0,1,1,1,1,1,0,0}, \tag{3.21}$$

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$$I_{25} = r_3 G_{1,1,0,0,1,1,1,0,0}, I_{26} = r_2 G_{1,1,0,1,0,1,1,0,0}, I_{27} = \frac{m_1^2 r_4 G_{2,1,0,1,1,0,1,0,0}}{\epsilon}, \tag{3.22}$$

$$I_{28} = \frac{m_1^2 r_5 G_{1,2,0,1,1,0,1,0,0}}{\epsilon}, I_{29} = \frac{m_3^2 r_2 G_{1,1,0,2,1,0,1,0,0}}{\epsilon}, I_{30} = \frac{m_3^2 r_3 G_{1,1,0,1,2,0,1,0,0}}{\epsilon}, \tag{3.23}$$

$$I_{31} = \frac{r_1 G_{1,1,0,1,1,0,2,0,0}}{\epsilon}, I_{32} = r_7 G_{1,1,0,1,1,0,1,0,0}, \tag{3.24}$$

$$I_{33} = -\frac{\left(st + m_1^2 m_3^2 - m_2^2 m_4^2\right) G_{1,1,0,1,1,0,2,0,0}}{2\epsilon} - \frac{\left(s + m_1^2 - m_2^2\right) m_3^2 G_{1,1,0,2,1,0,1,0,0}}{\epsilon}$$
$$- \frac{m_1^2 \left(s + m_3^2 - m_4^2\right) G_{2,1,0,1,1,0,1,0,0}}{\epsilon} + \frac{s m_1^2 m_3^2 G_{2,1,0,2,1,0,1,0,0}}{\epsilon^2}, \tag{3.25}$$

$$I_{34} = \frac{m_1^2 r_4 G_{1,2,0,1,1,1,0,0,0}}{\epsilon}, I_{35} = \frac{r_1 G_{1,1,1,0,1,0,2,0,0}}{\epsilon}, I_{36} = \frac{r_2 G_{1,1,1,0,2,0,1,-1,0}}{\epsilon}, \tag{3.26}$$

$$I_{37} = \frac{m_4^2 r_2 G_{1,1,1,0,2,1,0,0,0}}{\epsilon}, I_{38} = s r_2 G_{1,1,1,1,0,1,1,0,0}, I_{39} = \frac{m_3^2 r_2 G_{1,1,1,2,1,0,0,0,0}}{\epsilon}, \tag{3.27}$$

$$I_{40} = \frac{r_4 G_{0,0,1,0,1,1,2,0,0}}{\epsilon}, I_{41} = \frac{s m_3^2 G_{0,0,2,0,1,1,2,0,0}}{\epsilon^2} - \frac{3\left(s + m_3^2 - m_4^2\right) G_{0,0,1,0,1,1,2,0,0}}{\epsilon}, \tag{3.28}$$

$$I_{42} = \frac{r_3 G_{0,1,0,0,1,1,2,0,0}}{\epsilon}, I_{43} = \frac{t m_1^2 G_{0,2,0,0,1,1,2,0,0}}{\epsilon^2} - \frac{3\left(t + m_1^2 - m_4^2\right) G_{0,1,0,0,1,1,2,0,0}}{\epsilon}, \tag{3.29}$$

$$I_{44} = \frac{r_2 G_{0,1,0,1,0,1,2,0,0}}{\epsilon}, I_{45} = \frac{m_1^2 m_2^2 G_{0,2,0,1,0,1,2,0,0}}{\epsilon^2} - \frac{3\left(-s + m_1^2 + m_2^2\right) G_{0,1,0,1,0,1,2,0,0}}{\epsilon}, \tag{3.30}$$

$$I_{46} = \frac{r_5 G_{0,1,0,1,1,0,2,0,0}}{\epsilon}, I_{47} = \frac{t m_2^2 G_{0,2,0,1,1,0,2,0,0}}{\epsilon^2} - \frac{3\left(t + m_2^2 - m_3^2\right) G_{0,1,0,1,1,0,2,0,0}}{\epsilon}, \tag{3.31}$$

$$I_{48} = \frac{r_2 G_{0,1,1,0,0,1,2,0,0}}{\epsilon}, I_{49} = \frac{s m_1^2 G_{0,1,1,0,0,2,2,0,0}}{\epsilon^2} - \frac{3\left(s + m_1^2 - m_2^2\right) G_{0,1,1,0,0,1,2,0,0}}{\epsilon}, \tag{3.32}$$

$$I_{50} = \frac{r_5 G_{0,1,1,0,1,0,2,0,0}}{\epsilon}, I_{51} = \frac{t m_3^2 G_{0,1,1,0,2,0,2,0,0}}{\epsilon^2} - \frac{3\left(t - m_2^2 + m_3^2\right) G_{0,1,1,0,1,0,2,0,0}}{\epsilon}, \tag{3.33}$$

$$I_{52} = \frac{m_2^2 m_4^2 G_{0,2,1,0,2,1,0,0,0}}{\epsilon^2}, I_{53} = \frac{s m_2^2 G_{0,1,2,1,0,2,0,0,0}}{\epsilon^2}, I_{54} = \frac{m_2^2 m_3^2 G_{0,2,1,1,2,0,0,0,0}}{\epsilon^2}, \tag{3.34}$$

$$I_{55} = \frac{r_4 G_{1,0,0,1,1,0,2,0,0}}{\epsilon}, I_{56} = \frac{s m_4^2 G_{2,0,0,1,1,0,2,0,0}}{\epsilon^2} - \frac{3\left(s - m_3^2 + m_4^2\right) G_{1,0,,1,1,0,2,0,0}}{\epsilon}, \tag{3.35}$$

$$I_{57} = \frac{r_4 G_{1,0,1,0,1,0,2,0,0}}{\epsilon}, I_{58} = \frac{m_3^2 m_4^2 G_{1,0,1,0,2,0,2,0,0}}{\epsilon^2} - \frac{3\left(-s + m_3^2 + m_4^2\right) G_{1,0,1,0,1,0,2,0,0}}{\epsilon}, \tag{3.36}$$

$$I_{59} = \frac{s m_4^2 G_{2,0,1,0,2,1,0,0,0}}{\epsilon^2}, I_{60} = \frac{s^2 G_{1,0,2,1,0,2,0,0,0}}{\epsilon^2}, I_{61} = \frac{s m_3^2 G_{2,0,1,1,2,0,0,0,0}}{\epsilon^2}, \tag{3.37}$$

$$I_{62} = \frac{r_3 G_{1,1,0,0,1,0,2,0,0}}{\epsilon}, I_{63} = \frac{t m_4^2 G_{1,1,0,0,2,0,2,0,0}}{\epsilon^2} - \frac{3\left(t - m_1^2 + m_4^2\right) G_{1,1,0,0,1,0,2,0,0}}{\epsilon}, \tag{3.38}$$

$$I_{64} = \frac{m_1^2 m_4^2 G_{1,2,0,0,2,1,0,0,0}}{\epsilon^2}, I_{65} = \frac{r_2 G_{1,1,0,1,0,0,2,0,0}}{\epsilon}, \tag{3.39}$$

$$I_{66} = \frac{s m_2^2 G_{1,1,0,2,0,0,2,0,0}}{\epsilon^2} - \frac{3\left(s - m_1^2 + m_2^2\right) G_{1,1,0,1,0,0,2,0,0}}{\epsilon}, \tag{3.40}$$

$$I_{67} = \frac{s m_1^2 G_{2,1,0,2,0,1,0,0,0}}{\epsilon^2}, I_{68} = \frac{m_1^2 m_3^2 G_{1,2,0,1,2,0,0,0,0}}{\epsilon^2}, I_{69} = \frac{m_3^2 G_{0,0,2,0,2,0,1,0,0}}{\epsilon^2}, \tag{3.41}$$

$$I_{70} = \frac{t G_{0,2,0,0,2,0,1,0,0}}{\epsilon^2}, I_{71} = \frac{m_2^2 G_{0,2,0,2,0,0,1,0,0}}{\epsilon^2}, I_{72} = \frac{m_4^2 G_{2,0,0,0,2,0,1,0,0}}{\epsilon^2}, \tag{3.42}$$

$$I_{73} = \frac{s G_{0,0,2,0,0,2,1,0,0}}{\epsilon^2}, I_{74} = \frac{m_1^2 G_{0,2,0,0,0,2,1,0,0}}{\epsilon^2}. \tag{3.43}$$

In order to define the UT basis, besides the roots appearing in the one-loop integrals, we need other 6 roots, $r_6 \sim r_{11}$,

$$r_6^2 = s^2 + 2st + t^2 - 2sm_1^2 - 2tm_1^2 + m_1^4 - 2sm_3^2 - 2tm_3^2 + 2m_1^2m_3^2 + m_3^4 - 4m_2^2m_4^2, \quad (3.44)$$

$$r_7^2 = s^2 + 2st + t^2 - 2sm_2^2 - 2tm_2^2 + m_2^4 - 4m_1^2m_3^2 - 2sm_4^2 - 2tm_4^2 + 2m_2^2m_4^2 + m_4^4, \quad (3.45)$$

$$r_8^2 = s^2t^2 - 2s^2tm_1^2 + s^2m_1^4 + 2stm_1^2m_4^2 - 2sm_1^4m_4^2 - 2stm_2^2m_4^2 + 2sm_1^2m_2^2m_4^2$$
$$- 4sm_1^2m_3^2m_4^2 + m_1^4m_4^4 - 2m_1^2m_2^2m_4^4 + m_2^4m_4^4, \quad (3.46)$$

$$r_9^2 = s^2t^2 - 2s^2tm_2^2 + s^2m_2^4 - 2stm_1^2m_3^2 + 2stm_2^2m_3^2 + 2sm_1^2m_2^2m_3^2 - 2sm_2^4m_3^2$$
$$+ m_1^4m_3^4 - 2m_1^2m_2^2m_3^4 + m_2^4m_3^4 - 4sm_2^2m_3^2m_4^2, \quad (3.47)$$

$$r_{10}^2 = s^2t^2 - 2s^2tm_3^2 + 2stm_2^2m_3^2 - 4sm_1^2m_2^2m_3^2 + s^2m_3^4 - 2sm_2^2m_3^4 + m_2^4m_3^4$$
$$- 2stm_2^2m_4^2 + 2sm_2^2m_3^2m_4^2 - 2m_2^2m_3^2m_4^2 + m_2^4m_4^4, \quad (3.48)$$

$$r_{11}^2 = s^2t^2 - 2stm_1^2m_3^2 + m_1^4m_3^4 - 2s^2tm_4^2 + 2stm_1^2m_4^2 - 4sm_1^2m_2^2m_4^2 + 2sm_1^2m_3^2m_4^2$$
$$- 2m_1^4m_3^2m_4^2 + s^2m_4^4 - 2sm_1^2m_4^4 + m_1^4m_4^4. \quad (3.49)$$

Here we remark on how we find this UT integral bases:

1. Use the standard IBP programs [71, 72], we derived the analytic differential equation for the master integrals. For instance, with FIRE6 [72], the computation took about two days with a node of 50 cores. The resulting differential equation is saved for the latter computation to determine some sub-sector the UT integral candidates.

2. Apply the Baikov leading singularity method to determine the UT integrals in the sectors with 7, 6 and 5 propagators.

   - (Top sector) The top sector is $(1,1,1,1,1,1,1,0,0)$ and there are 4 master integrals in this sector. From the experience between UT integrals and the UV finiteness condition, we tend to use

   $$G_{1,1,1,1,1,1,1,0,0}, \quad G_{1,1,1,1,1,1,1,-1,0}, \quad G_{1,1,1,1,1,1,1,0,-1}, \quad G_{1,1,1,1,1,1,1,-1,-1} \quad (3.50)$$

   as UT integral candidates. If necessary, some subsector integrals would also be used.

   From the Baikov leading singularity example (2.22) in the previous section, we estimate that $sr_1G_{1,1,1,1,1,1,1,0,0}$ is a UT integral. To make a UT integral candidate from $G_{1,1,1,1,1,1,1,-1,0}$, based on the experience from [27], we can consider the Baikov integral with the sub-maximal cut. Given $G_{1,1,1,1,1,1,1,-1,0}$, the ansatz for a UT integral can be set as,

   $$f_1G_{1,1,1,1,1,1,1,-1,0} + f_2G_{0,1,1,1,1,1,1,0,0} + f_3G_{1,0,1,1,1,1,1,0,0} + f_4G_{1,1,0,1,1,1,1,0,0}$$
   $$+ f_5G_{1,1,1,0,1,1,1,0,0} + f_6G_{1,1,1,1,0,1,1,0,0} + f_7G_{1,1,1,1,1,0,1,0,0} + f_8G_{1,1,1,1,1,1,0,0,0}.$$

   We use right-to-left Baikov representation on this ansatz to simplify the calculation of leading singularities. Require the leading singularities to be rational

numbers, and then the coefficients $f_1 \sim f_8$ are fixed up to some constants:

$$f_1 = c_1 r_2 s, \qquad f_2 = -c_1 r_2 m_3^2, \qquad f_3 = c_2 r_4 s, \qquad f_4 = -c_1 r_2 m_4^2$$
$$f_5 = c_3 r_{11}, \qquad f_6 = c_4 r_2 s, \qquad f_7 = c_5 r_{10}, \qquad f_8 = c_6 r_2 r_4 \,.$$

$c_1 \sim c_6$ are rational numbers, and $c_1 \neq 0$ since this ansatz is from $G_{1,1,1,1,1,1,1,-1,0}$. We can set $c_2 \sim c_6$ to 0, and $c_1 = 1$ for the simplicity. Finally, we get a UT integral candidate:

$$I_3 = r_2 \left(-m_3^2 G_{0,1,1,1,1,1,1,0,0} - m_4^2 G_{1,1,0,1,1,1,1,0,0} + s G_{1,1,1,1,1,1,1,-1,0} \right). \quad (3.51)$$

With the leading singularity analysis in the Baikov representation, it is also easy to see that

$$\tilde{I}_4 \equiv s G_{1,1,1,1,1,1,1,-1,-1} + \frac{1}{2} s \left(s - m_1^2 - m_2^2 \right) G_{1,1,1,1,1,1,1,-1,0}$$
$$+ \frac{1}{2} s \left(s - m_3^2 - m_4^2 \right) G_{1,1,1,1,1,1,1,0,-1} \quad (3.52)$$

has constant leading singularities, from the *maximal cut* of the Baikov representation. However, later on, from the differential equation computation, we see $\tilde{I}_4$ itself is not a UT integral. To upgrade $\tilde{I}_4$ to a UT integral, subsector integrals should be added. It is a nontrivial computation to determine those subsector integrals, so we keep $\tilde{I}_4$ at this stage and later describe how to get subsector integrals.

