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Quantum field theoretic representation of Wilson surfaces. Part I. Higher coadjoint orbit theory

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Abstract: This is the first of a series of two papers devoted to the partition function realization of Wilson surfaces in strict higher gauge theory. A higher version of the Kirillov-Kostant-Souriau theory of coadjoint orbits is presented based on the derived geometric framework, which has shown its usefulness in 4-dimensional higher Chern-Simons theory. An original notion of derived coadjoint orbit is put forward. A theory of derived unitary line bundles and Poisson structures on regular derived orbits is constructed. The proper derived counterpart of the Bohr-Sommerfeld quantization condition is then identified. A version of derived prequantization is proposed. The difficulties hindering a full quantization, shared with other approaches to higher quantization, are pinpointed and a possible way-out is suggested. The theory we elaborate provide the geometric underpinning for the field theoretic constructions of the companion paper.

Keywords: Differential and Algebraic Geometry, Sigma Models, Topological Field Theories, Wilson, 't Hooft and Polyakov loops

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Contents

1 Introduction

Wilson loops were introduced by Wilson in 1974 [1] as a natural set of gauge invariant variables suitable for the description of the non perturbative regime of quantum chromodynamics. Since then, they have been widely employed in lattice gauge theory.

In the loop formulation of gauge theory [2, 3], the quantum Hilbert space consists of gauge invariant wave functionals on the gauge field configuration space. According to a theorem of Giles [4], Wilson loops constitute a basis of the Hilbert space allowing to switch from the gauge field to the loop representation.

Wilson loops are fundamental constitutive elements of a canonical formulation of quantum gravity as a gauge theory, known as loop quantum gravity [5], and their incorporation has led to the very powerful spin network and foam approaches of this latter [6].

Wilson loops are relevant also in condensed matter physics at low energy, specifically in the study of topologically ordered phases of matter described by topological quantum field theories. In models of fractional quantum Hall states as well as lattice models such as Kitaev's toric code [7], fractional braiding statistics between quasiparticles emerges through the correlation function of a pair of Wilson loops forming a Hopf link [8, 9].

Wilson loops depend on the topology of the underlying knots and, as shown in Witten's foundational work [10], they can be employed to study knot topology in 3-dimensional Chern-Simons (CS) theory using basic techniques of quantum field theory. CS correlators of Wilson loop operators provide a variety of knot and link invariants.

Higher gauge theory is a generalization of ordinary gauge theory where gauge fields are higher degree forms [11, 12]. It is considered to be a promising candidate for the description of the dynamics of the higher dimensional extended objects occurring in supergravity and string theory thought to be the basic constituents of matter and mediators of fundamental interactions (see [13] for an updated general overview). Higher gauge theory is relevant also in spin foam theory [14] and condensed matter physics [15].

Wilson surfaces [16–23], 2-dimensional counterparts of Wilson loops, emerge naturally in theories with higher form gauge fields such as those mentioned in the previous paragraph and are so expected to be relevant in the analysis of various basic aspects of them for reasons analogous to those for which Wilson loops are.

In 4 spacetime dimensions, particle-like excitations cannot braid and have only ordinary bosonic/fermionic statistics. Fractional braiding statistics can still occur through the braiding of either a point-like and a loop-like or two loop-like excitations. This has been adequately described through the correlation functions of Wilson loops and surfaces in BF type topological quantum field theories [24–26].

Wilson surfaces also should be a basic element of any field theoretic approach to 4-dimensional 2-knot topology [27, 28]. Based on Witten's paradigm, it should be possible to study surface knot topology in 4-dimensions computing correlators of Wilson surfaces in an appropriate 4-dimensional version of CS theory using again techniques of quantum field theory [29–31].

The aim of the present two-part study is constructing a 2-dimensional topological sigma model whose quantum partition function yields a Wilson surface in strict higher gauge theory on the same lines as the 1-dimensional topological sigma model providing a Wilson loop in ordinary gauge theory. The path we have in mind is described in the next subsections.

1.1 Wilson loops as partition functions

The idea of representing a given Wilson loop as the partition function of a 1-dimensional sigma model has a long history. In the context of 4-dimensional Yang-Mills theory, this formulation can be traced back to the work of Balachandran et al. [32]. The approach was subsequently developed by Alekseev et al. in [33] and Diakonov and Petrov in [34, 35]. More recently, it was applied to the canonical quantization of CS theory by Elitzur et al. in [36]. The functional integral expression of a Wilson loop holds in fact in general for any gauge theory in any dimension. Below, we briefly outline the principles on which this theoretical framework is based. See also [37–39] for clear illustrations of the underlying theory and some of its most significant applications.

The definition of ordinary Wilson loops in gauge theory is well-known. In a gauge theory with gauge group G, a Wilson loop $W_R(C)$ depends on a representation R of G and an oriented loop C in the spacetime manifold M and is given by the gauge invariant trace in R of the holonomy of the gauge field ω along C,

$$W_R(C) = \mathrm{tr}_R \,\mathrm{Pexp}\left(-\int_C \omega\right). \tag{1.1}$$

This description of $W_R(C)$ is intrinsically quantum mechanical [10]. Expression (1.1) clearly indicates that $W_R(C)$ may be identified with the partition function of some auxiliary fictitious quantum system. The representation space of R corresponds to the Hilbert space $\mathcal{H}$ of the system, the trace over R to the usual trace over $\mathcal{H}$ and the gauge field ω specifying holonomy to the Hamiltonian operator H governing time-evolution. The correspondence established in this way is summarized schematically by

$$R \longleftrightarrow \mathcal{H}, \qquad \omega \longleftrightarrow iH. \tag{1.2}$$

From such a standpoint, $W_R(C)$ takes so the form

$$W_R(C) = \mathrm{tr}_{\mathcal{H}} \,\mathrm{Texp}\left(-i\int_C H\right). \tag{1.3}$$

The inherent quantum mechanical nature of the Wilson loop $W_R(C)$ rules out any possibility of performing semiclassical path integral manipulations in gauge theory with

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Wilson loop insertions based solely on the expression (1.1). If we still want to achieve a description of such a kind, an independent fully semi-classical functional integral expression of $W_R(C)$ is required.

The description of $W_R(C)$ alluded to in the previous paragraph must necessarily be based upon a 1-dimensional field theory on the loop C compatible with the correspondence (1.2) and its implications upon quantization. The gauge group G should further act as a symmetry group to account for the gauge invariance of $W_R(C)$. Therefore, it is plausible that this field theory is a 1-dimensional sigma model featuring a G-valued auxiliary bosonic field g coupled to the gauge field ω acting as a background field. The expression of $W_R(C)$ we are aiming to should then have the schematic form

$$W_R(C) = \int \mathscr{D}g \exp\left(i\, S_R(g, \omega)\right), \tag{1.4}$$

where $S_R(g, \omega)$ is a gauge-invariant action functional of g and the restriction of ω to C depending upon the representation R given by an integral on C of a local Lagrangian density.

1.2 Wilson loops and coadjoint orbits

The quantum system underlying the partition function realization of a Wilson loop can be described quite explicitly as we review below.

We assume that G is a compact semisimple Lie group and that R is an irreducible representation of G. R is uniquely characterized up to equivalence by its highest weight λ and so we write $R = R_\lambda$. As is well-known, $\lambda \in \Lambda_{\mathsf{wG}}{}^+$, the lattice of dominant weights of G in the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G.

In general, with any element $\lambda \in \mathfrak{g}^*$ there is associated the coadjoint orbit $\mathscr{O}_\lambda = \{\mathrm{Ad}^* \gamma(\lambda) | \gamma \in \mathsf{G}\}$. $\mathscr{O}_\lambda$ is a homogeneous space: $\mathscr{O}_\lambda = \mathsf{G}/\mathsf{G}_\lambda$, where G_λ is the stabilizer subgroup of λ. G is in this way structured as a principal G_λ-bundle over $\mathscr{O}_\lambda$. Forms on $\mathscr{O}_\lambda$ are thus representable as forms on G which are horizontal and invariant with respect to the multiplicative right G_λ-action.

The left multiplicative action of G on itself induces owing to its commutativity with the right G_λ-action a G-action on the coadjoint orbit $\mathscr{O}_\lambda$. This action constitutes a primal property of $\mathscr{O}_\lambda$.

In Kirillov-Kostant-Souriau (KKS) theory [40], the coadjoint orbit $\mathscr{O}_\lambda$ is promoted to a symplectic manifold by equipping it with the symplectic 2-form

$$\nu_\lambda = \frac{1}{2}\langle \lambda, [\gamma^{-1} d\gamma, \gamma^{-1} d\gamma]\rangle, \tag{1.5}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$. In this way, $\mathscr{O}_\lambda$ is endowed with a Poisson bracket structure $\{\cdot, \cdot\}$. ν_λ is invariant under the G-action. This latter is actually Hamiltonian. Its moment map $q_\lambda : \mathfrak{g} \to \mathrm{C}^\infty(\mathscr{O}_\lambda)$ is given by

$$q_\lambda(x) = \langle \mathrm{Ad}^* \gamma(\lambda), x\rangle \tag{1.6}$$

with $x \in \mathfrak{g}$ and satisfies the Poisson bracket

$$\{q_\lambda(x), q_\lambda(y)\} = q_\lambda([x, y]) \tag{1.7}$$

for $x, y \in \mathfrak{g}$. As q is a Lie algebra morphism, the Hamiltonian governing the classical system underlying the Wilson loop $W_{R_\lambda}(C)$ is assumably

$$H_\lambda = q_\lambda(\varsigma^*\omega), \tag{1.8}$$

where $\varsigma : C \to M$ is the embedding of C in the spacetime manifold M. The quantization of the coadjoint orbit $\mathcal{O}_\lambda$ can now be carried in two distinct though related ways. Below, we restrict for simplicity to the case where λ is regular, that is the stabilizer subgroup G_λ of λ is a maximal torus of G.

The techniques of geometric quantization [41–43] can be applied to the coadjoint orbit $\mathcal{O}_\lambda$ if the Bohr-Sommerfeld quantization condition is satisfied requiring that the cohomology class $[\nu_\lambda/2\pi] \in H^2(\mathcal{O}_\lambda, \mathbb{R})$ lies in the image of $H^2(\mathcal{O}_\lambda, \mathbb{Z})$. This happens precisely when $\lambda \in \Lambda_{\mathrm{w}\mathsf{G}}$, the weight lattice of G, just the situation we are interested in. By the integrality of $\nu_\lambda/2\pi$, there is a unitary line bundle $\mathcal{L}_\lambda$ on $\mathcal{O}_\lambda$ with curvature form $i\nu_\lambda$. $\mathcal{L}_\lambda$ is endowed with a canonical Hermitian metric and the G-invariant unitary connection

$$D_\lambda = d - i\langle \lambda, \gamma^{-1} d\gamma \rangle. \tag{1.9}$$

It turns out that there is a G-invariant complex structure J_λ on $\mathcal{O}_\lambda$ compatible with ν_λ. J_λ and ν_λ constitute in this way an invariant Kaehler structure on $\mathcal{O}_\lambda$ providing a complex polarization. Since the curvature $i\nu_\lambda$ of $\mathcal{L}_\lambda$ equals the Kaehler form, $\mathcal{L}_\lambda$ is also a holomorphic line bundle. All the elements of geometric quantization are in place in this way. The quantum Hilbert space is then identified with the space of holomorphic sections of $\mathcal{L}_\lambda$

$$\mathcal{H}_\lambda = H^0_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_\lambda). \tag{1.10}$$

The Hilbert inner product is the one naturally induced by the Hermitian structure of $\mathcal{L}_\lambda$ and the symplectic form ν_λ. The quantization map is constructed based on the connection D_λ. A unitary G-action on $\mathcal{H}_\lambda$ is associated with the G-action of $\mathcal{O}_\lambda$. The quantization of the moment map q_λ is the corresponding infinitesimal generator. Geometric quantization provides in this manner the quantum set-up required for expressing the Wilson loop $W_{R_\lambda}(C)$ according to (1.3).

The Borel-Weil-Bott theorem [44, 45] connects the quantization of $\mathcal{O}_\lambda$ described above to the representation R_λ. It states that $H^0_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_\lambda) \neq 0$ precisely when $\lambda \in {\Lambda_{\mathrm{w}\mathsf{G}}}^+$ and that in that case under the G-action $H^0_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_\lambda)$ is the representation space of the representation R_λ.

The methods of functional integral quantization [46] can also be applied to $\mathcal{O}_\lambda$. To this end, we need to begin with an action S_λ. This has the standard form $S_\lambda = \int_C (\varpi_\lambda - H_\lambda)$, where ϖ_λ is a symplectic potential of ν_λ satisfying $\nu_\lambda = d\varpi_\lambda$,

$$\varpi_\lambda = -\langle \lambda, \gamma^{-1} d\gamma \rangle, \tag{1.11}$$

and H is the Hamiltonian (1.8). The action thus reads

$$S_\lambda(g, \omega) = \int_C \langle \lambda, \mathrm{Ad}\, g^{-1}(\varsigma^*\omega) + g^{-1} dg \rangle \tag{1.12}$$

after a conventional overall sign redefinition. Here, g is a G valued field on the closed oriented curve C. Hence, the action S_λ is a functional on the mapping space $\mathrm{Map}(C, \mathsf{G})$

that generically describes a 1-dimensional sigma model with target space G. Since we are interested in the coadjoint orbit $\mathcal{O}_\lambda = \mathsf{G}/\mathsf{G}_\lambda$ instead than G, S_λ, or better its exponentiated form $\exp(iS_\lambda)$ relevant in functional integral quantization, should rather be a functional on the mapping space $\mathrm{Map}(C, \mathsf{G}/\mathsf{G}_\lambda)$. This requires that $S_\lambda(g', \omega) = S_\lambda(g, \omega) \bmod 2\pi\mathbb{Z}$ for $g' = gv$ with $v \in \mathrm{Map}(C, \mathsf{G}_\lambda)$ resulting again in the condition that $\lambda \in \Lambda_{\mathrm{w}\mathsf{G}}$. The quantization of $\mathcal{O}_\lambda$ is now given formally by the partition function

$$Z_\lambda(C) = \int_{\mathrm{Map}(C,\mathrm{G}/\mathrm{G}_\lambda)} \mathscr{D}g \exp\left(i\, S_\lambda(g, \omega)\right). \tag{1.13}$$

The Wilson loop $W_{R_\lambda}(C)$ equals $Z_\lambda(C)$ up to an overall normalization in accordance with (1.4). This identification can be tested in a number of ways by verifying that $Z_\lambda(C)$ enjoys the properties which $W_{R_\lambda}(C)$ does. In fact, as a functional of the gauge field ω, $Z_\lambda(C)$ is gauge invariant. Furthermore, when ω is flat, $Z_\lambda(C)$ is also invariant under smooth variations of the embedding ς as a consequence of certain Schwinger-Dyson relations.

The 1-dimensional sigma model summarily described in the previous paragraph has properties analogous to those of 3-dimensional CS theory. It is in fact a Schwarz type topological quantum field theory. In this paper, we shall call it topological coadjoint orbit (TCO) model for reference.

1.3 Wilson surfaces as partition functions

The natural question arises about the degree to which the above analysis can be extended and adapted to Wilson surfaces. In this paper, we shall consider Wilson surfaces in the simplest version of higher gauge theory, the strict one. As this is the only form of higher gauge theory that we shall deal with, we shall omit the specification 'strict' in the rest of our discussion.

Higher gauge symmetry hinges on Lie group crossed modules. For the purpose of this brief outline, it is sufficient to recall that a Lie group crossed module M consists of two Lie groups G, E together with two structure maps relating them and enjoying certain properties [47, 48]. A higher gauge theory with structure crossed module M features a 1-form gauge field ω and a 2-form gauge field $\varOmega$ valued respectively in the Lie algebras $\mathfrak{g}$, $\mathfrak{e}$ of G, E [49, 50].

A Wilson surface is naturally given by a straightforward extension of the familiar Wilson formula (1.1),

$$W_R(N) = \mathrm{tr}_R\, \mathrm{Sexp}\left(-\int_N \varOmega\right). \tag{1.14}$$

Above, N is an oriented closed surface. tr_R denotes an invariant trace. Sexp signifies sweep ordered exponentiation, a 2-dimensional counterpart of path ordered exponentiation.[1] $\mathrm{Sexp}\left(-\int_N \varOmega\right)$ depends on two choices: the gauge of $\varOmega$ and the marking of N by

[1]The notion of sweep ordering is purely heuristic and is used here only to stress its analogy to path ordering. Strictly speaking, in higher gauge theory it is not possible to define 1- and 2-dimensional holonomies independently. $\mathrm{Sexp}\left(-\int_N \varOmega\right)$ depends not only on the 2-form gauge field $\varOmega$ but secretly also on the 1-form gauge field ω, even though this is not shown by our notation, and is well defined only if ω, $\varOmega$ satisfy the so-called vanishing fake curvature condition. See [18, 19] and in particular section 2.1 of [20] for a rigorous treatment of this point.

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a homotopically trivial base loop (analogous to the pointing of a loop by a base point). The invariance of tr_R ensures that the value of $W_R(N)$ is unaffected by the variation of $\mathrm{Sexp}\left(-\int_N \Omega\right)$ resulting from a change of such data. The above elements are only qualitatively sketched. A more precise definition together with an in-depth analysis of their properties can be found in refs. [21–23].

Based on the characterization of Wilson loops illustrated in subsection 1.1, it is natural to ask whether a Wilson surface can be expressed as the partition function of an auxiliary quantum system in a manner analogous to that a Wilson loop does. Assuming anyway that this is indeed possible, the issue is then posed about the precise description of the partition function as the Hilbert space trace of the evolution operator of the system and the path integral of an associated sigma model on the lines of subsection 1.1. The 2-dimensional nature of the underlying surface entails in any case that we have to investigate this matter in the realm of 2-dimensional quantum field theory.

1.4 Our approach to Wilson surfaces

The problem of obtaining a functional integral realizations of a Wilson surface, raised at the end of the previous subsection, has already been tackled in the literature from different perspectives [51–53]. In this subsection, we shall outline our approach to the subject firmly framed in higher gauge theory.

As already anticipated, higher gauge symmetry rests on Lie group crossed modules. Our handling of crossed modules is based on the derived set-up originally worked out in refs. [54, 55]. It is in essence a superfield formalism providing an efficient way of encoding most of the structural features of crossed modules and proceeds by associating with any crossed module a derived Lie group, a graded Lie group with a structure determined by that of the crossed module. At the infinitesimal level, higher gauge symmetry is described by Lie algebra crossed modules. With any such module there is similarly attached a derived Lie algebra. The whole derived set-up is compatible with Lie differentiation.

The relevant fields of a higher gauge theory are just crossed module valued inhomogeneous form fields. They can be dealt with in the derived framework in a very elegant and compact manner as derived fields, that is derived Lie group and algebra valued fields. The resulting derived field formalism allows one to cast any higher gauge theory as a derived gauge theory, basically an ordinary gauge theory with the derived group as gauge group. The higher gauge fields and gauge transformations, once expressed in derived form, can then be manipulated very much as their ordinary counterparts. In this manner, by highlighting the close correspondence of the higher to the ordinary setting, the derived formalism enables one to import many ideas and techniques of the latter to the former.

The derived field formalism has been successfully applied in ref. [31] to the formulation of 4-dimensional CS theory, a higher gauge theoretic enhancement of familiar 3-dimensional CS theory. The tight formal relationship of the 4-dimensional theory to the 3-dimensional one brought out by the derived design has shown itself to be very useful in the analysis of the properties of the model. It is reasonable to expect that the derived set-up could be the most appropriate formal framework also for the investigation of Wilson surfaces construed as higher gauge theoretic extension of Wilson loops.

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Our approach to the realization of Wilson surfaces as partition functions consists therefore in extending the ordinary geometric or functional integral quantization schemes of the coadjoint orbits reviewed in subsection 1.2 to a higher crossed module theoretic setting by relying on the derived formal set-up. This however is not simply a matter of a straightforward derived rewriting of these well-established approaches to coadjoint orbit quantization. There are in fact very basic elements of such schemes which do not have any fitting higher counterparts for reasons which we are going to survey momentarily. These issues are inevitably going to come to the surface in some form also in the derived approach.

To the best of our knowledge, there are no obvious counterparts of the notions of coadjoint action and orbit for Lie group crossed modules. Further, there is no fully developed representation theory and no analog of the highest weight theorem for crossed modules. While the derived formulation should provide in principle the definition of these higher objects and describe their properties, the way this is achieved in practice is far from clear.

Lie crossed modules belong to the realm of higher Lie theory. So, the geometric quantization of a derived coadjoint orbit, no matter the way it is conceived, presumably must be formulated in the framework of multisymplectic geometry [56]. Higher geometric quantization of multisymplectic manifolds is however a subject not fully understood yet (see however refs. [57–60] for a variety of approaches to this issue). Since a 2-dimensional field theory must be the end result of quantization as already noticed, geometric quantization may alternatively be based on a symplectic loop space [61, 62], proceeding via transgression from a finite dimensional 2-plectic space on the lines of ref. [63]. The infinite dimensional geometry involved in this approach is however problematic to deal with.

The uncertainties affecting a workable theory of derived coadjoint orbits render problematic also the construction of the derived TCO sigma model on which the functional integral quantization of one such orbit should be based, if as expected crucial elements of the orbit's geometry are required by the model's formulation.

Higher gauge theory possesses in addition to a gauge symmetry also a gauge for gauge symmetry. The latter should emerge in the derived TCO model as a novel gauge symmetry with non counterpart in the ordinary model and requiring a special handling.

In fact, all the themes discussed above eventually emerge and are dealt with in the derived formulation, but in a novel and unified manner.

1.5 Plan of the endeavour

The present endeavour is naturally divided in two parts, henceforth referred to as I and II, of which the present paper is the first.

In I, a higher version of the KKS theory of coadjoint orbits is presented based on the derived geometric framework. An original notion of derived coadjoint orbit is proposed. A theory of derived unitary line bundles and Poisson structures on regular derived orbits is constructed. The proper derived counterpart of the Bohr-Sommerfeld quantization condition is then identified. A version of derived prequantization is put forward. The difficulties hindering a full quantization are discussed and a possible way-out is suggested. The theory elaborated and the results obtained, mainly of a geometric nature, provide a basic underpinning for the field theoretic constructions of II.

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In II, the derived TCO sigma model is presented and studied. Its manifold symmetries are described. Its quantization is analyzed in the functional integral framework. Strong evidence is provided that the model does indeed underlie the partition function realization of a Wilson surface. The origination of the vanishing fake curvature condition is explained and homotopy invariance for flat derived gauge field is shown. The model's Hamiltonian formulation is elaborated and through this the its close relationship to the derived KKS theory developed in I is highlighted.

2 Part I: derived KKS theory

The present paper, which constitutes part I of our endeavour, is devoted to derived KKS theory. In this section, we provide an introductory overview of this subject and an outlook on future developments.

Our presentation of derived KKS theory employs the language of graded differential geometry. The reader is referred to the appendices for useful background and e.g. ref. [64] for a thorough exposition of this subject. The graded geometric set-up is naturally suited for the description of the higher geometric structures dealt with in this paper. It subsumes the ordinary differential geometric one, but at the same times it enriches and broadens it allowing for a series of non standard constructions which otherwise would not be possible.

The organizing principle of our construction of derived KKS theory is operational calculus, a formal extension of classic Cartan calculus that has found a wide range of applications in differential geometry and topology [65]. The operational framework furnishes indeed an efficient and elegant means of describing the basic geometry of the principal bundles occurring in KKS theory.

2.1 Plan of the part I

Paper I is organized in a number of sections and appendices as follows.

In section 3, we survey the basic notions of the derived theory of Lie group crossed modules and review the main results of the derived field formalism used throughout the present work.

In section 4, we provide an overview of standard KKS theory and geometric quantization of coadjoint orbits. Our presentation of the subject is unconventional and partial: it relies on an operational description and touches the subject of quantization only marginally. It is however designed in a way that directly points to the derived extension constructed in the following section.

In section 5, we construct a derived KKS theory drawing inspiration from ordinary KKS theory as exposed in section 4, exploiting the advantages provided by the operational set-up and employing the full power of the derived approach. In particular, we introduce an appropriate definition of derived coadjoint orbit. We further elaborate a suitable notion of derived prequantum line bundle and connection thereof and use a connection's curvature to construct a derived presymplectic structure satisfying the appropriate Bohr-Sommerfeld quantization condition. We also lay the foundations for derived orbit prequantization.

The section ends with the determination of the derived counterpart of the classic KKS symplectic structure.

Finally, in the appendices of section A, we collect basic notions of graded geometry and operation theory used throughout the paper.

2.2 Overview of derived KKS theory

Since the remarks of subsection 1.4 are merely qualitative, we provide in this subsection a somewhat more formal introduction to derived KKS theory to more precisely delineate the subject and facilitate the reading of the paper. A more rigorous and complete analysis of the material surveyed below is available in the main body of this paper.

As we indicated earlier, a Lie group crossed module M features a pair of Lie groups E and G, the module's source and target groups. To these there are added an equivariant morphism $\tau : \mathsf{E} \to \mathsf{G}$ and an action $\mu : \mathsf{G} \times \mathsf{E} \to \mathsf{E}$ of G on E by automorphisms, the module's target and action structure maps, with certain natural properties.

The derived Lie group $\mathrm{D}\mathsf{M}$ of M is a graded Lie group built out of G and the degree shifted variant $\mathfrak{e}[1]$ of the Lie algebra $\mathfrak{e}$ of E, viz the semidirect product

$$\mathrm{D}\mathsf{M} = \mathfrak{e}[1] \rtimes_{\dot{\mu}} \mathsf{G}, \tag{2.1}$$

where here and below the dot $\dot{}$ denotes Lie differentiation with respect to the relevant variable.

The derived formalism on which our formulation of higher KKS theory is based is graded geometric in nature. A derived map Φ is a map from the degree shifted tangent bundle $T[1]X$ of the relevant manifold X into either the derived Lie group $\mathrm{D}\mathsf{M}$ or its Lie algebra $\mathrm{D}\mathfrak{m}$ or one of its degree shifted modifications. It always has two components ϕ and $\varPhi$. ϕ is valued in either the Lie group G or its Lie algebra $\mathfrak{g}$ or a degree shifted variant of it; $\varPhi$ in either the Lie algebra $\mathfrak{e}$ of E or a degree shifted variant thereof. Moreover, one has $\deg \varPhi = \deg \phi + 1$. A degree 1 nilpotent derived differential d extending the ordinary de Rham differential d is also available.

The derived formalism can be employed to describe the derived group $\mathrm{D}\mathsf{M}$ itself. Analogously to any ordinary Lie group, $\mathrm{D}\mathsf{M}$ is characterized by a derived group variable Γ and the associated derived Maurer-Cartan form $\Sigma = \Gamma^{-1}\mathrm{d}\Gamma$.

With any element $\varLambda \in \mathfrak{e}$, there is associated a crossed submodule $\mathrm{Z}\mathsf{M}_\varLambda$ of M, the centralizer crossed module of $\varLambda$, which is the largest crossed submodule of M whose associated adjoint and μ actions both leave $\varLambda$ invariant.

In derived theory, we attach derived groups $\mathrm{D}\mathsf{M}$ and $\mathrm{D}\mathrm{Z}\mathsf{M}_\varLambda$ to M and $\mathrm{Z}\mathsf{M}_\varLambda$ respectively. Since $\mathrm{Z}\mathsf{M}_\varLambda$ is a crossed submodule of M, $\mathrm{D}\mathrm{Z}\mathsf{M}_\varLambda$ is a subgroup of $\mathrm{D}\mathsf{M}$. The derived coadjoint orbit of $\varLambda$ is the derived homogeneous space

$$\mathcal{O}_\varLambda = \mathrm{D}\mathsf{M}/\mathrm{D}\mathrm{Z}\mathsf{M}_\varLambda. \tag{2.2}$$

$\mathcal{O}_\varLambda$ is a non negatively graded manifold of degree 1, as both the derived groups $\mathrm{D}\mathsf{M}$ and $\mathrm{D}\mathrm{Z}\mathsf{M}_\varLambda$ themselves are. The notion of derived coadjoint orbit given here manifestly

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replicates in a derived key that of ordinary coadjoint orbit as homogeneous space recalled in subsection 1.2.

A Lie group crossed module M is said to be compact if the target Lie group G is compact. For a compact crossed module, a notion of maximal toral crossed submodule can be defined analogous to that of maximal torus of a compact group. Indeed, when J is a maximal toral crossed submodule of a compact crossed module M, the target group T of J is a maximal torus of the target group G of M. The regular elements $\Lambda \in \mathfrak{e}$ are those for which the centralizer ZM_Λ is a maximal toral crossed submodule J of M and so the associated derived coadjoint orbit $\mathcal{O}_\Lambda$ is the regular homogeneous space $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$, just as in the ordinary theory of subsection 1.2.

The study of derived coadjoint orbits is most effectively done by studying general derived homogeneous spaces of the form $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{M}'$, where M' is a crossed submodule of a Lie group crossed module M. The operational approach advocated in refs. [54, 55] turns out to be quite natural and useful to this end, as $\mathrm{D}\mathsf{M}$ can be regarded as a principal $\mathrm{D}\mathsf{M}'$-bundle over $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{M}'$, whose basic geometry is aptly described by a $\mathrm{D}\mathsf{M}'$-operation. In this way, derived maps on $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{M}'$ of a given kind can be regarded as basic maps on $\mathrm{D}\mathsf{M}$ of the same kind. The operational set-up allows further to efficiently construct such basic maps by employing the derived variable Γ and Maurer Cartan element Σ of $\mathrm{D}\mathsf{M}$.

When M is a compact Lie group crossed module and J is a maximal toral crossed submodule of M, the regular derived homogeneous space $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$ can be considered in particular. A derived theory of unitary line bundles and connections thereof can then be worked out. Indeed, as for an ordinary compact Lie group with a maximal torus, characters of J can be defined and with any character β of J one can associate a derived unitary line bundle $\mathcal{L}_\beta$ on $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$ whose typical fiber is a proper derived version $\mathrm{D}\mathbb{C}$ of the complex line $\mathbb{C}$. A derived unitary connection A of $\mathcal{L}_\beta$ is a derived $\mathsf{U}(1)$ gauge field with certain properties in the underlying $\mathrm{D}\mathsf{J}$-operation. A has a 1-form component a and a 2-form component A. Its curvature $\mathrm{B} = \mathrm{d}\mathrm{A}$ has therefore a 2-form component b and 3-form component B.

The curvature B of a connection A of a derived line bundle $\mathcal{L}_\beta$ is d-closed by virtue of the derived Bianchi identity. In this way, $-i\mathrm{B}$ constitutes a derived presymplectic form, which can be used to construct a derived Poisson structure $\{\cdot,\cdot\}_{\mathrm{A}}$ on a suitable space of derived Hamiltonian functions $\mathrm{DF}\textsc{nc}_{\mathrm{A}}(\mathrm{D}\mathsf{M})$. The functions of $\mathrm{DF}\textsc{nc}_{\mathrm{A}}(\mathrm{D}\mathsf{M})$ are basic and therefore represent functions on $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$. A derived presymplectic structure of this kind belongs to the realm of multisymplectic geometry, since the 3-form component $-iB$ is closed in the usual sense. They further obey a derived Bohr-Sommerfeld quantization condition, as they arise from connections of $\mathcal{L}_\beta$.

For a suitably non singular derived presymplectic structure $-i\mathrm{B}$ of the above type, it is possible to define a natural derived prequantization map. There exists a distinguished subspace $\mathrm{DF}\textsc{nc}_{\mathrm{A}\mathsf{h}}(\mathrm{D}\mathsf{M})$ of the derived Hamiltonian function space $\mathrm{DF}\textsc{nc}_{\mathrm{A}}(\mathrm{D}\mathsf{M})$ closed under derived Poisson bracketing constituted by the prequantizable functions. The prequantization map assigns to any function $\mathrm{F} \in \mathrm{DF}\textsc{nc}_{\mathrm{A}\mathsf{h}}(\mathrm{D}\mathsf{M})$ a first order differential operator $\widehat{\mathrm{F}}$

acting on a space $\mathrm{D}\Omega^0{}_{\mathsf{h}}(\mathcal{L}_\beta)$ of 0-form derived sections of $\mathcal{L}_\beta$. The map is such that

$$[\widehat{\mathrm{F}}, \widehat{\mathrm{G}}] = i\widehat{\{\mathrm{F}, \mathrm{G}\}}_{\mathrm{A}} \tag{2.3}$$

for any two functions $\mathrm{F}, \mathrm{G} \in \mathrm{DF}_{\mathrm{NC}_{\mathrm{A}\mathsf{h}}}(\mathsf{DM})$. However, there is no prequantum Hilbert space structure with respect to which the operators yielded by derived prequantization are formally Hermitian.