We remark that the $D = 4$ leading singularity analysis of integrals in this sector was calculated in the ref. [73].

- (6-propagator sector) It is straightforward to find the UT candidates for 6-propagator sectors by the Baikov leading singularity analysis. For example, the sector $(0, 1, 1, 1, 1, 1, 1, 0, 0)$ contains one master integral $G_{0,1,1,1,1,1,1,0,0}$. By a maximal cut of the loop by loop Baikov representation of $G_{0,1,1,1,1,1,1,0,0}$, it's easy to get a UT integral candidate,

$$I_5 = r_9 G_{0,1,1,1,1,1,1,0,0}. \quad (3.53)$$

- (5-propagator sector) We take the sector $(1, 1, 0, 1, 1, 0, 1, 0, 0)$ as an example. This sector contains 7 master integrals, which is the sector with the largest number of master integrals in this family. To find all the corresponding 7 UT integrals would be challenging. From the loop-by-loop Baikov leading singularity analysis, it is easy to see that

$$I_{32} = \frac{r_7 G_{1,1,0,1,1,0,1,0,0}}{\epsilon} \quad (3.54)$$

is a UT integral candidate. Furthermore, based on the algorithm in [27], we can look for reducible integrals in the super sectors of the sector $(1, 1, 0, 1, 1, 0, 1, 0, 0)$. In this way, we obtained 6 UT integrals candidates, namely I_j, $j = 27, \ldots, 32$. To get the last UT candidate for this sector, we start with $\tilde{I}_{33} = s m_1^2 m_3^2 G_{2,1,0,2,1,0,1,0,0}/\epsilon^2$, an integral with two double propagators. Later on, from the differential equation, we upgrade $\tilde{I}_{33}$ to a UT integral.

3. For sectors with 3 or 4 propagators, it is relatively easy to find the UT integral candidates. We apply the strategy from [21, 22] to add double propagators and an extra factor ϵ^{-1} for each bubble sub-diagram, in order to get UT candidates.

4. Then we transfer the original DE to a DE for the 74 UT integral candidates. The computation for this transformation is difficult, due to the complicated kinematics. Here our strategy in the transformation, is to set s, t, m_1, m_2, m_3 and m_4 as some random integer values while keep ϵ analytically. Then we can identify the entries of the differential equation matrix which are *not* proportional to ϵ. Explicitly, such entries are located in the row and columns of the new differential equation corresponding to $\tilde{I}_4$ and $\tilde{I}_{33}$. This indicates that $\tilde{I}_4$ and $\tilde{I}_{33}$ should be upgraded to UT real integrals.

We observe that entries in the DE matrix's 33-rd row, which correspond to $\tilde{I}_{33}$, are either proportional to ϵ or has the form $a + b\epsilon$. For the latter form, a simple transformation like that in [58],

$$I_{33} \equiv \tilde{I}_{33} + \sum_{j=27,j\neq33}^{74} I_j \int \left(A_{33,j}^{(n)}\big|_{\epsilon\to0}\right)dx_n \qquad (3.55)$$

can remove all the ϵ^0 terms in the 33rd row. Here $x_1, \ldots, x_6$ stand for the variables s, t, m_1, m_2, m_3 and m_4. $A_{i,j}^{(n)}$ is the differential equation matrix element for the derivative in x_n. To apply this transformation, we need to first evaluate $A_{i,j}^{(n)}$ analytically for $i = 33$. The resulting integral,

$$I_{33} = -\frac{\left(st + m_1^2 m_3^2 - m_2^2 m_4^2\right) G_{1,1,0,1,1,0,2,0,0}}{2\epsilon} - \frac{\left(s + m_1^2 - m_2^2\right) m_3^2 G_{1,1,0,2,1,0,1,0,0}}{\epsilon}$$
$$- \frac{m_1^2 \left(s + m_3^2 - m_4^2\right) G_{2,1,0,1,1,0,1,0,0}}{\epsilon} + \frac{sm_1^2 m_3^2 G_{2,1,0,2,1,0,1,0,0}}{\epsilon^2} \qquad (3.56)$$

is indeed a UT integral.

The upgrading of $\tilde{I}_4$ is more involved. The DE matrix's 4-th row contains several entries which are not linear in ϵ, so a transformation like (3.55) is not enough to make the DE canonical. Here our strategy is an induction over the number of mass parameters. We take the limit when some or all of the mass parameter to zero, and reduce $\tilde{I}_4$ to the known UT integral basis [21, 27, 74] in this limit. The reduction coefficients are not all rational constants, so we can by-hand add some integrals to $\tilde{I}_4$ to make the coefficients to be constants, in the partially massless or completely massless limit. After this, the resulting integral should be "closer" to a UT integral and indeed the corresponding DE matrix row is completely linear in ϵ. Then we use a similarity transformation like (3.55) again, to make the whole differential equation matrix proportional to ϵ.

The computer readable UT integrals definitions, as well as the corresponding definition of the square roots, can be found in the "output" folder of the auxiliary files, named "dbox4m_UT.txt" and "rootdef.txt", respectively.

3.3 Canonical differential equations and the alphabet

With $\tilde{I}_4$ and $\tilde{I}_{33}$ upgraded, numerically, we see that the differential equation is canonical. Then the next computation is to get the analytic canonical differential equation. Here we apply the finite-field package FINITEFLOW [10] for this computation, since it is well known that the canonical DE usually has much simpler coefficients than the ordinary DE and the finite field reconstruction is extremely efficient. For this family, the canonical differential equation reconstruction with FINITEFLOW only takes about 14 seconds on a workstation with 50 cores. The analytic expressions of the differential equations can be found in the "output" folder of the auxiliary files, named " UTDE.txt", which is a list of 6 differential equations, with respect to the kinematic variables s, t, and $m_1^2 \sim m_4^2$, respectively.

Schematically, the canonical differential equations for the UT basis can be written as

$$\partial_{x_i} I = \epsilon(\partial_{x_i}\tilde{A})I, \tag{3.57}$$

where

$$\tilde{A} = \sum_i a_i \log(W_i). \tag{3.58}$$

Here a_i's are matrices of rational numbers irrelevant to x_i's. W_i are the symbol letters. After integrating the expressions of the differential equations, we derived $\tilde{A}$. We found that it contains 68 symbol letters. Among the letters, $W_1 \sim W_{18}$ are even letters, which are polynomials of the 6 kinematic variables. The rest 50 letters are odd under the simultaneous sign change of all square roots $r_1, \ldots, r_{11}$. Letters $W_{19} \sim W_{34}$ contains one square root each, and $W_{35} \sim W_{68}$ contains 2 square roots each. Specifically, the even letters are

$$
\begin{aligned}
&W_1 = m_1^2,\ W_2 = m_2^2,\ W_3 = m_3^2,\ W_4 = m_4^2,\ W_5 = s,\ W_6 = t,\\
&W_7 = r_5^2,\ W_8 = r_3^2,\ W_9 = r_2^2,\ W_{10} = r_4^2,\ W_{11} = r_7^2,\ W_{12} = r_6^2,\\
&W_{13} = r_1^2,\ W_{14} = r_8^2,\ W_{15} = r_9^2,\ W_{16} = r_{10}^2,\ W_{17} = r_{11}^2,\\
&W_{18} = s^2t + st^2 - stm_1^2 - stm_2^2 + sm_1^2m_2^2 - stm_3^2 - sm_1^2m_3^2 - tm_1^2m_3^2 + m_1^4m_3^2\\
&\qquad + tm_2^2m_3^2 - m_1^2m_2^2m_3^2 + m_1^2m_3^4 - stm_4^2 + tm_1^2m_4^2 - sm_2^2m_4^2 - tm_2^2m_4^2\\
&\qquad - m_1^2m_2^2m_4^2 + m_2^4m_4^2 + sm_3^2m_4^2 - m_1^2m_3^2m_4^2 - m_2^2m_3^2m_4^2 + m_2^2m_4^4,
\end{aligned}
\tag{3.59}
$$

The odd letters that contain one square root are,

$$
\begin{aligned}
&W_{19} = \frac{f_{19} + r_6}{f_{19} - r_6}, \qquad W_{20} = \frac{f_{20} + r_7}{f_{20} - r_7}, \qquad\qquad W_{21} = \frac{f_{21} + r_2}{f_{21} - r_2}, \qquad W_{22} = \frac{f_{22} + r_4}{f_{22} - r_4},\\
&W_{23} = \frac{f_{23} + r_5}{f_{23} - r_5}, \qquad W_{24} = \frac{f_{24} + r_3}{f_{24} - r_3}, \qquad\qquad W_{25} = \frac{f_{25} + r_1}{f_{25} - r_1},\\
&W_{26} = \frac{f_{26} + (m_1^2 - m_2^2)\,r_2}{f_{26} - (m_1^2 - m_2^2)\,r_2}, \qquad\qquad W_{27} = \frac{f_{27} + (m_3^2 - m_4^2)\,r_4}{f_{27} - (m_3^2 - m_4^2)\,r_4},\\
&W_{28} = \frac{f_{28} + (m_2^2 - m_3^2)\,r_5}{f_{28} - (m_2^2 - m_3^2)\,r_5}, \qquad\qquad W_{29} = \frac{f_{29} + (m_1^2 - m_4^2)\,r_3}{f_{29} - (m_1^2 - m_4^2)\,r_3},\\
&W_{30} = \frac{f_{30} + (m_1^2m_3^2 - m_2^2m_4^2)\,r_1}{f_{30} - (m_1^2m_3^2 - m_2^2m_4^2)\,r_1},
\end{aligned}
\tag{3.60}
$$

$$W_{31} = \frac{f_{31} + \left(st - sm_3^2 + m_2^2 m_3^2 - m_2^2 m_4^2\right) r_{10}}{f_{31} - \left(st - sm_3^2 + m_2^2 m_3^2 - m_2^2 m_4^2\right) r_{10}},$$

$$W_{32} = \frac{f_{32} + \left(-st + m_1^2 m_3^2 + sm_4^2 - m_1^2 m_4^2\right) r_{11}}{f_{32} - \left(-st + m_1^2 m_3^2 + sm_4^2 - m_1^2 m_4^2\right) r_{11}},$$

$$W_{33} = \frac{f_{33} + \left(st - sm_1^2 + m_1^2 m_4^2 - m_2^2 m_4^2\right) r_8}{f_{33} - \left(st - sm_1^2 + m_1^2 m_4^2 - m_2^2 m_4^2\right) r_8},$$

$$W_{34} = \frac{f_{34} + \left(-st + sm_2^2 + m_1^2 m_3^2 - m_2^2 m_3^2\right) r_9}{f_{34} - \left(-st + sm_2^2 + m_1^2 m_3^2 - m_2^2 m_3^2\right) r_9},$$

where

$$
\begin{aligned}
& f_{19} = -s - t + m_1^2 + m_3^2, \ f_{20} = -s - t + m_2^2 + m_4^2, \ f_{21} = s - m_1^2 - m_2^2, \\
& f_{22} = s - m_3^2 - m_4^2, \ f_{23} = t - m_2^2 - m_3^2, \ f_{24} = t - m_1^2 - m_4^2, \\
& f_{25} = st - m_1^2 m_3^2 - m_2^2 m_4^2, \\
& f_{26} = -sm_1^2 + m_1^4 - sm_2^2 - 2m_1^2 m_2^2 + m_2^4, \\
& f_{27} = -sm_3^2 + m_3^4 - sm_4^2 - 2m_3^2 m_4^2 + m_4^4, \\
& f_{28} = -tm_2^2 + m_2^4 - tm_3^2 - 2m_2^2 m_3^2 + m_3^4, \\
& f_{29} = -tm_1^2 + m_1^4 - tm_4^2 - 2m_1^2 m_4^2 + m_4^4, \\
& f_{30} = -stm_1^2 m_3^2 + m_1^4 m_3^4 - stm_2^2 m_4^2 - 2m_1^2 m_2^2 m_3^2 m_4^2 + m_2^4 m_4^4, \\
& f_{31} = s^2 t^2 - 2s^2 t m_3^2 + 2stm_2^2 m_3^2 - 2sm_1^2 m_2^2 m_3^2 + s^2 m_3^4 - 2sm_2^2 m_3^4 + m_2^4 m_3^4 \\
& \qquad - 2stm_2^2 m_4^2 + 2sm_2^2 m_3^2 m_4^2 - 2m_2^4 m_3^2 m_4^2 + m_2^4 m_4^4, \\
& f_{32} = s^2 t^2 - 2stm_1^2 m_3^2 + m_1^4 m_3^4 - 2s^2 t m_4^2 + 2stm_1^2 m_4^2 - 2sm_1^2 m_2^2 m_4^2 \\
& \qquad + 2sm_1^2 m_3^2 m_4^2 - 2m_1^4 m_3^2 m_4^2 + s^2 m_4^4 - 2sm_1^2 m_4^4 + m_1^4 m_4^4, \\
& f_{33} = s^2 t^2 - 2s^2 t m_1^2 + s^2 m_1^4 + 2stm_1^2 m_4^2 - 2sm_1^4 m_4^2 - 2stm_2^2 m_4^2 + 2sm_1^2 m_2^2 m_4^2 \\
& \qquad - 2sm_1^2 m_3^2 m_4^2 + m_1^4 m_4^4 - 2m_1^2 m_2^2 m_4^4 + m_2^4 m_4^4, \\
& f_{34} = s^2 t^2 - 2s^2 t m_2^2 + s^2 m_2^4 - 2stm_1^2 m_3^2 + 2stm_2^2 m_3^2 + 2sm_1^2 m_2^2 m_3^2 - 2sm_2^4 m_3^2 \\
& \qquad + m_1^4 m_3^4 - 2m_1^2 m_2^2 m_3^4 + m_2^4 m_3^4 - 2sm_2^2 m_3^2 m_4^2.
\end{aligned}
\tag{3.61}
$$