Let M again be a compact Lie group crossed module and equipped with a non singular bilinear pairing $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{e} \to \mathbb{R}$ invariant under the adjoint and μ G-actions. If $\varLambda \in \mathfrak{e}$ is a regular element, the associated derived coadjoint orbit is $\mathcal{O}_\varLambda = \mathsf{DM}/\mathsf{DJ}$ for some maximal toral crossed submodule J of M. If $\varLambda$ satisfies furthermore a certain quantization condition, $\xi_\varLambda := \langle \cdot, \varLambda \rangle$ is an element of the dual integral lattice of the maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. $\xi_\varLambda$ yields in turn a character $\beta_\varLambda$ of J and so a derived line bundle $\mathcal{L}_\varLambda := \mathcal{L}_{\beta_\varLambda}$. $\mathcal{L}_\varLambda$ possesses a canonical connection $\mathrm{A}_\varLambda$. The curvature $\mathrm{B}_\varLambda$ of $\mathrm{A}_\varLambda$ furnishes a derived symplectic structure with 2- and 3-form component

$$-ib_\varLambda = \frac{1}{2}\langle [\sigma, \sigma], \varLambda \rangle, \tag{2.4}$$

$$-iB_\varLambda = \langle \dot{\tau}(\varLambda), \dot{\mu}(\sigma, \varSigma) \rangle, \tag{2.5}$$

where σ and $\varSigma$ are the degree 1- and 2-form components of the Maurer-Cartan form Σ. This is the derived KKS presymplectic structure, the sought for extension of the classic KKS symplectic structure (1.5).

2.3 Outlook

Though we have gone a long way toward reaching the goal of constructing a higher KKS theory, as essential geometrical underpinning of the realization of Wilson surfaces as partition functions, a few basic issues remain unsolved.

Derived geometric prequantization ostensibly does not admit a prequantum Hilbert space structure with respect to which the operators prequantizing derived Hamiltonian functions are formally Hermitian. This is due to the fact that the manifold on which the would be wave functions should be defined is a non negatively graded one of positive degree. On manifolds of this kind, only Dirac delta distributional integral forms can be integrated [66]. The derived formulation does not yield anything of this kind. The question arises about whether this is an essential impossibility or else new elements can be added to the theory allowing for the construction of a natural prequantum Hilbert space structure.

A definition of a derived analog of polarization, assuming that such a thing exists at all, is still to be achieved. The absence of a prequantum Hilbert space structure precludes in any case going beyond derived geometric prequantization into quantization proper along the familiar lines of ordinary quantization.

We believe that the limitations pointed out above of which derived prequantization suffers indicate that the appropriate geometric quantization of derived KKS theory cannot have some kind of quantum mechanical model, albeit exotic, as its end result but a two dimensional quantum field theory. This is in line with standard expectations to the extent

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to which derived KKS theory can be regarded as some kind of categorification of the ordinary theory. The derived TCO model studied in paper II is an attempt to concretize these intuitions.

The derived KKS theory formulated in this paper provides the geometric backdrop against which the derived TCO model is built and by virtue of which it has the form it does. In turn, the TCO model furnishes the physical motivation for the elaboration of the derived KKS set-up carried out in the present paper I.

3 Derived geometric framework

In this section, we review the derived geometric framework originally elaborated in refs. [54, 55]. By design, the derived set-up allows reformulating any theory whose symmetry is specified by a Lie group crossed module as one with a symmetry codified by a graded group, the associated derived group, e.g. 4-dimensional higher CS theory [21]. The derived framework also enables one to structure the higher KKS theory worked out in section 5 on the model of the standard theory as presented in section 4 and lies at the heart of the construction of the higher TCO model in close analogy to the ordinary one in II. By its many virtues, it will employed throughout our endeavour.

The derived set-up belongs to the realm of graded differential geometry in a deeper way than the set-up of employed in standard KKS and TCO theory. In fact, some of the structures featured in it cannot be expressed in the language of ordinary differential geometry in any straightforward if cumbersome way.

3.1 Lie group and algebra crossed modules and invariant pairings

Crossed modules encode the symmetry of higher gauge theory both at the finite and the infinitesimal level. A geometric formulation of higher KKS and TCO theory must necessarily set forth from them. In this subsection, we review the theory of Lie group and algebra crossed modules and module morphisms. A more comprehensive treatment complete with detailed definitions and relevant relations is provided in refs. [47, 48] and the appendices of ref. [31].

The structure of finite Lie crossed module abstracts and extends the set-up consisting of a Lie group G and a normal subgroup E of G acted upon by G by conjugation. A Lie group crossed module M features indeed two Lie groups E, G together with an action $\mu : \mathsf{G} \times \mathsf{E} \to \mathsf{E}$ of G on E by automorphisms and an equivariant morphism map $\tau : \mathsf{E} \to \mathsf{G}$ intertwining μ and the adjoint action of E. E, G and τ, μ are called the source and target groups and the target and action structure maps of M, respectively. We shall often write $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ to specify the crossed module through its data.

There exist many examples of Lie group crossed modules. In particular, Lie groups and automorphisms, representations and central extensions of Lie groups can be described as instances of Lie group crossed modules. We mention here two basic examples of crossed modules for illustrative purposes. Other examples will be presented later. They are defined for any Lie group G. The first is the inner automorphism crossed module of G, $\mathrm{I\scriptstyle NN}\,\mathsf{G} = (\mathsf{G}, \mathsf{G}, \mathrm{id}_{\mathsf{G}}, \mathrm{Ad}_{\mathsf{G}})$. The second is the (finite) coadjoint action crossed module of G, $\mathrm{A\scriptstyle D}^*\,\mathsf{G} =$

$(\mathfrak{g}^*, \mathsf{G}, 1_{\mathsf{G}}, \mathrm{Ad}_{\mathsf{G}}{}^*)$, where $\mathfrak{g}$ is the Lie algebra of G and its dual space $\mathfrak{g}^*$ is viewed as an Abelian group and $1_{\mathsf{G}} : \mathfrak{g}^* \to \mathsf{G}$ is the trivial morphism.

Crossed module morphisms will occur only occasionally in our analysis, but they nevertheless play an important role at some point in it. A morphism of finite crossed modules is a map of crossed modules respecting the module structure expressing so a relationship of likeness of the modules involved. More explicitly, a morphism $\beta : \mathsf{M}' \to \mathsf{M}$ of Lie group crossed modules consists of two group morphisms $\varPhi : \mathsf{E}' \to \mathsf{E}$ and $\phi : \mathsf{G}' \to \mathsf{G}$ intertwining in the obvious way the structure maps τ', μ', τ, μ. We shall normally write $\beta : \mathsf{M}' \to \mathsf{M} = (\varPhi, \phi)$ to indicate the constituent morphisms of the crossed module morphism.

A crossed module morphism $\rho : \mathrm{I{\scriptsize NN}}\,\mathsf{G}' \to \mathrm{I{\scriptsize NN}}\,\mathsf{G}$ reduces to a group morphism $\chi : \mathsf{G}' \to \mathsf{G}$. A crossed module morphism $\alpha : \mathrm{A{\scriptsize D}}^*\,\mathsf{G}' \to \mathrm{A{\scriptsize D}}^*\,\mathsf{G}$ is specified similarly by a group morphism $\lambda : \mathsf{G}' \to \mathsf{G}$ and an intertwiner $\varLambda : \mathfrak{g}'^* \to \mathfrak{g}^*$ of $\mathrm{Ad}_{\mathsf{G}'}{}^*$ to $\mathrm{Ad}_{\mathsf{G}}{}^* \circ \lambda$.

The structure of infinitesimal Lie crossed module axiomatizes likewise the set-up consisting of a Lie algebra $\mathfrak{g}$ and an ideal $\mathfrak{e}$ of $\mathfrak{g}$ equipped with the adjoint action of $\mathfrak{g}$. It is therefore the differential version of that of finite Lie crossed module. A Lie algebra crossed module $\mathfrak{m}$ consists so of two Lie algebras $\mathfrak{e}$, $\mathfrak{g}$ together with an action $m : \mathfrak{g} \times \mathfrak{e} \to \mathfrak{e}$ of $\mathfrak{g}$ on $\mathfrak{e}$ by derivations and an equivariant morphism $t : \mathfrak{e} \to \mathfrak{g}$ intertwining m and the adjoint action of $\mathfrak{e}$. $\mathfrak{e}$, $\mathfrak{g}$ and t, m are called the source and target algebras and the target and action structure maps of $\mathfrak{m}$, respectively. We shall often write $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, t, m)$ to specify the crossed module through its data.

Many examples of Lie algebra crossed modules and crossed module morphisms are also available. They match precisely with the examples of Lie group crossed modules and crossed module morphisms recalled above. Ordinary Lie algebras and derivations, representations and central extensions of Lie algebras can be described as instances of Lie algebra crossed modules. In particular, there are two crossed modules defined for any Lie algebra $\mathfrak{g}$ corresponding to the inner automorphism and coadjoint action crossed modules introduced above. The first is the inner derivation crossed module of $\mathfrak{g}$, $\mathrm{I{\scriptsize NN}}\,\mathfrak{g} = (\mathfrak{g}, \mathfrak{g}, \mathrm{id}_{\mathfrak{g}}, \mathrm{ad}\,_{\mathfrak{g}})$. The second is the (infinitesimal) coadjoint action crossed module of $\mathfrak{g}$, $\mathrm{A{\scriptsize D}}^*\,\mathfrak{g} = (\mathfrak{g}^*, \mathfrak{g}, 0_{\mathfrak{g}}, \mathrm{ad}_{\mathfrak{g}}{}^*)$, where $\mathfrak{g}^*$ is regarded as an Abelian algebra and $0_{\mathfrak{g}} : \mathfrak{g}^* \to \mathfrak{g}$ is the vanishing Lie algebra morphism.

A morphism of infinitesimal Lie crossed modules is a map of crossed modules preserving the module structure. It describes a way such crossed modules are concordant. They constitute therefore the differential counterpart of the morphisms of finite Lie group crossed modules introduced above. More explicitly, a morphism $p : \mathfrak{m}' \to \mathfrak{m}$ of Lie algebra crossed modules consists of two algebra morphisms $h : \mathfrak{g}' \to \mathfrak{g}$ and $H : \mathfrak{e}' \to \mathfrak{e}$ intertwining in the appropriate sense the structure maps t', m', t, m. We shall use often the notation $p : \mathfrak{m}' \to \mathfrak{m} = (H, h)$ to indicate constituent morphisms of the crossed module morphism.

A crossed module morphism $r : \mathrm{I{\scriptsize NN}}\,\mathfrak{g}' \to \mathrm{I{\scriptsize NN}}\,\mathfrak{g}$ reduces to an algebra morphism $x : \mathfrak{g}' \to \mathfrak{g}$. A crossed module morphism $a : \mathrm{A{\scriptsize D}}^*\,\mathfrak{g}' \to \mathrm{A{\scriptsize D}}^*\,\mathfrak{g}$ is specified likewise by an algebra morphism $l : \mathfrak{g}' \to \mathfrak{g}$ and an intertwiner $L : \mathfrak{g}'^* \to \mathfrak{g}^*$ of $\mathrm{ad}_{\mathfrak{g}'}{}^*$ to $\mathrm{ad}_{\mathfrak{g}}{}^* \circ l$.

Lie differentiation plays the same important role in Lie crossed module theory as it does in Lie group theory. With any Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ there is associated the Lie algebra crossed module $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, \dot{\tau}, \dot{\mu})$, where $\mathfrak{e}$, $\mathfrak{g}$ are the Lie algebras of Lie groups E, G respectively and the dot notation $\dot{}$ denotes Lie differentiation along the relevant Lie group, much as a Lie algebra is associated with a Lie group.[2] Similarly, with any Lie group crossed module morphism $\beta : \mathsf{M}' \to \mathsf{M} = (\Phi, \phi)$ there is associated the Lie algebra crossed module morphism $\dot\beta : \mathfrak{m}' \to \mathfrak{m} = (\dot\Phi, \dot\phi)$, just as a Lie algebra morphism is associated with a Lie group morphism.

As examples, we mention that the Lie algebra crossed modules of the Lie group crossed modules $\mathrm{I{\scriptsize NN}}\,\mathsf{G}$ and $\mathrm{A{\scriptsize D}}^*\,\mathsf{G}$ we introduced above for any Lie group G are precisely $\mathrm{I{\scriptsize NN}}\,\mathfrak{g}$ and $\mathrm{A{\scriptsize D}}^*\,\mathfrak{g}$, respectively, as expected.

Crossed modules with invariant pairing enter in many higher gauge theoretic constructions. Indeed, invariant pairings play in higher gauge theory a role similar to that of invariant traces in ordinary gauge theory and are basic structural elements of e.g. kinetic terms enjoying the appropriate symmetries. For similar reasons, they appear prominently also in derived KKS and TCO theory.

Following ref. [31], we define an invariant pairing on a Lie algebra crossed module $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, t, m)$ as a non singular bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{e} \to \mathbb{R}$ enjoying the invariance property

$$\langle \mathrm{ad}\, z(x), X \rangle + \langle x, m(z, X) \rangle = 0 \tag{3.1}$$

for $z, x \in \mathfrak{g}$, $X \in \mathfrak{e}$ and obeying the symmetry relation

$$\langle t(X), Y \rangle = \langle t(Y), X \rangle. \tag{3.2}$$

for $X, Y \in \mathfrak{e}$. Other definitions of invariant pairing involving only either $\mathfrak{g}$ or $\mathfrak{e}$ turn out to be both unnatural and non viable.[3]

A Lie algebra crossed module with invariant pairing is a Lie algebra crossed module $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, t, m)$ furnished with an invariant pairing $\langle \cdot, \cdot \rangle$ of the kind defined in the previous paragraph.

A Lie group crossed module with invariant pairing is a Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ whose associated Lie algebra crossed module $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, \dot\tau, \dot{}\mu\dot{})$ is one with invariant pairing. The invariance property however is required to hold not only at the infinitesimal level as in eq. (3.1) but also at the finite one, viz

$$\langle \mathrm{Ad}\, a(x), \mu\dot{}(a, X) \rangle = \langle x, X \rangle \tag{3.3}$$

for $a \in \mathsf{G}$, $x \in \mathfrak{g}$, $X \in \mathfrak{e}$.

A Lie algebra crossed module $\mathfrak{m}$ with invariant pairing is balanced, that is such $\dim \mathfrak{g} = \dim \mathfrak{e}$, by the non singularity of the pairing. This is not too restrictive. In fact, any Lie algebra crossed module $\mathfrak{m}$ can always be trivially extended to a balanced crossed module $\mathfrak{m}^c$ [29]. Similarly a Lie group crossed module M with invariant pairing is balanced, as $\dim \mathsf{G} = \dim \mathsf{E}$. Further, any Lie group crossed module M can always be trivially extended to a balanced crossed module M^c.

[2] Note here that μ has three Lie differentials, $\dot{}\mu$, $\mu\dot{}$, $\dot{}\mu\dot{}$, according to whether the G, the E and both the G and E arguments are subject to differentiation, respectively [31].

[3] Viewing a Lie algebra crossed module $\mathfrak{m}$ as a strict 2-term L_∞ algebra $\mathfrak{l}$ as described in ref. [48], an invariant pairing on $\mathfrak{m}$ corresponds to a cyclic structure on $\mathfrak{l}$.

3.2 Derived Lie groups and algebras

The notion of derived Lie group of a Lie group crossed module and the corresponding infinitesimal notion of derived Lie algebra of a Lie algebra crossed module were originally introduced in refs. [54, 55].

The formal set-up of derived Lie groups and algebras is an elegant and convenient way of handling certain structural elements of the Lie group and algebra crossed modules appearing in higher gauge theory. It is a compact superfield formalism not unlike the analogous formalisms broadly used in supersymmetric field theories and particularly suited for a higher gauge theoretic setting.

The derived Lie group of a Lie group crossed module does not fully encode this latter, but it only describes an approximation of it in the sense of synthetic geometry. In fact, the target map of the crossed module is not involved in the definition of the derived group, nor could it be because, roughly speaking, the approximation is such to push the range of the target map away out of reach. Only the action map of the crossed module and its properties not implicating the target map are presupposed. All the algebraic structure hinging on the target map is in this way forgotten by the derived construction. Similar considerations apply to the derived Lie algebra of a Lie algebra crossed module. The reader is referred to ref. [54] for a more precise discussion.

Consider a Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$. The derived Lie group $\mathrm{D}\mathsf{M}$ of M is the semidirect product group

$$\mathrm{D}\mathsf{M} = \mathfrak{e}[1] \rtimes_{\mu\dot{}} \mathsf{G}, \tag{3.4}$$

where $\mathfrak{e}[1]$ is regarded as a G-module through the G-action $\mu\dot{}$. $\mathrm{D}\mathsf{M}$ is therefore a graded Lie group concentrated in degrees 0, 1.

Each element $\mathrm{P} \in \mathrm{D}\mathsf{M}$ has the formal representation

$$\mathrm{P}(\alpha) = \mathrm{e}^{\alpha P}\, p \tag{3.5}$$

with $\alpha \in \mathbb{R}[-1]$,[4] where $p \in \mathsf{G}$, $P \in \mathfrak{e}[1]$. p, P are called the components of P. In this description, the group operations of $\mathrm{D}\mathsf{M}$ read as

$$\mathrm{PQ}(\alpha) = \mathrm{e}^{\alpha(P+\mu\dot{}(p,Q))}\, pq, \tag{3.6}$$

$$\mathrm{P}^{-1}(\alpha) = \mathrm{e}^{-\alpha\mu\dot{}(p^{-1},P)}\, p^{-1}, \tag{3.7}$$

where $\mathrm{P}, \mathrm{Q} \in \mathrm{D}\mathsf{M}$ are any two group elements with $\mathrm{P}(\alpha) = \mathrm{e}^{\alpha P}\, p$, $\mathrm{Q}(\alpha) = \mathrm{e}^{\alpha Q}\, q$.

A morphism $\beta : \mathsf{M}' \to \mathsf{M}$ of Lie group crossed modules induces by means of its constituent group morphisms $\Phi : \mathsf{E}' \to \mathsf{E}$, $\phi : \mathsf{G}' \to \mathsf{G}$ a Lie group morphism $\mathrm{D}\beta : \mathrm{D}\mathsf{M}' \to \mathrm{D}\mathsf{M}$.[5]

[4] In this paper, $\alpha \in \mathbb{R}[p]$ has degree p, because its geometrical degree is tacitly considered. In refs. [54, 55], $\alpha \in \mathbb{R}[p]$ has degree $-p$, because its algebraic degree is considered instead. In this paper, we work with the geometric degree rather than the algebraic one, as is more natural in a geometrical framework as the one illustrated here. See appendix A.1 for more details.

[5] This property might be expressed in the language of category theory by characterizing D as a functor of the appropriate Lie crossed module and group categories, but we shall not do so here.

The notion of derived Lie group has an evident infinitesimal counterpart. Consider a Lie algebra crossed module $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, t, m)$. The derived Lie algebra of $\mathfrak{m}$ is the semidirect product algebra

$$\mathrm{D}\mathfrak{m} \simeq \mathfrak{e}[1] \rtimes_m \mathfrak{g}, \tag{3.8}$$

where $\mathfrak{e}[1]$ is regarded as a $\mathfrak{g}$-module through the $\mathfrak{g}$-action m. $\mathrm{D}\mathfrak{m}$ is so a graded Lie algebra concentrated in degree 0, 1.

Analogously, each element $\mathrm{U} \in \mathrm{D}\mathfrak{m}$ has the formal representation

$$\mathrm{U}(\alpha) = u + \alpha U \tag{3.9}$$

with $\alpha \in \mathbb{R}[-1]$, where $u \in \mathfrak{g}$, $U \in \mathfrak{e}[1]$ are the components of U. The Lie bracket read in this set-up as

$$[\mathrm{U}, \mathrm{V}](\alpha) = [u, v] + \alpha(m(u, V) - m(v, U)) \tag{3.10}$$

with $\mathrm{U}, \mathrm{V} \in \mathrm{D}\mathfrak{m}$ any two algebra elements with $\mathrm{U}(\alpha) = u + \alpha U$, $\mathrm{V}(\alpha) = v + \alpha V$.

A morphism $p : \mathfrak{m}' \to \mathfrak{m}$ of Lie algebra crossed modules induces through its underlying algebra morphisms $H : \mathfrak{e}' \to \mathfrak{e}$, $h : \mathfrak{g}' \to \mathfrak{g}$, a Lie algebra morphism $\mathrm{D}p : \mathrm{D}\mathfrak{m}' \to \mathrm{D}\mathfrak{m}$, analogously to the finite case.

The derived construction introduced above is fully compatible with Lie differentiation. This property is in fact essential for its viability. If $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ is a Lie group crossed module and $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, \dot{\tau}, \dot{\mu})$ is its associated Lie algebra crossed module, then $\mathrm{D}\mathfrak{m}$ is the Lie algebra of $\mathrm{D}\mathsf{M}$. Further, if $\beta : \mathsf{M}' \to \mathsf{M}$ is a Lie group crossed module morphism and $\dot{\beta} : \mathfrak{m}' \to \mathfrak{m}$ is the corresponding Lie algebra crossed module morphism, then $\dot{\mathrm{D}}\beta = \mathrm{D}\dot{\beta}$.

3.3 Derived field formalism

In this subsection, we shall survey the main spaces of Lie group and algebra crossed module valued fields using a derived field framework.

We assume that the fields propagate on a general non negatively graded manifold X. Later, we shall add the restriction that X is an ordinary orientable and compact manifold, possibly with boundary. To include also differential forms without renouncing to a convenient graded geometric description, the fields will be maps from the shifted tangent bundle $T[1]X$ of X into some graded target manifold T. Below, we denote by $\mathrm{M{\scriptstyle AP}}(T[1]X, T)$ the space of non negative internal degree internal maps from $T[1]X$ into T. The more restricted space $\mathrm{Map}(T[1]X, T)$ of ordinary maps from $T[1]X$ to T can be also considered and treated much in the same way. See appendices A.1, A.2 for more details.

The fields we shall consider will be valued either in the derived Lie group $\mathrm{D}\mathsf{M}$ of a Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ or in the derived Lie algebra $\mathrm{D}\mathfrak{m}$ of the associated Lie algebra crossed module $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, \dot{\tau}, \dot{\mu})$ (cf. subsection 3.2). A more comprehensive treatment of this kind of fields is given in ref. [54].

We consider first $\mathrm{D}\mathsf{M}$-valued fields. Fields of this kind are elements of the mapping space $\mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{D}\mathsf{M})$. If $\mathrm{U} \in \mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{D}\mathsf{M})$, then

$$\mathrm{U}(\alpha) = \mathrm{e}^{\alpha U}\, u \tag{3.11}$$

with $\alpha \in \mathbb{R}[-1]$, where $u \in \textsc{Map}(T[1]X, \mathsf{G})$, $U \in \textsc{Map}(T[1]X, \mathfrak{e}[1])$. u, U are the components of U. $\textsc{Map}(T[1]X, \mathrm{D}\mathsf{M})$ has a Lie group structure induced by that of $\mathrm{D}\mathsf{M}$: if $\mathrm{U} \in \textsc{Map}(T[1]X, \mathrm{D}\mathsf{M})$, $\mathrm{V} \in \textsc{Map}(T[1]X, \mathrm{D}\mathsf{M})$, then

$$\mathrm{UV}(\alpha) = \mathrm{e}^{\alpha(U+\mu^{\cdot}(u,V))}\, uv, \qquad \mathrm{U}^{-1}(\alpha) = \mathrm{e}^{-\alpha\mu^{\cdot}(u^{-1},U))}\, u^{-1}. \tag{3.12}$$

Next, we consider first $\mathrm{D}\mathfrak{m}$-valued fields. Fields of this kind are elements of the mapping space $\textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m})$. If $\Phi \in \textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m})$, then

$$\Phi(\alpha) = \phi + \alpha\mathit{\Phi} \tag{3.13}$$

with $\alpha \in \mathbb{R}[-1]$, where $\phi \in \textsc{Map}(T[1]X, \mathfrak{g})$, $\mathit{\Phi} \in \textsc{Map}(T[1]X, \mathfrak{e}[1])$. Again, ϕ, $\mathit{\Phi}$ are the components of Φ. $\textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m})$ has a Lie algebra structure induced by that of $\mathrm{D}\mathfrak{m}$: if $\Phi \in \textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m})$, $\Psi \in \textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m})$, then

$$[\Phi, \Psi](\alpha) = [\phi, \psi] + \alpha({}^{\cdot}\mu^{\cdot}(\phi, \mathit{\Psi}) - {}^{\cdot}\mu^{\cdot}(\psi, \mathit{\Phi}))\,. \tag{3.14}$$

$\textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m})$ is the virtual Lie algebra of $\textsc{Map}(T[1]X, \mathrm{D}\mathsf{M})$. (For an explanation of this terminology, see ref. [54]).

As it turns out, the $\mathrm{D}\mathfrak{m}$-valued fields introduced above are not enough for our proposes. One also needs to incorporate fields that are valued in the degree shifted linear spaces $\mathrm{D}\mathfrak{m}[p]$ with p some integer. Together, they constitute the mapping space $\textsc{Map}(T[1]X, \mathrm{SD}\mathfrak{m})$.[6] If $\Phi \in \textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m}[p])$, then

$$\Phi(\alpha) = \phi + (-1)^p\alpha\mathit{\Phi} \tag{3.15}$$

with components $\phi \in \textsc{Map}(T[1]X, \mathfrak{g}[p])$, $\mathit{\Phi} \in \textsc{Map}(T[1]X, \mathfrak{e}[p+1])$. There is a bilinear bracket that associates with a pair of fields $\Phi \in \textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m}[p])$, $\Psi \in \textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m}[q])$ a field $[\Phi, \Psi] \in \textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m}[p+q])$ given by

$$[\Phi, \Psi](\alpha) = [\phi, \psi] + (-1)^{p+q}\alpha({}^{\cdot}\mu^{\cdot}(\phi, \mathit{\Psi}) - (-1)^{pq}\,{}^{\cdot}\mu^{\cdot}(\psi, \mathit{\Phi}))\,. \tag{3.16}$$

$\textsc{Map}(T[1]X, \mathrm{SD}\mathfrak{m})$ becomes in this way a graded Lie algebra. This contains the Lie algebra $\textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m})$ as its degree 0 subalgebra.

An adjoint action of $\textsc{Map}(T[1]X, \mathrm{D}\mathsf{M})$ on the Lie algebra $\textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m})$ and more generally on the graded Lie algebra $\textsc{Map}(T[1]X, \mathrm{SD}\mathfrak{m})$ is defined. For $\mathrm{U} \in \textsc{Map}(T[1]X, \mathrm{D}\mathsf{M})$, $\Phi \in \textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m}[p])$, one has

$$\mathrm{Ad}\,\mathrm{U}(\Phi)(\alpha) = \mathrm{Ad}\,u(\phi) + (-1)^p\alpha(\mu^{\cdot}(u, \mathit{\Phi}) - {}^{\cdot}\mu^{\cdot}(\mathrm{Ad}\,u(\phi), U)), \tag{3.17}$$

$$\mathrm{Ad}\,\mathrm{U}^{-1}(\Phi)(\alpha) = \mathrm{Ad}\,u^{-1}(\phi) + (-1)^p\alpha\mu^{\cdot}(u^{-1}, \mathit{\Phi} + {}^{\cdot}\mu^{\cdot}(\phi, U)). \tag{3.18}$$

The adjoint action preserves Lie brackets as in ordinary Lie theory. Indeed, for $\mathrm{U} \in \textsc{Map}(T[1]X, \mathrm{D}\mathsf{M})$, $\Phi \in \textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m}[p])$, $\Psi \in \textsc{Map}(T[1]X, \mathrm{D}\mathfrak{m}[q])$,

$$[\mathrm{Ad}\,\mathrm{U}(\Phi), \mathrm{Ad}\,\mathrm{U}(\Psi)] = \mathrm{Ad}\,\mathrm{U}([\Phi, \Psi]). \tag{3.19}$$

[6] Throughout this paper, we denote by $\mathrm{S}E$ the degree extended form of a possibly graded vector space E. Explicitly, one has $\mathrm{S}E = \bigoplus_{p=-\infty}^{\infty} E[p]$. In refs. [54, 55], the very same object is denoted instead as $\mathrm{Z}E$, a notation that we shall employ in later sections with a different meaning.

As is well-known, in the graded geometric formulation we adopt, the nilpotent de Rham differential d is a degree 1 homological vector field on $T[1]X$, $d^2 = 0$. d induces a natural degree 1 derived differential d on the graded vector space $\mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{SD}\mathfrak{m})$. Concisely, $\mathrm{d} = d + \dot{\tau} d/d\alpha$. In more ore explicit terms, for $\Phi \in \mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{D}\mathfrak{m}[p])$, the field $\mathrm{d}\Phi \in \mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{D}\mathfrak{m}[p+1])$ reads as

$$\mathrm{d}\Phi(\alpha) = d\phi + (-1)^p \dot{\tau}(\varPhi) + (-1)^{p+1} \alpha d\varPhi. \tag{3.20}$$

It can be straightforwardly verified that

$$\mathrm{d}[\Phi, \Psi] = [\mathrm{d}\Phi, \Psi] + (-1)^p [\Phi, \mathrm{d}\Psi] \tag{3.21}$$

for $\Phi \in \mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{D}\mathfrak{m}[p])$, $\Psi \in \mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{D}\mathfrak{m}[q])$ and that

$$\mathrm{d}^2 = 0. \tag{3.22}$$

In this way, $\mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{SD}\mathfrak{m})$ becomes a differential graded Lie algebra.

On several occasions, the pull-backs $\mathrm{dUU}^{-1}, \mathrm{U}^{-1}\mathrm{dU} \in \mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{D}\mathfrak{m}[1])$ of the Maurer-Cartan forms of $\mathrm{D}\mathsf{M}$ by a $\mathrm{D}\mathsf{M}$ field $\mathrm{U} \in \mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{D}\mathsf{M})$ will enter our considerations. For these, there exist explicit expressions,

$$\mathrm{dUU}^{-1}(\alpha) = duu^{-1} + \dot{\tau}(U) - \alpha\big(dU + \tfrac{1}{2}[U, U] - \dot{\mu}\dot{}(duu^{-1} + \dot{\tau}(U), U)\big)\,, \tag{3.23}$$

$$\mathrm{U}^{-1}\mathrm{dU}(\alpha) = \mathrm{Ad}\,u^{-1}\big(duu^{-1} + \dot{\tau}(U)\big) - \alpha\mu\dot{}\big(u^{-1}, dU + \tfrac{1}{2}[U, U]\big)\,. \tag{3.24}$$

By the relation $\mathrm{d} = d + \dot{\tau} d/d\alpha$, (3.23), (3.24) follow from (3.11) and the variational identities $\delta\, \mathrm{e}^{\alpha X}\, \mathrm{e}^{-\alpha X} = \frac{\exp(\alpha\,\mathrm{ad}\,X) - 1}{\alpha\,\mathrm{ad}\,X} \delta(\alpha X)$, $\mathrm{e}^{-\alpha X}\, \delta\, \mathrm{e}^{\alpha X} = \frac{1 - \exp(-\alpha\,\mathrm{ad}\,X)}{\alpha\,\mathrm{ad}\,X} \delta(\alpha X)$ with $\delta = \dot{\tau} d/d\alpha$, owing to the nilpotence of α.

Next, let the Lie group crossed module M be equipped with an invariant pairing $\langle \cdot, \cdot \rangle$ (cf. subsection 3.1). A pairing on the graded Lie algebra $\mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{SD}\mathfrak{m})$ is induced in this way: for $\Phi \in \mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{D}\mathfrak{m}[p])$, $\Psi \in \mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{D}\mathfrak{m}[q])$

$$(\Phi, \Psi) = \langle \phi, \varPsi \rangle + (-1)^{pq} \langle \psi, \varPhi \rangle. \tag{3.25}$$

Note that $(\Phi, \Psi) \in \mathrm{M{\scriptstyle AP}}(T[1]X, \mathbb{R}[p+q+1])$. The field pairing $(\cdot, \cdot)$ therefore has degree 1. $(\cdot, \cdot)$ is bilinear. More generally, when scalars with non trivial grading are involved, the left and right brackets (and) behave as if they had respectively degree 0 and 1. For instance, $(c\Phi, \Psi) = c(\Phi, \Psi)$ whilst $(\Phi, \Psi c) = (-1)^k (\Phi, \Psi)c$ if the scalar c has degree k. $(\cdot, \cdot)$ is further graded symmetric,

$$(\Phi, \Psi) = (-1)^{pq} (\Psi, \Phi). \tag{3.26}$$

$(\cdot, \cdot)$ is also non singular.

The field pairing $(\cdot, \cdot)$ has several other properties which make it a very natural ingredient in the field theoretic constructions of later sections. First, $(\cdot, \cdot)$ is $\mathrm{D}\mathsf{M}$-invariant. If $\Phi \in \mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{D}\mathfrak{m}[p])$, $\Psi \in \mathrm{M{\scriptstyle AP}}(T[1]X, \mathrm{D}\mathfrak{m}[q])$, we have

$$(\mathrm{Ad}\,\mathrm{U}(\Phi), \mathrm{Ad}\,\mathrm{U}(\Psi)) = (\Phi, \Psi) \tag{3.27}$$

for $\mathsf{U} \in \mathrm{Map}(T[1]X, \mathrm{D}\mathsf{M})$. By Lie differentiation, $(\cdot, \cdot)$ enjoys also $\mathrm{D}\mathfrak{m}$ invariance. This latter, however, admits a graded extension, because of which

$$([\Xi, \Phi], \Psi) + (-1)^{pr}(\Phi, [\Xi, \Psi]) = 0 \tag{3.28}$$

for $\Xi \in \mathrm{Map}(T[1]M, \mathrm{D}\mathfrak{m}[r])$.

Second, $(\cdot, \cdot)$ is compatible with the derived differential d, i. e. the de Rham vector field d differentiates $(\cdot, \cdot)$ through d,

$$d(\Phi, \Psi) = (\mathrm{d}\Phi, \Psi) + (-1)^{p}(\Phi, \mathrm{d}\Psi). \tag{3.29}$$

Let M, M' be Lie group crossed modules with associated Lie algebra crossed modules $\mathfrak{m}$, $\mathfrak{m}'$. Suppose that M' is a submodule of M and that, consequently, $\mathfrak{m}'$ is a submodule of $\mathfrak{m}$. As $\mathrm{D}\mathsf{M}'$ is a Lie subgroup of $\mathrm{D}\mathsf{M}$, $\mathrm{Map}(T[1]X, \mathrm{D}\mathsf{M}')$ is a Lie subgroup of $\mathrm{Map}(T[1]X, \mathrm{D}\mathsf{M})$. Similarly, as $\mathrm{D}\mathfrak{m}'$ is a Lie subalgebra of $\mathrm{D}\mathfrak{m}$, $\mathrm{Map}(T[1]X, \mathrm{D}\mathfrak{m}')$ is a Lie subalgebra of $\mathrm{Map}(T[1]X, \mathrm{D}\mathfrak{m})$. What is more, $\mathrm{Map}(T[1]X, \mathrm{SD}\mathfrak{m}')$ is a differential graded Lie subalgebra of $\mathrm{Map}(T[1]X, \mathrm{SD}\mathfrak{m})$, since it is invariant under the action of d as is evident from (3.20).

In concrete field theoretic analyses, one deals with functionals of the relevant derived fields on some compact manifold X. These are given as integrals on $T[1]X$ of certain functions of $\mathrm{Fun}(T[1]X)$ constructed using the derived fields. Integration is carried out using the Berezinian ϱ_X of X.

3.4 Ordinary geometric framework as a special case

The geometric framework employed in ordinary gauge theory as well as in the formulation of ordinary KKS theory and the TCO model is in fact a special case of the derived geometric framework. We devote this final subsection to the illustration of this point.

Let G be a Lie group. There exists a unique Lie group crossed module with target group G and trivial source group $\mathsf{E} = 1$, since the target morphism τ and the action map μ in this case can be only the trivial ones. With a harmless abuse of notation, we shall denote this crossed module also by G, since it codifies the Lie group structure of G in a manner equivalent to the usual one. Similarly, for a Lie algebra $\mathfrak{g}$, there exists a unique Lie algebra crossed module with target algebra $\mathfrak{g}$ and trivial source algebra $\mathfrak{e} = 0$, since the target morphism t and the action map m again can be only the trivial ones. We shall denote this crossed module also by $\mathfrak{g}$, since it provides an equivalent codification of the Lie algebra structure of $\mathfrak{g}$. This crossed module reinterpretation of ordinary Lie theory is compatible with Lie differentiation: $\mathfrak{g}$ is the Lie algebra of the Lie group G if and only if $\mathfrak{g}$ is the Lie algebra crossed module of the Lie group crossed module G.

We note however that while a Lie algebra $\mathfrak{g}$ can support an invariant pairing $\langle \cdot, \cdot \rangle$ as such, it does not support any invariant pairing $\langle \cdot, \cdot \rangle$ of the kind defined in subsection 3.1 as a Lie algebra crossed module because of the vanishing of the source Lie algebra of this latter.

The derived functor D relates the crossed module and the ordinary formulation of Lie theory. From (3.4) and (3.5)–(3.7) with $\mathfrak{e} = 0$, it appears that $\mathrm{D}\mathsf{G} \simeq \mathsf{G}$ for any Lie group

G: the derived Lie group $\mathrm{D}\mathsf{G}$ of G as a Lie group crossed module is just G as a Lie group. From (3.8) and (3.9), (3.10) with $\mathfrak{e} = 0$, it similarly appears that $\mathrm{D}\mathfrak{g} \simeq \mathfrak{g}$ for any Lie algebra $\mathfrak{g}$: the derived Lie algebra $\mathrm{D}\mathfrak{g}$ of $\mathfrak{g}$ as a Lie algebra crossed module is just $\mathfrak{g}$ as a Lie algebra.

For a Lie group G with Lie algebra $\mathfrak{g}$, the mapping spaces $\mathrm{MAP}(T[1]X, \mathrm{D}\mathsf{G})$ and $\mathrm{MAP}(T[1]X, \mathsf{G})$ are in this way identified and so are also the mapping spaces $\mathrm{MAP}(T[1]X, \mathrm{D}\mathfrak{g})$ and $\mathrm{MAP}(T[1]X, \mathfrak{g})$ and their degree shifted versions. Further, the derived adjoint action defined in (3.17), (3.18) reduces to the ordinary one and likewise the derived differential d defined in (3.20) reproduced the ordinary de Rham differential.

4 Standard KKS theory, a review

The Kirillov-Kostant-Souriau (KKS) construction provides the coadjoint orbit of an element of the dual of a Lie algebra with a natural symplectic structure, which under certain integrality conditions can be quantized. Since our aim is to generalize the KKS construction to a crossed module theoretic setting in a derived perspective, it is appropriate to begin our path by reviewing the standard KKS theory. Our survey of this latter is biased on one hand and incomplete on the other. It is biased, because it is purposefully patterned in a way that plainly alludes to the derived extension presented in section 5, based as it is on an operational description, a formal refinement of the classic Cartan calculus, which is not strictly necessary in the ordinary theory but it is essential in the higher derived version. It is also incomplete, because we have deliberately avoided to delve in depth into quantization proper, aware that the problem we aim to eventually solve is to some extent related, although not fully equivalent, to one of quantization of a 2-plectic manifold, a delicate issue that is not completely settled yet in the literature and whose solution lies beyond the scope of the present work [58–60, 63].

The main notions of the operational framework mentioned in the previous paragraph are briefly reviewed in appendix A.3. An exhaustive exposition of the subject is furnished in ref. [65]. In outline, if P is a manifold and $\mathfrak{f}$ is a Lie algebra, an $\mathfrak{f}$-operation on P is a collection of graded derivations of $\mathrm{Fun}(T[1]P)$ formed by a degree -1 derivation j_x and a degree 0 derivation l_x for each $x \in \mathfrak{f}$ and a degree 1 derivation d obeying the six Cartan relations (A.5)–(A.8). If P is a manifold carrying a right action of a Lie group F with Lie algebra $\mathfrak{f}$, in particular a principal F-bundle, then an $\mathfrak{f}$-operation on P is defined such that j_x, l_x are the contraction and Lie derivative along the vertical vector field $S_x \in \mathrm{Vect}(P)$ of the action corresponding to $x \in \mathfrak{f}$ and d is the Rham differential of P. The operational framework furnishes a powerful method for the description of the geometry of the base manifolds of principal bundles, the so called basic geometry, in particular of homogeneous manifolds such as the coadjoint orbits

4.1 Coadjoint orbits

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. For any element $\lambda \in \mathfrak{g}^*$ of the dual vector space of $\mathfrak{g}$, the coadjoint orbit $\mathcal{O}_\lambda$ of λ is the submanifold of $\mathfrak{g}^*$ spanned by the coadjoint action of G on λ. Explicitly, $\mathcal{O}_\lambda = \{\mathrm{Ad}^* \gamma(\lambda) | \gamma \in \mathsf{g}\}$, where Ad^* denotes the coadjoint representation of G, the dual of the adjoint representation Ad with respect to the canonical duality pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$.

The orbit $\mathcal{O}_\lambda$ is a homogeneous G-manifold. Indeed,

$$\mathcal{O}_\lambda = \mathsf{G}/\mathrm{Z}\mathsf{G}_\lambda, \tag{4.1}$$

where $\mathrm{Z}\mathsf{G}_\lambda$ is the invariance subgroup of λ in G. Suppose that G is compact and connected. Then, $\mathrm{Z}\mathsf{G}_\lambda$ always contains a maximal torus T of G. If $\mathrm{Z}\mathsf{G}_\lambda = \mathsf{T}$, λ is said to be regular. In that case, $\mathcal{O}_\lambda = \mathsf{G}/\mathsf{T}$. Below, we shall rely heavily on the homogeneous space description of $\mathcal{O}_\lambda$ concentrating on the regular case.

4.2 Operational description of homogeneous spaces

The fact that coadjoint orbits are instances of homogeneous spaces opens the possibility of an operational formulation of KKS theory. We shall however consider the problem we intend to study from a wider perspective as follows. If G is a Lie group and G' is a subgroup of G, then G can be regarded as a principal G'-bundle over the homogeneous space G/G'. As such, G is amenable to the operational description, from which much information about G/G' can be extracted.

For the operational analysis, we need to begin with appropriate coordinates of the shifted tangent bundle $T[1]\mathsf{G}$. By the isomorphism $T[1]\mathsf{G} \simeq \mathsf{G} \times \mathfrak{g}[1]$, we can use as coordinates of $T[1]\mathsf{G}$ variables $\gamma \in \mathsf{G}$ and $\sigma \in \mathfrak{g}[1]$. They obey

$$\gamma^{-1} d\gamma = \sigma, \quad d\sigma = -\frac{1}{2}[\sigma, \sigma], \tag{4.2}$$

where d is the de Rham differential regarded as a homological vector field on $T[1]\mathsf{G}$. σ is therefore identified with the Maurer-Cartan form of G.

The right G'-action of G induces an action on $T[1]\mathsf{G}$, which reads as

$$\gamma^{\mathsf{R}q} = \gamma q, \quad \sigma^{\mathsf{R}q} = \mathrm{Ad}\, q^{-1}(\sigma) \tag{4.3}$$

with $q \in \mathsf{G}'$ in terms of the coordinates γ, σ. From here, we can readily read off the action of the derivations of the associated right $\mathfrak{g}'$-operation of G on γ, σ,

$$\gamma^{-1} j_{\mathsf{R}x}\gamma = 0, \quad \gamma^{-1} l_{\mathsf{R}x}\gamma = x, \quad j_{\mathsf{R}x}\sigma = x, \quad l_{\mathsf{R}x}\sigma = -[x, \sigma], \tag{4.4}$$

where $x \in \mathfrak{g}'$.

As a group, G is characterized also by a left G-action, which we shall write in right form for convenience. In terms of the coordinates γ, σ, it reads as

$$\gamma^{\mathsf{L}e} = e^{-1}\gamma, \quad \sigma^{\mathsf{L}e} = \sigma \tag{4.5}$$

with $e \in \mathsf{G}$. Its main feature is the invariance of the Maurer-Cartan form. Below, we shall employ extensively the associated left $\mathfrak{g}$-operation. The derivations of this latter act as

$$j_{\mathsf{L}h}\gamma\gamma^{-1} = 0, \quad l_{\mathsf{L}h}\gamma\gamma^{-1} = -h, \quad j_{\mathsf{L}h}\sigma = -\mathrm{Ad}\,\gamma^{-1}(h), \quad l_{\mathsf{L}h}\sigma = 0 \tag{4.6}$$

with $h \in \mathfrak{g}$.

The right G'- and left G-actions commute, as is evident from their coordinate expressions. Consequently, the derivations of the right $\mathfrak{g}'$- and left $\mathfrak{g}$-operations also commute in the graded sense.

Though automorphism symmetry is rarely mentioned in standard presentations of KKS theory, it is nevertheless a feature of the KKS set-up inherent in its bundle theoretic nature. We consider also this aspect in the above wider perspective. The automorphisms of the principal G'-bundle G are the fiber preserving invertible maps of G into itself compatible the right G'-action. Concretely, they are maps $\psi \in \mathrm{Map}(\mathsf{G}, \mathsf{G}')$ with certain basicness properties under the right G'-action and so naturally constitute a distinguished subgroup $\mathrm{Aut}_{\mathsf{G}'}(\mathsf{G})$ of the infinite dimensional Lie group $\mathrm{Map}(\mathsf{G}, \mathsf{G}')$. $\mathrm{Aut}_{\mathsf{G}'}(\mathsf{G})$ is selected by the condition

$$\psi^{\mathsf{R}q} = q^{-1}\psi q \tag{4.7}$$

with $q \in \mathsf{G}'$, where $\psi^{\mathsf{R}q}$ denotes the pull-back of ψ by the right G'-action, i.e. $\psi^{\mathsf{R}q}(\gamma) = \psi(\gamma^{\mathsf{R}q^{-1}})$. In the right $\mathfrak{g}'$-operation, so, ψ obeys

$$j_{\mathsf{R}x}\psi\psi^{-1} = 0, \quad j_{\mathsf{R}x}\psi\psi^{-1} = -x + \mathrm{Ad}\,\psi(x) \tag{4.8}$$

for $x \in \mathfrak{g}'$, where we view ψ as a map of $\mathrm{Map}(T[1]\mathsf{G}, \mathsf{G}')$ relying on the identity $\mathrm{Map}(\mathsf{G}, \mathsf{G}') = \mathrm{Map}(T[1]\mathsf{G}, \mathsf{G}')$.

The automorphism group $\mathrm{Aut}_{\mathsf{G}'}(\mathsf{G})$ of the G'-bundle G acts on G, the action being a left one. In terms of the coordinates γ, σ, the action of an automorphism $\psi \in \mathrm{Aut}_{\mathsf{G}'}(\mathsf{G})$ on $T[1]\mathsf{G}$ reads as

$${}^{\psi}\gamma = \gamma\psi^{-1}, \quad {}^{\psi}\sigma = \mathrm{Ad}\,\psi(\sigma) - d\psi\psi^{-1}. \tag{4.9}$$

The second relation follows form the first one and the first relation (4.2). Notice that $({}^{\psi}\gamma)^{\mathsf{R}q} = {}^{\psi^{\mathsf{R}q}}\gamma^{\mathsf{R}q}$ as required by compatibility.

By (4.5) and (4.9), the left G- and automorphism actions of the bundle G commute and consequently are automatically compatible. In fact, automorphisms transform trivially under the action being

$$\psi^{\mathsf{L}e} = \psi \tag{4.10}$$

for $e \in \mathsf{G}$. In the left $\mathfrak{g}$-operation, therefore, for $h \in \mathfrak{g}$ we have

$$j_{\mathsf{L}h}\psi\psi^{-1} = 0, \quad j_{\mathsf{L}h}\psi\psi^{-1} = 0. \tag{4.11}$$

4.3 Unitary line bundles on a regular homogeneous space

The construction of the appropriate prequantum line bundle is an essential step of the geometric prequantization of KKS theory. In this subsection, we examine this problem from a broader point of view again. We consider a homogeneous space of the form G/T, where G is a compact Lie group and T is a maximal torus of G, modelling a regular coadjoint orbit and for this reason called regular. Relying on the operational set-up of subsection 4.2, we then describe the unitary line bundles over G/T, their sections and their unitary connections.

Recall that a character of T is a Lie group morphism of T into $\mathsf{U}(1)$, so an element $\xi \in \mathrm{Hom}(\mathsf{T}, \mathsf{U}(1))$. With the character ξ, we can associate a unitary line bundle $\mathcal{L}_\xi \to \mathsf{G}/\mathsf{T}$, viz $\mathcal{L}_\xi = \mathsf{G} \times_\xi \mathbb{C}$. By definition, $\mathsf{G} \times_\xi \mathbb{C} = \mathsf{G} \times \mathbb{C}/\sim_\xi$, where $\sim_\xi$ denotes the equivalence relation on $\mathsf{G} \times \mathbb{C}$ yielded by the identification

$$(\gamma, z) \sim_\xi (\gamma q^{-1}, \xi(q) z) \tag{4.12}$$

with $\gamma \in \mathsf{G}$, $z \in \mathbb{C}$ and $q \in \mathsf{T}$.

A p-form section of $\mathcal{L}_\xi$ is a map $s \in \mathrm{Map}(T[1]\mathsf{G}, \mathbb{C}[p])$ obeying

$$j_{\mathsf{R}x} s = 0, \quad l_{\mathsf{R}x} s = \dot{\xi}(x) s \tag{4.13}$$

with $x \in \mathfrak{t}$ in the right $\mathfrak{t}$-operation, where $\dot{\xi} \in \mathrm{Hom}(\mathfrak{t}, \mathfrak{u}(1))$ denotes the Lie differential of ξ. p-form section of $\mathcal{L}_\xi$ form a space $\Omega^p(\mathcal{L}_\xi)$.

The operational framework allows for a total space description of $\mathsf{U}(1)$ gauge theory, in particular of connections. A unitary connection of the line bundle $\mathcal{L}_\xi$ is a map $a \in \mathrm{Map}(T[1]\mathsf{G}, \mathfrak{u}(1)[1])$ obeying

$$j_{\mathsf{R}x} a = -\dot{\xi}(x), \quad l_{\mathsf{R}x} a = 0 \tag{4.14}$$

for $x \in \mathfrak{t}$. The connection a defines a covariant derivative d_a of p-form sections of $\mathcal{L}_\xi$. The curvature of a is the map $b \in \mathrm{Map}(T[1]\mathsf{G}, \mathfrak{u}(1)[2])$ satisfying

$$b = da, \quad db = 0 \tag{4.15}$$

the second relation being the Bianchi identity. b obeys

$$j_{\mathsf{R}x} b = 0, \quad l_{\mathsf{R}x} b = 0. \tag{4.16}$$

By (4.15), (4.16), $-ib$ is a closed 2-form on G/T. The unitary connections of $\mathcal{L}_\xi$ form an affine space $\mathrm{Conn}(\mathcal{L}_\xi)$.

The de Rham cohomology class $c(\mathcal{L}_\xi) = [-ib/2\pi] \in H^2(\mathsf{G}/\mathsf{T}, \mathbb{R})$ of the normalized curvature $-ib/2\pi$ of a connection a of $\mathcal{L}_\xi$ does not depend on a and so characterizes the line bundle $\mathcal{L}_\xi$ topologically. $c(\mathcal{L}_\xi)$ lies in a cohomology lattice, the image of the integral cohomology $H^2(\mathsf{G}/\mathsf{T}, \mathbb{Z})$ in $H^2(\mathsf{G}/\mathsf{T}, \mathbb{R})$.

The gauge transformations of the line bundle $\mathcal{L}_\xi$ can be straightforwardly characterized in the operational framework too. A gauge transformations is a map $u \in \mathrm{Map}(\mathsf{G}, \mathsf{U}(1))$ obeying the basicness conditions

$$j_{\mathsf{R}x} u u^{-1} = 0, \quad l_{\mathsf{R}x} u u^{-1} = 0 \tag{4.17}$$

for $x \in \mathfrak{t}$, where we view u as a map of $\mathrm{Map}(T[1]\mathsf{G}, \mathsf{U}(1))$ by virtue of the identity $\mathrm{Map}(T[1]\mathsf{G}, \mathsf{U}(1)) = \mathrm{Map}(\mathsf{G}, \mathsf{U}(1))$. Gauge transformations form a subgroup $\mathrm{Gau}(\mathcal{L}_\xi)$ of the infinite dimensional Lie group $\mathrm{Map}(\mathsf{G}, \mathsf{U}(1))$.

A gauge transformation $u \in \mathrm{Gau}(\mathcal{L}_\xi)$ act on a p-form section s of $\mathcal{L}_\xi$ as

$${}^u s = us. \tag{4.18}$$

It acts also on a connection a of $\mathcal{L}_\xi$ in the familiar manner

$${}^u a = a - du u^{-1}. \tag{4.19}$$

The curvature b of a is then gauge invariant

$${}^u b = b. \tag{4.20}$$

4.4 Poisson structures on a regular homogeneous space

In KKS theory, the prequantization of a coadjoint orbit requires that its symplectic structure matches the curvature of a suitable prequantum line bundle. In this subsection, working from a broader perspective as done before, we study the presymplectic structures on a regular homogeneous space G/T of the kind considered in subsection 4.3 which enjoy this essential compatibility property as well as the conjoined Poisson bracket structures and Hamiltonian function algebras.

Below, we shall rely to a large extent on the Cartan calculus of G, the $\mathrm{Vect}(\mathsf{G})$-operation on G featuring the contractions j_V and Lie derivatives l_V along the vector fields $V \in \mathrm{Vect}(\mathsf{G})$ and the de Rham differential d. The derivations $j_{\mathsf{R}x}$, $l_{\mathsf{R}x}$ with $x \in \mathfrak{t}$ of the right $\mathfrak{t}$-operation are just the derivations $j_{S_{\mathsf{R}x}}$, $l_{S_{\mathsf{R}x}}$ associated with the vertical vector field $S_{\mathsf{R}x}$ of the right T-action corresponding to x and similarly for the derivations $j_{\mathsf{L}h}$, $l_{\mathsf{L}h}$ with $h \in \mathfrak{g}$ of the left $\mathfrak{g}$-operation with regard to the vertical vector fields $S_{\mathsf{L}h}$ of the left G-action corresponding to h.

The closed 2-form $-ib$ associated with the curvature of a unitary connection a of the unitary line bundle $\mathcal{L}_\xi$ on G/T of a character ξ of T constitutes a presymplectic 2-form on G/T and therefore define a Poisson bracket structure on an associated Hamiltonian function algebra. In the operational description, the Hamiltonian functions are the degree 0 elements $f \in \mathrm{Fun}(T[1]\mathsf{G})$ which are invariant under the right $\mathfrak{t}$-action, so that

$$l_{\mathsf{R}x} f = 0 \tag{4.21}$$

for $x \in \mathfrak{t}$, and have the property that there is a vector field $P_f \in \mathrm{Vect}(\mathsf{G})$ with

$$df - ij_{P_f} b = 0. \tag{4.22}$$

The Hamiltonian vector field P_f of f is defined modulo vector fields $V \in \mathrm{Vect}(\mathsf{G})$ such that $j_V b = 0$ and obeys $j_{l_{S_{\mathsf{R}x}} P_f} b = 0$. The Hamiltonian functions form a degree 0 subalgebra $\mathrm{Fun}_a(T[1]\mathsf{G})$ of $\mathrm{Fun}(T[1]\mathsf{G})$ depending on a. The Poisson bracket of two functions $f, g \in \mathrm{Fun}_a(T[1]\mathsf{G})$ is given by

$$\{f, g\}_a = -ij_{P_g} j_{P_f} b. \tag{4.23}$$

By the right invariance condition property (4.21), the Hamiltonian function algebra $\mathrm{Fun}_a(T[1]\mathsf{G})$ can be identified with a subalgebra $\mathrm{Fun}_a(\mathsf{G}/\mathsf{T})$ of $\mathrm{Fun}(\mathsf{G}/\mathsf{T})$ and $\{\cdot,\cdot\}_a$ with a Poisson bracket structure on $\mathrm{Fun}_a(\mathsf{G}/\mathsf{T})$.

With standard KKS theory in mind, we concentrate now on the case where the connection a of $\mathcal{L}_\xi$ is invariant under the left G-action so that

$$l_{\mathsf{L}h} a = 0 \tag{4.24}$$

for $h \in \mathfrak{g}$. The associated curvature b is then also invariant

$$l_{\mathsf{L}h} b = 0. \tag{4.25}$$

We can then expect that the Poisson bracket structure $\{\cdot,\cdot\}_a$ will exhibit left symmetry properties. In fact, the left G-action is Hamiltonian. The Hamiltonian function $q_a(h) \in$

$\mathrm{Fun}_a(T[1]\mathsf{G})$ corresponding the Lie algebra element $h \in \mathfrak{g}$ is

$$q_a(h) = -ij_{\mathsf{L}h}a. \tag{4.26}$$

The Hamiltonian property of $q_a(h)$ follows from the left invariance of a and the commutativity of the right T- and left G-actions, which imply that $q_a(h)$ satisfies (4.21) and (4.22) with $P_{q_a(h)} = S_{\mathsf{L}h}$. Crucially, the map $q_a : \mathfrak{g} \to \mathrm{Fun}_a(T[1]\mathsf{G})$ is equivariant and constitutes a representation of $\mathfrak{g}$, as for $h, k \in \mathfrak{g}$,

$$l_{\mathsf{L}h}q_a(k) = q_a([h,k]) = \{q_a(h), q_a(k)\}_a, \tag{4.27}$$

a relation that characterizes q_a as the moment map of the left G-symmetry.

The Poisson bracket structure $\{\cdot,\cdot\}_a$ has simple gauge invariance properties. Since the connection a is restricted to be left invariant by (4.24), the gauge transformations allowed must be correspondingly left invariant, viz

$$l_{\mathsf{L}h}uu^{-1} = 0 \tag{4.28}$$

for $h \in \mathfrak{g}$. It is immediate that the Poisson bracket $\{\cdot,\cdot\}_a$ is gauge invariant, since it is defined via (4.22), (4.23) in terms of the curvature b of a which is gauge invariant. It is also readily checked from (4.26), (4.28) that the Hamiltonian functions $q_a(h)$, $h \in \mathfrak{g}$, are gauge invariant.

We conclude this subsection by observing that the Poisson bracket structure $\{\cdot,\cdot\}_a$ in general is not induced by a genuine symplectic structure on G/T unless the right T-action vertical vector fields $S_{\mathsf{R}x}$ with $x \in \mathfrak{g}$ are the only vector fields $V \in \mathrm{Vect}(\mathsf{G})$ such that $j_V b = 0$. In KKS theory, this is the situation customarily considered because in such a case $\mathrm{Fun}_a(\mathsf{G}/\mathsf{T}) = \mathrm{Fun}(\mathsf{G}/\mathsf{T})$ and standard geometric prequantization is possible.

4.5 Prequantization

The datum on which prequantization is based is the Poisson structure of the regular homogeneous space G/T associated with a unitary connection a of the line bundle $\mathcal{L}_\xi$ of a character ξ of T. Prequantization requires that the underlying presymplectic structure $-ib$ is a symplectic one. We therefore assume that the curvature b of a is non singular, so that $j_V b = 0$ with $V \in \mathrm{Vect}(\mathsf{G})$ only if $V = S_{\mathsf{R}x}$ for some $x \in \mathfrak{g}$.

Prequantization takes its start by setting

$$\widehat{f}s = ij_{P_f}\mathrm{d}_a s + fs \tag{4.29}$$

for any Hamiltonian function $f \in \mathrm{Fun}_a(\mathsf{G}/\mathsf{T})$ and 0-form section $s \in \Omega^0(\mathcal{L}_\xi)$. It is immediately checked using (4.13) and (4.21) that $\widehat{f}s \in \Omega^0(\mathcal{L}_\xi)$. $\widehat{f}$ is manifestly an endomorphism of the vector space $\Omega^0(\mathcal{L}_\xi)$. The symplectic structure $-ib$ being proportional to the curvature b of the connection a ensures that

$$[\widehat{f}, \widehat{g}] = i\widehat{\{f,g\}}_a \tag{4.30}$$

for $f, g \in \mathrm{Fun}_a(\mathsf{G}/\mathsf{T})$. The basic requirement of prequantization is thus met.

By virtue of (4.26), for $h \in \mathfrak{g}$ the operator $\widehat{q}_a(h)$ associated with the left G-symmetry moment map $q_a(h)$ takes the form

$$\widehat{q}_a(h) = il_{\mathsf{L}h}, \tag{4.31}$$

in agreement with the interpretation of $\widehat{q}_a$ as infinitesimal generator of the symmetry. From (4.27) and (4.30), one has

$$[\widehat{q}_a(h), \widehat{q}_a(k)] = i\widehat{q}_a([h,k]) \tag{4.32}$$

for $h, k \in \mathfrak{g}$, as expected.

The space $\Omega^0(\mathcal{L}_\xi)$ of 0-form sections of $\mathcal{L}_\xi$ has a Hilbert structure: for any two sections $s, t \in \Omega^0(\mathcal{L}_\xi)$

$$\langle s, t\rangle = \frac{1}{n!}\int_{T[1](\mathsf{G}/\mathsf{T})} \varrho_{\mathsf{G}/\mathsf{T}}(-ib)^n s^* t, \tag{4.33}$$

where $n = \dim(\mathsf{G}/\mathsf{T})/2$. The operator $\widehat{f}$ associated with each Hamiltonian function $f \in \mathrm{Fun}_a(\mathsf{G}/\mathsf{T})$ is formally Hermitian with respect to this Hilbert structure by virtue of the unitarity of the connection a.

4.6 Coadjoint orbit quantization and Borel-Weil theorem

The presymplectic structures studied in subsection 4.4 obey a Bohr-Sommerfeld type quantization condition since they come from the curvature of connections of line bundles. This important fact plays an important role in KKS theory as it circumscribes the range of quantizable coadjoint orbits as symplectic manifolds. This introduces us to the central part of our review of KKS theory: the quantization of coadjoint orbits and the related Borel-Weil theorem.

The basic data of the KKS construction are a compact semisimple Lie group G and a regular element $\lambda \in \mathfrak{g}^*$ of the dual of the Lie algebra $\mathfrak{g}$ of G. So, the coadjoint orbit $\mathcal{O}_\lambda = \mathsf{G}/\mathsf{T}$ for some maximal torus T of G. We can so rely on the results we obtained in the previous subsections.

The Lie algebra $\mathfrak{t}$ of T is a maximal toral subalgebra of $\mathfrak{g}$. By restricting λ to $\mathfrak{t}$, we obtain an element $\lambda \in \mathfrak{t}^*$ denoted in the same way for simplicity. If $\lambda/2\pi \in \Lambda_{\mathsf{G}}{}^*$, the dual of the integral lattice Λ_{G} of $\mathfrak{t}$, then there exists a character $\xi_\lambda \in \mathrm{Hom}(\mathsf{T}, \mathsf{U}(1))$ given by

$$\xi_\lambda(\mathrm{e}^t) = \mathrm{e}^{i\langle\lambda,t\rangle} \tag{4.34}$$

for $t \in \mathfrak{t}$. With λ, we can thus associate a unitary line bundle $\mathcal{L}_\lambda := \mathcal{L}_{\xi_\lambda}$ using the construction of subsection 4.3.

$\mathcal{L}_\lambda$ is equipped with a canonical unitary connection

$$a_\lambda = -i\langle\lambda, \sigma\rangle. \tag{4.35}$$

By (4.15) and (4.2), the curvature of a_λ is

$$b_\lambda = \frac{i}{2}\langle\lambda, [\sigma, \sigma]\rangle. \tag{4.36}$$

Both a_λ and b_λ are left invariant, since σ is on account of (4.5). The presymplectic structure $-ib_\lambda$ is furthermore non singular on G/T and thus symplectic. This is the KKS symplectic structure of $\mathcal{O}_\lambda$, the KKS λ-structure. The Hamiltonian functions of the left G-symmetry also have a simple expression

$$q_\lambda(h) = \langle \mathrm{Ad}\,\gamma^*(\lambda), h\rangle. \tag{4.37}$$

The whole above structure has natural invariance properties under the automorphism group action of the principal T-bundle G. With any automorphism $\psi \in \mathrm{Aut}_{\mathsf{T}}(\mathsf{G})$, there is associated a map $u_\lambda \in \mathrm{MAP}(T[1]\mathsf{G}, \mathsf{U}(1))$ by

$$u_\lambda = \exp\left(-i\int_{1_{\mathsf{G}}}^{\cdot}\langle\lambda, d\psi\psi^{-1}\rangle\right). \tag{4.38}$$

As $d\psi\psi^{-1} \in \mathrm{MAP}(T[1]\mathsf{G}, \mathfrak{t}[1])$ represents an element of $\Omega^1(\mathsf{G}, \mathfrak{t})$ with periods in the integral lattice Λ_{G} and $\lambda \in \Lambda_{\mathsf{G}}{}^*$, u_λ is singlevalued as required. By (4.8), u_λ obeys relations (4.17) and so is a gauge transformation, $u_\lambda \in \mathrm{Gau}(\mathcal{L}_\lambda)$. A group morphism $\psi \to u_\lambda$ from the automorphism group $\mathrm{Aut}_{\mathsf{T}}(\mathsf{G})$ of G to the gauge transformation group $\mathrm{Gau}(\mathcal{L}_\lambda)$ of $\mathcal{L}_\lambda$ is established by (4.38) in this way.