The odd letters that contain 2 square roots are

$$W_{35} = \frac{f_{35} + r_2 r_4}{f_{35} - r_2 r_4}, \quad W_{36} = \frac{f_{36} + r_2 r_5}{f_{36} - r_2 r_5}, \quad W_{37} = \frac{f_{37} + r_2 r_6}{f_{37} - r_2 r_6}, \quad W_{38} = \frac{f_{38} + r_2 r_3}{f_{38} - r_2 r_3}, \tag{3.62}$$

$$W_{39} = \frac{f_{39} + r_2 r_7}{f_{39} - r_2 r_7}, \quad W_{40} = \frac{f_{40} + r_3 r_4}{f_{40} - r_3 r_4}, \quad W_{41} = \frac{f_{41} + r_4 r_6}{f_{41} - r_4 r_6}, \quad W_{42} = \frac{f_{42} + r_4 r_5}{f_{42} - r_4 r_5}, \tag{3.63}$$

$$W_{43} = \frac{f_{43} + r_4 r_7}{f_{43} - r_4 r_7}, \quad W_{44} = \frac{f_{44} + r_2 r_9}{f_{44} - r_2 r_9}, \quad W_{45} = \frac{f_{45} + r_2 r_8}{f_{45} - r_2 r_8}, \quad W_{46} = \frac{f_{46} + r_1 r_2}{f_{46} - r_1 r_2}, \tag{3.64}$$

$$W_{47} = \frac{f_{47} + r_2 r_{10}}{f_{47} - r_2 r_{10}}, \quad W_{48} = \frac{f_{48} + r_2 r_{11}}{f_{48} - r_2 r_{11}}, \quad W_{49} = \frac{f_{49} + r_4 r_{11}}{f_{49} - r_4 r_{11}}, \quad W_{50} = \frac{f_{50} + r_4 r_{10}}{f_{50} - r_4 r_{10}}, \tag{3.65}$$

$$W_{51} = \frac{f_{51} + r_1 r_4}{f_{51} - r_1 r_4}, \quad W_{52} = \frac{f_{52} + r_4 r_8}{f_{52} - r_4 r_8}, \quad W_{53} = \frac{f_{53} + r_4 r_9}{f_{53} - r_4 r_9}, \quad W_{54} = \frac{f_{54} + r_3 r_5}{f_{54} - r_3 r_5}, \tag{3.66}$$

$$W_{55} = \frac{f_{55} + r_5 r_6}{f_{55} - r_5 r_6}, \quad W_{56} = \frac{f_{56} + r_3 r_6}{f_{56} - r_3 r_6}, \quad W_{57} = \frac{f_{57} + r_5 r_7}{f_{57} - r_5 r_7}, \quad W_{58} = \frac{f_{58} + r_3 r_7}{f_{58} - r_3 r_7}, \tag{3.67}$$

$$W_{59} = \frac{f_{59} + r_1 r_5}{f_{59} - r_1 r_5}, \quad W_{60} = \frac{f_{60} + r_5 r_9}{f_{60} - r_5 r_9}, \quad W_{61} = \frac{f_{61} + r_5 r_{10}}{f_{61} - r_5 r_{10}}, \quad W_{62} = \frac{f_{62} + r_1 r_3}{f_{62} - r_1 r_3}, \quad (3.68)$$

$$W_{63} = \frac{f_{63} + r_3 r_8}{f_{63} - r_3 r_8}, \quad W_{64} = \frac{f_{64} + r_3 r_{11}}{f_{64} - r_3 r_{11}}, \quad W_{65} = \frac{f_{65} + r_1 r_8}{f_{65} - r_1 r_8}, \quad W_{66} = \frac{f_{66} + r_1 r_9}{f_{66} - r_1 r_9}, \quad (3.69)$$

$$W_{67} = \frac{f_{67} + r_1 r_{10}}{f_{67} - r_1 r_{10}}, \quad W_{68} = \frac{f_{68} + r_1 r_{11}}{f_{68} - r_1 r_{11}}. \quad (3.70)$$

The definition of the polynomials $f_{35} \sim f_{68}$ is in the appendix B.

The computer readable results of the symbol letters and $\tilde{A}$ can be found in the "output" folder of the auxiliary files, named "letterdef.txt" and "Atilde.txt", respectively.

3.4 Symbol structures

The result of the canonical differential equations (3.58) are derived in the last subsection. The integration to get the analytic expressions for the UT integrals is not easy due to the boundary condition. However, deriving the corresponding symbol letters are relatively easier, which can already help us to acquire interesting properties of the analytic expressions. The integrals' symbols can be easily derived from (3.58), using (2.11), or equivalently, the recursion

$$\mathcal{S}(I_i^{(m+1)}) = \sum_k (a_k)_{ij} \Big(\mathcal{S}(I_j^{(m)}) \otimes S[W_k] \Big), \quad (3.71)$$

with the lowest order boundary condition

$$\mathcal{S}(I_i^{(0)}) = I_i^{(0)}. \quad (3.72)$$

Having the specific results for the constant matrices a_i defined in (3.58), in order to derive the symbols for the integrals, we also need the results for $I_i^{(0)}$. These are the coefficients at the order ϵ^{-2L}, being rational constants, where L is the loop order.

The coefficients $I_i^{(0)}$ can be derived analytically from the infrared and collinear regions of Feynman integrals. The resulting coefficients at ϵ^{-4} order, namely $I_i^{(0)}$ satisfy that $I_4^{(0)} = -\frac{3}{2}$, $I_i^{(0)} = 1$ for

$$i \in \{18, 33, 41, 43, 45, 47, 49, 51 \sim 54, 56, 58 \sim 61, 63, 64, 66 \sim 74\}, \quad (3.73)$$

and $I_i^{(0)} = 0$ for other choices of i. These coefficients are consistent with the numeric results obtained using FIESTA [72, 75, 76].

With results of $I_i^{(0)}$ and a_i defined in (3.58), we can use (3.71) and (3.72) to derive the symbols of the UT integrals and higher ϵ orders. Using this method, we derived the symbol letters of the UT integrals up to the weight-4 order. The result can be found in "output" folder of the auxiliary files, named "UTSymbols.txt", where "UTSymbol[k]" stands for $I_i^{(k)}$. From the results, we have observed the following phenomena: firstly, the complexity of the symbol expressions goes rapidly up while the order increases. At the weight-1 order , the symbols are quite simple. There are only 28 integrals having non-zero symbols at this order with totally 64 terms. For example, we have $\mathcal{S}(I_4^{(1)}) = 3S[W_5]$ with 1 term. Among the symbols at this order, the ones with most terms are

$$\mathcal{S}(I_{18}^{(1)}) = S[W_1] - S[W_2] + S[W_3] - S[W_4] - 2S[W_5] \quad (3.74)$$

and

$$\mathcal{S}(I_{33}^{(1)}) = -S[W_1] + S[W_2] - S[W_3] + S[W_4] - 2S[W_5], \tag{3.75}$$

with 5 terms each. At the weight-2 order, there are 56 integrals of non-zero symbols with totally 360 terms. The symbols having the most number of terms are $\mathcal{S}(I_{18}^{(2)})$ and $\mathcal{S}(I_{33}^{(2)})$, with 31 terms each. At the weight-3 order, there are 56 integrals of non-zero symbols with totally 2898 terms. The symbols having the most number of terms are also $\mathcal{S}(I_{18}^{(3)})$ and $\mathcal{S}(I_{33}^{(3)})$, with 213 terms each. At the weight-2 order, there are 74 integrals of non-zero symbols with totally 35673 terms. The symbols having the most number of terms are again $\mathcal{S}(I_{18}^{(4)})$ and $\mathcal{S}(I_{33}^{(4)})$, with 2139 terms each.

Secondly, many integrals have vanishing symbols at certain ϵ orders. At the weight-1 order, we have shown that only 28 integrals are with non-zero symbols. Integrals with the zero symbol at the weight-0 order also have the zero symbol at the order the weight-1 order. The integrals with the zero symbol at weight 1 and 2, are I_i for

$$i \in \{1 \sim 3, 5 \sim 17, 19 \sim 32, 34 \sim 40, 42, 44, 46, 48, 50, 55, 57, 62, 65\}. \tag{3.76}$$

At the weight-2 order only 18 integrals have the zero symbol. They are I_i for

$$i \in \{1 \sim 3, 5 \sim 9, 17, 19, 20, 22 \sim 26, 32, 38\}. \tag{3.77}$$

These integral also have the zero symbol at the weight-3 order. At the weight-4 order, all integrals have nonzero symbols. Note that, (3.77) is a subset of (3.76).

Thirdly, letters cannot appear at arbitrary positions of the symbols. To state this, we the following terminology: the symbols of UT integrals consists of terms proportional to some $S[W_{i_1}, W_{i_2}, \cdots]$, of which we call W_{i_1} the first letter (or letter at the first entry), W_{i_2} the second letter (or letter at second entry) and so on. As we can see from the results, among the 68 symbol letters, only $W_1 \sim W_6$ can appear at the first entries, which are exactly the 6 kinematic variables s, t, m_1^2, m_2^2, m_3^2 and m_4^2. The letters appearing at the second entries are $W_1 \sim W_6$ and $W_{21} \sim W_{30}$.

Moreover, some pairs of letters can never be next to each other in the symbols. We can explain this from (3.71). For constant matrices a_i and a_j, if $a_i\, a_j = a_j\, a_i = 0$, then according to (3.71), W_i and W_j cannot appear at adjacent entries, which means, symbols of the UT integrals contain no term like $S[\cdots, W_i, W_j, \cdots]$ or $S[\cdots, W_j, W_i, \cdots]$, at all orders of ϵ. Then we call that the adjacency between letters W_i and W_j is forbidden. Among all possible adjacencies of 68 letters, which are $68 \times 69/2 = 2346$ in total, only 919 of them are allowed and the rest 1427 are forbidden. The explicit allowed and forbidden letter pairs can be found in "output" folder of the auxiliary files, named "AdjLetters.txt" and "NonAdjLetters.txt", respectively.

3.5 Check with known dual conformal invariant integral result

Some of the symbol derived from canonical differential equations in section 3.4 can be verified with results from the dual conformal invariance (DCI) point of view. (An introduction to DCI integrals is to be given in the next section.) We take I_1 (3.10), which is the scalar

double-box integral (proportional to $G[1, 1, 1, 1, 1, 1, 1, 0, 0]$) as an example. Such a ladder integral has been derived previously (see refs. [20, 70]) as

$$f^{(L)} = \sum_{m=L}^{2L} \frac{m!(-1)^m (\log(-z\bar{z}))^{2L-m}}{L!(m-L)!(2L-m)!} \left(\text{Li}_m(z) - \text{Li}_m(\bar{z}) \right). \tag{3.78}$$

From this result we can derive the corresponding symbols.

Taking the loop number $L = 2$, we get the scalar double box UT integral $f^{(2)}$ at the right hand side. In (3.78), the integral is in two variables z and $\bar{z}$. They are related to the kinematic variables s, t, and $m_1 \sim m_4$ as

$$\frac{z\bar{z}}{(1-z)(1-\bar{z})} = \frac{x_{13}^2 x_{57}^2}{x_{15}^2 x_{37}^2}, \quad \frac{1}{(1-z)(1-\bar{z})} = \frac{x_{17}^2 x_{35}^2}{x_{15}^2 x_{37}^2}, \tag{3.79}$$

where

$$x_{13}^2 = m_1^2, \ x_{57}^2 = m_3^2, \ x_{15}^2 = s, \ x_{37}^2 = t, \ x_{17}^2 = m_4^2, \ x_{35}^2 = m_2^2. \tag{3.80}$$

Thus, we are able to derive the symbol expressions for $f^{(2)}$ in terms of z and $\bar{z}$ and use the relations above to rewrite it in terms of s, t, and $m_1 \sim m_4$. Before this, we need to mention that the DCI result (3.78) is derived at dimension $d = 4$ and this integral is UV and IR finite. From the expression (3.78), we can derive $\mathcal{S}(f^{(2)})$. In the right hand side of (3.78), the presenting transcendental functions are logarithm functions $\log(z)$ and poly logarithm functions $\text{Li}_m(z)$. The corresponding symbols are

$$\mathcal{S}(\log(z)) = S[z], \tag{3.81}$$

and

$$\mathcal{S}(\text{Li}_m(z)) = -S[1 - z, \underbrace{z, \cdots, z}_{(m-1)}]. \tag{3.82}$$

To derive the symbol $\mathcal{S}(f^{(2)})$ from the symbols of above functions, we may need the following properties of symbols: for two functions f_1 and f_2 whose symbols are

$$\mathcal{S}(f_i) = \sum_j c_{ij} S_{ij}[\ldots], \tag{3.83}$$

where $i = 1, 2$, we have the following relations

$$\mathcal{S}(f_1 + f_2) = \mathcal{S}(f_1) + \mathcal{S}(f_2), \tag{3.84}$$

$$\mathcal{S}(f_1 f_2) = \sum_{i,j} c_{1i} c_{2j} \ \text{Shuffle}(S_{1i}[\ldots], S_{2j}[\ldots]). \tag{3.85}$$

The operation *shuffle* for two symbol monomials $S[a_1, \ldots, a_m]$ and $[b_1, \ldots, b_n]$ results in a summation of symbol monomials with a sequence of $(m + n)$ letters which are some permutations of $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ keeping the relative orders of a's and b's unchanged.

Using (3.81)∼(3.85), as well as (3.78), we can calculate the symbol of $f^{(2)}$ as

$$\begin{aligned}
\mathcal{S}(f^{(2)}) = {}& S[1-z,z,z,\bar{z}] + S[1-z,z,\bar{z},z] - S[1-z,z,\bar{z},\bar{z}] + S[1-z,\bar{z},z,z] \\
& - S[1-z,\bar{z},z,\bar{z}] - S[1-z,\bar{z},\bar{z},z] + S[z,1-z,z,z] - S[z,1-z,z,\bar{z}] \\
& - S[z,1-z,\bar{z},z] - S[z,z,1-z,z] + S[z,z,1-\bar{z},\bar{z}] + S[z,1-\bar{z},z,z] \\
& + S[z,1-\bar{z},\bar{z},z] - S[z,1-\bar{z},\bar{z},\bar{z}] - S[z,\bar{z},1-z,z] + S[z,\bar{z},1-\bar{z},\bar{z}] \\
& + S[1-\bar{z},z,z,\bar{z}] + S[1-\bar{z},z,\bar{z},z] - S[1-\bar{z},z,\bar{z},\bar{z}] + S[1-\bar{z},\bar{z},z,z] \\
& - S[1-\bar{z},\bar{z},z,z] - S[1-\bar{z},\bar{z},\bar{z},z] + S[\bar{z},1-z,z,z] - S[\bar{z},1-z,z,z] \\
& - S[\bar{z},1-z,\bar{z},z] - S[\bar{z},z,1-z,z] + S[\bar{z},z,1-\bar{z},\bar{z}] + S[\bar{z},1-\bar{z},z,z] \\
& + S[\bar{z},1-\bar{z},\bar{z},z] - S[\bar{z},1-\bar{z},\bar{z},\bar{z}] - S[\bar{z},\bar{z},1-z,z] + S[\bar{z},\bar{z},1-\bar{z},\bar{z}].
\end{aligned} \tag{3.86}$$

This result can also be found in "output" folder of the auxiliary files, named "f2Symbol.txt".

In order to compare with the symbol from DCI with that from the canonical differential equations of the section 3.4, we consider the relations between the two languages introduced in (3.79) and (3.80). After some simple calculations, we get the relations of the corresponding symbols,

$$\begin{aligned}
\text{dlog}(z) &= \frac{1}{2}\Big(\text{dlog}W_1 + \text{dlog}W_3 - \text{dlog}W_2 - \text{dlog}W_4 - \text{dlog}W_{25}\Big), \\
\text{dlog}(1-z) &= \frac{1}{2}\Big(\text{dlog}W_5 + \text{dlog}W_6 - \text{dlog}W_2 - \text{dlog}W_4\Big) + \frac{1}{4}\Big(\text{dlog}W_{30} - \text{dlog}W_{25}\Big), \\
\text{dlog}(\bar{z}) &= \frac{1}{2}\Big(\text{dlog}W_1 + \text{dlog}W_3 + \text{dlog}W_{25} - \text{dlog}W_2 - \text{dlog}W_4\Big), \\
\text{dlog}(1-\bar{z}) &= \frac{1}{2}\Big(\text{dlog}W_5 + \text{dlog}W_6 - \text{dlog}W_2 - \text{dlog}W_4\Big) + \frac{1}{4}\Big(\text{dlog}W_{25} - \text{dlog}W_{30}\Big),
\end{aligned} \tag{3.87}$$

where the letters W_i are defined in section 3.3. With these relations we can rewrite $\mathcal{S}(f^{(2)})$ in terms of W_i's, considering the symbol property that

$$S[a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_m] = \sum_j c_j S[a_1, \ldots, a_{i-1}, b_{ij}, a_{i+1}, \ldots, a_m], \tag{3.88}$$

if

$$\text{dlog}(a_i) = \sum_j c_j \text{dlog}(b_{ij}). \tag{3.89}$$

After the variable replacement using (3.87) to (3.89), $\mathcal{S}(f^{(2)})$ is written in a summation of 474 terms of symbols formed by letters $W_1 \sim W_6$, W_{25} and W_{30}. This expression is identical to the symbol of $I_1^{(4)}$ calculated in the section 3.4.

4 The symbology from twistor geometries of DCI integrals

In this section, we move to the study of the alphabet as well as certain properties of the symbols from the so-called Schubert problem, namely by studying geometric configurations in momentum twistor space associated with (leading singularities of) DCI integrals. This is part of a on-going program towards understanding the symbology of DCI integrals, first

from cluster algebras (directly related to quiver diagrams of the kinematics) [19, 20, 38] and then from Schubert problems or twistor geometries as mentioned above [40]. Most recently, [41] extends the twistor-geometry method to non-DCI integrals. In the following, we will first give a quick review of DCI integrals, focusing on those relevant for our studies, and then explain the complete alphabet from twistor geometries, as well as certain properties of the symbol. We will mainly follow the analysis in [40] for predicting the alphabet of 10-point double-box integral, which corresponds to the alphabet in section 3.3 after taking non-DCI limits. Readers can refer to [40, 41] for a systematic discussion of the method, for both DCI and general Feynman integrals.

4.1 Review of relevant DCI integrals

Recall that for a n-point massless planar integral, we can first introduce dual points $\{x_i\}$ such that $x_{i+1} - x_i = p_i$ with $x_{n+1} = x_1$ to make the momentum conservation manifest, and then the integral only depends on planar variables, which are also the first entries:

$$x_{i,j}^2 := (x_i - x_j)^2 = (p_i + \cdots + p_{j-1})^2.$$

It is convenient to further introduce momentum twistors $Z_i \in \mathbb{P}^3$ such that each dual point x_i is associated with a bi-twistor $Z_{i-1} \wedge Z_i$ and planar variables become

$$(x_i - x_j)^2 = \frac{Z_{i-1} \wedge Z_i \wedge Z_{j-1} \wedge Z_j}{(Z_{i-1} \wedge Z_i \wedge I_\infty)(Z_{j-1} \wedge Z_j \wedge I_\infty)} = \frac{\langle i-1\,i\,j-1\,j \rangle}{\langle i-1\,i\,\infty \rangle \langle j-1\,j\,\infty \rangle},$$

where $\langle ijkl \rangle := \det(Z_i Z_j Z_k Z_l)$ and I_∞ is the *infinity bi-twistor*. In terms of momentum twistors, the massless conditions $(x_{i+1} - x_i)^2 = p_i^2 = 0$ are automatically solved, and conformal transformations of dual momenta become SL(4) linear transformations of $\{Z_i\}$ together with rescaling $Z_i \to t_i Z_i$. We say that an integral is dual conformal invariant if it only depends on cross-ratios of planar variables (or four-brackets $\langle i-1\,i\,j-1\,j \rangle$) and do not depend on the infinity bitwistor I_∞.

Note that for a general planar integral with m (possibly massive) legs, the kinematics can be described by a subset of $n \geq m$ dual points as above, by identifying each massive momentum with two (or more) massless legs [18]. For example, a one-mass triangle kinematics depends on 3 of 4 dual points, e.g. x_2, x_3, x_4 (but not x_1), and a four-mass box kinematics depends on 4 of 8 dual points, e.g. x_2, x_4, x_6, x_8. Moreover, if we start with a DCI integral which has at least one dual point which is not null separated from adjacent ones, we can arrive at a non-DCI integral by simply sending it to infinity, or identifying this bi-twistor with I_∞. As explained in [19], a general DCI kinematics with $n-2m$ massless legs and m massive legs has dimension $3n-2m-15$.[1] By sending a point (between two massive legs) to infinity, this is exactly the dimension of a non-DCI kinematics with $m-1$ massive legs and $n-2m$ massless ones: we have 3 degree of freedoms for each massless leg, and 4 for each massive one, minus the dimension of Poincare group -10 and an overall scaling, $3 \times (n-2m) + 4 \times (m-1) - 10 - 1$.

---

[1]Except for the special case $n = 8, m = 4$ which has dimension 2 instead of 1.

For our purpose, we start with a DCI kinematics that depends on five generic dual points (fully massive pentagon) with $n = 10, m = 5$:
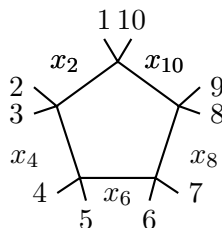
which depends on $3 \times 10 - 2 \times 5 - 15 = 5$ cross-ratios

$$u_1 = \frac{x_{4,10}^2 x_{6,8}^2}{x_{6,10}^2 x_{4,8}^2}, \ \ u_2 = \frac{x_{2,8}^2 x_{4,6}^2}{x_{2,6}^2 x_{4,8}^2}, \ \ v_1 = \frac{x_{8,10}^2 x_{4,6}^2}{x_{4,8}^2 x_{6,10}^2}, v_2 = \frac{x_{2,4}^2 x_{6,8}^2}{x_{2,6}^2 x_{4,8}^2}, \ \ u_3 = \frac{x_{2,10}^2 x_{4,6}^2}{x_{2,6}^2 x_{4,10}^2}. \quad (4.1)$$

At one-loop level, any integral with this pentagon kinematics can be reduced to five four-mass boxes

$$I_{4\mathrm{m}}(4,6,8,10), \ \ I_{4\mathrm{m}}(2,4,6,10), \ \ I_{4\mathrm{m}}(2,4,8,10), \ \ I_{4\mathrm{m}}(2,6,8,10), \ \ I_{4\mathrm{m}}(2,4,6,8), \quad (4.2)$$

where each DCI four-mass box integral gives

$$I_{4\mathrm{m}}(a,b,c,d) = \ \ \raisebox{-1em}{\includegraphics{box}} \ \ = \frac{1}{\Delta_{abcd}} \left(\mathrm{Li}_2(1 - z_{abcd}) - \mathrm{Li}_2(1 - \bar{z}_{abcd}) + \frac{1}{2}\log(v_{abcd})\log\left(\frac{z_{abcd}}{\bar{z}_{abcd}}\right) \right).$$

We have seen that I_{4m} only depends two DCI variables

$$u_{abcd} = \frac{x_{a,b}^2 x_{c,d}^2}{x_{a,c}^2 x_{b,d}^2} =: z_{abcd}\bar{z}_{abcd}, \quad v_{abcd} = \frac{x_{a,d}^2 x_{b,c}^2}{x_{a,c}^2 x_{b,d}^2} =: (1 - z_{abcd})(1 - \bar{z}_{abcd}),$$

and it involves a square root,

$$\Delta_{abcd} := \sqrt{(1 - u_{abcd} - v_{abcd})^2 - 4u_{abcd}v_{abcd}}.$$

Note that their $\{u, v\}$ are related to the five cross-ratios (4.1) as

$$\{u_1, v_1\}, \left\{u_3, \frac{v_2}{u_1}\right\}, \left\{\frac{v_1 v_2}{u_1 u_2}, \frac{u_3}{u_2}\right\}, \left\{\frac{v_1}{u_2}, \frac{u_1 u_3}{u_2}\right\}, \{u_2, v_2\}$$

respectively. In order to arrive at non-DCI kinematics, one can send any one of the five dual points to infinity and identify the resulting massive corners with four massive legs of four-mass non-DCI kinematics. As shown in the following subsection, this limit induces a map between cross-ratios (4.1) and kinematics variables $\{s, t, m_i^2\}_{i=1,\cdots,4}$, and reveals connections between DCI and non-DCI letters for one-loop integrals. In fact, this is nothing but the well-known example that in such limits, a DCI four-box becomes a (non-DCI) massive triangle integral (literally the box with a point at infinity).

Similarly, we can move to two loops, and the prototype of DCI integrals are 10-point double-box of the form:
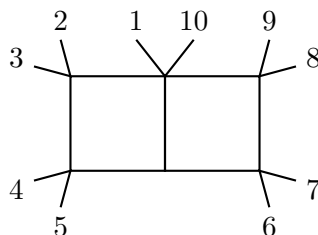
Note that this DCI integral has not been computed directly, though a similar 9-point double box (with leg 10 removed) has been computed. However, alphabets of such integrals can be predicted by considering twistor geometries associated with leading singularities (or the Schubert problems) as first proposed by N. Arkani-Hamed and worked out for numerous one- and two-loop examples in [40]. We will not review the details of such a method, but just to outline the basic idea as follows.

The Schubert problem concerns geometric configurations of intersecting lines in momentum twistor space, which are associated with maximal cuts, or leading singularities of (four-dimensional) integrals [29]. By representing each loop and external dual point as a line in twistor space, each cut propagator corresponds to intersecting two lines, and it is a beautiful problem for determining the intersections on internal or external lines. Whenever there are at least four points on such a line, the only DCI quantities one can write down are cross-ratios of such points. Remarkably, as shown in [40], such cross-ratios lead to symbol letters of DCI integrals! The classical example include A_3, E_6 alphabets for $n = 6, 7$ [16] and the $9 + 9$ algebraic letters for $n = 8$ two-loop integrals [77].

It is straightforward to apply Schubert-problem-based method to 10-point double box (similar computation has been done for 9-point case in [40]), as we will demonstrate soon. Here we first give the result for its leading singularity, which is proportional to (inverse of)

$$\Delta_2 = \sqrt{(1 - u_1 - u_2 + u_1 u_3)^2 - 4 v_1 v_2}.$$

As we will see, by sending certain dual point x_i to infinity, this gives remaining square roots for non-DCI integrals, in addition to those from one-loop leading singularities.

4.2 The alphabet explained by limits of DCI integrals

4.2.1 Even letters

To reduce the DCI kinematics to non-DCI four-mass kinematics, we take the non-DCI limit $x_2 \to \infty$ in the DCI fully massive pentagon, then we get a non-DCI box
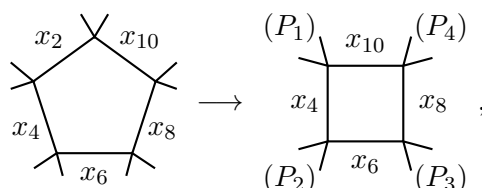
and we can represent massive legs by $P_1 = x_{10,4}$, $P_2 = x_{4,6}$, $P_3 = x_{6,8}$ and $P_4 = x_{8,10}$. In this limit, finite planar variables become

$$
\begin{aligned}
x_{6,10}^2 &\to (P_1 + P_2)^2 = s, & x_{4,8}^2 &\to (P_2 + P_3)^2 = t, \\
x_{4,10}^2 &\to P_1^2 = m_1^2, & x_{4,6}^2 &\to P_2^2 = m_2^2, & x_{6,8}^2 &\to P_3^2 = m_3^2, & x_{8,10}^2 &\to P_4^2 = m_4^2,
\end{aligned}
\tag{4.3}
$$

and other infinite $x_{2,*}^2$ cancel as $x_{2,i}^2/x_{2,j}^2 \to 1$ in cross-ratios $\{u_{abcd}, v_{abcd}\}$. For example,

$$
u_{24810} \to \frac{x_{8,10}^2}{x_{4,10}^2} \to \frac{m_4^2}{m_1^2}, \quad v_{24810} \to \frac{x_{4,8}^2}{x_{4,10}^2} \to \frac{t}{m_1^2},
\tag{4.4}
$$

and we can also see that

$$
\Delta_{24810} \to \frac{1}{m_1^2}\sqrt{(m_1^2 - t - m_4^2)^2 - 4tm_4^2} = \frac{r_3}{m_1^2},
\tag{4.5}
$$

so Δ_{24810} becomes r_3 up to a factor m_1^2! Similarly, the square roots of five four-mass boxes in eq. (4.2) give $\{r_1, r_2, r_3, r_4, r_5\}$ (defined in eq. (3.8)) in this limit respectively.

Note that we can also set another x_i to ∞ with certain identification of massive legs, then the same four-mass square root may correspond to another one-loop square root. For example, sending $x_8 \to \infty$, Δ_{24810} gives r_5 by identifying $P_1 = x_{6,10}$, $P_2 = x_{10,2}$, $P_3 = x_{2,4}$ and $P_4 = x_{4,6}$. In addition, under any identification Δ_{24810} always gives r_1 when sending $x_6 \to \infty$. The same four-mass square root may also correspond to different forms of the same square root under different non-DCI limits. For example, if we take $x_4 \to \infty$ and set $P_1 = x_{10,2}, P_2 = x_{2,6}, P_3 = x_{6,8}, P_4 = x_{8,10}$, then

$$
\Delta_{24810} \to \frac{1}{x_{2,8}^2}\sqrt{(x_{2,8}^2 - x_{2,10}^2 - x_{8,10}^2)^2 - x_{2,10}x_{8,10}} \to \frac{1}{t}\sqrt{(t - m_1^2 - m_4^2)^2 - 4m_1^2 m_4^2} = \frac{r_3}{t},
\tag{4.6}
$$

Two limits $x_2 \to \infty$ and $x_4 \to \infty$ coincidentally give the same square root r_3 since the four-mass box $I_{4m}(2,4,8,10)$ becomes the same triangle
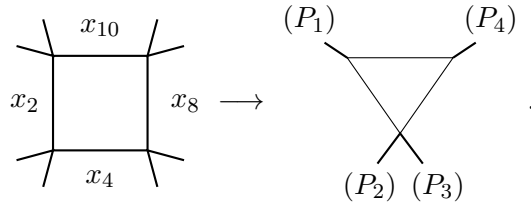
In a word, five four-mass square roots always give the set $\{r_1, \ldots, r_5\}$, no matter how we take the limit $x_i \to \infty$ and how we identify external massive legs.