For an automorphism $\psi \in \mathrm{Aut}_{\mathsf{T}}(\mathsf{G})$, let ${}^{\psi}a_\lambda$ be the connection given by (4.35) with σ replaced by its transform ${}^{\psi}\sigma$ (cf. eqs. (4.9)). Then, ${}^{\psi}a_\lambda = {}^{u_\lambda}a_\lambda$ where the expression in the right hand side is given by (4.19). Consequently, one has ${}^{\psi}b_\lambda = {}^{u_\lambda}b_\lambda = b_\lambda$ by (4.20). Similarly, one has ${}^{\psi}q_\lambda = q_\lambda$.

By the above findings, it appears that $\mathcal{L}_\lambda$ is a prequantum line bundle on G/T. A prequantum Hilbert space structure is then defined as described in subsection 4.5. To get the quantum Hilbert space, one needs a polarization. This is obtained by endowing G/T with a complex structure J realized as an integrable complex splitting of the complexified tangent bundle $T_{\mathbb{C}}(\mathsf{G}/\mathsf{T})$. Because of the left G-invariance of the whole geometric set-up, it is enough to provide the splitting at the identity coset T of G/T. Since $T_{\mathbb{C}\mathsf{T}}(\mathsf{G}/\mathsf{T}) \simeq (\mathfrak{g}\ominus\mathfrak{t})\otimes\mathbb{C}$, the complexification of the orthogonal complement of $\mathfrak{t}$ with respect to the Cartan form of $\mathfrak{g}$, we may choose $(\mathfrak{g}\ominus\mathfrak{t})^{1,0} = \bigoplus_{\alpha\in\Delta_+}\mathfrak{e}_\alpha$, $(\mathfrak{g}\ominus\mathfrak{t})^{0,1} = \bigoplus_{\alpha\in\Delta_+}\mathfrak{e}_{-\alpha}$, where Δ_+ is a set of positive roots of $\mathfrak{g}$ and the $\mathfrak{e}_\alpha$ are the root spaces of $\mathfrak{g}\otimes\mathbb{C}$. Further, J is compatible with the symplectic structure $-ib_\lambda$ in the sense that $-ib_\lambda(\cdot, J\cdot)$ constitutes a left G-invariant Kaehler metric on G/T whose Kaehler form is precisely $-ib_\lambda$. In this way, once endowed with the complex structure J, G/T is a Kaehler manifold. $\mathcal{L}_\lambda$ then turns out to be a holomorphic line bundle on G/T. With the above polarization of G/T available, it is possible to define the quantum Hilbert space $\mathscr{H}_\lambda$ as the space of square integrable holomorphic sections of $\mathcal{L}_\lambda$,

$$\mathscr{H}_\lambda = H^0_{\bar\partial}(\mathsf{G}/\mathsf{T}, \mathcal{L}_\lambda), \tag{4.39}$$

provided that this latter is non vanishing. According to the Borel-Weil-Bott theorem [44, 45], $H^0_{\bar\partial}(\mathsf{G}/\mathsf{T}, \mathcal{L}_\lambda)$ is non zero precisely when $\lambda \in \Lambda_{\mathsf{wG}}{}^+$, the lattice of dominant weights of G. The highest weight theorem establishes a one-to-one correspondence between $\Lambda_{\mathsf{wG}}{}^+$ and the set of equivalence classes of irreducible representations of G. The theorem provides in this way the grounding for the relation between coadjoint orbit geometric quantization and partition function description of Wilson loops to be discussed in II.

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5 Derived KKS theory

In this section, working systematically within the derived framework of section 3, we shall elaborate a derived KKS theory on the model of the standard KKS theory expounded in section 4. In ref. [31], we showed that higher 4-dimensional CS theory admits a derived formulation that very closely parallels that of its familiar ordinary 3-dimensional counterpart. The idea is to follow the same path and formulate higher KKS theory by reproducing the constructions of its ordinary counterpart in a derived perspective. This approach will lead us far but it will not solve all the problems. The decisive step of geometric quantization, the construction of a prequantum Hilbert space and a polarization, remains elusive even though the immediate reasons for this seem clear.

In outline, we propose a definition of a derived coadjoint orbit as an instance of derived homogeneous space. In general, a derived homogeneous space is a graded manifold of the form $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{M}'$, where M is a Lie group crossed module and M' is a crossed submodule of M and $\mathrm{D}\mathsf{M}$ and $\mathrm{D}\mathsf{M}'$ are their derived Lie groups, which is very aptly described in an appropriate operational framework (cf. appendix A.3). We concentrate on the important regular case, where M' is a maximal toral crossed submodule of M. We then show that with any character β of J, there is associated a derived unitary line bundle $\mathcal{L}_\beta$ on $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$ and define the notion of derived connection of $\mathcal{L}_\beta$ and curvature thereof. The curvature of a connection provides a derived presymplectic form on $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$ obeying an integrality condition. Since a regular derived coadjoint orbit is just a manifold such as $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$ equipped with a derived prequantum line bundle such as $\mathcal{L}_\beta$, this analysis indicates which features the derived KKS symplectic form of a regular derived coadjoint orbit with a built in Bohr-Sommerfeld quantization condition should have. This is not sufficient for a complete derived KKS geometric quantization, but it paves the way to other more conventional quantization schemes such as that of derived TCO theory in section 5 of II.

The content of this section is mostly technical. The nature of the topics covered makes this essentially unavoidable. We collect anyway the relevant component identities at the end of each subsection, to make a first reading easier.

5.1 Derived coadjoint orbits and regular elements

Our construction of derived KKS theory must necessarily begin with the definition of the notions of derived coadjoint orbit and regularity. As anticipated, we shall do so by taking the corresponding notions of ordinary KKS theory as a model (cf. subsection 4.1) and using firmly the derived set-up of section 3 as our framework.

In ordinary KKS theory, a coadjoint orbit always refers to some element $\lambda \in \mathfrak{g}^*$ of the dual space of the Lie algebra $\mathfrak{g}$ of the given Lie group G. In derived KKS theory, the relevant Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ is as a rule equipped with an invariant pairing $\langle \cdot, \cdot \rangle$ (cf. subsection 3.1). In this way, $\mathfrak{e}$, as a vector space, can be identified with $\mathfrak{g}^*$. Further, by (3.3), the G-action of $\mathfrak{e}$, $\dot{\mu}$, is dual with respect to the pairing to the adjoint G-action of $\mathfrak{g}$, Ad. It is so sensible that a derived coadjoint orbit should refer to some element $\Lambda \in \mathfrak{e}$. It could do so even in the absence of an invariant pairing, $\dot{\mu}$ playing the role of the coadjoint action.

With the above considerations in mind, we introduce a few pertinent concepts of crossed module theory. Let $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ be a Lie group crossed module and let $\mathsf{M}' = (\mathsf{E}', \mathsf{G}')$ be a crossed submodule of M. We can associate with M' two subgroups of G and one of E. The centralizer $\mathrm{Z}\mathsf{G}'$ of G' is the subgroup of G constituted by those elements $a \in \mathsf{G}$ such that $aba^{-1} = b$ for $b \in \mathsf{G}'$. Similarly, the μ-centralizer of $\mu\,\mathrm{Z}\mathsf{E}'$ of E' is the subgroup of G of the elements $a \in \mathsf{G}$ such that $\mu(a, B) = B$ for $B \in \mathsf{E}'$. Finally, the μ-trivializer $\mu\,\mathrm{F}\mathsf{G}'$ of G' is the Lie subgroup of E formed by those elements $A \in \mathsf{E}$ obeying $\mu(b, A) = A$ for $b \in \mathsf{G}'$. It can be easily verified that the structure $\mathrm{Z}\mathsf{M}' = (\mu\,\mathrm{F}\mathsf{G}', \mathrm{Z}\mathsf{G}' \cap \mu\,\mathrm{Z}\mathsf{E}')$ is a crossed submodule of M, that we call the centralizer crossed module of M'.

The case we have in mind features a crossed module M and a crossed submodule M' of M that is a submodule of its centralizer $\mathrm{Z}\mathsf{M}'$. This property implies restrictively that both E' and G' are Abelian and that the G'-action on E' is trivial.

We now have at our disposal all the elements required for the definition of the derived coadjoint orbit of an element $\varLambda \in \mathfrak{e}$. We denote by $\mathsf{E}_\varLambda$ and $\mathsf{G}_\varLambda$ the subgroups of E and G constituted by the elements of the form $\mathrm{e}^{x\varLambda}$ and $\mathrm{e}^{y\dot{\tau}(\varLambda)}$ with $x, y \in \mathbb{R}$, respectively. Then, the structure $\mathsf{M}_\varLambda = (\mathsf{E}_\varLambda, \mathsf{G}_\varLambda)$ is a crossed submodule of M, the 1-parameter submodule generated by $\varLambda$. The centralizer crossed module of $\mathsf{M}_\varLambda$, which we shall call the centralizer crossed submodule of $\varLambda$ in M for clarity, is $\mathrm{Z}\mathsf{M}_\varLambda = (\mathrm{Z}\mathsf{E}_\varLambda, \mu\,\mathrm{Z}\mathsf{E}_\varLambda)$, where $\mathrm{Z}\mathsf{E}_\varLambda$ and $\mu\,\mathrm{Z}\mathsf{E}_\varLambda$ are here the subgroups of E and G of the elements $A \in \mathsf{E}$ and $a \in \mathsf{G}$ such that $\mathrm{Ad}\,A(\varLambda) = \varLambda$ and $\dot{\mu}(a, \varLambda) = \varLambda$, respectively. As $\mathsf{M}_\varLambda$ is a crossed submodule of $\mathrm{Z}\mathsf{M}_\varLambda$, this is a case of the kind considered in the previous paragraph.

The derived KKS coadjoint orbit of $\varLambda$ is the derived homogeneous space

$$\mathscr{O}_\varLambda = \mathrm{D}\mathsf{M}/\mathrm{D}\mathrm{Z}\mathsf{M}_\varLambda. \tag{5.1}$$

Since the centralizer $\mathrm{Z}\mathsf{M}_\varLambda$ of $\mathsf{M}_\varLambda$ can be considered as the stability crossed submodule of $\varLambda$, $\mathscr{O}_\varLambda$ can legitimately be regarded as a derived model of the coadjoint orbit of $\varLambda$ in M in analogy to a coadjoint orbit of ordinary KKS theory. From the general definition of derived group, eq. (3.4), and the characterization of the crossed module $\mathrm{Z}\mathsf{M}_\varLambda$ given in the previous paragraph, it follows readily that

$$\mathscr{O}_\varLambda = \mathfrak{e}/\mathrm{Z}\mathfrak{e}_\varLambda[1] \times \mathsf{G}/\mu\,\mathrm{Z}\mathsf{E}_\varLambda, \tag{5.2}$$

where $\mathrm{Z}\mathfrak{e}_\varLambda$ is the Lie algebra of $\mathrm{Z}\mathsf{E}_\varLambda$.

In ordinary KKS theory, one considers mainly the coadjoint orbits of regular Lie algebra elements for simplicity. When the relevant group is compact, the regular elements are those whose stability group is a maximal torus. In derived KKS theory, regular elements are defined in a similar way upon introducing suitable notions of compact crossed module and maximal toral crossed submodule thereof.

A Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ is called compact if G is a compact group. A maximal toral crossed submodule J of a compact Lie group crossed module M consists of subgroups T and H of G and E, respectively, with the following properties: (i) T is a maximal torus of G; (ii) for $A \in \mathsf{H}$, $\tau(A) \in \mathsf{T}$; (iii) for $a \in \mathsf{T}$ and $A \in \mathsf{H}$, one has $\mu(a, A) = A$; (iv) for fixed T, H is maximal with the above properties. In fact, it can be shown that $\mathsf{H} = \mu\,\mathrm{F}\,\mathsf{T}$, where $\mu\,\mathrm{F}\,\mathsf{T}$ is the μ-trivializer of T, the subgroup of E constituted by the elements $A \in \mathsf{E}$ such that $\mu(a, A) = A$ for $a \in \mathsf{T}$. Therefore, T determines J and so J may be referred to as the maximal toral crossed submodule of T in M.

A maximal toral crossed submodule $\mathsf{J} = (\mathsf{H}, \mathsf{T})$ of M has properties which make it akin to a familiar Lie theoretic maximal torus. First, J with the induced target and action structure maps is a crossed submodule of M as indicated by its name. Second, by virtue of the Peiffer identity, H is automatically an Abelian subgroup of E, though not necessarily maximally Abelian, rendering J a fully Abelian object. Third, J is maximal, since T is and H is fully determined by T.

Suppose so that the Lie group crossed module M is compact. An element $\Lambda \in \mathfrak{e}$ is said to be regular if the centralizer $\mathrm{Z}\mathsf{M}_\Lambda$ of Λ is a maximal toral crossed submodule of M. In this case, from (5.1), the derived coadjoint orbit of Λ is $\mathcal{O}_\Lambda = \mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$. By the special Abelian features of J, $\mathcal{O}_\Lambda$ is expected to enjoy a particularly simple description. On account of (5.2), we have $\mathcal{O}_\Lambda = \mathfrak{e}/\mathfrak{h}[1] \times \mathsf{G}/\mathsf{T}$.

As an illustration we present a few examples. With any pair of a Lie group G and a normal subgroup N of G, there is associated a Lie group crossed module $\mathrm{I{\scriptsize NN}}_{\mathsf{G}}\,\mathsf{N} = (\mathsf{N}, \mathsf{G}, \iota, \kappa)$, where ι is the inclusion map of N into G and κ is the left conjugation action of G on N. For $\Lambda \in \mathfrak{n}$, the centralizer crossed submodule of Λ in $\mathrm{I{\scriptsize NN}}_{\mathsf{G}}\,\mathsf{N}$ is the crossed module $\mathrm{I{\scriptsize NN}}_{\mathrm{Z}_{\mathsf{G}}\Lambda}(\mathsf{N} \cap \mathrm{Z}_{\mathsf{G}}\Lambda)$, where $\mathrm{Z}_{\mathsf{G}}\Lambda$ is the subgroup of G of the elements $a \in \mathsf{G}$ such that $\mathrm{Ad}\,a(\Lambda) = \Lambda$. If G is compact and T is a maximal torus of G, the maximal toral crossed submodule of T in $\mathrm{I{\scriptsize NN}}_{\mathsf{G}}\,\mathsf{N}$ is the crossed module $\mathrm{I{\scriptsize NN}}_{\mathsf{T}}(\mathsf{N} \cap \mathsf{T})$. Λ is regular precisely when $\mathrm{Z}_{\mathsf{G}}\Lambda = \mathsf{T}$ for some maximal torus T, i.e. when Λ is regular as an element of $\mathfrak{g}$ in the usual sense. In such a case, we have $\mathcal{O}_\Lambda = \mathfrak{n}/\mathfrak{n} \cap \mathfrak{t}[1] \times \mathsf{G}/\mathsf{T}$.

With any central extension $1 \longrightarrow \mathsf{C} \xrightarrow{\iota} \mathsf{Q} \xrightarrow{\pi} \mathsf{G} \longrightarrow 1$ of Lie groups, there is associated a Lie group crossed module $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G}) = (\mathsf{Q}, \mathsf{G}, \pi, \alpha)$, where the action α is given by $\alpha(a, A) = \sigma(a)A\sigma(a)^{-1}$ for $a \in \mathsf{G}$, $A \in \mathsf{Q}$ with $\sigma : \mathsf{G} \to \mathsf{Q}$ a section of the projection π, i.e. $\pi \circ \sigma = \mathrm{id}_{\mathsf{G}}$, whose choice is immaterial. If $\Lambda \in \mathfrak{q}$, there is an induced Lie group central extension $1 \longrightarrow \mathsf{C} \xrightarrow{\iota} \mathrm{Z}_{\mathsf{Q}}\Lambda \xrightarrow{\pi} \sigma\,\mathrm{Z}_{\mathsf{G}}\Lambda \longrightarrow 1$, where $\mathrm{Z}_{\mathsf{Q}}\Lambda$ and $\sigma\,\mathrm{Z}_{\mathsf{G}}\Lambda$ are the subgroups of Q and G of the elements $A \in \mathsf{Q}$ and $a \in \mathsf{G}$ such that $\mathrm{Ad}\,A(\Lambda) = \Lambda$ and $\mathrm{Ad}\,\sigma(a)(\Lambda) = \Lambda$, respectively. The centralizer crossed submodule of Λ in $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G})$ is so the crossed module $\mathsf{C}(\mathrm{Z}_{\mathsf{Q}}\Lambda \xrightarrow{\pi} \sigma\,\mathrm{Z}_{\mathsf{G}}\Lambda)$. If G is compact and T is a maximal torus of G, there is an induced group sequence $1 \longrightarrow \mathsf{C} \xrightarrow{\iota} \sigma\,\mathrm{F}_{\mathsf{Q}}\mathsf{T} \xrightarrow{\pi} \mathsf{T} \longrightarrow 1$, where $\sigma\,\mathrm{F}_{\mathsf{Q}}\mathsf{T}$ is the subgroup of Q of the elements $A \in \mathsf{A}$ such that $\sigma(a)A\sigma(a)^{-1} = A$ for $a \in \mathsf{T}$. This sequence is generally not exact, as π may fail to be onto. It is if σ can be chosen such that $\sigma : \mathsf{T} \to \mathsf{Q}$ is a group morphism for one and so all maximal tori T. In that case, the maximal toral crossed submodule of T in $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G})$ is the crossed module $\mathsf{C}(\sigma\,\mathrm{F}_{\mathsf{Q}}\mathsf{T} \xrightarrow{\pi} \mathsf{T})$. Λ is regular precisely when $\sigma\,\mathrm{Z}_{\mathsf{G}}\Lambda = \mathsf{T}$ for some maximal torus T. This being so depends ultimately on the form of the projection π. If Λ is regular, we have $\mathcal{O}_\Lambda = \mathfrak{q}/\sigma\,\mathrm{F}_{\mathsf{Q}}\mathfrak{t}[1] \times \mathsf{G}/\mathsf{T}$, where $\sigma\,\mathrm{F}_{\mathsf{Q}}\mathfrak{t}$ is the Lie algebra of $\sigma\sigma\,\mathrm{F}_{\mathsf{Q}}\mathsf{T}$.

A Lie group crossed module $\mathsf{D}(\rho) = (\mathsf{V}, \mathsf{G}, 1_{\mathsf{G}}, \rho)$ can be associated with the data consisting of a Lie group G, a vector space V regarded as an Abelian group, the trivial morphism 1_{G} of V into G and a representation ρ of G in V. In this case, the centralizer

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crossed submodule of an element $\varLambda \in \mathfrak{v}$ in $\mathsf{D}(\rho)$ is the crossed module $\mathsf{D}(\rho|_{\mathrm{INV}_{\mathsf{G},\rho}\varLambda})$ where $\mathrm{INV}_{\mathsf{G},\rho}\varLambda$ is the invariance subgroup of $\varLambda$ in G. If G is compact and T is maximal torus of G, the maximal toral crossed submodule of T in $\mathsf{D}(\rho)$ is the crossed module $\mathsf{D}(\iota_{\mathsf{T},\mathsf{V}_\mathsf{T}})$, where V_T is the subspace of V spanned by the elements $X \in \mathsf{V}$ such that $\rho(a)X = X$ for $a \in \mathsf{T}$ and $\iota_{\mathsf{T},\mathsf{V}_\mathsf{T}}$ is the trivial representation of T in V_T. Thus, $\varLambda$ is regular only provided that the representation ρ is trivial. Clearly, the derived KKS theory of the crossed modules $\mathsf{D}(\rho)$ is kind of trivial, making them less interesting than the ones treated in the previous two paragraphs.

In the next several subsections, we shall work out the main elements of derived KKS theory from a general standpoint. The model crossed modules treated above can be refereed to for illustration of the abstract notions and exemplification of the most significant relations of the theory. It turns out that in most case this is a fairly straightforward task, which for this reason is left to the reader. In subsection 5.10, which is the culmination of our formal development and contains the results of derived KKS theory most relevant in paper II, we shall present in great detail further examples.

5.2 Operational description of derived homogeneous spaces

As in the standard theory expounded in section 4, the fact that derived coadjoint orbits are instances of derived homogeneous spaces allows for the application of operational methods in derived KKS theory. In this subsection patterned on subsection 4.2, we take a general standpoint. We consider a Lie group crossed module M and a crossed submodule M' of M and working in the derived framework regard $\mathrm{D}\mathsf{M}$ as a principal $\mathrm{D}\mathsf{M}'$-bundle over the derived homogeneous space $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{M}'$. We then offer an operational description of $\mathrm{D}\mathsf{M}$ as such, using which the geometry of $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{M}'$ can be studied as the basic geometry of $\mathrm{D}\mathsf{M}$. Later, we shall adapt the resulting analysis to special choices of M'.

The basic data of the operation that we are going to construct are the derived Lie algebra $\mathrm{D}\mathfrak{m}'$ of $\mathfrak{m}'$ and the internal function algebra $\mathrm{FUN}(T[1]\,\mathrm{D}\mathsf{M})$ of $T[1]\,\mathrm{D}\mathsf{M}$ (cf. appendix A.1). Because of the graded nature of the derived group $\mathrm{D}\mathsf{M}'$, the ordinary algebra $\mathrm{Fun}(T[1]\,\mathrm{D}\mathsf{M})$ cannot in fact be consistently employed.

The detailed specification of the operation requires that appropriate coordinates for the shifted tangent bundle $T[1]\,\mathrm{D}\mathsf{M}$ are used. By the isomorphism $T[1]\,\mathrm{D}\mathsf{M} \simeq \mathrm{D}\mathsf{M} \times \mathrm{D}\mathfrak{m}[1]$, it is natural to employ coordinates adapted to the two Cartesian factors $\mathrm{D}\mathsf{M}$, $\mathrm{D}\mathfrak{m}[1]$, which we may think of as base and fiber coordinates of $T[1]\,\mathrm{D}\mathsf{M}$. These are conveniently encoded in derived variables $\Gamma \in \mathrm{D}\mathsf{M}$, $\Sigma \in \mathrm{D}\mathfrak{m}[1]$ in an index free manner.

The action of the de Rham derivation d is expressed naturally via that of the corresponding derived differential d (cf. subsection 3.3). As the derived coordinates Γ, Σ we use reflect the isomorphism $T[1]\,\mathrm{D}\mathsf{M} \simeq \mathrm{D}\mathsf{M} \times \mathrm{D}\mathfrak{m}[1]$, d acts on Γ as

$$\Gamma^{-1}\mathrm{d}\Gamma = \Sigma, \tag{5.3}$$

identifying Σ as the derived Maurer-Cartan form associated with Γ. The action of d on Σ is mandated by the requirement of nilpotence of d and reduces to the derived Maurer-Cartan equation

$$\mathrm{d}\Sigma = -\tfrac{1}{2}[\Sigma, \Sigma]. \tag{5.4}$$

These identities are the derived counterpart of relations (4.2) of the standard theory.

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The right DM'-action of DM has an obvious expression in terms of the derived coordinates Γ, Σ. On the base coordinate Γ, the action reads as

$$\Gamma^{\mathsf{R}\mathrm{Q}} = \Gamma\mathrm{Q} \tag{5.5}$$

with $\mathrm{Q} \in \mathrm{DM}'$. The resulting action on the fiber coordinate Σ is then determined by (5.3) and takes the form

$$\Sigma^{\mathsf{R}\mathrm{Q}} = \mathrm{Ad}\,\mathrm{Q}^{-1}(\Sigma) + \mathrm{Q}^{-1}\mathrm{d}\mathrm{Q}. \tag{5.6}$$

These relations correspond to relations (4.3) in the standard theory. Note however the appearance in (5.6) of an inhomogeneous term $\mathrm{Q}^{-1}\mathrm{dQ}$ with no counterpart in (4.3) due to the special form of the derived differential d (cf. eq. (3.20)), which allows $\mathrm{Q}^{-1}\mathrm{dQ}$ to be non vanishing.

In the operational set-up associated with the right DM'-action, Γ behaves as

$$\Gamma^{-1} j_{\mathsf{R}\mathrm{X}}\Gamma = 0, \tag{5.7}$$

$$\Gamma^{-1} l_{\mathsf{R}\mathrm{X}}\Gamma = \mathrm{X} \tag{5.8}$$

with $\mathrm{X} \in \mathrm{D}\mathfrak{m}'$. By virtue of (5.3), Σ then satisfies

$$j_{\mathsf{R}\mathrm{X}}\Sigma = \mathrm{X}, \tag{5.9}$$

$$l_{\mathsf{R}\mathrm{X}}\Sigma = \mathrm{dX} - [\mathrm{X}, \Sigma]. \tag{5.10}$$

These relations answer to relations (4.4) of the standard theory as appears from inspection. The appearance in (5.10) of a non vanishing inhomogeneous term dX with no counterpart in (4.4) due to the special form of the derived differential d is again to be noticed.

For the sake of concreteness and completeness and later use, we express the above relations in terms of the components γ, $\mathit{\Gamma}$, σ, $\mathit{\Sigma}$ of the derived coordinates Γ, Σ. This material constitutes a reference table, which the hurried reader can skip, if he/she wishes so, proceeding directly to the next subsection.

The defining relation (5.3) and the Maurer-Cartan equation (5.4) of the components γ, $\mathit{\Gamma}$, σ, $\mathit{\Sigma}$ of Γ, Σ read as

$$\mathrm{Ad}\,\gamma^{-1}\left(d\gamma\gamma^{-1} + \dot{\tau}(\mathit{\Gamma})\right) = \sigma, \tag{5.11}$$

$$\mu\dot{}\left(\gamma^{-1}, d\mathit{\Gamma} + \tfrac{1}{2}[\mathit{\Gamma}, \mathit{\Gamma}]\right) = \mathit{\Sigma}, \tag{5.12}$$

$$d\sigma = -\tfrac{1}{2}[\sigma, \sigma] + \dot{\tau}(\mathit{\Sigma}), \tag{5.13}$$

$$d\mathit{\Sigma} = -\dot{}\mu\dot{}(\sigma, \mathit{\Sigma}). \tag{5.14}$$

The component expressions of the right DM'-action, given in eqs. (5.5) and (5.6), involve besides the coordinate components γ, $\mathit{\Gamma}$, σ, $\mathit{\Sigma}$ the components q, Q of the derived group parameter Q. They take the form

$$\gamma^{\mathsf{R}q,Q} = \gamma q, \tag{5.15}$$

$$\mathit{\Gamma}^{\mathsf{R}q,Q} = \mathit{\Gamma} + \mu\dot{}(\gamma, Q), \tag{5.16}$$

$$\sigma^{\mathsf{R}q,Q} = \mathrm{Ad}\,q^{-1}\left(\sigma + \dot{\tau}(Q)\right), \tag{5.17}$$

$$\mathit{\Sigma}^{\mathsf{R}q,Q} = \mu\dot{}\left(q^{-1}, \mathit{\Sigma} + \dot{}\mu\dot{}(\sigma, Q) + \tfrac{1}{2}[Q, Q]\right). \tag{5.18}$$

The component expressions of the structure relations (5.7)–(5.10) of the associated right $\mathrm{D}\mathfrak{m}'$-operation involve in addition to the coordinate components γ, $\varGamma$, σ, $\varSigma$ the components x, X of the derived algebra parameter X in similar fashion and read explicitly as

$$\gamma^{-1} j_{\mathsf{R}x,X} \gamma = 0, \tag{5.19}$$

$$\dot{\mu}(\gamma^{-1}, j_{\mathsf{R}x,X} \varGamma) = 0, \tag{5.20}$$

$$\gamma^{-1} l_{\mathsf{R}x,X} \gamma = x, \tag{5.21}$$

$$\dot{\mu}(\gamma^{-1}, l_{\mathsf{R}x,X} \varGamma) = X, \tag{5.22}$$

$$j_{\mathsf{R}x,X} \sigma = x, \tag{5.23}$$

$$j_{\mathsf{R}x,X} \varSigma = X, \tag{5.24}$$

$$l_{\mathsf{R}x,X} \sigma = -[x, \sigma] + \dot{\tau}(X), \tag{5.25}$$

$$l_{\mathsf{R}x,X} \varSigma = -\dot{\mu}\dot{}(x, \varSigma) + \dot{\mu}\dot{}(\sigma, X). \tag{5.26}$$

5.3 Target kernel symmetry

The target kernel symmetry of a derived homogeneous space is the counterpart of the left symmetry of an ordinary homogeneous space described in subsection 4.2. Its properties however turn out to be a bit more involved. The symmetry is the topic of this subsection.

Every Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ is characterized as such by a canonical crossed submodule, the target kernel crossed module $\mathsf{M}_\tau = (\ker\tau, \mathsf{G})$. With this there is associated a distinguished symmetry of the homogeneous space $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{M}'$ of the kind studied in subsection 5.2.

The size and nature of the target kernel crossed module M_τ depends on crossed module M to an important extent. For an illustration of this point, we shall refer here to the model crossed modules described in detail in the last part of subsection 5.1. If M the crossed module $\mathrm{I{\small NN}}_{\mathsf{G}}\,\mathsf{N}$ associated with a normal subgroup N of the Lie group G, $\ker\tau$ is the trivial group 1; M_τ then reduces to the crossed module $(1, \mathsf{G})$, which codifies the group G as a crossed module. If M is the crossed modules $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G})$ yielded by a central extension $1 \longrightarrow \mathsf{C} \xrightarrow{\iota} \mathsf{Q} \xrightarrow{\pi} \mathsf{G} \longrightarrow 1$, $\ker\tau$ is just the central subgroup $\iota(\mathsf{C})$; M_τ is the identified with the crossed module $(\iota(\mathsf{C}), \mathsf{G})$ with trivial target and action maps. Finally, if M is the crossed modules $\mathsf{D}(\rho)$ corresponding to a representation ρ of G in a vector space V, $\ker\tau$ is the whole space V; M_τ therefore is the crossed module (V, G), that is the whole crossed module M.

As a manifold, $\mathrm{D}\mathsf{M}$ is endowed with a left $\mathrm{D}\mathsf{M}_\tau$-action, which we shall write in right form for convenience. This has a fairly simple expression in terms derived coordinates Γ, Σ. On the base coordinate Γ, the action reads as

$$\Gamma^{\mathsf{LE}} = \mathrm{E}^{-1}\Gamma \tag{5.27}$$

with $\mathrm{E} \in \mathrm{D}\mathsf{M}_\tau$. The action on the fiber coordinate Σ is trivial

$$\Sigma^{\mathsf{LE}} = \Sigma. \tag{5.28}$$

This follows readily from (5.3) noting that M_τ is by construction the largest crossed submodule of M such that $\mathrm{E}^{-1}d\mathrm{E} = 0$ for $\mathrm{E} \in \mathrm{D}\mathsf{M}_\tau$, by (3.24). Comparison of (5.27), (5.28) with (4.5) shows clearly that the target kernel symmetry of the derived KKS theory answers to the left symmetry of the ordinary one as we anticipated. The analogy is not complete however: while in the derived theory the target kernel group DM_τ is a proper subgroup of the ambient group DM in general, in the ordinary one the left symmetry group and the ambient group are the same, viz G.

The left DM_τ-action is aptly described infinitesimally by an associated left $\mathrm{D}\mathfrak{m}_\tau$-operation. The derivations j_{LH}, l_{LH} thereof are indexed by a derived target kernel Lie algebra parameter $\mathrm{H} \in \mathrm{D}\mathfrak{m}_\tau$. They act as

$$j_{\mathsf{LH}}\Gamma\Gamma^{-1} = 0, \tag{5.29}$$
$$l_{\mathsf{LH}}\Gamma\Gamma^{-1} = -\mathrm{H} \tag{5.30}$$

on the coordinate Γ and

$$j_{\mathsf{LH}}\Sigma = -\operatorname{Ad}\Gamma^{-1}(\mathrm{H}), \tag{5.31}$$
$$l_{\mathsf{LH}}\Sigma = 0 \tag{5.32}$$

on the coordinate Σ. Straightforward inspection reveals that relations (5.29)–(5.32) of the derived theory are the counterpart of relations (4.6) of the ordinary one, as expected.

The left DM_τ-action manifestly commutes with the right DM'-action of eqs. (5.5), (5.6). It descends so to a DM_τ-action on the homogeneous space DM/DM'. The commutativity of the DM_τ- and DM'-actions is reflected by the graded commutativity of the derivations j_{LH}, l_{LH} of eqs. (5.29)–(5.32) and the derivations j_{RX}, l_{RX} of the right $\mathrm{D}\mathfrak{m}'$-operation given by eqs. (5.7)–(5.10).

For completeness and later reference, we rewrite the above relations in terms of the components γ, $\mathit{\Gamma}$, σ, $\mathit{\Sigma}$ of the derived coordinates Γ, Σ. As in the analogous previous circumstances, the uninterested reader can skip directly to the next subsection, if he/she wishes so.

The component expression of the left DM_τ-action shown in eqs. (5.27), (5.28) involves beside the coordinate components γ, $\mathit{\Gamma}$, σ, $\mathit{\Sigma}$ the components e, E of the derived Lie group parameter E. They take the form

$$\gamma^{\mathsf{L}e,E} = e^{-1}\gamma, \tag{5.33}$$
$$\mathit{\Gamma}^{\mathsf{L}e,E} = \dot{\mu}(e^{-1}, \mathit{\Gamma} - E), \tag{5.34}$$
$$\sigma^{\mathsf{L}e,E} = \sigma, \tag{5.35}$$
$$\mathit{\Sigma}^{\mathsf{L}e,E} = \mathit{\Sigma}. \tag{5.36}$$

The component expressions of the structure relations (5.29)–(5.32) of the associated left $\mathrm{D}\mathfrak{m}_\tau$-operation involve in addition to the components γ, $\mathit{\Gamma}$, σ, $\mathit{\Sigma}$ the components h, H

of the derived Lie algebra parameter H in similar fashion. Explicitly, they take the form

$$j_{\mathsf{L}h,H}\gamma\gamma^{-1} = 0, \tag{5.37}$$

$$j_{\mathsf{L}h,H}\varGamma = 0, \tag{5.38}$$

$$l_{\mathsf{L}h,H}\gamma\gamma^{-1} = -h, \tag{5.39}$$

$$l_{\mathsf{L}h,H}\varGamma = -H - \dot{}\mu\dot{}(h,\varGamma), \tag{5.40}$$

$$j_{\mathsf{L}h,H}\sigma = -\operatorname{Ad}\gamma^{-1}(h), \tag{5.41}$$

$$j_{\mathsf{L}h,H}\varSigma = -\mu\dot{}(\gamma^{-1}, H + \dot{}\mu\dot{}(h,\varGamma)), \tag{5.42}$$

$$l_{\mathsf{L}h,H}\sigma = 0, \tag{5.43}$$

$$l_{\mathsf{L}h,H}\varSigma = 0. \tag{5.44}$$

5.4 Derived automorphism symmetry

The automorphisms of the principal DM'-bundle DM studied in subsection 5.2 are the fiber preserving internal mappings of DM compatible with the right DM'-action. They are internal maps $\Psi \in \mathrm{MAP}(\mathsf{DM},\mathsf{DM}')$ with certain covariance properties under the right DM'-action and so naturally constitute a distinguished subgroup $\mathrm{AUT}_{\mathsf{DM}'}(\mathsf{DM})$ of the infinite dimensional Lie group $\mathrm{MAP}(\mathsf{DM},\mathsf{DM}')$. The associated derived automorphism symmetry is the subject of this subsection.

The full content of the derived automorphism group $\mathrm{AUT}_{\mathsf{DM}'}(\mathsf{DM})$ is specified by the requirement that automorphisms $\Psi \in \mathrm{AUT}_{\mathsf{DM}'}(\mathsf{DM})$ transform under the right DM'-action of DM according to

$$\Psi^{\mathsf{RQ}} = \mathrm{Q}^{-1}\Psi\mathrm{Q} \tag{5.45}$$

for $\mathrm{Q} \in \mathsf{DM}'$ (cf. subsection 5.2). (5.45) replicates in derived theory relation (4.7) of ordinary theory. It characterizes Ψ as a section of an $\mathrm{Ad}_{\mathsf{DM}'}$-bundle over DM/DM' associated to the DM'-bundle DM. In the corresponding right $\mathrm{D}\mathfrak{m}'$-operation, the Ψ consequently obey

$$j_{\mathsf{RX}}\Psi\Psi^{-1} = 0, \tag{5.46}$$

$$l_{\mathsf{RX}}\Psi\Psi^{-1} = -\mathrm{X} + \operatorname{Ad}\Psi(\mathrm{X}) \tag{5.47}$$

with $\mathrm{X} \in \mathrm{D}\mathfrak{m}'$. In the above expressions and similar ones below, right composition of Ψ with the bundle projection $T[1]\,\mathsf{DM} \simeq \mathsf{DM} \times \mathrm{D}\mathfrak{m}[1] \rightarrow \mathsf{DM}$ is tacitly understood. (5.46), (5.47) are evidently the derived theory counterpart of the ordinary theory relations (4.8).

The derived automorphism group $\mathrm{AUT}_{\mathsf{DM}'}(\mathsf{DM})$ acts on DM. An automorphism $\Psi \in \mathrm{AUT}_{\mathsf{DM}'}(\mathsf{DM})$ acts on the base coordinate Γ of DM as

$${}^{\Psi}\Gamma = \Gamma\Psi^{-1}. \tag{5.48}$$

This action yields a corresponding action on the fiber coordinate Σ of DM, viz

$${}^{\Psi}\Sigma = \operatorname{Ad}\Psi(\Sigma) - \mathrm{d}\Psi\Psi^{-1}, \tag{5.49}$$

on account of relation (5.3). It is immediate to see from (5.5) that the right DM'-action on $\mathrm{AUT}_{\mathsf{DM}'}(\mathsf{DM})$ given in eq. (5.45) is determined by the compatibility requirement that

$({}^{\Psi}\Gamma)^{\mathsf{RQ}} = {}^{\Psi^{\mathsf{RQ}}}\Gamma^{\mathsf{RQ}}$ with $\mathrm{Q} \in \mathsf{DM}'$ for the action (5.48). The formal correspondence of relations (5.48), (5.48) and (4.9) in the derived and ordinary theory is again evident.

The left target kernel DM_τ-action also extends to the derived automorphism group $\mathrm{AUT}_{\mathsf{DM}'}(\mathsf{DM})$ (cf. subsection 5.3). The extension is trivial: Ψ is left invariant,

$$\Psi^{\mathsf{LE}} = \Psi. \tag{5.50}$$

for $\mathrm{E} \in \mathsf{DM}_\tau$. In the left $\mathsf{D}\mathfrak{m}_\tau$-operation yielded by the DM_τ-action one has so

$$j_{\mathsf{LH}}\Psi\Psi^{-1} = 0, \tag{5.51}$$

$$l_{\mathsf{LH}}\Psi\Psi^{-1} = 0 \tag{5.52}$$

with $\mathrm{H} \in \mathsf{D}\mathfrak{m}_\tau$. Relations (5.50)–(5.52) obviously reproduce in the derived theory relations (4.10), (4.11) of the ordinary theory.

The DM_τ-action (5.50) is determined by the requirement of compatibility with the automorphism action (5.48) on the base coordinate Γ demanding that $({}^{\Psi}\Gamma)^{\mathsf{LE}} = {}^{\Psi^{\mathsf{LE}}}\Gamma^{\mathsf{LE}}$ with $\mathrm{E} \in \mathsf{DM}_\tau$ for all automorphisms $\Psi \in \mathrm{AUT}_{\mathsf{DM}'}(\mathsf{DM})$, analogously to the right DM'-action.

Again, for the sake of concreteness and completeness and later use, we provide a reference list of the component expression of the above relations. The hurried reader can once more skip to the next subsection.

The component expression of the right DM'-action on automorphisms, given by eq. (5.45), involves in addition to the components ψ, $\mathit{\Psi}$ of the automorphism Ψ the components q, Q of the derived group parameter Q,

$$\psi^{\mathsf{R}q,Q} = q^{-1}\psi q, \tag{5.53}$$

$$\mathit{\Psi}^{\mathsf{R}q,Q} = \dot{\mu}(q^{-1}, \mathit{\Psi} - Q + \dot{\mu}(\psi, Q)). \tag{5.54}$$

The component expressions of the structure relations (5.46), (5.47) of the right $\mathsf{D}\mathfrak{m}'$-operation involves correspondingly the components x, X of the derived algebra parameter X and read as

$$j_{\mathsf{R}x,X}\psi\psi^{-1} = 0, \tag{5.55}$$

$$j_{\mathsf{R}x,X}\mathit{\Psi} = 0, \tag{5.56}$$

$$l_{\mathsf{R}x,X}\psi\psi^{-1} = -x + \operatorname{Ad}\psi(x), \tag{5.57}$$

$$l_{\mathsf{R}x,X}\mathit{\Psi} = -\dot{}\mu\dot{}(x, \mathit{\Psi}) - X + \dot{\mu}(\psi, X). \tag{5.58}$$

Relations (5.48), (5.49) describing the automorphism action in coordinate form read componentwise as

$${}^{\psi,\mathit{\Psi}}\gamma = \gamma\psi^{-1}, \tag{5.59}$$

$${}^{\psi,\mathit{\Psi}}\mathit{\Gamma} = \mathit{\Gamma} - \dot{\mu}(\gamma\psi^{-1}, \mathit{\Psi}), \tag{5.60}$$

$${}^{\psi,\mathit{\Psi}}\sigma = \operatorname{Ad}\psi(\sigma) - d\psi\psi^{-1} - \dot{\tau}(\mathit{\Psi}), \tag{5.61}$$

$${}^{\psi,\mathit{\Psi}}\mathit{\Sigma} = \dot{\mu}(\psi, \mathit{\Sigma}) - d\mathit{\Psi} - \tfrac{1}{2}[\mathit{\Psi}, \mathit{\Psi}] - \dot{}\mu\dot{}(\operatorname{Ad}\psi(\sigma) - d\psi\psi^{-1} - \dot{\tau}(\mathit{\Psi}), \mathit{\Psi}). \tag{5.62}$$

The component expression of the left DM_τ-action relation (5.50) reads as follows

$$\psi^{\mathsf{L}e,E} = \psi, \tag{5.63}$$
$$\varPsi^{\mathsf{L}e,E} = \varPsi. \tag{5.64}$$

The component expression of the structure relations (5.51), (5.52) in the associated $\mathrm{D}\mathfrak{m}_\tau$-operation takes consequently the form

$$j_{\mathsf{L}h,H}\psi\psi^{-1} = 0, \tag{5.65}$$
$$j_{\mathsf{L}h,H}\varPsi = 0, \tag{5.66}$$
$$l_{\mathsf{L}h,H}\psi\psi^{-1} = 0, \tag{5.67}$$
$$l_{\mathsf{L}h,H}\varPsi = 0. \tag{5.68}$$

5.5 The derived unitary line bundle of a character

In this subsection, the operational framework of subsection 5.2. describing a derived homogeneous space DM/DM' is adapted to the case occurring in the derived KKS theory of regular coadjoint orbits (cf. subsection 5.1) where M is compact and M' is a maximal toral crossed submodule J of M. On a regular derived homogeneous space further geometrical objects can be considered such as derived unitary line bundles, their sections and their unitary connections.

With any Lie group L, there is associated its inner automorphism crossed module $\mathrm{I{\scriptsize NN}}\,\mathsf{L} = (\mathsf{L}, \mathsf{L}, \mathrm{id}_\mathsf{L}, \mathrm{Ad}_\mathsf{L})$, where Ad_L denotes the conjugation action of L on itself. Similarly, with any Lie algebra $\mathfrak{l}$, there is associated its inner derivation crossed module $\mathrm{I{\scriptsize NN}}\,\mathfrak{l} = (\mathfrak{l}, \mathfrak{l}, \mathrm{id}_\mathfrak{l}, \mathrm{ad}_\mathfrak{l})$, where $\mathrm{ad}_\mathfrak{l}$ denotes the adjoint action of $\mathfrak{l}$ on itself. If $\mathfrak{l}$ is the Lie algebra of L, then $\mathrm{I{\scriptsize NN}}\,\mathfrak{l}$ is the Lie algebra crossed module of $\mathrm{I{\scriptsize NN}}\,\mathsf{L}$.

The inner automorphism crossed module $\mathrm{I{\scriptsize NN}}\,\mathsf{U}(1)$ of the Lie group $\mathsf{U}(1)$ and the Lie algebra crossed module $\mathrm{I{\scriptsize NN}}\,\mathfrak{u}(1)$ of the Lie algebra $\mathfrak{u}(1)$ feature in the theory of crossed module characters outlined momentarily. For these, the conjugation action $\mathrm{Ad}_{\mathsf{U}(1)}$ and the adjoint action $\mathrm{ad}_{\mathfrak{u}(1)}$ appearing in their definition are trivial due to the Abelian nature of $\mathsf{U}(1)$ and $\mathfrak{u}(1)$, respectively.

Let J be a maximal toral crossed submodule of the Lie group crossed module M. A character of J is a Lie group crossed module morphism $\beta : \mathsf{J} \to \mathrm{I{\scriptsize NN}}\,\mathsf{U}(1)$ (cf. subsection 3.1). The component $\xi : \mathsf{T} \to \mathsf{U}(1)$ of β is then an ordinary character of the maximal torus T of G. Further, the component $\varXi : \mathsf{H} \to \mathsf{U}(1)$ is determined by ξ as $\varXi = \xi \circ \tau$. There exists in this way a one-to-one correspondence between the character set of J and that of T.

Having defined the notion of character, we consider next a compact Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ (cf. subsection 5.1) and a maximal toral crossed submodule $\mathsf{J} = (\mathsf{H}, \mathsf{T})$ of M equipped with a character $\beta = (\varXi, \xi)$.

Consider the complex vector space

$$\mathrm{D}\mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}[1]. \tag{5.69}$$

The coordinate $\mathrm{Z} \in \mathrm{D}\mathbb{C}$ has therefore a component expansion of the form

$$\mathrm{Z}(\alpha) = z + \alpha Z \tag{5.70}$$

with $\alpha \in \mathbb{R}[-1]$, where $z \in \mathbb{C}$, $Z \in \mathbb{C}[1]$. Conjugation in $\mathrm{D}\mathbb{C}$ is defined such that Z^* has components z^*, Z^*. $\mathrm{D}\mathbb{C}$ has further a natural field structure with

$$\mathrm{WZ}(\alpha) = wz + \alpha(zW + wZ), \qquad \mathrm{Z}^{-1}(\alpha) = z^{-1} - \alpha z^{-2} Z. \tag{5.71}$$

For this reason, we shall call $\mathrm{D}\mathbb{C}$ the derived complex line though it is in fact 2-dimensional.

The derived Lie group $\mathrm{D}\,\mathrm{I}\textsc{nn}\,\mathsf{U}(1)$ acts multiplicatively on $\mathrm{D}\mathbb{C}$ via an embedding ι : $\mathrm{D}\,\mathrm{I}\textsc{nn}\,\mathsf{U}(1) \to \mathrm{D}\mathbb{C}$. The coordinate expression of ι reads as

$$\iota(\mathrm{P})(\alpha) = p + \alpha p P, \tag{5.72}$$

where $\mathrm{P} \in \mathrm{D}\,\mathrm{I}\textsc{nn}\,\mathsf{U}(1)$. Likewise, the derived $\mathrm{D}\,\mathrm{I}\textsc{nn}\,\mathfrak{u}(1)$ acts multiplicatively on $\mathrm{D}\mathbb{C}$ via the associated embedding $\imath : \mathrm{D}\,\mathrm{I}\textsc{nn}\,\mathfrak{u}(1) \to \mathrm{D}\mathbb{C}$ given by

$$\imath(\mathrm{Y})(\alpha) = y + \alpha Y \tag{5.73}$$

in coordinate form, where $\mathrm{Y} \in \mathrm{D}\,\mathrm{I}\textsc{nn}\,\mathfrak{u}(1)$. In what follows, ι and $\imath$ will be tacitly understood for brevity.

Via the character β, $\mathrm{D}\mathbb{C}$ carries the left action of $\mathrm{D}\,\mathsf{J}$ defined by

$$\rho_\beta(\mathrm{Q})(Z) = \mathrm{D}\beta(\mathrm{Q})^{-1}\mathrm{Z} \tag{5.74}$$

with $\mathrm{Q} \in \mathrm{D}\,\mathsf{J}$, where $\mathrm{D}\beta$ is defined in subsection 3.2. Writing Z and Q in the forms (5.70) and (3.11) respectively, $\rho_\beta(\mathrm{Q})(\mathrm{Z})$ can be expressed in components as

$$\rho_\beta(\mathrm{Q})(\mathrm{Z})(\alpha) = \xi(q)^{-1} z + \alpha \xi(q)^{-1}(Z - \dot{\Xi}(Q) z), \tag{5.75}$$

where $\Xi = \xi \circ \tau$.

We introduce next the vector bundle on $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$ defined as

$$\mathcal{L}_\beta = \mathrm{D}\mathsf{M} \times_\beta \mathrm{D}\mathbb{C} = \mathrm{D}\mathsf{M} \times \mathrm{D}\mathbb{C}/\sim_\beta, \tag{5.76}$$

where $\sim_\beta$ is the equivalence relation

$$(\Gamma, \mathrm{Z}) \sim_\beta (\Gamma\mathrm{Q}, \rho_\beta(\mathrm{Q}^{-1})(\mathrm{Z})) \tag{5.77}$$

in coordinate form with $\mathrm{Q} \in \mathrm{D}\,\mathsf{J}$. $\mathcal{L}_\beta$ is the derived unitary line bundle associated with the character β. $\mathcal{L}_\beta$ is termed in this way by its analogy to the ordinary unitary line bundle of a character (cf. subsection 4.3), though $\mathcal{L}_\beta$ is strictly speaking a rank 2 vector bundle and so not a line bundle in the usual sense.

The definitions of and results about sections and connections of $\mathcal{L}_\beta$ provided next elegantly replicate in derived theory the analogous notions of ordinary one reviewed in subsection (4.3), showing the effectiveness of the derived approach.

A derived degree p $\mathbb{C}$-valued function is defined naturally as an internal map $\mathrm{S} \in \textsc{Map}(T[1]\,\mathrm{D}\mathsf{M}, \mathrm{D}\mathbb{C}[p])$. It has a component expansions of the form

$$\mathrm{S}(\alpha) = s + (-1)^p \alpha S \tag{5.78}$$

with $\alpha \in \mathbb{R}[-1]$, where $s \in \textsc{Map}(T[1]\,\mathsf{DM}, \mathbb{C}[p])$, $S \in \textsc{Map}(T[1]\,\mathsf{DM}, \mathbb{C}[p+1])$. A derived differential $\mathrm{d} : \textsc{Map}(T[1]\,\mathsf{DM}, \mathrm{D}\mathbb{C}[p]) \to \textsc{Map}(T[1]\,\mathsf{DM}, \mathrm{D}\mathbb{C}[p+1])$ is defined through the component expression

$$\mathrm{dS}(\alpha) = ds + (-1)^p S + (-1)^{p+1} \alpha dS. \tag{5.79}$$

d is nilpotent as is easily checked.

In the operational framework we are employing, the space $\mathrm{D}\Omega^p(\mathcal{L}_\beta)$ of derived p-form sections of the derived line bundle $\mathcal{L}_\beta$ can be identified with the subspace of the space of derived degree p $\mathbb{C}$-valued functions $\textsc{Map}(T[1]\,\mathsf{DM}, \mathrm{D}\mathbb{C}[p])$ spanned by the elements which are basic with respect to the action ρ_β defining $\mathcal{L}_\beta$. A derived function $\mathrm{S} \in \textsc{Map}(T[1]\,\mathsf{DM}, \mathrm{D}\mathbb{C}[p])$ is basic if it satisfies the horintality and covariance conditions

$$j_{\mathsf{RX}}\mathrm{S} = 0, \tag{5.80}$$

$$l_{\mathsf{RX}}\mathrm{S} = \mathrm{D}\dot{\beta}(\mathrm{X})\mathrm{S} \tag{5.81}$$

for $\mathrm{X} \in \mathrm{D}\mathfrak{j}$. Compare the above derived relations with the corresponding ones of the ordinary case in eq. (4.13).

The unitary connections of the derived line bundle $\mathcal{L}_\beta$ can also be described in the operational formalism. A derived unitary connection of $\mathcal{L}_\beta$ is an internal derived function $\mathrm{A} \in \textsc{Map}(T[1]\,\mathsf{DM}, \mathrm{D}\,\textsc{Inn}\,\mathfrak{u}(1)[1])$ satisfying for $\mathrm{X} \in \mathrm{D}\mathfrak{j}$

$$j_{\mathsf{RX}}\mathrm{A} = -\,\mathrm{D}\dot{\beta}(\mathrm{X}), \tag{5.82}$$

$$l_{\mathsf{RX}}\mathrm{A} = -\mathrm{d}\,\mathrm{D}\dot{\beta}(\mathrm{X}), \tag{5.83}$$

where d is the derived differential of DM. The curvature of A is the derived function $\mathrm{B} \in \textsc{Map}(T[1]\,\mathsf{DM}, \mathrm{D}\,\textsc{Inn}\,\mathfrak{u}(1)[2])$ defined by

$$\mathrm{B} = \mathrm{dA}. \tag{5.84}$$

By construction, B obeys the derived Bianchi identity

$$\mathrm{dB} = 0. \tag{5.85}$$

Further, from (5.82), (5.83), B satisfies

$$j_{\mathsf{RX}}\mathrm{B} = 0 \tag{5.86}$$

$$l_{\mathsf{RX}}\mathrm{B} = 0. \tag{5.87}$$

(5.86)–(5.87) indicate that B is a 2-form section of the derived line bundle $\mathcal{L}_1$ of the trivial character $1 : \mathsf{J} \to \mathrm{Inn}\,\mathsf{U}(1)$, which we may describe as an $\mathrm{Inn}\,\mathsf{U}(1)$-valued derived 2-form on $\mathsf{DM}/\,\mathsf{DJ}$. The operational conditions (5.82), (5.83) specifying a connection extend the ordinary requisites (4.14). The right hand side of (5.83) does not vanish again by the special nature of the derived differential d (cf. eq. (3.20)). Conversely, the definition (5.84), the Bianchi identity (5.85) and the operational relations (5.86), (5.87) specifying a connection's curvature exactly match their ordinary counterparts of eqs. (4.15) and (4.16). The unitary connections of $\mathcal{L}_\beta$ form an affine space $\textsc{Conn}(\mathcal{L}_\beta)$.

When a connection A of $\mathcal{L}_\beta$ is assigned, the derived covariant differential of a p-form section S of $\mathcal{L}_\beta$ can be defined,

$$\mathrm{d}_{\mathrm{A}}\mathrm{S} = \mathrm{dS} + \mathrm{AS}. \tag{5.88}$$

By virtue of (5.80), (5.81) and (5.82), (5.83), $\mathrm{d}_{\mathrm{A}}\mathrm{S}$ is a $p+1$-form section of $\mathcal{L}_\beta$ as required. The derived Ricci identity holds,

$$\mathrm{d}_{\mathrm{A}}\mathrm{d}_{\mathrm{A}}\mathrm{S} = \mathrm{BS}. \tag{5.89}$$

In the operational framework, it is further possible to characterize the derived gauge transformations of the derived line bundle $\mathcal{L}_\beta$. A gauge transformation of $\mathcal{L}_\beta$ can be defined as an internal derived function $\mathrm{U} \in \mathrm{M{\small AP}}(T[1]\,\mathsf{D}\mathsf{M}, \mathrm{D\,I{\small NN}}\,\mathsf{U}(1))$ satisfying the horizontality and conjugation covariance conditions

$$j_{\mathsf{R}\mathrm{X}}\mathrm{UU}^{-1} = 0, \tag{5.90}$$

$$l_{\mathsf{R}\mathrm{X}}\mathrm{UU}^{-1} = 0 \tag{5.91}$$

for $\mathrm{X} \in \mathsf{D}\mathfrak{j}$. It is useful to compare the derived conditions (5.90), (5.91) with the corresponding relations (4.17) of the ordinary theory. Derived gauge transformations constitute a distinguished subgroup $\mathrm{G{\small AU}}(\mathcal{L}_\beta)$ of the mapping group $\mathrm{M{\small AP}}(T[1]\,\mathsf{D}\mathsf{M}, \mathrm{D\,I{\small NN}}\,\mathsf{U}(1))$.

A gauge transformation U acts on a degree p section S of $\mathcal{L}_\beta$ as

$${}^{\mathrm{U}}\mathrm{S} = \mathrm{US}. \tag{5.92}$$

U also acts on a connection A of $\mathcal{L}_\beta$ as

$${}^{\mathrm{U}}\mathrm{A} = \mathrm{A} - \mathrm{dUU}^{-1}. \tag{5.93}$$

The curvature B of A is then gauge invariant

$${}^{\mathrm{U}}\mathrm{B} = \mathrm{B}. \tag{5.94}$$

By construction, the action is such that

$$\mathrm{d}_{{}^{\mathrm{U}}\mathrm{A}}{}^{\mathrm{U}}\mathrm{S} = {}^{\mathrm{U}}(\mathrm{d}_{\mathrm{A}}\mathrm{S}) = \mathrm{U}\mathrm{d}_{\mathrm{A}}\mathrm{S} \tag{5.95}$$

as usual. The derived gauge transformation properties of sections, connections and curvature thereof of the bundle $\mathcal{L}_\beta$ shown in eqs. (5.92)–(5.94) are totally analogous to their ordinary counterparts (4.18)–(4.20).

As in earlier similar instances, for concreteness and completeness, we express the above relations in terms of the components of the fields involved. This will also exemplify how of the derived framework effectively encodes non trivial higher gauge theoretic relations. The uninterested reader can skip directly to the paragraph following eq. (5.125).

Consider a p-form section S of $\mathcal{L}_\beta$ with components s, S. In terms of these latter, the horizontality and covariance conditions (5.80), (5.81) take the form

$$j_{\mathsf{R}x,X}\,s = 0, \tag{5.96}$$

$$j_{\mathsf{R}x,X}\,S = 0, \tag{5.97}$$

$$l_{\mathsf{R}x,X}\,s = \dot{\xi}(x)s, \tag{5.98}$$

$$l_{\mathsf{R}x,X}\,S = \dot{\xi}(x)S + (-1)^p\dot{\Xi}(X)s, \tag{5.99}$$

where x, X are the components of the Lie algebra element X.

A connection A of $\mathcal{L}_\beta$ can similarly be expressed through its components a, A. By (5.82), (5.83), a, A obey

$$j_{\mathsf{R}x,X}a = -\dot{\xi}(x), \tag{5.100}$$

$$j_{\mathsf{R}x,X}A = -\dot{\Xi}(X), \tag{5.101}$$

$$l_{\mathsf{R}x,X}a = -\dot{\Xi}(X), \tag{5.102}$$

$$l_{\mathsf{R}x,X}A = 0. \tag{5.103}$$

By virtue of (5.84) the components of the curvature B of A, b, B, are expressible in terms of a, A according to

$$b = da - A, \tag{5.104}$$

$$B = dA, \tag{5.105}$$

where d denotes the de Rham differential of $\mathrm{D}\mathsf{M}$. The Bianchi identity (5.85) obeyed by B translates into the relations

$$db + B = 0, \tag{5.106}$$

$$dB = 0, \tag{5.107}$$

By (5.86), (5.87), b, B satisfy furthermore

$$j_{\mathsf{R}x,X}b = 0, \tag{5.108}$$

$$j_{\mathsf{R}x,X}B = 0, \tag{5.109}$$

$$l_{\mathsf{R}x,X}b = 0, \tag{5.110}$$

$$l_{\mathsf{R}x,X}B = 0. \tag{5.111}$$

We denote by $d_{a,A}s$, $d_{a,A}S$ the components of the covariant differential $\mathrm{d_A S}$ of a p-form section S with respect to an assigned connection A of $\mathcal{L}_\beta$. Using (5.88), $d_{a,A}s$, $d_{a,A}S$ are found to be given by

$$d_{a,A}s = ds + as + (-1)^p S, \tag{5.112}$$

$$d_{a,A}S = dS + aS + (-1)^p As. \tag{5.113}$$

Note that $d_{a,A}s$, $d_{a,A}S$ are not separately linear is s, S. The component notation used here, though suggestive, must therefore be used with some care. The Ricci identity (5.89) can correspondingly be written as

$$d_{a,A}d_{a,A}s = bs, \tag{5.114}$$

$$d_{a,A}d_{a,A}S = bS + (-1)^p Bs. \tag{5.115}$$

A gauge transformation U of $\mathcal{L}_\beta$ can similarly be expressed through its components u, U. By (5.90), (5.91), u, U obey

$$j_{\mathsf{R}x,X}uu^{-1} = 0, \tag{5.116}$$

$$j_{\mathsf{R}x,X}U = 0, \tag{5.117}$$

$$l_{\mathsf{R}x,X}uu^{-1} = 0, \tag{5.118}$$

$$l_{\mathsf{R}x,X}U = 0. \tag{5.119}$$

On account of (5.92), the action of a gauge transformation U on a p-form section S of $\mathcal{L}_\beta$ takes in components the form

$$^{u,U}s = us, \tag{5.120}$$

$$^{u,U}S = u(S + (-1)^p Us). \tag{5.121}$$

By (5.93), the action of U on a connection A $\mathcal{L}_\beta$ reads in component form as

$$^{u,U}a = a - duu^{-1} - U, \tag{5.122}$$

$$^{u,U}A = A - dU. \tag{5.123}$$

Similarly, by (5.94), the action of U on the curvature B of A is componentwise

$$^{u,U}b = b, \tag{5.124}$$

$$^{u,U}B = B. \tag{5.125}$$

We conclude this subsection with an examination of basic differential topological issues concerning derived unitary line bundles. Is there a topological classification of such bundles based on a cohomological characterization of their curvature analogous to that of ordinary line bundle? This is a far reaching question whose full answer lies beyond the scope of the present work. We shall limit ourselves to the following considerations.

We start with the following premises. First, the derived line bundle $\mathcal{L}_\beta$ of a character β of J is a graded vector bundle on the graded manifold $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$. Second, a unitary connection A of $\mathcal{L}_\beta$ is a non ordinary $\mathrm{DInn}\,\mathfrak{u}(1)[1]$-valued internal map on the graded manifold $T[1]\,\mathrm{D}\mathsf{M}$ with special properties. Similarly, the curvature B of A is a non ordinary $\mathrm{D\small{INN}}\,\mathfrak{u}(1)[2]$-valued internal map on $T[1]\,\mathrm{D}\mathsf{M}$. The bundle theoretic set-up we are concerned with here, therefore, is a non standard one. This renders the application of standard differential topological results problematic. We may try to tackle the issue anyway from the derived standpoint on which our whole approach is based.

The characters β of the maximal toral crossed submodule J are in one-to-one correspondence with the characters ξ of the maximal torus T of G. These constitute a lattice, the dual lattice $\Lambda_{\mathsf{G}}{}^*$ of the integer lattice Λ_{G} of $\mathfrak{t}$. The derived line bundles $\mathcal{L}_\beta$ of the characters β thus form a discrete family of vector bundles on $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$ organized as a kind of lattice isomorphic in some sense to $\Lambda_{\mathsf{G}}{}^*$.

Below, we shall consider derived cohomology, that is the cohomology of the complex of $\mathrm{Inn}\,\mathsf{U}(1)$-valued forms on $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$ and the derived differential d, and so the terms 'closed' and 'exact' will tacitly refer to such complex. We saw earlier that, due to (5.86), (5.87), the curvature B of a unitary connection A of a derived line bundle $\mathcal{L}_\beta$ is an $\mathrm{Inn}\,\mathsf{U}(1)$-valued derived 2-form. By (5.85), B is further closed. If A, A′ are connections of $\mathcal{L}_\beta$, then their difference $\mathrm{A}'-\mathrm{A}$ is a derived 1-form on account of (5.82), (5.83) and consequently, by (5.84), the difference of the curvatures B, B′ of A, A′, $\mathrm{B}' - \mathrm{B} = \mathrm{d}(\mathrm{A}' - \mathrm{A})$, is an exact derived 2-form. Hence, the degree 2 derived cohomology class $c(\mathcal{L}_\beta) = [-i\mathrm{B}/2\pi]_{\mathrm{d}}$ does not depend on the connection A underlying B and so constitutes a differential topological property of the line bundle $\mathcal{L}_\beta$, the derived Chern class of $\mathcal{L}_\beta$, much in analogy to the ordinary line bundle theory.