For two-loop cases, we again send $x_2 \to \infty$, then
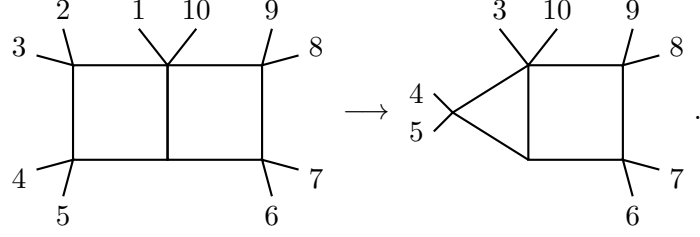
Now we have different choices to label the massive corners as the non-DCI massive legs. For example, if we still identify $P_1 = x_{10,4}$, $P_2 = x_{4,6}$, $P_3 = x_{6,8}$ and $P_4 = x_{8,10}$, then $\Delta_2 \to \frac{1}{st} r_9$. Similarly, if we identify $P_3 = x_{10,4}$, $P_4 = x_{4,6}$, $P_1 = x_{6,8}$ and $P_2 = x_{8,10}$, $\Delta_2 \to \frac{1}{st} r_{11}$.

Here we list all the proper limits and corresponding identifications to get the other square roots:

r_6 : sending $x_6 \to \infty$ with the identification $P_1 = x_{10,2}$, $P_2 = x_{2,4}$, $P_3 = x_{4,8}$ and $P_4 = x_{8,10}$,

r_7 : sending $x_6 \to \infty$ with the identification $P_4 = x_{10,2}$, $P_1 = x_{2,4}$, $P_2 = x_{4,8}$ and $P_3 = x_{8,10}$;

r_8 : sending $x_8 \to \infty$ with the identification $P_2 = x_{10,2}$, $P_3 = x_{2,4}$, $P_4 = x_{4,6}$ and $P_1 = x_{6,10}$,

r_{10} : sending $x_8 \to \infty$ with the identification $P_4 = x_{10,2}$, $P_1 = x_{2,4}$, $P_2 = x_{4,6}$ and $P_3 = x_{6,10}$;

Note that we can also reproduce the six two-loop roots starting from other double-box integrals with the same topology but the external legs cyclically rotated.

A nice consequence of such identifications is that the roots always take the form $r_i^2 := A_i^2 - 4B_i$, where A_i and B_i are polynomials in kinematic variables. We we record them as

$$
\begin{aligned}
r_1^2 &= (st - m_1^2 m_3^2 - m_2^2 m_4^2)^2 - 4 m_1^2 m_2^2 m_3^2 m_4^2, \\
r_2^2 &= (s - m_1^2 - m_2^2)^2 - 4 m_1^2 m_2^2, \quad r_3^2 = (t - m_1^2 - m_4^2)^2 - 4 m_1^2 m_4^2, \\
r_4^2 &= (s - m_3^2 - m_4^2)^2 - 4 m_3^2 m_4^2, \quad r_5^2 = (t - m_2^2 - m_3^2)^2 - 4 m_2^2 m_3^2, \\
r_6^2 &= (s + t - m_1^2 - m_3^2)^2 - 4 m_2^2 m_4^2, \quad r_7^2 = (s + t - m_2^2 - m_4^2)^2 - 4 m_1^2 m_3^2, \\
r_8^2 &= ((m_1^2 - m_2^2) m_4^2 + s(t - m_1^2))^2 - 4 s m_1^2 m_3^2 m_4^2, \\
r_9^2 &= ((m_2^2 - m_1^2) m_3^2 + s(t - m_2^2))^2 - 4 s m_2^2 m_3^2 m_4^2, \\
r_{10}^2 &= ((m_3^2 - m_4^2) m_2^2 + s(t - m_3^2))^2 - 4 s m_1^2 m_2^2 m_3^2, \\
r_{11}^2 &= ((m_4^2 - m_3^2) m_1^2 + s(t - m_4^2))^2 - 4 s m_1^2 m_2^2 m_4^2,
\end{aligned}
\tag{4.7}
$$

where $r_2, \cdots, r_5$ form a cyclic orbit, so do r_6, r_7 and $r_8, \cdots, r_{11}$. We can write "one-loop" square roots r_i^2 for $i = 1, \ldots, 5$ as $A_i^2 - 4B_i$ in different ways, e.g.

$$
r_3^2 = (t - m_1^2 - m_4^2)^2 - 4 m_1^2 m_4^2 = (m_1^2 - t - m_4^2)^2 - 4 t m_4^2,
\tag{4.8}
$$

which reflect the fact that there are different choices to take limits for the one-loop case. Note that no matter how we write it, B_i is always a monomial which is *positive* when all kinematic variables are positive. Furthermore, since the DCI square root is *positive* for any positive DCI kinematic point, these square roots stay positive in non-DCI limit for such positive regions.

We have one more comment on the W_{18} and its positivity. Note that there exists a DCI letter

$$\tilde{W} = u_1^2 u_2 u_3 - u_1^2 u_2 - u_1^2 u_3^2 - u_1^2 u_3 v_2 + u_1^2 u_3 - u_1 u_2^2 - u_1 u_2 u_3 v_1 + u_1 u_2 u_3 + u_1 u_2 v_1$$
$$+ u_1 u_2 v_2 + u_1 u_2 + u_1 u_3 v_1 v_2 + u_1 u_3 v_1 + u_1 u_3 v_2 - u_1 u_3 + u_1 v_1 v_2 - u_1 v_1 + u_2 v_1 v_2$$
$$- u_2 v_2 - v_1^2 v_2 - v_1 v_2^2 + v_1 v_2 \tag{4.9}$$

whose non-DCI limits are always proportional to W_{18}, independent of the dual point x_i we send to infinity. For example, the DCI letter becomes

$$\tilde{W} \to -\frac{m_2^2 m_3^2}{s^3 t^2} W_{18}, \ \tilde{W} \to -\frac{m_3^2}{m_2^4 s^2} W_{18}, \tag{4.10}$$

when sending $x_2 \to \infty$ or $x_4 \to \infty$ respectively, and similar for the other 3 non-DCI limits. The crucial point is that $\tilde{W}$ is positive definite in the positive region $G_+(4, 10)$, as can be checked by any positive parameterization [78]. It is believed that $\tilde{W}$ is a letter of 10-point double-box integral.

4.2.2 Odd letters

Next we study algebraic or odd letters, i.e. those that involve square roots, and their log flip signs when we take flip the sign of a square root. The discussion above directly motivates us to relate odd letters that only involve one square root, i.e. $W_{19}, W_{20}, \ldots, W_{34}$, to similar odd letters for DCI integrals. For one-loop case, it is well known that odd letters of four-mass box (and higher-loop generalizations such as ladder integrals) take the form $z_i/\bar{z}_i$ and $(1 - z_i)/(1 - \bar{z}_i)$. In the "non-DCI" limit $x_2 \to \infty$ with $P_1 = x_{10,4}$, $P_2 = x_{4,6}$, $P_3 = x_{6,8}$ and $P_4 = x_{8,10}$, the five z's obtained from limits of one-loop case take the form

$$z_1 = \frac{1}{2} + \frac{m_1^2 m_3^2 - m_2^2 m_4^2 - r_1}{2st}, \quad z_2 = \frac{1}{2} + \frac{m_2^2 - s + r_2}{2m_1^2}, \quad z_3 = \frac{1}{2} + \frac{m_4^2 - t + r_3}{2m_1^2},$$
$$z_4 = \frac{1}{2} + \frac{-m_3^2 + m_4^2 - r_4}{2s}, \qquad z_5 = \frac{1}{2} + \frac{m_2^2 - m_3^2 - r_5}{2t}, \tag{4.11}$$

and $\bar{z}_i := z_i(r_i \to -r_i)$. Remarkably, we find that the 10 odd letters from four-mass boxes, $z_i/\bar{z}_i$ and $(1 - z_i)/(1 - \bar{z}_i)$ for $i = 1, \cdots, 5$ are nothing but (multiplicative combinations of) $W_{21}, \ldots, W_{30}$; precisely we have

$$W_{21} = \frac{z_2}{\bar{z}_2}, W_{22} = \frac{1 - z_4}{1 - \bar{z}_4} \frac{\bar{z}_4}{z_4}, W_{23} = \frac{1 - z_5}{1 - \bar{z}_5} \frac{\bar{z}_5}{z_5}, W_{24} = \frac{\bar{z}_3}{z_3}, W_{25} = \frac{1 - z_1}{1 - \bar{z}_1} \frac{\bar{z}_1}{z_1}, W_{26} = \left(\frac{1 - z_2}{1 - \bar{z}_2}\right)^2 \frac{\bar{z}_2}{z_2},$$
$$W_{27} = \frac{1 - z_4}{1 - \bar{z}_4} \frac{z_4}{\bar{z}_4}, W_{28} = \frac{1 - \bar{z}_5}{1 - z_5} \frac{\bar{z}_5}{z_5}, W_{29} = \left(\frac{1 - \bar{z}_3}{1 - z_3}\right)^2 \frac{z_3}{\bar{z}_3}, W_{30} = \frac{1 - \bar{z}_1}{1 - z_1} \frac{\bar{z}_1}{z_1}. \tag{4.12}$$

For the remaining 6 odd letters with single square roots obtained from two-loop diagrams, they cannot be interpreted as odd letters of a four-mass box, but it is straightforward to introduce analogous z's

$$z_6 = \frac{1}{2} + \frac{s - m_3^2 + r_6}{2\left(t - m_1^2\right)}, \qquad z_7 = \frac{1}{2} + \frac{t - m_4^2 + r_7}{2\left(s - m_2^2\right)},$$

$$z_8 = \frac{1}{2} + \frac{(m_1^2 - m_2^2)m_4^2 + r_8}{2s\left(t - m_1^2\right)}, \quad z_9 = \frac{1}{2} + \frac{(m_2^2 - m_1^2)m_3^2 + r_9}{2s\left(t - m_2^2\right)}, \tag{4.13}$$

$$z_{10} = \frac{1}{2} + \frac{(m_3^2 - m_4^2)m_2^2 + r_{10}}{2s\left(t - m_3^2\right)}, \quad z_{11} = \frac{1}{2} + \frac{(m_4^2 - m_3^2)m_1^2 + r_{11}}{2s\left(t - m_4^2\right)}$$

which nicely simplify them as (note there is no letters of the form $(1 - z)/(1 - \bar{z})$):

$$W_{19} = \frac{\bar{z}_6}{z_6},\ W_{20} = \frac{\bar{z}_7}{z_7},\ W_{31} = \left(\frac{z_{10}}{\bar{z}_{10}}\right)^2,\ W_{32} = \left(\frac{\bar{z}_{11}}{z_{11}}\right)^2,\ W_{33} = \left(\frac{z_8}{\bar{z}_8}\right)^2,\ W_{34} = \left(\frac{\bar{z}_9}{z_9}\right)^2.$$

Thus all odd letters with single square root can be explained from those for DCI integrals.

Now we turn to the other 34 odd letters $W_{35}, \ldots, W_{68}$ involving two square roots, which take the form

$$\frac{a + r_i r_j}{a - r_i r_j}$$

with a being generally involved polynomials in kinematic variables, and we call them mixed algebraic letters to distinguish them from those letters with only one square root. Mixed algebraic letters only show up as two-loop letters. We can generate them from the approach of Schubert problems and the alphabet of 10-point double-box integral above.

To be explicit, we consider 5 lines, (12), (34), (56), (78), (9 10) in momentum twistor space, which correspond to dual points $\{x_2, x_4, x_6, x_8, x_{10}\}$ respectively. Following the procedure in [40], for each DCI four-mass-box $I_{4m}(i, j, k, l)$, two solutions $(AB)_{i,j,k,l}^{\pm}$ from its maximal cut produce 8 intersections on its 4 external lines, which will be denoted as $\{\alpha_{i,j,k,l}^{\pm}, \beta_{i,j,k,l}^{\pm}, \gamma_{i,j,k,l}^{\pm}, \delta_{i,j,k,l}^{\pm}\}$ on $\{(i{-}1 i), (j{-}1 j), (k{-}1 k), (l{-}1 l)\}$ respectively. Furthermore, after cutting all propagators of the 10-point double-box integral, i.e. looking for two lines (CD) and (EF) such that (CD) intersects with (12), (34), (56) (EF), and (EF) intersects with (CD), (56), (78), (9 10), we have two pairs of solutions $\{(CD)_i, (EF)_i\}_{i=\pm}$, and each external line contains two new intersections, which we denote them as $\{\epsilon_k^{\pm}\}_{k=2,4,6,8,10}$ on x_k. Note that no matter $i = +$ or $i = -$, three lines $(CD)_i$ $(EF)_i$ and (56) share a same intersection ϵ_6^i, thus (56) has only two new intersections on it as well.
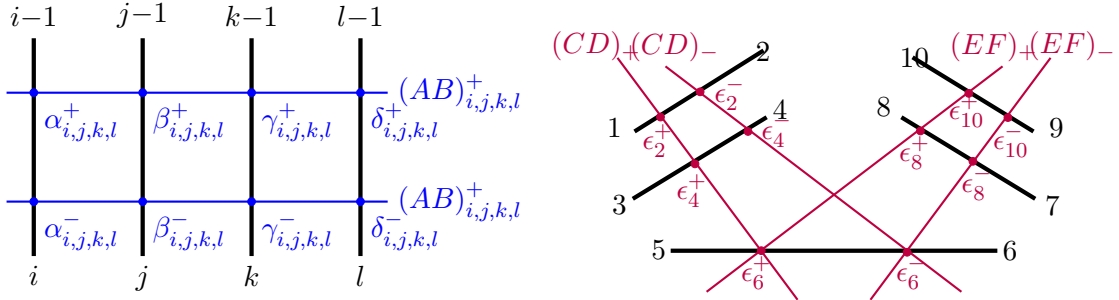
Similar to the 9-point case in [40], now we can construct possible DCI letters for 10-point two-loop double-box integral from the intersections on external lines, once a line has four distinct points on it. The upshot is that all 34 mixed algebraic letters here are non-DCI limits of certain DCI letters constructed in this approach!

For instance, to generate W_{35}, which involves two one-loop square roots r_2 and r_4. We consider four distinct intersections

$$\{X_1 = \alpha^+_{2,4,6,10}, X_2 = \alpha^-_{2,4,6,10}, X_3 = \alpha^+_{2,6,8,10}, X_4 = \alpha^-_{2,6,8,10}\} \tag{4.14}$$

on the line (12). Since each X_i is an individual momentum twistor, we can construct the following cross-ratio

$$\frac{(X_1, X_3)(X_2, X_4)}{(X_1, X_4)(X_2, X_3)} := \frac{\langle X_1 X_3 I \rangle \langle X_2 X_4 I \rangle}{\langle X_1 X_4 I \rangle \langle X_2 X_3 I \rangle}$$

from this configuration. Here I is a reference line (bitwistor) in momentum twistor space that does not intersect with (12). It can be directly checked that such a cross-ratio is DCI and actually independent of I. Moreover, after taking its non-DCI limit $x_2 \to \infty$ and identifying $P_1 = x_{10,4}$, $P_2 = x_{4,6}$, $P_3 = x_{6,8}$, $P_4 = x_{8,10}$, this cross-ratio yields the mixed algebraic letter W_{35}! Furthermore, W_{37}, involving a one-loop square root r_2 and a two-loop square root r_6, can also be constructed from this approach. In this case we take four points on (56) as

$$\{X_1 = \epsilon^+_6, X_2 = \epsilon^-_6, X_3 = \gamma^+_{2,4,6,10}, X_4 = \gamma^-_{2,4,6,10}\}, \tag{4.15}$$

and consider $\frac{(X_1,X_3)(X_2,X_4)}{(X_1,X_4)(X_2,X_3)}$ again. Non-DCI limit $x_6 \to \infty$ of this letter with the identification $P_1 = x_{10,2}$, $P_2 = x_{2,4}$, $P_3 = x_{4,8}$ and $P_4 = x_{8,10}$ then gives W_{37} as we want.

In the rest of this subsection, we list all the configurations we use to construct 34 mixed algebraic letters. We first classify 34 mixed algebraic letters according to different square roots, e.g. letters with only one-loop square roots and letters with various two-loop roots. For each group we fix a non-DCI limit and the corresponding identification of external legs, and present one possible choice $\{X_1, X_2, X_3, X_4\}$ to construct each individual letter. We stick to the cross-ratio $\frac{(X_1,X_3)(X_2,X_4)}{(X_1,X_4)(X_2,X_3)}$ from each configuration and consider its non-DCI limit in each case. Note that the configurations we present here are not unique and these mixed algebraic letters can be generated by many different choices.

1. letters with only one-loop square roots $r_1 \sim r_5$: this group of letters consists of W_{35}, W_{36}, W_{38}, W_{40}, W_{42} and W_{54} with only triangle roots, and W_{46}, W_{51}, W_{59}, W_{62} with r_1. To generate them, we keep the non-DCI limit $x_2 \to \infty$ and the identification $P_1 = x_{10,4}$, $P_2 = x_{4,6}$, $P_3 = x_{6,8}$, $P_4 = x_{8,10}$. As we have pointed out, five square roots of DCI four-mass boxes in (4.1) give $\{r_1, r_2, r_3, r_4, r_5\}$ respectively. Then the configurations we need are

$$W_{35} : \{\alpha^+_{2,4,6,10}, \alpha^-_{2,4,6,10}, \alpha^+_{2,6,8,10}, \alpha^-_{2,6,8,10}\}, \quad W_{36} : \{\alpha^+_{2,4,6,10}, \alpha^-_{2,4,6,10}, \alpha^+_{2,4,6,8}, \alpha^-_{2,4,6,8}\}$$
$$W_{38} : \{\alpha^+_{2,4,6,10}, \alpha^-_{2,4,6,10}, \alpha^+_{2,4,8,10}, \alpha^-_{2,4,8,10}\}, \quad W_{40} : \{\alpha^+_{2,4,8,10}, \alpha^-_{2,4,8,10}, \alpha^+_{2,6,8,10}, \alpha^-_{2,6,8,10}\}$$
$$W_{42} : \{\alpha^+_{2,6,8,10}, \alpha^-_{2,6,8,10}, \alpha^+_{2,4,6,8}, \alpha^-_{2,4,6,8}\}, \quad W_{54} : \{\alpha^+_{2,4,8,10}, \alpha^-_{2,4,8,10}, \alpha^+_{2,4,6,8}, \alpha^-_{2,4,6,8}\},$$

for those with only triangle square roots, and

$$W_{46} : \{\delta_{4,6,8,10}^+, \delta_{4,6,8,10}^-, \delta_{2,4,6,10}^+, \delta_{2,4,6,10}^-\}, \quad W_{51} : \{\delta_{4,6,8,10}^+, \delta_{4,6,8,10}^-, \delta_{2,6,8,10}^+, \delta_{2,6,8,10}^-\}$$
$$W_{59} : \{\gamma_{4,6,8,10}^+, \gamma_{4,6,8,10}^-, \gamma_{2,4,6,8}^+, \gamma_{2,4,6,8}^-\}, \quad W_{62} : \{\gamma_{4,6,8,10}^+, \gamma_{4,6,8,10}^-, \gamma_{2,4,8,10}^+, \gamma_{2,4,8,10}^-\}$$

for those with one triangle square root $r_2 \sim r_5$ and one r_1.

2. letters with r_6 or r_7: letters involving r_6 consists of W_{37}, W_{41}, W_{55}, W_{56}, and similarly those with r_7 are W_{39}, W_{43}, W_{57} and W_{58}. Sending $x_6 \to \infty$ with the identification $P_1 = x_{10,2}$, $P_2 = x_{2,4}$, $P_3 = x_{4,8}$ and $P_4 = x_{8,10}$ (note that under this limit five square roots of DCI four-mass boxes (4.1) give $\{r_4, r_2, r_1, r_3, r_5\}$ respectively), we can then recover the four letters from (56) as

$$W_{37} : \{\gamma_{2,4,6,10}^+, \gamma_{2,4,6,10}^-, \epsilon_6^+, \epsilon_6^-\}, \qquad W_{41} : \{\beta_{4,6,8,10}^+, \beta_{4,6,8,10}^-, \epsilon_6^+, \epsilon_6^-\},$$
$$W_{55} : \{\gamma_{2,4,6,8}^+, \gamma_{2,4,6,8}^-, \epsilon_6^+, \epsilon_6^-\}, \qquad W_{56} : \{\beta_{2,6,8,10}^+, \beta_{2,6,8,10}^-, \epsilon_6^+, \epsilon_6^-\}.$$

On the other hand, if we keep $x_6 \to \infty$ but change the identification to $P_4 = x_{10,2}$, $P_1 = x_{2,4}$, $P_2 = x_{4,8}$ and $P_3 = x_{8,10}$ (five square roots of DCI four-mass boxes (4.1) give $\{r_5, r_3, r_1, r_4, r_2\}$ respectively), we will get the four letters with r_7 from the four configurations instead.

3. letters with r_8 or r_{10}: letters involving r_8 consists of W_{45}, W_{52}, W_{63} W_{65}. Sending $x_8 \to \infty$ with the identification $P_2 = x_{10,2}$, $P_3 = x_{2,4}$, $P_4 = x_{4,6}$ and $P_1 = x_{6,10}$ (five square roots of DCI four-mass boxes (4.1) give $\{r_3, r_1, r_5, r_2, r_4\}$ respectively), we can then recover these four letters from

$$W_{45} : \{\gamma_{2,6,8,10}^+, \gamma_{2,6,8,10}^-, \epsilon_8^+, \epsilon_8^-\}, \qquad W_{52} : \{\delta_{2,4,6,8}^+, \delta_{2,4,6,8}^-, \epsilon_8^+, \epsilon_8^-\},$$
$$W_{63} : \{\gamma_{4,6,8,10}^+, \gamma_{4,6,8,10}^-, \epsilon_8^+, \epsilon_8^-\}, \qquad W_{65} : \{\delta_{2,4,6,10}^+, \delta_{2,4,6,10}^-, \epsilon_{10}^+, \epsilon_{10}^-\}.$$

On the other hand, if we keep $x_8 \to \infty$ but change the identification to $P_4 = x_{10,2}$, $P_1 = x_{2,4}$, $P_2 = x_{4,6}$ and $P_3 = x_{6,10}$ (five square roots of DCI four-mass boxes (4.1) give $\{r_5, r_1, r_3, r_4, r_2\}$ respectively), we will get the four letters W_{47}, W_{50}, W_{61} and W_{67} with r_{10} from the four configurations instead.

4. letters with r_9 or r_{11}: letters involving r_9 consists of W_{44}, W_{53}, W_{60} and W_{66}. Sending $x_2 \to \infty$ with the identification $P_1 = x_{10,4}$, $P_2 = x_{4,6}$, $P_3 = x_{6,8}$, $P_4 = x_{8,10}$ (same limit and identification as that for the first group), we can then recover these four letters from

$$W_{44} : \{\alpha_{2,4,6,10}^+, \alpha_{2,4,6,10}^-, \epsilon_2^+, \epsilon_2^-\}, \qquad W_{53} : \{\alpha_{2,6,8,10}^+, \alpha_{2,6,8,10}^-, \epsilon_2^+, \epsilon_2^-\},$$
$$W_{60} : \{\alpha_{2,4,6,8}^+, \alpha_{2,4,6,8}^-, \epsilon_2^+, \epsilon_2^-\}, \qquad W_{66} : \{\delta_{4,6,8,10}^+, \delta_{4,6,8,10}^-, \epsilon_{10}^+, \epsilon_{10}^-\}.$$

On the other hand, if we change the identification to $P_3 = x_{10,4}$, $P_4 = x_{4,6}$, $P_1 = x_{6,8}$ and $P_2 = x_{8,10}$ (five square roots of DCI four-mass boxes (4.1) give $\{r_1, r_4, r_5, r_2, r_3\}$), we recover letters W_{48}, W_{49}, W_{64} and W_{68} with r_{11}.

Note that the appearance of algebraic letters with two-loop square roots has certain pattern. For instance, on the 17th row of the DE ($I_{17} = r_6 G_{0,1,1,0,1,1,1,0,0}$, which is finite), there are five off-diagonal elements which are just the five odd letters $\{W_{19}, W_{37}, W_{41}, W_{55}, W_{56}\}$ with r_6. Furthermore, these are exactly the last entries of I_{17} at $\mathcal{O}(\epsilon)$! Similar phenomenon applies to double-triangle I_{32} with r_7 and box-triangles I_5, I_6, I_7, I_8 with $r_8, \cdots, r_{11}$, respectively: at leading order, there are exactly 5 last entries, namely those 5 odd letters containing r_i, for each of these integrals.

Last but not least, we note that the product of numerator and denominator of each mixed algebraic letter is always proportional to the last rational letter, which was shown to be the Gram determinant, and other factors of this product are just kinematic variables $m_1^2, \ldots, m_4^2, s, t$. For example, the product of numerator and denominator of W_{36} is

$$(f_{36} + r_2 r_5)(f_{36} - r_2 r_5) = 4 m_2^2 W_{18}. \tag{4.16}$$

Therefore, any mixed odd letters are positive in the positive region.

4.3 Properties of the symbols

In this section, we investigate some analytic structures of these integrals from their symbols, which can be nicely understood from DCI integrals.

From canonical differential equations $dI_i = \epsilon \sum_j d\log A_{ij} I_j$, we can construct symbols of integrals in the basis by

$$I_a^{(w)} = \sum_{i_w} I_{i_w}^{(w-1)} \otimes A_{ai_w} = \sum_{i_1, \ldots, i_w} I_{i_1}^{(0)} A_{i_2 i_1} \otimes A_{i_3 i_2} \otimes \cdots \otimes A_{i_w i_{w-1}} \otimes A_{ai_w}, \tag{4.17}$$

where $I_k^{(w)}$ is the series coefficient of $I_k(\epsilon) = \sum_{w \geq 0} I_k^{(w)} \epsilon^{w-4}$. One can easily check the solution (4.17) by series expansion of the canonical differential equations.

The starting point is to understand branch points, or the first entries of the symbol, and corresponding physical discontinuities. For any planar integral, branch points must correspond to *planar variables* $s_{i,\ldots,k} := (p_i + p_{i+1} + \cdots + p_k)^2 = 0$. In our cases, this means that first entries of I_a should be

$$\{s, t, m_1^2, m_2^2, m_3^2, m_4^2\},$$

which can be directly verified from $\sum_a A_{ai} I_i^{(0)}$. The corresponding physical discontinuity for the channel $x = 0$ is

$$\text{Disc}_{x=0}(I_a^{(w)}) = \sum_{i_1, \ldots, i_w} I_{i_1}^{(0)} \text{Disc}_{x=0}(\log A_{i_2 i_1}) A_{i_3 i_2} \otimes \cdots \otimes A_{ai_w}. \tag{4.18}$$

Then we consider double discontinuities. The corresponding physical constrains are Steinmann relations, which say that the double discontinuities taken in overlapping channels of any planar integrals should vanish. In our cases, the only overlapping channels are $s = 0$ and $t = 0$. We consider the combination

$$X_{ac} = \sum_b A_{bc} \otimes A_{ab}, \tag{4.19}$$

such that $I_i^{(w)}$ can be written as $\sum_{p,q} X_{p,q} \otimes Y_{i,p,q}$ where $Y_{i,p,q}$ are symbols of weight-$(w-2)$, then

$$\mathrm{Disc}_{s=0}\,\mathrm{Disc}_{t=0}(I_i^{(w)}) = \sum_{p,q} \mathrm{Disc}_{s=0}\,\mathrm{Disc}_{t=0}(X_{p,q})Y_{i,p,q}. \tag{4.20}$$

The double discontinuity of X_{ac} is just

$$\mathrm{Disc}_{s=0}\,\mathrm{Disc}_{t=0} X_{ac} = \sum_b \mathrm{Disc}_{s=0}(\log A_{ab})\,\mathrm{Disc}_{t=0}(\log A_{bc}), \tag{4.21}$$

the product of two discontinuity matrix. We check that both $\mathrm{Disc}_{s=0}\,\mathrm{Disc}_{t=0} X_{ac}$ and $\mathrm{Disc}_{t=0}\,\mathrm{Disc}_{s=0} X_{ac}$ vanish.