The lattice nature of the family of derived line bundles $\mathcal{L}_\beta$ for varying character β indicates their derived Chern classes $\mathrm{c}(\mathcal{L}_\beta)$ belong to some kind of lattice in the 2nd derived cohomology. We do not know whether there is a way of characterizing this cohomological lattice in terms of some form integrality of derived periods, in analogy to the ordinary theory.

The derived closedness condition of the curvature B of a derived line bundle $\mathcal{L}_\beta$ can be written through its components b, B and reads as in (5.106), (5.107). In customary de Rham cohomology, B is so exact whilst b is not even closed. Hence, the derived Chern class $\mathrm{c}(\mathcal{L}_\beta)$ cannot be straightforwardly framed in de Rham cohomology.

A relationship of the theory of derived line bundles with connection to the theory of twisted Hermitian line bundles of bundle gerbes proposed by C. Rogers in ref. [56] is conceivable but remains to be elucidated. The point is that our approach to derived line bundles, based as it is on an operational set-up, is essentially a total space one while Rogers's is a base space one. Clarifying the nature of the relations between the two standpoints, if a relation does exist, would require further work.

5.6 Derived Poisson structure of a derived line bundle

In subsection 5.5, we built the derived unitary line bundle $\mathcal{L}_\beta$ of a given character β on the homogeneous space DM/DJ. By (5.84)–(5.87), the curvature B of a unitary connection A of $\mathcal{L}_\beta$ is a d-closed $\textsc{Inn}\,\mathfrak{u}(1)$-valued derived 2-form. $-i\mathrm{B}$ can so be thought of as a derived presymplectic structure on DM/DJ. One can construct an associated derived Poisson structure duplicating in the derived framework the standard theory reviewed in subsection 4.4 as we shall show below.

Below, we shall systematically use the internal Cartan calculus of DM (cf. subsection A.3). This features a set of graded derivations of $\textsc{Fun}(T[1]\,\mathsf{DM})$ including the degree -1 contractions j_V and Lie derivatives l_V along the vector fields $V \in \textsc{Vect}(\mathsf{DM})$, where $\textsc{Vect}(\mathsf{DM})$ is the Lie algebra of internal vector fields of DM, and the de Rham differential d and obeying relations (A.5)–(A.8). The derivations $j_{\mathsf{R}\mathrm{X}}$, $l_{\mathsf{R}\mathrm{X}}$ with $\mathrm{X} \in \mathrm{D}\mathfrak{j}$ of the right DJ-operation are just $j_{S_{\mathsf{R}\mathrm{X}}}$, $l_{S_{\mathsf{R}\mathrm{X}}}$, where $S_{\mathsf{R}\mathrm{X}}$ is the vertical vector field of the DJ-action corresponding to X.

Next, we consider the set $\textsc{Vect}_{\mathsf{iA}}(\mathsf{DM})$ of all vector fields $V \in \textsc{Vect}(\mathsf{DM})$ leaving B invariant, that is satisfying

$$l_V \mathrm{B} = 0, \tag{5.126}$$

and the set $\textsc{Vect}_{\mathsf{kA}}(\mathsf{DM})$ of all vector fields $V \in \textsc{Vect}(\mathsf{DM})$ obeying

$$j_V \mathrm{B} = 0. \tag{5.127}$$

It is straightforward to verify that $\textsc{Vect}_{\mathsf{iA}}(\mathsf{DM})$ is a Lie subalgebra of $\textsc{Vect}(\mathsf{DM})$ and that $\textsc{Vect}_{\mathsf{kA}}(\mathsf{DM})$ is a Lie ideal of $\textsc{Vect}_{\mathsf{iA}}(\mathsf{DM})$. We can thus construct the quotient Lie algebra $\textsc{Vect}_{\mathsf{qA}}(\mathsf{DM}) = \textsc{Vect}_{\mathsf{iA}}(\mathsf{DM})/\textsc{Vect}_{\mathsf{kA}}(\mathsf{DM})$.

Let $\textsc{Vect}_{\mathsf{v}}(\mathsf{DM})$ be the Lie subalgebra of $\textsc{Vect}(\mathsf{DM})$ of the vertical vector fields $S_{\mathsf{R}\mathrm{X}}$ with $\mathrm{X} \in \mathrm{D}\mathfrak{j}$. $\textsc{Vect}_{\mathsf{v}}(\mathsf{DM})$ is a Lie subalgebra of $\textsc{Vect}_{\mathsf{iA}}(\mathsf{DM})$, as $l_{\mathsf{R}\mathrm{X}}\mathrm{B} = 0$ for $\mathrm{X} \in \mathrm{D}\mathfrak{j}$ by (5.87). The Lie derivatives $l_{S_{\mathsf{R}\mathrm{X}}}$ are thus derivations of the Lie algebra $\textsc{Vect}_{\mathsf{iA}}(\mathsf{DM})$

preserving the Lie ideal $\mathrm{VECT}_{\mathsf{kA}}(\mathsf{DM})$. They so induce derivations $l_{\mathsf{q}S_{\mathsf{RX}}}$ of the Lie algebra $\mathrm{VECT}_{\mathsf{qA}}(\mathsf{DM})$ defined in the previous paragraph. $\mathrm{VECT}_{\mathsf{bA}}(\mathsf{DM}) = \bigcap_{\mathrm{X}\in\mathsf{Dj}} \ker l_{\mathsf{q}S_{\mathsf{RX}}}$ is consequently a Lie subalgebra of $\mathrm{VECT}_{\mathsf{qA}}(\mathsf{DM})$. Since we aim eventually to construct a structure on the derived homogeneous space DM/DJ, $\mathrm{VECT}_{\mathsf{bA}}(\mathsf{DM})$ is the relevant vector field algebra. Explicitly, $\mathrm{VECT}_{\mathsf{bA}}(\mathsf{DM})$ consists of the vector fields $V \in \mathrm{VECT}(\mathsf{DM})$ obeying (5.126) and defined modulo vector fields $V' \in \mathrm{VECT}(\mathsf{DM})$ obeying (5.127) with the property that $l_{S_{\mathsf{RX}}}V$ obeys (5.127) for all $\mathrm{X} \in \mathsf{Dj}$.

The derived real line is the real vector space $\mathrm{D}\mathbb{R} = \mathbb{R} \oplus \mathbb{R}[1]$. $\mathrm{D}\mathbb{R}$ has properties completely analogous to those of the derived complex line $\mathrm{D}\mathbb{C}$ studied earlier in subsection 5.5, in particular it is a field (cf. eqs. (5.70), (5.70)). The space $\mathrm{MAP}(T[1]\,\mathsf{DM}, \mathrm{D}\mathbb{R}[p])$ of degree p derived $\mathbb{R}$-valued functions is then available. It has a component reduction analogous to that of its complex counterpart (cf. eq. (5.78)) and is acted upon by the derived differential d (cf. eq. (5.79)). Below, we shall concentrate on the space $\mathrm{DFNC}(\mathsf{DM}) = \mathrm{MAP}(T[1]\,\mathsf{DM}, \mathrm{D}\mathbb{R}[0])$ of derived functions of DM. An element $\mathrm{F} \in \mathrm{DFNC}(\mathsf{DM})$ is therefore a derived field: $\mathrm{F}(\alpha) = f + \alpha F$ with $f \in \mathrm{MAP}(T[1]\,\mathsf{DM}, \mathbb{R}[0])$, $F \in \mathrm{MAP}(T[1]\,\mathsf{DM}, \mathbb{R}[1])$. $\mathrm{DFNC}(\mathsf{DM})$ has a natural algebra structure induced by the field structure of $\mathrm{D}\mathbb{R}$. There exists a special mapping $\varpi : \mathrm{DFNC}(\mathsf{DM}) \to \mathrm{DFNC}(\mathsf{DM})$ defined componentwise as $\varpi\mathrm{F}(\alpha) = f$. ϖ is an algebra morphism. Its range $\mathrm{DFNC}_{\varpi}(\mathsf{DM})$ is therefore a subalgebra of $\mathrm{DFNC}(\mathsf{DM})$, which by construction is isomorphic to $\mathrm{MAP}(T[1]\,\mathsf{DM}, \mathbb{R}[0])$. In what follows, the functions of $\mathrm{DFNC}_{\varpi}(\mathsf{DM})$ will be called short and $\mathrm{DFNC}_{\varpi}(\mathsf{DM})$ will be referred to as the short subalgebra.

By the isomorphism $\mathrm{DFNC}_{\varpi}(\mathsf{DM}) \simeq \mathrm{MAP}(T[1]\,\mathsf{DM}, \mathbb{R}[0])$ noticed above and by the further isomorphism $\mathrm{MAP}(T[1]\,\mathsf{DM}, \mathbb{R}[0]) \simeq \mathrm{MAP}(\mathsf{DM}, \mathbb{R}[0])$, for any $\mathrm{F} \in \mathrm{DFNC}_{\varpi}(\mathsf{DM})$ and $V \in \mathrm{VECT}(\mathsf{DM})$, the product $\mathrm{F}V = f\mathrm{V} \in \mathrm{VECT}(\mathsf{DM})$ is defined, rendering $\mathrm{VECT}(\mathsf{DM})$ a $\mathrm{DFNC}_{\varpi}(\mathsf{DM})$-module. This restricted module structure is responsible for the special features of the derived Poisson set-up.

Again, as we aim eventually to obtain a structure on DM/DJ, we restrict the range of derived functions we consider to the subset $\mathrm{DFNC}_{\mathsf{b}}(\mathsf{DM})$ of $\mathrm{DFNC}(\mathsf{DM})$ of the basic ones. This is formed by the derived functions $\mathrm{F} \in \mathrm{DFNC}(\mathsf{DM})$ obeying

$$j_{\mathsf{RX}}\mathrm{F} = 0, \tag{5.128}$$

$$l_{\mathsf{RX}}\mathrm{F} = 0 \tag{5.129}$$

for $\mathrm{X} \in \mathsf{Dj}$. (5.129) answers to relation (4.21) of the standard theory, where a corresponding restriction is imposed. The counterpart of (5.128) does no appear in this latter as it is automatically fulfilled for grading reasons. $\mathrm{DFNC}_{\mathsf{b}}(\mathsf{DM})$ constitutes a subalgebra of $\mathrm{DFNC}(\mathsf{DM})$, as the j_{RX}, l_{RX} are derivations. Since $\varpi j_{\mathsf{RX}} = j_{\mathsf{RX}}\varpi$, $\varpi l_{\mathsf{RX}} = l_{\mathsf{RX}}\varpi$, $\mathrm{DFNC}_{\mathsf{b}}(\mathsf{DM})$ is invariant under ϖ and consequently $\mathrm{DFNC}_{\varpi\mathsf{b}}(\mathsf{DM}) = \varpi\,\mathrm{DFNC}_{\mathsf{b}}(\mathsf{DM})$ is the short subalgebra of $\mathrm{DFNC}_{\mathsf{b}}(\mathsf{DM})$.

Suppose that $\mathrm{F} \in \mathrm{DFNC}_{\mathsf{b}}(\mathsf{DM})$ and $P_{\mathrm{F}} \in \mathrm{VECT}(\mathsf{DM})$ obey the equation

$$\mathrm{dF} - ij_{P_{\mathrm{F}}}\mathrm{B} = 0. \tag{5.130}$$

Then, as is straightforward enough to check, $P_{\mathrm{F}} \in \mathrm{VECT}_{\mathsf{iA}}(\mathsf{DM})$, P_{F} is determined by eq. (5.130) mod $\mathrm{VECT}_{\mathsf{kA}}(\mathsf{DM})$ and $l_{S_{\mathsf{RX}}}P_{\mathrm{F}} \in \mathrm{VECT}_{\mathsf{kA}}(\mathsf{DM})$ for all $\mathrm{X} \in \mathsf{Dj}$. Thus, $P_{\mathrm{F}} \in$

$\mathrm{V\small{ECT}}_{\mathsf{b}\mathrm{A}}(\mathsf{DM})$ and as such P_{F} is uniquely determined by F. If the vector field P_{F} exists, F is said to be a Hamiltonian derived function and P_{F} is called the Hamiltonian vector field of F as (5.130) is clearly a derived extension of (4.22), the relation defining Hamiltonian functions and vector fields in the standard theory. We denote by $\mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM})$ the set of the Hamiltonian functions of $\mathrm{DF\small{NC}}_{\mathsf{b}}(\mathsf{DM})$ and by $\mathrm{V\small{ECT}}_{\mathrm{A}}(\mathsf{DM})$ that of the Hamiltonian vector fields of $\mathrm{V\small{ECT}}_{\mathsf{b}\mathrm{A}}(\mathsf{DM})$. We denote similarly by $\mathrm{DF\small{NC}}_{\varpi\mathrm{A}}(\mathsf{DM})$ the set of the short functions of $\mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM})$ and by $\mathrm{V\small{ECT}}_{\varpi\mathrm{A}}(\mathsf{DM})$ that of the associated vector fields of $\mathrm{V\small{ECT}}_{\mathrm{A}}(\mathsf{DM})$.

$\mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM})$ is only a vector subspace of $\mathrm{DF\small{NC}}_{\mathsf{b}}(\mathsf{DM})$, while $\mathrm{DF\small{NC}}_{\varpi\mathrm{A}}(\mathsf{DM})$ is a subalgebra of $\mathrm{DF\small{NC}}_{\varpi\mathsf{b}}(\mathsf{DM})$. The reason for this somewhat surprising difference in their algebraic properties can be ultimately traced back to the fact that $\mathrm{V\small{ECT}}(\mathsf{DM})$ is not a $\mathrm{DF\small{NC}}(\mathsf{DM})$-module but only a $\mathrm{DF\small{NC}}_{\varpi}(\mathsf{DM})$-module. We shall come back to this point momentarily.

It can also be seen that $\mathrm{V\small{ECT}}_{\mathrm{A}}(\mathsf{DM})$ is a Lie subalgebra of $\mathrm{V\small{ECT}}_{\mathsf{b}\mathrm{A}}(\mathsf{DM})$ and $\mathrm{V\small{ECT}}_{\varpi\mathrm{A}}(\mathsf{DM})$ is a Lie ideal of $\mathrm{V\small{ECT}}_{\mathrm{A}}(\mathsf{DM})$.

Mimicking a similar construction of Poisson theory, we may define a bracket $\{\cdot,\cdot\}_{\mathrm{A}}$: $\mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM}) \times \mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM}) \to \mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM})$ as follows. It can be shown that if F,G are Hamiltonian derived functions and $P_{\mathrm{F}}, P_{\mathrm{G}}$ are the associated Hamiltonian vector fields, then $-ij_{P_{\mathrm{G}}}j_{P_{\mathrm{F}}}\mathrm{B}$ is also a Hamiltonian function and $[P_{\mathrm{F}}, P_{\mathrm{G}}]$ is the associated Hamiltonian vector field. Taking relation (4.23) of the standard theory as a model, we then define the derived Poisson bracket

$$\{\mathrm{F},\mathrm{G}\}_{\mathrm{A}} = -ij_{P_{\mathrm{G}}}j_{P_{\mathrm{F}}}\mathrm{B}. \tag{5.131}$$

This bracket is bilinear and antisymmetric. Unlike its ordinary counterpart, however, it fails to satisfy the Jacobi identity. One has indeed

$$\{\mathrm{F},\{\mathrm{G},\mathrm{H}\}_{\mathrm{A}}\}_{\mathrm{A}} + \{\mathrm{G},\{\mathrm{H},\mathrm{F}\}_{\mathrm{A}}\}_{\mathrm{A}} + \{\mathrm{H},\{\mathrm{F},\mathrm{G}\}_{\mathrm{A}}\}_{\mathrm{A}} = \langle \mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathrm{A}} \tag{5.132}$$

for $\mathrm{F},\mathrm{G},\mathrm{H} \in \mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM})$, the Jacobiator in the right hand side being given by

$$\langle \mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathrm{A}} = i\mathrm{d}j_{P_{\mathrm{H}}}j_{P_{\mathrm{G}}}j_{P_{\mathrm{F}}}\mathrm{B}. \tag{5.133}$$

Note that, even though the derived Lie group DM is a non negatively graded manifold and $-i\mathrm{B}$ is a degree 2 derived function, $\langle \mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathrm{A}}$ is generally non vanishing because the degree 3 component B of B is not necessarily annihilated by $j_{P_{\mathrm{H}}}j_{P_{\mathrm{G}}}j_{P_{\mathrm{F}}}$. By virtue of (5.130) and d-exactness, however, the Hamiltonian vector field $P_{\langle \mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathrm{A}}}$ of $\langle \mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathrm{A}}$ vanishes and so $\langle \mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathrm{A}}$ is central,

$$\{\langle \mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathrm{A}},\cdot\}_{\mathrm{A}} = 0. \tag{5.134}$$

$\mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM})$ equipped with the bracket $\{\cdot,\cdot\}_{\mathrm{A}}$ can so be described as a twisted Lie algebra. The connection A is called simple if the Jacobiator vanishes.

It is remarkable that the twisted Lie bracket $\{\cdot,\cdot\}_{\mathrm{A}}$ just introduced restricts to a honest Poisson bracket on the short subalgebra $\mathrm{DF\small{NC}}_{\varpi\mathrm{A}}(\mathsf{DM}) \subset \mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM})$. Indeed, on $\mathrm{DF\small{NC}}_{\varpi\mathrm{A}}(\mathsf{DM})$ the bracket $\{\cdot,\cdot\}_{\mathrm{A}}$ turns out to be Leibniz in both arguments and the Jacobiator $\langle\cdot,\cdot,\cdot\rangle_{\mathrm{A}}$ to vanish identically. As a Lie algebra, $\mathrm{DF\small{NC}}_{\varpi\mathrm{A}}(\mathsf{DM})$ is a Lie ideal of

$\mathrm{DFNC}_{\mathrm{A}}(\mathrm{DM})$. With this we mean that the Lie bracket $\{\mathrm{F},\mathrm{G}\}_{\mathrm{A}}$ is short when at least one of its two arguments is and that the Jacobiator $\langle\mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathrm{A}}$ vanishes when at least one of its three arguments is short.

The function space $\mathrm{DFNC}_{\mathrm{A}}(\mathrm{DM})$ together with the short function algebra $\mathrm{DFNC}_{\varpi\mathrm{A}}(\mathrm{DM}) \subset \mathrm{DFNC}_{\mathrm{A}}(\mathrm{DM})$ and the bracket $\{\cdot,\cdot\}_{\mathrm{A}}$ are the basic elements of the derived Poisson structure associated with the unitary connection A of $\mathcal{L}_\beta$. It possibly exemplifies a higher form of Poisson geometrical structure, though its precise framing in Poisson theory remains to be elucidated.

A remarkable property of the derived Poisson structure of A is its gauge invariance. This follows immediately from the gauge invariance of B itself (cf. eq. (5.94)).

Next, we express the most relevant relations obtained above in components to make the relationship of derived to ordinary Poisson theory clearer. Consider a derived functions $\mathrm{F} \in \mathrm{DFNC}_{\mathsf{b}}(\mathrm{DM})$. The basicness conditions (5.128), (5.129) read in components as $j_{\mathsf{R}x,X} f = 0$, $l_{\mathsf{R}x,X} f = 0$, $j_{\mathsf{R}x,X} F = 0$, $l_{\mathsf{R}x,X} F = 0$. Further, when $\mathrm{F} \in \mathrm{DFNC}_{\mathrm{A}}(\mathrm{DM})$, the Hamiltonian relation (5.130) takes the form

$$df + F - ij_{P_{f,F}} b = 0, \tag{5.135}$$
$$dF - ij_{P_{f,F}} B = 0. \tag{5.136}$$

For $\mathrm{F},\mathrm{G} \in \mathrm{DFNC}_{\mathrm{A}}(\mathrm{DM})$, the Lie bracket $\{\mathrm{F},\mathrm{G}\}_{\mathrm{A}}$ has a component expression of the form $\{\mathrm{F},\mathrm{G}\}_{\mathrm{A}}(\alpha) = \{f,g\}_{a,A} + \alpha\{F,G\}_{a,A}$. The notation used here is merely suggestive and should be handled with care, since in general $\{f,g\}_{a,A}$, $\{F,G\}_{a,A}$ both depend on f, F, g, G, as the derived expression (5.131) gives

$$\{f,g\}_{a,A} = -ij_{P_{g,G}} j_{P_{f,F}} b, \tag{5.137}$$
$$\{F,G\}_{a,A} = -ij_{P_{g,G}} j_{P_{f,F}} B. \tag{5.138}$$

When $\mathrm{F} \in \mathrm{DFNC}_{\varpi\mathrm{A}}(\mathrm{DM})$, we have that $\{\mathrm{F},\mathrm{G}\}_{\mathrm{A}} \in \mathrm{DFNC}_{\varpi\mathrm{A}}(\mathrm{DM})$ and therefore $\{F,G\}_{a,A} = 0$. The Poisson bracket $\{\mathrm{F},\mathrm{G}\}_{\mathrm{A}}$ reduces in this way to the degree 0 component $\{f,g\}_{a,A}$ only.

For $\mathrm{F},\mathrm{G},\mathrm{H} \in \mathrm{DFNC}_{\mathrm{A}}(\mathrm{DM})$, the Jacobiator $\langle\mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathrm{A}}$ has the component structure $\langle\mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathrm{A}}(\alpha) = \langle f,g,h\rangle_{a,A} + \alpha\langle F,G,H\rangle_{a,A}$, where again the notation used is merely suggestive for reasons already explained. The Jacobiator components $\langle f,g,h\rangle_{a,A}$, $\langle F,G,H\rangle_{a,A}$ are given by

$$\langle f,g,h\rangle_{a,A} = -ij_{P_{h,H}} j_{P_{g,G}} j_{P_{f,F}} B, \tag{5.139}$$
$$\langle F,G,H\rangle_{a,A} = idj_{P_{h,H}} j_{P_{g,G}} j_{P_{f,F}} B. \tag{5.140}$$

Notice that they depend only on the degree 3 components B of B. A term of the form $idj_{P_{h,H}} j_{P_{g,G}} j_{P_{f,F}} b$ does not appear in the right hand side of (5.139), since it vanishes identically for grading reasons.

The notion of derived Poisson structure worked out above is reminiscent of that of pre-2-symplectic Lie 2-algebra proposed by Rogers in [67]. Roughly, the latter is obtained from the former by replacing the derived gauge field and curvature A and B by their degree 2 and 3 components A and B, respectively.

We consider the set $\mathrm{V\small{ECT}}_{\mathsf{i}A}(\mathsf{DM})$ of all vector fields $V \in \mathrm{V\small{ECT}}(\mathsf{DM})$ satisfying

$$l_V B = 0 \tag{5.141}$$

and the set $\mathrm{V\small{ECT}}_{\mathsf{k}A}(\mathsf{DM})$ of all vector fields $V \in \mathrm{V\small{ECT}}(\mathsf{DM})$ obeying

$$j_V B = 0. \tag{5.142}$$

Again, $\mathrm{V\small{ECT}}_{\mathsf{i}A}(\mathsf{DM})$ is a Lie subalgebra of $\mathrm{V\small{ECT}}(\mathsf{DM})$ and $\mathrm{V\small{ECT}}_{\mathsf{k}A}(\mathsf{DM})$ is a Lie ideal of $\mathrm{V\small{ECT}}_{\mathsf{i}A}(\mathsf{DM})$. It is therefore possible to construct the quotient Lie algebra $\mathrm{V\small{ECT}}_{\mathsf{q}A}(\mathsf{DM}) = \mathrm{V\small{ECT}}_{\mathsf{i}A}(\mathsf{DM})/\mathrm{V\small{ECT}}_{\mathsf{k}A}(\mathsf{DM})$. The Lie derivatives $l_{S_{\mathsf{R}\mathrm{X}}}$ with $\mathrm{X} \in \mathsf{Dj}$ induce derivations $l_{\mathsf{q}S_{\mathsf{R}\mathrm{X}}}$ of $\mathrm{V\small{ECT}}_{\mathsf{q}A}(\mathsf{DM})$ and hence $\mathrm{V\small{ECT}}_{\mathsf{b}A}(\mathsf{DM}) = \bigcap_{\mathrm{X}\in\mathsf{Dj}} \ker l_{\mathsf{q}S_{\mathsf{R}\mathrm{X}}}$ is a Lie subalgebra of $\mathrm{V\small{ECT}}_{\mathsf{q}A}(\mathsf{DM})$.

The relevant function space is again the basic space $\mathrm{DF\small{NC}}_{\mathsf{b}}(\mathsf{DM})$ decomposed as the direct sum of its degree 0 and 1 components $\mathrm{DF\small{NC}}_{\mathsf{b}0}(\mathsf{DM})$, $\mathrm{DF\small{NC}}_{\mathsf{b}1}(\mathsf{DM})$. The Hamiltonian functions are the elements of $F \in \mathrm{DF\small{NC}}_{\mathsf{b}1}(\mathsf{DM})$ for which there exists a vector field $P_F \in \mathrm{V\small{ECT}}_{\mathsf{b}A}$, necessarily unique, such that

$$dF - ij_{P_F} B = 0. \tag{5.143}$$

We denote by $\mathrm{DF\small{NC}}^*_{A1}(\mathsf{DM})$ the subspace of $\mathrm{DF\small{NC}}_{\mathsf{b}1}(\mathsf{DM})$ they form and by $\mathrm{DF\small{NC}}^*_{A}(\mathsf{DM}) = \mathrm{DF\small{NC}}_{\mathsf{b}0}(\mathsf{DM}) \oplus \mathrm{DF\small{NC}}^*_{A1}(\mathsf{DM})$ the corresponding subspace of $\mathrm{DF\small{NC}}_{\mathsf{b}}(\mathsf{DM})$.

We finally introduce a unary, a binary and a trinary bracket $\{\cdot\}_A$, $\{\cdot,\cdot\}_A$, $\{\cdot,\cdot,\cdot\}_A$ on $\mathrm{DF\small{NC}}^*_{A}(\mathsf{DM})$ whose only non zero instances are

$$\{f\}_A = df, \tag{5.144}$$

$$\{F,G\}_A = -ij_{P_G} j_{P_F} B, \tag{5.145}$$

$$\{F,G,H\}_A = ij_{P_H} j_{P_G} j_{P_F} B \tag{5.146}$$

with $f \in \mathrm{DF\small{NC}}_{\mathsf{b}0}(\mathsf{DM})$, $F,G,H\, \mathrm{DF\small{NC}}^*_{A1}(\mathsf{DM})$. By means of a -1 degree shift placing $\mathrm{DF\small{NC}}_{\mathsf{b}0}(\mathsf{DM})$, $\mathrm{DF\small{NC}}^*_{A1}(\mathsf{DM})$ respectively in degree -1, 0, the graded vector space $\mathrm{DF\small{NC}}^*_{A}(\mathsf{DM})$ equipped with the brackets $\{\cdot\}_A$, $\{\cdot,\cdot\}_A$, $\{\cdot,\cdot,\cdot\}_A$ is a semistrict Lie 2-algebra [48].

By (5.136), we have a mapping $\lambda : \mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM}) \to \mathrm{DF\small{NC}}^*_{A1}(\mathsf{DM})$ given by $\lambda(\mathrm{F}) = F$ defining a subspace $\mathrm{DF\small{NC}}^{\lambda}_{\mathrm{A}}(\mathsf{DM}) = \mathrm{DF\small{NC}}_{\mathsf{b}0}(\mathsf{DM}) \oplus \lambda(\mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM}))$ of $\mathrm{DF\small{NC}}^*_{A}(\mathsf{DM})$. $\mathrm{DF\small{NC}}^{\lambda}_{\mathrm{A}}(\mathsf{DM})$ is a Lie 2-subalgebra of $\mathrm{DF\small{NC}}^*_{A}(\mathsf{DM})$.[7] Further,

$$\lambda(\{\mathrm{F},\mathrm{G}\}_{\mathrm{A}}) = \{F,G\}_{a,A} = \{F,G\}_A, \tag{5.147}$$

$$\lambda(\langle\mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathrm{A}}) = \{\{F,G,H\}_A\}_A. \tag{5.148}$$

This clarifies the relationship between the two formulations.

[7]Note here that for $f \in \mathrm{DF\small{NC}}_{\mathsf{b}0}(\mathsf{DM})$ one has $\{f\}_A \in \lambda(\mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM}))$, as $df = \lambda(\mathrm{Q}_f)$ where $\mathrm{Q}_f \in \mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM})$ with $\mathrm{Q}_f(\alpha) = -f + \alpha df$.

5.7 Hamiltonian nature of the target kernel symmetry

As detailed in subsection 5.3, the derived group DM is characterized by the left action of the derived target kernel group DM_τ. The left DM_τ-action commutes with the right DJ-action. It therefore descends on one on the regular homogeneous manifold DM/DJ and, for each character β, the derived unitary line bundle $\mathcal{L}_\beta$. In this subsection, we examine the left target kernel symmetry from the standpoint of the derived Poisson theory of subsection 5.6 and show its Hamiltonian nature. Our analysis is patterned again on the corresponding one of the standard formulation, reviewed in subsection 4.4, where the left symmetry can be similarly shown to be Hamiltonian.

At the infinitesimal level, the left DM_τ-action is codified by the associated left $\mathsf{D}\mathfrak{m}_\tau$-operation. A derived unitary connection A of $\mathcal{L}_\beta$ is left invariant if

$$l_{\mathsf{LH}}\mathrm{A} = 0. \tag{5.149}$$

This property implies the left invariance of the associated curvature B,

$$l_{\mathsf{LH}}\mathrm{B} = 0 \tag{5.150}$$

on account of (5.84). (5.149), (5.150) answer to and have the same meaning as that of the invariance relations (4.24), (4.25) of the standard theory.

A derived gauge transformation U of $\mathcal{L}_\beta$ is invariant if

$$l_{\mathsf{LH}}\mathrm{U}\mathrm{U}^{-1} = 0 \tag{5.151}$$

just as in the standard theory, see eq. (4.28). If A is an invariant connection and U is an invariant gauge transformation, then ${}^{\mathrm{U}}\mathrm{A}$ is also an invariant connection.

As the curvature B of an invariant unitary connection A is also invariant, the derived Poisson structure associated with A is expected to have special properties with regard to the DM_τ target kernel symmetry. In particular, the natural question arises about whether the DM_τ-action is Hamiltonian. It turns out that it is, as we show next. We recall here that the derivations j_{LH}, l_{LH} with $\mathrm{H} \in \mathsf{D}\mathfrak{m}_\tau$ are just the Cartan calculus derivations $j_{S_{\mathsf{LH}}}$, $l_{S_{\mathsf{LH}}}$ considered in subsection 5.6, where S_{LH} is the vertical vector field of the left DM_τ-action corresponding to H.

The expression of the moment map of the left DM_τ-action is analogous to that of the moment map of the left G-action in the standard theory in eq. (4.26). It is the degree 0 derived function valued map $\mathrm{Q}_\mathrm{A} : \mathsf{D}\mathfrak{m}_\tau \to \mathrm{DF}\textsc{nc}_\mathsf{b}(\mathsf{DM})$ given by

$$\mathrm{Q}_\mathrm{A}(\mathrm{H}) = -ij_{\mathsf{LH}}\mathrm{A} \tag{5.152}$$

with $\mathrm{H} \in \mathsf{D}\mathfrak{m}_\tau$. By (5.82), (5.83) $\mathrm{Q}_\mathrm{A}(\mathrm{H})$ satisfies the basicness conditions (5.128), (5.129). Thus, $\mathrm{Q}_\mathrm{A}(\mathrm{H}) \in \mathrm{DF}\textsc{nc}_\mathsf{b}(\mathsf{DM})$ as indicated and so $\mathrm{Q}_\mathrm{A}(\mathrm{H})$ defines a derived function on DM/DJ as required. Further, by the invariance of A, Q_A obeys the moment map equation

$$d\mathrm{Q}_\mathrm{A}(\mathrm{H}) - ij_{\mathsf{LH}}\mathrm{B} = 0. \tag{5.153}$$

$\mathrm{Q}_\mathrm{A}(\mathrm{H})$ is therefore a Hamiltonian derived function with Hamiltonian vector field $P_{\mathrm{Q}_\mathrm{A}(\mathrm{H})} = S_{\mathsf{LH}}$.

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Q_{A} obeys the derived equivariance identity

$$l_{\mathsf{LH}}\mathrm{Q}_{\mathrm{A}}(\mathrm{K}) - \mathrm{Q}_{\mathrm{A}}([\mathrm{H},\mathrm{K}]) = 0 \tag{5.154}$$

for $\mathrm{H},\mathrm{K} \in \mathrm{D}\mathfrak{m}_{\mathcal{T}}$. (5.154) formally reproduces the corresponding relation of the standard theory, eq. (4.27), only in part. In fact, in the present derived Poisson set-up, $l_{\mathsf{LH}}\mathrm{Q}_{\mathrm{A}}(\mathrm{K}) \neq \{\mathrm{Q}_{\mathrm{A}}(\mathrm{H}),\mathrm{Q}_{\mathrm{A}}(\mathrm{K})\}_{\mathrm{A}}$, in general in contrast to ordinary Poisson theory. The defect map $\mathrm{C}_{\mathrm{A}} : \mathsf{DM}_{\mathcal{T}} \times \mathsf{DM}_{\mathcal{T}} \to \mathrm{DF}\textsc{nc}_{\mathsf{b}}(\mathsf{DM})$ which measures the failure for such an equality to hold is given by

$$\mathrm{C}_{\mathrm{A}}(\mathrm{H},\mathrm{K}) = -i\mathrm{d}j_{\mathsf{LK}}j_{\mathsf{LH}}\mathrm{A} \tag{5.155}$$

with $\mathrm{H},\mathrm{K} \in \mathrm{D}\mathfrak{m}_{\mathcal{T}}$. From (5.82), (5.83) again, $\mathrm{C}_{\mathrm{A}}(\mathrm{H},\mathrm{K})$ satisfies the basicness requirements (5.128), (5.129) and therefore $\mathrm{C}_{\mathrm{A}}(\mathrm{H},\mathrm{K}) \in \mathrm{DF}\textsc{nc}_{\mathsf{b}}(\mathsf{DM})$ defining a function on DM/DJ. The defect relation

$$l_{\mathsf{LH}}\mathrm{Q}_{\mathrm{A}}(\mathrm{K}) = \{\mathrm{Q}_{\mathrm{A}}(\mathrm{H}),\mathrm{Q}_{\mathrm{A}}(\mathrm{K})\}_{\mathrm{A}} - \mathrm{C}_{\mathrm{A}}(\mathrm{H},\mathrm{K}) \tag{5.156}$$

holds by construction. Through (5.156), we can reexpress relation (5.154) suggestively in the form

$$\{\mathrm{Q}_{\mathrm{A}}(\mathrm{H}),\mathrm{Q}_{\mathrm{A}}(\mathrm{K})\}_{\mathrm{A}} - \mathrm{Q}_{\mathrm{A}}([\mathrm{H},\mathrm{K}]) = \mathrm{C}_{\mathrm{A}}(\mathrm{H},\mathrm{K}). \tag{5.157}$$

Analogously to the Poisson Jacobiator encountered in subsection 5.6, $\mathrm{C}_{\mathrm{A}}(\mathrm{H},\mathrm{K})$ is generally non vanishing because the degree 2 component A of A is not necessarily annihilated by $j_{\mathsf{LK}}j_{\mathsf{LH}}$. By (5.130) and d-exactness, however, $\mathrm{C}_{\mathrm{A}}(\mathrm{H},\mathrm{K})$ is a Hamiltonian derived function and its Hamiltonian vector field $P_{\mathrm{C}_{\mathrm{A}}(\mathrm{H},\mathrm{K})}$ vanishes. Hence, $\mathrm{C}_{\mathrm{A}}(\mathrm{H},\mathrm{K})$ is central,

$$\{\mathrm{C}_{\mathrm{A}}(\mathrm{H},\mathrm{K}),\cdot\}_{\mathrm{A}} = 0. \tag{5.158}$$

An invariant connection A is called strict if $\mathrm{C}_{\mathrm{A}} = 0$. For a strict connection, the equivariance relation (5.157) takes a more familiar form analogous to that relation (4.27) does in ordinary Poisson theory.

By (5.93) and (5.152), under an invariant gauge transformation U, the moment map Q_{A} varies by a d-exact term

$${}^{\mathrm{U}}\mathrm{Q}_{\mathrm{A}}(\mathrm{H}) = \mathrm{Q}_{{}^{\mathrm{U}}\mathrm{A}}(\mathrm{H}) = \mathrm{Q}_{\mathrm{A}}(\mathrm{H}) - i\mathrm{d}(j_{\mathsf{LH}}\mathrm{U}\mathrm{U}^{-1}), \tag{5.159}$$

thanks to the invariance relation (5.151). It is immediately checked that the moment map properties (5.153), (5.154) are compatible with the gauge transformation action (5.159).

It is interesting to examine the form the above relations take in components. By the defining relation (5.152), the moment map components read as

$$q_{a,A}(h,H) = -ij_{\mathsf{L}h,H}a, \tag{5.160}$$
$$Q_{a,A}(h,H) = -ij_{\mathsf{L}h,H}A. \tag{5.161}$$

The moment map equation (5.153) yields the pair of equations

$$dq_{a,A}(h,H) + Q_{a,A}(h,H) - ij_{\mathsf{L}h,H}b = 0, \tag{5.162}$$
$$dQ_{a,A}(h,H) - ij_{\mathsf{L}h,H}B = 0. \tag{5.163}$$

By (5.155), the defect map components are given by

$$c_{a,A}(h,H,k,K) = ij_{\mathsf{L}k,K}j_{\mathsf{L}h,H}A, \tag{5.164}$$

$$c_{a,A}(h,H,k,K) = -idj_{\mathsf{L}k,K}j_{\mathsf{L}h,H}A. \tag{5.165}$$

Notice that they depend only on the degree 2 components A of A. A term of the form $-idj_{\mathsf{L}k,K}j_{\mathsf{L}h,H}a$ does not appear in the right hand side of (5.176), since it vanishes identically for grading reasons. The defect equivariance relation (5.157) assume componentwise the shape

$$\{q_{a,A}(h,H), q_{a,A}(k,K)\}_{a,A} - q_{a,A}([h,k], \dot{\mu}(h,K) - \dot{\mu}(k,H)) = c_{a,A}(h,H,k,K), \tag{5.166}$$

$$\{Q_{a,A}(h,H), Q_{a,A}(k,K)\}_{a,A} - Q_{a,A}([h,k], \dot{\mu}(h,K) - \dot{\mu}(k,H)) = C_{a,A}(h,H,k,K), \tag{5.167}$$

where the derived Poisson bracket components are defined as in eqs. (5.137), (5.138).

From (5.159), the gauge variation of the moment map components are

$$^{u,U}q_{a,A}(h,H) = q_{a,A}(h,H) - id(j_{\mathsf{L}h,H}uu^{-1}) + ij_{\mathsf{L}h,H}U, \tag{5.168}$$

$$^{u,U}Q_{a,A}(h,H) = Q_{a,A}(h,H) - idj_{\mathsf{L}h,H}U. \tag{5.169}$$

The above notion of derived moment map is related to that of homotopy moment map worked out by Callies et al. [68]. The relationship between the two can be understood in terms of that intercurring between the derived Poisson and pre-2-symplectic structures described at the end of subsection 5.6. Again, as we outline next, the latter is obtained from the former roughly by replacing the derived gauge field and curvature A and B by their degree 2 and 3 components A and B, respectively.

Similarly to the derived case, we assume that the degree 2 component A of the relevant connection A is invariant so that

$$l_{\mathsf{L}h,H}A = 0 \tag{5.170}$$

for $h \in \mathfrak{g}$, $H \in \ker \dot{\tau}[1]$. As a consequence, by (5.105), also the degree 3 component B of the associated curvature B is

$$l_{\mathsf{L}h,H}B = 0. \tag{5.171}$$

The homotopy moment map $Q_A : \mathrm{D}\mathfrak{m}_\tau \to \mathrm{DF}\textsc{nc}_{\mathsf{b}1}(\mathrm{D}M)$ is defined as

$$Q_A(h,H) = -ij_{\mathsf{L}h,H}A \tag{5.172}$$

with $h \in \mathfrak{g}$, $H \in \ker \dot{\tau}[1]$ analogously to 5.152. The basicness of $Q_A(h,H)$ is checked using relations (5.101), (5.103). $Q_A(h,H)$ is Hamiltonian with Hamiltonian vector field $S_{\mathsf{L}h,H}$ in the pre-2-plectic Lie 2-algebra structure of A as

$$dQ_A(h,H) - ij_{\mathsf{L}h,H}B = 0, \tag{5.173}$$

so that $Q_A(h,H)$ is a Hamiltonian function.

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The defect map $R_A : \mathsf{DM}_\tau \times \mathsf{DM}_\tau \to \mathrm{DF{\small NC}}_{\mathsf{b}0}(\mathsf{DM})$ associated with the moment map Q_A is given by

$$R_A(h,H,k,K) = -ij_{\mathsf{L}k,K}j_{\mathsf{L}h,H}A. \tag{5.174}$$

The basicness of $R_A(h,H,k,K)$ is checked easily using again relations (5.101), (5.103). The moment map obeys further the relations

$$\begin{aligned}&\{Q_A(h,H),Q_A(k,K)\}_A - Q_A([h,k],\dot{}\mu\dot{}(h,K)-\dot{}\mu\dot{}(k,H)) \\ &\quad = dR_A(h,H,k,K),\end{aligned} \tag{5.175}$$

$$\begin{aligned}&\{Q_A(h,H),Q_A(k,K),Q_A(l,L)\}_A \\ &\quad = R_A(h,H,[k,l],\dot{}\mu\dot{}(k,L)-\dot{}\mu\dot{}(l,K)) + R_A(k,K,[l,h],\dot{}\mu\dot{}(l,H)-\dot{}\mu\dot{}(h,L)) \\ &\qquad + R_A(l,L,[h,k],\dot{}\mu\dot{}(h,K)-\dot{}\mu\dot{}(k,H)).\end{aligned} \tag{5.176}$$

Relations (5.175), (5.176) characterize the homotopy moment map Q_A in the pre-2-plectic Lie 2-algebra set-up [68].

In subsection 5.6, we exhibited a mapping $\lambda : \mathrm{DF{\small NC}}_{\mathrm{A}}(\mathsf{DM}) \to \mathrm{DF{\small NC}}^*_{A1}(\mathsf{DM})$ connecting the Hamiltonian derived and pre-2-plectic function spaces preserving the bracket structure in the appropriate sense (cf eqs. (5.147), (5.148)). λ also relates the derived and homotopy moment map data in a consistent manner,

$$\lambda(\mathrm{Q}_{\mathrm{A}}(\mathrm{H})) = Q_{a,A}(h,H) = Q_A(h,H), \tag{5.177}$$

$$\lambda(C_{\mathrm{A}}(\mathrm{H},\mathrm{K})) = \{R_A(h,H,k,K)\}_A. \tag{5.178}$$

This clarifies again the relationship between our constructions and those of ref. [68] .

5.8 Derived prequantization

In derived KKS theory, prequantization is implemented along lines analogous to those of ordinary KKS prequantization as reviewed in subsection 4.5. Derived prequantization is however more involved than its ordinary counterpart for a number of reasons. In this subsection, we illustrate its construction and the problems which affect it.

The foundation on which derived prequantization rests is the derived Poisson structure of the regular derived homogeneous space DM/DJ associated with a unitary connection A of the derived line bundle $\mathcal{L}_\beta$ of a character β of J introduced and studied in subsection 5.6. This however is not a genuine Poisson structure. The derived Poisson bracket $\{\cdot,\cdot\}_{\mathrm{A}}$ has a generally non trivial Jacobiator and the Hamiltonian derived function space $\mathrm{DF{\small NC}}_{\mathrm{A}}(\mathsf{DM})$ is not an algebra. It is therefore necessary to impose a suitable condition on A and to restrict the range of Hamiltonian functions to an appropriate subspace $\mathrm{DF{\small NC}}_{\mathrm{A}\mathsf{h}}(\mathsf{DM})$ of the space $\mathrm{DF{\small NC}}_{\mathrm{A}}(\mathsf{DM})$. Upon doing so, $\{\cdot,\cdot\}_{\mathrm{A}}$ becomes a genuine Lie bracket as required by prequantization. It can then be shown that with any function $\mathrm{F} \in \mathrm{DF{\small NC}}_{\mathrm{A}\mathsf{h}}(\mathsf{DM})$ there is associated an endomorphism $\widehat{F}$ of a certain subspace $\mathrm{D}\Omega^0{}_{\mathsf{h}}(\mathcal{L}_\beta)$ of the space $\mathrm{D}\Omega^0(\mathcal{L}_\beta)$ of 0-form sections of $\mathcal{L}_\beta$ such that for $\mathrm{F},\mathrm{G} \in \mathrm{DF{\small NC}}_{\mathrm{A}\mathsf{h}}(\mathsf{DM})$, one has $[\widehat{\mathrm{F}},\widehat{\mathrm{G}}] = i\widehat{\{\mathrm{F},\mathrm{G}\}_{\mathrm{A}}}$. $\mathcal{L}_\beta$ gets so interpreted as the derived prequantum line bundle.

In this way, a derived KKS prequantization is defined associating an operator with any suitably restricted Hamiltonian derived function such that the resulting operator commutator structure is fully compatible with the derived Poisson bracket structure. However, there apparently is no natural derived prequantum Hilbert space structure on DM/DJ with respect to which the operators yielded by derived prequantization are formally Hermitian. This seems to be in line with the findings of several higher prequantization schemes available in the literature, see in particular refs. [57, 59, 60].

Derived prequantization requires the underlying derived presymplectic structure to be symplectic as in the ordinary setting. We thus assume that the curvature B of the connection A is non singular. In this way, if $V \in \mathrm{V\small{ECT}}(\mathsf{DM})$ and $j_V \mathrm{B} = 0$, then $V = S_{\mathrm{RX}}$ for some $\mathrm{X} \in \mathsf{Dj}$ pointwise on DM.

For any Hamiltonian derived function $\mathrm{F} \in \mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM})$ and derived 0-form section $\mathrm{S} \in \mathrm{D}\Omega^0(\mathcal{L}_\beta)$, we set

$$\widetilde{\mathrm{F}}\mathrm{S} = ij_{P_{\mathrm{F}}}\mathrm{d}_{\mathrm{A}}\mathrm{S} + \mathrm{FS}, \tag{5.179}$$

where P_{F} is the Hamiltonian vector field of F (cf. eq. (5.130)). It can be readily verified that $\widetilde{\mathrm{F}}\mathrm{S} \in \mathrm{D}\Omega^0(\mathcal{L}_\beta)$ using (5.80), (5.81) and (5.128), (5.129). $\widetilde{\mathrm{F}}$ is clearly an endomorphism of the vector space $\mathrm{D}\Omega^0(\mathcal{L}_\beta)$. By the structural analogy of eq. (5.179) to relation (4.29) providing the prequantization of Hamiltonian functions in ordinary KKS theory, $\widetilde{\mathrm{F}}$ can naively be thought of as the derived prequantization of F. Unfortunately, we have $[\widetilde{\mathrm{F}}, \widetilde{\mathrm{G}}] \neq i\widetilde{\{\mathrm{F},\mathrm{G}\}}_{\mathrm{A}}$ for $\mathrm{F},\mathrm{G} \in \mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM})$ in general.[8] Our simpleminded derived prequantization approach requires therefore appropriate modifications in order to be viable.

We assume in what follows that the connection A of $\mathcal{L}_\beta$ of 1-form type, that is with the property that $j_V j_W \mathrm{A} = 0$ for any two vector fields $V, W \in \mathrm{V\small{ECT}}(\mathsf{DM})$. The curvature B of A is then of 2-form type, i.e. one has $j_V j_W j_Z \mathrm{B} = 0$ for any $V, W, Z \in \mathrm{V\small{ECT}}(\mathsf{DM})$. Furthermore, A is both simple and strict (cf. subsections 5.6, 5.7).

A derived function $\mathrm{F} \in \mathrm{DF\small{NC}}_{\mathsf{b}}(\mathsf{DM})$ is said to be pure if $j_V \mathrm{F} = 0$ for any vector field $V \in \mathrm{V\small{ECT}}(\mathsf{DM})$. The pure derived functions constitute a subalgebra $\mathrm{DF\small{NC}}_{\mathsf{bh}}(\mathsf{DM})$ of $\mathrm{DF\small{NC}}_{\mathsf{b}}(\mathsf{DM})$. Similarly, a 0-form section $\mathrm{S} \in \mathrm{D}\Omega^0(\mathcal{L}_\beta)$ is said to be pure if $j_V \mathrm{S} = 0$ for any vector field $V \in \mathrm{V\small{ECT}}(\mathsf{DM})$. The pure 0-from sections form a subspace $\mathrm{D}\Omega^0{}_{\mathsf{h}}(\mathcal{L}_\beta)$ of $\mathrm{D}\Omega^0(\mathcal{L}_\beta)$.

Let $\mathrm{DF\small{NC}}_{\mathrm{A}\mathsf{h}}(\mathsf{DM}) = \mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM}) \cap \mathrm{DF\small{NC}}_{\mathsf{bh}}(\mathsf{DM})$ be the space of pure Hamiltonian derived functions. For any function pair $\mathrm{F},\mathrm{G} \in \mathrm{DF\small{NC}}_{\mathrm{A}\mathsf{h}}(\mathsf{DM})$, one has $\{\mathrm{F},\mathrm{G}\}_{\mathrm{A}} \in \mathrm{DF\small{NC}}_{\mathrm{A}\mathsf{h}}(\mathsf{DM})$. Further, the restriction of the twisted Lie bracket $\{\cdot,\cdot\}_{\mathrm{A}}$ to $\mathrm{DF\small{NC}}_{\mathrm{A}\mathsf{h}}(\mathsf{DM})$ is a genuine Lie bracket, since the connection A is simple and so the Jacobiator $\langle\cdot,\cdot,\cdot\rangle_{\mathrm{A}}$ of $\{\cdot,\cdot\}_{\mathrm{A}}$ vanishes identically (cf. eqs. (5.132), (5.133)). Note that, like $\mathrm{DF\small{NC}}_{\mathrm{A}}(\mathsf{DM})$, $\mathrm{DF\small{NC}}_{\mathrm{A}\mathsf{h}}(\mathsf{DM})$ depends on the connection A and is a vector space but not an algebra and so $\{\cdot,\cdot\}_{\mathrm{A}}$ is a Lie but not a Poisson bracket.

It can be readily shown that $\widetilde{\mathrm{F}}\mathrm{S} \in \mathrm{D}\Omega^0{}_{\mathsf{h}}(\mathcal{L}_\beta)$ for any $\mathrm{F} \in \mathrm{DF\small{NC}}_{\mathrm{A}\mathsf{h}}(\mathsf{DM})$ and $\mathrm{S} \in \mathrm{D}\Omega^0{}_{\mathsf{h}}(\mathcal{L}_\beta)$ as a consequence of A being of 1-form type. Let $\widehat{\mathrm{F}}$ be the restriction of $\widetilde{\mathrm{F}}$ to

[8] We have in fact that

$$\begin{aligned}[\widetilde{\mathrm{F}}, \widetilde{\mathrm{G}}] = {} & i\widetilde{\{\mathrm{F},\mathrm{G}\}}_{\mathrm{A}} - \mathrm{d}_{\mathrm{A}} j_{P_{\mathrm{G}}} j_{P_{\mathrm{F}}} \mathrm{d}_{\mathrm{A}} \\ & + ((j_{P_{\mathrm{G}}} j_{P_{\mathrm{F}}} \mathrm{A}) + i(j_{P_{\mathrm{F}}}\mathrm{G}) - i(j_{P_{\mathrm{G}}}\mathrm{F}))\mathrm{d}_{\mathrm{A}} + (j_{P_{\mathrm{G}}}\mathrm{B}) j_{P_{\mathrm{F}}} - (j_{P_{\mathrm{F}}}\mathrm{B}) j_{P_{\mathrm{G}}}. \end{aligned} \tag{5.180}$$

The terms in the right hand side beyond the first one do not vanish in general. A relation analogous to the above one holds also in ordinary KKS theory, where however the extra terms can be shown to vanish identically for grading reasons.

$\mathrm{D}\Omega^0{}_{\mathsf{h}}(\mathcal{L}_\beta)$. Then, $\widehat{\mathrm{F}}$ is an endomorphism of the vector space $\mathrm{D}\Omega^0{}_{\mathsf{h}}(\mathcal{L}_\beta)$. Furthermore, the commutation relation

$$[\widehat{\mathrm{F}},\widehat{\mathrm{G}}] = i\widehat{\{\mathrm{F},\mathrm{G}\}}_{\mathrm{A}} \tag{5.181}$$

holds for any two functions $\mathrm{F},\mathrm{G} \in \mathrm{DF\textsc{nc}}_{\mathrm{A}\mathsf{h}}(\mathsf{DM})$. Note that (5.181) is consistent with the Jacobi property of the endomorphism commutator because $\{\cdot,\cdot\}_{\mathrm{A}}$ is a Lie bracket on $\mathrm{DF\textsc{nc}}_{\mathrm{A}\mathsf{h}}(\mathsf{DM})$. (5.181) reproduces the commutation relation (4.30) of the ordinary theory.

The target kernel symmetry moment maps are pure Hamiltonian derived functions. By (5.152) and the 1-form property of the connection A it is indeed evident that $\mathrm{Q}_{\mathrm{A}}(\mathrm{H}) \in \mathrm{DF\textsc{nc}}_{\mathsf{bh}}(\mathsf{DM})$ for $\mathrm{H} \in \mathrm{D}\mathfrak{m}_\tau$. Since $\mathrm{Q}_{\mathrm{A}}(\mathrm{H}) \in \mathrm{DF\textsc{nc}}_{\mathrm{A}}(\mathsf{DM})$, $\mathrm{Q}_{\mathrm{A}}(\mathrm{H}) \in \mathrm{DF\textsc{nc}}_{\mathrm{A}\mathsf{h}}(\mathsf{DM})$ and so the operators $\widehat{\mathrm{Q}}_{\mathrm{A}}(\mathrm{H})$ is defined. From (5.152), (5.153) and (5.179), its expression is very simple

$$\widehat{\mathrm{Q}}_{\mathrm{A}}(\mathrm{H}) = il_{\mathrm{LH}}. \tag{5.182}$$

The commutation relation (5.181) takes for the $\widehat{\mathrm{Q}}_{\mathrm{A}}(\mathrm{H})$ the expected form

$$[\widehat{\mathrm{Q}}_{\mathrm{A}}(\mathrm{H}),\widehat{\mathrm{Q}}_{\mathrm{A}}(\mathrm{K})] = i\widehat{\mathrm{Q}}_{\mathrm{A}}([\mathrm{H},\mathrm{K}]). \tag{5.183}$$

with $\mathrm{H},\mathrm{K} \in \mathrm{D}\mathfrak{m}_\tau$. (5.182), (5.183) replicate in derived KKS theory the ordinary KKS relations (4.31), (4.32).

We conclude this subsection explaining why no natural derived prequantum Hilbert space structure on DM/DJ can be defined in derived KKS theory with respect to which the operators yielded by derived prequantization are formally Hermitian. Duplicating the standard expression of the prequantum Hilbert inner product of ordinary KKS theory given in eq. (4.33) in our derived setting furnishes the following tentative expression for the derived prequantum Hilbert inner product of two sections $\mathrm{S},\mathrm{T} \in \mathrm{D}\Omega^0{}_{\mathsf{h}}(\mathcal{L}_\beta)$:

$$\langle \mathrm{S},\mathrm{T}\rangle = \frac{1}{n!}\int_{T[1](\mathsf{DM}/\mathsf{DJ})} \varrho_{\mathsf{DM}/\mathsf{DJ}}(-iB)^n \mathrm{S}^*\mathrm{T} \qquad \text{(tentative)}. \tag{5.184}$$

There are a minor and a major problem with this formula: the existence of an appropriate value of the half dimension n and the well-definedness of integration, respectively. If $\mathsf{M} = (\mathsf{E},\mathsf{G})$ and $\mathsf{J} = (\mathsf{H},\mathsf{T})$, the degree of the Berezinian $\varrho_{\mathsf{DM}/\mathsf{DJ}}$ is $d = -\dim\mathsf{G} + \dim\mathsf{T} + \dim\mathsf{E} - \dim\mathsf{H}$. To have a degree 0 integrand, we should so set $n = -d/2$, as B has degree 2. However, unlike ordinary KKS theory, $-d/2$ is not guaranteed to be a positive integer as far as we can see. Further, it is known that on a non negatively graded manifold of degree 1 such as DM/DJ, integration is defined and convergent only if the integrand is a Dirac type distribution in the coordinates of positive even degree (in supergeometry forms of this kind are called integral [66]). Unfortunately, the integrand in the right hand side of eq. (5.184) does not have this property. In conclusion, the prequantum Hilbert structure of ordinary KKS theory does not have a viable derived analog.

5.9 Paths to derived quantization

The analysis of subsection 5.8 indicates that conventionally designed derived KKS prequantization falls short of meeting its prerequisites in spite of reproducing some of the basic features of ordinary KKS prequantization. The non existence of a derived prequantum Hilbert structure precludes the successful completion of the geometric quantization program of derived KKS theory as ordinarily envisaged. At this point, it is therefore important to explore the range of quantization options at our disposal.

In the incomplete prequantization framework of subsection 5.8, we assumed that the underlying derived connection A was of 1-form type and we restricted to the space of pure Hamiltonian derived functions $\mathrm{DF}_{\mathrm{NC}_{\mathrm{A}\mathsf{h}}}(\mathsf{DM})$ in order to dispose of the Jacobiator $\langle\cdot,\cdot,\cdot\rangle_{\mathrm{A}}$ and so deal with a genuine Lie algebra structure $\{\cdot,\cdot\}_{\mathrm{A}}$, a minimal requirement for any operator/wave-function type quantization scheme such as geometric quantization. If we do not impose such restriction on the connection and the Hamiltonian functions, we face a twisted Lie algebra structure $\{\cdot,\cdot\}_{\mathrm{A}}$. Multiple advancements in quantum mechanics and string theory have revealed that twisted Poisson structure featuring non trivial Jacobiators are physically relevant and that their quantization must be tackled. (See [69] for a updated review and extensive referencing.) Since operator/wave-function quantization is not viable in this case, alternative quantization schemes must be sought.

To the best of our knowledge, there exist three main approaches to quantisation of a twisted Poisson manifold. The first approach, developed in [70], is an extension of the standard framework of deformation quantisation. The basic idea informing it is that the non trivial Poisson Jacobiator leads to a non associative star product algebra, which is explicitly constructed. The method suffers the usual problem of being based on purely formal power series in $\hbar$. The second approach, originally proposed is [71], is an adaptation of the classic technique of symplectic realisation of Poisson theory and consists in embedding the relevant twisted Poisson manifold into a symplectic manifold of twice the dimension. In this way, standard operator/wave-function based geometric quantisation is again possible. Ineliminable spurious extra variables with no evident interpretation must however be introduced in order to absorb the Jacobiator into the higher dimensional symplectic structure. In [72], a third approach was suggested, which can be described as higher geometric quantization. Its basic principle is replacing the usual quantum line bundle and the connection with curvature equal to the symplectic form by a quantum bundle gerbe and a connection with curvature equal to the Jacobiator form. Correspondingly, the customary quantum Hilbert space of sections of the line bundle and its operator algebra get substituted by a 2-Hilbert space of sections of the gerbe, a categorified Hilbert space structured as a monoidal category, and its endofunctor category. Although this approach provides a higher quantum Hilbert space formulation naturally incorporating non associativity, its interpretation in more conventional physical terms remains to be elucidated.

The natural question arises about to what extent one or the other of the three approaches to quantization of twisted Poisson manifolds outlined in the previous paragraph can be utilized for the quantization of the derived Poisson manifold DM/DJ. An exhaustive answer to it would require an in-depth analysis that lies beyond the scope of the present work. Here, we shall limit ourselves to the following considerations.

The three methods have been formulated for ordinary manifolds. DM/DJ is instead a genuinely graded manifold. Hence, derived KKS quantization would require a graded geometric extension of such approaches, which to the best of our knowledge is presently lacking. As the findings of subsection 5.8 show, such extension might not be straightforward or even possible.

The Hamiltonian derived function space $\mathrm{DF}\textsc{nc}_{\mathrm{A}}(\mathsf{DM})$ does not contain only functions meant as 0-forms on DM/DJ but also 1-forms. In fact, it is a subspace of $\mathrm{DF}\textsc{nc}(\mathsf{DM}) = \mathrm{M}\textsc{ap}(T[1]\,\mathsf{DM}, \mathrm{D}\mathbb{R}[0])$ consisting of inhomogeneous forms of form degree up to 1 obeying the basicness conditions (5.128), (5.129) and the Hamiltonian condition (5.130). Furthermore, it is not an algebra, but only a vector space. This makes the first two approaches apparently hardly workable leaving utilizing the third as an open option.

On the whole, in fact, of the three approaches the third one appears to be the most likely adaptable to the derived KKS framework. This indeed involves in an essential way derived line bundles with connection. Geometric objects of this kind, as observed at the end of subsection 5.5, are likely related to Roger's twisted line bundles of bundle gerbes [56]. We shall not however attempt to follow this strategy.

The issues affecting derived KKS geometric quantization exposed in subsection 5.8 do not by themselves imply that geometric quantization is outright impossible in the derived KKS set-up, but only that the problem of geometric quantization in that context cannot be coped with by a straightforward extension or adaptation to the derived setting of the basic techniques used to solve the corresponding problem in the ordinary KKS set-up.

In paper II, we shall provide an indirect solution of the problem of geometric quantization of derived KKS theory. Specifically, we shall elaborate a derived 2-dimensional TCO sigma model and present substantial evidence that such model is the appropriate derived counterpart of the ordinary 1-dimensional TCO quantum mechanical model. Since that latter provides a quantization scheme for ordinary KKS theory equivalent to geometric quantization, it is conceivable that the former may furnish a quantization framework of derived KKS theory taking the place of geometric quantization.

The derived TCO model is based on derived KKS geometry much as the ordinary TCO model is on ordinary KKS geometry. This justifies the construction of the derived KKS theory that we have carried out in this section and shall complete in the next subsection.

5.10 Derived KKS theory in the regular case

In subsections 5.5–5.7, we studied the derived unitary line bundles and their connections and the associated derived Poisson structures on a regular derived homogeneous space, finding that the target kernel symmetry is Hamiltonian. As in ordinary KKS theory, a full fledged derived KKS theory takes shape when the underlying homogeneous space is a coadjoint orbit. In this subsection, we expound its construction in full detail.

The results presented below provide a concrete illustration of the abstract theory of subsections 5.5–5.7 by applying it to the description of regular derived coadjoint orbits and their derived Poisson structures. Most importantly, however, they prepare the ground and provide the necessary geometric underpinning for the quantization of such orbits and the construction of the associated derived TCO sigma model presented in paper II.

The derived presymplectic structure underlying a derived Poisson structure stems from the curvature of a unitary connection of a derived line bundle and so satisfies a quantization condition. This property plays an important role in derived KKS theory, as in the ordinary theory, since it limits the range of quantizable coadjoint orbits as derived symplectic manifolds.

The data of the construction we are presenting are (cf. subsection 5.1): a compact Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$, a maximal toral crossed submodule $\mathsf{J} = (\mathsf{H}, \mathsf{T})$ of M, an invariant pairing $\langle \cdot, \cdot \rangle$ of M and an element $\Lambda \in \mathfrak{e}$ satisfying the following two admissibility conditions. First, $\mathsf{J} = \mathsf{ZM}_\Lambda$, where M_Λ is the 1-parameter crossed submodule of M generated by Λ and ZM_Λ is its centralizer crossed module; second, the map $\xi_\Lambda : \mathsf{T} \to \mathsf{U}(1)$ defined by

$$\xi_\Lambda(\mathrm{e}^x) = \mathrm{e}^{i\langle x, \Lambda\rangle} \tag{5.185}$$

with $x \in \mathfrak{t}$ is a character of the maximal torus T of G. The first admissibility requirement implies that Λ is a regular element of $\mathfrak{e}$ and that the derived coadjoint orbit of Λ is $\mathcal{O}_\Lambda = \mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$. The second entails that the restriction of the mapping $x \to \langle x, \Lambda\rangle/2\pi$ to the integer lattice Λ_G of T belongs to the dual integral lattice ${\Lambda_\mathsf{G}}^*$ of Λ_G, an integrality property.

By equipping M with an invariant pairing, we are tacitly assuming that the crossed module M is balanced (cf. subsection 3.1). The theory developed hitherto does not require such restriction at any point. Demanding it, however, involves only a seeming loss of generality. Let us discuss this point in some depth.