Steinmann relations constrain first-two entries of the symbol of $I_a^{(w)}$, we could further consider the same constrains on any adjacent entries of the symbol, which are known as the extended Steinmann relations. For the symbol of $I_a^{(w)}$, these constrains are

$$\sum_{i_1,\ldots,\widehat{i_k}\ldots,i_w} I_{i_1}^{(0)} A_{i_2 i_1} \otimes \cdots \otimes \widehat{A_{i_k i_{k-1}}} \otimes \widehat{A_{i_{k+1} i_k}} \otimes \cdots \otimes A_{a i_w} \mathrm{Disc}_{s=0}\,\mathrm{Disc}_{t=0} X_{i_{k+1},i_{k-1}} = 0 \tag{4.22}$$

and

$$\sum_{i_1,\ldots,\widehat{i_k}\ldots,i_w} I_{i_1}^{(0)} A_{i_2 i_1} \otimes \cdots \otimes \widehat{A_{i_k i_{k-1}}} \otimes \widehat{A_{i_{k+1} i_k}} \otimes \cdots \otimes A_{a i_w} \mathrm{Disc}_{t=0}\,\mathrm{Disc}_{s=0} X_{i_{k+1},i_{k-1}} = 0, \tag{4.23}$$

for any k and the only overlapping channels $s = 0$ and $t = 0$. Therefore, Steinmann relations for X also guarantee the extended Steinmann relations of $I_a^{(w)}$.

In [39], it's conjectured from first entry condition and Steinmann relations that the first-two entries of DCI integrals can only be linear combination of one-loop box functions and some trivial log log functions. For any order of ϵ of $I_a^{(w)}$, we also find that the first-two entries of I_i can only be linear combinations of

$$\{\log(s)^2, \log(t)^2, \log(m_i^2)\log(s), \log(m_i^2)\log(t), \log(m_i^2)\log(m_j^2), F(z_k, \bar{z}_k)\}_{i,j=1,\ldots,4,k=1,\ldots 5}, \tag{4.24}$$

where $F(z_k, \bar{z}_k)$ are (normalized) four-mass box function whose symbol is

$$\mathcal{S}[F(z, \bar{z})] := ((1-z)(1-\bar{z})) \otimes \frac{z}{\bar{z}} - (z\bar{z}) \otimes \frac{1-z}{1-\bar{z}}. \tag{4.25}$$

Therefore, the first two entries are also box functions (after taking one dual point to infinity) except log log functions.

5 Summary and outlook

In this paper, we study multi-loop Feynman integrals and their symbology. The symbols of UT integrals, in principle, can be derived from the differential equations of the UT integral basis. As shown in this paper, the symbols of some integrals can be very complicated. Also, the symbols are with many "mysterious" structures. We provide insights for their symbology by considering closely-related DCI integrals in $D = 4$, whose symbology can be

explained in terms of Schubert problems. Meanwhile, the corresponding symbol structures can also be analyzed. This gives us an insight about some properties of the symbols, before we comply the canonical differential equation computation, in future studies for new Feynman integrals.

To explain the above specifically, in this paper, we studied a cutting-edge example, the two-loop double box diagram with 4 different external masses. We determine its UT integral basis mainly using loop-by-loop analysis of leading singularity in Baikov representation. The analytic derivation of corresponding differential equations is nontrivial and we use the package FINITEFLOW based on the finite-field method for this computation.

From the differential equations, we derived the symbols of the UT integrals from order ϵ^{-4} to ϵ^0. The symbols consist of 68 letters. Among the letters, there are 18 even letters being rational functions, 16 odd letters containing one square root each, and 34 odd letters containing 2 square roots each. The symbols for the 74 integrals are derived using the differential equations. We found their properties that: 1) The letters that can appear at the first and the second entries of the symbols are limited. 2) A large number of the possible letter pairs that can appear at two adjacent entries are forbidden, read from the letter structure of differential equations.
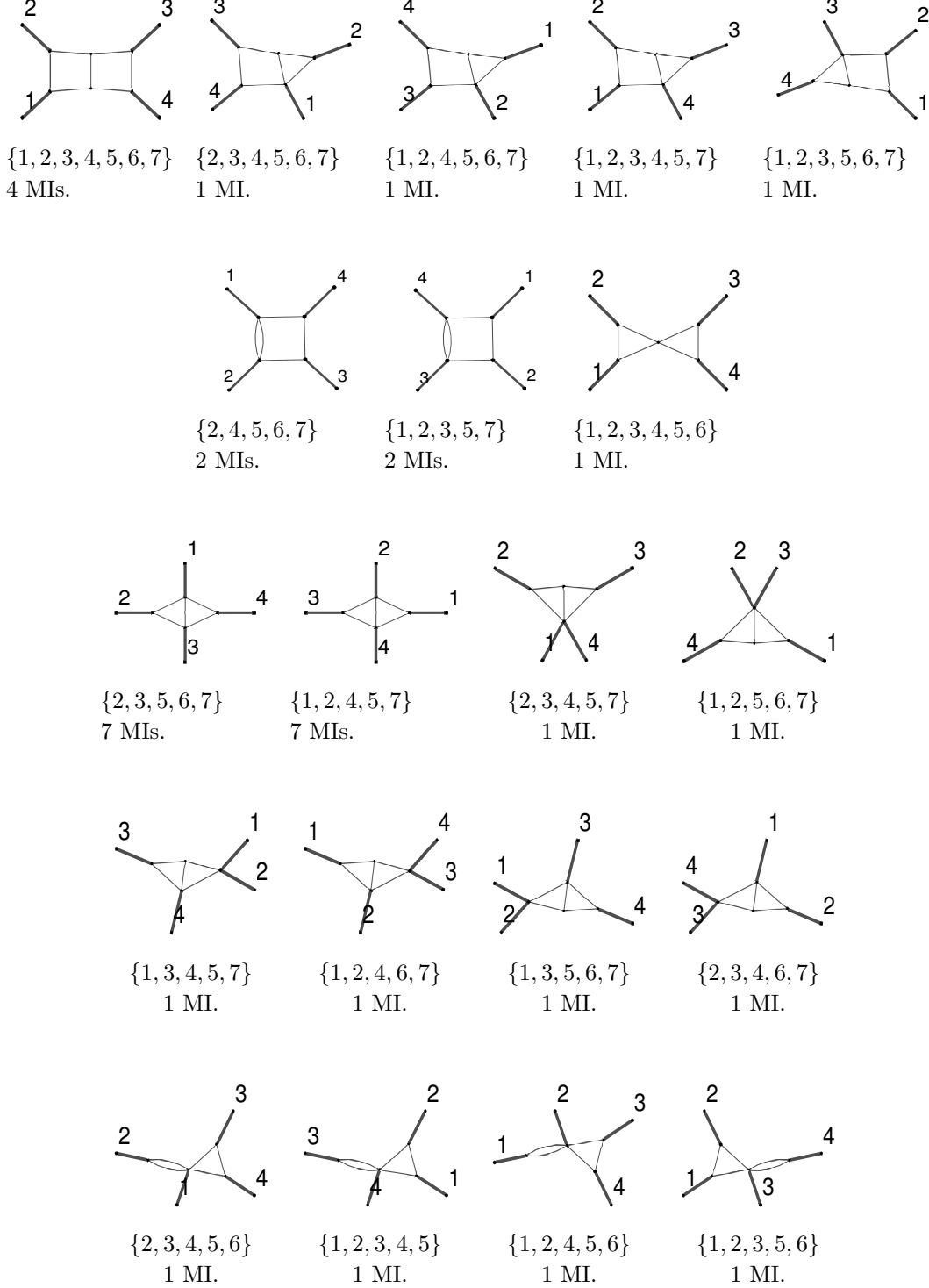
We have shown that all symbol letters of these integrals have clear origin from those of DCI integrals, which have been studied extensively and exhibit rich mathematical structures. Both even and odd letters are obtained by taking a dual point to infinity, of the letters for one- and two-loop DCI integrals with generic pentagon kinematics. Not only do the square roots correspond to one- and two-loop leading singularities, but we determine all complicated mixed odd letters from associated twistor geometries, which are also "last entries" for corresponding finite, two-loop integrals. We find it satisfying that one CAN obtain all letters by considering mathematical structure of DCI integrals without actually computing them, and we leave a more systematic study of this phenomenon to a future work. Last but not least, we find that important properties of DCI integrals nicely carry over to UT integrals we consider in this paper: the first two entries come from one-loop DCI box functions, and extended Steinmann relations further constrain any two adjacent entries to any order in ϵ. It would be highly desirable to understand further how structures of their symbols follow from properties of these DCI integrals.

Acknowledgments
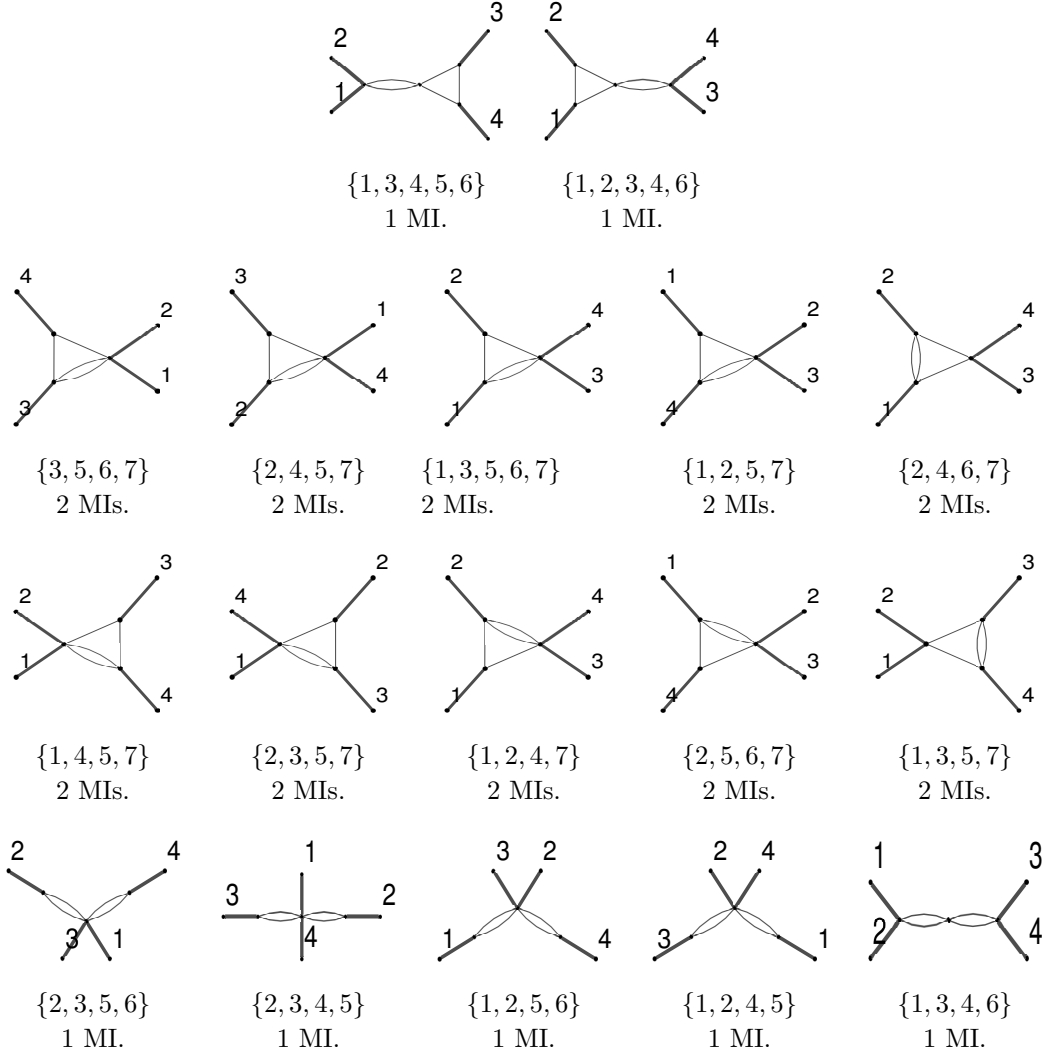
A Diagram of the two-loop master integrals

In this appendix, we list the diagrams for the two-loop Feynman integrals with four external massive legs of every sector with master integrals. They are shown in figure 3.

$\{1,2,3,4,5,6,7\}$
4 MIs.

$\{2,3,4,5,6,7\}$
1 MI.

$\{1,2,4,5,6,7\}$
1 MI.

$\{1,2,3,4,5,7\}$
1 MI.

$\{1,2,3,5,6,7\}$
1 MI.

$\{2,4,5,6,7\}$
2 MIs.

$\{1,2,3,5,7\}$
2 MIs.

$\{1,2,3,4,5,6\}$
1 MI.

$\{2,3,5,6,7\}$
7 MIs.

$\{1,2,4,5,7\}$
7 MIs.

$\{2,3,4,5,7\}$
1 MI.

$\{1,2,5,6,7\}$
1 MI.

$\{1,3,4,5,7\}$
1 MI.

$\{1,2,4,6,7\}$
1 MI.

$\{1,3,5,6,7\}$
1 MI.

$\{2,3,4,6,7\}$
1 MI.

$\{2,3,4,5,6\}$
1 MI.

$\{1,2,3,4,5\}$
1 MI.

$\{1,2,4,5,6\}$
1 MI.

$\{1,2,3,5,6\}$
1 MI.

$\{1, 3, 4, 5, 6\}$
1 MI.

$\{1, 2, 3, 4, 6\}$
1 MI.

$\{3, 5, 6, 7\}$
2 MIs.

$\{2, 4, 5, 7\}$
2 MIs.

$\{1, 3, 5, 6, 7\}$
2 MIs.

$\{1, 2, 5, 7\}$
2 MIs.

$\{2, 4, 6, 7\}$
2 MIs.

$\{1, 4, 5, 7\}$
2 MIs.

$\{2, 3, 5, 7\}$
2 MIs.

$\{1, 2, 4, 7\}$
2 MIs.

$\{2, 5, 6, 7\}$
2 MIs.

$\{1, 3, 5, 7\}$
2 MIs.

$\{2, 3, 5, 6\}$
1 MI.

$\{2, 3, 4, 5\}$
1 MI.

$\{1, 2, 5, 6\}$
1 MI.

$\{1, 2, 4, 5\}$
1 MI.

$\{1, 3, 4, 6\}$
1 MI.

B Polynomials for the definition of the symbol letters

The definition of polynomials $f_{35} \sim f_{68}$ that appear in the odd letters with 2 square roots, shown in (3.62) $\sim$ (3.70), are as follows.

$$f_{35} = -s^2 - 2st + sm_1^2 + sm_2^2 + sm_3^2 + m_1^2 m_3^2 - m_2^2 m_3^2 + sm_4^2 - m_1^2 m_4^2 + m_2^2 m_4^2, \quad \text{(B.1)}$$

$$f_{36} = -st + tm_1^2 - sm_2^2 - tm_2^2 - m_1^2 m_2^2 + m_2^4 + sm_3^2 - m_1^2 m_3^2 - m_2^2 m_3^2 + 2m_2^2 m_4^2, \quad \text{(B.2)}$$

$$f_{37} = -s^2 - st + 2sm_1^2 + tm_1^2 - m_1^4 + sm_2^2 - tm_2^2 + m_1^2 m_2^2 + sm_3^2 - m_1^2 m_3^2$$
$$- m_2^2 m_3^2 + 2m_2^2 m_4^2, \quad \text{(B.3)}$$

$$f_{38} = -st - sm_1^2 - tm_1^2 + m_1^4 + tm_2^2 - m_1^2 m_2^2 + 2m_1^2 m_3^2 + sm_4^2 - m_1^2 m_4^2 - m_2^2 m_4^2, \quad \text{(B.4)}$$

$$f_{39} = -s^2 - st + sm_1^2 - tm_1^2 + 2sm_2^2 + tm_2^2 + m_1^2 m_2^2 - m_2^4 + 2m_1^2 m_3^2 + sm_4^2$$
$$- m_1^2 m_4^2 - m_2^2 m_4^2, \quad \text{(B.5)}$$

$$f_{40} = -st + sm_1^2 + tm_3^2 - m_1^2 m_3^2 - sm_4^2 - tm_4^2 - m_1^2 m_4^2 + 2m_2^2 m_4^2 - m_3^2 m_4^2 + m_4^4, \quad \text{(B.6)}$$

$$f_{41} = -s^2 - st + sm_1^2 + 2sm_3^2 + tm_3^2 - m_1^2 m_3^2 - m_3^4 + sm_4^2 - tm_4^2 - m_1^2 m_4^2$$
$$+ 2m_2^2 m_4^2 + m_3^2 m_4^2, \quad \text{(B.7)}$$
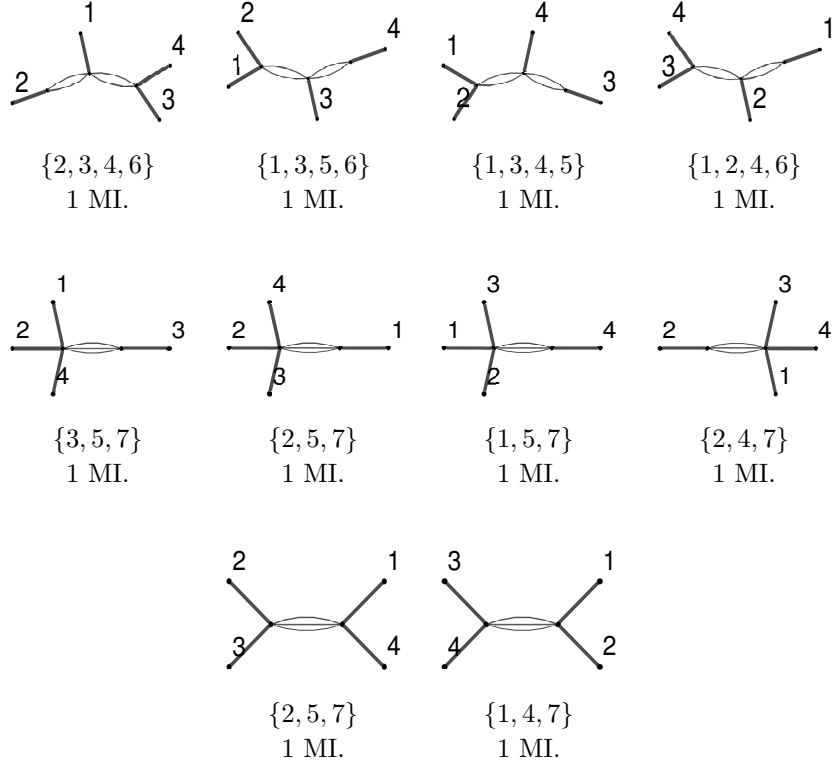
Figure 3. Diagrams for the two-loop master integrals with four external massive legs.

$$f_{42} = -st + sm_2^2 - sm_3^2 - tm_3^2 + 2m_1^2 m_3^2 - m_2^2 m_3^2 + m_3^4 + tm_4^2 - m_2^2 m_4^2 - m_3^2 m_4^2, \quad \text{(B.8)}$$

$$f_{43} = -s^2 - st + sm_2^2 + sm_3^2 - tm_3^2 + 2m_1^2 m_3^2 - m_2^2 m_3^2 + 2sm_4^2 + tm_4^2 - m_2^2 m_4^2$$
$$+ m_3^2 m_4^2 - m_4^4, \quad \text{(B.9)}$$

$$f_{44} = -s^2 t + stm_1^2 - s^2 m_2^2 - stm_2^2 - sm_1^2 m_2^2 + sm_2^4 + sm_1^2 m_3^2 - m_1^4 m_3^2 + sm_2^2 m_3^2$$
$$+ 2m_1^2 m_2^2 m_3^2 - m_2^4 m_3^2 + 2sm_2^2 m_4^2, \quad \text{(B.10)}$$

$$f_{45} = -s^2 t - s^2 m_1^2 - stm_1^2 + sm_1^4 + stm_2^2 - sm_1^2 m_2^2 + 2sm_1^2 m_3^2 + sm_1^2 m_4^2 - m_1^4 m_4^2$$
$$+ sm_2^2 m_4^2 + 2m_1^2 m_2^2 m_4^2 - m_2^4 m_4^2, \quad \text{(B.11)}$$

$$f_{46} = -s^2 t + stm_1^2 + stm_2^2 - 2sm_1^2 m_2^2 + sm_1^2 m_3^2 - m_1^4 m_3^2 + m_1^2 m_2^2 m_3^2 + sm_2^2 m_4^2$$
$$+ m_1^2 m_2^2 m_4^2 - m_2^4 m_4^2, \quad \text{(B.12)}$$

$$f_{47} = -s^2 t + stm_1^2 + stm_2^2 - 2sm_1^2 m_2^2 + s^2 m_3^2 - sm_1^2 m_3^2 - 2sm_2^2 m_3^2 - m_1^2 m_2^2 m_3^2$$
$$+ m_2^4 m_3^2 + sm_2^2 m_4^2 + m_1^2 m_2^2 m_4^2 - m_2^4 m_4^2, \quad \text{(B.13)}$$

$$f_{48} = -s^2 t + stm_1^2 + stm_2^2 - 2sm_1^2 m_2^2 + sm_1^2 m_3^2 - m_1^4 m_3^2 + m_1^2 m_2^2 m_3^2 + s^2 m_4^2$$
$$- 2sm_1^2 m_4^2 + m_1^4 m_4^2 - sm_2^2 m_4^2 - m_1^2 m_2^2 m_4^2, \quad \text{(B.14)}$$

$$f_{49} = -s^2 t + stm_3^2 + sm_1^2 m_3^2 - m_1^2 m_3^4 - s^2 m_4^2 - stm_4^2 + sm_1^2 m_4^2 + 2sm_2^2 m_4^2$$
$$- sm_3^2 m_4^2 + 2m_1^2 m_3^2 m_4^2 + sm_4^4 - m_1^2 m_4^4, \quad \text{(B.15)}$$

$$f_{50} = -s^2 t - s^2 m_3^2 - stm_3^2 + 2sm_1^2 m_3^2 + sm_2^2 m_3^2 + sm_3^4 - m_2^2 m_3^4 + stm_4^2 + sm_2^2 m_4^2$$
$$- sm_3^2 m_4^2 + 2m_2^2 m_3^2 m_4^2 - m_2^2 m_4^4, \quad \text{(B.16)}$$

$$f_{51} = -s^2t + stm_3^2 + sm_1^2m_3^2 - m_1^2m_3^4 + stm_4^2 + sm_2^2m_4^2 - 2sm_3^2m_4^2 + m_1^2m_3^2m_4^2$$
$$+ m_2^2m_3^2m_4^2 - m_2^2m_4^4, \tag{B.17}$$

$$f_{52} = -s^2t + s^2m_1^2 + stm_3^2 - sm_1^2m_3^2 + stm_4^2 - 2sm_1^2m_4^2 + sm_2^2m_4^2 - 2sm_3^2m_4^2$$
$$- m_1^2m_3^2m_4^2 + m_2^2m_3^2m_4^2 + m_1^2m_4^4 - m_2^2m_4^4, \tag{B.18}$$

$$f_{53} = -s^2t + s^2m_2^2 + stm_3^2 + sm_1^2m_3^2 - 2sm_2^2m_3^2 - m_1^2m_3^4 + m_2^2m_3^4 + stm_4^2$$
$$- sm_2^2m_4^2 - 2sm_3^2m_4^2 + m_1^2m_3^2m_4^2 - m_2^2m_3^2m_4^2, \tag{B.19}$$

$$f_{54} = -2st - t^2 + tm_1^2 + tm_2^2 - m_1^2m_2^2 + tm_3^2 + m_1^2m_3^2 + tm_4^2 + m_2^2m_4^2 - m_3^2m_4^2, \tag{B.20}$$

$$f_{55} = -st - t^2 + tm_1^2 - sm_2^2 + tm_2^2 - m_1^2m_2^2 + sm_3^2 + 2tm_3^2 - m_1^2m_3^2 + m_2^2m_3^2$$
$$- m_3^4 + 2m_2^2m_4^2, \tag{B.21}$$

$$f_{56} = -st - t^2 + sm_1^2 + 2tm_1^2 - m_1^4 + tm_3^2 - m_1^2m_3^2 - sm_4^2 + tm_4^2 + m_1^2m_4^2$$
$$+ 2m_2^2m_4^2 - m_3^2m_4^2, \tag{B.22}$$

$$f_{57} = -st - t^2 + sm_2^2 + 2tm_2^2 - m_2^4 - sm_3^2 + tm_3^2 + 2m_1^2m_3^2 + m_2^2m_3^2 + tm_4^2$$
$$- m_2^2m_4^2 - m_3^2m_4^2, \tag{B.23}$$

$$f_{58} = -st - t^2 - sm_1^2 + tm_1^2 + tm_2^2 - m_1^2m_2^2 + 2m_1^2m_3^2 + sm_4^2 + 2tm_4^2 + m_1^2m_4^2$$
$$- m_2^2m_4^2 - m_4^4, \tag{B.24}$$

$$f_{59} = -st^2 + stm_2^2 + stm_3^2 + tm_1^2m_3^2 - 2tm_2^2m_3^2 + m_1^2m_2^2m_3^2 - m_1^2m_3^4 + tm_2^2m_4^2$$
$$- m_2^4m_4^2 + m_2^2m_3^2m_4^2, \tag{B.25}$$

$$f_{60} = -st^2 + 2stm_2^2 - sm_2^4 + stm_3^2 + tm_1^2m_3^2 + sm_2^2m_3^2 - tm_2^2m_3^2 - m_1^2m_2^2m_3^2$$
$$+ m_2^4m_3^2 - m_1^2m_3^4 - m_2^2m_3^4 + 2m_2^2m_3^2m_4^2, \tag{B.26}$$

$$f_{61} = -st^2 + stm_2^2 + 2stm_3^2 + sm_2^2m_3^2 - tm_2^2m_3^2 + 2m_1^2m_2^2m_3^2 - m_2^4m_3^2 - sm_3^4$$
$$+ m_2^2m_3^4 + tm_2^2m_4^2 - m_2^4m_4^2 - m_2^2m_3^2m_4^2, \tag{B.27}$$

$$f_{62} = -st^2 + stm_1^2 + tm_1^2m_3^2 - m_1^4m_3^2 + stm_4^2 - 2tm_1^2m_4^2 + tm_2^2m_4^2 + m_1^2m_2^2m_4^2$$
$$+ m_1^2m_3^2m_4^2 - m_2^2m_4^4, \tag{B.28}$$

$$f_{63} = -st^2 + 2stm_1^2 - sm_1^4 + stm_4^2 + sm_1^2m_4^2 - tm_1^2m_4^2 + m_1^4m_4^2 + tm_2^2m_4^2$$
$$- m_1^2m_2^2m_4^2 + 2m_1^2m_3^2m_4^2 - m_1^2m_4^4 - m_2^2m_4^4, \tag{B.29}$$

$$f_{64} = -st^2 + stm_1^2 + tm_1^2m_3^2 - m_1^4m_3^2 + 2stm_4^2 + sm_1^2m_4^2 - tm_1^2m_4^2 - m_1^4m_4^2$$
$$+ 2m_1^2m_2^2m_4^2 - m_1^2m_3^2m_4^2 - sm_4^4 + m_1^2m_4^4, \tag{B.30}$$

$$f_{65} = -s^2t^2 + s^2tm_1^2 + stm_1^2m_3^2 - sm_1^4m_3^2 - stm_1^2m_4^2 + 2stm_2^2m_4^2 - sm_1^2m_2^2m_4^2$$
$$+ 2sm_1^2m_3^2m_4^2 - m_1^4m_3^2m_4^2 + m_1^2m_2^2m_3^2m_4^2 + m_1^2m_2^2m_4^4 - m_2^4m_4^4, \tag{B.31}$$

$$f_{66} = -s^2t^2 + s^2tm_2^2 + 2stm_1^2m_3^2 - stm_2^2m_3^2 - sm_1^2m_2^2m_3^2 - m_1^4m_3^4 + m_1^2m_2^2m_3^4$$
$$+ stm_2^2m_4^2 - sm_2^4m_4^2 + 2sm_2^2m_3^2m_4^2 + m_1^2m_2^2m_3^2m_4^2 - m_2^4m_3^2m_4^2, \tag{B.32}$$

$$f_{67} = -s^2t^2 + s^2tm_3^2 + stm_1^2m_3^2 - stm_2^2m_3^2 + 2sm_1^2m_2^2m_3^2 - sm_1^2m_3^4 - m_1^2m_2^2m_3^4$$
$$+ 2stm_2^2m_4^2 - sm_2^2m_3^2m_4^2 + m_1^2m_2^2m_3^2m_4^2 + m_2^4m_3^2m_4^2 - m_2^4m_4^4, \tag{B.33}$$

$$f_{68} = -s^2t^2 + 2stm_1^2m_3^2 - m_1^4m_3^4 + s^2tm_4^2 - stm_1^2m_4^2 + stm_2^2m_4^2 + 2sm_1^2m_2^2m_4^2$$
$$- sm_1^2m_3^2m_4^2 + m_1^4m_3^2m_4^2 + m_1^2m_2^2m_3^2m_4^2 - sm_2^2m_4^4 - m_1^2m_2^2m_4^4 \tag{B.34}$$

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