Any Lie group crossed module M can always be extended to a balanced crossed module M^c. In fact, if $\mathsf{W} = (\mathsf{I}, \mathsf{K})$ is a trivial Abelian crossed module[9] such that $\dim \mathsf{E} + \dim \mathsf{I} = \dim \mathsf{G} + \dim \mathsf{K}$, then the product crossed module[10] $\mathsf{M}^c = \mathsf{M} \times \mathsf{W}$ is a balanced crossed module containing M as a submodule. M^c depends on the choice of W and so is not uniquely defined. There is however a minimal choice of W and so of M^c for which either I or K are the trivial group 1. The following discussion does not assume however that M^c is minimal.

An element $\Lambda \in \mathfrak{e}$ can be viewed as an element $\Lambda \in \mathfrak{e} \oplus \mathfrak{n}$, which we denote by the same symbol. The 1-parameter submodules M_Λ and ${\mathsf{M}^c}_\Lambda$ of Λ in M and M^c, respectively, are trivially related as ${\mathsf{M}^c}_\Lambda = \mathsf{M}_\Lambda \times 1_\mathsf{W}$, where $1_\mathsf{W} = (1_\mathsf{I}, 1_\mathsf{K})$ is a trivial Abelian crossed module. Correspondingly, the centralizer crossed submodules ZM_Λ and ${\mathsf{ZM}^c}_\Lambda$ of Λ in M and M^c are simply related as ${\mathsf{ZM}^c}_\Lambda = \mathsf{ZM}_\Lambda \times \mathsf{W}$.

The derived Lie groups of M^c and ${\mathsf{ZM}^c}_\Lambda$ factorize as $\mathrm{D}\mathsf{M}^c = \mathrm{D}\mathsf{M} \times \mathrm{D}\mathsf{W}$, $\mathrm{D}{\mathsf{ZM}^c}_\Lambda = \mathrm{D}\mathsf{ZM}_\Lambda \times \mathrm{D}\mathsf{W}$ by the definition of M^c and the factorization of ${\mathsf{ZM}^c}_\Lambda$ shown in the previous paragraph.[11] As a consequence, the derived coadjoint orbit $\mathcal{O}_\Lambda$ of Λ has the homogeneous

[9] A trivial Abelian Lie group crossed module is a Lie group crossed module $\mathsf{W} = (\mathsf{I}, \mathsf{K}, \upsilon, \nu)$, where I, K are Abelian Lie groups and the target and action maps υ and ν are trivial: $\upsilon(Z) = 1_\mathsf{K}$ and $\nu(z, Z) = Z$ for $z \in \mathsf{K}$, $Z \in \mathsf{I}$.

[10] Let $\mathsf{M}_1 = (\mathsf{E}_1, \mathsf{G}_1, \tau_1, \mu_1)$, $\mathsf{M}_2 = (\mathsf{E}_2, \mathsf{G}_2, \tau_2, \mu_2)$ be Lie group crossed modules. The product of M_1, M_2 is the Lie group crossed module $\mathsf{M}_1 \times \mathsf{M}_2 = (\mathsf{E}_1 \times \mathsf{E}_2, \mathsf{G}_1 \times \mathsf{G}_2, \tau_1 \times \tau_2, \mu_1 \times \mu_2)$ with target and action maps $\tau_1 \times \tau_2$, $\mu_1 \times \mu_2$ given by $\tau_1 \times \tau_2(A_1 \times A_2) = \tau_1(A_1) \times \tau_2(A_2)$ and $\mu_1 \times \mu_2(a_1 \times a_2, A_1 \times A_2) = \mu_1(a_1, A_1) \times \mu_2(a_2, A_2)$ for $a_1 \in \mathsf{G}_1$, $a_2 \in \mathsf{G}_2$, $A_1 \in \mathsf{E}_1$, $A_2 \in \mathsf{E}_2$. The product crossed modules $\mathsf{M}_1 \times 1_2 = (\mathsf{E}_1 \times 1_{\mathsf{E}_2}, \mathsf{G}_1 \times 1_{\mathsf{G}_2})$, $1_1 \times \mathsf{M}_2 = (1_{\mathsf{E}_1} \times \mathsf{E}_2, 1_{\mathsf{G}_1} \times \mathsf{G}_2)$, where $1_1 = (1_{\mathsf{E}_1}, 1_{\mathsf{G}_1})$, $1_2 = (1_{\mathsf{E}_2}, 1_{\mathsf{G}_2})$ are trivial Abelian crossed modules, are crossed submodules of $\mathsf{M}_1 \times \mathsf{M}_2$ isomorphic to M_1, M_2, respectively.

[11] For any two Lie group crossed modules M_1, M_2 with product $\mathsf{M}_1 \times \mathsf{M}_2$, the derived Lie groups $\mathrm{D}\mathsf{M}_1$, $\mathrm{D}\mathsf{M}_2$, $\mathrm{D}(\mathsf{M}_1 \times \mathsf{M}_2)$ stand in the relation $\mathrm{D}(\mathsf{M}_1 \times \mathsf{M}_2) = \mathrm{D}\mathsf{M}_1 \times \mathrm{D}\mathsf{M}_2$, as is straightforward to verify.

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space realization $\mathcal{O}_\Lambda = \mathsf{DM}^c/\mathsf{DZM}^c{}_\Lambda$ alternative to the basic realization $\mathcal{O}_\Lambda = \mathsf{DM}/\mathsf{DZM}_\Lambda$ given in (5.1).

If the crossed module M is compact and the trivial Abelian crossed module W is taken to be compact, the balanced crossed module M^c is compact as well. If the group K is further connected, then for any maximal toral crossed submodule J of M the crossed module $\mathsf{J}^c = \mathsf{J} \times \mathsf{W}$ is a maximal toral crossed submodule of M^c. In such a case, as $\mathsf{ZM}^c{}_\Lambda = \mathsf{ZM}_\Lambda \times \mathsf{W}$, if Λ is regular with respect to the crossed module structure of M, it is also regular with respect to that of M^c.

The above discussion shows that we are allowed to substitute the crossed module M with a balanced extension M^c without changing the geometry the derived KKS theory describes.

The inclusion of an invariant pairing in our construction constitutes a fresh element of the theory not considered up to this point. Given its crucial ilportance, the natural question arises about the existence of an invariant pairing on a balanced crossed module M. We cannot solve this issue in full generality. Existence can nevertheless be proven under fairly broad assumptions.

To state our result, we need to introduce a few basic notions of crossed module theory. A Lie group crossed module M (not necessarily balanced) is said to be inert on the target kernel if $\mu\dot{}(a, X) = X$ for $a \in \mathsf{G}$ and $X \in \ker \dot{\tau}$. M is called inert on the target cokernel if $\operatorname{Ad} a(x) = x$ for $a \in \mathsf{G}$ and $x \in \mathfrak{g}/\operatorname{im}\dot{\tau}$. It is not difficult to verify that if M is inert on the target kernel, respectively cokernel, then any balanced extension M^c of M also is.

If M is a balanced crossed module that is compact and inert on the target kernel and cokernel, then M admits and invariant pairing $\langle \cdot, \cdot \rangle$. The key point of the proof is the existence of a left and right invariant normalized Haar measure ω_{G} on the target group G ensured by the compactness of this latter. By the classic method of averaging on G, ω_{G} allows one to construct an invariant pairing from a given non invariant one.[12] We observe however that the above conditions on M are only sufficient for the occurrence of an invariant pairing, which may exist even when such conditions are not met.

As an illustration, we examine again the model crossed modules introduced in subsection 5.1. Consider the Lie group crossed module $\mathrm{I\scriptsize{NN}}_{\mathsf{G}}\,\mathsf{N} = (\mathsf{N}, \mathsf{G}, \iota, \kappa)$ associated with a normal subgroup N of a Lie group G. Since $\dim \mathsf{N} \leqslant \dim \mathsf{G}$, $\mathrm{I\scriptsize{NN}}_{\mathsf{G}}\,\mathsf{N}$ is not balanced in general. A minimal choice of a balanced extension of $\mathrm{I\scriptsize{NN}}_{\mathsf{G}}\,\mathsf{N}$ would be $\mathrm{I\scriptsize{NN}}_{\mathsf{G}}\,\mathsf{N}^c = (\mathsf{N} \times \mathsf{I}, \mathsf{G})$, where I is an Abelian group such that $\dim \mathsf{I} = \dim \mathsf{G} - \dim \mathsf{N}$. Since $\ker \dot{\iota} = 0_{\mathfrak{n}}$ in the present case, $\mathrm{I\scriptsize{NN}}_{\mathsf{G}}\,\mathsf{N}$ and thus $\mathrm{I\scriptsize{NN}}_{\mathsf{G}}\,\mathsf{N}^c$ are trivially inert on the target kernel. Conversely, $\mathrm{I\scriptsize{NN}}_{\mathsf{G}}\,\mathsf{N}$ and $\mathrm{I\scriptsize{NN}}_{\mathsf{G}}\,\mathsf{N}^c$ are not inert on the target cokernel in general, because the adjoint action of

[12] In outline the proof runs as follows. The singular value decomposition theorem furnishes a simple expression of $\dot{\tau}$: $\dot{\tau} = \sum_i t_i u_i \otimes U_i{}^t$, where the scalars $t_i \geqslant 0$ and the vectors $u_i \in \mathfrak{g}$ and $U_i \in \mathfrak{e}$ constitute orthonormal bases of $\mathfrak{g}$ and $\mathfrak{e}$ with respect to G-invariant inner products $(\cdot,\cdot)_{\mathfrak{g}}$ and $(\cdot,\cdot)_{\mathfrak{e}}$ of $\mathfrak{g}$ and $\mathfrak{e}$, the G-actions being Ad and $\mu\dot{}$ for $\mathfrak{g}$ and $\mathfrak{e}$, respectively. Using the u_i and U_i, we can construct a non singular bilinear map $\langle \cdot, \cdot \rangle_0 : \mathfrak{g} \times \mathfrak{e} \to \mathbb{R}$ by setting $\langle x, X \rangle_0 = \sum_i (x, u_i)_{\mathfrak{g}} (U_i, X)_{\mathfrak{e}}$ with $x \in \mathfrak{g}$, $X \mathfrak{e}$. $\langle \cdot, \cdot \rangle_0$ enjoys the symmetry property (3.2) but not the invariance property (3.3) in general. This can be enforced by averaging on G, that is by setting $\langle x, X \rangle = \int_{\mathsf{G}} d\omega_{\mathsf{G}}(g) \langle \operatorname{Ad} g(x), \mu\dot{}(g, X) \rangle_0$. The property of inertness on the target kernel and cokernel are sufficient to ensure the non singularity of $\langle \cdot, \cdot \rangle$.

G on $\mathfrak{g}/\mathfrak{n}$ needs not be trivial. They are only if $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{n}$. In such a case, provided G is compact, an invariant pairing can be constructed on $\mathrm{I{\scriptstyle NN}}_{\mathsf{G}}\,\mathsf{N}^c$.

Consider next the Lie group crossed module $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G}) = (\mathsf{Q}, \mathsf{G}, \pi, \alpha)$ associated with a central extension $1 \longrightarrow \mathsf{C} \xrightarrow{\iota} \mathsf{Q} \xrightarrow{\pi} \mathsf{G} \longrightarrow 1$. Since $\dim \mathsf{Q} \geqslant \dim \mathsf{G}$, $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G})$ is not balanced in general. A minimal choice of a balanced extension of $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G})$ would be $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G})^c = (\mathsf{Q}, \mathsf{G} \times \mathsf{K})$, where K is an Abelian group such that $\dim \mathsf{K} = \dim \mathsf{Q} - \dim \mathsf{G}$, e.g. $\mathsf{K} = \mathsf{C}$. Since $\dot{\iota}(\mathfrak{c})$ is central in $\mathfrak{q}$ and $\dot{\alpha}(a, \cdot) = \mathrm{Ad}\,\sigma(a)$ for a section $\sigma : \mathsf{G} \to \mathsf{Q}$ of π, $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G})$ and thus $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G})^c$ are inert on the target kernel. $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G})$ and $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G})^c$ are trivially inert on the target cokernel since $\operatorname{im} \dot{\pi} = \mathfrak{g}$. In this way, when G is compact, the existence of an invariant pairing on $\mathsf{C}(\mathsf{Q} \xrightarrow{\pi} \mathsf{G})^c$ is ensured.

Consider finally the Lie group crossed modules $\mathsf{D}(\rho) = (\mathsf{V}, \mathsf{G}, 1_{\mathsf{G}}, \rho)$, where ρ is a representation of Lie group G on the vector space V. $\mathsf{D}(\rho)$ needs not be balanced in general. A minimal balanced extension of $\mathsf{D}(\rho)$ is of the form $\mathsf{D}(\rho)^c = (\mathsf{V} \times \mathsf{I}, \mathsf{G})$ or $\mathsf{D}(\rho)^c = (\mathsf{V}, \mathsf{G} \times \mathsf{K})$ depending on whether $\dim \mathsf{V} \leqslant \dim \mathsf{G}$ or $\dim \mathsf{V} \geqslant \dim \mathsf{G}$ for Abelian groups I or K of suitable dimensions, respectively. Since $\ker \dot{1}_{\mathsf{G}} = \mathsf{V}$, $\mathsf{D}(\rho)$ and $\mathsf{D}(\rho)^c$ are inert on the target kernel only if ρ is trivial. Since $\operatorname{im} \dot{1}_{\mathsf{G}} = 0_{\mathfrak{g}}$, $\mathsf{D}(\rho)$ and $\mathsf{D}(\rho)^c$ are inert on the target cokernel only if $\mathfrak{g}$ is Abelian. For G compact, under such restrictions, then the existence of an invariant pairing on $\mathsf{D}(\rho)^c$ is ensured. Clearly the set-up emerging here is rather trivial.

The character (5.185) enters the derived KKS theory much as the character (4.34) enters the standard one. Indeed, as explained in subsection 5.5, ξ_Λ defines a character of J, $\beta_\Lambda = (\Xi_\Lambda, \xi_\Lambda)$ where $\Xi_\Lambda = \xi_\Lambda \circ \tau$, and therefore a derived unitary line bundle $\mathcal{L}_\Lambda := \mathcal{L}_{\beta_\Lambda}$ on $\mathrm{D}\mathsf{M}/\mathrm{D}\mathsf{J}$ through the geometrical construction we detailed there.

We are going to equip the derived line bundle $\mathcal{L}_\Lambda$ with a canonical unitary connection A_Λ built upon the above data only. This connection is analogous to the canonical connection (4.35) of a regular dual integral lattice element of ordinary KKS theory and is indeed the appropriate derived enhancement of this latter. In the framework of subsection 5.2, the components a_Λ, A_Λ of A_Λ are expressed in terms of the Maurer-Cartan elements σ, Σ of $\mathrm{D}\mathsf{M}$ and read as

$$a_\Lambda = -i\langle \sigma, \Lambda \rangle, \tag{5.186}$$

$$A_\Lambda = -i\langle \dot{\tau}(\Lambda), \Sigma \rangle. \tag{5.187}$$

Employing relations (5.23)–(5.26), it is readily checked that a_Λ, A_Λ obey relations (5.100)–(5.103) and so are the components of $\mathcal{L}_\Lambda$.

The curvature B_Λ of the connection A_Λ is the object that interests us most because of its eventual relation to the derived KKS symplectic structure. It is as expected a derived enhancement of the curvature of the ordinary KKS theory canonical connection shown in (4.36). The components b_Λ, B_Λ of B_λ can be readily obtained from the components a_Λ, A_Λ of A_Λ given in (5.186), (5.187) employing relations (5.104), (5.105),

$$b_\Lambda = \frac{i}{2}\langle [\sigma, \sigma], \Lambda \rangle, \tag{5.188}$$

$$B_\Lambda = i\langle \dot{\tau}(\Lambda), \dot{\mu}\dot{}(\sigma, \Sigma) \rangle. \tag{5.189}$$

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It can be checked exploiting identities (5.13), (5.14) that b_Λ, B_Λ obey the Bianchi identities (5.106), (5.107) as required. It can also be verified using relations (5.23)–(5.26) that they obey relations (5.108)–(5.111).

The analysis of subsection 5.6 shows that $-i\mathrm{B}_\Lambda$ constitutes a derived presymplectic structure on DM/DJ. As $j_V\mathrm{B} = 0$ for a vector field $V \in \mathrm{V\small{ECT}}(\mathsf{DM})$ only if $V = S_{\mathrm{RX}}$ for some $\mathrm{X} \in \mathsf{Dj}$ pointwise on DM, $-i\mathrm{B}_\Lambda$ turns out to be non singular and so a derived symplectic structure, the derived KKS Λ-structure. With it, there is associated a derived Poisson structure with brackets $\{\cdot,\cdot\}_\Lambda = \{\cdot,\cdot\}_{\mathrm{A}_\Lambda}$.

Just as the standard KKS set-up of subsection 4.6 is fully left invariant, the derived KKS set-up constructed above is invariant under the left target kernel symmetry dealt with in subsection 5.3. The connection A_Λ defined componentwise by eqs. (5.186), (5.187) is invariant as a consequence of the invariance of the Maurer-Cartan map Σ. Indeed, relations (5.43), (5.44) evidently imply that A satisfies the invariance condition (5.149). In this way, also the associated curvature B_Λ, given componentwise by eqs. (5.188), (5.189), satisfies (5.150) and so is invariant.

For reasons detailed in subsection 5.7 in full generality, in derived KKS theory the target kernel DM_τ-symmetry action on DM/DJ is Hamiltonian with respect to the derived KKS Λ-structure similarly to its counterpart in ordinary KKS theory. The components of the associated derived moment map Q_A are

$$q_\Lambda(h,H) = \langle h, \dot{\mu}(\gamma,\Lambda)\rangle\,, \tag{5.190}$$

$$Q_\Lambda(h,H) = \langle h, [\Gamma, \dot{\mu}(\gamma,\Lambda)]\rangle\,. \tag{5.191}$$

This moment map manifestly is the proper derived counterpart of the ordinary KKS theory moment map (4.37). Using (5.11), (5.12) and (5.37)–(5.40), it is straightforward to verify relations (5.162), (5.163). The defect map components (5.164), (5.165) vanish as a consequence of (5.41), (5.42) and so the connection A_Λ turns out to be strict. The Poisson bracket relations (5.166), (5.167) thus hold strictly, i.e. with vanishing right hand sides.

The structure described above enjoys natural invariance properties under the derived automorphism group action of the principal DJ-bundle DM similarly again to the ordinary KKS theory surveyed in subsection 4.6. Indeed, we can associate with any automorphism Ψ of DM a gauge transformation U_Λ of the derived line bundle $\mathcal{L}_\Lambda$. This gauge transformation is analogous to the gauge transformation (4.38) one associates with an automorphism of ordinary KKS theory and is indeed again a derived extension of this latter. In the description of automorphisms expounded in subsection 5.4, the components u_Λ, U_Λ of U_Λ are expressed in terms of the components ψ, $\varPsi$ of Ψ as follows,

$$u_\Lambda = \exp\left(-i\int_{1_{\mathsf{G}}}^{\cdot}\langle d\psi\psi^{-1},\Lambda\rangle\right), \tag{5.192}$$

$$U_\Lambda = -i\langle\dot{\tau}(\Lambda),\varPsi\rangle. \tag{5.193}$$

Since $d\psi\psi^{-1} \in \mathrm{M\small{AP}}(T[1]\,\mathsf{DM},\mathfrak{t}[1])$ can be equated to an element of $\Omega^1(\mathsf{G},\mathfrak{t})$ with periods in the lattice Λ_{G}, u_Λ is singlevalued as required by virtue of the integrality property of Λ. The choice of 1_{G} as base point of integration is conventional but natural. We are assuming here that G is connected. Employing relations (5.55)–(5.58), it is readily checked that u_Λ, U_Λ obey relations (5.116)–(5.119) and thus are the components of a gauge transformation of $\mathcal{L}_\Lambda$.

The gauge transformation U_Λ is invariant as a consequence of the invariance of automorphism map Ψ, as relation (5.67), (5.68) entail that U_Λ obeys the invariance condition (5.151).

A mapping $\Psi \to \mathrm{U}_\Lambda$ from the automorphism group $\mathrm{A{\scriptsize UT}}_{\mathrm{D}\mathsf{J}}(\mathrm{D}\mathsf{M})$ of $\mathrm{D}\mathsf{M}$ to the gauge transformation group $\mathrm{G{\scriptsize AU}}(\mathcal{L}_\Lambda)$ of $\mathcal{L}_\Lambda$ is established by (5.192), (5.193). As is easily checked, this mapping is a group morphism as a consequence of the identity of ZM_Λ and J.

For an automorphism $\Psi \in \mathrm{A{\scriptsize UT}}_{\mathrm{D}\mathsf{J}}(\mathrm{D}\mathsf{M})$, let ${}^{\Psi}\mathrm{A}_\Lambda$ be the connection whose components ${}^{\psi,\Psi}a_\Lambda$, ${}^{\psi,\Psi}A_\Lambda$ are given by (5.186), (5.187) with σ, $\varSigma$ replaced by their transforms ${}^{\psi,\Psi}\sigma$, ${}^{\psi,\Psi}\varSigma$ (cf. eqs. (5.61), (5.62)). Then, ${}^{\psi,\Psi}a_\Lambda = {}^{u_\Lambda,U_\Lambda}a_\Lambda$, ${}^{\psi,\Psi}A_\Lambda = {}^{u_\Lambda,U_\Lambda}A_\Lambda$, where ${}^{u_\Lambda,U_\Lambda}a_\Lambda$, ${}^{u_\Lambda,U_\Lambda}A_\Lambda$ are given in terms of u_Λ, U_Λ by (5.122), (5.123). Consequently, the curvature B_Λ is automorphism invariant, ${}^{\psi,\Psi}b_\Lambda = b_\Lambda$, ${}^{\psi,\Psi}B_\Lambda = B_\Lambda$. The moment map Q_Λ is also invariant, ${}^{\psi,\Psi}q_\Lambda = q_\Lambda$, ${}^{\psi,\Psi}Q_\Lambda = Q_\Lambda$.

We concluded this subsection by noting that the connection A_Λ is of 1-form type as is immediate to realize by inspection of (5.186), (5.187). Since its curvature B_Λ is also non singular as already noticed, the incomplete derived prequantization procedure described in subsection 5.8 can be implemented in the present case.

5.11 Conclusions

In this section, we have gone a long way toward a complete and satisfactory formulation of higher KKS theory. We have done so in the derived framework, that seems to be ideally suited for this purpose. As we have seen, derived prequantization suffers certain limitation which we have detailed. We believe that the reason for this is ultimately that the appropriate geometric quantization of derived KKS theory, whatever form it takes, cannot have some kind of quantum mechanical model, albeit exotic, as its end result but a two dimensional quantum field theory. This seems to give further support to standard expectations that derived KKS theory can be regarded as some kind of categorification of the ordinary theory. The TCO model elaborated in paper II, whose geometric foundation is provided precisely by derived KKS theory, is an attempt to concretize the above intuitions.

A Appendices

The following appendices review certain nonstandard notions whose knowledge is tacitly assumed throughout the paper. Our presentation has no pretence of rigour and completeness and serves mainly the purpose of facilitating the reading.

A.1 Generalities on the graded geometric set-up

In this paper, we adhere to a graded geometric perspective. This provides an especially natural language for the description the higher geometric structures encountered along the way. The reader is referred to e.g. ref. [64] for a readable introduction to graded differential geometry and its applications. Here, we shall limit ourselves to discuss a few conceptual issues.

Adopting the most common standpoint of the physical literature, we shall describe a graded manifold X using local coordinates. As well-known, these divide into body and soul coordinates x^a and ξ^r characterized by integer degrees. The coordinates should be regarded as formal parameters. The topology and geometry of X is encoded in the transition functions relating sets of coordinates x^a, ξ^r and $\tilde{x}^a$, $\tilde{\xi}^r$. These are the degree 0 smooth functions $f^a(x)$, $f^r{}_{r_1\ldots r_p}(x)$ appearing in the coordinate change relations

$$\tilde{x}^a = f^a(x), \tag{A.1}$$

$$\tilde{\xi}^r = \textstyle\sum_{p\geqslant 0} f^r{}_{r_1\ldots r_p}(x)\xi^{r_1}\cdots\xi^{r_p}. \tag{A.2}$$

The summation occurring in (A.2) is in principle infinite. To avoid to be embroiled in subtle problems concerning the proper treatment of such formal expressions, we shall tacitly consider only non negatively graded manifolds unless otherwise stated. For these, the soul coordinate degree $\deg\xi^r$ are all strictly positive and so all the summations such as the above finite.

At several points in our analysis the distinction between ordinary and internal functions and maps, albeit technical, will play an important role. To the reader's convenience, we recall briefly the difference between these two types of maps. Let X and Y be graded manifolds and $\varphi : Y \to X$ a smooth map. In terms of local coordinates x^a, ξ^r and y^i, η^h of X and Y, φ has an expression of the form

$$\varphi^a(y,\eta) = \textstyle\sum_{p\geqslant 0}\varphi^a{}_{h_1\ldots h_p}(y)\eta^{h_1}\cdots\eta^{h_p}, \tag{A.3}$$

$$\varphi^r(y,\eta) = \textstyle\sum_{p\geqslant 0}\varphi^r{}_{h_1\ldots h_p}(y)\eta^{h_1}\cdots\eta^{h_p}, \tag{A.4}$$

where $\varphi^a{}_{h_1\ldots h_p}(y)$, $\varphi^r{}_{h_1\ldots h_p}(y)$ are smooth functions. If φ is an ordinary map, the $\varphi^a{}_{h_1\ldots h_p}(y)$, $\varphi^r{}_{h_1\ldots h_p}(y)$ have all degree 0. If instead φ is internal, the $\varphi^a{}_{h_1\ldots h_p}(y)$, $\varphi^r{}_{h_1\ldots h_p}(y)$ may have non zero internal degree. As above, the summations occurring in (A.3), (A.4) have generally an infinite range. For the non negatively graded manifolds which we restrict to, the degrees of the $\phi^a{}_{h_1\ldots h_p}(y)$, $\phi^r{}_{h_1\ldots h_p}(y)$ are allowed to take only non negative values and the summations are again all finite. We distinguish the ordinary and internal map sets by employing the generic notation Fun and Map for the former and Fun and Map for the latter. In section 3, the derived set-up is formulated mostly in the internal case, since internal functions provide a broader function range. The ordinary case can be treated anyway essentially in the same way.

We can associate a clone manifold $X^+ = \mathrm{M{\scriptsize AP}}(*,X)$ with any graded manifold X, where $*$ is the singleton manifold. Inspection of (A.3), (A.4) for the case where $Y = *$ reveals that X^+ provides a description of the 'range' of any assigned local coordinate system x^a, ξ^r of X, since the points of the range are in bijective correspondence with a subset of maps of X^+. Working with X^+ furnishes in this manner an index free way of encoding coordinate expressions. While this does not constitute by itself a sufficient reason for a systematic application of cloning in general, it is when X is endowed with a linear structure that the usefulness of cloning becomes apparent as we illustrate next.

Let E be a negatively graded vector space. Then, the dual E^* of E is a positively graded vector space. The coordinate functions of E, as elements of E^*, have thus positive degrees. With any basis x_a of E, there is associated a full set of coordinate functions of E, namely the basis x^a of E^* dual to x_a. A basis change $x_a \mapsto \tilde{x}_a$ in E induces a linear transformation $x^a \mapsto \tilde{x}^a$. In this way, upon treating the x^a as formal parameters, E can be viewed as a positively graded manifold. So, while E is negatively graded as a vector space, it is positively graded as a manifold. Because of this sign mismatch between vector and geometric grading, the geometric structure of E cannot be directly described through the vector structure. However, since the clone manifold E^+ of E is naturally a positively graded vector space with a vector grading content matching the geometric one of E,[13] the geometric properties of E can be expressed in principle as vector properties of E^+ in a very convenient and natural manner. Similar considerations apply to negatively graded vector bundles.

Though the difference between a vector space or bundle E and its clone E^+ matters if one is to have the correct grading matching as explained above, E and E^+ may be harmlessly confused in practice in most instances. We decided so, albeit with some hesitation, that keeping the distinction between E and E^+ manifest may burden notation unnecessarily. Throughout this paper, therefore, we shall not discriminate notationally between E and E^+. It will be clear from context which is which. In our analysis, we shall mainly focus on graded geometric aspects and correspondingly it will be the clone space that will be tacitly considered. At any rate, the reader not interested in fine technical points such as this and content with mere formal manipulations can ignore cloning altogether.

A.2 Differential forms as graded functions

Differential forms on a manifold X can be described as graded functions on the shifted tangent bundle $T[1]X$. We have indeed a graded algebra isomorphism $\Omega^*(X) \simeq \mathrm{Fun}(T[1]X)$. This well-known property will be extensively exploited in the present work. In the graded geometric framework we are adopting, it is definitely more natural to treat forms as graded functions. This standpoint has also the advantage of extending the range of manipulations which can be performed with forms. We shall also consider internal differential forms and deal with these as internal functions exploiting the isomorphism $\boldsymbol{\Omega}^*(X) \simeq \textsc{Fun}(T[1]X)$.

A.3 The operational approach

The operational approach underlies many of the constructions of ordinary and derived KKS theory as well as the related ordinary and derived TCO models for its aptness to describe the basic geometry of principal bundles, in particular homogeneous spaces, and provides a efficient framework for the application of standard cohomological methods such as transgression. For this reason, we review its basic notions here. An exhaustive exposition of the subject and its applications to topology and geometry can be found in ref. [65].

[13] The difference between E and E^+ can be illustrated somewhat more explicitly as follows. Let E^0 the ungraded vector space underlying E and ${x^0}_a$ the basis of E^0 corresponding to a basis x_a of E. Then, E consists of the linear combinations $c^a x_a$ with $c^a \in \mathbb{R}$, while E^+ of the formal expressions $x^a \otimes {x^0}_a$.

Suppose that $\mathfrak{f}$ is a Lie algebra and A an associative commutative graded algebra. An $\mathfrak{f}$-operation on A consists of a degree -1 derivation j_x and a degree 0 derivation l_x for every $x \in \mathfrak{f}$ and a degree 1 derivation d of A obeying the relations

$$[j_x, j_y] = 0, \tag{A.5}$$

$$[l_x, j_y] = j_{[x,y]}, \qquad [l_x, l_y] = l_{[x,y]}, \tag{A.6}$$

$$[j_x, d] = l_x, \qquad [l_x, d] = 0, \tag{A.7}$$

$$[d, d] = 0 \tag{A.8}$$

for $x, y \in \mathfrak{f}$. All commutators shown above are graded.

Suppose P is a manifold, $\mathrm{Vect}(P)$ the Lie algebra of vector fields of P and $\mathrm{Fun}(T[1]P)$ the graded algebra of functions on the shifted tangent bundle $T[1]P$ of P. Then, there exists a canonical $\mathrm{Vect}(P)$-operation on $\mathrm{Fun}(T[1]P)$, customarily called Cartan calculus of P. Under the isomorphism $\mathrm{Fun}(T[1]P) \simeq \Omega^*(P)$ recalled in appendix A.2, this consists of the familiar degree -1 contractions j_V and degree 0 Lie derivatives l_V along the vector fields $V \in \mathrm{Vect}(P)$ and the de Rham differential d, all realized as graded vector fields on $T[1]P$. The action of such derivations extends to the mapping spaces $\mathrm{Map}(T[1]P, X)$, where X is a graded manifold, as an action on functions on $T[1]P$ using coordinates of X.

Suppose that F is a Lie group, B is a manifold and P is a principal F-bundle over B. With the right action of F on P, there is associated an $\mathfrak{f}$-operation. comprising the contractions j_x and Lie derivatives l_x along the vertical vector fields $S_x \in \mathrm{Vect}(P)$ of the right F-action of P associated with the Lie algebra elements $x \in \mathfrak{f}$ and the degree 1 de Rham differential d of P. Since $B = P/\mathsf{F}$, the operational set-up allows for an elegant description of the differential geometry of B. The function algebra $\mathrm{Fun}(T[1]B)$ of $T[1]B$ can be identified with the basic subalgebra of $\mathrm{Fun}(T[1]P)$, the joint kernel of the derivations j_x and l_x with $x \in \mathfrak{f}$. Similarly, the mapping space $\mathrm{Map}(T[1]B, X)$ can be identified with a basic subspace of $\mathrm{Map}(T[1]P, X)$.

The operational approach, can be adapted straightforwardly to the case where the principal bundle P, its base B and its structure group F are positively graded manifolds. One important aspect distinguishing an operational set-up of graded manifolds from an ordinary one is that by consistency the underlying graded algebra is the internal function algebra $\mathrm{F{\scriptstyle UN}}(T[1]P)$ rather than the ordinary function algebra $\mathrm{Fun}(T[1]P)$, because of the graded nature of $\mathfrak{f}$. Besides, the whole formal apparatus works much in the same way.

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